HOMEWORK 5
MATH 4070, FALL 2025

Problems to type in TEX

1. For each of the following, **prove or disprove** that the statement holds for any choice of square matrix A.
   (a) The squares of the eigenvalues of A are eigenvalues of A^2.

   Let λ be an eigenvalue of A. By definition, there exists a eigenvector v such that $Av = \lambda v$. We can find A^2 by multiplying both sides by A, $A(Av) = A(\lambda v) \implies A^2 v = \lambda(Av) \implies A^2 v = \lambda^2 v$. We know v is non-zero, so we see the squares of the eigenvalues of A are the eigenvalues of A^2. Thus, the statement is true.

   (b) The squares of the singular values of A are singular values of A^2.

   We'll disprove this by counterexample. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. We then find $A^*A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. This tells us the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 0$. Then we can find the singular values $s_1 = \sqrt{1} = 1$ and $s_2 = \sqrt{0} = 0$. Then lets try find the singular values for A^2. First, $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. We can already see a problem: no matter what, the singular values will always be 0 even though we found a 1 for A. Thus, the statement can't be true.

   (c) The product of the singular values is the absolute value of the determinant.

   Let $s_1, \ldots, s_n$ be the singular values of A and $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A^*A such that $\lambda_i = s_i^2$ for $1 \leq i \leq n$. The product of the eigenvalues of a matrix is its determinant, so $det(A^*A) = \prod_{i=1}^n s_i^2$. Then, by the properties of determinant, $det(A^*A) = \overline{det(A)}det(A) \implies |det(A)|^2 = \prod_{i=1}^n s_i^2 \implies |det(A)| = \prod_{i=1}^n s_i$. Thus, the statement is true.

   (d) If A is Hermitian, then its singular values are the same as its eigenvalues.

   We'll prove this by counter example. Let $A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$ be hermitian with eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$. First, find the eigenvalues of $A^*A = A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$, which are $\lambda_1 = 1$ and $\lambda_2 = 4$. This gives the singular values $s_1 = \sqrt{1} = 1$ and $s_2 = \sqrt{4} = 2$. These singular values are obviously different from the eigenvalues of A. Thus, the statement is false.

Problems to do in MATLAB

2. Consider the linear system $Ax = b$ where
$$A = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 3 & -1 & 2 & -2 \end{bmatrix} \qquad \text{and} \qquad b = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$
   (a) Use the QR method to find a least-squares solution of small norm.
   (b) Use the SVD method to find the least-squares solution of minimum norm.
   (c) Use your SVD from (b) to find $A^\dagger$, and verify that $A^\dagger b$ agrees with your answer to (b).

3. As this problem demonstrates, you have to be careful solving equations when the coefficient matrix has small positive singular values. A linear transformation $A \in \mathbb{R}^{20\times 10}$ and its output $b \in \mathbb{R}^{20}$ were observed under noisy conditions, and then saved in the file `HW5_num3.mat`. You can use `load` to get them in your workspace.
   (a) Use `lsqminnorm` to find the least-squares solution $\hat{x}$ of $Ax = b$. What are $\|A\hat{x} - b\|$ and $\|\hat{x}\|$?
   (b) What does MATLAB think the rank of A is, and what do singular values indicate it really is?
   (c) Find the best approximation A_r of A whose rank is what you think it should be. Verify that A_r is very close to A.
   (d) Find $A_r^\dagger$, and use it to find the least-squares solution $\hat{x}_r$ of $A_r x = b$.
   (e) What are $\|A\hat{x}_r - b\|$ and $\|\hat{x}_r\|$?
   (f) Which of $\hat{x}$ and $\hat{x}_r$ do you think is more believable as the least-squares solution of $Ax = b$? Why?