HOMEWORK 4
MATH 4070, FALL 2025

Throughout this homework set, $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

Problems to type in TEX

1. Let $x \in \mathbb{F}^n$ be nonzero.
   (a) Show that $xx^* \in \mathbb{F}^{n\times n}$ has eigenvalues $\|x\|^2$ (with multiplicity one) and 0 (with multiplicity $n-1$).
   Hint: Apply the spectral theorem, and think about rank and trace.

   First lets find $xx^* = \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix} \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix} = \begin{bmatrix} x_1^2 & \dots & x_1x_n \\ \dots & x_2^2 & \dots \\ x_nx_1 & \dots & \dots \end{bmatrix}$.
   Let $A = xx^*$. Because the resulting matrix is symmetric, $A = A^T$. Applying the Spectral Theorem, A must be orthogonal diagonalizable and unitarily diagonalizable since $A = A^T$ and $A = A^* \implies A = (xx^*)^* = xx^* = A$ respectively.

   Now consider, $Ax = (xx^*)x = x(x^*x)$. We know $x^*x = ||x||^2$, which is a scalar. So $Ax = ||x||^2x$. This is now the definition of an eigenvalue and eigenvector $\implies Av = \lambda v$, where $\lambda = ||x||^2$ and $v = x$. Thus, because x is nonzero, $\lambda_1 = ||x||^2$ is an eigenvalue of A with eigenvector x, thus has multiplicity one.

   Taking a look at the columns of A, we can see each column can be represented as xx_i $\forall i \in [1, \dots, n]$, otherwise a scalar multiple of x. This means $rank(A) = 1$, since A is spanned by a single vector x. By Rank-Nullity, $rank(A) + ker(A) = n$, thus $ker(A) = n - 1$. By the definition of the kernel, its the set of all vectors v such that $Av = 0$. This is the eigenspace for the eigenvalue $\lambda_2 = 0$. We know there $n-1$ vectors, so its multiplicity is $n - 1$.

   Double checking using trace since the trace of a matrix is the sum of its eigenvalues: $tr(A) = x_1^2 + x_2^2 + \dots + x_n^2 = ||x||^2$, which matches what we got.

   (b) Now assume $\|x\| = 1$, and **use (a)** to find the eigenvalues of the corresponding Householder matrix $H = I - 2xx^*$ together with their multiplicities.

   Let $H = I - 2A$. Since $|x| = 1$, from (a), $\lambda_1 = 1$ (with multiplicity one) and $\lambda_2 = 0$ (with multiplicity $n-1$). Let v be an eigenvector of A with eigenvalue λ_A such that $Av = \lambda_A v$. Lets substitute $Hv = \lambda_H v$ using those values: $Hv = (I - 2A)v = Iv - 2Av = v - 2(\lambda_A v) = (1 - 2\lambda_A)v$ where $\lambda_H = 1 - 2\lambda_A$. This tells us the eigenvalues of H. For $\lambda_A = 1$ (multiplicity one), $\lambda_H = 1 - 2 = -1$ with the corresponding eigenvector x. For $\lambda_A = 1$ (multiplicity $n-1$), $\lambda_H = 1 - 0 = 1$ with the corresponding eigenspace all vectors v such that $Av = 0$. Thus, the eigenvalues of the Householder matrix H are $\lambda_1 = -1$ (with multiplicity one) and $\lambda_2 = 1$ (with multiplicity $n-1$).

Double checking using trace again: $tr(H) = tr(I - 2xx^*) = tr(I) - 2tr(xx^*) = n - 2\|x\|^2 = n - 2(1) = n - 2$ or $(1) * -1 + (n-1) * 1$, which matches our values.

2. Let $P \in \mathbb{F}^{n\times n}$ be a matrix that satisfies $P^2 = P$. Show that $\sigma(P) \subseteq \{0, 1\}$.
Hint: What happens when you hit an eigenvector with P?
Note: σ is the spectrum of a matrix (ie, set of all eigenvalues).

We want to show that all eigenvalues that exist in P are either 0 or 1. By definition of an eigenvalue and eigenvector, $Pv = \lambda v$. Apply P to both sides, $P(Pv) = P(\lambda v) \implies P^2 v = \lambda P v \implies Pv = \lambda(\lambda v) \implies Pv = \lambda^2 v$. Substituting this into our original equation, $Pv = \lambda v \implies \lambda^2 v = \lambda v \implies \lambda^2 v - \lambda v = 0 \implies \lambda(\lambda - 1) = 0$. Thus, $\lambda = 0$ or $\lambda = 1$. Therefore, the set of all eigenvalues $\sigma(P)$ must be a subset of $\{0, 1\}$.

Problems to do in MATLAB

3. MATLAB has a hard time row reducing the matrix
$$A = \begin{bmatrix} -4 & -5 & 3 & -1 \\ -3 & 0 & 0 & -4 \\ -1 & -5 & 3 & 3 \\ -5 & 3 & -2 & 0 \end{bmatrix}.$$
   (a) What does `rref(A)` suggest is the rank of A?
   (b) Find invertible S and diagonal D that diagonalize A.
   Hint: Use `eig`.
   (c) What does `diag(D)` suggest is the rank of A?
   (d) **Use (b)** to find an explicit nonzero vector with integer entries in $\ker A$.
   Hint: For an appropriate eigenvector x, check out x/x_1.
   (e) Explain how you know `rref(A)` is incorrect.

4. Consider the quadratic form $Q\colon \mathbb{R}^4 \to \mathbb{R}$ given by
$$Q(x) = 3x_1x_2 + 5x_1x_3 + 7x_1x_4 + 7x_2x_3 + 5x_2x_4 + 3x_3x_4.$$
Find $\max_{\|x\|=1} Q(x)$ and $\min_{\|y\|=1} Q(y)$, and find corresponding unit vectors x and y at which the max and min are attained.