HOMEWORK 3
MATH 4070, FALL 2025

Problems to type in $\TeX$

1. Let A be a matrix in $\mathbb{F}^{n\times n}$, where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Show that A is invertible if and only if it admits a QR-factorization $A = QR$ where $Q \in \mathbb{F}^{n\times n}$ is unitary and $R \in \mathbb{F}^{n\times n}$ is upper triangular with **strictly positive** entries on the diagonal.

   (A is invertible $\implies$ A has a QR-factorization):
   Assume A is invertible, we need to show A has a QR-factorization. We'll find the QR decomposition of A using Gram-Schmidt. Let the columns of A be $a_1, \dots, a_n \in \mathbb{F}^n$. So, $A = [a_1 \dots a_n]$, where A's columns form a basis for $\mathbb{F}$ since A is invertible (its columns are linearly independent). We can now use Gram-Schmidt on the columns of A to get the orthonormal set of vectors $\{q_1. \dots, q_n\}$. From Gram-Schmidt:
   $u_k = a_k - \sum_{i=1}^{k-1} \langle a_k, q_i\rangle q_i$, for $k = 2, \dots, n$
   $q_k = \frac{u_k}{||u_k||}$
   $||u_k||q_k = a_k - \sum_{i=1}^{k-1} \langle a_k, q_i\rangle q_i \implies a_k = |u_k|q_k + \sum_{i=1}^{k-1} \langle a_k, q_i\rangle q_i$
   So, in general: $a_k = \langle a_k, q_1\rangle q_1 + \cdots + \langle a_k, q_{k-1}\rangle q_{k-1} + ||u_k||q_k$
   Using this, we can construct Q and R. Let Q be the orthonormal vectors we constructed with Gram-Schmidt: $Q = [q_1 \dots q_n]$. Since the set of vectors we constructed are orthonormal by the Gram-Schmidt construction, Q is a unitary matrix because its square. Next, we need an R s.t. $A = QR$ and R is upper triangular with strictly positive entries on the diagonal. The equation $a_k = \sum_{i=1}^{n} q_i r_{ik}$ represents the k-th column of the product QR. This allows us to define the entries of R using that equation and the calculation for a_k using Gram-Schmidt:
   $r_{ik} = 0$, for $i > k$. Thus, R is upper triangular.
   $r_{kk} = ||u_k||$, for $i = k$. Since the columns of A are linearly independent, $u_k \neq 0$, so $||u_k|| > 0$. Thus, strictly positive entries on the diagonal.
   $r_{ik} = \langle a_k, q_i\rangle$, for $i < k$.
   So, we have constructed $A = QR$ where Q unitary and R is an upper triangular matrix with strictly positive entries on its diagonal. Thus, the forward is true.

   (A has a QR-factorization $\implies$ A is invertible):
   By the def of invertible, its determinant must be non-zero. So, $\det(A) = \det(QR)$, we want to prove $\det(QR)$ is non-zero. Then, using the property $\det(XY) = \det(X)\det(Y)$, $\det(A) = \det(QR) \implies \det(A) = \det(Q)\det(R)$. Starting with $\det(Q)$, we know Q is unitary so $\det(Q^*Q) = \det(I) \implies \det(Q)\det(Q^*) = 1 \implies \det(Q)\overline{\det(Q)} = 1 \implies |\det(Q)|^2 = 1 \implies |\det(Q)| = 1$. Therefore, $\det(Q)$ is non-zero. Then working on $\det(R)$, we

know the determinant of an upper triangle is the product of its diagonal entries. However, we know R's diagonal entries are all strictly positive. Therefore, $\det(R)$ must be non-zero. Thus, we can conclude $\det(A) \neq 0$ because $\det(A) = \det(Q)\det(R)$ with $\det(Q)$ and $\det(R)$ being non-zero. A square matrix is invertible if and only if its determinant is non-zero, so A is invertible. Thus, the reverse is true.

Since both directions are proven, the statement: A is invertible if and only if it admits a QR-factorization $A = QR$ where $Q \in \mathbb{F}^{n\times n}$ is unitary and $R \in \mathbb{F}^{n\times n}$ is upper triangular with strictly positive entries on the diagonal, must be correct.

2. Find a least-squares solution by hand with the normal equations:

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $y = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. Least-squares tells us: $A^TAx = A^Ty$. Finding the transpose: $A^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$. Then we combine:
$A^TA = \begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix}$, $A^Ty = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$. So, $A^TAx = A^Ty \implies \begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$. This gives us the equations:
$4x_1 + 10x_2 = 2 \implies 2x_1 + 5x_2 = 1$
$10x_1 + 30x_2 = 4 \implies 5x_1 + 15x_2 = 2$
Isolating x_1 in the first equation: $x_1 = \frac{1-5x_2}{2}$
Plugging it into the second equation:
$\frac{5(1-5x_2)}{2} + 15x_2 = 2 \implies 5 - 25x_2 + 30x_2 = 4 \implies 5x_2 = -1 \implies x_2 = -\frac{1}{5}$.
Plugging that into the isolated x_1 equation: $x_1 = \frac{1-5(-\frac{1}{5})}{2} = 1$.
Thus, the least squares solution is $x_1 = 1$, and $x_2 = -\frac{1}{5}$

Problems to do in MATLAB

3. Consider the matrix and column vectors

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ -1 & 1 & 0 & 2 \\ 3 & -1 & 2 & -2 \\ 0 & 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix}.$$

(a) Find a subset of $\{a_1, a_2, a_3, a_4\}$ that forms a basis for $\operatorname{im} A$.

(b) Use your answer to (a) and apply the method of inverting a Gram matrix to find the matrix of orthogonal projection onto $\operatorname{im} A$.

(c) Use a QR-factorization of A to find an orthonormal basis for $\operatorname{im} A$.

(d) Use your answer to (c) to find the matrix of orthogonal projection onto $\operatorname{im} A$ again. Did you get the same answer as in (b)?

4. A healthy child's systolic blood pressure p (in millimeters of mercury) and weight w (in pounds) are approximately related by the equation

$$\beta_0 + \beta_1 \ln w = p,$$

for some real parameters β_0 and β_1. Use the method of least-squares with the following experimental data to estimate the systolic blood pressure of a healthy child weighing 100 pounds. Specifically, estimate β_0 and β_1 with the QR method from class, and apply the result.

w	44	61	81	113	141
p	91	98	103	110	112

Hint: In MATLAB, `log` gives the natural logarithm.

5. To measure the takeoff performance of an airplane, the horizontal position of the plane was measured every second, from $t = 0$ to $t = 12$. The positions (in feet) were: 0, 8.8, 29.9, 62.0, 104.7, 159.1, 222.0, 294.5, 380.4, 471.1, 571.7, 686.8, and 809.2.

(a) Find the least-squares cubic curve $y = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3$ that best fits the data. Specifically, use a Cholesky decomposition to solve the normal equations.

(b) Same as (a), but use a QR factorization. (Same answer, right?)

(c) Use the results of (a) and (b) to estimate the horizontal position when $t = 4.5$ seconds.