CS1231S Tutorial 6: Induction and recursion

National University of Singapore

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More challenging questions are indicated by an asterisk (*). When asked to prove a statement by induction, one may use regular or Strong Mathematical Induction.

Questions for discussion on the LumiNUS Forum

Answers to these questions will not be provided.

> **Theorem.** Every non-negative integer is interesting.
>
> *Proof.* We prove this by contradiction.
> 1. Suppose some non-negative integer is uninteresting.
> 2. By the Well-Ordering Principle, there is a smallest uninteresting non-negative integer.
> 3. The smallest uninteresting non-negative integer is *highly* interesting.
> 4. An integer cannot be both interesting and uninteresting; so we have the required contradiction. ☐

D1. What is wrong with the following proof that $2^n = 1$ for all $n \in \mathbb{Z}_{\geqslant 0}$?

1. For each $n \in \mathbb{Z}_{\geqslant 0}$, let $P(n)$ be the proposition " $2^n = 1$ ".
2. (Base step) $P(0)$ is true because $2^0 = 1$.
3. (Induction step)
   3.1. Let $k \in \mathbb{Z}_{\geqslant 0}$ such that $P(0), P(1) \ldots, P(k)$ are true, i.e., that
   $$2^0 = 2^1 = \cdots = 2^k = 1.$$
   3.2. Then $\quad 2^{k+1} = \dfrac{2^k \times 2^k}{2^{k-1}}$
   3.3. $\quad = \dfrac{1 \times 1}{1}$ by the induction hypothesis;
   3.4. $\quad = 1.$
   3.5. Thus $P(k+1)$ is true.
4. Hence $\forall n \in \mathbb{Z}_{\geqslant 0}\ P(n)$ is true by Strong MI.

D2.* Abelard (a twelfth-century Parisian logician) and Eloise (the niece of a canon of Notre Dame) are playing games. Each game has a fixed length, say $n \in \mathbb{Z}_{\geqslant 0}$. In the game, the players take turns to play a move, starting with Eloise. A play of the game thus looks like
$$(x_1, x_2, \ldots, x_n),$$
where $x_1, x_3, \ldots$ are the moves by Eloise, and $x_2, x_4, \ldots$ are the moves by Abelard. When a player plays a move x_i, she/he is able to see all the previous moves $x_1, x_2, \ldots, x_{i-1}$ in the game. The rules of the game, set out before the game begins, consist of a set R: Eloise wins if and only if the play of the game $(x_1, x_2, \ldots, x_n)$ is an element of R. There is no draw.

Show by induction on n that no matter what n and R are, one of the players can guarantee a win. (A proof of this was first published by Ernst Zermelo in 1912.)

(Hint: a player can play a move to get into a winning position if and only if the current position is already winning for her/him.)

D3. A *well-order* on a set A is a total order on A with respect to which every nonempty subset of A has a smallest element. Our Well-Ordering Principle states that the usual order on $\mathbb{Z}_{\geqslant 0}$ is a well-order. Under this order, below any number there are only finitely many numbers. Find a well-order without this property, i.e., find a well-order on $\mathbb{Z}_{\geqslant 0}$ with respect to which some number has infinitely many numbers below it.

D4. Peter needs to climb a flight of stairs of n steps, where $n \in \mathbb{Z}_{\geqslant 1}$. He can go up 1 or 2 steps with each stride. Let s_n be the number of ways in which Peter can climb n steps. (So $s_2 = 2$ for example, since he can climb 2 steps in 1 stride going up 2 steps, or in 2 strides each going up 1 step.)

(a) Express s_n in terms of $s_1, s_2, \ldots, s_{n-1}$.

(b) What is the sequence $s_1, s_2, \ldots$?

D5.* (D.R. Hofstadter) The following rules govern which strings of letters Douglas can write down.

(I) Douglas can write down MI.

(II) After writing down xI for some string x, Douglas can write down xIU.

(III) After writing down Mx for some string x, Douglas can write down Mxx.

(IV) After writing down xIIIy for some strings x, y, Douglas can write down xUy.

(V) After writing down xUUy for some strings x, y, Douglas can write down xy.

(VI) Douglas cannot write down a string unless it is allowed by one of the rules above.

(a) According to these rules, can Douglas write down MUIIU? Prove that your answer is correct.

(b) According to these rules, can Douglas write down MU? Prove that your answer is correct. (Hint: count the number of I's in the strings Douglas can write down. Note that if $[a] \in \mathbb{Z}_3$ and $[a] \neq [0]$, then $[a] + [a] \neq [0]$.)

Tutorial questions

1. Prove by induction that for all $n \in \mathbb{Z}_{\geqslant 1}$,

$$1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}\, n(n+1)(2n+1).$$

2. Let $x \in \mathbb{R}_{\geqslant -1}$. Prove by induction that $1 + nx \leqslant (1+x)^n$ for all $n \in \mathbb{Z}_{\geqslant 1}$.

3. Prove by induction that 3 divides $n^3 + 11n$ for all $n \in \mathbb{Z}_{\geqslant 1}$.

4. Let a be an odd integer. Prove by induction that 2^{n+2} divides $a^{2^n} - 1$ for all $n \in \mathbb{Z}_{\geqslant 1}$. Note that $a^{b^c} = a^{(b^c)}$ by convention. Here you may use without proof the fact that the product of any two consecutive integers is even.

5.* Prove by induction that

$$\forall n \in \mathbb{Z}_{\geqslant 8}\ \exists x, y \in \mathbb{Z}_{\geqslant 0}\ (n = 3x + 5y).$$

(In other words, any integer-valued transaction over 8 dollars can be carried out using only 3-dollar and 5-dollar coins.)

6.* Prove by induction that every positive integer can be written as a sum of *distinct* non-negative integer powers of 2, i.e.,

$$\forall n \in \mathbb{Z}_{\geqslant 1}\ \exists \ell \in \mathbb{Z}_{\geqslant 1}\ \exists i_1, i_2, \ldots, i_\ell \in \mathbb{Z}_{\geqslant 0}\ (i_1 < i_2 < \cdots < i_\ell \wedge n = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_\ell}).$$

(Hint: think in terms of binary representations.)

Definition 8.2.2. The *Fibonacci sequence* $F_0, F_1, F_2, \ldots$ is defined by setting

$$F_0 = 0 \quad \text{and} \quad F_1 = 1 \quad \text{and} \quad F_{n+2} = F_{n+1} + F_n$$

for each $n \in \mathbb{Z}_{\geqslant 0}$.

7. Show that $F_{n+4} = 3F_{n+2} - F_n$ for all $n \in \mathbb{Z}_{\geqslant 0}$.

8. Show by induction that $F_{n+1}^2 - F_{n+1}F_n - F_n^2 = (-1)^n$ for every $n \in \mathbb{Z}_{\geqslant 0}$.

9. Let $a_0, a_1, a_2, \ldots$ be the sequence satisfying

$$a_0 = 0, \quad a_1 = 2, \quad a_2 = 7, \quad \text{and} \quad a_{n+3} = a_{n+2} + a_{n+1} + a_n$$

for all $n \in \mathbb{Z}_{\geqslant 0}$. Prove by induction that $a_n < 3^n$ for all $n \in \mathbb{Z}_{\geqslant 0}$.

10. Define a set S recursively as follows.

    (1) $2 \in S$. (base clause)

    (2) If $x \in S$, then $3x \in S$ and $x^2 \in S$. (recursion clause)

    (3) Membership for S can always be demonstrated by (finitely many) successive applications of clauses above. (minimality clause)

    Which of the numbers $0, 6, 15, 16, 36$ are in S? Which are not?

11. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 3, 5, 7, 9\}$. Define a set S recursively as follows.

    (1) $A, B \in S$. (base clause)

    (2) If $X, Y \in S$, then $X \cap Y \in S$ and $X \cup Y \in S$ and $X \setminus Y \in S$. (recursion clause)

    (3) Membership for S can always be demonstrated by (finitely many) successive applications of clauses above. (minimality clause)

    For each of the following sets, determine whether it is in S, and use one sentence to explain your answer.

    (a) $C = \{2, 4, 7, 9\}$.

    (b) $D = \{2, 3, 4, 5\}$.

1. Prove by induction that for all $n \in \mathbb{Z}_{\geqslant 1}$,

$$1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1).$$

1. For each $n \in \mathbb{Z}_{\geq 1}$, let $P(n)$ be the proposition "$1^2+2^2+\ldots+n^2 = \frac{1}{6}n(n+1)(2n+1)$".
2. (Base step) $P(1)$ is true because

$1^2 = \frac{1}{6}(1)(1+1)(2(1)+1)$

LHS $= 1^2 = 1$.
RHS $= \frac{1}{6}(1)(1+1)(2(1)+1) = 1 =$ LHS.

3. (Induction step)

not n, cannot assume for all k, else circular $\Rightarrow$ proving for all k.

3.1 Let $k \in \mathbb{Z}_{\geq 1}$ such that $P(k)$ is true, such that $1^2+2^2+\ldots+k^2 = \frac{1}{6}(k)(k+1)(2k+1)$

3.2 Then, $1^2+2^2+\ldots+k^2+(k+1)^2$

3.3 $= \frac{1}{6}(k)(k+1)(2k+1) + (k+1)^2$ by induction hypothesis $P(k)$

3.4
$$\begin{aligned}
&= (k+1)\left(\tfrac{1}{6}(k)(2k+1) + (k+1)\right) \\
&= (k+1)\left(\tfrac{1}{6}(2k^2+k) + k+1\right) \\
&= (k+1)\left(\tfrac{2}{6}k^2 + \tfrac{1}{6}k + k + 1\right) \\
&= (k+1)\left(\tfrac{2}{6}k^2 + \tfrac{7}{6}k + 1\right) \\
&= \tfrac{1}{6}(k+1)(2k^2+7k+6) \\
&= \tfrac{1}{6}(k+1)(k+2)(2k+3) \\
&= \tfrac{1}{6}(k+1)((k+1)+1)(2(k+1)+1)
\end{aligned}$$
by basic algebra.

$(k+2)(2k+3)$
$= 2k^2+3k+4k+6$
$= 2k^2+7k+6$

to make clearer.

3.5. So $P(k+1)$ is true.

4. Hence $\forall n \in \mathbb{Z}_{\geq 1}$ $P(n)$ is true by MI.

2. Let $x \in \mathbb{R}_{\geqslant -1}$. Prove by induction that $1 + nx \leqslant (1+x)^n$ for all $n \in \mathbb{Z}_{\geqslant 1}$.

1. For each $n \in \mathbb{Z}_{\geq 1}$, let $P(n)$ be the proposition "$1+nx \leq (1+x)^n$"

defining predicate.

2. (base step) $P(1)$ is true because

can write $=$, also satisfies inequality.

$1+(1)x \leq (1+x)^1$

$1+x \leq 1+x$.

for some.

3. (induction step) let $k \in \mathbb{Z}_{\geq 1}$ such that $P(k)$ is true, such that $1+kx \leq (1+x)^k$

Assume $P(k)$ is true for some $k \in \mathbb{Z}_{\geq 1}$

3.1. Then,

$1+(k+1)x = 1+kx+x$ as $k \geq 1$ and $x^2 \geq 0$

3.2. $\leq (1+x)^k + x$ by induction hypothesis $P(k)$

how?

3.3. $\leq (1+x)^k \cdot (1+x)$ $(1+x) \geq 0 \Rightarrow$ impt for inequality.

3.4. $= (1+x)^{k+1}$

3.5. So $P(k+1)$ is true.

$(1+x)^{k+1}$
$= (1+x)^k \cdot (1+x)$
$= (1+x)^k + x$ cannot assume.

Can do from the other side:

$(1+x)^{k+1}$
$= (1+x)^k(1+x)$
$\geq (1+kx)(1+x)$ by induction hypothesis
$= 1+(k+1)x+kx^2$. & $1+x \geq 0$
$\geq 1+(k+1)x$ as $k \geq 1$ and $x^2 \geq 0$.

4. Hence $\forall n \in \mathbb{Z}_{\geq 1} \forall x \in \mathbb{R}_{\geq -1} \left(1+nx \leq (1+x)^n\right)$ is true by mathematical induction.

$3 \mid n^3+11n.$

3. Prove by induction that 3 divides $n^3 + 11n$ for all $n \in \mathbb{Z}_{\geqslant 1}$.

1. For each $n \in \mathbb{Z}_{\geq 1}$, let $P(n)$ be the proposition " $3 \mid (n^3+11n)$ ".
2. (base step) $P(1)$ is true because
$$3 \mid (1^3 + 11(1))$$
$$3 \mid 12. \quad = 3 \times 4.$$
3. (Induction step) let $k \in \mathbb{Z}_{\geq 1}$, such that $P(k)$ is true, such that $3 \mid (k^3 + 11k)$.

$k^3+11k = 3m$ (defn of divisibility)

3.1 Then,
$$\begin{aligned}
&(k+1)^3 + 11(k+1) \\
&= (k+1)((k+1)^2+11) \\
&= (k+1)(k^2+2k+1+11) \\
&= (k+1)(k^2+2k+12) \\
&= k^3 + 2k^2 + 12k + k^2 + 2k + 12 \\
&= k^3 + 3k^2 + 14k + 12 \\
&= (k^3 + 11k) + (3k^2 + 3k + 12)
\end{aligned}$$

by closure, is integer

3.2. By induction hypothesis $P(k)$, since $3 \mid (k^3+11k)$ and $3 \mid (3k^2+3k+12)$, then $3 \mid (k+1)^3 + 11(k+1)$.

divisibility holds for sums.

$3m + 3(k^2+k+4)$
$= 3(m+k^2+k+4)$
by definition

3.3. So $P(k+1)$ is true.

4. Hence $\forall n \in \mathbb{Z}_{\geq 1}$ $(3 \mid (n^3+11n))$ is true by mathematical induction.

proof $P(k)$ is true, then $P(k+2)$ is true. ⇒ odd integer

4. Let a be an odd integer. Prove by induction that 2^{n+2} divides $a^{2^n} - 1$ for all $n \in \mathbb{Z}_{\geqslant 1}$. Note that $a^{b^c} = a^{(b^c)}$ by convention. Here you may use without proof the fact that the product of any two consecutive integers is even. $l(l+1)$ is even.

1. For each $n \in \mathbb{Z}_{\geq 1}$, let $P(n)$ be the proposition " $2^{n+2} \mid a^{2^n} - 1$ "
2. (base step) $P(1)$ is true because

$$2^{1+2} \mid a^{2^{(1)}} - 1$$
$$2^3 \mid a^2 - 1$$
$$8 \mid a^2 - 1$$
$$8 \mid (2b+1)^2 - 1$$ by definition of odd, $b \in \mathbb{Z}$, b is even.
$$8 \mid 4b^2 + 4b + \cancel{1} - 1$$ even $= 2m$
$$\therefore\ 8 \mid \underbrace{4(b^2+b)}_{8m}$$ $4l(l+1)$

3. (Induction step) let $k \in \mathbb{Z}_{\geq 1}$, such that $P(k)$ is true and $2^{k+2} \mid a^{2^k} - 1$
3.1 Then,

$$2^{k+1+2} \mid a^{2(k+1)} - 1$$
$$2^{k+3} \mid a^{2k+2} - 1$$
$$2^{k+3} \mid (2b+1)^{2k+2} - 1$$
$$\mid (2b+1)^{2k} \cdot (2b+1)^2 - 1$$

$$a^{2^k} - 1 = 2^{k+2} m.$$

Then,

$$a^{2^{k+1}} - 1 = a^{2^k \times 2} - 1$$
$$= (a^{2^k})^2 - 1$$
$$= ((a^{2^k} - 1) + 1)^2 - 1$$
$$= ((2^{k+2} m)^2 + 2 \times 2^{(k+2)} m + 1) - 1$$
$$= 2^{2k+4} m^2 + 2^{k+3} m.$$
$$= 2^{k+3} (\underbrace{2^{k+1} m^2 + m}_{\mathbb{Z}})$$
$$2^{(k+1)+2} \mid (a^{2^{k+1}} - 1)$$

$\therefore P(k+1)$ is true.

Similar to
$\gcd(a,b) = ax + by$
$x, y \in \mathbb{Z}$
can be negative.

5.* Prove by induction that

$$\forall n \in \mathbb{Z}_{\geqslant 8}\ \exists x, y \in \mathbb{Z}_{\geqslant 0}\ (n = 3x + 5y).$$

(In other words, any integer-valued transaction over 8 dollars can be carried out using only 3-dollar and 5-dollar coins.)

1. For each $n \in \mathbb{Z}_{\geq 8}$, let $P(n)$ be the proposition "$\exists x, y, \in \mathbb{Z}_{\geq 0}\ (n = 3x + 5y)$"

2. (Base case) $P(8)$ is true because when $x = 1$ and $y = 1$,
$n = 3(1) + 5(1)$
$n = 3 + 5.$
$n = 8$ ✓

3. (Induction case) let $k \in \mathbb{Z}_{\geq 1}$, such that $P(k)$ is true and $\exists x, y, \in \mathbb{Z}_{\geq 0}\ (k = 3x + 5y)$

3.1 Then,
$k+1 = 3x' + 5y'.$

Same pattern:
$8 = 3(1) + 5(1)$
$9 = 3(3) + 5(0)$ ($+2$, -1)
$10 = 3(0) + 5(2)$ (-3, $+2$)
$11 = 3(2) + 5(1)$ ($+2$, -1)
$12 = 3(4) + 5(0).$ ($+2$, -1.)

3.2. By induction hypothesis $P(k)$,
$3x + 5y + 1 = 3x' + 5y'$
$1 = 3x' + 5y' - 3x - 5y.$
$1 = 3(x' - x) + 5(y' - y)$

let $a = x' - x$, $b = y' - y$, $a \in \mathbb{Z}$, $b \in \mathbb{Z}$.
$1 = 3a + 5b.$ — Sad :c

2 cases: $y > 0$ decrease y by 1, increase x by 3.
Then $k+1 = (3x + 5y) + 1$
$= 3(x+2) + 5(y-1)$
As $y > 0$, $y - 1 \in \mathbb{Z}_{\geq 0}$
$\therefore P(k+1)$ is true.

$y = 0$ - Increase y by 2, decrease x by 3.
Then $k = 3x + 5 \times 0 = 3x'$
$\therefore x = \frac{k}{3} \geq \frac{8}{3}$, $k \geq 8$
$\therefore x \geq 3.$

Thus, $k + 1 = 3x + 1$
$= 3(x-3) + 5 \times 2.$
$\in \mathbb{Z}_{\geq 0}.$

$\therefore P(k+1)$ is true

Strong MI., odd-even

6.* Prove by induction that every positive integer can be written as a sum of *distinct* non-negative integer powers of 2, i.e.,

$$\forall n \in \mathbb{Z}_{\geqslant 1}\ \exists \ell \in \mathbb{Z}_{\geqslant 1}\ \exists i_1, i_2, \ldots, i_\ell \in \mathbb{Z}_{\geqslant 0}\ (i_1 < i_2 < \cdots < i_\ell \wedge n = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_\ell}).$$

(Hint: think in terms of binary representations.)

1. For each $n \in \mathbb{Z}_{\geq 1}$, let $P(n)$ be the proposition
"$\exists \ell \in \mathbb{Z}_{\geq 1}\ \exists i_1, i_2, \ldots, i_\ell \in \mathbb{Z}_{\geq 0}\ (i_1 < i_2 < \ldots < i_\ell \wedge n = 2^{i_1} + 2^{i_2} + \ldots + 2^{i_\ell})$"

2. (base clause) $P(1)$ is true because $1 = 2^0$

$P(1) \wedge P(2) \wedge \ldots \wedge P(k) \Rightarrow P(k+1)$ is true.

3. (induction clause) Let $k \in \mathbb{Z}_{\geq 1}$, such that $P(k)$ is true and $k = 2^{i_1} + \ldots + 2^{i_\ell}$.
3.1 Then,
$k+1 = 2^{i_1} + \ldots + 2^{i_\ell} + 1$ by induction hypothesis $P(k)$
$= 2^{i_1} + \ldots + 2^{i_\ell} + 2^0$ by base clause.
$= 2^0 + 2^{i_2} + \ldots + 2^{i_\ell + 1}$
3.2. So, $P(k+1)$ is true.

4. Hence, the proposition is true by MI.

k is odd. $k+1 = 2m$.
k is even $k+1 = 2m+1$.

$m \leq k$.
either case:
$2m \leq k+1$.
$m \geq 1$.

$P(m)$ is true $\rightarrow 1 \leq m \leq k$.

Case 1: $k+1 = 2m$.
$k+1 = 2m$.
$= 2(2^{i_1} + \ldots + 2^{i_\ell})$
$= 2^{i_1+1} + 2^{i_2+1} + \ldots 2^{i_\ell+1}$
$i_1+1 < i_2+1 < \ldots < i_\ell+1$
$\therefore P(k+1)$ is true

Case 2: $k+1 = 2m+1$.
$= 2(2^{i_1} + 2^{i_2} + \ldots + 2^{i_\ell}) + 1$.
$= 2^0 + 2^{i_1+1} + \ldots + 2^{i_\ell+1}$
$\therefore 0 < i_1+1 < i_2+1 < \ldots < i_\ell+1$.
distinct
$\therefore P(k+1)$ is true.

Definition 8.2.2. The *Fibonacci sequence* $F_0, F_1, F_2, \ldots$ is defined by setting

$$F_0 = 0 \quad \text{and} \quad F_1 = 1 \quad \text{and} \quad F_{n+2} = F_{n+1} + F_n$$

not necessary for MI, use recursive defn of Fib.

for each $n \in \mathbb{Z}_{\geqslant 0}$.

7. Show that $F_{n+4} = 3F_{n+2} - F_n$ for all $n \in \mathbb{Z}_{\geqslant 0}$.

$F_{n+4} = F_{n+3} + F_{n+2}$
$= (F_{n+2} + F_{n+1}) + F_{n+2}$
$= 2F_{n+2} + F_{n+1}$
$= 3F_{n+2} - F_{n+2} + F_{n+1}$
$= 3F_{n+2} - (F_{n+1} + F_n) + F_{n+1}$
$= 3F_{n+2} - F_n$

1. For each $n \in \mathbb{Z}_{\geq 0}$, let $P(n)$ be the proposition " $F_{n+4} = 3F_{n+2} - F_n$ "
2. (base case) $P(0)$ and $P(1)$ are true because

$F_{0+4} = 3F_{0+2} - F_0$
$F_4 = 3F_2 - 0$
$= 3F_2.$
$= F_2 + (F_1 + F_0) + F_2.$
$= (F_2 + F_1) + F_2.$
$= F_3 + F_2$ //

$F_{1+4} = 3F_{1+2} - F_1$
$F_5 = 3F_3 - F_1$
$= 2F_3 - F_1 + F_3$
$= F_3 + (F_2 + F_1) - F_1 + F_3.$
$= F_3 + F_2 + F_1 - F_1 + F_3.$
$= (F_3 + F_2) + F_3$
$= F_4 + F_3.$ //

$F_4 = F_3 - F_2.$
$F_5 = F_4 - F_3.$

3. (Induction case) Let $k \in \mathbb{Z}_{\geq 0}$ such that $P(0), P(1), \ldots P(k+1)$ are true
3.1 Then,

$F_{(k+2)+4} = F_{k+6}$
$= F_{k+5} + F_{k+4}$ by defn of F_{k+6}
$= F_{(k+1)+4} + F_{k+4}$
$= 3F_{k+1+2} - F_{k+1}$
$+ 3F_{k+2} - F_k$ as $P(k)$ and $P(k+1)$ are true.
$= 3F_{k+3} - F_{k+1} + 3F_{k+2} - F_k$
$= 3F_{k+3} \quad \cancel{- F_{k+1}}$
$+ 2F_{k+2} + (\cancel{F_{k+1} + F_k}) \cancel{- F_k}.$ by defn of F_{k+2}
$= 3F_{k+3} + 2F_{k+2}$
$= 3F_{k+3} + 3F_{k+2} - F_{k+2}.$
$= 3(F_{k+3} + F_{k+2}) - F_{k+2}.$
$= 3F_{k+4} - F_{k+2}.$ by defn of $F_{k+4}.$
$= 3F_{(k+2)+2} - F_{k+2}$ //

get $3F_{k+2+2} - F_{k+2}.$
$3F_{k+4} - F_{k+2}$
$3(F_{k+3} + F_{k+2}) - F_{k+2}$
$3F_{k+3} + 3F_{k+2} - F_{k+2}.$

3.2. so $P(k+2)$ is true

4. Hence, " $F_{n+4} = 3F_{n+2} - F_n$ " is true by strong MI.

In terms of $F_{k+2}^2 - F_{k+2}F_k - F_k^2$.
ab terms.

$$a^2 - ab - b^2.$$

base case for $P(k+2)$ may not make sense

8. Show by induction that $F_{n+1}^2 - F_{n+1}F_n - F_n^2 = (-1)^n$ for every $n \in \mathbb{Z}_{\geq 0}$.

1. For each $n \in \mathbb{Z}_{\geq 0}$, def. $P(n)$ be the proposition " $F_{n+1}^2 - F_{n+1}F_n - F_n^2 = (-1)^n$ "

2. (base step) $P(0)$ ~~and $P(1)$~~ are true because

$$F_{0+1}^2 - F_{0+1}F_0 - F_0^2 = (-1)^0$$
$$F_1^2 - F_1 \cdot F_0 - F_0^2 = 1$$
$$(1)^2 - (1)(0) - (0)^2 = 1.$$
$$1 = 1 \text{ //}$$

$$F_{1+1}^2 - F_{1+1}F_1 - F_1^2 = (-1)^1$$
$$F_2^2 - F_2 F_1 - F_1^2 = -1$$
Since $F_2 = F_1 - F_0 = 1 - 0 = 1.$
$$(1)^2 - (1)(1) - (1)^2 = -1$$
$$1 - 1 - 1 = -1$$
$$-1 = -1 \text{ //}$$

3. (Induction step) let $k \in \mathbb{Z}_{\geq 0}$ such that $P(0), P(1), \dots, P(k+1)$ are true,

3.1 Then,

$$F_{k+2+1}^2 - F_{k+2+1} \cdot F_{k+2} - F_{k+2}^2 = (-1)^{k+2}$$
$$F_{k+3}^2 - F_{k+3} \cdot F_{k+2} - F_{k+2}^2 = (-1)^{k+1}(-1)^1$$

replace $k+2$ by $k+1$, $k+1$ by k

$$= (F_{k+2} + F_{k+1})^2 - (F_{k+2} + F_{k+1})(F_{k+2}) - (F_{k+2})^2$$
$$= \cancel{F_{k+2}^2} + \cancel{2}F_{k+2}F_{k+1} + F_{k+1}^2 - \cancel{F_{k+2}^2} - \cancel{F_{k+2}F_{k+1}} - F_{k+2}^2$$
$$= -F_{k+2}^2 + F_{k+2}F_{k+1} + F_{k+1}^2$$
$$= -(F_{k+2}^2 - F_{k+2}F_{k+1} - F_{k+1}^2)$$
$$= (-1)^{k+1}(-1)^1 \quad \text{as } P(k+1) \text{ is true}$$

3.2. So $P(k+2)$ is true

4. By induction, statement is true.

The answer proof $k+1$ & k.

strong MI.

9. Let $a_0, a_1, a_2, \ldots$ be the sequence satisfying

n-bonacci no.

$$a_0 = 0, \quad a_1 = 2, \quad a_2 = 7, \quad \text{and} \quad a_{n+3} = a_{n+2} + a_{n+1} + a_n$$

base step.

for all $n \in \mathbb{Z}_{\geqslant 0}$. Prove by induction that $a_n < 3^n$ for all $n \in \mathbb{Z}_{\geqslant 0}$.

1. For each $n \in \mathbb{Z}_{\geq 0}$, let $P(n)$ be the proposition "$a_n < 3^n$"
2. (base case) $P(0), P(1), P(2)$ are true as

$$a_0 = 0 < 3^0 = 1, \quad a_1 = 2 < 3^1 = 3, \quad a_2 = 7 < 3^2 = 9$$

3. (induction case) let $k \in \mathbb{Z}_{\geq 0}$, such that $P(0), P(1), \ldots P(k+2)$ are true.

3.1 Then,

$$\begin{aligned} P(k+3) = a_{k+3} &< 3^{k+3} \\ &= a_{k+2} + a_{k+1} + a_k \quad \text{by def}^n \\ &< 3^{k+2} + 3^{k+1} + 3^k \quad \text{as } P(k+2), P(k+1), P(k) \text{ are true, by induction hypothesis} \\ &= 3^k \cdot 3^2 + 3^k \cdot 3 + 3^k \\ &= 3^k(3^2 + 3 + 1) \\ &= 3^k(13) \\ &< 3^k(27) \\ &= 3^k \cdot 3^3 \\ &= 3^{k+3} \end{aligned}$$

or $< 3^{k+2} + 3^{k+2} + 3^{k+2}$

$3 \times 3^{k+2}$

3^{k+3}

3.2. So, $P(k+3)$ is true

4. Hence, statement is true by strong MI.

$q = \frac{m}{n}$ $\quad n, m \in \mathbb{Z}$.

$P(x)$ $\quad x \in \mathbb{Q}$

$P(\frac{1}{n}) \wedge P(\frac{k}{n}) \rightarrow P(\frac{k+1}{n})$

$P(\frac{m}{1}) \wedge P(\frac{m}{k}) \rightarrow P(\frac{m}{k+1})$

$\frac{m}{n}$

$n \backslash m$	1	2	3	4
	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{1}$	...
1				
2	$\frac{1}{2}$	,		
3	$\frac{1}{3}$			
4	$\frac{1}{4}$			
	⋮			

10. Define a set S recursively as follows.

(1) $2 \in S$. $S = \{2, 6, 4, 18, 36, 12, 16, \dots\}$. (base clause)

(2) If $x \in S$, then $3x \in S$ and $x^2 \in S$. (recursion clause)

not discussed in detail (3) Membership for S can always be demonstrated by (finitely many) successive applications of clauses above. (minimality clause)

Which of the numbers $0, 6, 15, 16, 36$ are in S? Which are not?

In S: $6, 36, 16$

Not in S: $0, 15$.

$0 \notin S$ because all $x \in S$ satisfy $x \geq 2$

$\Rightarrow$ by structural induction.

$15 \notin S$ because no $x \in S$ is odd.

$\Rightarrow$ by structural induction.

$P(2)$ is true

& $\forall x \in S\,(P(x) \Rightarrow P(3x) \wedge P(x^2))$ is true

11. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 3, 5, 7, 9\}$. Define a set S recursively as follows.

(1) $A, B \in S$. (base clause)

(2) If $X, Y \in S$, then $X \cap Y \in S$ and $X \cup Y \in S$ and $X \setminus Y \in S$. (recursion clause)

(3) Membership for S can always be demonstrated by (finitely many) successive applications of clauses above. (minimality clause)

For each of the following sets, determine whether it is in S, and use one sentence to explain your answer.

(a) $C = \{2, 4, 7, 9\}$.

(b) $D = \{2, 3, 4, 5\}$.

Possibilities:

$X \cap Y$: $\{1, 3, 5\}$.

$X \cup Y$: $\{1, 2, 3, 4, 5, 7, 9\}$.

$X \setminus Y$: $\{2, 4\}$

$\{1, 3, 5, 7, 9\}$

$\{1, 2, 3, 4, 5\}$.

$X \setminus Y$

$\{7, 9\}$

a) $C \in S$.

$X \cup Y$.

$\{2, 4, 7, 9\}$.

$(A \setminus B) \cup (B \setminus A)$

b) $D \notin S$.

It is impossible to combine sets such that 1 is not part of set A. Since 1 is part of both sets in base clause

In any case, 1 must come w/ 3.

Structural Induction over S. $P(A)$ and $P(B)$ is true.

$\forall x \in S\,(P(x) \wedge P(Y) \Rightarrow$

$P(X \cap Y) \wedge P(X \cup Y) \wedge P(X \setminus Y))$

is true.

$1 \notin D$ and $3 \in D$ but $\forall x \in S\,(1 \in x \Leftrightarrow 3 \in x)$.