EXERCISES FOR CHAPTER 4: VECTOR SPACES ASSOCIATED WITH MATRICES

1. For each of the following $m \times n$ matrices,

RREF

(i) find a basis for the row space and a basis for the column space;

(ii) extend the basis for the row space in (i) to a basis for $\mathbb{R}^m$; add missing corresponding row for identity matrix

(iii) extend the basis for the column space in (i) to a basis for $\mathbb{R}^n$; add missing corresponding col.

(iv) find a basis for the nullspace; Solution set of RREF of $Ax = 0$.

(v) find the rank and nullity of the matrix and hence verify the Dimension Theorem for Matrices; and rank + nullity = no. of col.

no. of pivot col. dim of nullspace

(vi) determine if the matrix has full rank. → longest diagonal of 1 for RREF.

dimension of a space = how many free variables. = no. of pivot col.

(a) $\boldsymbol{A} = \begin{pmatrix} 1 & 4 & 0 & 5 & 2 \\ 2 & 1 & 0 & 3 & 0 \\ -1 & 3 & 0 & 2 & 2 \\ 1 & -1 & 1 & -1 & 1 \end{pmatrix}$, (b) $\boldsymbol{B} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ -1 & 3 & 6 \\ 2 & 1 & 0 \\ 3 & 1 & -1 \end{pmatrix}$,

(c) $\boldsymbol{C} = \begin{pmatrix} 2 & 1 & 4 & 1 & 2 \\ 4 & 2 & 2 & 3 & 2 \\ 2 & 1 & -2 & 2 & 0 \\ 6 & 3 & 6 & 4 & 4 \end{pmatrix}$, (d) $\boldsymbol{D} = \begin{pmatrix} 1 & 4 & 5 & 8 \\ -1 & 4 & 3 & 0 \\ 2 & 0 & 2 & 1 \end{pmatrix}$.

span basis of I^n span $\mathbb{R}^n$.

2. Let W be the subspace of $\mathbb{R}^5$ spanned by the following vectors:

$$\boldsymbol{u}_1 = (1, -2, 0, 0, 3), \quad \boldsymbol{u}_2 = (2, -5, -3, -2, 6), \quad \boldsymbol{u}_3 = (0, 5, 15, 10, 0), \quad \boldsymbol{u}_4 = (2, 1, 15, 8, 6).$$

(a) Find a basis for W. RREF of the vectors put together $\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$ or $(u_1\ u_2\ u_3\ u_4)$

(b) What is $\dim(W)$? free variables.

(c) Extend the basis for W found in Part (a) to a basis for $\mathbb{R}^5$. add missing rows from corresponding identity matrix row.

3. For each of the following vector space $V = \text{span}(S)$, find a subset S' of S such that S' forms a basis for V. $\begin{pmatrix} 1 & 2 & \cdots \\ 0 & -1 \\ 1 & 0 \\ 3 & 1 \end{pmatrix} \xrightarrow{RREF}$ ✓ Take col. of RREF. combine all them, do RREF

(a) $S = \{(1, 0, 1, 3), (2, -1, 0, 1), (-1, 3, 5, 12), (0, 1, 2, 5), (3, -1, 1, 4)\}$.

(b) $S = \{(1, 0, 1, 3, 4), (2, 1, -2, 1, 0), (-3, -2, 5, 1, 4), (0, 5, 2, 1, 1), (0, 4, 6, 6, 9)\}$.

4. Find a basis for the following subspace of $\mathbb{R}^5$:

$$V = \{(a + b + 3c + 3d, b + 2c + d, a + c + 2d, -a - b - 3c - 3d, a + c + 2d) \mid a, b, c, d \in \mathbb{R}\}.$$

5. Let $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & -1 & 1 & 1 \\ 1 & 1 & 2 & 0 & 3 \\ -1 & 0 & 2 & -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$. ← R.

(a) Show that $\boldsymbol{R}$ is the reduced row-echelon form of $\boldsymbol{A}$. RREF of A.

(b) Let

v_1 v_2 v_3 $S = (v_1^T, v_2^T, v_3^T)$

$$S = \{(1,0,-1,1,1), (1,1,2,0,3), (-1,0,2,-1,0)\},$$

$$T = \{(1,0,0,1,2), (0,1,0,-1,-1), (0,0,1,0,1)\}.$$

(Sol'n set of RREF of S = span(T).

(i) Explain why S is a basis for span(T).

(ii) Find the transition matrix from S to T. $C \cdot S = T$?

6. Let $V = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4\}$ where

$$\boldsymbol{u}_1 = (1,1,1,1,1), \quad \boldsymbol{u}_2 = (1,a,a,a,a), \quad \boldsymbol{u}_3 = (1,a,a^2,a,a^2), \quad \boldsymbol{u}_4 = (1,a^3,a,2a-a^3,a)$$

RREF. no. of pivot col.

for some constant a. Find a basis for V and determine the dimension of V.

7. Let $V = \text{span}\{(1,1,0,0), (-1,0,1,0)\}$ and $W = \text{span}\{(-1,2,3,0), (2,-1,2,-1)\}$. Find a basis for $V + W$. (See Question 3.20.) RREF of (V+W).

8. Let $V = \{a(1,2,0,0) + b(0,-1,1,0) + c(0,0,0,1) \mid a,b,c \in \mathbb{R}\}$.

(a) Find a 4×4 matrix $\boldsymbol{A}$ such that the row space of $\boldsymbol{A}$ is V. — take vectors as rows.

(b) Find a 4×4 matrix $\boldsymbol{B}$ such that the column space of $\boldsymbol{B}$ is V. — take vectors as col.

(c) Find a 4×4 matrix $\boldsymbol{C}$ such that the nullspace of $\boldsymbol{C}$ is V. — take vector as row

9. Let $\boldsymbol{A}$ be a 3×4 matrix. Suppose that $x_1 = 1, x_2 = 0, x_3 = -1, x_4 = 0$ is a solution to a non-homogeneous linear system $\boldsymbol{Ax} = \boldsymbol{b}$ and that the homogeneous system $\boldsymbol{Ax} = \boldsymbol{0}$ has a general solution $x_1 = t - 2s$, $x_2 = s + t$, $x_3 = s$, $x_4 = t$ where s, t are arbitrary parameters.

(a) Find a basis for the nullspace of $\boldsymbol{A}$ and determine the nullity of $\boldsymbol{A}$. basis = $\{x_1, x_2, x_3, x_4\}$ nullity = 4.

(b) Find a general solution for the system $\boldsymbol{Ax} = \boldsymbol{b}$. $(\ )t + (\ )s$.

(c) Write down the reduced row-echelon form of $\boldsymbol{A}$. determine from gen. sol'n.

(d) Find a basis for the row space of $\boldsymbol{A}$ and determine the rank of $\boldsymbol{A}$. RREF of row vectors → rank = pivot col

(e) Do we have enough information for us to find the column space of $\boldsymbol{A}$? yes.

10. Let $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 & \boldsymbol{a}_5 \end{pmatrix}$ be a 4×5 matrix such that the columns $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ are linearly independent while $\boldsymbol{a}_4 = \boldsymbol{a}_1 - 2\boldsymbol{a}_2 + \boldsymbol{a}_3$ and $\boldsymbol{a}_5 = \boldsymbol{a}_2 + \boldsymbol{a}_3$.

(a) Determine the reduced row-echelon form of $\boldsymbol{A}$. (*Hint*: The linear relations between columns will not be changed by row operations. In this question, the fifth column of $\boldsymbol{A}$ is the sum of the second and the third columns of $\boldsymbol{A}$. Then the fifth column of the reduced row-echelon form $\boldsymbol{R}$ is still the sum of the second and the third columns of $\boldsymbol{R}$.)

gen sol

(b) Find a basis for the row space of $\boldsymbol{A}$ and a basis for the column space of $\boldsymbol{A}$. RREF row vec / RREF col. vec

11. For each of the following $\boldsymbol{A}$ and $\boldsymbol{b}$, solve the linear system $\boldsymbol{Ax} = \boldsymbol{b}$. Show that $\boldsymbol{b}$ belongs to the column space of $\boldsymbol{A}$ by expressing $\boldsymbol{b}$ as a linear combination of the columns of $\boldsymbol{A}$.

(a) $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 5 \\ 2 & 3 & 4 \\ 0 & 1 & -1 \\ 1 & 1 & 2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 16 \\ 13 \\ -4 \\ 7 \end{pmatrix}$. RREF (A | b)

(b) $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 2 & -1 & 1 \\ 1 & 1 & -1 & 2 & 0 \\ 3 & 1 & 0 & 0 & -1 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} -1 \\ 9 \\ 4 \end{pmatrix}$.

12. For each of the following cases, write down a matrix with the required property or explain why no such matrix exists.

(a) The column space contains vectors $(1,0,0)^{\mathrm{T}}$, $(0,0,1)^{\mathrm{T}}$ and the row space contains vectors $(1,1)$, $(1,2)$. No.

(b) The column space $= \mathbb{R}^4$ and the row space $= \mathbb{R}^3$. No.

(c) The column space = the row space = span$\{(1,2,3)\}$. ?

(d) A 2×2 matrix with the column space = the nullspace. $\vec{0}$.

13. For each of the following, find the largest possible value for the rank of the matrix $\boldsymbol{A}$ and the smallest possible value for the nullity of $\boldsymbol{A}$.

(a) $\boldsymbol{A}$ is 5×5. rank = 5. nullity = 0.

(b) $\boldsymbol{A}$ is 4×6. rank = 6. nullity = 0.

(c) $\boldsymbol{A}$ is 8×3. rank = 3. nullity = 0.

14. Consider a 2×2 matrix $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. What can you say about the values of a, b, c, d for each of the following cases?

(a) rank$(\boldsymbol{A}) = 0$. $\vec{0}$.

(b) rank$(\boldsymbol{A}) = 2$. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(c) rank$(\boldsymbol{A}) = 1$. either $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

15. Determine the possible rank and nullity of each of the following matrices:

(a) $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & a \\ 1 & a & 1 \\ a & 1 & 1 \end{pmatrix}$, $a=1$ $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ $r=1$ $n=2$. $a=0$ $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ $r=3$ $n=0$

(b) $\boldsymbol{B} = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & c \\ d & e & f \end{pmatrix}$, $r = 0, 1, 2.$ $n = 3, 2, 1$

where a, b, c, d, e, f are real numbers.

$2 = 2\cdot 1 + 2 - 1 |2|3$

16. Let $\boldsymbol{X}_n = (x_{ij})$ be a $(2n+1) \times (2n+1)$ matrix such that

$$x_{ij}^{rc} = \begin{cases} 1 & \text{if } i = j \text{ or } i = 2n+2-j \\ 0 & \text{otherwise.} \end{cases}$$

$X_1: \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ $X_2: \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix}$

(a) Write down $\boldsymbol{X}_1, \boldsymbol{X}_2$ and find their rank and nullity. $r=2$ $n=1$, $r=2$ $n=3$

(b) In general, what are rank$(\boldsymbol{X}_n)$ and nullity $(\boldsymbol{X}_n)$? $r=2$ $n=2n-1$

17. Let $\boldsymbol{A}$ be a 3×3 matrix. Describe geometrically the solution set of the linear system $\boldsymbol{Ax} = \boldsymbol{0}$ for each of the cases when rank$(\boldsymbol{A}) = 0, 1, 2, 3$. rank=0 ⇒ 3D space 3 ⇒ point 1 ⇒ plane 2 ⇒ line

18. Prove Theorem 4.1.11.

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be row equivalent matrices. Prove the following statements.

(a) A given set of columns of $\boldsymbol{A}$ is linear independent if and only if the set of corresponding columns of $\boldsymbol{B}$ is linearly independent. (*Hint*: Read Question 3.30.)

(b) A given set of columns of $\boldsymbol{A}$ forms a basis for the column space of $\boldsymbol{A}$ if and only if the set of corresponding columns of $\boldsymbol{B}$ forms a basis for the column space of $\boldsymbol{B}$.

19. Let $\boldsymbol{B}$ be an $m \times n$ matrix. If there exists an $n \times m$ matrix $\boldsymbol{C}$ such that $\boldsymbol{BC} = \boldsymbol{I}$, then $\boldsymbol{C}$ is called a *right inverse* of $\boldsymbol{B}$.

(a) Let $\boldsymbol{B} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$. Solve each of the following linear systems: $(B \mid I)$ $(I \mid B^{-1})$

(i) $\boldsymbol{B}\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, RREF (ii) $\boldsymbol{B}\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, (iii) $\boldsymbol{B}\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Hence, or otherwise, find a right inverse of $\boldsymbol{B}$. Is the right inverse unique? unique.

(b) Give an example of a nonzero matrix that has no right inverse. det $\neq 0 \Rightarrow$ zero rows.

(c) Show that an $m \times n$ matrix $\boldsymbol{B}$ has a right inverse if and only if $\operatorname{rank}(\boldsymbol{B}) = m$. det $= 0 \Rightarrow$ no zero rows.

20. Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are two matrices such that $\boldsymbol{AB} = \boldsymbol{0}$. Show that the column space of $\boldsymbol{B}$ is $\subseteq$ contained in the nullspace of $\boldsymbol{A}$. $Ax=0$ - if $AB=0$, $x=B$, then nullspace of A = soln space of $Ax=0$ $\supseteq$ col. space of B.

21. Show that there is no matrix whose row space and nullspace both contain the same nonzero vector.

row of $A \times$ col. of x = 0 $\Rightarrow$ $A \neq x$. soln of $Ax=0$

22. Let $\boldsymbol{A}$ be an $m \times n$ matrix and $\boldsymbol{P}$ an $m \times m$ matrix.

(a) If $\boldsymbol{P}$ is invertible, show that $\operatorname{rank}(\boldsymbol{PA}) = \operatorname{rank}(\boldsymbol{A})$. invertible $\Leftrightarrow$ no non-zero row $\Leftrightarrow$ all pivot cols $\therefore$ if any col of A is non-pivot, PA col will be non-pivot $\therefore$ rank(PA) = rank(A)

(b) Give an example such that $\operatorname{rank}(\boldsymbol{PA}) < \operatorname{rank}(\boldsymbol{A})$. P have zero rows.

(c) Suppose $\operatorname{rank}(\boldsymbol{PA}) = \operatorname{rank}(\boldsymbol{A})$. Is it true that $\boldsymbol{P}$ must be invertible? Justify your answer. ?

23. Let $\boldsymbol{A}$ and]$\boldsymbol{B}$ be two matrices of the same size. Show that

$$\underbrace{\operatorname{rank}(\boldsymbol{A}+\boldsymbol{B})}_{\rho \text{ of } A+B} \le \underbrace{\operatorname{rank}(\boldsymbol{A})}_{\rho \text{ of } A} + \underbrace{\operatorname{rank}(\boldsymbol{B})}_{\rho \text{ of } B}.$$

✓ $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x = b$.

24. Let $\boldsymbol{A}$ be an $m \times n$ matrix. Suppose the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ is consistent for all $\boldsymbol{b} \in \mathbb{R}^m$. Show that the linear system $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{0}$ has only the trivial solution. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} y = 0$, if consistent, $A = I$. $\therefore A^T = I$. $(I \mid 0)$ = trivial y.

25. Let $\boldsymbol{A}$ be an $m \times n$ matrix.

(a) Show that the nullspace of $\boldsymbol{A}$ is equal to the nullspace of $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$.

(b) Show that $\operatorname{nullity}(\boldsymbol{A}) = \operatorname{nullity}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A})$ and $\operatorname{rank}(\boldsymbol{A}) = \operatorname{rank}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A})$.

(c) Is it true that $\operatorname{nullity}(\boldsymbol{A}) = \operatorname{nullity}(\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}})$? Justify your answer. ?

(d) Is it true that $\operatorname{rank}(\boldsymbol{A}) = \operatorname{rank}(\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}})$? Justify your answer.

26. Determine which of the following statements are true. Justify your answer.

(a) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are two row equivalent matrices, then the row space of $\boldsymbol{A}^{\mathrm{T}}$ and the row space of $\boldsymbol{B}^{\mathrm{T}}$ are the same.

(b) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are two row equivalent matrices, then the column space of $\boldsymbol{A}^{\mathrm{T}}$ and the column space of $\boldsymbol{B}^{\mathrm{T}}$ are the same.

(c) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are two row equivalent matrices, then the nullspace of $\boldsymbol{A}^{\mathrm{T}}$ and the nullspace of $\boldsymbol{B}^{\mathrm{T}}$ are the same.

(d) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are two matrices of the same size, then $\operatorname{rank}(\boldsymbol{A}+\boldsymbol{B}) = \operatorname{rank}(\boldsymbol{A}) + \operatorname{rank}(\boldsymbol{B})$.

(e) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are two matrices of the same size, then $\operatorname{nullity}(\boldsymbol{A}+\boldsymbol{B}) = \operatorname{nullity}(\boldsymbol{A}) + \operatorname{nullity}(\boldsymbol{B})$.

(f) If $\boldsymbol{A}$ is an $n \times m$ matrix and $\boldsymbol{B}$ is an $m \times n$ matrix, then $\operatorname{rank}(\boldsymbol{AB}) = \operatorname{rank}(\boldsymbol{BA})$.

?

(g) If $\boldsymbol{A}$ is an $n \times m$ matrix and $\boldsymbol{B}$ is an $m \times n$ matrix, then $\text{nullity}(\boldsymbol{AB}) = \text{nullity}(\boldsymbol{BA})$.