HOMEWORK 6
MATH 4070, FALL 2025

Problems to type in TEX

1. If $0 < p \le q \le \infty$, show that $\|x\|_p \ge \|x\|_q$ for every $x \in \mathbb{F}^n$. You may use anything stated before this fact in the notes from Oct 27 2025.

   Hint: When $q < \infty$ and $x \ne 0$, first prove it for $y = \frac{x}{\|x\|_q}$.

   Let $x = [x_1, \ldots, x_n]$. First, consider the case where x is the zero vector. This means $\|x\|_p = 0$ for all $p > 0$. Thus, we can conclude $\|x\|_p \ge \|x\|_q$ since $0 < p \le q \le \infty$. Now, consider when x is a non-zero vector. We will split this into two cases:

   case 1 ($q = \infty$): We want to show $\|x\|_p \ge \|x\|_\infty$ for $0 < p < \infty$. By the definition of a inifity norm, $\|x\|_\infty = \max_{1 \le i \le n}|x_i|$. Let j be the index such that $|x_j| = \|x\|_\infty$. Expanding the p-norm, $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$, the sum must include the term $|x_j|^p$. Since all values are non-negative, the sum must be at least as large as that term: $\sum_{i=1}^n |x_i|^p \ge |x_j|^p \implies (\sum_{i=1}^n |x_i|^p)^{1/p} \ge (|x_j|^p)^{1/p} \implies \|x\|_p \ge |x_j| \implies \|x\|_p \ge \|x\|_\infty$. Thus, this case is proven.

   case 2 ($q < \infty$): We want to show $\|x\|_p \ge \|x\|_q$ for $0 < p \le q < \infty$. Let the q normalized vector y be $y = \frac{x}{\|x\|_q} \implies x = y\|x\|_q$, this is possible since x is non-zero, thus $\|x\|_q > 0$. Thus, it must be true that $\|y\|_q = 1$. Let $y = [y_1 \ldots y_n]$ such that $\|y\|_q = (\sum_{i=1}^n |y_i|^q)^{1/q} = 1$. This means each y_i must follow $|y_i| \le 1$. This then allows us to say $|y_i|^p \ge |y_i|^q$, because $p \le q$. Substituting this in our original sum, $\sum_{i=1}^n |y_i|^p \ge 1 \implies (\sum_{i=1}^n |y_i|^p)^{1/p} \ge 1$. Then, by the definition of a p-norm, $(\sum_{i=1}^n |y_i|^p)^{1/p} \ge 1 \implies \|y\|_p \ge 1$. Substitute the value of y, $\|y\|_p \ge 1 \implies \|\frac{x}{\|x\|_q}\|_p \ge 1$. We know $\frac{1}{\|x\|_q}$ is a constant, so we rearrange it to the other side such that $\|\frac{x}{\|x\|_q}\|_p \ge 1 \implies \|x\|_p \ge \|x\|_q$. Thus, this case is proven.

   Since both cases and zero vectors are proven, the statement must be true.

2. If $1 \le p \le q \le \infty$, show that $\|x\|_p \le n^{\frac{1}{p} - \frac{1}{q}}\|x\|_q$ for every $x \in \mathbb{F}^n$. (Here, we interpret $\frac{1}{q} = 0$ when $q = \infty$.) Together with Problem 1, this gives an explicit equivalence between p- and q-norms.

   Hint: Hölder's inequality

   Let $x = [x_1, \ldots, x_n]$. Consider when x is the zero vector, both sides would become 0 and the inequality $\|x\|_p \le n^{\frac{1}{p} - \frac{1}{q}}\|x\|_q \implies 0 \le 0$ holds. Now, consider when x is a non-zero vector and $p = q$. This means

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$\|x\|_p \leq n^{\frac{1}{p}-\frac{1}{q}}\|x\|_q \implies \|x\|_p \leq \|x\|_q$ and $\|x\|_q = \|x\|_p$, thus the inequality holds. Finally, consider when x is non-zero with the following cases:

case 1 ($1 \leq p < q = \infty$): We want to show $\|x\|_p \leq n^{1/p}\|x\|_\infty$. Following the definition of a infinite norm like we did in case 1 of problem 1, we can conclude $|x_i| \leq \|x\|_\infty$ for some max value $|x_i|$ in x at some index $i \in [1, \ldots, n]$. We know $p \geq 1$, so raising both sides to p gets us $|x_i|^p \leq (\|x\|_\infty)^p$. This inequality holds for all i, so we can take the sum: $\sum_{i=1}^{n} |x_i|^p \leq \sum_{i=1}^{n}(\|x\|_\infty)^p \implies (\|x\|_p)^p \leq n(\|x\|_\infty)^p \implies \|x\|_p \leq n^{1/p}\|x\|_\infty$. Thus, this case is proven.

case 2 ($1 \leq p < q < \infty$): We want to show $\|x\|_p \leq n^{\frac{1}{p}-\frac{1}{q}}\|x\|_q$. Hölder's inequality states for vectors $u, v \in \mathbb{F}^n$ and for $1 \leq s, t \leq \infty$ with $\frac{1}{s} + \frac{1}{t} = 1$, we have $\sum_{i=1}^{n}|u_i v_i| \leq |u|_t |v|_s$. We know $\|x\|_p^p = \sum_{i=1}^{n}|x_i|^p$. If we let $|x_i|^p = 1 * |x_i|^p$, we can apply Hölder's inequality such that $u_i = 1$ and $v_i = |x_i|^p$ for $\frac{1}{s} + \frac{1}{t} = 1$. This gives us, $\sum_{i=1}^{n}|x_i|^p \leq (\sum_{i=1}^{n}|1|^s)^{1/s}(\sum_{i=1}^{n}(|x_i|^p)^t)^{1/t} \implies \sum_{i=1}^{n}|x_i|^p \leq n^{1/s}(\sum_{i=1}^{n}|x_i|^{pt})^{1/t}$. This means we want $q = pt \implies t = \frac{q}{p} > 1$ since we have $p < q$. Now we need to find s, $\frac{1}{s} = 1 - \frac{1}{t} = 1 - \frac{p}{q} = \frac{q-p}{q} \implies s = \frac{q}{q-p}$, which is defined since $q > p \geq 1 \implies q - p > 0$. Substitute both values we found for s and t, $\sum_{i=1}^{n}|x_i|^p \leq n^{1/s}(\sum_{i=1}^{n}|x_i|^{pt})^{1/t} \implies \sum_{i=1}^{n}|x_i|^p \leq n^{\frac{q-p}{q}}(\sum_{i=1}^{n}|x_i|^q)^{p/q} \implies \sum_{i=1}^{n}|x_i|^p \leq n^{1-p/q}\|x\|_q^p$. Take the $1/p$ power to both sides, $(\sum_{i=1}^{n}|x_i|^p)^{1/p} \leq (n^{1-p/q}\|x\|_q^p)^{1/p} \implies \|x\|_p \leq n^{1/p-1/q}\|x\|_q$. Thus, this case is proven.

Since the statement holds for all cases, the statement must be true.

3. **Prove or disprove** each of the following.

   (a) For any $A \in \mathbb{F}^{m \times n}$, the function $p \mapsto \|A\|_p$ is decreasing on $[1, \infty]$, where $\|\cdot\|_p$ denotes the induced p-norm.

   For this statement to be true, $f(q) \leq f(p)$ for all values $1 \leq p \leq q \leq \infty$, where $f(p) : p \mapsto \|A\|_p$. We'll disprove by counterexample. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $p = 1$, and $q = 2$. The maximum absolute value column sum is $\|A\|_1 = 1$. The largest singular value of matrix is just the 2-norm, so we first find $A^T A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Then, find its corresponding eigenvalues with $\det(A^T A - \lambda I) = 0 \implies (1 - \lambda)(1 - \lambda) - 1 = 0 \implies \lambda^2 - 2\lambda = 0 \implies \lambda(\lambda - 2) = 0$, so $\lambda_1 = 0$ and $\lambda_2 = 2$. Thus, our singular values are $\sigma_1 = 0$ and $\sigma_2 = \sqrt{2}$. So, $\|A\|_2 = \sqrt{2}$. This goes against what we said at the start since $\|A\|_2 \not\leq \|A\|_1$. Thus, the statement is false.

   (b) For any $A \in \mathbb{F}^{m \times n}$, the function $p \mapsto \|A\|_{S,p}$ is decreasing on $[1, \infty]$, where $\|\cdot\|_{S,p}$ denotes the Schatten p-norm.

The Schatten p-norm of A is the p-norm of its vector of singular values. So, let A have the non-zero singular values $\sigma = [\sigma_1 \ldots \sigma_r]$ such that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r \geq 0$ and $\operatorname{rank}(A) = r$. We want to show $f(q) \leq f(p)$ for all values $1 \leq p \leq q \leq \infty$, where $f(p) : p \mapsto \|A\|_{S,p} \implies f(p) : p \mapsto \|\sigma\|_p$.

First, if A is the zero matrix, then all of its singular values with also be zero. Meaning all of its norms are a constant 0. Thus, $f(q) \leq f(p)$ holds. So, let A be a non-zero matrix. We'll prove it using the following cases:

case 1 ($q = \infty$): We want to show $f(\infty) \leq f(p)$ for all $1 \leq p \leq \infty$. By definition, $f(\infty) = \|A\|_{S,\infty} = \max_{1 \leq i \leq r} \sigma_i = \sigma_1$. So after expanding $f(p)$, we get $\sigma_1 \leq (\sum_{i=1}^r \sigma_i^p)^{1/p} \implies \sigma_1^p \leq \sum_{i=1}^r \sigma_i^p$. Since each term is non-negative and the sum also includes σ_1, this inequality must hold. Thus, this case holds.

case 2 ($q < \infty$): We want to show $f(q) \leq f(p) \implies \|\sigma\|_q \leq \|\sigma\|_p$ for all $1 \leq p \leq q < \infty$. Define a normalized vector $s = \frac{\sigma}{\|\sigma\|_q}$. By construction, $\|s\|_q = 1 \implies (\sum_{i=1}^r |s_i|^q)^{1/q} = 1 \implies \sum_{i=1}^r |s_i|^q = 1$. So, each term $|s_i|^q \leq 1 \implies |s_i| \leq 1$ for all i. Because $p \leq q$ and $|s_i| \leq 1$, we cans say $|s_i|^q \leq |s_i|^p$ for all i. Applying the sum we get $\sum_{i=1}^r |s_i|^q \leq \sum_{i=1}^r |s_i|^p \implies 1 \leq \|s\|_p^p \implies 1 \leq \|s\|_p$. Then, we substitute the value of s, $1 \leq \|s\|_p \implies 1 \leq \|\frac{\sigma}{\|\sigma\|_q}\|_p$. We know $\frac{1}{\|\sigma\|_q}$ is a constant, so we can rearrange it such that $1 \leq \|\frac{\sigma}{\|\sigma\|_q}\|_p \implies \|\sigma\|_q \leq \|\sigma\|_p \implies .$ Thus, this case is proven.

Since the statement holds for the zero matrix, and for non-zero matrices in both cases, the statement must be true.

4. In this problem, we denote $\|A\|_{S,p}$ for the Schatten p-norm of a matrix A, and $\|A\|_*$ for the nuclear norm.

   (a) Given $1 \leq p \leq q \leq \infty$ and a positive integer r, find explicit finite constants $C_1, C_2 > 0$ such that $C_1\|A\|_{S,p} \leq \|A\|_{S,q} \leq C_2\|A\|_{S,p}$ for every complex matrix A with $\operatorname{rank} A \leq r$. Your constants can depend on p, q, and r, but not on the choice of A.

   Let $k = \operatorname{rank}(A)$ and the singular values of A be $\sigma = [\sigma_1 \ldots \sigma_k]$. We already defined the Schattern p-norm as the singular values in Problem 3, using that we get $C_1\|\sigma\|_p \leq \|\sigma\|_q \leq C_2\|\sigma\|_p$.

   First, lets find C_2 using the second half of the inequality: $\|\sigma\|_q \leq C_2\|\sigma\|_p$. The inequality is trivial if $\|\sigma\|_p = 0$. We'll assume $\|\sigma\|_p > 0$. Define a normalized vector $s = \frac{\sigma}{\|\sigma\|_p}$. By construction, $\|s\|_p = 1 \implies (\sum_{i=1}^r |s_i|^p)^{1/p} = 1 \implies \sum_{i=1}^r |s_i|^p = 1$. Since $\sigma_i \geq 0$, it must be true that $s_i \geq 0$. This allows us to conclude that $0 \leq s_i \leq 1$ for all i. This, along with $q \geq p$, we can say $s_i^q \leq s_i^p$. Taking the sum,

$\sum_{i=1}^{k} s_i^q \leq \sum_{i=1}^{k} s_i^p \implies \sum_{i=1}^{k} s_i^q \leq 1 \implies \|s\|_q^q \leq 1 \implies \|s\|_q \leq 1$.
Then we substitute the value of s, $\|s\|_q \leq 1 \implies \|\frac{\sigma}{\|\sigma\|_p}\|_q \leq 1 \implies$
$\|\sigma\|_q \leq \|\sigma\|_p$. This inequality only holds when $C_2 = 1$, since $\|\sigma\|_q \leq$
$C_2\|\sigma\|_p \implies \|\sigma\|_q \leq \|\sigma\|_p$.

Now, lets find C_1 using the first half of the inequality $C_1\|\sigma\|_p \leq$
$\|\sigma\|_q \implies \|\sigma\|_p \leq \frac{1}{C_1}\|\sigma\|_q$. Lets take $\|\sigma\|_p = (\sum_{i=1}^{k} \sigma_i^p)^{1/p}$, using
Hölder's inequality for $u, v \in \mathbb{F}^n$ and for $1 \leq a, b \leq \infty$ with $\frac{1}{a} + \frac{1}{b} = 1$,
we have $\sum_{i=1}^{n} |u_i v_i| \leq |u|_a |v|_b$. Let $u_i = \sigma_i^p$ and $v_i = 1$, which tells
us $\sum_{i=1}^{k} \sigma_i^p \leq \|\sigma^p\|_a \|1\|_b$. Expanding the right side, $\|\sigma^p\|_a \|1\|_b =$
$(\sum_{i=1}^{k} \sigma_i^{pa})^{1/a} (k)^{1/b}$. So, we want $pa = q \implies a = q/p$ and $\frac{1}{b} =$
$1 - \frac{1}{a} = 1 - \frac{p}{q} = \frac{q-p}{q} \implies b = \frac{q}{q-p}$. Back to our inequality we'll substi-
tute what we found, $\sum_{i=1}^{k} \sigma_i^p \leq (\sum_{i=1}^{k} \sigma_i^{pa})^{1/a} (k)^{1/b} \implies \sum_{i=1}^{k} \sigma_i^p \leq$
$(\sum_{i=1}^{k} \sigma_i^q)^{p/q} (k)^{\frac{q-p}{q}} \implies \|\sigma\|_p^p \leq \|\sigma\|_q^p (k)^{\frac{q-p}{q}} \implies \|\sigma\|_p \leq \|\sigma\|_q (k)^{\frac{1}{p}-\frac{1}{q}}$.
This tells us $\frac{1}{C_1} = k^{\frac{1}{p}-\frac{1}{q}} \implies C_1 = k^{\frac{1}{q}-\frac{1}{p}}$. Since $q \geq p$, C_1 will be
non-increasing, so the minimum value will be r since $k \leq r$. So, $C_1 =$
$r^{\frac{1}{q}-\frac{1}{p}} \leq k^{\frac{1}{q}-\frac{1}{p}}$. Plugging it in, $\|A\|_{S,q} \geq k^{\frac{1}{q}-\frac{1}{p}} \|A\|_{S,p} \geq r^{\frac{1}{q}-\frac{1}{p}} \|A\|_{S,p}$.
So, our best case is $C_1 = r^{\frac{1}{q}-\frac{1}{p}}$.

(b) Given a positive integer n, find explicit finite constants $C_1, C_2 > 0$
such that $C_1\|A\|_* \leq \|A\|_F \leq C_2\|A\|_*$ for every $A \in \mathbb{F}^{n \times n}$. Your con-
stants can depend on n, but not on the choice of A.

We know $\|A\|_* = \|A\|_{S,1}$ and $\|A\|_F = \|A\|_{S,2}$. Substituting these val-
ues we get, $C_1\|A\|_{S,1} \leq \|A\|_{S,2} \leq C_2\|A\|_{S,1}$. This is very similar to the
results in (a), so we'll say $p = 1$, $q = 2$, and $r = n$ (max possible rank).

Following the same steps from (a) lets first find C_2 using the right
hand side: $\|A\|_{S,2} \leq C_2\|A\|_{S,1}$. This is easily found since (a) tells us
when $1 \leq p \leq q \leq \infty$ the constant $C_2 = 1$ for this type of equation.
We know p and q fit into this requirement, thus $C_2 = 1$.

Finally, finding C_1 is just as easy for the inequality $C_1\|A\|_{S,1} \leq \|A\|_{S,2}$.
Problem (a) tells us $C_1 = r^{\frac{1}{q}-\frac{1}{p}}$. If we substitute our values we get
$C_1 = n^{\frac{1}{2}-\frac{1}{1}} = n^{-1/2} = \frac{1}{\sqrt{n}}$.

Problems to do in MATLAB

5. (a) The file `barbara.pgm` contains a standard 512×512 grayscale test
image. Let $A \in \mathbb{R}^{512 \times 512}$ be the corresponding matrix with values
between 0 (black) and 255 (white). Find the Eckart–Young approxi-
mation $B \in \mathbb{R}^{512 \times 512}$ of rank 20, save the corresponding image as a
PGM file, and submit it with this problem.

   Hint: To get A, use `imread` and `double`. To turn B back into an
image, use `uint8` and `imwrite` (or export from `imshow`).

   (b) Compute the distance between A and B in each of the following norms:

   (i) max,
  (ii) sum,
 (iii) induced 1-norm,
 (iv) induced ∞-norm,
  (v) induced 2-norm,
 (vi) nuclear,
(vii) Frobenius.

(c) Find the smallest constant C such that $\|A - B\|_{S,p} \leq C$ for every $p \in [1, \infty]$, where $\| \cdot \|_{S,p}$ denotes the Schatten p-norm.