HOMEWORK 1
MATH 4070, FALL 2025

Problems to type in TEX

1. (a) In $\mathbb{R}^3$, prove that the set of vectors $(x, y, z)^\top$ which satisfy $x-y+4z = 0$ is a subspace.

   Let $V = \{(x, y, t)^T \in \mathbb{R}^3 | x - y + 4z = 0\}$. For V to be a subspace of $\mathbb{R}^3$, it must show:
   1. $0 \in V$
   2. for any $v, w \in V$, $v + w \in V$
   3. for any $v \in V$ and $\alpha \in \mathbb{F}$, $\alpha v \in V$

   1. $0 = (0, 0, 0)^T \implies (0) - (0) + 4(0) = 0$, which is true, so $0 \in V$.

   2. Let $v = (x_1, y_1, z_1)^T$ and $w = (x_2, y_2, z_2)^T$, s.t.
   $x_1 - y_1 + 4z_1 = 0$
   $x_2 - y_2 + 4z_2 = 0$
   Consider $v + w = (x_1 + x_2, y_1 + y_2, z_1 + z_2)^T$, s.t.
   $(x_1 + x_2) - (y_1 + y_2) + 4(z_1 + z_2) = 0$, rearrange
   $(x_1 - y_1 + 4z_1) + (x_2 - y_2 + 4z_2) = 0$
   $0 + 0 = 0$, therefore $v + w \in V$

   3. Let $v = (x, y, z)^T$ s.t.
   $x - y + 4z = 0$
   Consider $\alpha \in \mathbb{R}$, where $\alpha v = (\alpha x, \alpha y, \alpha z)^T$, s.t.
   $\alpha x - \alpha y + 4\alpha z = 0$, rearrange
   $\alpha(x - y + 4z) = 0 \implies x - y + 4z = 0$, therefore $\alpha v \in V$

   Because V matches all of the conditions, V is a subspace of $\mathbb{R}^3$.

   (b) In $\mathbb{R}^3$, prove that the set of vectors $(x, y, z)^\top$ which satisfy $x-y+4z = 1$ is not a subspace.

   Let $V = \{(x, y, t)^T \in \mathbb{R}^3 | x - y + 4z = 1\}$. For V to be a subspace of $\mathbb{R}^3$, it must show:
   1. $0 \in V$
   2. for any $v, w \in V$, $v + w \in V$
   3. for any $v \in V$ and $\alpha \in \mathbb{F}$, $\alpha v \in V$

   1. $0 = (0, 0, 0)^T \implies (0) - (0) + 4(0) = 1$, which is not true, so $0 \notin V$.

   Because V fails one of the conditions, it can not be a subspace of $\mathbb{R}^3$.

2. For the matrix below, find (a) a spanning set for the kernel, (b) a basis for the image, and (c) its rank:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{bmatrix}.$$

(a). $N(L) = \{v \in A : Lv = 0\} \leq A$, where $v = (v_1, v_2, v_3)^T$
Gives us the equations:
$v_1 - v_2 + 2v_3 = 0$
$-2v_1 + 2v_2 - 4v_3 = 0 \implies v_1 - v_2 + 2v_3 = 0$
Simplify:
$v_1 = v_2 - 2v_3$
Replace in v:
$(v_2 - 2v_3, v_2, v_3)^T \implies v_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$
which gives us our spanning set for the kernal: $\{(1, 1, 0)^T, (-2, 0, 1)^T\}$
Any vector in the kernel has the form above, hence is a linear combination of those two vectors, so they span $kerA$.

(b). Find row reduced echelon form of A:
$\begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{bmatrix} \to R_2 + 2R_1$
$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, done
This tells us there is only one pivot column (column 1).
Therefore our basis for $R(A) = \{(1, -2)^T\}$
We know this because every column of A is a linear combination of the pivot columns, because we used the RREF form. Also, pivot columns are independent because each introduces a new leading 1 in the RREF that was made. Thus by definition of a basis, that must be the basis for the image.

(c). $rankA = dim(R(A))$, which we know is 1

3. (a) The **trace** of a square matrix A is the sum $\operatorname{tr} A$ of its diagonal entries. Show that if A is $m \times n$ and B is $n \times m$, then $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.
Hint: Write out everything as a double sum.

AB is a $m \times m$ matrix, so its trace can be written as $\sum_{i=1}^{m} AB_{ii}$. The entry of the matrix AB can be written as $AB_{ik} = \sum_{j=1}^{n} A_{ij}B_{jk}$. If we plug this into our trace algorithm we get $tr(AB) = \sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}B_{ji}$

Similarly BA is a $n \times n$ matrix, so $tr(BA) = \sum_{j=1}^{n} BA_{jj}$. The entry of the matrix BA can be written as $BA_{jk} = \sum_{i=1}^{m} B_{ji}A_{ik}$. If we plug this into our trace algorithm we get $tr(BA) = \sum_{j=1}^{n}\sum_{i=1}^{m} B_{ji}A_{ij}$

Let $tr(AB) = tr(BA) \implies \sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}B_{ji} = \sum_{j=1}^{n}\sum_{i=1}^{m} B_{ji}A_{ij}$
B_{ji} and A_{ij} are scalars so they can be swapped.
$\sum_{i=1}^{m}\sum_{j=1}^{n} B_{ji}A_{ij} = \sum_{j=1}^{n}\sum_{i=1}^{m} B_{ji}A_{ij}$,

Our sums are finite, so they can be swapped as well.
$\sum_{j=1}^{n}\sum_{i=1}^{m} B_{ji}A_{ij} = \sum_{j=1}^{n}\sum_{i=1}^{m} B_{ji}A_{ij}$
Therefore $tr(AB) = tr(BA)$

(b) Show that similar matrices have the same trace.

Let A and B be similar matrices. Therefore $B = P^{-1}AP$ and A, B, and P are all square matrices of the same size (due to the rules of inverting matrices). So,
$tr(B) = tr(P^{-1}AP)$, focusing on the right side, we can rearrange using cycles $tr(P^{-1}AP) = tr(APP^{-1}) = tr(PP^{-1}A)$
$tr(B) = tr(PP^{-1}A)$, this can be simplified
$tr(B) = tr(IA) \implies tr(B) = tr(A)$, thus similar matrices have the same trace.

(c) Prove or disprove: If A is $m \times n$ and B is $n \times m$, then AB and BA have the same determinant.

By counter example let $A = (1, 2)$ and $B = (3, 4)^T$. $AB = (11) \implies det(AB) = 11$. $BA = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix} \implies det(BA) = (3)(8) - (4)(6) = 0.$
Therefore, $det(AB) \neq det(BA)$. Thus, by counter example, AB and BA don't have the same determinant if A is $m \times n$ and B is $n \times m$.

Problems to do in MATLAB

4. In a wind tunnel experiment, the force on a projectile due to air resistance was measured at different velocities:

Velocity (100 ft/sec)	0	2	4	6	8	10
Force (100 lb)	0	2.90	14.8	39.6	74.3	119

(a) Find an interpolating polynomial of degree 5 that expresses force as a function of velocity. Your polynomial should have the form

$$p(t) = c_0 + c_1t + c_2t^2 + c_3t^3 + c_4t^4 + c_5t^5.$$

Hint: Make a function that turns numbers into row vectors:

```
r = @(t) [1 t t^2 t^3 t^4 t^5].
```

Check out what happens when you type `r(0)`. Then try `[r(0);r(2)]`. What you just learned should make your life easier.
Another hint: If a linear system $Ax = b$ has a solution, then MATLAB will return one when you enter `A \ b`.

(b) Use your answer to (a) to estimate the force on the projectile when it is traveling at 750 ft/sec.

5. A function $L\colon \mathbb{R}^4 \to \mathbb{R}^4$ is linear and satisfies

$$\begin{aligned} L(0,1,1,1)^\top &= (-3,-2,-1,0)^\top \\ L(1,1,0,1)^\top &= (-1,0,-3,-2)^\top \\ L(1,1,0,0)^\top &= (1,1,-3,-3)^\top \\ L(0,1,0,-1)^\top &= (2,2,-2,-2)^\top \\ L(0,-1,1,-1)^\top &= (1,-2,3,0)^\top. \end{aligned}$$

(a) Find the matrix for L in the standard basis.
Hint: The naive way to do this is to find out what L does to the standard basis vectors. It turns out that the input vectors I gave you span $\mathbb{R}^4$, so each e_j can be written as a linear combination of them. Once you find coefficients for that expansion, you can use linearity to find Le_j.

(b) Use your answer to (a) to find $L(1,-2,-1,1)^\top$.

(c) Use similarity to find the matrix for L in the following basis:

$$\begin{aligned} b_1 &= (1,1,1,1)^\top \\ b_2 &= (1,-1,1,-1)^\top \\ b_3 &= (1,1,-1,-1)^\top \\ b_4 &= (1,-1,-1,1)^\top \end{aligned}$$

Hint: Use `inv` to get the inverse of a matrix.

(d) Without computing anything more, use your answer to (c) to determine if L is one-to-one.