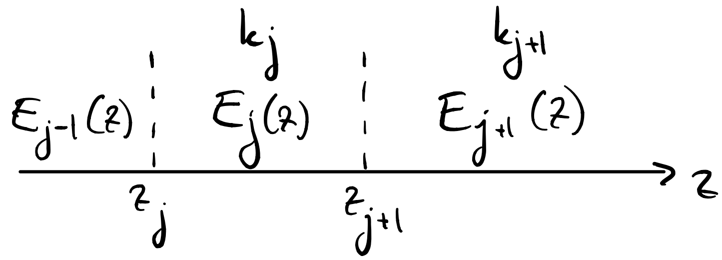

Figure 1 – Caption

1 Reflectometry

1.1 This

$$R(q_z) = R_F \left| \frac{1}{\rho_S} \int \left. \frac{\mathrm{d}\rho}{\mathrm{d}z} \right|_z e^{-iq_z z} \, \mathrm{d}z \right|^2$$

1.2 That

The propagation of light as a magneto-electric wave of wave vector $\mathbf{k}$ is given by,

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \qquad \mathbf{B}(\mathbf{r}, t) = \frac{\hat{\mathbf{k}} \times \mathbf{E}(\mathbf{r}, t)}{v}$$

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \qquad \omega\mathbf{B}(\mathbf{r}, t) = \mathbf{k} \times \mathbf{E}(\mathbf{r}, t)$$

In each layer j, there is a wave incident (I) and a reflected (R) wave,

$$\begin{aligned} \mathbf{E}_j(\mathbf{r}, t) &= \left[\mathbf{E}_{0j}^{(\mathrm{I})} \exp\left(i\mathbf{k}_j^{(\mathrm{I})} \cdot \mathbf{r}\right) + \mathbf{E}_{0j}^{(\mathrm{R})} \exp\left(i\mathbf{k}_j^{(\mathrm{R})} \cdot \mathbf{r}\right)\right] e^{-i\omega t} \\ \omega\mathbf{B}_j(\mathbf{r}, t) &= \left[\mathbf{k}_j^{(\mathrm{I})} \times \mathbf{E}_{0j}^{(\mathrm{I})} \exp\left(i\mathbf{k}_j^{(\mathrm{I})} \cdot \mathbf{r}\right) + \mathbf{k}_j^{(\mathrm{R})} \times \mathbf{E}_{0j}^{(\mathrm{R})} \exp\left(i\mathbf{k}_j^{(\mathrm{R})} \cdot \mathbf{r}\right)\right] e^{-i\omega t} \end{aligned}$$

Wave vector and group speed are related,

$$k_j^{(\mathrm{I,R})} v_j = \omega \qquad k_j^{(\mathrm{I,R})} = |\mathbf{k}_j^{(\mathrm{I,R})}|$$

$$n_j v_j = c \qquad \frac{n_j}{k_j^{(\mathrm{I,R})}} = \frac{c}{\omega} = \frac{\lambda}{2\pi}$$

The interface between layer $j-1$ and j is define as $\mathbf{r}_j$.

$$\mathbf{r}_j = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z_j\hat{\mathbf{z}}$$

The electric field component at the interface need to respect the following equation

$$\begin{aligned} \big(\mathbf{E}_{j-1}(\mathbf{r}_j, t)\big)_{x,y} &= \big(\mathbf{E}_j(\mathbf{r}_j, t)\big)_{x,y} \\ \epsilon_{j-1}\big(\mathbf{E}_{j-1}(\mathbf{r}_j, t)\big)_z &= \epsilon_j\big(\mathbf{E}_j(\mathbf{r}_j, t)\big)_z \end{aligned}$$

while the magnetic field components,

$$\frac{1}{\mu_{j-1}}\big(\mathbf{B}_{j-1}(\mathbf{r}_j,t)\big)_{x,y} = \frac{1}{\mu_j}\big(\mathbf{B}_j(\mathbf{r}_j,t)\big)_{x,y}$$
$$\big(\mathbf{B}_{j-1}(\mathbf{r}_j,t)\big)_z = \big(\mathbf{B}_j(\mathbf{r}_j,t)\big)_z$$

Note that, at the frequency of X rays, the relative magnetic permeabilty of matter is unity up to many orders of magnitude, (Snoek's limit)

$$\mu_j \sim \mu_0 \qquad\qquad \frac{\Delta\mu}{\mu_0} \ll \frac{\Delta\epsilon}{\epsilon_0}$$

As an example, the continuity of the x component of the electric field gives the following equation,

$$E^{(\mathrm{I})}_{0z,j-1}\exp\left(i\mathbf{k}^{(\mathrm{I})}_{j-1}\cdot\mathbf{r}\right) + E^{(\mathrm{R})}_{0z,j-1}\exp\left(i\mathbf{k}^{(\mathrm{R})}_{j-1}\cdot\mathbf{r}\right) = E^{(\mathrm{I})}_{0z,j}\exp\left(i\mathbf{k}^{(\mathrm{I})}_{j}\cdot\mathbf{r}\right) + E^{(\mathrm{R})}_{0z,j}\exp\left(i\mathbf{k}^{(\mathrm{R})}_{j}\cdot\mathbf{r}\right)$$

The x and y dependance must be the same for each term of this equation, meaning that, in each layer we must have

$$k_{x,j} = k^{(\mathrm{I})}_{x,j} = k^{(\mathrm{R})}_{x,j} \qquad\qquad k_{y,j} = k^{(\mathrm{I})}_{y,j} = k^{(\mathrm{R})}_{y,j}$$

and therefore

$$k_{z,j} = k^{(\mathrm{I})}_{z,j} = -k^{(\mathrm{R})}_{z,j}.$$

Between planes

$$k_{x,j-1} = k_{x,j} \qquad\qquad k_{y,j-1} = k_{y,j}$$

If we define θ_j as the angle between the interface and $\mathbf{k}_j$, we get the following relations (Snell's law).

$$k_{j-1}\cos\theta_{j-1} = k_j\cos\theta_j \qquad\qquad \frac{\cos\theta_j}{\cos\theta_{j-1}} = \frac{k_{j-1}}{k_j} = \frac{n_{j-1}}{n_j} = \left(\frac{\epsilon_{j-1}\mu_{j-1}}{\epsilon_j\mu_j}\right)^{1/2} \sim \left(\frac{\epsilon_{j-1}}{\epsilon_j}\right)^{1/2}$$

If the first incident beam is k_0 at an angle θ_0, this means,

$$k_0\cos\theta_0 = k_j\cos\theta_j \qquad\qquad \frac{k_0}{n_0} = \frac{k_j}{n_j}$$

We will choose the direction of propagation in order to have $k_y = 0$.

$$\begin{aligned} k_{z,j} &= k_j\sin\theta_j \\ &= k_0 n_j\big(1-\cos^2\theta_j\big)^{1/2} \\ &= k_0\left(n_j^2 - n_j^2\cos^2\theta_j\right)^{1/2} \\ &= k_0\left(n_j^2 - n_0^2\cos^2\theta_0\right)^{1/2} \end{aligned}$$

1.3 In plane polarisation

If the polarization of the electric field is in the propagration plane (xz), the components of the electric field for the incident and reflected waves are,

$$
\begin{aligned}
E_{0x,j}^{(\mathrm{I,p})} &= E_{0,j}^{(\mathrm{I,p})} \sin\theta_j & \qquad E_{0x,j}^{(\mathrm{R,p})} &= -E_{0,j}^{(\mathrm{R,p})} \sin\theta_j \\
E_{0y,j}^{(\mathrm{I,p})} &= 0 & E_{0y,j}^{(\mathrm{R,p})} &= 0 \\
E_{0z,j}^{(\mathrm{I,p})} &= E_{0,j}^{(\mathrm{I,p})} \cos\theta_j & E_{0z,j}^{(\mathrm{R,p})} &= E_{0,j}^{(\mathrm{R,p})} \cos\theta_j
\end{aligned}
$$

while the components of the magnetic field are,

$$
\begin{aligned}
B_{0x,j}^{(\mathrm{I,p})} &= 0 & \qquad B_{0x,j}^{(\mathrm{R,p})} &= 0 \\
\omega B_{0y,j}^{(\mathrm{I,p})} &= k_j^{(\mathrm{I,p})} E_{0,j}^{(\mathrm{I,p})} & \omega B_{0y,j}^{(\mathrm{R,p})} &= k_j^{(\mathrm{R,p})} E_{0,j}^{(\mathrm{R,p})} \\
B_{0z,j}^{(\mathrm{I,p})} &= 0 & B_{0z,j}^{(\mathrm{R,p})} &= 0
\end{aligned}
$$

The relevant equations are

$$
\begin{aligned}
\sin\theta_{j-1}\left(E_{0,j-1}^{(\mathrm{I,p})} e^{ik_{z,j-1}z_j} - E_{0,j-1}^{(\mathrm{R,p})} e^{-ik_{z,j-1}z_j}\right) &= \sin\theta_j \left(E_{0,j}^{(\mathrm{I,p})} e^{ik_{z,j}z_j} - E_{0,j}^{(\mathrm{R,p})} e^{-ik_{z,j}z_j}\right) \\
\epsilon_{j-1}\cos\theta_{j-1}\left(E_{0,j-1}^{(\mathrm{I,p})} e^{ik_{z,j-1}z_j} + E_{0,j-1}^{(\mathrm{R,p})} e^{-ik_{z,j-1}z_j}\right) &= \epsilon_j \cos\theta_j \left(E_{0,j}^{(\mathrm{I,p})} e^{ik_{z,j}z_j} + E_{0,j}^{(\mathrm{R,p})} e^{-ik_{z,j}z_j}\right) \\
k_{j-1}\left(E_{0,j-1}^{(\mathrm{I,p})} e^{ik_{z,j-1}z_j} + E_{0,j-1}^{(\mathrm{R,p})} e^{ik_{z,j-1}z_j}\right) &= k_j \left(E_{0,j}^{(\mathrm{I,p})} e^{ik_{z,j}z_j} + E_{0,j}^{(\mathrm{R,p})} e^{ik_{z,j}z_j}\right)
\end{aligned}
$$

Since,

$$
\frac{\epsilon_{j-1}\cos\theta_{j-1}}{\epsilon_j \cos\theta_j} = \frac{k_{j-1}}{k_j} \qquad \rightarrow \qquad \frac{\cos\theta_{j-1} k_j}{\cos\theta_j k_{j-1}} = \frac{\epsilon_j}{\epsilon_{j-1}}
$$

the third equation is equivalent to the second one.

1.4 Out of plane polarisation

If the polarization of the electric field is out of the propagration plane (y), the components of the electric field for the incident and reflected waves are,

$$
\begin{aligned}
E_{0x,j}^{(\mathrm{I,s})} &= 0 & \qquad E_{0x,j}^{(\mathrm{R,s})} &= 0 \\
E_{0y,j}^{(\mathrm{I,s})} &= E_{0,j}^{(\mathrm{I,s})} & E_{0y,j}^{(\mathrm{R,s})} &= E_{0,j}^{(\mathrm{R,s})} \\
E_{0z,j}^{(\mathrm{I,s})} &= 0 & E_{0z,j}^{(\mathrm{R,s})} &= 0
\end{aligned}
$$

while the components of the magnetic field are,

$$
\begin{aligned}
\omega B_{0x,j}^{(\mathrm{I,s})} &= k_j E_{0,j}^{(\mathrm{I,s})} \sin\theta_j & \qquad \omega B_{0x,j}^{(\mathrm{R,s})} &= -k_j E_{0,j}^{(\mathrm{R,s})} \sin\theta_j \\
B_{0y,j}^{(\mathrm{I,s})} &= 0 & B_{0y,j}^{(\mathrm{R,s})} &= 0 \\
\omega B_{0z,j}^{(\mathrm{I,s})} &= k_j E_{0,j}^{(\mathrm{I,s})} \cos\theta_j & \omega B_{0z,j}^{(\mathrm{R,s})} &= k_j E_{0,j}^{(\mathrm{R,s})} \cos\theta_j
\end{aligned}
$$

The relevant equations are then,

$$
\begin{aligned}
E^{(\mathrm{I,s})}_{0,j-1} e^{ik_{z,j-1}z_j} + E^{(\mathrm{R,s})}_{0,j-1} e^{-ik_{z,j-1}z_j} &= E^{(\mathrm{I,s})}_{0,j} e^{ik_{z,j}z_j} + E^{(\mathrm{R,s})}_{0,j} e^{-ik_{z,j}z_j} \\
k_{j-1}\sin\theta_{j-1}\left(E^{(\mathrm{I,s})}_{0,j-1} e^{ik_{z,j-1}z_j} - E^{(\mathrm{R,s})}_{0,j-1} e^{-ik_{z,j-1}z_j}\right) &= k_j\sin\theta_j\left(E^{(\mathrm{I,s})}_{0,j} e^{ik_{z,j}z_j} - E^{(\mathrm{R,s})}_{0,j} e^{-ik_{z,j}z_j}\right) \\
k_{j-1}\cos\theta_{j-1}\left(E^{(\mathrm{I,s})}_{0,j-1} e^{ik_{z,j-1}z_j} + E^{(\mathrm{R,s})}_{0,j-1} e^{-ik_{z,j-1}z_j}\right) &= k_j\cos\theta_j\left(E^{(\mathrm{I,s})}_{0,j} e^{ik_{z,j}z_j} + E^{(\mathrm{R,s})}_{0,j} e^{-ik_{z,j}z_j}\right)
\end{aligned}
$$

First and third equations are equivalent because,

$$
\frac{k_{j-1}\cos\theta_{j-1}}{k_j\cos\theta_j} = 1
$$

1.5 Sets of equations

$$
\begin{aligned}
\sin\theta_{j-1}\left(E^{(\mathrm{I,p})}_{0,j-1} e^{ik_{z,j-1}z_j} - E^{(\mathrm{R,p})}_{0,j-1} e^{-ik_{z,j-1}z_j}\right) &= \sin\theta_j\left(E^{(\mathrm{I,p})}_{0,j} e^{ik_{z,j}z_j} - E^{(\mathrm{R,p})}_{0,j} e^{-ik_{z,j}z_j}\right) \\
k_{j-1}\left(E^{(\mathrm{I})}_{0,j-1} e^{ik_{z,j-1}z_j} + E^{(\mathrm{R})}_{0,j-1} e^{ik_{z,j-1}z_j}\right) &= k_j\left(E^{(\mathrm{I})}_{0,j} e^{ik_{z,j}z_j} + E^{(\mathrm{R})}_{0,j} e^{ik_{z,j}z_j}\right)
\end{aligned}
$$

$$
\mathbf{S}^{(\mathrm{p})}_j(z)\cdot\mathbf{E}^{(\mathrm{p})}_j \equiv \begin{pmatrix} \sin\theta_j e^{ik_{z,j}z} & -\sin\theta_j e^{-ik_{z,j}z} \\ k_j e^{ik_{z,j}z} & k_j e^{-ik_{z,j}z} \end{pmatrix} \begin{pmatrix} E^{(\mathrm{I,p})}_{0,j} \\ E^{(\mathrm{R,p})}_{0,j} \end{pmatrix}
$$

$$
\begin{aligned}
E^{(\mathrm{I,s})}_{0,j-1} e^{ik_{z,j-1}z_j} + E^{(\mathrm{R,s})}_{0,j-1} e^{-ik_{z,j-1}z_j} &= E^{(\mathrm{I,s})}_{0,j} e^{ik_{z,j}z_j} + E^{(\mathrm{R,s})}_{0,j} e^{-ik_{z,j}z_j} \\
k_{j-1}\sin\theta_{j-1}\left(E^{(\mathrm{I,s})}_{0,j-1} e^{ik_{z,j-1}z_j} - E^{(\mathrm{R,s})}_{0,j-1} e^{-ik_{z,j-1}z_j}\right) &= k_j\sin\theta_j\left(E^{(\mathrm{I,s})}_{0,j} e^{ik_{z,j}z_j} - E^{(\mathrm{R,s})}_{0,j} e^{-ik_{z,j}z_j}\right)
\end{aligned}
$$

$$
\mathbf{S}^{(\mathrm{s})}_j(z)\cdot\mathbf{E}^{(\mathrm{s})}_j \equiv \begin{pmatrix} e^{ik_{z,j}z} & e^{-ik_{z,j}z} \\ k_j\sin\theta_j e^{ik_{z,j}z} & -k_j\sin\theta_j e^{-ik_{z,j}z} \end{pmatrix} \begin{pmatrix} E^{(\mathrm{I,s})}_{0,j} \\ E^{(\mathrm{R,s})}_{0,j} \end{pmatrix}
$$

For either polarisation the relation between $\mathbf{E}_j$ and $\mathbf{E}_{j-1}$ is given by the following equation,

$$
\mathbf{S}_{j-1}(z_j)\cdot\mathbf{E}_{j-1} = \mathbf{S}_j(z_j)\cdot\mathbf{E}_j
$$

We then compute $\mathbf{E}_j$ from $\mathbf{E}_{j-1}$ with

$$
\mathbf{E}_j = \mathbf{T}_j\cdot\mathbf{E}_{j-1}
$$

with

$$
\mathbf{T}_j \equiv \mathbf{S}^{-1}_j(z_j)\cdot\mathbf{S}_{j-1}(z_j)
$$

$$
\begin{aligned}
\mathbf{T}^{(\mathrm{p})}_j &= \frac{1}{2k_j\sin\theta_j} \begin{pmatrix} k_j e^{-ik_{z,j}z_j} & \sin\theta_j e^{-ik_{z,j}z_j} \\ -k_j e^{ik_{z,j}z_j} & \sin\theta_j e^{ik_{z,j}z_j} \end{pmatrix} \begin{pmatrix} \sin\theta_{j-1} e^{ik_{z,j-1}z_j} & -\sin\theta_{j-1} e^{-ik_{z,j-1}z_j} \\ k_{j-1} e^{ik_{z,j-1}z_j} & k_{j-1} e^{-ik_{z,j-1}z_j} \end{pmatrix} \\
&= \frac{1}{2k_{z,j}} \begin{pmatrix} \left(\frac{n_j}{n_{j-1}}k_{z,j-1} + \frac{n_{j-1}}{n_j}k_{z,j}\right) e^{-ik^-_{z,j}z_j} & \left(-\frac{n_j}{n_{j-1}}k_{z,j-1} + \frac{n_{j-1}}{n_j}k_{z,j}\right) e^{-ik^+_{z,j}z_j} \\ \left(-\frac{n_j}{n_{j-1}}k_{z,j-1} + \frac{n_{j-1}}{n_j}k_{z,j}\right) e^{ik^+_{z,j}z_j} & \left(\frac{n_j}{n_{j-1}}k_{z,j-1} + \frac{n_{j-1}}{n_j}k_{z,j}\right) e^{ik^-_{z,j}z_j} \end{pmatrix} \\
&= \frac{1}{2n_{j-1}n_jk_{z,j}} \begin{pmatrix} \left(n^2_{j-1}k_{z,j} + n^2_jk_{z,j-1}\right) e^{-ik^-_{z,j}z_j} & \left(n^2_{j-1}k_{z,j} - n^2_jk_{z,j-1}\right) e^{-ik^+_{z,j}z_j} \\ \left(n^2_{j-1}k_{z,j} - n^2_jk_{z,j-1}\right) e^{ik^+_{z,j}z_j} & \left(n^2_{j-1}k_{z,j} + n^2_jk_{z,j-1}\right) e^{ik^-_{z,j}z_j} \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\det\left\{\mathbf{T}_j^{(\mathrm{p})}\right\} &= \frac{\left(n_{j-1}^2 k_{z,j} + n_j^2 k_{z,j-1}\right)^2 - \left(n_{j-1}^2 k_{z,j} - n_j^2 k_{z,j-1}\right)^2}{4 n_{j-1}^2 n_j^2 k_{z,j}^2} \\
&= \frac{k_{z,j-1}}{k_{z,j}}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{T}_j^{(\mathrm{s})} &= \frac{1}{-2k_j \sin\theta_j} \begin{pmatrix} -k_j \sin\theta_j e^{-ik_{z,j}z_j} & -e^{-ik_{z,j}z_j} \\ -k_j \sin\theta_j e^{ik_{z,j}z_j} & e^{ik_{z,j}z_j} \end{pmatrix} \begin{pmatrix} e^{ik_{z,j-1}z_j} & e^{-ik_{z,j-1}z_j} \\ k_{j-1}\sin\theta_{j-1} e^{ik_{z,j-1}z_j} & -k_{j-1}\sin\theta_{j-1} e^{-ik_{z,j-1}z_j} \end{pmatrix} \\
&= \frac{1}{2k_{z,j}} \begin{pmatrix} k_{z,j}^{+} e^{-ik_{z,j}^{-}z_j} & k_{z,j}^{-} e^{-ik_{z,j}^{+}z_j} \\ k_{z,j}^{-} e^{ik_{z,j}^{+}z_j} & k_{z,j}^{+} e^{ik_{z,j}^{-}z_j} \end{pmatrix}
\end{aligned}
$$

with

$$
k_{z,j}^{\pm} = k_{z,j} \pm k_{z,j-1}
$$

Note that,

$$
\begin{aligned}
\det\left\{\mathbf{T}_j^{(\mathrm{s})}\right\} &= \frac{\left(k_{z,j} + k_{z,j-1}\right)^2 - \left(k_{z,j} - k_{z,j-1}\right)^2}{4k_{z,j}^2} \\
&= \frac{k_{z,j-1}}{k_{z,j}}
\end{aligned}
$$

$$
\mathbf{T}_j^{(\mathrm{p,s})} = \begin{pmatrix} p_j^{(\mathrm{p,s})} e^{-ik_{z,j}^{-}z_j} & m_j^{(\mathrm{p,s})} e^{-ik_{z,j}^{+}z_j} \\ m_j^{(\mathrm{p,s})} e^{ik_{z,j}^{+}z_j} & p_j^{(\mathrm{p,s})} e^{ik_{z,j}^{-}z_j} \end{pmatrix}
$$

with,

$$
\begin{aligned}
p_j^{(\mathrm{p})} &= \frac{k_{z,j} + k_{z,j-1}}{2k_{z,j}} & m_j^{(\mathrm{p})} &= \frac{k_{z,j} - k_{z,j-1}}{2k_{z,j}} \\
p_j^{(\mathrm{s})} &= \frac{n_{j-1}^2 k_{z,j} + n_j^2 k_{z,j-1}}{2 n_j n_{j-1} k_{z,j}} & m_j^{(\mathrm{s})} &= \frac{n_{j-1}^2 k_{z,j} - n_j^2 k_{z,j-1}}{2 n_j n_{j-1} k_{z,j}}
\end{aligned}
$$

We define the following vector and matrix

$$\mathbf{W}_j \equiv \mathbf{S}_j(z) \cdot \mathbf{E}_j$$

where

$$\mathbf{W}_j(\mathbf{r}) = \begin{pmatrix} E_{0x,j}(\mathbf{r}) \\ \epsilon_j E_{0z,j}(\mathbf{r}) \end{pmatrix} = E_{0,j}(\mathbf{r}) \begin{pmatrix} \sin\theta_j \\ \epsilon_j \cos\theta_j \end{pmatrix} \qquad \mathbf{E}_j = \begin{pmatrix} E_{0,j}^{(\mathrm{I})} \\ E_{0,j}^{(\mathrm{R})} \end{pmatrix} \qquad \mathbf{S}_j(z) = \begin{pmatrix} \alpha_j e^{ik_{z,j}z} & -\alpha_j e^{-ik_{z,j}z} \\ \beta_j e^{ik_{z,j}z} & \beta_j e^{-ik_{z,j}z} \end{pmatrix}$$

where we defined

$$\alpha_j \equiv \sin\theta_j = \left(1 - \frac{n_0^2\cos^2\theta_0}{n_j^2}\right)^{1/2} \qquad \beta_j \equiv \epsilon_j \cos\theta_j = \frac{n_j}{\mu_j}\lambda k_0 \cos\theta_0$$

so that,

$$\mathbf{W}_{j-1}(\mathbf{r}_j, t) = \mathbf{W}_j(\mathbf{r}_j, t)$$

$$\det\{\mathbf{S}_j\} = 2\alpha_j\beta_j = 2\epsilon_j \sin\theta_j \cos\theta_j = 2n_j^2 \sin\theta_j \cos\theta_j = 2n_j \sin\theta_j n_0 \cos\theta_0$$

Polarisation out of plane, with $k_y = 0$,

$$E_{0x,j}^{(\mathrm{I})} = 0 \qquad E_{0x,j}^{(\mathrm{R})} = 0$$
$$E_{0y,j}^{(\mathrm{I})} = E_{0,j}^{(\mathrm{I})} \qquad E_{0y,j}^{(\mathrm{R})} = E_{0,j}^{(\mathrm{R})}$$
$$E_{0z,j}^{(\mathrm{I})} = 0 \qquad E_{0z,j}^{(\mathrm{R})} = 0$$

$$\omega B_{0x,j}^{(\mathrm{I})} = k_j E_{0,j}^{(\mathrm{I})} \sin\theta_j \qquad \omega B_{0x,j}^{(\mathrm{R})} = -k_j E_{0,j}^{(\mathrm{R})} \sin\theta_j$$
$$B_{0y,j}^{(\mathrm{I})} = 0 \qquad B_{0y,j}^{(\mathrm{R})} = 0$$
$$\omega B_{0z,j}^{(\mathrm{I})} = k_j E_{0,j}^{(\mathrm{I})} \cos\theta_j \qquad \omega B_{0z,j}^{(\mathrm{R})} = k_j E_{0,j}^{(\mathrm{R})} \cos\theta_j$$

The relevant equations are

$$E_{0,j-1}^{(\mathrm{I})} e^{ik_{z,j-1}z_j} + E_{0,j-1}^{(\mathrm{R})} e^{-ik_{z,j-1}z_j} = E_{0,j}^{(\mathrm{I})} e^{ik_{z,j}z_j} + E_{0,j}^{(\mathrm{R})} e^{-ik_{z,j}z_j}$$
$$k_{j-1}\sin\theta_{j-1}\left(E_{0,j-1}^{(\mathrm{I})} e^{ik_{z,j-1}z_j} + E_{0,j-1}^{(\mathrm{R})} e^{-ik_{z,j-1}z_j}\right) = k_j \sin\theta_j \left(E_{0,j}^{(\mathrm{I})} e^{ik_{z,j}z_j} - E_{0,j}^{(\mathrm{R})} e^{-ik_{z,j}z_j}\right)$$

$$\mathbf{W}_j(\mathbf{r}) = \begin{pmatrix} E_{0y,j}(\mathbf{r}) \\ \omega B_{0x,j}(\mathbf{r}) \end{pmatrix} = E_{0,j}(\mathbf{r}) \begin{pmatrix} 1 \\ k_j \sin\theta_j \end{pmatrix} \qquad \mathbf{E}_j = \begin{pmatrix} E_{0,j}^{(\mathrm{I})} \\ E_{0,j}^{(\mathrm{R})} \end{pmatrix} \qquad \mathbf{S}_j(z) = \begin{pmatrix} e^{ik_{z,j}z} & e^{-ik_{z,j}z} \\ k_{z,j} e^{ik_{z,j}z} & -k_{z,j} e^{-ik_{z,j}z} \end{pmatrix}$$

with

$$k_{z,j} \equiv k_j \sin\theta_j = k_j \alpha_j = \frac{2\pi n_j \alpha_j}{\lambda}$$

For X-Rays both polarisation can pe approximate by the later one.
Lets note that $\det\{\mathbf{S}_j\} = -2k_{z,j}$.
In both polarisation what we have at the interface is,

$$\mathbf{W}_{j-1}(\mathbf{r}_j, t) = \mathbf{W}_j(\mathbf{r}_j, t)$$

In case of $n_j < \cos\theta_j$, $k_{z,j}$ is imaginary. We define $\kappa_z = \mathcal{I}\{k_z\}$ and,

$$\mathbf{S}_j(z) = \begin{pmatrix} e^{ik_{z,j}z} & e^{-ik_{z,j}z} \\ k_{z,j}e^{ik_{z,j}z} & -k_{z,j}e^{-ik_{z,j}z} \end{pmatrix} \qquad \mathbf{S}_j(z) = \begin{pmatrix} e^{-\kappa_{z,j}z} & e^{\kappa_{z,j}z} \\ i\kappa_{z,j}e^{-\kappa_{z,j}z} & -i\kappa_{z,j}e^{\kappa_{z,j}z} \end{pmatrix}$$

$$k_{z,j} \to i\kappa_{z,j}$$

1.6 Interface transfer

We define the transfer matrix as,

$$\mathbf{E}_j = \mathbf{T}_j \cdot \mathbf{E}_{j-1}$$

We can find this matrix with the help of the interface condition,

$$\mathbf{W}_j(\mathbf{r}_j) = \mathbf{W}_{j-1}(\mathbf{r}_j)$$
$$\mathbf{S}_j(\mathbf{r}_j) \cdot \mathbf{E}_j = \mathbf{S}_{j-1}(\mathbf{r}_j) \cdot \mathbf{E}_{j-1}$$

so that,

$$\mathbf{T}_j = \mathbf{S}_j^{-1}(\mathbf{r}_j) \cdot \mathbf{S}_{j-1}(\mathbf{r}_j)$$

Let's note that $\det\{\mathbf{T}_j\} = \frac{k_{j-1}}{k_j}$.

$$\begin{aligned}\mathbf{T}_j &= \frac{1}{2k_{z,j}} \begin{pmatrix} k_{z,j}e^{-ik_{z,j}z_j} & e^{-ik_{z,j}z_j} \\ k_{z,j}e^{ik_{z,j}z_j} & -e^{ik_{z,j}z_j} \end{pmatrix} \begin{pmatrix} e^{ik_{z,j-1}z_j} & e^{-ik_{z,j-1}z_j} \\ k_{z,j-1}e^{ik_{z,j-1}z_j} & -k_{z,j-1}e^{-ik_{z,j-1}z_j} \end{pmatrix} \\ &= \frac{1}{2k_{z,j}} \begin{pmatrix} (k_{z,j}+k_{z,j-1})e^{-i(k_{z,j}-k_{z,j-1})z_j} & (k_{z,j}-k_{z,j-1})e^{-i(k_{z,j}+k_{z,j-1})z_j} \\ (k_{z,j}-k_{z,j-1})e^{i(k_{z,j}+k_{z,j-1})z_j} & (k_{z,j}+k_{z,j-1})e^{i(k_{z,j}-k_{z,j-1})z_j} \end{pmatrix}\end{aligned}$$

Evanescant solution

$$\mathbf{T}_j = \frac{1}{2\kappa_{z,j}} \begin{pmatrix} (\kappa_{z,j}+\kappa_{z,j-1})e^{(\kappa_{z,j}-\kappa_{z,j-1})z_j} & (\kappa_{z,j}-\kappa_{z,j-1})e^{(\kappa_{z,j}+\kappa_{z,j-1})z_j} \\ (\kappa_{z,j}-\kappa_{z,j-1})e^{-(\kappa_{z,j}+\kappa_{z,j-1})z_j} & (\kappa_{z,j}+\kappa_{z,j-1})e^{-(\kappa_{z,j}-\kappa_{z,j-1})z_j} \end{pmatrix}$$

To go from the first interface to the last we just need to apply the matrices in sequence.

$$\mathbf{E}_N = \mathbf{L} \cdot \mathbf{E}_0 \qquad \mathbf{L} = \mathbf{T}_N \cdot \mathbf{T}_{N-1} \cdots \mathbf{T}_2 \cdot \mathbf{T}_1 = \prod_{n=1}^{N} \mathbf{T}_n$$

$$\begin{pmatrix} t \\ 0 \end{pmatrix} = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ r \end{pmatrix}$$

$$r = -\frac{L_{21}}{L_{22}} \qquad t = L_{11} - \frac{L_{12}L_{21}}{L_{22}} = \frac{\det\{\mathbf{L}\}}{L_{22}}$$

Let's note that $\det\{\mathbf{L}\} = \prod_{j=1}^{N} \frac{k_{j-1}}{k_j} = \frac{k_0}{k_N}$ if $k_j > 0$ and 0 otherwise.

1.7 Roughness

$$
\begin{aligned}
\mathbf{L} &= \int \mathrm{d}z\, p_j(z) \mathbf{T}_N \cdots \mathbf{T}_j(z) \cdots \cdot \mathbf{T}_1 \\
&= \mathbf{T}_N \cdots \left(\int \mathrm{d}z\, p_j(z) \mathbf{T}_j(z) \right) \cdots \cdot \mathbf{T}_1
\end{aligned}
$$

$$
p_j(z) = \frac{1}{\sigma_j \sqrt{2\pi}} \exp\left\{ -\frac{1}{2} \left(\frac{z - z_j}{\sigma_j} \right)^2 \right\}
$$

$$
\int \mathrm{d}z\, p_j(z) e^{i\left(k_{z,j} \pm k_{z,j-1}\right) z} = e^{i\left(k_{z,j} \pm k_{z,j-1}\right) z_j} e^{-\frac{1}{2}\left(k_{z,j} \pm k_{z,j-1}\right)^2 \sigma_j^2}
$$

$$
\begin{aligned}
& \int \mathrm{d}z\, p_j(z) \mathbf{T}_j(z) \\
= \frac{1}{2k_{z,j}} & \begin{pmatrix} \left(k_{z,j} + k_{z,j-1}\right) e^{-i\left(k_{z,j} - k_{z,j-1}\right) z_j} e^{-\frac{1}{2}\left(k_{z,j} - k_{z,j-1}\right)^2 \sigma_j^2} & \left(k_{z,j} - k_{z,j-1}\right) e^{-i\left(k_{z,j} + k_{z,j-1}\right) z_j} e^{-\frac{1}{2}\left(k_{z,j} + k_{z,j-1}\right)^2 \sigma_j^2} \\ \left(k_{z,j} - k_{z,j-1}\right) e^{i\left(k_{z,j} + k_{z,j-1}\right) z_j} e^{-\frac{1}{2}\left(k_{z,j} + k_{z,j-1}\right)^2 \sigma_j^2} & \left(k_{z,j} + k_{z,j-1}\right) e^{i\left(k_{z,j} - k_{z,j-1}\right) z_j} e^{-\frac{1}{2}\left(k_{z,j} - k_{z,j-1}\right)^2 \sigma_j^2} \end{pmatrix}
\end{aligned}
$$

$$
\int \mathrm{d}z\, p_j(z) \mathbf{T}_j(z) \approx \frac{1}{2k_{z,j}} \begin{pmatrix} 2k_{z,j} e^{-i\left(k_{z,j} - k_{z,j-1}\right) z_j} & \left(k_{z,j} - k_{z,j-1}\right) e^{-i\left(k_{z,j} + k_{z,j-1}\right) z_j} e^{-\frac{1}{2}\left(k_{z,j} + k_{z,j-1}\right)^2 \sigma_j^2} \\ \left(k_{z,j} - k_{z,j-1}\right) e^{i\left(k_{z,j} + k_{z,j-1}\right) z_j} e^{-\frac{1}{2}\left(k_{z,j} + k_{z,j-1}\right)^2 \sigma_j^2} & \left(k_{z,j} + k_{z,j-1}\right) e^{i\left(k_{z,j} - k_{z,j-1}\right) z_j} e^{-\frac{1}{2}\left(k_{z,j} - k_{z,j-1}\right)^2 \sigma_j^2} \end{pmatrix}
$$

1.8 Slope

$$
\mathbf{T}_j = \frac{1}{2k_{z,j}} \begin{pmatrix} \left(k_{z,j} + k_{z,j-1}\right) e^{-i\left(k_{z,j} - k_{z,j-1}\right) z_j} & \left(k_{z,j} - k_{z,j-1}\right) e^{-i\left(k_{z,j} + k_{z,j-1}\right) z_j} \\ \left(k_{z,j} - k_{z,j-1}\right) e^{i\left(k_{z,j} + k_{z,j-1}\right) z_j} & \left(k_{z,j} + k_{z,j-1}\right) e^{i\left(k_{z,j} - k_{z,j-1}\right) z_j} \end{pmatrix}
$$

$$
\begin{aligned}
k_{z,j} + k_{z,j-1} &\approx 2k_{z,j} \\
k_{z,j} - k_{z,j-1} &\approx \frac{\mathrm{d}k_z}{\mathrm{d}z} \Delta z_{j,j-1}
\end{aligned}
$$

$$
\mathbf{T}_j = \frac{1}{2k_{z,j}} \begin{pmatrix} 2k_{z,j} e^{-i \frac{\mathrm{d}k_z}{\mathrm{d}z} \Delta z_{j,j-1} z_j} & \frac{\mathrm{d}k_z}{\mathrm{d}z} \Delta z_{j,j-1} e^{-i 2k_{z,j} z_j} \\ \frac{\mathrm{d}k_z}{\mathrm{d}z} \Delta z_{j,j-1} e^{i 2k_{z,j} z_j} & 2k_{z,j} e^{i \frac{\mathrm{d}k_z}{\mathrm{d}z} \Delta z_{j,j-1} z_j} \end{pmatrix}
$$

$$
\mathbf{T}_j \approx \frac{1}{2k_{z,j}} \begin{pmatrix} 2k_{z,j} \left(1 - i \frac{\mathrm{d}k_z}{\mathrm{d}z} \Delta z_{j,j-1} z_j + \ldots\right) & \frac{\mathrm{d}k_z}{\mathrm{d}z} \Delta z_{j,j-1} e^{-i 2k_{z,j} z_j} \\ \frac{\mathrm{d}k_z}{\mathrm{d}z} \Delta z_{j,j-1} e^{i 2k_{z,j} z_j} & 2k_{z,j} \left(1 + i \frac{\mathrm{d}k_z}{\mathrm{d}z} \Delta z_{j,j-1} z_j + \ldots\right) \end{pmatrix}
$$

$$\mathbf{T}_j \approx \mathbb{1} + \frac{\mathrm{d}k_z}{\mathrm{d}z}\Delta z_{j,j-1} z_j \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} + \frac{1}{2k_{z,j}}\frac{\mathrm{d}k_z}{\mathrm{d}z}\Delta z_{j,j-1} \begin{pmatrix} 0 & e^{-i2k_{z,j}z_j} \\ e^{i2k_{z,j}z_j} & 0 \end{pmatrix}$$

$$\mathbf{T}_j \approx \mathbb{1} + \Delta k_{z,j,j-1} z_j \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} + \frac{\Delta k_{z,j,j-1}}{2k_{z,j}} \begin{pmatrix} 0 & e^{-i2k_{z,j}z_j} \\ e^{i2k_{z,j}z_j} & 0 \end{pmatrix}$$

$$\mathbf{T}_j \approx 2k_{z,j}\left(\mathbb{1} + \Delta k_{z,j,j-1} z_j \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\right) + \Delta k_{z,j,j-1} \begin{pmatrix} 0 & e^{-i2k_{z,j}z_j} \\ e^{i2k_{z,j}z_j} & 0 \end{pmatrix}$$

$$\prod_{j=0}^{N} \mathbf{T}_j = \mathbb{1} + \sum_{j=0}^{N} \frac{1}{2k_{z,j}}\frac{\mathrm{d}k_z}{\mathrm{d}z}\Delta z_{j,j-1} \begin{pmatrix} 0 & e^{-i2k_{z,j}z_j} \\ e^{i2k_{z,j}z_j} & 0 \end{pmatrix} + \mathcal{O}(\Delta z^2)$$

$$\prod_{j=0}^{N} \mathbf{T}_j = \mathbb{1} + \int_{\text{top}}^{\text{bottom}} \mathrm{d}z \, \frac{1}{2k_z(z)}\frac{\mathrm{d}k_z}{\mathrm{d}z} \begin{pmatrix} 0 & e^{-i2k_z(z)z} \\ e^{i2k_z(z)z} & 0 \end{pmatrix}$$

$$\prod_{j=0}^{N} \mathbf{T}_j = \mathbb{1} + \int \mathrm{d}z \, \frac{1}{2k_z(z)}\frac{\mathrm{d}k_z}{\mathrm{d}z} \begin{pmatrix} 0 & e^{-i2k_z(z)z} \\ e^{i2k_z(z)z} & 0 \end{pmatrix}$$

$$\prod_{j=0}^{N} \mathbf{T}_j = \mathbb{1} + \int \mathrm{d}z \, \frac{1}{2k_z(z)}\frac{\mathrm{d}k_z}{\mathrm{d}z} \begin{pmatrix} 0 & e^{-i2k_z(z)z} \\ e^{i2k_z(z)z} & 0 \end{pmatrix}$$

$$\prod_{j=0}^{N} \mathbf{T}_j = \mathbb{1} + \int_{k_{z,\text{top}}}^{k_{z,\text{bottom}}} \frac{\mathrm{d}k_z}{2k_z} \begin{pmatrix} 0 & e^{-i2k_z z(k_z)} \\ e^{i2k_z z(k_z)} & 0 \end{pmatrix}$$

1.9 Layer Transfer

We define the transfer matrix as,

$$\mathbf{W}_j(z_{j+1}) = \mathbf{M}_j \cdot \mathbf{W}_j(z_j)$$

We can find this matrix with the help **E**

$$\mathbf{S}_j(z_{j+1}) \cdot \mathbf{E}_j = \mathbf{M}_j \cdot \mathbf{S}_j(z_j) \cdot \mathbf{E}_j$$

So that,

$$\mathbf{M}_j = \mathbf{S}_j(z_{j+1}) \cdot \mathbf{S}_j^{-1}(z_j)$$

$$
\begin{aligned}
\mathbf{M}_j &= \begin{pmatrix} e^{ik_{z,j}z_{j+1}} & e^{-ik_{z,j}z_{j+1}} \\ k_{z,j}e^{ik_{z,j}z_{j+1}} & -k_{z,j}e^{-ik_{z,j}z_{j+1}} \end{pmatrix} \frac{1}{2k_{z,j}} \begin{pmatrix} k_{z,j}e^{-ik_{z,j}z_j} & e^{-ik_{z,j}z_j} \\ k_{z,j}e^{ik_{z,j}z_j} & -e^{ik_{z,j}z_j} \end{pmatrix} \\
&= \frac{1}{2k_{z,j}} \begin{pmatrix} \cos\left(k_{z,j}d_j\right) & ik_{z,j}^{-1}\sin\left(k_{z,j}d_j\right) \\ ik_{z,j}\sin\left(k_{z,j}d_j\right) & \cos\left(k_{z,j}d_j\right) \end{pmatrix}
\end{aligned}
$$

To go from the first interface to the last we just need to apply the matrices in sequence.

$$
\mathbf{L} = \mathbf{S}_N(z_N)^{-1} \cdot \mathbf{M}_{N-1} \cdot \mathbf{M}_{N-2} \cdots \mathbf{M}_2 \cdot \mathbf{M}_1 \cdot \mathbf{S}_0(z_1)
$$

$$
r = -e^{i2k_z(a)a} \frac{k_z(b) - k_z(a) + (b-a)k_z(a)k_z(b) - \int_a^b \mathrm{d}z\, k_z^2(z)}{k_z(b) + k_z(a) - (b-a)k_z(a)k_z(b) - \int_a^b \mathrm{d}z\, k_z^2(z)}
$$

2 Geometry

2.1 Vecteurs de base réseau direct

$\mathbf{a}_1$ $\mathbf{a}_2$ $\mathbf{a}_3$

2.2 Vecteurs de base réseau réciproque

On construit les vecteur de base du réseau réciproque de manière à ce que les $\mathbf{b}_i$ soient orthogonaux aux $\mathbf{a}_j$ pour $i \neq j$.

$$
\mathbf{b}_i = \frac{\pi}{V}\epsilon_{ijk}\mathbf{a}_j \times \mathbf{a}_k
$$

où on divise par deux car on compte $\mathbf{a}_j \times \mathbf{a}_k - \mathbf{a}_k \times \mathbf{a}_j$.

2.3 Produit scalaire espace direct

Le produits sclaire entre deux vecteurs de base du réseau cristalin implique les longueurs et l'angle mutuelle,

$$
\mathbf{a}_i \cdot \mathbf{a}_j = a_i a_j \cos\alpha_{ij}.
$$

2.4 Produit scalaire espace reciproque

Le produit scalaire de deux vecteurs de la base du réseau réciproque peut être exprimé à l'aide de produits des vecteurs de base du réseau direct,

$$
\mathbf{b}_i \cdot \mathbf{b}_j = \left(\frac{\pi}{V}\right)^2 \epsilon_{imn}\epsilon_{jpq}(\mathbf{a}_m \times \mathbf{a}_n) \cdot \left(\mathbf{a}_p \times \mathbf{a}_q\right).
$$

On utilise l'identité vectorielle suivante,

$$
(\mathbf{a}_m \times \mathbf{a}_n) \cdot \left(\mathbf{a}_p \times \mathbf{a}_q\right) = \left(\mathbf{a}_m \cdot \mathbf{a}_p\right)\left(\mathbf{a}_n \cdot \mathbf{a}_q\right) - \left(\mathbf{a}_m \cdot \mathbf{a}_q\right)\left(\mathbf{a}_n \cdot \mathbf{a}_p\right)
$$

pour exprimé le résultat en fonction de produit scalaire uniquement,

$$\mathbf{b}_i \cdot \mathbf{b}_j = \left(\frac{\pi}{V}\right)^2 \epsilon_{imn}\epsilon_{jpq}\left[\left(\mathbf{a}_m \cdot \mathbf{a}_p\right)\left(\mathbf{a}_n \cdot \mathbf{a}_q\right) - \left(\mathbf{a}_m \cdot \mathbf{a}_q\right)\left(\mathbf{a}_n \cdot \mathbf{a}_p\right)\right].$$

À l'aide de 2.3, on obtient donc,

$$\mathbf{b}_i \cdot \mathbf{b}_j = \left(\frac{\pi}{V}\right)^2 \epsilon_{imn}\epsilon_{jpq} a_m a_n a_p a_q \left[\cos\alpha_{mp}\cos\alpha_{nq} - \cos\alpha_{mq}\cos\alpha_{np}\right].$$

2.5 Produit scalaire croisé

Par construction le produit scalaire entre un vecteur de la base directe et un vecteur de la base réciproque est,

$$\mathbf{a}_i \cdot \mathbf{b}_j = 2\pi\delta_{ij}$$

2.6 Produit vectoriel espace direct

La définion des vecteurs de base du réseau réciproque contient le produit vectoriel des vecteurs de base du réseaud direct. Pour extraire, le résultat du produit on multiplie simplement par le tenseur de Levi-Civita,

$$\begin{aligned}
\mathbf{b}_i\epsilon_{imn} &= \frac{\pi}{V}\epsilon_{imn}\epsilon_{ijk}\mathbf{a}_j \times \mathbf{a}_k \\
&= \frac{\pi}{V}\left(\delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj}\right)\mathbf{a}_j \times \mathbf{a}_k \\
&= \frac{\pi}{V}\left(\mathbf{a}_m \times \mathbf{a}_n - \mathbf{a}_n \times \mathbf{a}_m\right) \\
&= \frac{2\pi}{V}\mathbf{a}_m \times \mathbf{a}_n
\end{aligned}$$

On peut alors isoler le résultat,

$$\mathbf{a}_m \times \mathbf{a}_n = \frac{V}{2\pi}\epsilon_{imn}\mathbf{b}_i$$

2.7 Produit vectoriel espace réciproque

Le produit vectorielle entre les vecteurs de base du réseau réciproque prend la forme d'un produit quadruple

$$\mathbf{b}_i \times \mathbf{b}_l = \left(\frac{\pi}{V}\right)^2 \epsilon_{ijk}\epsilon_{lmn}\left(\mathbf{a}_j \times \mathbf{a}_k\right) \times \left(\mathbf{a}_m \times \mathbf{a}_n\right)$$

qui peut être réexprimé sous la forme du produit triple.

$$\left(\mathbf{a}_j \times \mathbf{a}_k\right) \times \left(\mathbf{a}_m \times \mathbf{a}_n\right) = \left[\mathbf{a}_j, \mathbf{a}_k, \mathbf{a}_n\right]\mathbf{a}_m - \left[\mathbf{a}_j, \mathbf{a}_k, \mathbf{a}_m\right]\mathbf{a}_n$$

Le produit triple des vecteurs de base du réseau cristallin donne le volume de la maille élémentaire. Ce produit peut être fait dans les deux sens.

$$\left[\mathbf{a}_j, \mathbf{a}_k, \mathbf{a}_n\right] = V\epsilon_{jkn} - V\epsilon_{nkj}$$

Cela nous permet d'écrire le produit vectorielle de deux vecteur de base du réseau réciproque comme une combinaison linéaire des vecteur de base du réseau cristlin.

$$\mathbf{b}_i \times \mathbf{b}_l = \frac{(2\pi)^2}{V}\epsilon_{ilm}\mathbf{a}_m$$

2.8 Produit vectoriel croisé

$$
\begin{aligned}
\mathbf{a}_i \times \mathbf{b}_j &= \frac{\pi}{V}\epsilon_{jmn}\mathbf{a}_i \times (\mathbf{a}_m \times \mathbf{a}_n) \\
&= \frac{\pi}{V}\epsilon_{jmn}[(\mathbf{a}_i \cdot \mathbf{a}_n)\mathbf{a}_m - (\mathbf{a}_i \cdot \mathbf{a}_m)\mathbf{a}_n] \\
&= \frac{2\pi}{V}\epsilon_{jmn}(\mathbf{a}_i \cdot \mathbf{a}_n)\mathbf{a}_m \\
&= \frac{2\pi}{V}\epsilon_{jmn}a_i a_n \cos\alpha_{in}\mathbf{a}_m
\end{aligned}
$$

2.9 Volume cellule unité

Par définition le volume de la cellule unité est,

$$V = \mathbf{a}_1 \cdot (\mathbf{a}_2 \times \mathbf{a}_3)$$

Il est plus pratique d'exprimer le carré de ce volume,

$$
\begin{aligned}
V^2 &= (\mathbf{a}_1 \cdot (\mathbf{a}_2 \times \mathbf{a}_3))(\mathbf{a}_1 \cdot (\mathbf{a}_2 \times \mathbf{a}_3)) \\
&= \det\left\{ \begin{pmatrix} \mathbf{a}_1 \cdot \mathbf{a}_1 & \mathbf{a}_1 \cdot \mathbf{a}_2 & \mathbf{a}_1 \cdot \mathbf{a}_3 \\ \mathbf{a}_2 \cdot \mathbf{a}_1 & \mathbf{a}_2 \cdot \mathbf{a}_2 & \mathbf{a}_2 \cdot \mathbf{a}_3 \\ \mathbf{a}_3 \cdot \mathbf{a}_1 & \mathbf{a}_3 \cdot \mathbf{a}_2 & \mathbf{a}_3 \cdot \mathbf{a}_3 \end{pmatrix} \right\} \\
&= \det\left\{ \begin{pmatrix} a_1^2 & a_1 a_2 \cos\alpha_{12} & a_1 a_3 \cos\alpha_{13} \\ a_1 a_2 \cos\alpha_{12} & a_2^2 & a_2 a_3 \cos\alpha_{23} \\ a_1 a_3 \cos\alpha_{13} & a_2 a_3 \cos\alpha_{23} & a_3^2 \end{pmatrix} \right\} \\
&= a_1^2 a_2^2 a_3^2 (1 + 2\cos\alpha_{12}\cos\alpha_{23}\cos\alpha_{13} - \cos^2\alpha_{12} - \cos^2\alpha_{23} - \cos^2\alpha_{13})
\end{aligned}
$$

2.10 Vecteur du réseau direct

Les vecteurs du réseau direct peuvent être construits grâce au trio d'entier pqr :

$$\mathbf{R}_{pqr} = p\mathbf{a_1} + q\mathbf{a_2} + r\mathbf{a_3}.$$

2.11 Vecteur du réseau réciproque

Les vecteurs du réseau réciproque peuvent être construits grâce au trio d'entier hkl :

$$\mathbf{G}_{hkl} = h\mathbf{b_1} + k\mathbf{b_2} + l\mathbf{b_3}.$$

2.12 Produit scalaire direct

$$\mathbf{R}_{pqr} \cdot \mathbf{R}_{p'q'r'} = pp'\mathbf{a_1} \cdot \mathbf{a_1} + qq'\mathbf{a_2} \cdot \mathbf{a_2} + rr'\mathbf{a_3} \cdot \mathbf{a_3} + (pq' + qp')\mathbf{a_1} \cdot \mathbf{a_2} + (qr' + rq')\mathbf{a_2} \cdot \mathbf{a_3} + (rp' + pr')\mathbf{a_3} \cdot \mathbf{a_1}.$$

2.13 Produit scalaire réciproque

Le produit scalaire entre deux de ces vecteur s'exprime, en générale, de la manière suivante,

$$\mathbf{G}_{hkl} \cdot \mathbf{G}_{h'k'l'} = hh'\mathbf{b_1} \cdot \mathbf{b_1} + kk'\mathbf{b_2} \cdot \mathbf{b_2} + ll'\mathbf{b_3} \cdot \mathbf{b_3} + (hk' + kh')\mathbf{b_1} \cdot \mathbf{b_2} + (kl' + lk')\mathbf{b_2} \cdot \mathbf{b_3} + (lh' + hl')\mathbf{b_3} \cdot \mathbf{b_1}.$$

2.14 Produit scalaire croisé

$$\mathbf{R}_{pqr} \cdot \mathbf{G}_{hkl} = ph + qk + rl$$

2.15 Produit vectoriel direct

$$\begin{aligned}\mathbf{R} \times \mathbf{R}' &= p_i p'_j \mathbf{a}_i \times \mathbf{a}_j \\ &= \frac{V}{2\pi} \epsilon_{ijk} p_i p'_j \mathbf{b}_k\end{aligned}$$

2.16 Produit vectoriel réciproque

$$\begin{aligned}\mathbf{G} \times \mathbf{G}' &= h_i h'_j \mathbf{b}_i \times \mathbf{b}_j \\ &= \frac{(2\pi)^2}{V} \epsilon_{ijk} h_i h'_j \mathbf{a}_k\end{aligned}$$

2.17 Produit vectoriel croisé

$$\begin{aligned}\mathbf{G} \times \mathbf{R} &= h_i p_j \mathbf{b}_i \times \mathbf{a}_j \\ &= \frac{2\pi}{V} \epsilon_{jmn} h_i p_j a_i a_n \cos \alpha_{in} \mathbf{a}_m \\ &= \frac{2\pi}{V} \epsilon_{jmn} h_i p_j (\mathbf{a}_i \cdot \mathbf{a}_n) \mathbf{a}_m\end{aligned}$$

2.18 Angle entre deux vecteur du réseau réciproque

Grâce au produit sclaire, on peut être l'angle entre ces deux vecteurs.

$$\cos \theta_{hkl,h'k'l'} = \frac{\mathbf{G}_{hkl} \cdot \mathbf{G}_{h'k'l'}}{|\mathbf{G}_{hkl}||\mathbf{G}_{h'k'l'}|} = \frac{\mathbf{G}_{hkl} \cdot \mathbf{G}_{h'k'l'}}{(\mathbf{G}_{hkl} \cdot \mathbf{G}_{hkl})^{1/2} (\mathbf{G}_{h'k'l'} \cdot \mathbf{G}_{h'k'l'})^{1/2}}$$

2.19 Projection sur un vecteur du réseau réciproque

Projeté sur $\mathbf{G}_{HKL}$,

$$(\mathbf{G}_{hkl})_{\parallel \mathbf{G}_{HKL}} = \frac{\mathbf{G}_{HKL} \cdot \mathbf{G}_{hkl}}{\mathbf{G}_{HKL} \cdot \mathbf{G}_{HKL}} \mathbf{G}_{HKL}$$

Projeté sur le plan,

$$(\mathbf{G}_{hkl})_{\perp \mathbf{G}_{HKL}} = \mathbf{G}_{hkl} - \frac{\mathbf{G}_{HKL} \cdot \mathbf{G}_{hkl}}{\mathbf{G}_{HKL} \cdot \mathbf{G}_{HKL}} \mathbf{G}_{HKL}$$

Produit de phase $\mathbf{G} \cdot \mathbf{r}$

$$\mathbf{G} = h_i \mathbf{b}_i \qquad\qquad \mathbf{r} = \eta_j \mathbf{a}_j$$

$$\begin{aligned} \mathbf{G} \cdot \mathbf{r} &= h_i \eta_j \mathbf{b}_i \cdot \mathbf{a}_j \\ &= 2\pi h_i \eta_j \end{aligned}$$

3 Espace

On peut définir la position d'un réseau dans l'espace en stipulant la direction d'un vecteur du réseau réciproque selon $\hat{\mathbf{z}}$ et un vecteur du réseau cristalin selon $\hat{\mathbf{x}}$:

$$\mathbf{G}^{(z)} = G^{(z)} \hat{\mathbf{z}} \quad \text{et} \quad \mathbf{R}^{(x)} = R^{(x)} \hat{\mathbf{x}}$$

où

$$\mathbf{R}_{pqr} = p\mathbf{a_1} + q\mathbf{a_2} + r\mathbf{a_3}$$

Ces deux vecteur doivent être orthogonaux et donc

$$\mathbf{G}^{(z)} \cdot \mathbf{R}^{(x)} = 2\pi \Big(h^{(z)} p^{(x)} + k^{(z)} q^{(x)} + l^{(z)} r^{(x)} \Big) = G^{(z)} R^{(x)} \hat{\mathbf{z}} \cdot \hat{\mathbf{x}} = 0$$

On peut obtenir les composantes selon $\hat{\mathbf{z}}$ et $\hat{\mathbf{x}}$ d'un vecteur du réseau réciproque dans cette espace en le projetant sur $\mathbf{G}^{(z)}$, $\mathbf{R}^{(x)}$.

$$\mathbf{G}_{hkl} \cdot \hat{\mathbf{z}} = \frac{\mathbf{G}_{hkl} \cdot \mathbf{G}^{(z)}}{\left|\mathbf{G}^{(z)}\right|} \qquad\qquad \mathbf{G}_{hkl} \cdot \hat{\mathbf{x}} = \frac{\mathbf{G}_{hkl} \cdot \mathbf{R}^{(x)}}{\left|\mathbf{R}^{(x)}\right|}$$

On obtient la direction de $\hat{\mathbf{y}}$ à partir de $\mathbf{G}^{(z)}$, $\mathbf{R}^{(x)}$ de la manière suivante,

$$\hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{x}} = \frac{\mathbf{G}^{(z)} \times \mathbf{R}^{(x)}}{\left|\mathbf{G}^{(z)}\right| \left|\mathbf{R}^{(x)}\right|}$$

On peut donc écrire la rpojection selon $\hat{\mathbf{y}}$ grâce à un produit triple. On doit alors être en mesure d'effectuer un produit vectorielle entre deux vecteurs du réseau réciproque.

$$\mathbf{G}_{hkl} \cdot \hat{\mathbf{y}} = \frac{\mathbf{G}_{hkl} \cdot \Big(\mathbf{G}^{(z)} \times \mathbf{R}^{(x)} \Big)}{\left|\mathbf{G}^{(z)}\right| \left|\mathbf{R}^{(x)}\right|} = \frac{\mathbf{R}^{(x)} \cdot \Big(\mathbf{G}_{hkl} \times \mathbf{G}^{(z)} \Big)}{\left|\mathbf{G}^{(z)}\right| \left|\mathbf{R}^{(x)}\right|}$$

Pour ce faire il est plus pratique d'écrire ce produit grâce à une notation indicielle

$$\mathbf{G} = h_i \mathbf{b}_i \qquad\qquad \mathbf{G} \times \mathbf{G}' = h_i h'_l \mathbf{b}_i \times \mathbf{b}_l$$

On arrive à une conclusion similaire pour deux vecteurs du réseau réciproque.

$$\mathbf{G} \times \mathbf{G}' = \frac{4\pi^2}{V} \epsilon_{ilm} h_i h'_l \mathbf{a}_m$$

Les coeifficient devant $\mathbf{a}_m$ peuvent donc être considérés comme des composante pqr.

Les composante d'un vecteur $\mathbf{R}_{pqr}$ se trouve de manière similaire. D'abord pour les composante selon $\hat{\mathbf{z}}$ et $\hat{\mathbf{x}}$.

$$\mathbf{R}_{pqr} \cdot \hat{\mathbf{z}} = \frac{\mathbf{R}_{pqr} \cdot \mathbf{G}^{(z)}}{\left|\mathbf{G}^{(z)}\right|} \qquad\qquad \mathbf{R}_{pqr} \cdot \hat{\mathbf{x}} = \frac{\mathbf{R}_{pqr} \cdot \mathbf{R}^{(x)}}{\left|\mathbf{G}^{(x)}\right|}$$

La projection selon $\hat{\mathbf{y}}$ fait également intervenir un produit vectorielle, mais entre des vecteurs du réseau direct.

$$\mathbf{R}_{pqr} \cdot \hat{\mathbf{y}} = \frac{\mathbf{R}_{pqr} \cdot \left(\mathbf{G}^{(z)} \times \mathbf{R}^{(x)}\right)}{\left|\mathbf{G}^{(z)}\right|\left|\mathbf{R}^{(x)}\right|} = \frac{\mathbf{G}^{(z)} \cdot \left(\mathbf{R}^{(x)} \times \mathbf{R}_{pqr}\right)}{\left|\mathbf{G}^{(z)}\right|\left|\mathbf{R}^{(x)}\right|}$$

$$\mathbf{R} = p_i \mathbf{a}_i \qquad\qquad \mathbf{R} \times \mathbf{R}' = p_i p'_l \mathbf{a}_i \times \mathbf{a}_l$$

$$\mathbf{R} \times \mathbf{R}' = \frac{V}{2\pi} \epsilon_{ilk} p_i p'_l \mathbf{b}_k$$

Explicitly,

$$\begin{aligned}
\mathbf{T}_j &= \frac{1}{2\alpha_j\beta_j} \begin{pmatrix} \beta_j e^{-ik_{z,j}z_j} & \alpha_j e^{-ik_{z,j}z_j} \\ -\beta_j e^{ik_{z,j}z_j} & \alpha_j e^{ik_{z,j}z_j} \end{pmatrix} \cdot \begin{pmatrix} \alpha_{j-1} e^{ik_{z,j-1}z_j} & -\alpha_{j-1} e^{-ik_{z,j-1}z_j} \\ \beta_{j-1} e^{ik_{z,j-1}z_j} & \beta_{j-1} e^{-ik_{z,j-1}z_j} \end{pmatrix} \\
&= \frac{1}{2\alpha_j\beta_j} \begin{pmatrix} \left(\alpha_{j-1}\beta_j + \alpha_j\beta_{j-1}\right) e^{i\left(k_{z,j-1}-k_{z,j}\right)z_j} & \left(-\alpha_{j-1}\beta_j + \alpha_j\beta_{j-1}\right) e^{-i\left(k_{z,j-1}+k_{z,j}\right)z_j} \\ \left(-\alpha_{j-1}\beta_j + \alpha_j\beta_{j-1}\right) e^{i\left(k_{z,j-1}+k_{z,j}\right)z_j} & \left(\alpha_{j-1}\beta_j + \alpha_j\beta_{j-1}\right) e^{-i\left(k_{z,j-1}-k_{z,j}\right)z_j} \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\mathbf{T}(z) &= \frac{1}{2\alpha(z)\beta(z)} \begin{pmatrix} \beta(z) e^{-ik_z(z)z} & \alpha(z) e^{-ik_z(z)z} \\ -\beta(z) e^{ik_z(z)z} & \alpha(z) e^{ik_z(z)z} \end{pmatrix} \cdot \begin{pmatrix} \alpha(z-\delta z) e^{ik_z(z-\delta z)z} & -\alpha(z-\delta z) e^{-ik_z(z-\delta z)z} \\ \beta(z-\delta z) e^{ik_z(z-\delta z)z} & \beta(z-\delta z) e^{-ik_z(z-\delta z)z} \end{pmatrix} \\
&= \frac{1}{2\alpha(z)\beta(z)} \begin{pmatrix} (\alpha(z-\delta z)\beta(z) + \alpha(z)\beta(z-\delta z)) e^{i(k_z(z-\delta z)-k_z(z))z} & (-\alpha(z-\delta z)\beta(z) + \alpha(z)\beta(z-\delta z)) e^{-i(k_z(z-\delta} \\ (-\alpha(z-\delta z)\beta(z) + \alpha(z)\beta(z-\delta z)) e^{i(k_z(z-\delta z)+k_z(z))z} & (\alpha(z-\delta z)\beta(z) + \alpha(z)\beta(z-\delta z)) e^{-i(k_z(z-\delta z}
\end{aligned}$$

$$\begin{aligned}
\alpha(z-\delta z)\beta(z) + \alpha(z)\beta(z-\delta z) &= 2\alpha(z)\beta(z) - \beta(z)\frac{\mathrm{d}\alpha}{\mathrm{d}z}\delta z - \alpha(z)\frac{\mathrm{d}\beta}{\mathrm{d}z}\delta z \\
&= 2\alpha(z)\beta(z) - \frac{\mathrm{d}(\alpha\beta)}{\mathrm{d}z}\delta z
\end{aligned}$$

$$\begin{aligned}
-\alpha(z-\delta z)\beta(z) + \alpha(z)\beta(z-\delta z) &= \beta(z)\frac{\mathrm{d}\alpha}{\mathrm{d}z}\delta z - \alpha(z)\frac{\mathrm{d}\beta}{\mathrm{d}z}\delta z \\
&= \beta^2(z)\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\alpha}{\beta}\right)\delta z
\end{aligned}$$

$$k_z(z-\delta z) - k_z(z) = -\frac{\mathrm{d}k_z}{\mathrm{d}z}\delta z$$

$$k_z(z-\delta z) + k_z(z) = 2k_z(z) - \frac{\mathrm{d}k_z}{\mathrm{d}z}\delta z$$

$$
\begin{aligned}
\mathbf{T}(z) &= \frac{1}{2\alpha(z)\beta(z)} \begin{pmatrix} \left(2\alpha(z)\beta(z) - \frac{\mathrm{d}(\alpha\beta)}{\mathrm{d}z}\delta z\right) e^{-i\left(\frac{\mathrm{d}k_z}{\mathrm{d}z}\delta z\right)z} & \left(\beta^2(z)\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\alpha}{\beta}\right)\delta z\right) e^{-i\left(2k_z(z) - \frac{\mathrm{d}k_z}{\mathrm{d}z}\delta z\right)z} \\ \left(\beta^2(z)\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\alpha}{\beta}\right)\delta z\right) e^{i\left(2k_z(z) - \frac{\mathrm{d}k_z}{\mathrm{d}z}\delta z\right)z} & \left(2\alpha(z)\beta(z) - \frac{\mathrm{d}(\alpha\beta)}{\mathrm{d}z}\delta z\right) e^{i\left(\frac{\mathrm{d}k_z}{\mathrm{d}z}\delta z\right)z} \end{pmatrix} \\
&= \begin{pmatrix} e^{-i\left(\frac{\mathrm{d}k_z}{\mathrm{d}z}\delta z\right)z} & \\ & e^{i\left(\frac{\mathrm{d}k_z}{\mathrm{d}z}\delta z\right)z} \end{pmatrix} + \frac{1}{2\alpha(z)\beta(z)} \begin{pmatrix} \left(-\frac{\mathrm{d}(\alpha\beta)}{\mathrm{d}z}\delta z\right) e^{-i\left(\frac{\mathrm{d}k_z}{\mathrm{d}z}\delta z\right)z} & \left(\beta^2(z)\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\alpha}{\beta}\right)\delta z\right) e^{-i\left(2k_z(z) - \frac{\mathrm{d}k_z}{\mathrm{d}z}\delta z\right)z} \\ \left(\beta^2(z)\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\alpha}{\beta}\right)\delta z\right) e^{i\left(2k_z(z) - \frac{\mathrm{d}k_z}{\mathrm{d}z}\delta z\right)z} & \left(-\frac{\mathrm{d}(\alpha\beta)}{\mathrm{d}z}\delta z\right) e^{i\left(\frac{\mathrm{d}k_z}{\mathrm{d}z}\delta z\right)z} \end{pmatrix} \\
&= \begin{pmatrix} e^{-i\mathrm{d}k_z z} & \\ & e^{i\mathrm{d}k_z z} \end{pmatrix} + \frac{1}{2\alpha(z)\beta(z)} \begin{pmatrix} -\mathrm{d}(\alpha\beta) e^{-i\mathrm{d}k_z z} & \beta^2(z)\mathrm{d}\left(\frac{\alpha}{\beta}\right) e^{-i(2k_z(z) - \mathrm{d}k_z)z} \\ \beta^2(z)\mathrm{d}\left(\frac{\alpha}{\beta}\right) e^{i(2k_z(z) - \mathrm{d}k_z)z} & -\mathrm{d}(\alpha\beta) e^{i\mathrm{d}k_z z} \end{pmatrix}
\end{aligned}
$$

$$
\mathbf{T}(z) = \mathbb{I} + \frac{1}{2\alpha(z)\beta(z)} \begin{pmatrix} -\mathrm{d}(\alpha\beta) & \beta^2(z)\mathrm{d}\left(\frac{\alpha}{\beta}\right) e^{-i2k_z(z)z} \\ \beta^2(z)\mathrm{d}\left(\frac{\alpha}{\beta}\right) e^{i2k_z(z)z} & -\mathrm{d}(\alpha\beta) \end{pmatrix}
$$

$$
\delta\mathbf{T} = \frac{1}{2\alpha(z)\beta(z)} \begin{pmatrix} -\mathrm{d}(\alpha\beta) & \beta^2(z)\mathrm{d}\left(\frac{\alpha}{\beta}\right) e^{-i2k_z(z)z} \\ \beta^2(z)\mathrm{d}\left(\frac{\alpha}{\beta}\right) e^{i2k_z(z)z} & -\mathrm{d}(\alpha\beta) \end{pmatrix}
$$

$$
\prod_{z_j} \mathbf{T}(z_j) = \prod_{z_j} \left(\mathbb{I} + \delta\mathbf{T}(z_j)\right) \approx \mathbb{I} + \sum_{z_j} \delta\mathbf{T}(z_j) + \mathcal{O}(\delta z^2)
$$

For a sample with multiple interfaces

$$
\mathbf{E}_N = \left(\prod_{n=1}^{N} \cdot \mathbf{T}_n\right) \cdot \mathbf{E}_0
$$

Explicitly,

$$
E_{N,z} = T_{N,zy} T_{N-1,yx} T_{N-2,xw} \dots T_{3,dc} T_{2,cb} T_{1,ba} E_{0,a}
$$

This is a matrix screw product
The second polarisation

$$
\mathbf{M}_{j,j+1} = \begin{pmatrix} \cos\left(k_{z,j} d_j\right) & k_{z,j}^{-1} \sin\left(k_{z,j} d_j\right) \\ k_{z,j} \sin\left(k_{z,j} d_j\right) & \cos\left(k_{z,j} d_j\right) \end{pmatrix}
$$

$$
\mathbf{M}(z) = \begin{pmatrix} \cos\left(k_z(z) d\right) & k_z(z)^{-1} \sin\left(k_z(z) d\right) \\ k_z(z) \sin\left(k_z(z) d\right) & \cos\left(k_z(z) d\right) \end{pmatrix}
$$

$$
\begin{aligned}
\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}z} &= \begin{pmatrix} -\sin\left(k_z(z) d\right) \frac{\mathrm{d}k_z}{\mathrm{d}z} d & -k_z(z)^{-2} \sin\left(k_z(z) d\right) \frac{\mathrm{d}k_z}{\mathrm{d}z} + k_z(z)^{-1} \cos\left(k_z(z) d\right) \frac{\mathrm{d}k_z}{\mathrm{d}z} d \\ \sin\left(k_z(z) d\right) \frac{\mathrm{d}k_z}{\mathrm{d}z} + k_z(z) \cos\left(k_z(z) d\right) \frac{\mathrm{d}k_z}{\mathrm{d}z} d & -\sin\left(k_z(z) d\right) \frac{\mathrm{d}k_z}{\mathrm{d}z} d \end{pmatrix} \\
&= \begin{pmatrix} -\sin\left(k_z(z) d\right) & k_z(z)^{-1} \cos\left(k_z(z) d\right) \\ k_z(z) \cos\left(k_z(z) d\right) & -\sin\left(k_z(z) d\right) \end{pmatrix} \frac{\mathrm{d}k_z}{\mathrm{d}z} d + \begin{pmatrix} 0 & -k_z(z)^{-2} \sin\left(k_z(z) d\right) \\ \sin\left(k_z(z) d\right) & 0 \end{pmatrix} \frac{\mathrm{d}k_z}{\mathrm{d}z}
\end{aligned}
$$

$$\lim_{\delta z \to 0} \mathbf{M}(z) = \begin{pmatrix} 1 & \delta z \\ k_z^2(z)\delta z & 1 \end{pmatrix} = \mathbb{I} + \delta\mathbf{M}(z)$$

$$\delta\mathbf{M}(z) = \begin{pmatrix} 0 & 1 \\ k_z^2(z) & 0 \end{pmatrix} \delta z$$

$$\prod_{z_j} \mathbf{M}(z_j) = \prod_{z_j} \left(\mathbb{I} + \delta\mathbf{M}(z_j)\right) \approx \mathbb{I} + \sum_{z_j} \delta\mathbf{M}(z_j) + \mathcal{O}\left(\delta z^2\right)$$

$$\sum_{z_j} \delta\mathbf{M}(z_j) = \int_a^b \mathrm{d}z \begin{pmatrix} 0 & 1 \\ k_z^2(z) & 0 \end{pmatrix}$$

$$\begin{aligned} k_z^2(z) &= \frac{(2\pi)^2 n^2(z)\alpha^2(z)}{\lambda^2} = \frac{(2\pi)^2 n^2(z)}{\lambda^2}\left(1 - \frac{\cos^2\theta_0}{n^2(z)}\right) \\ &= \frac{(2\pi)^2}{\lambda^2}\left[n^2(z) - \cos^2\theta_0\right] \end{aligned}$$

$$\begin{aligned} \mathbf{T} &= \mathbf{S}^{-1}(b) \cdot \begin{pmatrix} 1 & b-a \\ \int_a^b \mathrm{d}z\, k_z^2(z) & 1 \end{pmatrix} \mathbf{S}(a) \\ &= \frac{1}{2k_z(b)} \begin{pmatrix} k_z(b)e^{-ik_z(b)b} & e^{-ik_z(b)b} \\ k_z(b)e^{ik_z(b)b} & -e^{ik_z(b)b} \end{pmatrix} \begin{pmatrix} 1 & b-a \\ \int_a^b \mathrm{d}z\, k_z^2(z) & 1 \end{pmatrix} \begin{pmatrix} e^{ik_z(a)a} & e^{-ik_z(a)a} \\ k_z(a)e^{ik_z(a)a} & -k_z(a)e^{-ik_z(a)a} \end{pmatrix} \\ &= \frac{1}{2k_z(b)} \begin{pmatrix} k_z(b)e^{-ik_z(b)b} & e^{-ik_z(b)b} \\ k_z(b)e^{ik_z(b)b} & -e^{ik_z(b)b} \end{pmatrix} \begin{pmatrix} e^{ik_z(a)a} + (b-a)k_z(a)e^{ik_z(a)a} & e^{-ik_z(a)a} - (b-a)k_z(a)e^{-ik_z(a)a} \\ e^{ik_z(a)a}\int_a^b \mathrm{d}z\, k_z^2(z) + k_z(a)e^{ik_z(a)a} & e^{-ik_z(a)a}\int_a^b \mathrm{d}z\, k_z^2(z) - k_z(a)e^{-ik_z(a)a} \end{pmatrix} \\ &= \frac{1}{2k_z(b)} \begin{pmatrix} k_z(b)e^{-ik_z(b)b} & e^{-ik_z(b)b} \\ k_z(b)e^{ik_z(b)b} & -e^{ik_z(b)b} \end{pmatrix} \begin{pmatrix} e^{ik_z(a)a}(1 + (b-a)k_z(a)) & e^{-ik_z(a)a}(1 - (b-a)k_z(a)) \\ e^{ik_z(a)a}\left(\int_a^b \mathrm{d}z\, k_z^2(z) + k_z(a)\right) & e^{-ik_z(a)a}\left(\int_a^b \mathrm{d}z\, k_z^2(z) - k_z(a)\right) \end{pmatrix} \\ &= \frac{1}{2k_z(b)} \left(\begin{matrix} k_z(b)e^{-ik_z(b)b}e^{ik_z(a)a}(1 + (b-a)k_z(a)) + e^{-ik_z(b)b}e^{ik_z(a)a}\left(\int_a^b \mathrm{d}z\, k_z^2(z) + k_z(a)\right) & k_z(b)e^{-ik_z(b)b}e^{-ik_z(a)a}(1 - \\ k_z(b)e^{ik_z(b)b}e^{ik_z(a)a}(1 + (b-a)k_z(a)) - e^{ik_z(b)b}e^{ik_z(a)a}\left(\int_a^b \mathrm{d}z\, k_z^2(z) + k_z(a)\right) & k_z(b)e^{ik_z(b)b}e^{-ik_z(a)a}(1 - \end{matrix}\right. \\ &= \frac{1}{2k_z(b)} \left(\begin{matrix} e^{-ik_z(b)b}e^{ik_z(a)a}\left(k_z(b)(1 + (b-a)k_z(a)) + k_z(a) + \int_a^b \mathrm{d}z\, k_z^2(z)\right) & e^{-ik_z(b)b}e^{-ik_z(a)a}\left(k_z(b)(1 - (b-a)k_z(a\right. \\ e^{ik_z(b)b}e^{ik_z(a)a}\left(k_z(b)(1 + (b-a)k_z(a)) - k_z(a) - \int_a^b \mathrm{d}z\, k_z^2(z)\right) & e^{ik_z(b)b}e^{-ik_z(a)a}\left(k_z(b)(1 - (b-a)k_z(a)\right. \end{matrix}\right. \end{aligned}$$

$$r = -\frac{L_{21}}{L_{22}} \qquad\qquad t = L_{11} - \frac{L_{12}L_{21}}{L_{22}}$$

$$r = -e^{i2k_z(a)a}\frac{k_z(b) - k_z(a) + (b-a)k_z(a)k_z(b) - \int_a^b \mathrm{d}z\, k_z^2(z)}{k_z(b) + k_z(a) - (b-a)k_z(a)k_z(b) - \int_a^b \mathrm{d}z\, k_z^2(z)}$$

We new define the propagation matrix as,

$$\mathbf{W}_j\left(\mathbf{r}_{j+1}, t\right) = \mathbf{M}_{j,j+1} \cdot \mathbf{W}_j\left(\mathbf{r}_j, t\right)$$

It can be easily shown that,

$$\mathbf{M}_{j,j+1} = \mathbf{S}_j\left(\mathbf{r}_{j+1}, t\right) \cdot \mathbf{S}_j^{-1}\left(\mathbf{r}_j, t\right)$$

The the parallele polarisation this yield,

$$\begin{aligned}
\mathbf{M}^{(1)}_{j,j+1} &= \frac{1}{2\alpha_j\beta_j}\begin{pmatrix} \alpha_j e^{ik_{z,j}z_{j+1}} & -\alpha_j e^{-ik_{z,j}z_{j+1}} \\ \beta_j e^{ik_{z,j}z_{j+1}} & \beta_j e^{-ik_{z,j}z_{j+1}} \end{pmatrix} \cdot \begin{pmatrix} \beta_j e^{-ik_{z,j}z_j} & \alpha_j e^{-ik_{z,j}z_j} \\ -\beta_j e^{ik_{z,j}z_j} & \alpha_j e^{ik_{z,j}z_j} \end{pmatrix} \\
&= \frac{1}{2\alpha_j\beta_j}\begin{pmatrix} \alpha_j\beta_j\left(e^{ik_{z,j}\left(z_{j+1}-z_j\right)} + e^{-ik_{z,j}\left(z_{j+1}-z_j\right)}\right) & \alpha_j^2\left(e^{ik_{z,j}\left(z_{j+1}-z_j\right)} - e^{-ik_{z,j}\left(z_{j+1}-z_j\right)}\right) \\ \beta_j^2\left(e^{ik_{z,j}\left(z_{j+1}-z_j\right)} - e^{-ik_{z,j}\left(z_{j+1}-z_j\right)}\right) & \alpha_j\beta_j\left(e^{ik_{z,j}\left(z_{j+1}-z_j\right)} + e^{-ik_{z,j}\left(z_{j+1}-z_j\right)}\right) \end{pmatrix} \\
&= \begin{pmatrix} \cos\left(k_{z,j}d_j\right) & \frac{\alpha_j}{\beta_j}\sin\left(k_{z,j}d_j\right) \\ \frac{\beta_j}{\alpha_j}\sin\left(k_{z,j}d_j\right) & \cos\left(k_{z,j}d_j\right) \end{pmatrix}
\end{aligned}$$

The second polarisation

$$\begin{aligned}
\mathbf{M}^{(2)}_{j,j+1} &= \frac{1}{-2k_{z,j}}\begin{pmatrix} e^{ik_{z,j}z_{j+1}} & e^{-ik_{z,j}z_{j+1}} \\ k_{z,j}e^{ik_{z,j}z_{j+1}} & -k_{z,j}e^{-ik_{z,j}z_{j+1}} \end{pmatrix} \cdot \begin{pmatrix} -k_{z,j}e^{-ik_{z,j}z_j} & -e^{-ik_{z,j}z_j} \\ -k_{z,j}e^{ik_{z,j}z_j} & e^{ik_{z,j}z_j} \end{pmatrix} \\
&= \begin{pmatrix} \cos\left(k_{z,j}d_j\right) & k_{z,j}^{-1}\sin\left(k_{z,j}d_j\right) \\ k_{z,j}\sin\left(k_{z,j}d_j\right) & \cos\left(k_{z,j}d_j\right) \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\gamma^{(1)} &= \frac{\beta}{\alpha} = \frac{\epsilon\cos\theta}{\sin\theta} = \frac{n^2\cos\theta}{\mu\sin\theta} = \frac{\lambda^2k^2\cos\theta}{\mu\sin\theta} \\
\gamma^{(2)} &= k\sin\theta
\end{aligned}$$

If it goes through,

$$\mathbf{E}_0 = A\begin{pmatrix} 1 \\ r \end{pmatrix} \qquad\qquad \mathbf{E}_2 = A\begin{pmatrix} t \\ 0 \end{pmatrix}$$

$$\mathbf{W}_{N+1}(z_N) = \mathbf{W}_N(z_N) = \mathbf{M}_{0,N} \cdot \mathbf{W}_0(z_0)$$

$$\mathbf{E}_N = \mathbf{S}^{-1}_{N+1}(z_N) \cdot \mathbf{M}_{0,N} \cdot \mathbf{S}_0(z_0) \cdot \mathbf{E}_0$$

$$\mathbf{L} = \mathbf{S}^{-1}_{N+1}(z_N) \cdot \mathbf{M}_{0,N} \cdot \mathbf{S}_0(z_0)$$

$$\begin{pmatrix} t \\ 0 \end{pmatrix} = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ r \end{pmatrix}$$

$$r = -\frac{L_{21}}{L_{22}} \qquad\qquad t = L_{11} - \frac{L_{12}L_{21}}{L_{22}}$$

If a plane reflects,

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ r \end{pmatrix}$$

$$r = -\frac{L_{11}}{L_{12}} \qquad\qquad r = -\frac{L_{21}}{L_{22}}$$

$$\mathbf{M}_{0,1} = \mathbf{S}_1(z_1) \cdot \mathbf{S}_1^{-1}(z_0)$$

Test 1 layer, case 1.

$$\begin{aligned}
\mathbf{SMS} &= \mathbf{S}_2^{-1}(z_1) \cdot \mathbf{M}_{0,1} \cdot \mathbf{S}_0(z_0) \\
&= \frac{1}{\alpha_0\beta_0} \begin{pmatrix} \beta_0 e^{-ik_{z,0}z_1} & \alpha_0 e^{-ik_{z,0}z_1} \\ -\beta_0 e^{ik_{z,0}z_1} & \alpha_0 e^{ik_{z,0}z_1} \end{pmatrix} \cdot \begin{pmatrix} \cos\left(k_{z,1}d_1\right) & \frac{\alpha_1}{\beta_1}\sin\left(k_{z,1}d_1\right) \\ \frac{\beta_1}{\alpha_1}\sin\left(k_{z,1}d_1\right) & \cos\left(k_{z,1}d_1\right) \end{pmatrix} \cdot \begin{pmatrix} \alpha_0 e^{ik_{z,0}z_0} & -\alpha_0 e^{-ik_{z,0}z_0} \\ \beta_0 e^{ik_{z,0}z_0} & \beta_0 e^{-ik_{z,0}z_0} \end{pmatrix}
\end{aligned}$$

Iterative

$$\begin{aligned}
\mathbf{M}^{(2)}_{j-1,j+1} &= \begin{pmatrix} \cos\left(k_{z,j}d_j\right) & k_{z,j}^{-1}\sin\left(k_{z,j}d_j\right) \\ k_{z,j}\sin\left(k_{z,j}d_j\right) & \cos\left(k_{z,j}d_j\right) \end{pmatrix} \cdot \begin{pmatrix} \cos\left(k_{z,j-1}d_{j-1}\right) & k_{z,j-1}^{-1}\sin\left(k_{z,j-1}d_{j-1}\right) \\ k_{z,j-1}\sin\left(k_{z,j-1}d_{j-1}\right) & \cos\left(k_{z,j-1}d_{j-1}\right) \end{pmatrix} \\
&= \Bigg(\begin{matrix} \cos\left(k_{z,j}d_j\right)\cos\left(k_{z,j-1}d_{j-1}\right) + \frac{k_{z,j-1}}{k_{z,j}}\sin\left(k_{z,j}d_j\right)\sin\left(k_{z,j-1}d_{j-1}\right) & k_{z,j-1}^{-1}\cos\left(k_{z,j}d_j\right)\sin\left(k_{z,j-1}d_{j-1}\right) + k \\ k_{z,j-1}\cos\left(k_{z,j}d_j\right)\sin\left(k_{z,j-1}d_{j-1}\right) + k_{z,j}\sin\left(k_{z,j}d_j\right)\cos\left(k_{z,j-1}d_{j-1}\right) & \cos\left(k_{z,j}d_j\right)\cos\left(k_{z,j-1}d_{j-1}\right) + \frac{k_{z,j}}{k_{z,j-1}} \end{matrix}
\end{aligned}$$

$$\begin{aligned}
\mathbf{M}_{0,N} &= \mathbf{M}_{N-1,N} \cdot \mathbf{M}_{0,N-1} \\
&= \begin{pmatrix} \cos\left(k_{z,N}d_N\right) & k_{z,N}^{-1}\sin\left(k_{z,N}d_N\right) \\ k_{z,N}\sin\left(k_{z,N}d_N\right) & \cos\left(k_{z,N}d_N\right) \end{pmatrix} \cdot \begin{pmatrix} a_{N-1} & b_{n-1} \\ c_{n-1} & a_{N-1} \end{pmatrix} \\
&= \begin{pmatrix} a_{N-1}\cos\left(k_{z,N}d_N\right) + c_{n-1}k_{z,N}^{-1}\sin\left(k_{z,N}d_N\right) & b_{N-1}\cos\left(k_{z,N}d_N\right) + a_{n-1}k_{z,N}^{-1}\sin\left(k_{z,N}d_N\right) \\ a_{N-1}k_{z,N}\sin\left(k_{z,N}d_N\right) + c_{n-1}\cos\left(k_{z,N}d_N\right) & b_{N-1}k_{z,N}\sin\left(k_{z,N}d_N\right) + a_{n-1}\cos\left(k_{z,N}d_N\right) \end{pmatrix} \\
&= \begin{pmatrix} a_{N-1}\left(1 - \frac{k_{z,N}^2 d_N^2}{2}\right) + c_{n-1}d_N & b_{N-1}\left(1 - \frac{k_{z,N}^2 d_N^2}{2}\right) + a_{n-1}d_N \\ a_{N-1}k_{z,N}^2 d_N + c_{n-1}\left(1 - \frac{k_{z,N}^2 d_N^2}{2}\right) & b_{N-1}k_{z,N}^2 d_N + a_{n-1}\left(1 - \frac{k_{z,N}^2 d_N^2}{2}\right) \end{pmatrix}
\end{aligned}$$

$$a_n = a_{n-1}\left(1 - \frac{k_{z,N}^2 d_N^2}{2}\right) + c_{n-1}d_n \qquad\qquad b_n = b_{n-1}\left(1 - \frac{k_{z,N}^2 d_N^2}{2}\right) + a_{n-1}d_n$$

$$c_{n-1} = \frac{a_n - a_{n-1}}{d_n} + a_{n-1}\frac{k_{z,N}^2 d_N}{2} \qquad\qquad a_{n-1} = \frac{b_n - b_{n-1}}{d_n}$$

$$c(x) = \frac{\mathrm{d}a}{\mathrm{d}x}$$

$$\begin{aligned}\mathbf{W}_0(z) &= \mathbf{S}_j(z)\cdot\mathbf{E}_0\\ &= \begin{pmatrix} A\alpha_0 e^{ik_{z,0}z} - rA\alpha_0 e^{-ik_{z,0}z}\\ A\beta_0 e^{ik_{z,0}z} + rA\beta_0 e^{-ik_{z,0}z}\end{pmatrix}\end{aligned}$$

$$\begin{aligned}\mathbf{W}_2(z) &= \mathbf{S}_j(z)\cdot\mathbf{E}_2\\ &= \begin{pmatrix} tA\alpha_0 e^{ik_{z,0}z}\\ tA\beta_0 e^{ik_{z,0}z}\end{pmatrix}\end{aligned}$$

$$\mathbf{M}^{(1)} = \begin{pmatrix} \cos\left(k_{z,1}d_1\right) & \frac{\alpha_1}{\beta_1}\sin\left(k_{z,1}d_1\right)\\ \frac{\beta_1}{\alpha_1}\sin\left(k_{z,1}d_1\right) & \cos\left(k_{z,1}d_1\right)\end{pmatrix}$$

$$\begin{aligned}\mathbf{W_2}(z_1) &= \mathbf{M}\cdot\mathbf{W_0}(z_0)\\ \begin{pmatrix} tA\alpha_0 e^{ik_{z,0}z}\\ tA\beta_0 e^{ik_{z,0}z}\end{pmatrix} &= \begin{pmatrix} \cos\left(k_{z,1}d_1\right) & \frac{\alpha_1}{\beta_1}\sin\left(k_{z,1}d_1\right)\\ \frac{\beta_1}{\alpha_1}\sin\left(k_{z,1}d_1\right) & \cos\left(k_{z,1}d_1\right)\end{pmatrix}\cdot\begin{pmatrix} A\alpha_0 e^{ik_{z,0}z} - rA\alpha_0 e^{-ik_{z,0}z}\\ A\beta_0 e^{ik_{z,0}z} + rA\beta_0 e^{-ik_{z,0}z}\end{pmatrix}\end{aligned}$$

$$\begin{pmatrix} t\alpha_0 e^{ik_{z,0}z}\\ t\beta_0 e^{ik_{z,0}z}\end{pmatrix} = \begin{pmatrix} \cos\left(k_{z,1}d_1\right) & \frac{\alpha_1}{\beta_1}\sin\left(k_{z,1}d_1\right)\\ \frac{\beta_1}{\alpha_1}\sin\left(k_{z,1}d_1\right) & \cos\left(k_{z,1}d_1\right)\end{pmatrix}\cdot\begin{pmatrix} \alpha_0 e^{ik_{z,0}z} - rA\alpha_0 e^{-ik_{z,0}z}\\ \beta_0 e^{ik_{z,0}z} + rA\beta_0 e^{-ik_{z,0}z}\end{pmatrix}$$

Index of refraction for X-Rays is smaller then 1,

$$n = 1 - \delta = 1 - \frac{1}{2\pi} r_e N \lambda^2$$

where $r_e = 2.818 \times 10^{-15}$ m is the electron radius, N is the electronic density, and λ is the wavelength.

$$v = \frac{c}{n} = \frac{\lambda}{n\nu} = \frac{2\pi}{n\nu k}$$

$$Z_j = -\frac{i}{k_j} \qquad\qquad Z_j^{-1} = ik_j$$

$$E_j(z) = E_j^{(r)} e^{ik_j z} + E_j^{(l)} e^{-ik_j z}$$

$$H_j(z) = \frac{1}{Z_j}\left(E_j^{(r)} e^{ik_j z} - E_j^{(l)} e^{-ik_j z}\right)$$

$$\mathbf{W}_j(z) \equiv \begin{pmatrix} E_j(z) \\ H_j(z) \end{pmatrix}$$

$$E_{j-1}(z_j) = E_j(z_j) \qquad\qquad H_{j-1}(z_j) = H_j(z_j)$$

This can also be writen,

$$\mathbf{W}_{j-1}(z_j) = \mathbf{W}_j(z_j)$$

where,

$$E_{j-1}^{(r)} e^{ik_{j-1}z_j} + E_{j-1}^{(l)} e^{-ik_{j-1}z_j} = E_j^{(r)} e^{ik_j z_j} + E_j^{(l)} e^{-ik_j z_j}$$
$$\frac{1}{Z_{j-1}}\left(E_{j-1}^{(r)} e^{ik_{j-1}z_j} - E_j^{(l)} e^{-ik_{j-1}z_j}\right) = \frac{1}{Z_j}\left(E_j^{(r)} e^{ik_j z_j} - E_j^{(l)} e^{-ik_j z_j}\right)$$

$$\mathbf{S}_{j-1}(z_j) \cdot \mathbf{E}_{j-1} = \mathbf{S}_j(z_j) \cdot \mathbf{E}_j$$

with,

$$\mathbf{S}_j(z) = \begin{pmatrix} e^{ik_j z} & e^{-ik_j z} \\ Z_j^{-1} e^{ik_j z} & -Z_j^{-1} e^{-ik_j z} \end{pmatrix} \qquad\qquad \mathbf{E}_j = \begin{pmatrix} E_j^{(r)} \\ E_j^{(l)} \end{pmatrix}$$

$$\mathbf{W}_j(z) = \mathbf{S}_j(z) \cdot \mathbf{E}_j$$

We define the propagation matrix as,

$$\mathbf{W}_j(z_{j+1}) = \mathbf{M}_{j,j+1} \cdot \mathbf{W}_j(z_j)$$

$$\mathbf{S}_j(z_{j+1}) \cdot \mathbf{E}_j = \mathbf{M}_{j,j+1} \cdot \mathbf{S}_j(z_j) \cdot \mathbf{E}_j$$

It can be easily shown that,

$$\mathbf{M}_{j,j+1} = \mathbf{S}_j(z_{j+1}) \cdot \mathbf{S}_j^{-1}(z_j)$$

$$\mathbf{S}_j^{-1}(z) = \frac{Z_j}{2} \begin{pmatrix} Z_j^{-1} e^{-ik_j z} & e^{-ik_j z} \\ Z_j^{-1} e^{ik_j z} & -e^{ik_j z} \end{pmatrix}$$

$$\mathbf{M}_{j,j+1} = \frac{Z_j}{2} \begin{pmatrix} e^{ik_j z_{j+1}} & e^{-ik_j z_{j+1}} \\ Z_j^{-1} e^{ik_j z_{j+1}} & -Z_j^{-1} e^{-ik_j z_{j+1}} \end{pmatrix} \cdot \begin{pmatrix} Z_j^{-1} e^{-ik_j z_j} & e^{-ik_j z_j} \\ Z_j^{-1} e^{ik_j z_j} & -e^{ik_j z_j} \end{pmatrix}$$

$$d_{j,j+1} = z_{j+1} - z_j$$

$$\begin{aligned} \mathbf{M}_{j,j+1} &= \frac{Z_j}{2} \begin{pmatrix} Z_j^{-1}\left(e^{ik_j d_{j,j+1}} + e^{-ik_j d_{j,j+1}}\right) & e^{ik_j d_{j,j+1}} - e^{-ik_j d_{j,j+1}} \\ Z_j^{-2}\left(e^{ik_j d_{j,j+1}} - e^{-ik_j d_{j,j+1}}\right) & Z_j^{-1}\left(e^{ik_j d_{j,j+1}} + e^{-ik_j d_{j,j+1}}\right) \end{pmatrix} \\ &= \begin{pmatrix} \cos\left(k_j d_{j,j+1}\right) & iZ_j \sin\left(k_j d_{j,j+1}\right) \\ iZ_j^{-1} \sin\left(k_j d_{j,j+1}\right) & \cos\left(k_j d_{j,j+1}\right) \end{pmatrix} \end{aligned}$$

If there is N layer

$$\mathbf{W}_N(z_N) = \mathbf{M}_{0,N} \cdot \mathbf{W}_0(z_0)$$

$$\boldsymbol{\mathcal{M}} = \mathbf{M}_{0,N} = \mathbf{M}_{N-1,N} \cdot \ldots \cdot \mathbf{M}_{1,2} \cdot \mathbf{M}_{0,1}$$

The wave above the layer is,

$$E_0(z) = Ae^{ik_0 z} + rAe^{-ik_0 z}$$

as the wave below,

$$E_N(z) = tAe^{ik_N z}$$

$$\mathbf{W}_0(0) = A \begin{pmatrix} 1 & 1 \\ Z_0^{-1} & -Z_0^{-1} \end{pmatrix} \begin{pmatrix} 1 \\ r \end{pmatrix} = \frac{A}{Z_0} \begin{pmatrix} Z_0(1+r) \\ 1-r \end{pmatrix}$$

$$\mathbf{W}_N(L) = A\begin{pmatrix} e^{ik_NL} & e^{-ik_NL} \\ Z_N^{-1}e^{ik_NL} & -Z_N^{-1}e^{-ik_NL}\end{pmatrix}\begin{pmatrix} t \\ 0\end{pmatrix} = \frac{Ate^{ik_NL}}{Z_N}\begin{pmatrix} Z_N \\ 1\end{pmatrix}$$

$$\frac{Z_0te^{ik_NL}}{Z_N}\begin{pmatrix} Z_N \\ 1\end{pmatrix} = \begin{pmatrix}\mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22}\end{pmatrix}\cdot\begin{pmatrix} Z_0(1+r) \\ 1-r\end{pmatrix}$$

$$\begin{aligned} Z_0te^{ik_NL} &= \mathcal{M}_{11}Z_0(1+r) + \mathcal{M}_{12}(1-r) \\ Z_0te^{ik_NL} &= \mathcal{M}_{21}Z_0Z_N(1+r) + \mathcal{M}_{22}Z_N(1-r)\end{aligned}$$

$$\begin{aligned} Z_0te^{ik_NL} &= \mathcal{M}_{11}Z_0 + \mathcal{M}_{12} + r(\mathcal{M}_{11}Z_0 - \mathcal{M}_{12}) \\ Z_0te^{ik_NL} &= \mathcal{M}_{21}Z_0Z_N + \mathcal{M}_{22}Z_N + r(\mathcal{M}_{21}Z_0Z_N - \mathcal{M}_{22}Z_N)\end{aligned}$$

$$\begin{aligned} Z_0te^{ik_NL} - r(\mathcal{M}_{11}Z_0 - \mathcal{M}_{12}) &= \mathcal{M}_{11}Z_0 + \mathcal{M}_{12} \\ Z_0te^{ik_NL} - rZ_N(\mathcal{M}_{21}Z_0 - \mathcal{M}_{22}) &= Z_N(\mathcal{M}_{21}Z_0 + \mathcal{M}_{22})\end{aligned}$$

$$\begin{pmatrix} Z_0e^{ik_NL} & \mathcal{M}_{12} - \mathcal{M}_{11}Z_0 \\ Z_0e^{ik_NL} & Z_N(\mathcal{M}_{22} - \mathcal{M}_{21}Z_0)\end{pmatrix}\begin{pmatrix} t \\ r\end{pmatrix} = \begin{pmatrix}\mathcal{M}_{11}Z_0 + \mathcal{M}_{12} \\ Z_N(\mathcal{M}_{21}Z_0 + \mathcal{M}_{22})\end{pmatrix}$$

$$\begin{aligned}\begin{pmatrix} t \\ r\end{pmatrix} &= \frac{1}{Z_0e^{ik_NL}(\mathcal{M}_{22}Z_N - \mathcal{M}_{21}Z_0Z_N) - Z_0e^{ik_NL}(\mathcal{M}_{12} - \mathcal{M}_{11}Z_0)} \\ &\quad\times\begin{pmatrix} Z_N(\mathcal{M}_{22} - \mathcal{M}_{21}Z_0) & \mathcal{M}_{11}Z_0 - \mathcal{M}_{12} \\ -Z_0e^{ik_NL} & Z_0e^{ik_NL}\end{pmatrix}\begin{pmatrix}\mathcal{M}_{11}Z_0 + \mathcal{M}_{12} \\ Z_N(\mathcal{M}_{21}Z_0 + \mathcal{M}_{22})\end{pmatrix}\end{aligned}$$

$$\begin{aligned}\begin{pmatrix} t \\ r\end{pmatrix} &= \frac{1}{Z_0e^{ik_NL}}\frac{1}{Z_N(\mathcal{M}_{22} - \mathcal{M}_{21}Z_0) - (\mathcal{M}_{12} - \mathcal{M}_{11}Z_0)} \\ &\quad\times\begin{pmatrix} Z_N(\mathcal{M}_{22} - \mathcal{M}_{21}Z_0)(\mathcal{M}_{11}Z_0 + \mathcal{M}_{12}) + Z_N(\mathcal{M}_{11}Z_0 - \mathcal{M}_{12})(\mathcal{M}_{21}Z_0 + \mathcal{M}_{22}) \\ -Z_0e^{ik_NL}(\mathcal{M}_{11}Z_0 + \mathcal{M}_{12}) + Z_0e^{ik_NL}(\mathcal{M}_{21}Z_0Z_N + \mathcal{M}_{22}Z_N)\end{pmatrix}\end{aligned}$$

$$\begin{aligned}& Z_N(\mathcal{M}_{22} - \mathcal{M}_{21}Z_0)(\mathcal{M}_{11}Z_0 + \mathcal{M}_{12}) + Z_N(\mathcal{M}_{22} + \mathcal{M}_{21}Z_0)(\mathcal{M}_{11}Z_0 - \mathcal{M}_{12}) \\ &= 2Z_NZ_0(\mathcal{M}_{22}\mathcal{M}_{11} - \mathcal{M}_{21}\mathcal{M}_{12}) \\ &= 2Z_NZ_0\det\{\boldsymbol{\mathcal{M}}\}\end{aligned}$$

$$t = \frac{2ie^{-ik_NL}k_0\det\{\boldsymbol{\mathcal{M}}\}}{-\mathcal{M}_{21} + k_0k_N\mathcal{M}_{12} + i(k_0\mathcal{M}_{22} + k_N\mathcal{M}_{11})}$$

$$r = \frac{\mathcal{M}_{21} + k_0 k_N \mathcal{M}_{12} + i(k_0 \mathcal{M}_{22} - k_N \mathcal{M}_{11})}{-\mathcal{M}_{21} + k_0 k_N \mathcal{M}_{12} + i(k_0 \mathcal{M}_{22} + k_N \mathcal{M}_{11})}$$

Let's write it down in fonction of z for a slice d thick.

$$\begin{aligned}
\mathbf{T}(z) &= \frac{1}{2k_z(z_+)} \begin{pmatrix} k_z(z_+)e^{-ik_z(z_+)z} & e^{-ik_z(z_+)z} \\ k_z(z_+)e^{ik_z(z_+)z} & -e^{ik_z(z_+)z} \end{pmatrix} \begin{pmatrix} e^{ik_z(z_-)z} & e^{-ik_z(z_-)z} \\ k_z(z_-)e^{ik_z(z_-)z} & -k_z(z_-)e^{-ik_z(z_-)z} \end{pmatrix} \\
&= \frac{1}{2k_z(z_+)} \begin{pmatrix} 2k_z(z)e^{-ik_z'(z)\Delta z z} & k_z'(z)\Delta z e^{-i2k_z(z)z} \\ k_z'(z)\Delta z e^{i2k_z(z)z} & 2k_z(z)e^{ik_z'(z)\Delta z z} \end{pmatrix}
\end{aligned}$$

$$k_z(z_+) + k_z(z_-) = 2k_z(z) \qquad\qquad k_z(z_+) - k_z(z_-) = \frac{\mathrm{d}k_z}{\mathrm{d}z}\Delta z = k_z'(z)\Delta z$$

$$\begin{aligned}
\lim_{\Delta z \to 0} \mathbf{T}(z) &= \frac{1}{2k_z(z)} \begin{pmatrix} 2k_z(z)(1 - ik_z'(z)\Delta z z) & k_z'(z)\Delta z e^{-i2k_z(z)z} \\ k_z'(z)\Delta z e^{i2k_z(z)z} & 2k_z(z)(1 + ik_z'(z)\Delta z z) \end{pmatrix} \\
&= \mathbb{I} + \frac{1}{2k_z(z)} \begin{pmatrix} -i2k_z(z)k_z'(z)z & k_z'(z)e^{-i2k_z(z)z} \\ k_z'(z)e^{i2k_z(z)z} & i2k_z(z)k_z'(z)z \end{pmatrix} \Delta z \\
&= \mathbb{I} + \frac{1}{2k_z(z)} \begin{pmatrix} -i2k_z(z)z & e^{-i2k_z(z)z} \\ e^{i2k_z(z)z} & i2k_z(z)z \end{pmatrix} k_z'(z)\Delta z
\end{aligned}$$

$$\delta\mathbf{T} = \frac{1}{2k_z(z)} \begin{pmatrix} -i2k_z(z)z & e^{-i2k_z(z)z} \\ e^{i2k_z(z)z} & i2k_z(z)z \end{pmatrix} k_z'(z)\delta z$$

$$\prod_{z_j} \mathbf{T}(z_j) = \prod_{z_j} \left(\mathbb{I} + \delta\mathbf{T}(z_j)\right) \approx \mathbb{I} + \sum_{z_j} \delta\mathbf{T}(z_j) + \mathcal{O}(\delta z^2)$$

$$k_z(z) = \frac{2\pi}{\lambda}\left(n^2(z) - n_0^2\cos^2\theta_0\right)^{1/2}$$

$$\frac{\mathrm{d}k_z}{\mathrm{d}z} = \left(\frac{2\pi}{\lambda}\right)^2 \frac{n(z)}{k_z(z)} \frac{\mathrm{d}n}{\mathrm{d}z}$$

$$\begin{aligned}
\mathbf{T}_j &= \frac{1}{2k_{z,j}} \begin{pmatrix} k_{z,j}e^{-ik_{z,j}z_j} & e^{-ik_{z,j}z_j} \\ k_{z,j}e^{ik_{z,j}z_j} & -e^{ik_{z,j}z_j} \end{pmatrix} \begin{pmatrix} e^{ik_{z,j-1}z_j} & e^{-ik_{z,j-1}z_j} \\ k_{z,j-1}e^{ik_{z,j-1}z_j} & -k_{z,j-1}e^{-ik_{z,j-1}z_j} \end{pmatrix} \\
&= \frac{1}{2k_{z,j}} \begin{pmatrix} \left(k_{z,j} + k_{z,j-1}\right)e^{-i\left(k_{z,j}-k_{z,j-1}\right)z_j} & \left(k_{z,j} - k_{z,j-1}\right)e^{-i\left(k_{z,j}+k_{z,j-1}\right)z_j} \\ \left(k_{z,j} - k_{z,j-1}\right)e^{i\left(k_{z,j}+k_{z,j-1}\right)z_j} & \left(k_{z,j} + k_{z,j-1}\right)e^{i\left(k_{z,j}-k_{z,j-1}\right)z_j} \end{pmatrix}
\end{aligned}$$

$$\mathbf{T}_j \approx \begin{pmatrix} e^{-i\left(k_{z,j}-k_{z,j-1}\right)z_j} & \frac{k_{z,j}-k_{z,j-1}}{k_{z,j}+k_{z,j-1}}e^{-i\left(k_{z,j}+k_{z,j-1}\right)z_j} \\ \frac{k_{z,j}-k_{z,j-1}}{k_{z,j}+k_{z,j-1}}e^{i\left(k_{z,j}+k_{z,j-1}\right)z_j} & e^{i\left(k_{z,j}-k_{z,j-1}\right)z_j} \end{pmatrix}$$

$$\mathbf{T}_{j+1} = \frac{1}{2k_{z,j+1}} \begin{pmatrix} \left(k_{z,j+1} + k_{z,j}\right) e^{-i\left(k_{z,j+1} - k_{z,j}\right) z_{j+1}} & \left(k_{z,j+1} - k_{z,j}\right) e^{-i\left(k_{z,j+1} + k_{z,j}\right) z_{j+1}} \\ \left(k_{z,j+1} - k_{z,j}\right) e^{i\left(k_{z,j+1} + k_{z,j}\right) z_{j+1}} & \left(k_{z,j+1} + k_{z,j}\right) e^{i\left(k_{z,j+1} - k_{z,j}\right) z_{j+1}} \end{pmatrix}$$

$$\mathbf{T}_{j+1} \cdot \mathbf{T}_j = \frac{1}{4k_{z,j}k_{z,j+1}} \Bigg(\begin{matrix} \left(k_{z,j+1} + k_{z,j}\right)\left(k_{z,j} + k_{z,j-1}\right) e^{-i\left(k_{z,j+1} - k_{z,j}\right) z_{j+1}} e^{-i\left(k_{z,j} - k_{z,j-1}\right) z_j} + \left(k_{z,j+1} - k_{z,j}\right)\left(k_{z,j} - k_{z,j-1}\right) e^{-i\left(k_{z,j\cdots\right.}} \\ \left(k_{z,j+1} - k_{z,j}\right)\left(k_{z,j} + k_{z,j-1}\right) e^{i\left(k_{z,j+1} + k_{z,j}\right) z_{j+1}} e^{-i\left(k_{z,j} - k_{z,j-1}\right) z_j} + \left(k_{z,j+1} + k_{z,j}\right)\left(k_{z,j} - k_{z,j-1}\right) e^{i\left(k_{z,j+1}\cdots\right.} \end{matrix}$$

$$\begin{aligned}
11 &= \left(k_{z,j+1} + k_{z,j}\right)\left(k_{z,j} + k_{z,j-1}\right) e^{-i\left(k_{z,j+1} - k_{z,j}\right) z_{j+1}} e^{-i\left(k_{z,j} - k_{z,j-1}\right) z_j} \\
&\quad + \left(k_{z,j+1} - k_{z,j}\right)\left(k_{z,j} - k_{z,j-1}\right) e^{-i\left(k_{z,j+1} + k_{z,j}\right) z_{j+1}} e^{i\left(k_{z,j} + k_{z,j-1}\right) z_j} \\
12 &= \left(k_{z,j+1} + k_{z,j}\right)\left(k_{z,j} - k_{z,j-1}\right) e^{-i\left(k_{z,j+1} - k_{z,j}\right) z_{j+1}} e^{-i\left(k_{z,j} + k_{z,j-1}\right) z_j} \\
&\quad + \left(k_{z,j+1} - k_{z,j}\right)\left(k_{z,j} + k_{z,j-1}\right) e^{-i\left(k_{z,j+1} + k_{z,j}\right) z_{j+1}} e^{i\left(k_{z,j} - k_{z,j-1}\right) z_j} \\
21 &= \left(k_{z,j+1} - k_{z,j}\right)\left(k_{z,j} + k_{z,j-1}\right) e^{i\left(k_{z,j+1} + k_{z,j}\right) z_{j+1}} e^{-i\left(k_{z,j} - k_{z,j-1}\right) z_j} \\
&\quad + \left(k_{z,j+1} + k_{z,j}\right)\left(k_{z,j} - k_{z,j-1}\right) e^{i\left(k_{z,j+1} - k_{z,j}\right) z_{j+1}} e^{i\left(k_{z,j} + k_{z,j-1}\right) z_j} \\
22 &= \left(k_{z,j+1} - k_{z,j}\right)\left(k_{z,j} - k_{z,j-1}\right) e^{i\left(k_{z,j+1} + k_{z,j}\right) z_{j+1}} e^{-i\left(k_{z,j} + k_{z,j-1}\right) z_j} \\
&\quad + \left(k_{z,j+1} + k_{z,j}\right)\left(k_{z,j} + k_{z,j-1}\right) e^{i\left(k_{z,j+1} - k_{z,j}\right) z_{j+1}} e^{i\left(k_{z,j} - k_{z,j-1}\right) z_j}
\end{aligned}$$

$$\begin{aligned}
11 &= \left(k_{z,j+1} + k_{z,j}\right)\left(k_{z,j} + k_{z,j-1}\right) e^{-i\left(k_{z,j+1} z_{j+1} - k_{z,j}\left(z_{j+1} - z_j\right) - k_{z,j-1} z_j\right)} \\
&\quad + \left(k_{z,j+1} - k_{z,j}\right)\left(k_{z,j} - k_{z,j-1}\right) e^{-i\left(k_{z,j+1} z_{j+1} + k_{z,j}\left(z_{j+1} - z_j\right) - k_{z,j-1} z_j\right)} \\
12 &= \left(k_{z,j+1} + k_{z,j}\right)\left(k_{z,j} - k_{z,j-1}\right) e^{-i\left(k_{z,j+1} z_{j+1} - k_{z,j}\left(z_{j+1} - z_j\right) + k_{z,j-1} z_j\right)} \\
&\quad + \left(k_{z,j+1} - k_{z,j}\right)\left(k_{z,j} + k_{z,j-1}\right) e^{-i\left(k_{z,j+1} z_{j+1} + k_{z,j}\left(z_{j+1} - z_j\right) + k_{z,j-1} z_j\right)} \\
21 &= \left(k_{z,j+1} - k_{z,j}\right)\left(k_{z,j} + k_{z,j-1}\right) e^{i\left(k_{z,j+1} + k_{z,j}\right) z_{j+1}} e^{-i\left(k_{z,j} - k_{z,j-1}\right) z_j} \\
&\quad + \left(k_{z,j+1} + k_{z,j}\right)\left(k_{z,j} - k_{z,j-1}\right) e^{i\left(k_{z,j+1} - k_{z,j}\right) z_{j+1}} e^{i\left(k_{z,j} + k_{z,j-1}\right) z_j} \\
22 &= \left(k_{z,j+1} - k_{z,j}\right)\left(k_{z,j} - k_{z,j-1}\right) e^{i\left(k_{z,j+1} + k_{z,j}\right) z_{j+1}} e^{-i\left(k_{z,j} + k_{z,j-1}\right) z_j} \\
&\quad + \left(k_{z,j+1} + k_{z,j}\right)\left(k_{z,j} + k_{z,j-1}\right) e^{i\left(k_{z,j+1} - k_{z,j}\right) z_{j+1}} e^{i\left(k_{z,j} - k_{z,j-1}\right) z_j}
\end{aligned}$$