HOMEWORK 7
MATH 4070, FALL 2025

Problems to type in $\TeX$

1. Let $A \in \mathbb{C}^{n\times n}$ be a matrix such that every $\lambda \in \sigma(A)$ satisfies $|\lambda - 1| < 1$. Show that there exists a matrix $\sqrt{A} \in \mathbb{C}^n$ such that $\sqrt{A}^2 = A$, namely,

$$\sqrt{A} = \sum_{k=0}^{\infty} \binom{1/2}{k} (A - I)^k.$$

You may use the fact that the binomial series $\sum_{k=0}^{\infty} \binom{1/2}{k} z^k$ has radius of convergence $R = 1$.

Let the coefficients of the series be $c_k = \binom{1/2}{k}$ and the matrix be $B = A - I$. This gives us $\sum_{k=0}^{\infty} c_k B^k$. This is similar to one we found in class. We know this type of power series must converge if the spectral radius is less than the radius of convergence of $\sum_{k=0}^{\infty} c_k z^k$. Applying this to $\sum_{k=0}^{\infty} c_k B^k$, we want $\rho(B) < R \implies \rho(A - I) < 1 \implies \max_{\mu\in\sigma(A-I)}|\mu| < 1$. Let $\lambda \in \sigma(A)$, by the spectral mapping theorem, the eigenvalues of $A - I$ are stated as thus: $\sigma(A-I) = \{\lambda-1|\lambda \in \sigma(A)\}$. The problem tells us that every $\lambda \in \sigma(A)$ satisfies $|\lambda-1| < 1$, this allows us to say $\max_{\mu\in\sigma(A-I)}|\mu| < 1 \implies \max_{\lambda\in\sigma(A)}|\lambda-1| < 1$. Thus, this matrix converges. Now, we need to verify $\sqrt{A}^2 = A$. Take $\sqrt{A}^2 = (\sum_{i=0}^{\infty} c_i B^i)(\sum_{j=0}^{\infty} c_j B^j)$. We know both series converge, so we can use the Cauchy product: $(\sum_{i=0}^{\infty} c_i B^i)(\sum_{j=0}^{\infty} c_j B^j) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}(c_k B^k)(c_{n-k}B^{n-k}) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} c_k c_{n-k} B^n$. We know $c_k c_{n-k} = \binom{1/2}{k}\binom{1/2}{n-k} = \binom{1}{n}$, so substituting we get $\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{1}{n}B^n$. We let $d_n = \sum_{k=0}^{n}\binom{1}{n}$ be the coefficient for the series. The values of d_n are as follows: $d_0 = 1$, $d_1 = 1$, $d_{n\geq 2} = 0$. This then tells us $\sum_{n=0}^{\infty} d_n B^n = B^0 + B^1 + 0 + \cdots + 0 = B + I = (A - I) + I = A$. Thus, $\sqrt{A}^2 = A$. Our statement is then proven.

2. Let $f(z) = \sum_{k=0}^{\infty} c_k z^k$ be a power series with radius of convergence $R \in (0, \infty]$, and suppose all the coefficients c_k are **real**. Given any $A \in \mathbb{C}^{n\times n}$ with $\rho(A) < R$, prove that $f(A)^* = f(A^*)$.

Hint: Do something like on page 2 of the Nov 12 notes.

We know $f(A) = \sum_{k=0}^{\infty} c_k A^k$ converges under the condition $\rho(A) < R$. We then first want to prove $f(A^*)$ also converges. By the complex conjugate, if λ is an eigenvalue of A then $\bar{\lambda}$ is an eigenvalue of A^*. So, $\rho(A^*) = \max_{\lambda\in\sigma(A)}|\bar{\lambda}|$. We know $|\lambda| = |\bar{\lambda}|$, so $\max_{\lambda\in\sigma(A)}|\bar{\lambda}| = \max_{\lambda\in\sigma(A)}|\lambda| = \rho(A)$. Thus, $\rho(A) = \rho(A^*) < R$, so $f(A^*)$ converges under the same conditions. Let $f_N(z) = \sum_{k=0}^{N} c_k z^k$ be the N-th Taylor polynomial of $f(z)$. By definition of a function, $f(A) = \lim_{N\to\infty} f_N(A)$ and $f(A^*) =$

$\lim_{N\to\infty} f_N(A^*)$. So, we want to show $(f_N(A))^* = f_N(A^*)$. Left hand side first, $(f_N(A))^* = \left(\sum_{k=0}^{N} c_k A^k\right)^* = \sum_{k=0}^{N} \bar{c}_k (A^*)^k$. All coefficients are real, $\sum_{k=0}^{N} \bar{c}_k (A^*)^k = \sum_{k=0}^{N} c_k (A^*)^k$. This is the exact definition of the right side, so they are equal. Now taking the limit, $f(A)^* = (\lim_{N\to\infty} f_N(A))^* = \lim_{N\to\infty}(f_N(A))^* = \lim_{N\to\infty} f_N(A^*) = f(A^*)$. Thus, the statement is proven.

3. Prove that $\cos(A) = \frac{1}{2}\left(e^{iA} + (e^{iA})^*\right)$ for every Hermitian $A \in \mathbb{C}^{n\times n}$.

   You may use the fact that $e^{iz} = \cos z + i\sin(z)$ for all $z \in \mathbb{C}$, where $\cos z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k}$ and $\sin z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1}$ are given by power series with radius of convergence $R = \infty$.

   We'll do this by expanding the right side. First, $\frac{1}{2}\left(e^{iA}+(e^{iA})^*\right) = \frac{1}{2}(\cos(A) + i\sin(A) + (\cos(A) + i\sin(A))^*) = \frac{1}{2}(\cos(A) + i\sin(A) + \cos(A)^* - i\sin(A)^*)$. We know $f(A^*) = (f(A))^*$ holds for any A and real coefficients, which allows us to say $\cos(A^*) = \cos(A)^*$ and $\sin(A^*) = \sin(A)^*$. Applying this to the rhs, $\frac{1}{2}(\cos(A) + i\sin(A) + \cos(A^*) - i\sin(A^*))$. Since A is hermitian, $A = A^*$, $\frac{1}{2}(\cos(A) + i\sin(A) + \cos(A^*) - i\sin(A^*)) = \frac{1}{2}(\cos(A) + i\sin(A) + \cos(A) - i\sin(A)) = \cos(A)$. Thus, the statement must hold.

4. **Prove or disprove**: There exists a matrix $A \in \mathbb{C}^{n\times n}$ such that A is not invertible but $\sin(A)$ is invertible.

   This is false. We know $\sin(A) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} A^{2k+1} = A - \frac{A^3}{3!} + \frac{A^5}{5!} - \dots$. Let λ be an eigenvalue of A with eigenvector $v \neq 0$. By definition, $Av = \lambda v$. Then, for any integer $m > 0$, we know $A^m v = \lambda^m v$. Then we apply sin() such that, $\sin(A)v = (A - \frac{A^3}{3!} + \frac{A^5}{5!} - \cdots)v = (\lambda - \frac{\lambda^3}{3!} + \frac{\lambda^5}{5!} - \cdots)v = \sin(\lambda)v$. So, $\sin(\lambda)$ is an eigenvalue of $\sin(A)$. For contradiction, lets say there exists an A that is not invertible but $\sin(A)$ is invertible. If A is not invertible, then one of its eigenvalues must be 0. We've shown then that $\sin(0) = 0$ must be an eigenvalue of $\sin(A)$. Since an eigenvalue of $\sin(A)$ is 0, it must not be invertible. A contradiction, since our initial assumption was $\sin(A)$ was invertible. Therefore, the assumption must be false.

Problems to do in MATLAB

5. For the specific matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

   estimate $\sqrt{A}$ using a partial sum of the power series from Problem 1 that ends the summation at $k = 100$. Verify that the square of your estimate is nearly A.

6. A system of real-valued functions

$$\overline{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

satisfies the differential equations

$$\begin{cases} 4x_1'(t) &= 17x_1(t) - 86x_2(t) + 122x_3(t) - 50x_4(t) \\ 4x_2'(t) &= -12x_1(t) + 50x_2(t) - 82x_3(t) + 28x_4(t) \\ 4x_3'(t) &= -5x_1(t) + 22x_2(t) - 36x_3(t) + 12x_4(t) \\ 4x_4'(t) &= 16x_1(t) - 70x_2(t) + 106x_3(t) - 41x_4(t) \end{cases}$$

At time $t = 0$, the system has value

$$\begin{bmatrix} x_1(0) & x_2(0) & x_3(0) & x_4(0) \end{bmatrix}^\top = \begin{bmatrix} -1 & \frac{2}{3} & \frac{9}{2} & -\frac{6}{7} \end{bmatrix}^\top.$$

Use `expm` to find the value of the system at time $t = k \cdot \Delta t$ with $\Delta t = 0.25$ for all $k = 1, \ldots, 120$. Give your answer in the form of the matrix

$$\begin{bmatrix} \overline{x}(\Delta t) & \overline{x}(2 \cdot \Delta t) & \cdots & \overline{x}(120 \cdot \Delta t) \end{bmatrix} \in \mathbb{R}^{4\times 120}.$$