HOMEWORK 2
MATH 4070, FALL 2025

Throughout this document, $C([a,b])$ is defined as the real vector space of continuous functions $f\colon [a,b] \to \mathbb{R}$. It is equipped with the usual inner product $\langle f,g\rangle = \int_a^b f(t)g(t)\,dt$, and the distance from f to g is defined as $\|f-g\|$, where $\|\cdot\|$ is the norm given by $\langle\cdot,\cdot\rangle$.

Problems to type in TEX

1. Given any unit vector $u \in \mathbb{C}^n$, define the corresponding Householder matrix
$$H_u = I - 2uu^*.$$
Prove that this matrix is both self-adjoint and unitary. (This was stated in class, and now you have to prove it.)

   First, lets prove that H_u is self-adjoint. We defined self-adjoint in class as when a square matrix $A \in \mathbb{F}^{nxn}$, A is self-adjoint when $A = A^*$. Therefore we want to show $H_u = H_u^*$. Let $H_u^* = (I-2uu^*)^* = I^* - 2(uu^*)^*$, then by properties of adjoints ($I = I^*$ and $(AB)^* = B^*A^*$), $I^* - 2(uu^*)^* = I - 2uu^* = H_u$. Thus, $H_u = H_u^*$. Therefore, H_u is self-adjoint.

   Lastly, lets prove that H_u is unitary. We have many definitions for unitary, the easiest is for some matrix $U \in \mathbb{F}^{nxn}$, U is unitary when $U^*U = UU^* = I$. Therefore, we want to show $H_uH_u^* = I$. Using our previous findings, we know $H_uH_u^* = (I-2uu^*)(I-2uu^*) = I - 4uu^* + 4uu^*uu^*$. We know u is a unit column vector, so u^*u is a scalar with itself, which has to be 1 since its a unit vector. So, $I - 4uu^* + 4u(u^*u)u^* = I - 4uu^* + 4uu^* = I$. Thus, $H_uH_u^* = I$. Therefore, H_u is unitary.

2. (a) Show that the (infinite) set of functions
$$\left\{\pi^{-1/2}\cos(kt) : k = 1,2,\dots\right\} \cup \left\{\pi^{-1/2}\sin(kt) : k = 1,2,\dots\right\} \cup \left\{(2\pi)^{-1/2}\right\}$$
is orthonormal in $C([0,2\pi])$.

   We know that for a set of functions to be orthonormal it must be orthogonal and normal. The inner product of functions in $C([0,2\pi])$ is $\langle f,g\rangle = \int_0^{2\pi} f(t)g(t)dt$. Let $f_k(t) = \pi^{-1/2}\cos(kt)$ for $k \geq 1$, $g_k(t) = \pi^{-1/2}\sin(kt)$ for $k \geq 1$, and $c_0(t) = (2\pi)^{-1/2}$.

   First, lets prove normality by showing the inner product of any function with itself is one.
   $\langle f_k(t), f_k(t)\rangle = \int_0^{2\pi} \pi^{-1/2}\cos(kt)\pi^{-1/2}\cos(kt)dt = \int_0^{2\pi} \frac{\cos^2(kt)}{\pi}dt = \frac{1}{2\pi}\int_0^{2\pi} 1 + \cos(2kt)dt = \frac{1}{2\pi}[[t] + [\frac{\sin(2kt)}{2k}]]_0^{2\pi} = \frac{1}{2\pi}(2\pi + \frac{\sin(4k\pi)}{2k}) = 1 + \frac{\sin(4k\pi)}{4k\pi} = 1$

$\langle g_k(t), g_k(t)\rangle = \int_0^{2\pi} \pi^{-1/2}\sin(kt)\pi^{-1/2}\sin(kt)dt = \int_0^{2\pi} \frac{\sin^2(kt)}{\pi}dt = \frac{1}{2\pi}\int_0^{2\pi} 1-\cos(2kt)dt = \frac{1}{2\pi}[[t]-[\frac{\sin(2kt)}{2k}]]_0^{2\pi} = \frac{1}{2\pi}(2\pi - \frac{\sin(4k\pi)}{2k}) = 1 - \frac{\sin(4k\pi)}{4k\pi} = 1$
$\langle c_0(t), c_0(t)\rangle = \int_0^{2\pi}(2\pi)^{-1/2}(2\pi)^{-1/2}dt = \int_0^{2\pi}\frac{1}{2\pi}dt = \frac{2\pi}{2\pi} = 1$
All functions have a norm of 1, thus it is normal.

Lastly, lets prove orthogonality by showing the inner product of any two distinct functions in the set is zero. Let f_m and g_m be an exact copy of their respective function, but the constant k is replaced with a similar constant m (for the sake of distinct).

(for $k \neq m$): $\langle f_k, f_m\rangle = \int_0^{2\pi}(\pi^{-1/2}\cos(kt))(\pi^{-1/2}\cos(mt))dt = \frac{1}{\pi}\int_0^{2\pi}\cos(kt)\cos(mt)dt = \frac{1}{2\pi}\int_0^{2\pi}(\cos((k-m)t)+\cos((k+m)t)dt = \frac{1}{2\pi}[\frac{\sin((k-m)t)}{k-m}+\frac{\sin((k+m)t)}{k+m}]_0^{2\pi}$,
this leads to a similar result we got in our norm proof. Because $(k\pm m)$ is some whole constant in $\sin(c2\pi)$, where c is some whole constant, which will always result in 0. Thus this is 0 (we will use this logic later as well).

(for $k \neq m$): $\langle g_k, g_m\rangle = \int_0^{2\pi}(\pi^{-1/2}\sin(kt))(\pi^{-1/2}\sin(mt))dt = \frac{1}{\pi}\int_0^{2\pi}\sin(kt)\sin(mt)dt = \frac{1}{2\pi}\int_0^{2\pi}(\cos((k+m)t)+\cos((k-m)t)dt = 0$, same as previous.

(for any $k, m \geq 1$ and $k \neq m$): $\langle f_k, g_m\rangle = \frac{1}{\pi}\int_0^{2\pi}\cos(kt)\sin(mt)dt = \frac{1}{2\pi}\int_0^{2\pi}(\sin((k+m)t)+\sin((k-m)t)dt = \frac{1}{2\pi}[\frac{\cos((k+m)t)}{k+m}+\frac{\cos((k-m)t)}{k-m}]_0^{2\pi} = \frac{1}{2\pi}[(\frac{1}{k+m}+\frac{1}{k-m}) - (\frac{1}{k+m}+\frac{1}{k-m})] = 0$.

(for any $k, m \geq 1$ and $k = m$): $\langle f_k, g_m\rangle = \frac{1}{\pi}\int_0^{2\pi}\cos(kt)\sin(kt)dt = \frac{1}{2\pi}\int_0^{2\pi}\sin(2kt)dt = \frac{1}{2\pi}[\frac{-\cos(2kt)}{2k}]_0^{2\pi} = \frac{1}{2\pi}[\frac{-1}{2k} - (\frac{-1}{2k})] = 0$. Similar to what we did with $\sin(c2\pi)$, but for $\cos(c2\pi)$ instead, which results in 1 for 2π and 0.

$\langle c_0, f_k\rangle = \int_0^{2\pi}(2\pi)^{-1/2}(\pi^{-1/2}\cos(kt))dt = \frac{1}{\sqrt{2}\pi}\int_0^{2\pi}\cos(kt)dt = \frac{1}{\sqrt{2}\pi}[\frac{\sin(kt)}{k}]_0^{2\pi} = 0$

$\langle c_0, g_k\rangle = \int_0^{2\pi}(2\pi)^{-1/2}(\pi^{-1/2}\sin(kt))dt = \frac{1}{\sqrt{2}\pi}\int_0^{2\pi}\sin(kt)dt = \frac{1}{\sqrt{2}\pi}[\frac{-\cos(kt)}{k}]_0^{2\pi} = \frac{1}{\sqrt{2}\pi}[\frac{-1}{k} - (\frac{-1}{k})] = 0$

We have shown that the inner product of any function in the set with itself is 1 (normality) and the inner product of any two distinct functions in the set is 0 (orthogonality). Therefore, the set is orthonormal in $C([0, 2\pi])$.

(b) For any positive integer N, the set of **trigonometric polynomials** of degree at most N consists of all linear combinations of the (finite) set

$$\{\pi^{-1/2}\cos(kt) : k = 1, 2, \ldots, N\} \cup \{\pi^{-1/2}\sin(kt) : k = 1, 2, \ldots, N\} \cup \{(2\pi)^{-1/2}\}.$$

Given a continuous function $f\colon [0, 2\pi] \to \mathbb{R}$, let g_N be the trigonometric polynomial of degree at most N that is closest to f in $C([0, 2\pi])$.

Express $g_N(t)$ in terms of the **Fourier coefficients** of f:

$$a_0 := \frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt, \qquad a_k := \frac{1}{\pi}\int_0^{2\pi} f(t)\cos(kt)\,dt, \qquad b_k := \frac{1}{\pi}\int_0^{2\pi} f(t)\sin(kt)\,dt, \qquad k = 1, 2, \ldots.$$

As usual, you should prove that your formula works!

The trigonometric polynomial $g_N(t)$ of degree at most N that is closest to $f(t)$ is the orthogonal projection of f onto the subspace of such polynomials, lets call this subspace P_N. The set of functions is $B_N = \{u_0(t), u_k(t), v_k(t) : k = 1, \ldots, N\}$ where $u_0 = (2\pi)^{-1/2}$, $u_k = \pi^{-1/2}\cos(kt)$, and $v_k = \pi^{-1/2}\sin(kt)$, is an orthonormal basis for P_N.

Thus, we can say the orthogonal projection of f onto P_N is given by $g_N(t) = \langle f, u_0\rangle u_0(t) + \sum_{k=1}^{N}\big(\langle f, u_k\rangle u_k(t) + \langle f, v_k\rangle v_k(t)\big)$. Now we can move on to expressing $g_N(t)$ in terms of the Fourier coefficients:

$\langle f, u_0\rangle u_0(t) = \left(\int_0^{2\pi} f(t)\frac{1}{\sqrt{2\pi}}dt\right)\frac{1}{\sqrt{2\pi}} = \frac{1}{2\pi}\int_0^{2\pi} f(t)dt = a_0$

$\langle f, u_k\rangle u_k(t) = \left(\int_0^{2\pi} f(t)\frac{\cos(kt)}{\sqrt{\pi}}dt\right)\frac{\cos(kt)}{\sqrt{\pi}} = \left(\frac{1}{\pi}\int_0^{2\pi} f(t)\cos(kt)dt\right)\cos(kt) = a_k\cos(kt)$

$\langle f, v_k\rangle v_k(t) = \left(\int_0^{2\pi} f(t)\frac{\sin(kt)}{\sqrt{\pi}}dt\right)\frac{\sin(kt)}{\sqrt{\pi}} = \left(\frac{1}{\pi}\int_0^{2\pi} f(t)\sin(kt)dt\right)\sin(kt) = b_k\sin(kt)$

Substituting: $g_N(t) = a_0 + \sum_{k=1}^{N}\big(a_k\cos(kt) + b_k\sin(kt)\big)$.

To prove that g_N is the trigonometric polynomial closest to f, there must be no other trigonometric polynomial $p(t)$ that minimizes the distance better then $g_N(t)$. Using squared distance, we can represent this as $||f-p||^2 \geq ||f-g_N||^2$.

We can say $||f-p||^2 = \int_0^{2\pi}(f(t)-p(t))^2dt$. However we don't really have any info on $f(t)$ and $p(t)$, so we'll include $g_N(t)$ to help: $\int_0^{2\pi}(f(t)-p(t))^2dt = \int_0^{2\pi}(f(t)-g_N(t)+g_N(t)-p(t))^2dt = ||(f-g_N)+(g_N-p)||^2$. Then by the law of cosines, $||(f-g_N)+(g_N-p)||^2 = ||f-g_N||^2 + 2\langle f-g_N, g_N-p\rangle + ||g_N-p||^2$. Since g_N and p are both trigonometric polynomials in the subspace P_N, their difference (g_N-p) must also be in the subspace P_N. Also, by the definition of an orthogonal projection, $(f-g_N)$ is orthogonal to every vector in the subspace P_N. So, because (g_N-p) is in P_N, and $(f-g_N)$ is orthogonal to everything in P_N, we must have: $\langle f-g_N, g_N-p\rangle = 0$. Thus, $||f-p||^2 = ||f-g_N||^2 + ||g_N-p||^2$. Since $||g_N-p||^2 = \int_0^{2\pi}(g_N(t)-p(t))^2dt$ is the integral of a non-negative function, we know that $||g_N-p||^2 \geq 0$. Thus, the inequality shows that the distance from f to any other polynomial p is greater than or equal to the distance from f to g_N. Therefore, g_N is the closest polynomial. Furthermore, equality holds only when $\|g_N-p\|^2 = 0$, which implies $g_N = p$, proving that g_N is the unique closest polynomial.

3. In each case below, V is an inner product space over $\mathbb{F}$, and $W \leq V$ is a finite-dimensional subspace. Find (with proof) a nice description of $W^{\perp}$, and a nice formula for orthogonal projection onto W.
Note: These instructions are intentionally vague. I want you to find and state a clear description of the truth, with proof.
   (a) $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $V = \mathbb{F}^{n\times n}$, $W = \operatorname{span}\{I\}$

   A matrix $A \in V$ is in $W^{\perp}$ if and only if it is orthogonal to every matrix in W. So we want $\langle A, I_n\rangle = 0$ for each $A \in \mathbb{F}^{nxn}$ by the definition of orthogonal. We know $\langle A, I_n\rangle = tr(I_n A) = tr(A)$, thus we want $tr(A) = 0$. So, $W^{\perp} = \{A \in \mathbb{F}^{nxn} : tr(A) = 0\}$.

   Using the standard formula for projection: $P_W(v) = \frac{\langle v,w\rangle}{||w||^2}w$. Here, $v = A$ and $w = I$. Substituting: $\frac{\langle A,I\rangle}{||I||^2}I = \frac{tr(A)}{n}I$. To verify, $P_W(A) \in W$ (it is a scalar multiple of I) and $A - P_W(A) \in W^{\perp}$, since $tr(A - P_W(A)) = tr(A) - tr(\frac{tr(A)}{n}I) = tr(A) - \frac{tr(A)}{n}tr(I) = tr(A) - \frac{tr(A)}{n}n = 0$.. Both conditions are met, so the formula is correct.

   (b) $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $V = \mathbb{F}^n$, $W = \{x \in \mathbb{F}^n : \sum_{i=1}^n x_i = 0\}$

   Rewrite $\sum_{i=1}^n x_i = \langle x, 1\rangle = 0$. Thus, we can describe W as the set of all vectors that are orthogonal to the vector 1. By definition, the set of all vectors orthogonal to a given set of vectors is the orthogonal complement of the space they span. Therefore, $W = (span\{1\})^{\perp}$. This then tells us $W^{\perp} = ((span\{1\})^{\perp})^{\perp} = span\{1\}$.

   Using the standard formula for projection (we'll use $W^{\perp}$ onto W): $P_{W^{\perp}}(v) = \frac{\langle v,w\rangle}{||w||^2}w$. Here, $v = x$ and $w = 1$. Substituting: $\frac{\langle x,1\rangle}{||1||^2}1 = \frac{\sum_{i=1}^n x_i}{\langle 1,1\rangle}1 = \frac{\sum_{i=1}^n x_i}{n}1$. Take note that $\frac{\sum_{i=1}^n x_i}{n}$ is the average of all components in x, call it $\bar{x}$. So, $P_{W^{\perp}}(x) = (\bar{x}, \ldots, \bar{x})$. From decomposition $x = P_{W^{\perp}}(x) + P_W(x)$, so $P_W(x) = x - P_{W^{\perp}}(x) = x - (\bar{x}, \ldots, \bar{x}) = (x_1 - \bar{x}, x_2 - \bar{x}, \ldots, x_n - \bar{x})$. Next lets prove it. We just have to check if $P_W(x) \in W$ and $x - P_W(x) \in W^{\perp}$. First, lets check if $P_W(x) \in W$: $\sum_{i=1}^n (x_i - \bar{x}) = (\sum_{i=1}^n x_i) - \sum_{i=1}^n \bar{x} = n\bar{x} - n\bar{x} = 0$, thus it is in W. Second, lets check if $x - P_W(x) \in W^{\perp}$: $x - P_W(x) = x - (x - \bar{x}1) = \bar{x}1$, this is a scalar multiple of 1, so it is in $span\{1\} = W^{\perp}$. Both conditions are met, so the formula is correct.

   (c) $\mathbb{F} = \mathbb{R}$, $V = \mathbb{R}^{n\times n}$, $W = \{A \in \mathbb{R}^{n\times n} : A^{\top} = A\}$

   A matrix $B \in V$ is in $W^{\perp}$ if and only if it is orthogonal to every matrix in W. We know all $A \in W$ are symmetric since, by definition, $A^{\top} = A$. So we want a matrix $B \in W^{\perp}$ such that $\langle B, A\rangle = 0$. We know $\langle B, A\rangle = tr(B^T A)$, thus we want $tr(B^T A) = 0$. This must mean B is a skew matrix, ie, $B^T = -B$. To prove this, by the transpose property of traces $tr(B^T A) = tr((B^T A)^T) = tr(A^T B)$. Then, since A is symmetric, $tr(A^T B) = tr(AB)$. Then by the multiplication property of traces $tr(AB) = tr(BA)$. Going back, since B is skew symmetric $tr(B^T A) = tr(-BA) = -tr(BA)$. Since $\langle B, A\rangle = -tr(BA) = tr(BA)$, $\langle B, A\rangle$ can only be 0. Thus, $W^\perp = \{B \in \mathbb{F}^{nxn} : B^T = -B\}$.

We can't use the standard formula anymore since W isn't one dimensional anymore. Let $P_W(C)$ be the orthogonal projection onto W, where $C \in V$. By decomposition, $C = P_W(C) + (C - P_W(C))$, where $P_W(C) \in W$ and $(C - P_W(C)) \in W^\perp$. Let $A = P_W(C)$ and $B = C - P_W(C)$, such that $C = A + B$. We want to find what A is. Taking the transpose, $C^T = (A + B)^T = A^T + B^T$. Since $A \in W$ and $B \in W^\perp$, A is a symmetric matrix and B must be a symmetric skew matrix respectively. Thus, $C^T = A - B$. Combining C and C^T we get, $C + C^T = A + B + A - B = 2A \implies A = \frac{C+C^T}{2}$. This tells us $P_W(C) = \frac{C+C^T}{2}$. To verify, $\frac{C+C^\top}{2}$ is symmetric since the transpose $(\frac{C+C^\top}{2})^T = \frac{C^T+C}{2}$ is equivalent, thus is in W. Also, $C - \frac{C+C^\top}{2} = \frac{C-C^\top}{2}$ is skew-symmetric, since the transpose $(\frac{C-C^\top}{2})^T = \frac{C^T-C}{2}$ is the negative, thus is in $W^\perp$. Both conditions for a projection are met, so $P_W(C) = \frac{C+C^T}{2}$.

Problems to do in MATLAB

4. Consider the function $f\colon [0, 2\pi] \to \mathbb{R}$ given by $f(t) = e^{\sin t}$. On a single set of axes, plot f together with the nearest trigonometric polynomial of degree at most N for $N = 1, 3, 5$. Make sure you can tell which function is which at a glance. Submit a PDF of the resulting figure as an additional file with this problem.

   Hint 1: MATLAB can do numerical integration with `integral`. If it gives you any array size errors, you might have defined a function with something like "`fg = @(t) f(t)*g(t)`", and in that case you should change it to "`fg = @(t) f(t).*g(t)`".

   Hint 2: There are two ways to plot things in MATLAB: `fplot` and `plot`. The first uses functions, and the second uses vectors. It is easier to add vectors than functions in MATLAB, so you might prefer the second. I'll let you figure out the details.

5. Consider the function $f\colon [-1, 1] \to \mathbb{R}$ given by $f(t) = e^{\sin(\pi t)}$. Given N, let g_N be the polynomial of degree at most N that is closest to f in $C([-1, 1])$. (This is not like the last problem: g_N is an honest polynomial, not a trigonometric polynomial.) On a single set of axes, plot f together with g_N for $N = 1, 3, 5$. Do this without computing any orthonormal bases; instead, use the method of inverting a Gram matrix for a basis. Make sure you can tell which function is which at a glance. Submit a PDF of the resulting figure as an additional file with this problem.