MA1521
Calculus for Computing

Notes by CHAN Heng Huat (NUS-AY2021/2022 Semester 1)

Contents

References

The main reference is "Thomas' Calculus, 14-th edition". The supplementary reference is "Stewart's Calculus, Early Transcendentals, 9-th edition".

1 Functions

1.1 Real numbers and intervals

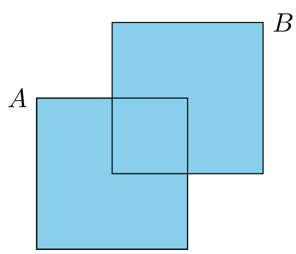
A **set** is a collection of **distinct** objects. In this course, a set will be denoted by capital letters $A, B, C, \ldots$. We use the notation

$$A = \{a, b, c, \ldots\}$$

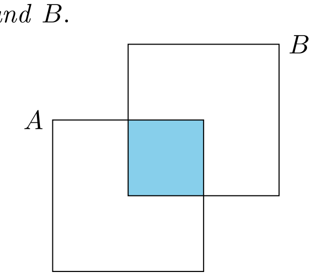
to represent a set. The objects $a, b, c, \ldots$ between "{" and "}" are called the **elements** of the set A. The notation $a \in A$ is used to represent the statement "a is an element of A". The notation $a \notin A$ if a is used to shorten the statement a is *not* an element of A. If every element of A is an element of B, we say A is a **subset** of B, and write $A \subseteq B$. We write $A = B$ if $A \subseteq B$ and $B \subseteq A$.

Given two sets A and B, we can define some operations.

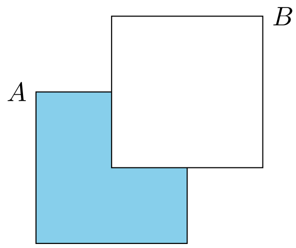
- Union: $A \cup B$ is the set consisting of the elements which are in A *or* B.



- Intersection: $A \cap B$ is the set consisting of the elements which are in both A *and* B.



- Difference: $A \backslash B$ (or $A - B$) is the set consisting of the elements which are in A but not in B.



- Product: $A \times B$ is the set consisting of the elements of the form (a, b) for which $a \in A$ and $b \in B$. When $A = \{1, 2, 3\}$ and $B = \{1, 2\}$, the set $A \times B$ is given by

$$A \times B = \{(1,1), (1,2), (2,1), (2,2), (3,1), (3,2)\}.$$

There are some basic notations we will use often for this course:

(a) $\mathbf{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$: the set of integers.
(b) $\mathbf{Z}^+ = \{1, 2, 3, \ldots\}$: the set of positive integers.
(c) $\mathbf{Q} = \{m/n \mid m, n \in \mathbf{Z}, n \neq 0\}$: the set of rational numbers.

Observe that $\mathbf{N} \subset \mathbf{Z} \subset \mathbf{Q}$.

The most important set in this course is the set of real numbers, denoted by $\mathbf{R}$. In this course, we will interpret real numbers as points on a line (called the real number line). On a straight line, we choose a point that we mark as 0 and call this point the origin. We take a convenient unit of length and to each number r, we assign a point P on the line whose distance is r measured to the right of 0 if r is positive and to the left of 0 if r is negative. Real numbers can be represented by points on a number line with a fixed origin O that represents the number 0. A "real number" r is positive if it is represented by a point to the right of 0. It is negative if it lies to the left of 0.

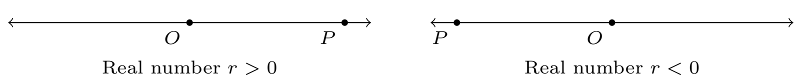
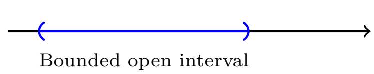


Figure 1.1 The real number line

There are several important subsets of the set of real numbers. The first subset which we will encounter is the open intervals. Let $a, b \in \mathbf{R}$. A set of the form

$$\{a < x < b | x \in \mathbf{R}\}$$

is a bounded open interval and is denoted by (a, b). We may think of an open interval as a line segment on the number line which does not include the end points.



An unbounded open interval is of the form

$$\{x > a | x \in \mathbf{R}\}$$

and

$$\{x < b | x \in \mathbf{R}\}$$

and their notations are (a, ∞) and $(-\infty, b)$ respectively.

Unbounded open interval

There are other important subsets of $\mathbf{R}$. They are the bounded closed intervals

$$[a, b] = \{a \leq x \leq b | x \in \mathbf{R}\}$$

and the unbounded closed intervals

$$[a, \infty) = \{x \geq a | x \in \mathbf{R}\}$$

and

$$(-\infty, b] = \{x \leq b | x \in \mathbf{R}\}.$$

Bounded closed interval

Unbounded closed interval

A bounded interval of the form

$$(a, b] = \{a < x \leq b | x \in \mathbf{R}\}$$

is neither open nor closed.

An interval that is neither open nor closed

We will refer to a set I of the form $a, b)$, $(a, b]$, $[a, b$, $[a, b]$, (a, ∞), $[a, \infty)$, $(-\infty, b)$, $(-\infty, b]$ as **interval**.

Given a set A in $\mathbf{R}$. If $a \in I$ and there is an open interval $\mathcal{I} \subset A$ that contains a, then we say that a is an **interior point** of A.

EXAMPLE 1.1

If $A = (1, 3]$ then the interior points are points in the interval $(1, 3)$.

A **boundary point** of A is a point b if every open interval that contains b has non-empty intersection with A and $A^c = \mathbf{R} - A$.[1]

EXAMPLE 1.2

If $A = (1, 3]$, then its boundary points are 1 and 3. A point in $(1, 3)$ is not a boundary point because if $x \in (1, 3)$ then we can find an open interval containing x that is a subset of A. To see that 1 is a boundary point, let I be an interval containing 1. Then we can find a sufficiently small $d > 0$ such that $(1 - d, 1 + d)$ is contained in I. The number $1 - d/2$ is not in A while the number $1 + d/2$ is in A. So I, which contains $(1 - d, 1 + d)$ has non-empty intersection with A and $A^c = \mathbf{R} - A$. This shows that 1 is a boundary point. Similarly, 3 is also a boundary point. Note that a boundary point may or may not belong to A.

1.2 Functions; Domain and Range

Let A and B be two sets. If there is a rule that assigns each element of A to a **unique** element in B, we call the rule a **function**. A function is usually denoted by f. The unique element in B that $a \in A$ is assigned to by f is called the **image** of a. Mathematically, we write $b = f(a)$. Here is the definition of a function:

DEFINITION 1.1 A function f from a set D to a set Y is a rule that assigns a unique value $f(x)$ in Y to each x in D

The set D of all possible input values is called the **domain** of the function. The set of all output values of $f(x)$ as x varies throughout D is called the **range** of the function. In this course, we will assume that **all our functions have domains and ranges which are subsets of R**. Most of the time, these subsets are intervals.

EXAMPLE 1.3

Find the domain and the range of the function

$$g(x) = \frac{2x}{3x - 1}.$$

[1] In Thomas' Calculus, boundary points of an interval are the end points of the interval (see p. AP-3).

Solution

The function is defined at every real x except when $3x - 1 = 0$, that is, when $x = 1/3$. Hence the domain is $\mathbf{R}\backslash\{\frac{1}{3}\}$.

To find the range of $g(x)$, let $y = g(x)$. Then $y = \dfrac{2x}{3x-1}$ implies that $x = \dfrac{y}{3y-2}$. It shows that y can take any real number except when $3y - 2 = 0$. Hence the range is $\mathbf{R}\backslash\{\frac{2}{3}\}$.

1.3 Graphs of Functions

DEFINITION 1.2 The set

$$\mathbf{R} \times \mathbf{R} = \{(a, b)|a \in \mathbf{R}, b \in \mathbf{R}\}$$

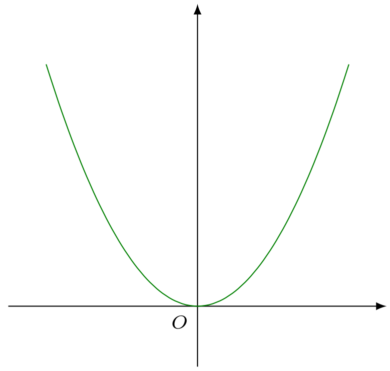
is called the Cartesian plane.

DEFINITION 1.3 If f is a function with domain D, its **graph** consists of the points in the Cartesian plane whose coordinates are of the form $(x, f(x))$. We use the notation $G(f)$ to represent the graph of f and write

$$G(f) := \{(x, f(x)) \mid x \in D\}.$$

The graph of $G(f)$ is a subset of the Cartesian plane $(\mathbf{R} \times \mathbf{R})$. The graph $G(f)$ allows us to have a pictorial view of f.

EXAMPLE 1.4



From the above graph of f, we observe that the domain of f is $\mathbf{R}$, and the range of f is $\mathbf{R}^+ \cup \{0\}$, the set of non-negative real numbers. The graph of a function allows us to "see" not only the domain and range, but also other behaviors.

1.4 Operations on functions

Let f and g be functions with domain A and B, respectively. We define

$$\begin{aligned} (f+g)(x) &:= f(x)+g(x) \quad \text{domain} = A \cap B, \\ (f-g)(x) &:= f(x)-g(x) \quad \text{domain} = A \cap B, \\ (fg)(x) &:= f(x)g(x) \quad \text{domain} = A \cap B, \\ (f/g)(x) &:= f(x)/g(x) \quad \text{domain} = \{x \in A \cap B \mid g(x) \neq 0\}\ . \end{aligned}$$

We can define **composite** of f and g by

$$(f \circ g)(x) := f(g(x)).$$

Note that $f \circ g$ is NOT the same as fg. The domain of $f \circ g$ is the set of elements x in the domain of g such that $g(x)$ is in the domain of f. In other words, if B is the domain of g and A is the domain of f, then the domain of $f \circ g$ is the set $\{x \in B \mid g(x) \in A\}$.

EXAMPLE 1.5

If $f(x) = x+5$ and $g(x) = x^2 - 3$, what is the composite $f \circ g(x)$?

Solution

$f \circ g(x) = f(g(x)) = f(x^2-3) = (x^2-3)+5 = x^2+2.$

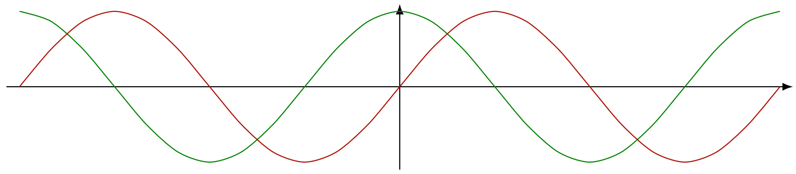


Figure 1.2 Graphs of $y = \sin x$ and $y = \cos x$.

1.5 Functions on $\mathbb{R}$

In this section, we recall some functions which you have already encountered before taking this course.

A polynomial of degree n is a function of the form

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

with $a_j \in \mathbf{R}, 0 \leq j \leq n$ and $a_n \neq 0$. Examples of such functions are $x, x^2, x^3 + 2x + 1$.

A rational function is a function which is of the form $P(x)/Q(x)$ where $P(x)$ and $Q(x)$ are polynomials of degrees n and m respectively.

We next introduce the algebraic functions. A function Y is said to be algebraic if it is a solution of a polynomial equation

$$Y^m + P_{m-1}(x)Y^{m-1} + \cdots + P_1(x)Y + P_0(x) = 0.$$

For example, $Y = \sqrt{x}$ is algebraic as it is the solution to $Y^2 - x = 0$. Other examples of algebraic functions are $Y = \sqrt[3]{x^2+1} + x$ and $Y = x^2 + \sqrt{x + \sqrt{x^2+1}}$.

Function in x which cannot be expressed as a solution of a polynomial equation with polynomial coefficients is called a transcendental function. Examples of such functions are the trigonometric functions such as $\sin x$, $\cos x$ or the exponential functions $a^x, a > 0$ or the logarithmic function $\log_a x$. We now recall what these functions are.

The trigonometric functions $\sin x$, $\cos x$ and $\tan x$ are defined as ratios of the sides of a right angled triangle. For example, if x is one of the angles (not the right angle) then $\sin x$ (where x is measured in radians) is defined to be the ratio of the side opposite the angle x to that of the hypotenuse. The graph of the sine and cosine function are shown as follow:

REMARK 1.1 A real number r is a **zero of a function** $f(x)$ if $f(r) = 0$. One way to see that $\sin x$ is not algebraic is to observe that it has infinitely many zeroes. It cannot be an algebraic function because an algebraic function can have only finitely many zeroes.

Next, we discuss briefly the exponential functions. We know how to compute

a^n for real positive number a and integers n, namely,

$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}.$$

When n is negative, we define

$$a^n = 1/a^{-n}.$$

We also know how to define $a^{p/q}$ when p and q are integers. The number $a^{p/q}$ is the solution b of the equation $b^q = a^p$. But when x is an irrational number such as $\sqrt{2}$ or π, how do we define a^x? We will have to be vague at this point and treat a^x as the value of a^{r_n}, with $r_n \in \mathbf{Q}$ as r_n "approaches" x. The function a^x satisfies the law of exponents, namely, $a^{u+v} = a^u a^v$ when $u, v \in \mathbf{R}$. Here are some examples of the graphs of exponential functions:

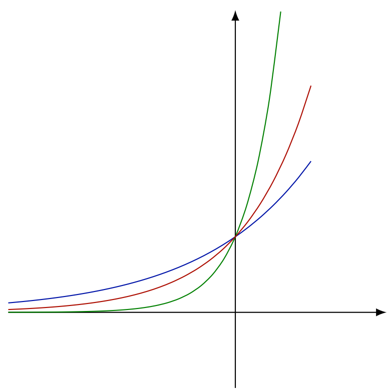


Figure 1.3 Graphs of $y = 10^x, y = 3^x, y = 2^x$

The next function we wish to highlight is the absolute function $|x|$. This function is defined differently on disjoint sets in the domain. It is given by The absolute value function $f(x) = |x|$ is defined by

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

The graph is:

The absolute function $|x|$ is an important function. Note that

$$a \leq |a|$$

since $|a| = \pm a$. Next,

$$|ab| = |a||b|$$

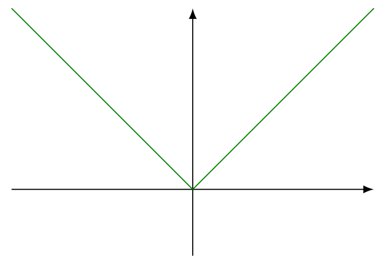


Figure 1.4 Graphs of $y = |x|$.

for all real numbers x. This follows from the following computations:

$$|ab|^2 = (\pm ab)^2 = a^2b^2 = |a|^2|b|^2,$$

which implies that

$$|ab| = \pm|a||b|.$$

But $|ab|$ and $|a||b|$ are both positive, and thus, $|ab| = |a||b|$. An important inequality satisfied by $|x|$ is the triangle inequality given by

THEOREM 1.1

$$|a+b| \leq |a| + |b|. \tag{1.1}$$

Proof
The inequality is true if either $a = 0$ or $b = 0$.
To prove the inequality, we observe that

$$\begin{aligned} |a+b|^2 &= |a+b||a+b| = |(a+b)^2| = (a+b)^2 \\ &= a^2 + b^2 + 2ab \leq a^2 + b^2 + 2|ab| = |a|^2 + |b|^2 + 2|a||b| = (|a| + |b|)^2. \end{aligned}$$

Since $|a+b|$ and $|a|+|b|$ are both positive (note that $a \neq 0$ and $b \neq 0$), the above inequality implies that

$$|a+b| \leq |a| + |b|.$$

□

REMARK 1.2 If $u > 0$ and $v > 0$, then $u^2 \geq v^2$ implies that $u \geq v$. This follows because $u^2 \geq v^2$ implies that $u^2 - v^2 \geq 0$, or $(u-v)(u+v) \geq 0$. Since $u + v > 0$, this previous inequality yields $u - v \geq 0$ or $u \geq v$.

The next function we define is called the integer floor function, denoted by $\lfloor x \rfloor$. This is defined as the greatest integer less than x. For example, $\lfloor 2.4 \rfloor = 2$ and $\lfloor -0.2 \rfloor = -1$. The graph of the function is given as follows:

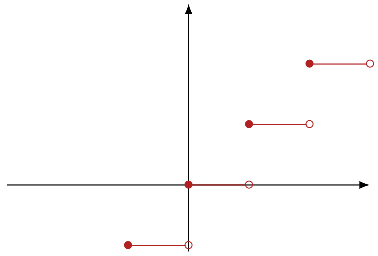


Figure 1.5 Graph of $y = \lfloor x \rfloor$.

DEFINITION 1.4 A function $f(x)$ is **one to one** on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

EXAMPLE 1.6

The function $f(x) = x^2$ for $x \in (0, \infty)$ is one to one. It is not one to one on $\mathbf{R}$.

DEFINITION 1.5 Suppose that f is a one to one function on a domain D with range R. The inverse function, denoted by f^{-1}, is defined by

$$f^{-1}(a) = b \quad \text{if } f(b) = a.$$

The domain of f^{-1} is R and the range of f^{-1} is D.

REMARK 1.3 The inverse $f^{-1}(x)$ IS NOT $1/f(x)$.

REMARK 1.4 Let f be a one to one function on a domain D. The composition $f^{-1} \circ f(x) = x$. To see this, let $b = f^{-1}(f(x))$. This means that $f(x) = f(b)$. But since f is one to one, $x = b$. Similarly, $f \circ f^{-1}(y) = y$. This is because if $y \in R$, then $y = f(x)$ for some $x \in D$. Therefore $f \circ f^{-1}(f(x)) = f(x) = y$, where we have used $f^{-1} \circ f(x) = x$.

EXAMPLE 1.7

The inverse function of $\sin x$ is denoted by $\sin^{-1} x$ and it is called "arcsine" of x. The inverse function of e^x is denoted by $\ln x$, the natural logarithm of x.

REMARK 1.5 In Calculus, the definition of $\ln x$ comes first after introducing integrals. The function e^x is then defined as the inverse of $\ln x$.

EXAMPLE 1.8

Find the inverse of $f(x) = \dfrac{x}{2} + 1$, expressed as a function of x.

Solution

The inverse of $f(x)$ is $f^{-1}(x) = 2x - 2$.

1.6 Special classes of functions

DEFINITION 1.6 A function is said to be **increasing** on an interval I if

$$f(x_1) < f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I.$$

It is called **decreasing** on I if

$$f(x_1) > f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I.$$

EXAMPLE 1.9

The function $f(x) = mx + c$, with $m > 0$ is an increasing function. The function $f(x) = mx + c$ with $m < 0$ is a decreasing function.

EXAMPLE 1.10

The function $f(x) = x^3$ is increasing on $\mathbf{R}$.

Solution

Suppose $a^3 = b^3$ and $a \neq b$. Then $a^3 - b^3 = 0$ implies that

$$(a-b)(a^2+ab+b^2) = 0.$$

But

$$a^2+ab+b^2 = \left(a+\frac{b}{2}\right)^2 + \frac{3b^2}{4} > 0$$

since a and b cannot be both zero or else $a = b$. Therefore, since $a \neq b$,

$$(a-b)(a^2+ab+b^2) \neq 0,$$

contradiction to our hypothesis. We are therefore forced to conclude that if $a^3 = b^3$, then $a = b$. [2]

DEFINITION 1.7 A function is **even** if it satisfies

$$f(-x) = f(x) \quad \text{for all } x.$$

EXAMPLE 1.11

The functions $f(x) = x^2$ and $g(x) = \cos x$ are both even functions.

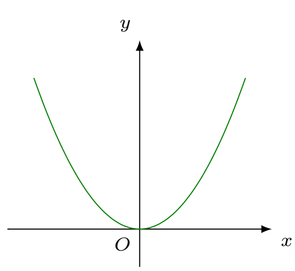


Figure 1.6 The graph of an even function

[2] This is an example of "proof by contradiction". We assume $a^3 = b^3$ and $a \neq b$ to arrive at a contradiction, forcing us to conclude that if $a^3 = b^3$, a must be equal to b. I use complex numbers during lecture. A more direct proof is to take cube root on both sides of $a^3 = b^3$.

DEFINITION 1.8 A function is **odd** if it satisfies

$$f(-x) = -f(x) \quad \text{for all } x.$$

EXAMPLE 1.12

The functions $f(x) = x^3$ and $g(x) = \sin x$ are both odd functions.

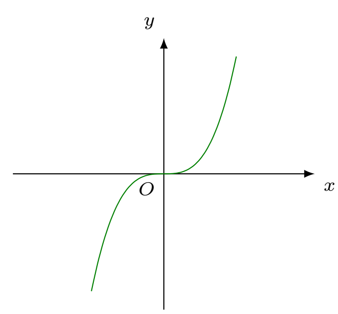
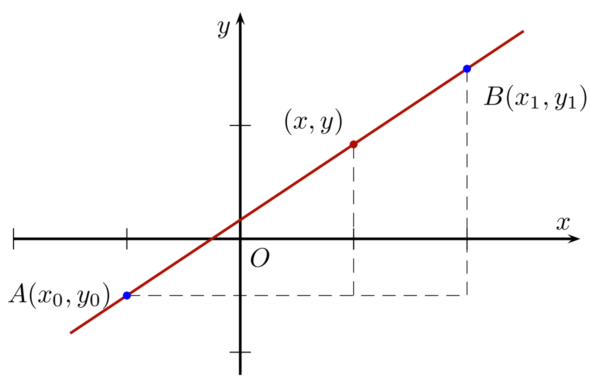


Figure 1.7 The graph of an odd function

2 Limits

2.1 The straight line

In Chapter 1, we saw how the graph presents the function. Let's now look at a special case: how to determine the function of a straight line on the xy-plane? We know that a straight line is uniquely determined by two distinct points. Consider the straight line passing through $A(x_0, y_0)$ and $B(x_1, y_1)$:



Let (x, y) be an arbitrary point on the line. Then by using similar triangles, we see that

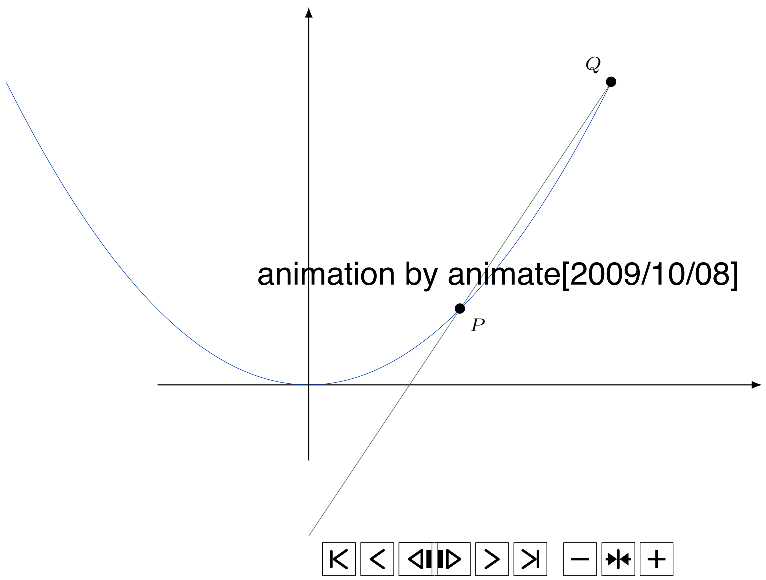
$$\frac{y - y_0}{x - x_0} = \frac{y_1 - y_0}{x_1 - x_0}.$$

Hence the equation of the line is

$$y = m(x - x_0) + y_0,$$

where the number $m = \dfrac{y_1 - y_0}{x_1 - x_0}$ is called the **slope** (or **gradient**) of the line.

2.2 The tangent line of a curve at a point



Suppose P and Q are two points on the graph of $f(x) = x^2$. As Q approaches P, the "secant line" joining Q and P eventually touches the curve $y = x^2$ at P. The resulting line is called the tangent line of $y = x^2$ at $(1, 1)$. We now compute the slope of the tangent line. To do so, we note that if the coordinate of Q is (a, a^2), then the slope of the secant line joining P and Q is given by

$$\frac{a^2 - 1}{a - 1}.$$

Note that as Q approaches P, this slope varies. When Q arrives at P, the secant line is the tangent line of the curve at P. The slope of this tangent line is therefore the expression

$$\frac{a^2 - 1}{a - 1}$$

as a approaches 1. We say that the slope of the tangent line is the "limit" of the slopes of the secant lines and we represent the slope of the tangent line as

$$m = \lim_{a \to 1} \frac{a^2 - 1}{a - 1}.$$

Since $\dfrac{a^2 - 1}{a - 1} = a + 1$, we conclude that the slope of the tangent line is 2, which is the value of $a + 1$ as a approaches 1. In other words, the slope of the tangent line of $y = x^2$ at $(1, 1)$ is 2. This is our first encounter of "limit" in this course. A more precise definition of limit will be given later.

EXAMPLE 2.1

Find the equation of the tangent line of $y = x^3$ at $(2, 8)$.

We now state the intuitive definition of Limit.

DEFINITION 2.1 (Intuitive definition of limit) We write

$$\lim_{x \to a} f(x) = L.$$

and say **the limit of $f(x)$, as x approaches a, equals L** if we can make the values of $f(x)$ *arbitrarily close* to L by taking x to be *sufficiently close* to a.

EXAMPLE 2.2

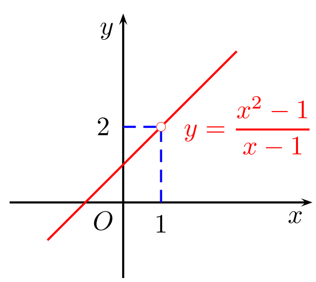
Let

$$f(x) = \frac{x^2 - 1}{x - 1}.$$

This is a function we encountered in the previous section. Note that this function is not defined at $x = 1$. However, we can guess the limit

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2,$$

because $\dfrac{x^2 - 1}{x - 1}$ is close to 2 by taking x as close to 1 (but not equal to 1) as possible.



We will discuss the precise definition of limit in the appendix.

2.3 Calculating limits using the limit laws

Let c be a real number. Then

$$\lim_{x \to a} c = c.$$

This is because c is independent of x, so the behavior of $f(x) = c$ is not affected by the behavior of x. It is trivial that as x is close to a, c is close to c.

Next, if $f(x) = x$, then

$$\lim_{x \to a} x = a.$$

We observe that as x is close to a, $f(x) = x$ is close to a.

We now state the limit laws. By using the limit laws and the above simple observations, we will be able to compute many simple limits.

THEOREM 2.1 (The Limit Laws) If L, M, c and k be real numbers and

$$\lim_{x \to c} f(x) = L \text{ and } \lim_{x \to c} g(x) = M,$$

then

(a) $\lim_{x \to c}(f(x) \pm g(x)) = L \pm M$,

(b) $\lim_{x \to c} f(x)g(x) = LM$,

(c) $\lim_{x \to c} kf(x) = kL$,

(d) $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, $M \neq 0$,

(e) $\lim_{x \to c}(f(x))^n = L^n$, n a positive integer,

(f) For odd positive integer n and $L \neq 0$, $\lim_{x \to c} f^{1/n}(x) = L^{1/n}$. For even positive integer n and $L > 0$, $\lim_{x \to c} f^{1/n}(x) = L^{1/n}$. If $L = 0$ and $n \neq 0$, $\lim_{x \to c}(f(x))^{1/n} = 0$.

EXAMPLE 2.3

Evaluate

$$\lim_{y \to 2} \frac{y+2}{y^2 + 5y + 6}$$

using The Limit Laws.

Solution

By the limit laws,

$$\lim_{y\to 2} \frac{y+2}{y^2+5y+6} = \frac{\lim_{y\to 2} y+2}{\lim_{y\to 2} y^2+5y+6} = \frac{4}{20} = \frac{1}{5}.$$

EXAMPLE 2.4

If $\lim_{x\to -2} \frac{f(x)}{x^2} = 1$, find $\lim_{x\to -2} f(x)$ and $\lim_{x\to -2} \frac{f(x)}{x}$.

Solution

The limits $\lim_{x\to -2} \frac{f(x)}{x^2} = 1$ and $\lim_{x\to -2} x^2 = 4$ implies that

$$\lim_{x\to -2} \frac{f(x)}{x^2} \lim_{x\to -2} x^2 = 4$$

but the left hand side is, by Limit Laws, $\lim_{x\to -2} f(x)$.

In a similar way, we conclude that

$$\lim_{x\to -2} \frac{f(x)}{x} = \lim_{x\to -2} \frac{f(x)}{x^2} \lim_{x\to -2} x = -2.$$

Using the limit laws, the following are true:

THEOREM 2.2 (Direct substitution property)

(a) If $P(x)$ is a polynomial, then

$$\lim_{x\to c} P(x) = P(c).$$

(b) If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x\to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

REMARK 2.1 Functions with the direct substitution property are said to be *continuous at a*. We will discuss continuous functions in Chapter 3

Note that we cannot apply Theorem 2.2(b) if $\lim_{x \to c} Q(x) = 0$. But it may still be possible that the limit $\lim_{x \to c} P(x)/Q(x)$ exists. We have seen such an example when we compute the slope of $y = x^2$ at $(1, 1)$. We now discuss two more examples when the limit of the denominator is 0.

EXAMPLE 2.5

Evaluate

$$\lim_{t \to 1} \frac{t^2 + t - 2}{t^2 - 1}.$$

Solution

The expression $\dfrac{t^2 + t - 2}{t^2 - 1}$ can be simplified as

$$\frac{t^2 + t - 2}{t^2 - 1} = \frac{(t-1)(t+2)}{(t-1)(t+1)} = \frac{t+2}{t+1}.$$

Now the denominator is no longer 0 as $t \to 1$ and we may apply Theorem 2.2 (b) and conclude that

$$\lim_{t \to 1} \frac{t^2 + t - 2}{t^2 - 1} = \frac{3}{2}.$$

EXAMPLE 2.6

Evaluate

$$\lim_{x \to -1} \frac{\sqrt{x^2 + 8} - 3}{x + 1}.$$

Solution

We try to remove the expression $x + 1$ in the denominator. We achieve this by multiplying the numerator and denominator of the expression in the limit by $\sqrt{x^2 + 8} + 3$. This yields

$$\frac{\sqrt{x^2 + 8} - 3}{x + 1} = \frac{x^2 + 8 - 9}{(x+1)(\sqrt{x^2 + 8} + 3)} = \frac{(x+1)(x-1)}{(x+1)(\sqrt{x^2 + 8} + 3)} = \frac{x - 1}{\sqrt{x^2 + 8} + 3}.$$

The limit is therefore $-\dfrac{2}{6} = -\dfrac{1}{3}$.

2.4 The Squeeze Theorem

THEOREM 2.3 (The Squeeze Theorem) Suppose $g(x) \leq f(x) \leq h(x)$ for all a in some open interval containing x except possibly at $x = a$. Suppose that

$$\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L.$$

then $\lim\limits_{x\to a} f(x) = L$.

EXAMPLE 2.7

Show that

$$\lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) = 0.$$

Solution

Note that $-1 \leq \sin(1/x) \leq 1$. Therefore $-x^2 \leq x^2\sin(1/x) \leq x^2$. Since $\lim\limits_{x\to 0} x^2 = 0$, by Squeeze Theorem, $\lim\limits_{x\to 0} x^2\sin(1/x) = 0$.

EXAMPLE 2.8

If $\sqrt{5-2x^2} \leq f(x) \leq \sqrt{5-x^2}$ for $-1 \leq x \leq 1$, find $\lim\limits_{x\to 0} f(x)$.

Solution

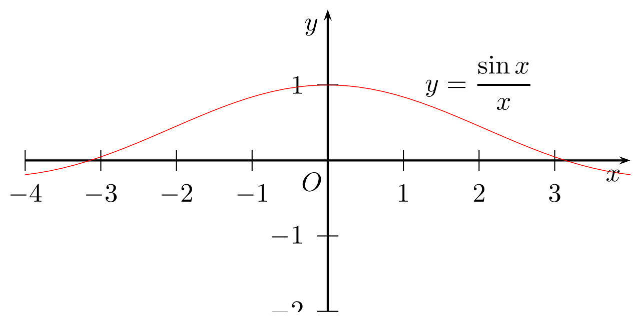
The limits $\lim\limits_{x\to 0}\sqrt{5-2x^2} = \sqrt{5}$ and $\lim\limits_{x\to 0}\sqrt{5-x^2} = \sqrt{5}$ and hence, $\lim\limits_{x\to 0} f(x) = \sqrt{5}$ by Squeeze Theorem.

One of the most interesting application of the squeeze theorem is to deduce the following limit:

THEOREM 2.4 Let θ be given in radians. Then

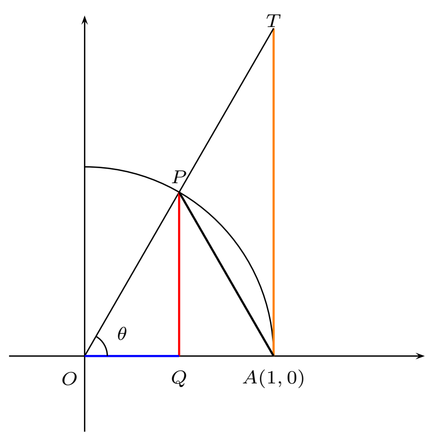
$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1.$$

Before we give the proof of the above identity, let us consider the graph of $(\sin x)/x$.



From the graph, the limit is 1. We now show this rigorously.

Proof
Consider the following diagram:



Note that the area of triangle OAP is less than the area of sector OAP, which is in turn less than the area of the triangle OAT. The area of triangle OAP is $\sin\theta/2$, the area of sector OAP is $\theta/2$ and the area of triangle OAT is $\tan\theta/2$. Therefore, we have

$$\frac{\sin\theta}{2} < \frac{\theta}{2} < \frac{\tan\theta}{2}.$$

Rearranging the above inequalities, we deduce that

$$\cos\theta < \frac{\sin\theta}{\theta} < 1.$$

Since $\lim_{\theta\to 0}\cos\theta = 1$, by the sandwich theorem, we deduce that

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1.$$

□

REMARK 2.2 We have taken for granted that $\lim\limits_{x\to 0}\cos x = 1$. This is true because $\lim\limits_{x\to 0}\sin x = 0$, which implies that $\lim\limits_{x\to 0}\cos x = \lim\limits_{x\to 0}\sqrt{1-\sin^2 x} = 1$. The limit $\lim\limits_{x\to 0}\sin x = 0$ follows from the inequality $|\sin x| \le x$ for $-\pi/2 < x < \pi/2$.

EXAMPLE 2.9

Show that $\lim_{h\to 0}\dfrac{\sin kt}{t}$.

Solution

Write $\dfrac{\sin kt}{t} = \dfrac{\sin kt}{kt}k$ and this tends to k as t tends to 0.

EXAMPLE 2.10

Show that $\lim\limits_{h\to 0}\dfrac{\cos h - 1}{h} = 0$.

Solution

We know that $\cos h = 1 - 2\sin^2(h/2)$. This means that

$$\frac{\cos h - 1}{h} = \frac{-2\sin^2(h/2)}{h} = -2\frac{\sin(h/2)}{h}\sin(h/2).$$

Now, $\lim\limits_{h\to 0}\dfrac{\sin(h/2)}{h} = 1/2$ by previous example, and $\lim\limits_{h\to 0}\sin(h/2) = 0$. Hence, $\lim\limits_{h\to 0}\dfrac{\cos h - 1}{h} = 0$.

EXAMPLE 2.11

Show that $\lim_{h\to 0} \dfrac{\sin(x+h)-\sin x}{h} = \cos x$.

Solution

Write

$$\sin(x+h)-\sin x = \sin x\cos h + \sin h\cos x - x = \sin x(\cos h - 1) + \sin h\cos x.$$

Therefore,

$$\frac{\sin(x+h)-\sin x}{h} = \sin x\frac{\cos h - 1}{h} + \cos x\frac{\sin h}{h}$$

and the limit follows from the previous two example.

2.5 The precise definition of limit

Up to this point, we only have a vague idea of what limits are. We also learn to use the limit laws and the sandwich theorem by assuming that they are true. In order to be sure that these results are true, we have to describe limits precisely. In this section, we will give the precise definition of limits and prove some limits directly using the definition. The theorems such as the limit laws and the sandwich theorem can all be established using the definition of limits. However, we will NOT give these proofs. Students interested to know more can enrol in modules that cover Mathematical Analysis.

DEFINITION 2.2 Let f be a function defined on some open interval that contains the number a, except possibly at a itself. We say that **the limit of $f(x)$ as x approaches a is L** and we write

$$\lim_{x\to a} f(x) = L.$$

if **for every number $\epsilon > 0$** there is a number $\delta_\epsilon > 0$ such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < |x-a| < \delta_\epsilon.$$

REMARK 2.3

1. We have seen the meaning of the expression

$$\lim_{x\to a} f(x) = L.$$

In order to say that

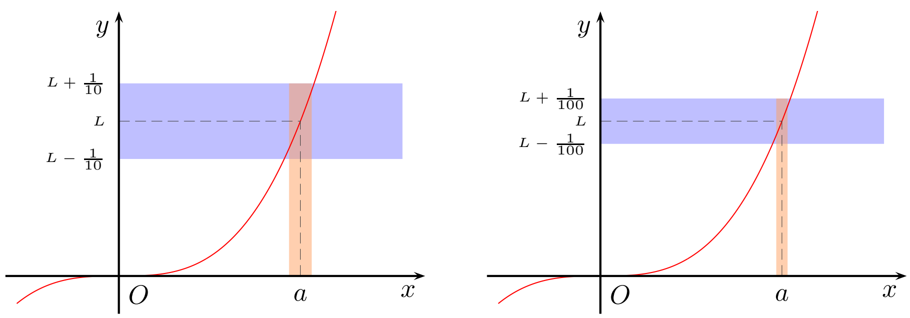
$$\lim_{x \to a} f(x) = L,$$

we must show that for every number $\epsilon > 0$ there is a number $\delta_\epsilon > 0$ such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < |x - a| < \delta_\epsilon.$$

2. The number δ_ϵ depends on ϵ, and it is not unique.

Let us consider a typical situation:



The above definition says that if for any arbitrarily small $\epsilon > 0$ we are able to find a $\delta_\epsilon > 0$ such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < |x - a| < \delta_\epsilon.$$

then we can conclude that

$$\lim_{x \to a} f(x) = L.$$

We now show how to use the precise definition of limit to verify that certain results involving limits hold.

EXAMPLE 2.12

Show that if c is real number, then

$$\lim_{x \to a} c = c.$$

Solution

The function $f(x) = c$ is independent of x. So $|f(x) - c| = 0 < \epsilon$ is true for all x and in particular for $|x - a| < \epsilon$. By the definition of limit, we conclude that $\lim\limits_{x \to c} c = c$.

EXAMPLE 2.13

Show that

$$\lim_{x \to 3}(3x - 7) = 2.$$

Solution

Let $\epsilon > 0$ be given. Our aim is to find a $\delta > 0$ such that

$$0 < |x - 3| < \delta$$

implies that

$$|(3x - 7) - 2| < \epsilon.$$

Let us "work backwards". Consider the right hand side

$$|3x - 7 - 2| = 3|x - 3|.$$

In order to have $3|x - 3| < \epsilon$, we need to have

$$|x - 3| < \epsilon/3.$$

In other words, we can set $\delta = \epsilon/3$ (or any positive number less than $\epsilon/3$). The proof can be written formally as follows:

Let $\epsilon > 0$ be given. Choose $\delta = \epsilon/3$. If $0 < |x - 3| < \delta$, then

$$|(3x - 7) - 2| = 3|x - 3| < 3\delta = \epsilon.$$

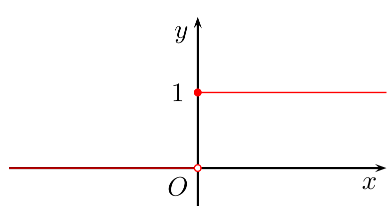
By definition of limit, we have

$$\lim_{x \to 3}(3x - 7) = 2.$$

2.5.1 One-sided limits and finite limits as $x \to \pm\infty$

Consider the *Heaviside function* H defined by

$$H(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$



We see from the graph that as x approaches 0 from the right, $H(x)$ approaches 1. We write

$$\lim_{x \to 0^+} H(x) = 1.$$

This is called the *right-hand limit of* $H(x)$ *as* x *approaches* 0. In general, we write

$$\lim_{x \to a^+} f(x) = L,$$

and say that the **right-hand limit of $f(x)$ as x approaches a from the right is equal to L** if we can make the values of $f(x)$ *arbitrarily close* to L by taking x to be *sufficiently close* to a and x is *greater than* a. The precise definition of the right-hand limit of $f(x)$ as x approaches a from the right is the following:

DEFINITION 2.3 We write $\lim_{x \to a^+} f(x) = L$ if for every $\epsilon > 0$, there exists a $\delta > 0$ (dependent on ϵ) such that if $a < x < a + \delta$, then $|f(x) - L| < \epsilon$.

As x approaches 0 from the left, $H(x)$ approaches 0 and we write

$$\lim_{x \to 0^-} H(x) = 0.$$

This is called the *left-hand limit of* $H(x)$ *as* x *approaches* 0. In general, we write

$$\lim_{x \to a^-} f(x) = L,$$

and say that the **left-hand limit of $f(x)$ as x approaches a^- is equal to L** if we can make the values of $f(x)$ *arbitrarily close* to L by taking x to be *sufficiently close* to a and x is *less than* a. The precise definition of the left-hand limit of $f(x)$ as x approaches a from the left of a is the following:

DEFINITION 2.4 We write $\lim_{x \to a^-} f(x) = L$ if for every $\epsilon > 0$, there exists a $\delta > 0$ (dependent on ϵ) such that if $a - \delta < x < a$, then $|f(x) - L| < \epsilon$.

The following is the relation between limit and one-sided limit:

THEOREM 2.5 Let $f(x)$ be a function. Then we have

$$\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L.$$

EXAMPLE 2.14

Let $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$. Does $\lim_{x\to 0} \dfrac{x}{|x|}$ exist? Justify your answer.

Solution

When $x > 0$, $|x| = x$ and $x/|x| = 1$. So $\lim_{x\to 0^+} x/|x| = 1$. When $x < 0$, $|x| = -x$ and $x/|x| = -1$ and therefore $\lim_{x\to 0^-} x/|x| = -1$. The left-hand and right-hand limits are different and therefore $\lim_{x\to 0} x/|x|$ does not exist.

The symbol for infinity is ∞. It does not represent a real number. We use ∞ to describe the behavior of a function when the values in its domain or range outgrow all finite bounds. For example, for the function $1/x$, when x is positive and becomes large, $1/x$ becomes small. Similarly when x becomes large and negative, $1/x$ becomes small. We write

$$\lim_{x\to\pm\infty} \frac{1}{x} = 0.$$

What we have discussed tells us that we expect the limit of $1/x$ as x tends to ∞ to be 0. In order to write this statement down, we need to use the precise definition of a limit of $f(x)$ when $x \to \infty$.

DEFINITION 2.5 We say that $\lim_{x\to\infty} f(x) = L$ if for every $\epsilon > 0$, there exists an M such that if $x > M$, then $|f(x) - L| < \epsilon$.

REMARK 2.4 To write $\lim_{x\to\infty} \dfrac{1}{x} = 0$, we let $M_\epsilon = 1/\epsilon$. For $x > M_\epsilon$, we have $1/x < 1/M_\epsilon = \epsilon$ and we use $\lim_{x\to\infty} \dfrac{1}{x} = 0$ to represent what we have just shown.

EXAMPLE 2.15

Find $\lim_{x\to\infty} \dfrac{\sin x}{x}$.

Solution

Note that $-1 \leq \sin x \leq 1$. Since $x \to \infty$, we may suppose that $x > 0$. Then

$-1/x \leq \sin x/x \leq 1/x$. Since $\lim\limits_{x\to\infty} 1/x = 0$, by Squeeze Theorem, $\lim\limits_{x\to\infty} \dfrac{\sin x}{x} = 0$.

EXAMPLE 2.16

Find $\lim\limits_{x\to\infty} \dfrac{1}{x}\lfloor x \rfloor$.

Solution

Note that $x - 1 < \lfloor x \rfloor \leq x$. This follows from the definition of the integer floor function. Since $x \to \infty$ we may suppose $x > 0$. Since $\lim\limits_{x\to\infty} (x-1)/x = 1$ and $\lim\limits_{x\to\infty} x/x = 1$, by Squeeze Theorem, $\lim\limits_{x\to\infty} \lfloor x \rfloor/x = 0$.

EXAMPLE 2.17

Find $\lim\limits_{x\to\infty} \dfrac{2+\sqrt{x}}{2-\sqrt{x}}$.

Solution

Since

$$\frac{2+\sqrt{x}}{2-\sqrt{x}} = \frac{2/\sqrt{x}+1}{2/\sqrt{x}-1},$$

and $\lim\limits_{x\to\infty} 1/\sqrt{x} = 0$, we conclude that $\lim\limits_{x\to\infty} \dfrac{2+\sqrt{x}}{2-\sqrt{x}} = -1$.

EXAMPLE 2.18

Find $\lim\limits_{x\to\infty} \left(\sqrt{x+9} - \sqrt{x+4}\right)$.

Solution

Rationalizing $\sqrt{x+9} - \sqrt{x+4}$, we find that

$$\sqrt{x+9} - \sqrt{x+4} = \frac{5}{\sqrt{x+9} + \sqrt{x+4}}.$$

This shows that the limit is 0 since $\lim_{x\to\infty} \frac{1}{\sqrt{x+9}+\sqrt{x+4}} = 0$.

3 Continuous Functions

3.1 Continuity

DEFINITION 3.1 A function f is **continuous at a** if

$$\lim_{x \to a} f(x) = f(a).$$

If f is *not* continuous at a, we say f is **discontinuous at a**.

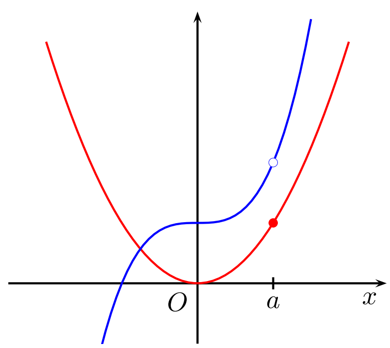
The definition above involves the following three statements to hold:

(i) $f(a)$ is defined (i.e., a is a number in the domain of f),
(ii) $\lim_{x \to a} f(x)$ exists, and
(iii) $\lim_{x \to a} f(x) = f(a)$.

Equivalently, we can use the ϵ, δ-definition: *A function f is continuous at a number a if f is defined at a and for every $\epsilon > 0$, there exists $\delta > 0$ such that*

$$|f(x) - f(a)| < \epsilon \quad \textit{whenever} \, |x - a| < \delta.$$

The following graph shows a function (in red) that is continuous at a and another function (in blue) that is discontinuous at a.



Let $h = x - a$. Then $x \to a$ if and only if $h \to 0$. We may use the alternative condition $\lim_{h \to 0} f(a + h) = f(a)$ for continuity.

EXAMPLE 3.1

The function $f(x) = \dfrac{x^2 - x - 2}{x - 2}$ has a point of discontinuity at $x = 2$. The $\lfloor x \rfloor$ has infinitely many points of discontinuity.

3.2 Continuity of a function on an interval

DEFINITION 3.2 A function f is **continuous from the right** at a number a if

$$\lim_{x \to a^+} f(x) = f(a),$$

and f is **continuous from the left** at a if

$$\lim_{x \to a^-} f(x) = f(a).$$

For example, at each $n \in \mathbf{Z}$, the floor function $\lfloor x \rfloor$ is continuous from the right but discontinuous from the left.

DEFINITION 3.3 A function is **continuous on an interval** if it is continuous at every number in the interval.

If f is only defined on one side of an endpoint of the interval, we understand that continuous at the endpoint to mean continuous from the right or continuous from the left. For example,

(i) f is continuous on $(a, b) \Leftrightarrow f$ is continuous at every $x \in (a, b)$;

(ii) f is continuous on $[a, b] \Leftrightarrow \begin{cases} f \text{ is continuous at every } x \in (a, b), \\ f \text{ is continuous from the right at } a, \\ f \text{ is continuous from the left at } b. \end{cases}$

THEOREM 3.1 (Basic properties of continuous function) Given two functions f and g which are continuous at a, the following functions are also continuous at a:

(i) $f \pm g$;

(ii) fg;
(iii) f/g, if $g(a) \neq 0$;
(iv) cf, where c is a constant.

3.3 Examples of continuous functions

Let $f(x) = c$ be a constant function. Take $a \in \mathbf{R}$. For each $\epsilon > 0$, choose $\delta = 1$, then

$$0 < |x - a| < \delta \Rightarrow |f(x) - f(a)| = |c - c| = 0 < \epsilon.$$

Therefore, $\lim_{x \to a} c = c$. It follows that every constant function is continuous on $\mathbf{R}$.

Let $f(x) = x$. Take $a \in \mathbf{R}$. For each $\epsilon > 0$, choose $\delta = \epsilon$, then

$$0 < |x - a| < \delta \Rightarrow |f(x) - f(a)| = |x - a| < \delta = \epsilon.$$

Therefore, $\lim_{x \to a} x = a$. So $f(x) = x$ is also continuous on $\mathbf{R}$.

The product law of limits says that if $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$ then

$$\lim_{x \to a} f(x)g(x) = LM.$$

Since $\lim_{x \to a} x = a$, applying the product law repeatedly we have

$$\lim_{x \to a} x^n = a^n \quad \text{for } n = 0, 1, 2, \ldots.$$

Therefore the function x^n is continuous at any real number a. Together with the sum law and scalar product law, we find that the polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

is continuous everywhere.

If $P(x)$ and $Q(x)$ are polynomials, then $P(x)/Q(x)$ is continuous whenever $Q(x) \neq 0$. In other words, the rational function $P(x)/Q(x)$ is continuous on its domain

$$D = \{x \in \mathbf{R} \mid Q(x) \neq 0\}.$$

We summarize our findings in the following:

THEOREM 3.2

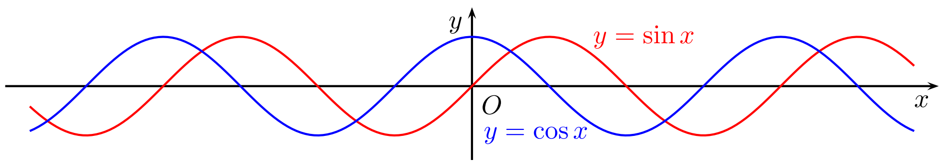
(a) A polynomial is continuous everywhere; and
(b) A rational function is continuous at points where it is defined.

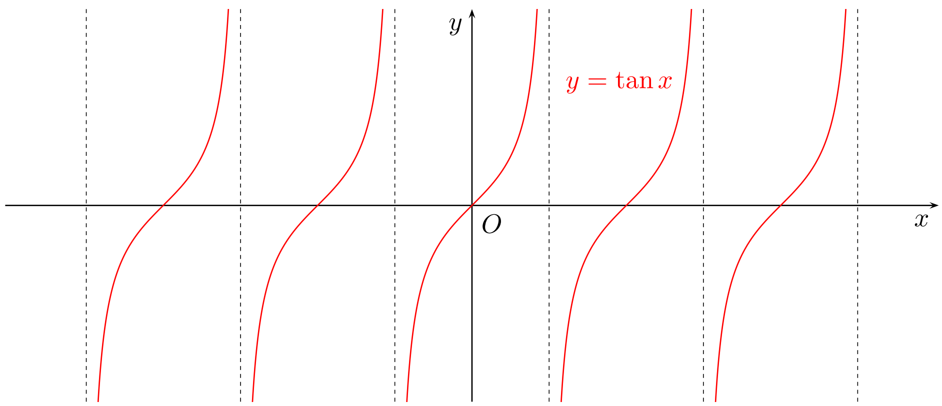
Besides the polynomials and rational functions, the **root function**,

$$f(x) = x^{1/n} \ (= \sqrt[n]{x}), \quad n \in \mathbf{Z}^+$$

is also continuous on the set where the n^{th} root of x is defined.

The **trigonometric functions** *sine* and *cosine functions* are both continuous on $\mathbf{R}$. The *tangent function* is continuous everywhere except at the points when $\cos x = 0$, namely, when $x = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \ldots$.





To prove that $f(x) = \sin x$ is continuous on $\mathbf{R}$, we only need to use the addition formula for sine and use the fact that

$$\lim_{x\to 0} \sin x = 0 \quad \text{and} \quad \lim_{x\to 0} \cos x = 1.$$

For the proof of these facts, see Remark 2.2.

We are now ready to show that $\sin x$ is continuous at any real number $a \in \mathbf{R}$. Observe that

$$\begin{aligned}
\lim_{h\to 0} \sin(a+h) &= \lim_{h\to 0} (\sin a \cos h + \cos a \sin h) \\
&= \sin a \cdot \lim_{h\to 0} \cos h + \cos a \cdot \lim_{h\to 0} \sin h \\
&= \sin a \cdot 1 + \cos a \cdot 0 \\
&= \sin a.
\end{aligned}$$

We have thus seen four classes of continuous functions, namely, the polynomials, the rational functions, functions of the form $x^{r/s}(s \neq 0$ and x is in the domain of $f(x) = x^{r/s})$ and the trigonometric functions. To construct more continuous functions from the known ones, we use the following Theorem:

THEOREM 3.3 If f is continuous at b and $\lim_{x\to a} g(x) = b$ then

$$\lim_{x\to a} f(g(x)) = f(b).$$

In other words,

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)).$$

COROLLARY 3.4 If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.

EXAMPLE 3.2

Evaluate $\lim_{x\to 1} \sin^{-1}\left(\frac{1}{2}\left(\frac{1-x^2}{1-x}\right)\right)$.

Solution

$\lim_{x\to 1} \frac{1-x^2}{2(1-x)} = \lim_{x\to 1} \frac{1+x}{2} = 1$. Therefore,

$$\lim_{x\to 0} \sin^{-1}\left(\frac{1}{2}\left(\frac{1-x^2}{1-x}\right)\right) = \sin^{-1}(1) = \frac{\pi}{2}.$$

EXAMPLE 3.3

Evaluate $\lim_{x\to 1} e^{\tan x}\sqrt{x+1}$.

Solution

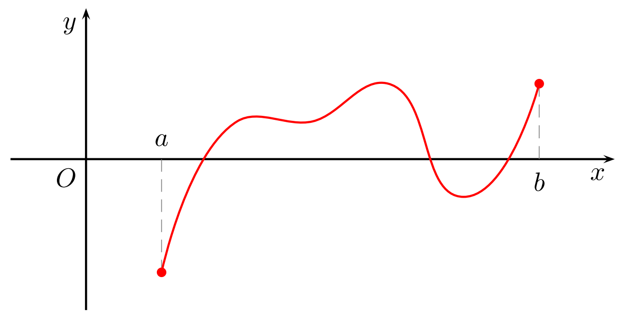
For this example, we can use direct substitution since $e^{\tan x}$ is continuous (being the composite of e^x and $\tan x$) and $\sqrt{x+1}$ is continuous (being the composite of $\sqrt{x}$ and $x+1$). The limit, using direct substitution, yields $e^{\tan 1}\sqrt{2}$.

3.4 The Intermediate Value Theorem

THEOREM 3.5 Let f be a function continuous on the closed interval $[a, b]$. If $f(a) < 0$ and $f(b) > 0$ (or $f(a) > 0$ and $f(b) < 0$) then there exists a number $c \in (a, b)$ such that $f(c) = 0$.

The proof of this result is beyond the scope of this course as it depends on the completeness property of the real number system and can be found in more advanced texts on Mathematical Analysis.

If f is a continuous function, then its graph has no hole or break. Now it is given that the value of f at a is negative, and that at b is positive. When $f(x)$ moves smoothly from negative to positive, its graph must cut the x-axis somewhere between a and b.



There is another form of Theorem 3.5. It is the following:

THEOREM 3.6 Suppose f is continuous on an interval I. Let $a, b \in I$ with $a < b$. If N is any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number $c \in [a, b]$ such that $f(c) = N$.

The above follows from Theorem 3.5. We let $F(x) = f(x) - N$. We are given that $f(a) \leq N \leq f(b)$. If $f(a) = N$ or $f(b) = N$ then we take $c = a$ and $c = b$ respectively. Next, suppose $f(a) < N < f(b)$. Then $F(a) = f(a) - N < 0$ and $F(b) = f(b) - N > 0$. By Theorem 3.5, we conclude that there is a c such that $F(c) = 0$, which means that there is a c such that $f(c) = N$.

EXAMPLE 3.4

A monk leaves the monastery at 7am and took his usual path to the bottom of the mountain, arriving at 7pm. The following morning he started at 7am at the bottom of the mountain and took the same path back, arriving at the monastery at 7pm. Use the intermediate value theorem to show that there is a point on the path that the monk would cross at exactly the same time of day on both days.

Solution

Let $h_1(t)$ and $h_2(t)$ be the height where the monk was at time t measured from the bottom of the mountain on the first day and second day respectively. Let $F(t) = h_1(t) - h_2(t)$. Suppose the monastery was at height H. Then $F(0) = h_1(0) - h_2(0) = H > 0$ and $F(12) = h_1(12) - h_2(12) = -H < 0$. Assuming the monk did not "teleport" which meant he was moving "continuously", we deduce from Theorem 3.5 that there exists t_0 such that $F(t_0) = 0$. This means that $h_1(t_0) = h_2(t_0)$, which implies that the monk was at the same spot on the path at time t_0 on both days.

EXAMPLE 3.5

Suppose that a function f is continuous on the closed interval $[0, 1]$ and that

$$0 \leq f(x) \leq 1$$

for every $x \in [0, 1]$. Show that there must be a number c in $[0, 1]$ such that $f(c) = c$. The number c is called a fixed point of f.

Solution

Let $g(x) = f(x) - x$. Note that $g(x)$ is continuous since $f(x)$ and x are continuous. Since $0 \leq f(x) \leq 1$, we deduce that $g(0) = f(0) - 0 = f(0) \geq 0$ and $g(1) = f(1) - 1 \leq 0$. If $g(0) = 0$ then we have $f(0) = 0$ and 0 is the fixed point we seek. If $g(1) = 0$, then $f(1) = 1$ and 1 is the fixed point of $f(x)$. So we may now suppose $0 < g(0)$ and $g(1) < 0$. By Theorem 3.5, there exists $c \in (0, 1)$ such that $g(c) = 0$. This implies that $f(c) = c$ and c is the fixed point of $f(x)$ on $[0, 1]$.

EXAMPLE 3.6

Explain why the equation $\cos x = x$ has at least a solution.

Solution

For this problem, we need to choose a suitable interval to work with. We choose $[0, \pi/2]$. Let $g(x) = \cos x - x$. Note that $g(x)$ is continuous on $\mathbf{R}$ since $\cos x$ and x are continuous on $\mathbf{R}$. Now, $g(0) = \cos 0 - 0 = 1 > 0$ and $g(\pi/2) = \cos \pi/2 - \pi/2 = -\pi/2 < 0$. Therefore, there exists $c \in [0, \pi/2]$ such that $g(c) = 0$, or in other words, this number c satisfies $\cos c = c$.

REMARK 3.1 If you have a calculator and your angle is in radians, by pressing your cosine button on your calculator repeatedly, you will observe that c is approximately $0.730 \cdots$.

4 Differentiation

4.1 The Derivative as a Function (Reference: Section 3.1)

In Chapter 2, we computed the slope of the tangent line of the curve $y = x^2$ at $P(a, a^2)$ by evaluating the limit

$$m := \lim_{h \to 0} \frac{(a+h)^2 - a^2}{h}.$$

We also derived the instantaneous velocity of an object with position function $s = f(t)$ at time $t = a$ by considering the limit

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Such limits suggest that we should study the quantity

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

carefully. This leads us to the following definition:

DEFINITION 4.1 The **derivative of a function f with respect to the variable x** is the function f' whose value at x is

$$f'(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.

Let $z = x + h$. Then $h \to 0$ if and only if $z \to x$. So we may

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}.$$

4.2 Calculating derivatives from the definition

4.2.1 Derivatives of product of functions and polynomials

The process of calculating a derivative is called **differentiation**. To emphasize that differentiating is an operation performed on $y = f(x)$, we use the notation

$$\frac{dy}{dx} = \frac{df(x)}{dx}$$

as other ways to denote $f'(x)$.

EXAMPLE 4.1

Suppose $f(x) = mx + c$, with $m, c \in \mathbf{R}$. Show that $f'(x) = m$.

Solution

The expression

$$\frac{f(x+h) - f(x)}{h} = \frac{mx + mh + c - mx - c}{h} = m.$$

Therefore

$$f'(x) = \lim_{h \to 0} \frac{m(x+h) + c - (mx + c)}{h} = m.$$

We now establish two important properties of differentiation which will help us to compute derivatives of more complicated functions.

THEOREM 4.1 Let $f(x)$ and $g(x)$ be two functions such that f and g have derivatives at $x = a$. Then

$$(f + g)'(a) = f'(a) + g'(a)$$

and

$$(f(a)g(a))' = f'(a)g(a) + f(a)g'(a).$$

Proof
The expression

$$\frac{f(a+h) + g(a+h) - f(a) - g(a)}{h} = \frac{f(a+h) - f(a)}{h} + \frac{g(a+h) - g(a)}{h}$$

tends to $f'(a) + g'(a)$ as $h \to 0$. Therefore, $(f + g)'(a) = f'(a) + g'(a)$.

To prove the second identity, let $h(x) = f(x)g(x)$. Then the expression

$$\begin{aligned}
&\frac{f(a+h)g(a+h) - f(a)g(a)}{h} \\
&\qquad = \frac{f(a+h)g(a+h) - f(a+h)g(a) + f(a+h)g(a) - f(a)g(a)}{h} \\
&\qquad = \frac{f(a+h)(g(a+h) - g(a))}{h} + g(a)\frac{f(a+h) - f(a)}{h},
\end{aligned}$$

which tends to $f(a)g'(a) + g(a)f'(a)$ as $h \to 0$. Therefore,

$$(fg)'(a) = f(a)g'(a) + f'(a)g(a).$$

$\square$

EXAMPLE 4.2

Let n be a positive integer. Show that if $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

Solution

The formula can be proved by induction using the second identity of Theorem 4.1. The case $n = 1$ is true since $(mx + c)' = m$, or $(x)' = 1$. Suppose we know that $(x^{n-1})' = (n-1)x^{n-2}$. Now,

$$(x^n)' = (x \cdot x^{n-1})' = x(n-1)x^{n-2} + x^{n-1} = nx^{n-1}$$

and the proof is complete.

REMARK 4.1 One can prove the above by using binomial theorem for positive integers, namely,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k.$$

Using the above identity, we find that

$$\frac{(x+h)^n - x^n}{h} = \frac{nx^{n-1}h + \cdots + nxh^{n-1} + h^n}{h}$$

which tends to nx^{n-1} as $h \to 0$. We can also prove the formula using the identity

$$(a^n - b^n) = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-1} + b^n).$$

Consider the following computations:

$$\begin{aligned}
\frac{(x+h)^n - x^n}{h} &= \frac{((x+h) - x)((x+h)^{n-1} + (x+h)^{n-2}x + \cdots + (x+h)x^{n-2} + x^{n-1})}{h} \\
&= (x+h)^{n-1} + (x+h)^{n-2}x + \cdots + (x+h)x^{n-2} + x^{n-1}.
\end{aligned}$$

This last expression tends to nx^{n-1} as $h \to 0$ which implies that the $(x^n)' = nx^{n-1}$.

Combining our knowledge of the derivative of x^n and the first identity of Theorem 4.1, we deduce that if $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, then

$$f'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1$$

and we now know how to differentiate any polynomials!

EXAMPLE 4.3

Find the derivative of the function $f(x) = x^2 - 8x + 9$.

Solution

The derivative of the polynomial $f(x) = x^2 - 8x + 9$ is $f'(x) = 2x - 8$.

Derivatives appear in physical world as "instantaneous velocity" of a moving object moving at a distance at time t.

EXAMPLE 4.4

Suppose a ball is dropped from a tower 450m above the ground. Find the instantaneous velocity of the ball after 5 seconds. We will take the distance $s(t)$ at time t as $4.9t^2$

Solution

The average velocity of the object in the interval $[t, t+h]$ is $\dfrac{s(t+h) - s(t)}{h}$. The instantaneous velocity of the ball at time t is

$$\lim_{h \to 0} \frac{s(t+h) - s(t)}{h} = (4.9t^2)' = 9.8t.$$

Therefore the instantaneous velocity of the ball at $t = 5$ is $(9.8) \times 5 = 49$ meter per second.

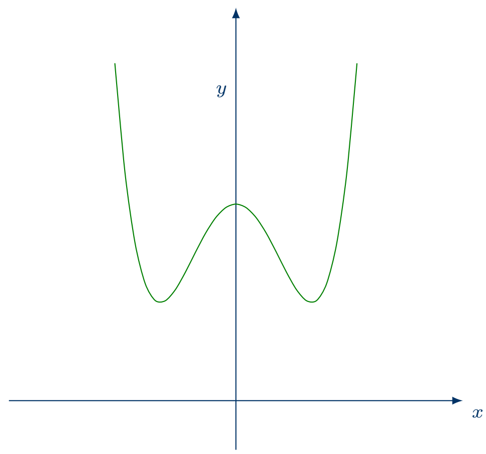
We begin our discussion of derivative of f at a by noting that it is the slope of the tangent line of the graph $y = f(x)$ at $(a, f(a))$. When the derivative $f'(a)$ is zero, it means that the tangent line at $(a, f(a))$ is a horizontal line. The following example is related to finding horizontal tangent lines of a point on the graph $G(f)$.

EXAMPLE 4.5

Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents?

Solution

Note that $f'(x) = 4x^3 - 4x$. Let $(a, f(a))$ be a point on $G(f)$ with horizontal tangent lines. This means that $f'(a) = 4a^3 - 4a = 0$ and therefore, $a = -1, 0, 1$. So the answer to the question is yea and that the horizontal tangent lines occur at $(-1, 1), (0, 2)$ and $(1, 1)$. The following diagram confirms our findings:



4.2.2 Derivatives of quotient of functions and rational functions

Let $g(x)$ be a function such that $u(x) = \dfrac{1}{g(x)}$ is differentiable at $x = a$. What is the value of $u'(a)$? We consider the expression

$$\frac{u(a+h) - u(a)}{h} = \frac{1}{h}\left(\frac{1}{u(a+h)} - \frac{1}{u(a)}\right) = \frac{u(a) - u(a+h)}{h}\frac{1}{u(a)u(a+h)}.$$

This tends to $-\dfrac{u'(a)}{u^2(a)}$ as $h \to 0$. Therefore,

$$\left(\frac{1}{u(x)}\right)' = -\frac{u'(x)}{u^2(x)}.$$

Combining the above with the second identity of Theorem 4.1, we conclude that

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)}{g(x)} + f(x)\left(-\frac{g'(x)}{g^2(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.$$

The above formula allows us to differentiate any rational functions at x whenever the derivatives of these functions exist.[1]

EXAMPLE 4.6

Find the derivative of

$$f(x) = \frac{2x+5}{3x-2}.$$

Solution

$$f'(x) = \frac{(3x-2)\times 2 - (2x+5)\times 3}{(3x-2)^2} = -\frac{19}{(3x-2)^2}.$$

4.2.3 Derivatives of trigonometric functions

We have seen that

$$\lim_{h\to 0}\frac{\sin h}{h} = 1.$$

A similar limit is the following:

$$\lim_{h\to 0}\frac{\cos h - 1}{h} = 0.$$

To prove the above limit, we observe that

$$\cos h - 1 = (\cos h - 1)\frac{(\cos h + 1)}{\cos h + 1} = \frac{\cos^2 h - 1}{\cos h + 1} = -\frac{\sin^2 h}{\cos h + 1}.$$

This gives

$$\lim_{h\to 0}\frac{\cos h - 1}{h} = \lim_{h\to 0} -\frac{\sin h}{\cos h + 1}\frac{\sin h}{h} = 0.$$

In this section, we will first show that

$$(\sin x)' = \cos x.$$

[1] The proof I gave here is different from the lecture slides.

We begin with

$\sin(x+h) - \sin x = \sin x \cos h + \sin h \cos x - \sin x = \sin x(\cos h - 1) + \sin h \cos x.$

Therefore,

$$\lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = \sin x \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h} = \cos x.$$

In a similar way, we can show that

EXAMPLE 4.7

$(\cos x)' = -\sin x.$

Solution

Since $\cos(x+h) = \cos x \cos h - \sin x \sin h$, we find that

$$\frac{\cos(x+h) - \cos x}{h} = \cos x \frac{\cos h - 1}{h} - \sin x \frac{\sin h}{h}.$$

Since $\lim_{h\to 0} \frac{\cos h - 1}{h} = 0$ and $\lim_{h\to 0} \frac{\sin h}{h} = 1$, we conclude that $(\cos x)' = -\sin x$.

We leave the following derivatives of other trigonometric functions as exercises: [2]

(iii)	$(\tan x)' = \sec^2 x,$	(iv)	$(\cot x)' = -\csc^2 x,$
(v)	$(\sec x)' = \sec x \tan x,$	(vi)	$(\csc x)' = -\csc x \cot x.$

EXAMPLE 4.8

Find f' if $f(x) = \dfrac{\sin x}{1 - \cos x}$.

Solution

$$f'(x) = \frac{(1-\cos x) \times \cos x - \sin x \times (-(-\sin x))}{(1-\cos x)^2} = \frac{\cos x - \cos^2 x - \sin^2 x}{(1-\cos x)^2} = \frac{1}{\cos x - 1}.$$

[2] See slides for the proofs of some of these identities or try proving them yourself.

4.2.4 Derivatives of exponential functions

Let $a > 0$ and $f(x) = a^x$. The expression

$$\frac{f(x+h) - f(x)}{h} = \frac{a^{x+h} - a^x}{h} = a^x \frac{a^h - 1}{h}.$$

This implies, as $h \to 0$, that

$$(a^x)' = L_a a^x,$$

where

$$L_a = \lim_{h \to 0} \frac{a^h - 1}{h}.$$

Note that L_a is dependent on a. Note that

$$L_a = \lim_{h \to 0} \frac{a^h - a^0}{h} = f'(0).$$

There is a special number "e" such that $L_e = 1$. With this chosen number e, the derivative of e^x becomes very simple, namely

$$(e^x)' = e^x.$$

It turns out that the number e can be defined as the number a such that $\lim\limits_{h \to 0} \dfrac{a^h - a^0}{h} = 1$. There are other definitions of e and we will encounter some of them later in the course.

4.3 The Chain Rule (Section 3.5)

Suppose we want to differentiate a function like

$$F(x) = \sin(x^2 + 1).$$

What should we do? Observe that $F(x)$ is the composite of functions

$$F(x) = f(g(x)),$$

where

$$f(x) = \sin x \quad \text{and} \quad g(x) = x^2 + 1,$$

and we know that $f'(x) = \cos x$ and $g'(x) = 2x$. The chain rule allows us to differentiate F using the derivatives of f and g.

THEOREM 4.2 (Chain Rule) If g is differentiable at a and f is differentiable at $g(a)$, then $F = f \circ g$ is differentiable at a. Moreover,

$$F'(a) = f'(g(a))g'(a).$$

In Leibniz notation, if we write $y = g(x)$ and $z = f(y)$, then the Chain Rule can be expressed as

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}.$$

EXAMPLE 4.9

Use chain rule to show that $(x^{-1})^m = -mx^{-m-1}$ for any positive integer m. This gives $(x^n)' = nx^{n-1}$ for negative integer n.

Solution

Let $f(x) = x^m$, $m \in \mathbf{Z}^+$ and $g(x) = \dfrac{1}{x}$. Quotient rule for differentiation gives us $g'(x) = -1/x^2$ and $(x^m)' = mx^{m-1}$. Thus by Chain rule,

$$(f(g(x)))' = m(1/x)^{m-1}(-1/x^2) = -mx^{-m-1}.$$

The Chain Rule can be generalized to the composite of three or more functions. Precisely, if $y = h(x)$, $z = g(y)$ and $w = f(z)$, then

$$\frac{dw}{dx} = \frac{dw}{dz}\frac{dz}{dy}\frac{dy}{dx}.$$

EXAMPLE 4.10

Find the derivative of $F(x) = \sin(x^2 + 1)$.

Solution

Using the Chain Rule, we find that

$$(\sin(x^2 + 1))' = \cos(x^2 + 1) \cdot 2x = 2x\cos(x^2 + 1).$$

EXAMPLE 4.11

Find the derivative of $F(x) = \sin(e^{1+x^2})$.

Solution

$$F'(x) = \cos(e^{1+x^2})e^{1+x^2}(2x).$$

We end this section with an intuitive justification of the chain rule. Let $F(x) = f(g(x))$. We wish to consider the expression

$$F(x+h) - F(x).$$

This is given by

$$f(g(x+h)) - f(g(x)) = (f(g(x+h)) - f(g(x)))\frac{g(x+h) - g(x)}{g(x+h) - g(x)}. \tag{4.1}$$

Let $\Delta u = g(x+h) - g(x)$. Note that since g is continuous at x, $\Delta \to 0$ as $h \to 0$.

From (4.1), we deduce that

$$\frac{F(x+h) - F(x)}{h} = \frac{f(g(x) + \Delta u) - f(g(x))}{\Delta u}\frac{g(x+h) - g(x)}{h}.$$

This implies that

$$F'(x) = f'(g(x))g'(x).$$

The above is not entirely rigorous because we have made the assumption that $g(x+h) - g(x)$ is never 0.

4.4 Derivatives of inverse functions and logarithms (Section 3.7)

In this section, we learn how to differentiate the inverse of a function.

THEOREM 4.3 If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain. The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}.$$

Here is an intuitive way of deriving the formula. Note that

$$f^{-1}(f(x)) = x.$$

By the chain rule, we find that

$$(f^{-1}(f(x)))' = (f^{-1})'(f(x))f'(x) = 1.$$

This yields

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}.$$

Set $b = f(x)$, or $x = f^{-1}(b)$. This yields

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}.$$

EXAMPLE 4.12

Find $(y)'$ if $y = \sin^{-1}(x)$.

Solution

Let $y = \sin^{-1}(x), -1 \leq x \leq 1$. Then

$$(y)' = \frac{1}{\cos(\sin^{-1}(x))} = \frac{1}{\sqrt{1 - \sin(\sin^{-1}(x))}} = \frac{1}{\sqrt{1 - x^2}},$$

which exists when $-1 < x < 1$.

EXAMPLE 4.13

Find $(y)'$ if $y = \tan^{-1}(x)$.

Solution

Since $(\tan x)' = \sec^2 x$,

$$(\tan^{-1} x)' = \frac{1}{\sec^2(\tan^{-1} x)} = \frac{1}{1 + x^2}$$

where we have used the identity $\sec^2 x = 1 + \tan^2 x$.

EXAMPLE 4.14

Find $(y)'$ if $y = \ln x$, the inverse of e^x.

Solution

Let $y = \ln x, x > 0$. Then

$$(y)' = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

when $x > 0$.

4.5 Higher derivatives

If f is differentiable then f' is also a function and so we may continue to differentiate f' to obtain $(f')'$. The function $(f')'$ written as f'' is called the **second derivative** of f. It is also written as

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}.$$

When $s(t)$ is the position function of an object that moves along a straight line, we know that its derivative represents the *velocity* $v(t)$. The second derivative of $s(t)$ is called the *acceleration* $a(t)$ of the object and it represents the instantaneous rate of change of velocity.

EXAMPLE 4.15

The position of a particle is given by the equation

$$s = t^3 - 6t^2 + 9t$$

where t is measured in seconds and s in meters. Find the acceleration at time t. What is the acceleration after 4 seconds?

Solution

Let $s(t)$ denote the position function of the particle. Then the acceleration of the particle at time t is given by $s''(t) = 6t - 12$. When $t = 4$, $s''(t) = 12$.

There is nothing to stop us from defining higher derivatives. We define

$$f''' = (f'')',$$

and also write

$$f^{(3)} = f'''.$$

In general we define $f^{(0)} := f$, and for integer $n \geq 1$,

$$f^{(n)} := \left(f^{(n-1)}\right)'.$$

EXAMPLE 4.16

If $f(x) = 1/x$, find $f^{(n)}$.

Solution

The first few derivatives gives $f' = -\frac{1}{x^2}, f'' = \frac{2}{x^3}, f^{(3)}(x) = -\frac{3!}{x^4}$. It appears that

$$f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}}$$

and this can be proved by induction. We leave the student to complete the proof.

EXAMPLE 4.17

Find $f', f'', f^{(3)}$ and $f^{(4)}$ if $f = \sin x$.

Solution

$f' = \cos x, f'' = -\sin x, f^{(3)} = -\cos x, f^{(4)} = \sin x$. The patterns repeat and it is possible to determine $f^{(n)}$ by just looking at the remainder of n when divided by 4.

4.6 Implicit Differentiation (Section 3.6)

We have learnt to differentiate *function* of the type $y = f(x)$. Suppose we are given an *equation*

$$F(x, y) = 0,$$

and we would like to compute $\frac{dy}{dx}$, what can we do?

We assumed that y can be implicitly expressed as a differentiable function of

x, and this method of obtaining y' (when y is not expressed as a function of x explicitly) is called the method of **implicit differentiation**.

1) Differentiate $f(x, y)$ with respect to x,
2) Solve the equation $\dfrac{d}{dx}f(x, y) = 0$ to express $\dfrac{dy}{dx}$ in terms of x and y.

EXAMPLE 4.18

Find $\dfrac{dy}{dx}$ if $y^2 - x = 0$.

Solution

We differentiate the left-hand-side and right-hand-side with respect to x to deduce using Chain Rule that

$$2y\frac{dy}{dx} - 1 = 0.$$

This gives

$$\frac{dy}{dx} = \frac{1}{2y}.$$

When $y = \sqrt{x}$, we have

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}.$$

When $y = -\sqrt{x}$, we have

$$\frac{dy}{dx} = -\frac{1}{2\sqrt{x}}.$$

REMARK 4.2 The above method allows us to derive

$$(x^{r/s})' = \frac{r}{s}x^{r/s-1}$$

when $r, s \in \mathbf{Z}$ and $s \neq 0$. This is a generalization of $(x^n)' = nx^{n-1}$. To prove the above, we let $y = x^{r/s}$. This gives $y^s = x^r$ and

$$sy^{s-1}\frac{dy}{dx} = rx^{r-1}.$$

Therefore,

$$\frac{dy}{dx} = \frac{r}{s}x^{r-1}y^{1-s} = \frac{r}{s}x^{r-1}x^{r/s(1-s)} = \frac{r}{s}x^{r/s-1}.$$

EXAMPLE 4.19

Use implicit differentiation to show that $(\tan^{-1}(x))' = \dfrac{1}{1+x^2}$.

Solution

Let $y = \tan^{-1} x$ then $\tan y = x$. Therefore, $\sec^2 y \dfrac{dy}{dx} = 1$. Therefore,

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

since $\tan y = x$.

5 Applications of Differentiation

5.1 Extreme values of functions (Section 4.1)

This section shows how to locate and identify extreme values of a continuous functions from its derivative. Once we learn how to do this, we will be able to solve a variety of optimization problems in which we can find the optimal way to do something in a given situation.

DEFINITION 5.1 Let f be a function with domain D. Then f has **absolute maximum** on D at a point a if

$$f(x) \leq f(a)$$

for all $x \in D$ and an **absolute minimum** on D at a if

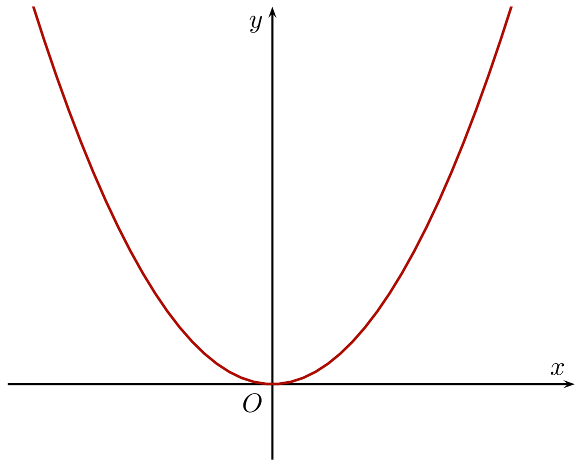
$$f(x) \geq f(a)$$

for all $x \in D$.

Absolute maximum and minimum values are called absolute extrema. They are also called global extrema, to distinguish them from local extrema to be defined in a subsequent section.

EXAMPLE 5.1

The absolute maximum of $y = x^2$ on $[-2, 2]$ occurs at $x = \pm 2$. The absolute minimum of $y = x^2$ on $[-2, 2]$ occurs at $x = 0$.



THEOREM 5.1 (The Extreme Value Theorem) If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m_1$, $f(x_2) = M$ and $m \leq f(x) \leq M$ for every other $x \in [a, b]$.

The proof of the Extreme Value Theorem requires a detailed knowledge of the real number system and will be covered in Mathematical Analysis.

5.2 Local (Relative) Extreme Values

DEFINITION 5.2 A function has a **local maximum** value at an interior point c within its domain D if $f(x) \leq f(c)$ for all $x \in D$ lying in some open interval containing c.

DEFINITION 5.3 A function has a **local minimum** value at an interior point c within its domain D if $f(x) \geq f(c)$ for all $x \in D$ lying in some open interval containing c.

DEFINITION 5.4 If the domain of f is the closed interval $[a, b]$, then a function has a **local maximum** value at the endpoint a (or b) if $f(x) \leq f(a)$ (or $f(x) \leq f(b)$) for all $x \in [a, a+\delta)$ (or $x \in (b-\delta, b]$), $\delta > 0$. Likewise, f has a local minimum at end point a (or b) if $f(x) \geq f(a)$ (or $f(x) \geq f(b)$) for all $x \in [a, a+\delta)$ (or $x \in (b-\delta, b]$), $\delta > 0$.

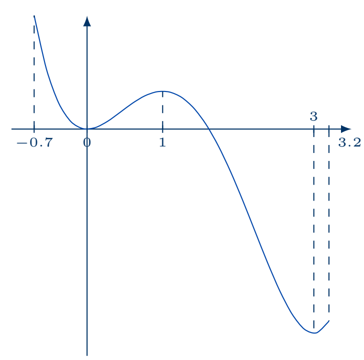
Local maximum values and local minimum values are called **local extrema** or **relative extrema**.

EXAMPLE 5.2

Let $f(x) = 3x^4 - 16x^3 + 18x^2$ for $-0.7 \leq x \leq 3.2$. Find the values of x which give the absolute maximum and minimum of $f(x)$ on $[-0.7, 3.2]$. Find the values of x when give the local maximum and minimum of $f(x)$.

Solution

Consider the following graph of f:



The function f has local maximum at end points $x = -0.7$ and $x = 3.2$. f has local minimum at $x = 0$ and $x = 3$ and local maximum at $x = 1$. The absolute maximum of f occurs when $x = -0.7$ and the absolute minimum of f occurs at $x = 3$.

EXAMPLE 5.3

When we cough, the trachea (windpipe) contracts to increase the velocity of the air going out. This raises the question of how much it should contract to maximize the velocity and whether it really contracts that much when we cough. Under reasonable assumptions about the elasticity of the tracheal wall and about how the air near the wall is slowed by friction, the average flow velocity v can be modeled by the equation

$$v = c(r_0 - r)r^2 \text{ cm/sec}, \quad \frac{r_0}{2} \le r \le r_0,$$

where r_0 is the rest radius of the trachea in centimeters and c is a positive constant whose value depends in part on the length of the trachea. Show that v is greatest when $r = \frac{2}{3}r_0$.

Solution

Let $v(r) = cr_0r^2 - cr^3$, $\frac{r_0}{2} \le r \le r_0$. Then $v'(r) = 2cr_0r - 3cr^2$. When $v'(r) = 0$, $r = 0$ or $2r_0/3$. Since $v(r_0/2) = cr_0^3/8 < v(2r_0/3)$ and $v(r_0) = 0$, we conclude that $v(r)$ is greatest when $r = 2r_0/3$.

THEOREM 5.2 (The first derivative theorem for local extreme values) If f has a local maximum and minimum value at an interior point c of its domain, and if f' is defined at c, then $f'(c) = 0$.

Proof
To prove that $f'(c) = 0$ at a local extremum, we show first that $f'(c)$ cannot be positive and second that $f'(c)$ cannot be negative.

To begin, suppose that f has a local maximum at $x = c$ so that $f(x) - f(c) \le 0$ for all values of x near enough to c. Since c is an interior point of f's domain, $f'(c)$ is defined by the two-sided limit

$$\lim_{x\to c} \frac{f(x) - f(c)}{x - c}.$$

When we examine these limits separately,

$$f'(c) = \lim_{x\to c^+} \frac{f(x) - f(c)}{x - c} \le 0.$$

Similarly,

$$f'(c) = \lim_{x\to c^-} \frac{f(x) - f(c)}{x - c} \ge 0.$$

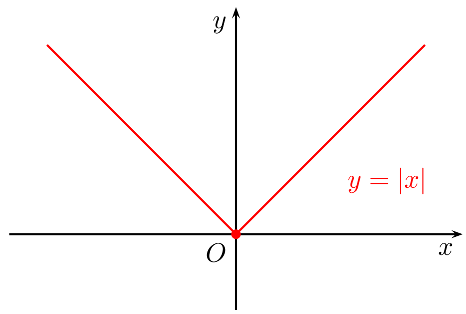
Therefore, $f'(c) = 0$. □

REMARK 5.1 The above result shows that the set of interior points of the domain D of f for which f has local extremum and f' exists is a subset of the zeroes of $f'(x) = 0$ in D. In other words, solving $f'(x) = 0$ would give us all local extrema of f at interior points of D.

The next example shows that even if $f'(c)$ does not exist, $(c, f(c))$ may still be a local maximum or a local minimum.

EXAMPLE 5.4

Consider the absolute value function $f(x) = |x|$.



Note that $f'(0)$ does not exist (or we say that f is not differentiable at 0) but $(0, 0)$ is a local minimum.

From the above examples, we see that if we want to find all local maximum and local minimum, we must search for those c such that either $f'(c) = 0$ or $f'(c)$ does not exist. We call such c a *critical number* of f.

DEFINITION 5.5 An interior point of the domain of a function where f' is zero or undefined is a **critical point** of f.

In order to find the absolute extrema of f on the closed interval $[a, b]$, we follow the following steps: **Closed Interval Method**:

i) Evaluate f at all critical points and endpoints.
ii) Take the largest and smallest of these values.

We have seen that if f is continuous and the derivatives of f exists on $[a, b]$ then to find critical points in (a, b), we need to find the values of x for which $f'(x) = 0$. Suppose c is a value such that $f'(c) = 0$, how do we determine if c is a local maximum, local minimum or neither? The quickest way is to check the values of f at $\ell = c - \delta$ and $r = c + \delta$ for some small $\delta > 0$. If $f(\ell) > f(c)$ and $f(r) > f(c)$ then c is a local minimum. If $f(\ell) < f(c)$ and $f(r) < f(c)$ then c is a local maximum.

But this method depends on the choice of δ. We will learn another method using the derivative of f after our discussion of the Mean Value Theorem in the next section.

5.3 Optimization Problems

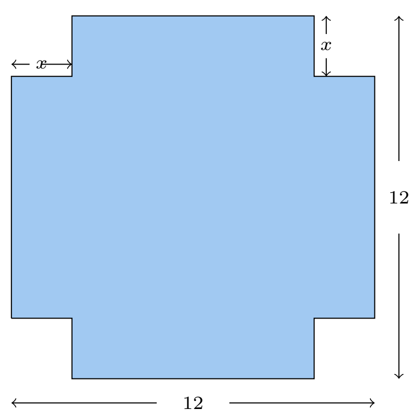
The methods we have learnt in this chapter for finding extreme values have practical applications in many areas of life. A business person wants to minimize costs and maximize profits. A traveler wants to minimize transportation time. In this section, we solve such problems such as maximizing areas, volumes and profits and minimizing distances, costs and times.

Here are the steps to follow in solving optimization problems:

1. Understand the problem.
2. Draw a diagram.
3. Introduce notation: Use a symbol for the quantity you want to maximize or minimize. Call it Q.
4. Express Q in terms of appropriate quantity (call it x) such as time, distance etc.. Write down the domain of this function.
5. Use the method illustrated previously to find maximum and minimum of the functions on the specified interval.

EXAMPLE 5.5

An open-top box is to be made by cutting small congruent squares from the corners of a 12-cm by 12-cm sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hod as much as possible?



Solution

The volume of the box $V(x)$ is given by $x \times (12-2x) \times (12-2x)$, or

$$V(x) = x(12-2x)^2 = x(144-48x+4x^2),$$

$0 \le x \le 6$. Now,

$$V'(x) = 144 - 96x + 12x^2.$$

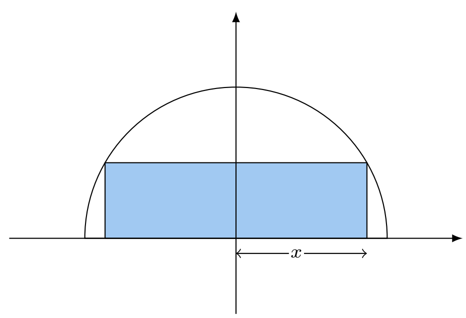
The solutions to $V'(x) = 0$ can be found by solving

$$x^2 - 8x + 12 = 0.$$

The equation has solutions $x = 2$ or 6. Since $V(0) = 0, V(6) = 0$ and $V(2) = 128$, we conclude that the box with largest volume occurs when $x = 2$ and its volume is 128 cm^3.

EXAMPLE 5.6

A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have and what are its dimensions?



Solution

Let the length of the rectangle be $2x$. Then the breadth of the rectangle is $\sqrt{9-x^2}$. The area $A(x)$ of the rectangle is $2x\sqrt{9-x^2}$. Instead of maximizing $A(x)$ for which $A(x) \ge 0$, we maximize $S(x) = A^2(x)$, where $0 \le x \le 3$. Note that

$$S'(x) = (4x^2(9-x^2))' = 72x - 16x^3.$$

Solving $S'(x) = 0$, we arrive at $x = 0$ or $x = 3/\sqrt{2}$. At the end point, $S(0) = 0$ and $S(3) = 0$. Therefore S must be an absolute maximum at $x = 3/\sqrt{2}$. Therefore the dimensions of the largest rectangle that can be inscribed in the circle is $6/\sqrt{2} \times 3/\sqrt{2}$.

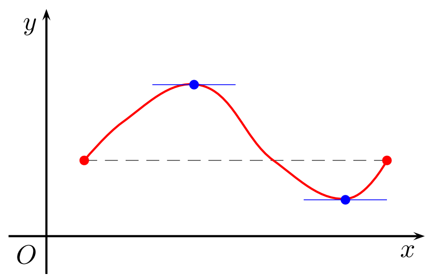
6 Rolle's Theorem and The Mean Value Theorem

6.1 The mean value theorem

THEOREM 6.1 (Rolle's Theorem) Let f be a function such that

(i) f is continuous on the closed interval $[a, b]$, and
(ii) f is differentiable on the open interval (a, b), and
(iii) $f(a) = f(b)$.

Then there is a number $c \in (a, b)$ such that $f'(c) = 0$.



Proof
There are two cases:

Case 1. Given that $f(a) = f(b) = N$, it could happen that Suppose $f(x) = N$ for all $x \in [a, b]$. In that case, we know that $f'(x) = 0$ since $f(x) = N$ is a constant on (a, b) and Rolle's Theorem is true.

Case 2. Suppose $f(a) = f(b) = N$ but $f(x) \neq N$ for some $x \in [a, b]$. There either $m < N$ or $M > N$ because N cannot be minimum or maximum at the same time. Suppose $m < N$. Then $f(x_1) = m$ is a local minimum at the interior point $x_1 \in (a, b)$. Therefore, $f'(x_1) = 0$. Rolle's Theorem holds. If $M > N$, then f attains local maximum at the interior point x_2 and $f'(x_2) = 0$ and Rolle's Theorem holds.

$\square$

EXAMPLE 6.1

The function $y = x^3 - x$ has zeroes at $-1, 0, 1$. By Rolle's Theorem, we expect to find two local extrema in the interval $[-1, 1]$. Indeed the points $c = -1/\sqrt{3} \in [-1, 0]$ and $d = 1/\sqrt{3} \in [0, 1]$ satisfies the equation $f'(x) = 0$.

We are now ready to state the Mean Value Theorem.

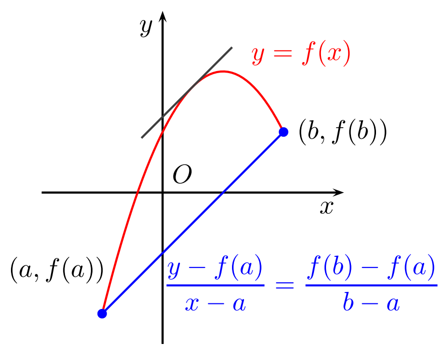
THEOREM 6.2 (Mean Value Theorem) Let f be a function such that

(i) f is continuous on the closed interval $[a, b]$, and
(ii) f is differentiable on the open interval (a, b).

Then there is a number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof
Consider the following graph:



Let $h(x)$ be the difference between $y = f(x)$ and the straight line passing through $(a, f(a))$ and $(b, f(b))$. More precisely, let

$$h(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}(x - a) + f(a)\right].$$

Then $h(a) = 0$ and $h(b) = 0$. By Rolle's Theorem, there exists $c \in (a, b)$ such that

$$h'(c) = 0.$$

This implies that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

and the Mean Value Theorem follows. $\square$

EXAMPLE 6.2

Let $f(x) = x^3 - x$. By the Mean Value Theorem, there exists c such that $f'(c) = (f(2) - f(0))/2 = 3$. This point c is $\dfrac{2\sqrt{3}}{3}$.

The Mean Value Theorem is an important result that can be used to solve interesting problems. We begin

EXAMPLE 6.3

Prove that for any real numbers a and b,

$$|\sin b - \sin a| \leq |b - a|.$$

Solution

Let $f(x) = \sin x$. Then $f'(x) = \cos x$. Note that

$$-1 \leq \cos x \leq 1,$$

or

$$|\cos x| \leq 1.$$

By the Mean Value Theorem, there exists $c \in (a, b)$ such that

$$\frac{\sin b - \sin a}{b - a} = f'(c) = \cos c.$$

Therefore

$$\left|\frac{\sin b - \sin a}{b - a}\right| = |\cos c| \leq 1.$$

This implies that

$$|\sin b - \sin a| \leq |b - a|$$

and the proof is complete.

6.2 L'Hôpital's Rule

The second application of the Mean Value Theorem that we are going to give is the proof of L'Hôpital's Rule. We will first state the rule with some examples before we give a sketch of the proof of the rule using the Mean Value Theorem.

THEOREM 6.3 (l'Hôpital's Rule) Suppose that $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I (except possibly at $x = a$). Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

EXAMPLE 6.4

Find the following limits:

(a) $\displaystyle \lim_{x \to 0} \frac{\sqrt{1-x} - 1 + \frac{x}{2}}{x^2}$,

(b) $\displaystyle \lim_{x \to 0} \frac{x - \sin x}{x^3}$.

Solution

(a) Let

$$f(x) = \sqrt{1-x} - 1 + \frac{x}{2} \text{ and } g(x) = x^2.$$

Note that $f(x) \to 0$ and $g(x) \to 0$ as $x \to 0$. Next,

$$f'(x) = -\frac{1}{2\sqrt{1-x}} + \frac{1}{2}$$

and

$$g'(x) = 2x$$

and $f'(x) \to 0$ and $g'(x) \to 0$ as $x \to 0$. We need to apply L'Hôpital's Rule again. Since $f''(x) = -\dfrac{1}{4(1-x)^{3/2}}$ and $g''(x) = 2$. we conclude that

$$\lim_{x \to 0} \frac{\sqrt{1-x} - 1 + \frac{x}{2}}{x^2} = \lim_{x \to 0} \frac{f''(x)}{g''(x)} = -\frac{1}{8}.$$

(b) Let $f(x) = x - \sin x$ and $g(x) = x^3$. Note that $f(x) \to 0$ and $g(x) \to 0$ as $x \to 0$. Next, $f'(x) = 1 - \cos x$ and $g'(x) = 3x^2$. Once again $f'(x) \to 0$ and $g'(x) \to 0$ as $x \to 0$. We need to apply L' Hôpital's Rule one more time. Note that $f''(x) = \sin x$ and $g''(x) = 6x$. Now,

$$\frac{f''(x)}{g''(x)} = \frac{\sin x}{6x} \to \frac{1}{6}$$

as $x \to 0$. Therefore,

$$\lim_{x\to 0} \frac{x - \sin x}{x^3} = \frac{1}{6}.$$

The L'Hôpital Rule can also be applied when the conditions on the limits in Theorem 6.3 are replaced by $\lim_{x\to a} f(x) = \pm\infty$, $\lim_{x\to a} g(x) = \pm\infty$. The following theorem gives the precise statement of the modification:

THEOREM 6.4 (l'Hôpital Rule ∞/∞) Suppose that $\lim_{x\to a} f(x) = \pm\infty$, $\lim_{x\to a} g(x) = \pm\infty$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I (except possibly at $x = a$). Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

EXAMPLE 6.5

Find the limit $\lim_{x\to 0^+} \frac{\ln(x^2 + 2x)}{\ln x}$.

Solution

Let $f(x) = \ln(x^2 + 2x)$ and $g(x) = \ln x$. Note that $f(x) \to -\infty$ and $g(x) \to -\infty$ when $x \to 0^+$. Next,

$$f'(x) = \frac{1}{x^2 + 2x}(2x + 2) \text{ and } g'(x) = \frac{1}{x}.$$

So,

$$\frac{f'(x)}{g'(x)} = \frac{2x + 2}{x^2 + 2x}\frac{1}{1/x} = \frac{2x + 2}{x + 2} \to 2$$

if $x \to 0^+$. Therefore, by L'Hôpital's Rule, $\lim_{x\to 0^+} \frac{\ln(x^2 + 2x)}{\ln x} = 1$.

EXAMPLE 6.6

Find the limit $\lim_{x\to 0^+}\left(1+\frac{1}{x}\right)^x$.

Solution

Let $w=\left(1+\frac{1}{x}\right)^x$. Then

$$\ln w = x\ln(1+1/x) = \ln(1+1/x)/(1/x).$$

Let $f(x)=\ln(1+1/x)$ and $g(x)=1/x$. Note that $f(x)\to\infty$ and $g(x)\to\infty$ as $x\to 0^+$. Next,

$$f'(x)=\frac{1}{1+1/x}\frac{-1}{x^2}, \text{ and } g'(x)=\frac{-1}{x^2}.$$

Therefore,

$$\frac{f'(x)}{g'(x)}=\frac{1}{1+1/x}=\frac{x}{x+1}\to 0$$

as $x\to 0^+$. Therefore,

$$\lim_{x\to 0^+}\ln w=\lim_{x\to 0^+}\frac{f'(x)}{g'(x)}=0,$$

or

$$\lim_{x\to 0^+} w = e^{\lim_{x\to 0^+}\ln w}=e^0=1.$$

The proof of the l'Hôpital's Rule requires Cauchy's Mean Value Theorem (Generalized Mean Value Theorem).

THEOREM 6.5 (Cauchy's Mean Value Theorem) Suppose functions f and g are continuous on $[a,b]$ and differentiable throughout (a,b), and also suppose $g'(x)\neq 0$ throughout (a,b). Then there exists a number c in (a,b) at which

$$\frac{f'(c)}{g'(c)}=\frac{f(b)-f(a)}{g(b)-g(a)}.$$

Proof

We first show that $g(b)\neq g(a)$.

If $g(b)=g(a)$, then by Rolle's theorem there exists a number $c\in(a,b)$ such that

$$g'(c)=0,$$

which contradicts the assumption that $g'(x)=0$ for all x. Therefore, $g(b)\neq g(a)$.

Next, let

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)} \cdot (g(x) - g(a)).$$

This function is continuous and differentiable wherever f and g are, and $F(a) = F(b) = 0$. Hence, by Rolle's theorem again, there exists a number $c \in (a, b)$ such that $F'(c) = 0$.

That is,

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}. \qquad \square$$

L'Hôpital's Rule now follows from Cauchy's Mean Value Theorem as follows:

Proof
We may assume that $f(a) = g(a) = 0$. Then f and g are continuous at a. Let

$$L = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Suppose x is to the right of a and $g'(x) \neq 0$. Apply Cauchy's mean value theorem on $[a, x]$. Then there exists $c \in (a, x)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(x) - f(a)}{g(x) - g(a)}.$$

But $f(a) = g(a) = 0$, the above becomes

$$\frac{f'(c)}{g'(c)} = \frac{f(x)}{g(x)}.$$

Therefore,

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{c \to a^+} \frac{f'(c)}{g'(c)} = L$$

(as $x \to a^+$, $c \to a^+$, because $c \in [a, x]$). Similarly if x is to the left of a, then

$$\lim_{x \to a^-} \frac{f(x)}{g(x)} = \lim_{c \to a^-} \frac{f'(c)}{g'(c)} = L.$$

Therefore,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{c \to a} \frac{f'(c)}{g'(c)}. \qquad \square$$

REMARK 6.1 From the proof, we see that the l'Hôpital's rule also holds for one-sided limits.

6.3 An important consequence of the Mean Value Theorem

One of the most interesting application of the Mean Value Theorem is the following theorem:

THEOREM 6.6 If f is continuous on $[a,b]$ and $f'(x)=0$ for all $x\in(a,b)$, then f is constant on $[a,b]$.

Proof
Let $\alpha,\beta\in(a,b)$, with $\beta>\alpha$. By the Mean Value Theorem, there exists $\gamma\in(\alpha,\beta)$ such that

$$\frac{f(\beta)-f(\alpha)}{\beta-\alpha}=f'(\gamma).$$

Since $f'(x)=0$ for all $x\in(a,b)$, $f'(\gamma)=0$. Therefore,

$$f(\beta)-f(\alpha)=f'(\gamma)(\beta-\alpha)=0.$$

Therefore $f(\beta)=f(\alpha)$. Since α and β are arbitrary points in (a,b), we conclude that $f(x)=c$ is a constant on (a,b). Now, $f(a)=\lim_{x\to a^+} f(x)=\lim_{x\to a^+} c=c$ and $f(b)=\lim_{x\to b^-} f(x)=\lim_{x\to b^-} c=c$. So f is a constant on $[a,b]$. □

COROLLARY 6.7 If f and g are continuous on $[a,b]$ and $f'(x)=g'(x)$ on (a,b), then $f(x)=g(x)+C$ on $[a,b]$ for a constant C.

Proof
Now $f'=g'$, $(f-g)'=0$ on (a,b). This implies that $f-g=c$ where $c\in\mathbf{R}$. Therefore $f=g+c$. □

EXAMPLE 6.7

Use the fact that $f'(x)=0$ for all x in some open interval implies that $f(x)=c$ for some $c\in\mathbf{R}$ to derive the following identity:

$$2\sin^{-1}(x)=\cos^{-1}(1-2x^2), x\geq 0.$$

Solution

Let $f = 2\sin^{-1} x$ and $g = \cos^{-1}(1-2x^2)$. Then

$$f' = \frac{1}{\sqrt{1-x^2}} \quad \text{and} \quad g' = -\frac{-4x}{\sqrt{1-(1-2x^2)^2}}.$$

The second derivative simplifies as

$$g' = \frac{4x}{2x\sqrt{1-x^2}} = \frac{2}{\sqrt{1-x^2}} = f'.$$

Therefore, $f - g = c$. Now $c = f(0) - g(0) = 0$ and $f = g$.

6.4 First and second derivative tests for local extrema

We will first establish the following result:

THEOREM 6.8 Suppose f is continuous on $[a, b]$ and differentiable on (a, b).

(a) If $f'(x) > 0$ at each point $x \in (a, b)$ then f is increasing on $[a, b]$.

(b) If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$.

Proof
Let $x_1, x_2 \in [a, b]$ such that $x_1 < x_2$.
We find, by the Mean Value Theorem, that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

for some $c \in (x_1, x_2)$. Since $f'(c) > 0$, we have $f(x_2) > f(x_1)$. □

REMARK 6.2 Note that the converse of Theorem 6.8 is *false*. That is, even if f is increasing on $[a, b]$, it may not be true that $f'(x) > 0$ on (a, b). For example, $f(x) = x^3$ is increasing on $\mathbf{R}$ but $f'(0) = 0$.

EXAMPLE 6.8

Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the open intervals on which f is increasing and on which f is decreasing.

Solution

Now, $f' = 3x^2 - 12$. So the critical points are $x = \pm 2$. If $x < -2$ then $x^2 > 4$ and $f' = 3x^2 - 12 > 0$ and so, f is increasing on $(-\infty, -2)$. If $-2 < x < 2$ then $3x^2 - 12 < 0$ and f is decreasing on $(-2, 2)$. If $x > 2$, $3x^2 - 12 > 0$ and f is increasing on $(2, \infty)$.

EXAMPLE 6.9

Prove that $e^x > 1 + x$ for $x > 0$.

Solution

First, observe that $e^x > 1^x = 1$ for $x > 0$ since $e > 1$. Let $f(x) = e^x - 1 - x$. Note that $f'(x) = e^x - 1 > 0$ by the above observation. So $f'(x)$ is increasing on $(0, \infty)$. Therefore $f(x) > f(0)$ if $x > 0$. This implies that $e^x - 1 - x > e^0 - 1 - 0$ and $e^x > 1 + x$ for $x > 0$.

We now state the first derivative test for local extrema.

THEOREM 6.9 (The First Derivative Test) Let c be a critical number of a continuous function f.

(a) If f' changes from positive to negative at c, then f has a local maximum at c.
(b) If f' changes from negative to positive at c, then f has a local minimum at c.
(c) If f' does not change sign at c (for example, if f' is positive on both sides of c or negative on both sides), then f has no local maximum or minimum at c.

We have seen that if $f' > 0$ on (a, b), the f is increasing on $[a, b]$ Let c be a point such that $f'(c) = 0$ and f'' exists at c. If $f'' > 0$ for an open interval I containing c, this would mean that f' is increasing on I. Since $f'(c) = 0$ this means that f' increases from negative to positve and therefore c is a local minimum. Similarly, if $f''(c) < 0$ then c is a local maximum. Determining the behavior of f at critical point in the interior of $[a, b]$ where second derivative exists at c is called the **second derivative test for local extrema**. Here is the precise statement:

THEOREM 6.10 (The Second Derivative Test) Suppose f is twice differentiable at c.

(a) If $f'(c) = 0$ and $f''(c) > 0$ then f is a local minimum at c.
(b) If $f'(c) = 0$ and $f''(c) < 0$ then f is a local maximum at c.

REMARK 6.3 If $f'(c) = 0$ and $f''(c) = 0$ then the test fails. The function f may have a local maximum, local minimum or neither.

EXAMPLE 6.10

Find the critical points of $f(x) = x^3 - 3x^2 + 2$. Use the second derivative test to find the local maximum and local minimum of $f(x)$ if they exist.

Solution

Note that $f'(x) = 3x^2 - 6x$. The critical points are $x = 0$ and 2. Next, $f''(x) = 6x - 6$. Since $f''(0) < 0$, 0 is a local maximum. Since $f''(2) > 0$, then 2 is a local minimum.

7 Integrals

7.1 Approximating area

Let $\{a_1, a_2, \cdots, a_n\}$ be a set of n real numbers. We use the symbol

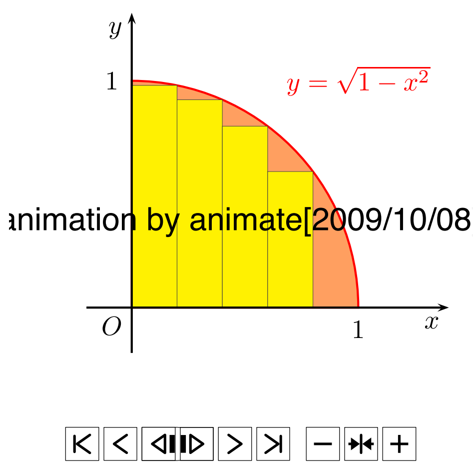
$$\sum_{j=1}^{n} a_j$$

to denote the finite sum $a_1 + a_2 + \cdots + a_n$. So, the formula for computing the sum of n consecutive integers starting from 1 is written as

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2}.$$

We can use finite sum to approximate area of certain regions.

Consider the following graph:



The area bounded by the curve and the x-axis is a quarter of a circle and we know that the area is $\pi/4$. Our aim is to approximate this area by finite sums. If we divide $[0, 1]$ into 5 equal subintervals, we find that the area of the 5 rectangles, denoted by

$$\frac{1}{5}\sum_{j=1}^{5}\sqrt{1-\frac{j^2}{5^2}},$$

is approximately 0.66. Since the area of the region is $\pi/4$, we conclude that

$$0.66 < \pi/4.$$

If we divide $[0,1]$ into n equal subintervals, we can approximate this area by

$$R_n = \frac{1}{n}\sum_{j=1}^{n}\sqrt{1-\frac{j^2}{n^2}}.$$

We see that as the number of intervals increases, our area of the rectangles approaches $\pi/4$. We use the symbol $\displaystyle\int_0^1 \sqrt{1-x^2}\,dx$ to denote the[1]

$$\lim_{n\to\infty} R_n.$$

We will next give a precise definition for $\displaystyle\int_a^b f(x)\,dx$.

7.2 Riemann Integral

Let $b > a$ and $I = [a,b]$. A partition P of I is a finite set $\{x_0, x_1, \cdots, x_{n-1}, x_n\}$ with $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ such that

$$[a,b] = I = \bigcup_{j=1}^{n}[x_{j-1}, x_j].$$

Given a partition P, we define the **diameter** of P as $\|P\| = \max_{1\le j\le n}|x_j - x_{j-1}|$.

Let $b > a$ and $I = [a,b]$. Let $P = \{x_0, x_1, \cdots, x_{n-1}, x_n\}$ be a partition of I. Let $f(x)$ be a continuous function on I. A **Riemann sum** of f associated with the partition P of $[a,b]$ is defined by

$$S_P = \sum_{j=1}^{n} f(c_j)(x_j - x_{j-1}) = \sum_{j=1}^{n} f(c_j)\Delta x_j,$$

where $c_j \in [x_{j-1}, x_j]$.

Let $f(x)$ be a function defined on a closed interval $[a,b]$. We say that a number J is the **definite integral of f over $[a,b]$** and that J is the limit of the **Riemann sums S_P** if the following condition is satisfied:

Given any $\epsilon > 0$, there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \cdots, x_n\}$ of $[a,b]$ with $\|P\| < \delta$ and any choice of c_j in $[x_{j-1}, x_j]$, we have

$$\left|\sum_{j=1}^{n} f(c_j)\ \Delta x_j - J\right| < \epsilon.$$

[1] The limit of a sequence will be discussed in a subsequent chapter.

The notation for the limit J is $\int_a^b f(x)\,dx$ and it is called the Riemann integral of f over $[a,b]$.

It is a fact that if f is continuous on $[a,b]$ then the Riemann integral of f over $[a,b]$, namely, $\int_a^b f(x)\,dx$ exists. [2] In other words, when f is continuous on $[a,b]$, then $S_P = \sum_{j=1}^{n} f(c_j)\Delta x_j$ converges to $\int_a^b f(x)\,dx$ when $\|P\| \to 0$. When this happens, we can use any partition P with $\|P\| \to 0$ to evaluate $\int_a^b f(x)\,dx$. In particular, we may let $P = \{a, a+(b-a)/n, a+2(b-a)/n, \cdots, b\}$ and $c_j = a + j(b-a)/n$.

EXAMPLE 7.1

Determine $\int_a^b c\,dx$ where c is a constant.

Solution

The Riemann Sum

$$S_P = \sum_{j=1}^{n} c\left(\frac{b-a}{n}\right) = c\frac{b-a}{n}n = c(b-a).$$

Therefore, $S_P \to c(b-a)$ as $\|P\| \to 0$. Hence,

$$\int_a^b c\,dx = c(b-a).$$

EXAMPLE 7.2

Determine $\int_a^b x\,dx$.

Solution

[2] See for example, *Principles of Mathematical Analysis* by W. Rudin (Theorem 6.8) .

The Riemann sum

$$
\begin{aligned}
S_P &= \sum_{j=1}^{n} \left(a + j\frac{b-a}{n} \right) \frac{b-a}{n} \\
&= \frac{a(b-a)}{n} \sum_{j=1}^{n} 1 + \frac{(b-a)^2}{n^2} \sum_{j=1}^{n} j \\
&= ab - a^2 + \frac{(b-a)^2}{n} \frac{n+1}{2} \\
&= \frac{2n(ab-a^2) + (a^2+b^2-2ab)n + (a^2+b^2-2ab)}{2n} \\
&= \frac{(b^2-a^2)}{2} + \frac{a^2+b^2-2ab}{2n} \to \frac{b^2-a^2}{2}
\end{aligned}
$$

as $n \to \infty$. Therefore,

$$\int_a^b x\,dx = \frac{b^2-a^2}{2}.$$

From the above two examples, we know that it is impossible to compute integrals by just using the definition of Riemann Integrals. More efficient methods are needed. But before we learn these methods, let us record the properties of the Riemann integrals that can be established using the precise definition of integral.

THEOREM 7.1 (Properties of Definite Integrals) Let f, g be functions which are integrable on the interval $[a, b]$. Then

(i) $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$,

(ii) $\displaystyle\int_a^a f(x)\,dx = 0$,

(iii) $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$, where k is a constant;

(iv) $\displaystyle\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$;

(v) $\displaystyle\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$;

(vi) If $f(x) \geq g(x)$ for all $a \leq x \leq b$, then $\displaystyle\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$;

(vii) If $m \leq f(x) \leq M$ for all $a \leq x \leq b$, then $\displaystyle m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$.

REMARK 7.1 Theorem 7.1 (vii) follows from Theorem 7.1 (vi) as follows:

$$m \leq f(x) \leq M$$

implies that

$$\int_a^b m\,dx \le \int_a^b f(x)\,dx \le \int_a^b M\,dx.$$

Using Example 7.1 in the integrals on the left and right of the above inequalities, we deduce that

$$(b-a)m \le \int_a^b f(x)\,dx \le (b-a)M\,dx,$$

which is Theorem 7.1 (vii).

EXAMPLE 7.3

Find an upper bound and a lower bound for $\displaystyle\int_0^1 \frac{1}{1+x^2}\,dx$.

Solution

Since $0 \le x \le 1$, we deduce that

$$\frac{1}{2} \le \frac{1}{1+x^2} \le 1.$$

By Theorem 7.1 (vii), we deduce that

$$\frac{1}{2} \le \int_0^1 \frac{1}{1+x^2}\,dx \le 1.$$

EXAMPLE 7.4

Let $f(x)$ be continuous on $[a,b]$. Show that

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

Solution

Note that

$$-|f(x)| \le f(x) \le |f(x)|.$$

By Theorem 7.1 (vi),

$$-\int_a^b |f(x)|\,dx \le \int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx.$$

Hence,

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

7.3 The Mean Value Theorem for Integral

Let f be a continuous function on $[a, b]$. By Extreme Value Theorem, we know that there exists $m = f(\alpha)$ and $M = f(\beta)$ such that the image of $[a, b]$ under f is $[m, M]$. From Theorem 7.1 (vii), we find that

$$m(b-a) \le \int_a^b f(t)\,dt \le M(b-a),$$

or in other words,

$$m = f(\alpha) \le \frac{1}{(b-a)}\int_a^b f(t)\,dt \le M = f(\beta).$$

This means that

$$\frac{1}{(b-a)}\int_a^b f(t)\,dt \in [m, M].$$

By the Intermediate Value Theorem, there exists a $c \in [\alpha, \beta] \subset [a, b]$ such that

$$f(c) = \frac{1}{(b-a)}\int_a^b f(t)\,dt.$$

This gives the following theorem known as the Mean Value Theorem for Integral:

THEOREM 7.2 If f is continuous on $[a, b]$, then at some point $c \in [a, b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(t)\,dt.$$

EXAMPLE 7.5

Show that if f is continuous on $[a, b]$, $a \neq b$, and if

$$\int_a^b f(t)\,dt = 0,$$

then $f(c) = 0$ for some $c \in [a, b]$.

7.4 Fundamental Theorem of Calculus, Part 1

THEOREM 7.3 If f is continuous on $[a, b]$ then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$. In other words,

$$F'(x) = \frac{d}{dx} \int_a^x f(t)\,dt = f(x).$$

Note that we have changed the variable in the integral from $f(x)\,dx$ to $f(t)\,dt$. We reserve the variable x for F. From now on, we will frequently write $\int_a^b f(t)dt$ instead of $\int_a^b f(x)\,dx$.

Proof
We first observe that

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h} \int_x^{x+h} f(t)\,dt.$$

By the Mean Value Theorem for integral, there exists $c_h \in (x, x+h)$ such that

$$f(c_h) = \frac{1}{h} \int_x^{x+h} f(t)\,dt = \frac{F(x+h) - F(x)}{h}.$$

As $h \to 0$, $c_h \to x$ and the right hand side tends to $F'(x)$. Therefore,

$$F'(x) = f(x).$$

□

EXAMPLE 7.6

Find $F'(x)$ if $F(x) = \int_a^x (t^3 + 1)\,dt$.

Solution

By Theorem 7.3, we find that

$$F'(x) = x^3 + 1.$$

EXAMPLE 7.7

Find $F'(x)$ if $F(x) = \int_1^{x^2} \cos t \, dt$.

Solution

Let $g(x) = \int_1^x \cos t \, dt$. Then $g'(x) = \cos x$. Since $F(x) = g(x^2)$, by the Chain Rule, we deduce that

$$F'(x) = (\cos x^2)(2x) = 2x \cos x^2.$$

EXAMPLE 7.8

Find $\lim_{x \to 0} \frac{1}{x^3} \int_0^x \frac{t^2}{t^4+1} \, dt$.

Solution

By L'Hôpital's rule,

$$\lim_{x \to 0} \frac{1}{x^3} \int_0^x \frac{t^2}{t^4+1} \, dt = \lim_{x \to 0} \frac{x^2/(x^4+1)}{3x^2} = \frac{1}{3}.$$

7.5 Fundamental Theorem of Calculus, Part 2

DEFINITION 7.1 A function F is called an antiderivative of f if

$$\frac{dF(x)}{dx} = F'(x) = f(x).$$

An antiderivative of a function f is not unique. In fact, if $F' = G' = f$, then $(F-G)' = 0$ which means that $F = G + k$ for some constant $k \in \mathbf{R}$. In other words, two antiderivatives of a function f differ by a constant $k \in \mathbf{R}$.

For example, when $n \geq 2$, the functions in the set

$$A_f = \left\{ \frac{x^{n+1}}{n+1} + k | k \in \mathbf{R} \right\}$$

are all antiderivatives of x^n since

$$\left(\frac{x^{n+1}}{n+1} + k \right)' = x^n.$$

We use the indefinite integral $\displaystyle\int f(x)\,dx$ to denote a "representative" of A_f. For example, you may think of $\displaystyle\int f(x)\,dx$ as the function $F(x) = \displaystyle\int_a^x f(t)\,dt, a \in \mathbf{R}$. If G is another antiderivative of f, then we write $G(x) = F(x) + k = \displaystyle\int f(x)\,dx + k$ for some $k \in \mathbf{R}$. This is why it is common to see the expression

$$\int f(x)\,dx = F(x) + k,$$

where $F(x)$ is an antiderivative of $f(x)$ and c is a constant. For example,

$$\int x\,dx = \frac{x^2}{2} + k.$$

Given a function $f(x)$, there is no effective algorithm for finding an antiderivative for $f(x)$. For example, we do know an alternative expression for the antiderivative of e^{x^2} other than $\displaystyle\int_0^x e^{t^2}\,dt$.

We usually record the integral of $f(x)$ after differentiating a function $F(x)$. For example, since

$$\frac{d \sin x}{dx} = \cos x,$$

we record the indefinite integral

$$\int \cos x\,dx = \sin x + k.$$

Here is a list of functions $f(x)$ where the indefinite integrals are known through differentiating some "basic" functions such as x^n and $\sin x$:

(a) $\displaystyle\int x^r\,dx = \frac{x^{r+1}}{r+1} + k, r \in \mathbf{Q}, r \neq -1,$

(b) $\displaystyle\int \cos x\,dx = \sin x + k,$

(c) $\displaystyle\int \sin x\,dx = -\cos x + k,$

(d) $\displaystyle\int \sec^2 x\, dx = \tan x + k.$

We will use these basic integrals and techniques of integration (which we will learn later) to find indefinite integrals of more complicated functions $f(x)$ such as $\sqrt{1-x^2}$. We are now ready to state Part 2 of the Fundamental Theorem of Calculus.

THEOREM 7.4 If f is continuous at every point of $[a, b]$ and G is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(t)\, dt = G(b) - G(a).$$

Proof
We have seen from Theorem 7.3 that the function $F(x) = \int_a^x f(t)\, dt$ satisfies $G' = f$. This means that F is an antiderivative of f. Since F is also an antiderivative of f, we conclude that

$$F(x) = G(x) + k$$

for some $k \in \mathbf{R}$. Note that $F(a) = 0$. Therefore $k = F(a) - G(a) = G(a)$. Hence,

$$G(x) = \int_a^x f(t)\, dt = G(x) - G(a).$$

□

EXAMPLE 7.9

Evaluate $\displaystyle\int_0^\pi \cos t\, dt.$

Solution

Note that $\int \cos x\, dx = \sin x + k$. Therefore,

$$\int_0^\pi \cos x\, dx = \sin \pi - \sin 0 = 0.$$

EXAMPLE 7.10

Evaluate $\displaystyle\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right)\, dt$.

Solution

The integral $\displaystyle\int \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) = \frac{3}{2}\frac{x^{3/2}}{3/2} + \frac{4}{x} = x^{3/2} + \frac{4}{x}$. Therefore,

$$\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right)\, dt = 4^{3/2} + \frac{4}{4} - 1 - 4 = 4.$$

7.6 The natural logarithm function

We have encountered functions such as e^x and a^x in early chapters and we still have a vague idea of the definition of these functions. We are now ready to know them better. We begin with the integral

$$\int_1^x \frac{1}{t}\, dt, x > 0.$$

First, note that $L(x)$ exists since $1/x$ is a continuous function on $[1, x]$ or $[x, 1]$ (in case $0 < x < 1$). We define $\ln x$ to be

$$\ln x = \int_1^x \frac{1}{t}\, dt, x > 0.$$

From Theorem 7.3, we conclude that

$$(\ln x)' = \frac{1}{x}. \tag{7.1}$$

Since $1/x > 0$ for $x > 0$, $\ln x$ is increasing on $(0, \infty)$ and therefore one to one. In other words, there is an inverse function for $\ln x$, which we call e^x, such that

$$e^{\ln x} = x, x > 0$$

and

$$\ln e^x = x, x \in \mathbf{R}.$$

Note that if $y = e^x$, then $\ln y = x$ and $\dfrac{1}{y}y' = 1$, or

$$y' = y = e^x.$$

In other words,

$$(e^x)' = e^x. \tag{7.2}$$

The identity (7.2) and (7.1) give us two additional basic indefinite integrals

$$\int e^x dx = e^x + k \text{ and } \int \frac{1}{x}\, dx = \ln x + k.$$

We now establish some identities using our new knowledge of $\ln x$ and e^x.

THEOREM 7.5 The functions $\ln x$ and e^x satisfy

$$\ln(zw) = \ln z + \ln w$$

and

$$e^{s+t} = e^s e^t.$$

Proof
We fix w and view $\ln(zw)$ as a function of z. Let $F(z) = \ln(zw)$. Then

$$F'(z) = \frac{1}{z}.$$

Next, let $G(z) = \ln z + \ln w$. Then

$$G'(z) = \frac{1}{z}.$$

Therefore,

$$F(z) = G(z) + k$$

for some $k \in \mathbf{R}$. Setting $z = 1$, we observe that $F(1) = \ln w = 0 + k$. Therefore

$$\ln(zw) = \ln z + \ln w.$$

To prove the second identity, we view e^{s+t} as a function of s, or we let $A(s) = e^{s+t}$. Then $\ln A(s) = s + t$. Next, let $U = e^s$ and $V = e^t$. Then $\ln U = s$ and $\ln V = t$. Therefore, $\ln A(s) = \ln U + \ln V = \ln(UV)$, by the previous identity. This implies that

$$e^{s+t} = A(s) = UV = e^s \cdot e^t.$$

□

With functions $\ln x$ and e^x defined, we define, for $a > 0$, the function

$$a^x = e^{x \ln a}.$$

Note that

$$(a^x)' = (\ln a) a^x.$$

Therefore

$$\int a^x\, dx = \frac{a^x}{\ln a} + k,\ \ a > 0.$$

8 Techniques of Integration

8.1 Techniques of integration

At present, we only two ways of computing a definite integral. One way is to use the definition (you do not want to try this) and the other way is to use integrals of functions which are derivatives of known functions. A third way would be to use identity to help us. For example, we can compute the following using trigonometric identities.

EXAMPLE 8.1

Find $\displaystyle\int \cos^2 x\,dx$.

Solution

Recall that $\cos 2x = \dfrac{\cos 2x + 1}{2}$. This implies that

$$\int \frac{\cos 2x + 1}{2}\,dx = \frac{\sin 2x}{4} + \frac{x}{2} + k.$$

All these methods are not helping us much in deriving many other indefinite integrals including "simple indefinite integrals" such as $\displaystyle\int \tan x\,dx$. In order to expand the list of indefinite integrals which we can evaluate, we need new techniques to be covered in this Chapter.

8.1.1 Substitution rule

THEOREM 8.1 Let f be a continuous function. If $x = g(t)$ is a function such that g' is continuous, then

$$\int f(x)\,dx = \int f(g(t))g'(t)\,dt.$$

REMARK 8.1 Before we sketch the proof, we recall that if $F(x)$ is a antiderivative of $f(x)$, then $F(x) = \int f(x)\,dx + k$. Note that the symbol $\int f(x)\,dx$ is used because we know that $\int_a^x f(t)\,dt$ is always an antiderivative of f. Conversely if $F(x) = \int f(x)\,dx + k$, then $F(x)$ is an antiderivative of f. Therefore, we can write

$$\frac{dF(x)}{dx} = f(x)$$

if and only if

$$F(x) = \int f(x)\,dx + k.$$

Proof
(Sketch)

Let F be an antiderivative of f. Then $F' = f$, and then

$$\frac{d}{dt}F(g(t)) = F'(g(t))g'(t) = f(g(t))g'(t).$$

This implies, using Remark 8.1, that

$$\int f(g(t))g'(t)\,dt = F(g(t)) + k.$$

Let $x = g(t)$. Then

$$\int f(x)\,dx = F(x) = F(g(t)) + k = \int f(g(t))g'(t)\,dt.$$

□

Note. The substitution rule converts the integral in x to an integral in t, and the answer will be a function in t. So in application, it usually requires $x = g(t)$ to be one to one so that we could convert the answer back to a function in x.

EXAMPLE 8.2

Find $\displaystyle\int \tan x\,dx$.

Solution

We first find an antiderivative for $\tan x$. We will not carry the constant in our computations.

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{1}{\cos x}(\cos x)'\,dx = -\int \frac{1}{v}\,dv = -\ln v + k = -\ln\cos x + k.$$

EXAMPLE 8.3

Evaluate $\displaystyle\int xe^{x^2}\,dx$.

Solution

$$\int xe^{x^2}\,dx = \frac{1}{2}\int e^{x^2}(x^2)'\,dx = \frac{1}{2}\int e^v\,dv = \frac{e^v}{2} + k = \frac{e^{x^2}}{2} + k.$$

REMARK 8.2 We cannot evaluate $\displaystyle\int e^{x^2}\,dx$ even though the integral looks "simple" and similar to the one we just evaluated.

In the above two examples, we express the integrals in terms of functions $g(x)$ where $g'(x)$ is also present in the integrals. We evaluate the integrals by using the substitution $v = g(x)$. Sometimes, this cannot be done as it is not obvious that we can find a function $g(x)$ for which $g'(x)$ is also present in the integrals. In such cases, we will need to let $x = h(t)$ for some functions h and use known identities to evaluate the integrals. A word of warning here is that when we let $x = h(t)$ to evaluate a definite integral, we have to be sure that $h^{-1}(x)$ exists in the interval of integration. We illustrate the difference between these two substitution procedures using the next example.

EXAMPLE 8.4

Evaluate $\displaystyle\int \frac{2x}{1+x^2}\,dx$.

Solution

The above can be evaluated in two ways. One writes
$$\int \frac{2x}{1+x^2}\,dx = \int \frac{1}{1+x^2}(x^2)'\,dx = \int \frac{1}{1+v}\,dv = \ln(1+v)+k = \ln(1+x^2)+k.$$

The second method is to set $x=\sqrt{u}$. This implies that $dx = \dfrac{1}{2\sqrt{u}}du$ and
$$\int \frac{2x}{1+^2}\,dx = \int \frac{2\sqrt{u}}{1+u}\frac{1}{2\sqrt{u}}\,du = \ln(1+u)+k = \ln(1+x^2)+k.$$

The antiderivative of $\dfrac{1}{\sqrt{1-x^2}}$ is $\sin^{-1}x$. The antiderivative of its reciprocal, namely $\sqrt{1-x^2}$, turns out to be more complicated.

EXAMPLE 8.5

Evaluate $\displaystyle\int \sqrt{1-x^2}\,dx$.

Solution

Let $x=\sin v$. Then $dx = \cos v\,dv$. Therefore,
$$\int \sqrt{1-x^2}\,dx = \int \cos v \cos v dv = \frac{\sin 2v}{4} + \frac{v}{2} + k = \frac{x\sqrt{1-x^2}}{2} + \frac{\sin^{-1}x}{2} + k.$$

8.1.2 Integration by parts (Reference: Section 8.2)

Recall the product law for differentiation:
$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).$$
This implies, from Remark 8.1, that
$$f(x)g(x) = \int f(x)g'(x)\,dx + \int g(x)f'(x)\,dx + k.$$
If we let $u=f(x)$ and $v=g(x)$, then the above becomes
$$\int u\,dv + \int v\,du = uv.$$

We will used the above formula to evaluate indefinite integrals. This technique is called **integration by parts**.

EXAMPLE 8.6

Find $\displaystyle\int \ln x\, dx$.

Solution

Let $u = \ln x$, $v = x$. Then
$$\int udv + \int vdu = x \ln x + k.$$
This implies that
$$\int \ln x dx + \int x \frac{1}{x} dx = x \ln x + k.$$
Therefore,
$$\int \ln x\, dx = x \ln x - x + k.$$

EXAMPLE 8.7

Find $\displaystyle\int e^x \sin x\, dx$.

Solution

Let $u = e^x$. Note that $\sin x dx = d(-\cos x)$. Let $v = -\cos x$. Then
$$\int udv + \int (-\cos x) de^x = -e^x \cos x + k.$$
This gives
$$\int e^x \sin x\, dx - \int e^x \cos x dx = -e^x \cos x + k.$$
Next, let $s = \sin x$. Observe that $e^x dx = de^x$ and so, we may let $t = e^x$. Then
$$\int s\, dt + \int t\, ds = e^x \sin x + \ell.$$
This yields
$$\int e^x \sin x dx + \int e^x \cos x dx = e^x \sin x + \ell.$$
Combining this with the previous identity, we deduce that
$$\int e^x \sin x\, dx = e^x \frac{\sin x - \cos x}{2} + k'.$$

EXAMPLE 8.8

Prove the reduction formula

$$\int \cos^n x\,dx = \frac{1}{n}\sin x\,\cos^{n-1} x + \frac{n-1}{n}\int \cos^{n-2} x\,dx,$$

where $n \geq 2$ is an integer.

Solution

Let $u = \cos^{n-1} x$ and observe that $\cos x dx = d\sin x$. Set $v = \sin x$. Then

$$\cos^{n-1} x \sin x = \int udv + \int vdu = \int \cos^n xdx + \int \sin x(n-1)\cos^{n-2} x(-\sin x)dx.$$

Simplifying the above using $\sin^2 x = 1 - \cos^2 x$ and rearranging, we arrive at the recursion formula.

8.1.3 Integration of rational functions by partial fractions

In this section, we will study indefinite integrals of the form

$$\int \frac{A(x)}{B(x)}\,dx,$$

where $A(x)$ and $B(x)$ are polynomials.

We may assume that the leading coefficient of $B(x)$ is 1. Such polynomial is called a **monic polynomial**. We also also assume that $A(x)$ and $B(x)$ have no common factor.

If the degree of $A(x)$ is greater than or equal to that of $B(x)$ we apply long division to obtain

$$A(x) = B(x)Q(x) + A_1(x),$$

where $A_1(x) = 0$ or $\deg A_1(x) < \deg B(x)$. We can then integrate

$$Q(x) + \frac{A_1(x)}{B(x)}.$$

From now on, we will assume that the degree of $A(x)$ is smaller than the degree of $B(x)$ in the expression $\dfrac{A(x)}{B(x)}$.

We now list the following facts without proof.

Fact 1. Every (non-constant) monic polynomial $B(x)$ (over $\mathbf{R}$) can be uniquely factorized into the product of linear factors and irreducible quadratic factors, say

$$B(x) = (x + a_1)^{k_1} \cdots (x + a_m)^{k_m} (x^2 + b_1 x + c_1)^{r_1} \cdots (x^2 + b_n x + c_n)^{r_n},$$

with $b_i^2 < 4c_i$ for $i = 1, \ldots, n$.

Fact 2. Suppose $\deg A(x) < \deg B(x)$. Then $\dfrac{A(x)}{B(x)}$ can be converted into a partial fraction as follows:

i) For each linear factor $(x + a)^k$ of $B(x)$, we have terms

$$\frac{A_1}{x + a} + \frac{A_2}{(x + a)^2} + \cdots + \frac{A_k}{(x + a)^k}.$$

ii) For each quadratic factor $(x^2 + bx + c)^r$ of $B(x)$, we have terms

$$\frac{B_1 x + C_1}{x^2 + bx + c} + \frac{B_2 x + C_2}{(x^2 + bx + c)^2} + \cdots + \frac{B_r x + C_r}{(x^2 + bx + c)^r}.$$

For example, $\dfrac{4x}{x^3 - x^2 - x + 1} = \dfrac{-1}{x + 1} + \dfrac{1}{x - 1} + \dfrac{2}{(x - 1)^2}$.

Integration of a rational function is thus reduced to integration of the following rational functions:

Linear factor. $\dfrac{1}{(x + a)^k}$, k is a positive integer.

$$\int \frac{1}{(x + a)^k}\, dx = \begin{cases} \ln|x + a| + K, & \text{if } k = 1, \\ \dfrac{(x + a)^{1-k}}{1 - k} + K, & \text{if } k \geq 2. \end{cases}$$

Quadratic factor. $\dfrac{Bx + C}{(x^2 + bx + c)^r}$, r is a positive integer, and $b^2 < 4c$.

Note that $x^2 + bx + c = (x + b/2)^2 + (c - b^2/4)$. Let $u = x + b/2$ and $d = \sqrt{c - b^2/4}$. Then

$$\frac{Bx + C}{(x^2 + bx + c)^r} = \frac{B(u - b/2) + C}{(u^2 + d^2)^r} = \frac{Bu}{(b^2 + d^2)^r} + \frac{-bB/2 + C}{(u^2 + d^2)^r}.$$

It suffices to integrate $\displaystyle\int \frac{1}{(u^2 + d^2)^r}\, du$ and $\displaystyle\int \frac{u}{(u^2 + d^2)^r}\, du$.

a) $$\int \frac{u}{(u^2 + d^2)^r}\, du = \frac{1}{2} \int \frac{d(u^2 + d^2)}{(u^2 + d^2)^r} = \begin{cases} \dfrac{1}{2} \ln(u^2 + d^2) + K, & \text{if } r = 1, \\ \dfrac{(u^2 + d^2)^{1-r}}{2(1 - r)} + K, & \text{if } r \geq 2. \end{cases}$$

b) $\displaystyle\int \frac{1}{(u^2 + d^2)^r}\, du$. Let $t = \tan^{-1}\left(\dfrac{u}{d}\right)$. Then

$$\int \frac{1}{(u^2 + d^2)^r}\, du = \frac{1}{d^{2r-1}} \int \cos^{2r-2} t\, dt.$$

The last term can be computed using the result in Example 8.8.

In particular, $\displaystyle\int \frac{1}{u^2+d^2}\,du = \frac{1}{d}\tan^{-1}\left(\frac{u}{d}\right) + K.$

EXAMPLE 8.9

Evaluate $\displaystyle\int \frac{x+4}{x^2+5x-6}\,dx.$

Solution

Let A and B be such that

$$\frac{x+4}{x^2+5x-6} = \frac{x+4}{(x-1)(x+6)} = \frac{A}{x+6} + \frac{B}{x-1}.$$

This yields

$$x+4 = A(x-1) + B(x-6). \tag{8.1}$$

Letting $x=1$ (8.1), we deduce that $B = 5/7$ and letting $x=6$ in (8.1), we find that $A = 2/7$. Therefore,

$$\int \frac{x+4}{x^2+5x-6}\,dx = \frac{2}{7}\ln(x+6) + \frac{5}{7}\ln(x-1) + c.$$

EXAMPLE 8.10

Evaluate $\displaystyle\int \frac{y^2+1+2y}{(y^2+1)^2}\,dx.$

Solution

We write the rational function as

$$\frac{y^2+1+2y}{(y^2+1)^2} = \frac{1}{y^2+1} + \frac{2y}{(y^2+1)^2}.$$

Therefore,

$$\int \frac{y^2+1+2y}{(y^2+1)^2}\,dx = \tan^{-1}(y) - \frac{1}{1+y^2} + k.$$

EXAMPLE 8.11

Evaluate $\displaystyle\int \frac{1}{1-\sin x}\,dx$.

Solution

Let $t = \tan(x/2)$. Then $\sin(x/2) = \dfrac{t}{\sqrt{1+t^2}}$ and $\cos(x/2) = \dfrac{1}{\sqrt{1+t^2}}$. This implies that

$$\sin x = 2\sin(x/2)\cos(x/2) = \frac{2t}{1+t^2}$$

and

$$\cos x = 2\cos^2(x/2) - 1 = \frac{1-t^2}{1+t^2}.$$

Next, $t = \tan(x/2)$ implies that

$$dt = \frac{1}{2}\sec^2(x/2)\,dx = \frac{1+t^2}{2}\,dx.$$

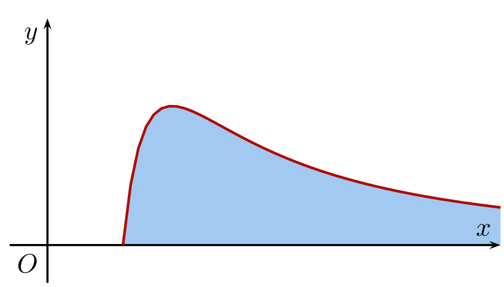
Therefore,

$$\int \frac{1}{1-\sin x}\,dx = \int \frac{1}{1-(2t/(1+t^2))}\frac{2}{1+t^2}\,dt = \int \frac{2}{(1-t)^2}\,dt = \frac{2}{1-\tan(x/2)} + k.$$

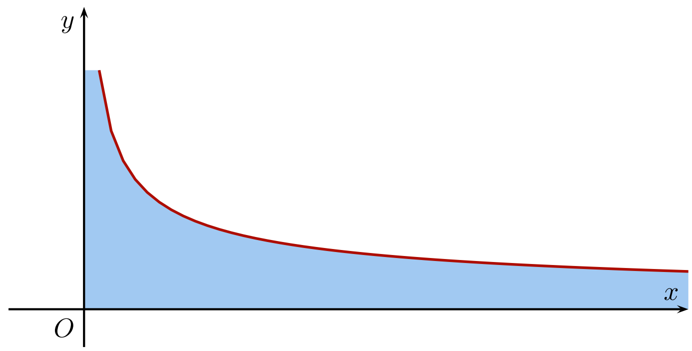
8.2 Improper Integrals

The definite integrals we have studied until now have two properties. Firstly the interval of integration $[a, b]$ be finite. Second.y, the values of the function (called the integrand) in the integral on $[a, b]$ is finite.

In practice, we may encounter problems that fail to meet one or both of these conditions. The integral for the area under the curve $y = (\ln x)/x^2$ from $x = 1$ to $x = \infty$ is an example for which the interval of integration is infinite.



The integral for the area under the curve of $y = 1/\sqrt{x}$ between $x = 0$ and $x = 1$ is an example for which the value of $f(x)$ at $x = 0$ is infinite. In both cases, the integrals are said to be **improper** and are calculated as limits. We will see that improper integrals play an important role when we study the convergence of certain infinite series in a latter Chapter.



8.2.1 Infinite Limits of Integration

DEFINITION 8.1 Integrals with infinite limits of integration are **improper integrals of Type I.**

1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx.$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx,$$

   where c is any real number.

In each case, if the limit is finite, we say that the improper integral converges and that the limit is the value of the improper integral. If the limit fails to exist, the improper integral diverges.

DEFINITION 8.2 Integrals of functions that become infinite at a point within the interval of integration are **improper integrals of Type II.**

1. If $f(x)$ is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx.$$

3. If $f(x)$ is discontinuous at c with $a < c < b$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In each case, if the limit is finite, we say that the improper integral converges and that the limit is the value of the improper integral. If the limit fails to exist, the improper integral diverges.

EXAMPLE 8.12

Evaluate the Type I improper integral

$$\int_1^\infty \frac{\ln x}{x^2}\,dx.$$

Solution

Let $u = \ln x$ and $dv = \dfrac{1}{x^2} = d(-1/x)$. This implies that

$$\int \frac{\ln x}{x^2}\,dx + \int -\frac{1}{x^2} dx = -\frac{\ln x}{x}.$$

Therefore,

$$\int_1^b \frac{\ln x}{x^2}\,dx = -\frac{\ln b}{b} - \frac{1}{b} + 1.$$

Letting $b \to \infty$, we conclude that

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = 1.$$

EXAMPLE 8.13

Evaluate the Type I improper integral

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+1}.$$

Solution

We need to evaluate

$$\int_0^{\infty} \frac{1}{1+x^2}\,dx$$

and

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx.$$

The integral

$$\int_0^{b} \frac{1}{1+x^2}\,dx = \tan^{-1} b.$$

Letting $b \to \infty$ yields

$$\int_0^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2}.$$

Similarly,

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx = \frac{\pi}{2}$$

and we conclude that

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \pi.$$

EXAMPLE 8.14

Evaluate the Type II improper integral

$$\int_0^{1} \frac{dx}{(x-1)^{2/3}}.$$

Solution

The integral

$$\int_0^b \frac{1}{(x-1)^{2/3}}\,dx = 3(b-1)^{1/3} + 3.$$

As $b \to 1^-$, we deduce that

$$\int_0^1 \frac{1}{(x-1)^{2/3}}\,dx = 3.$$

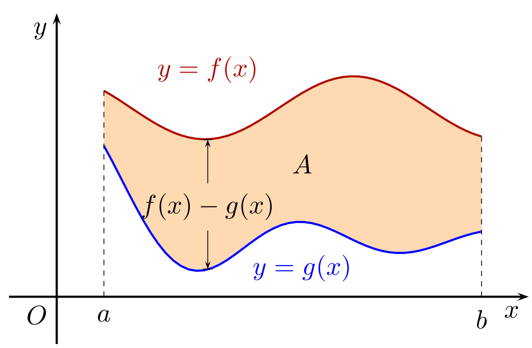
9 Applications of integrals

9.1 The area problem

Our study of definite integrals begins with the study of area bounded by the graph of a function and the x-axis. The following formula gives the area of the region bounded by graphs of two function within the interval $[a, b]$:

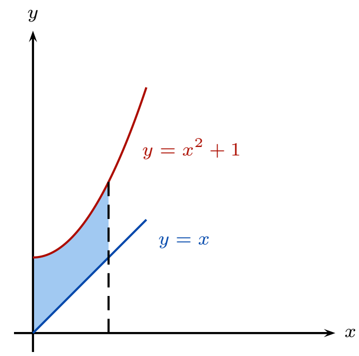
THEOREM 9.1 Let f and g be continuous function such that $f(x) \geq g(x)$ for all $x \in [a, b]$. Then the area A of the region bounded by the curves $y = f(x)$ and $y = g(x)$ from $x = a$ to $x = b$ is given by

$$A = \int_a^b (f(x) - g(x))\, dx.$$



EXAMPLE 9.1

Find the area of the region bounded above by $y = x^2 + 1$, bounded below by $y = x$ and bounded on the sides by $x = 0$ and $x = 1$.



Solution

The area is given by

$$\int_0^1 (x^2 + 1) - x\,dx = F(1) - F(0) = \frac{5}{6},$$

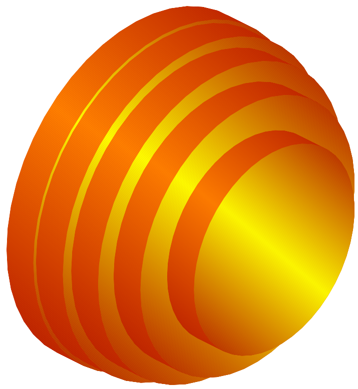
where

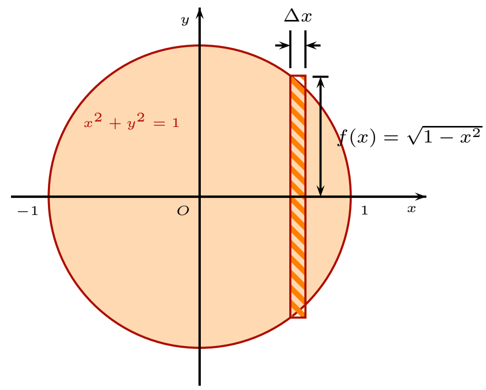
$$F(x) = \frac{x^3}{3} + x - \frac{x^2}{2}.$$

9.2 Volumes

We all know that the volume of the unit sphere is $\frac{4\pi}{3}$. But how do we show that this is the case?

When we evaluate the area of the circle, we use "small rectangles" to approximate the region bounded by $y = \sqrt{1 - x^2}$ and the x-axis from 0 to 1. We then increase the number of points in the partition of $[0, 1]$ and approximate the area with smaller rectangles. For the volume of solid which are generated from a curve by revolving the curve 360 degrees (or 2π radians) about an axis, we use solid disks instead. Suppose we want to estimate the volume of the hemisphere. The following diagram shows the approximation use several disks. As the number of disks increases we get better approximations.





Let the Δx be the width of each disk. Then the volume of the disk near point x is

$$\Delta V = \underbrace{\pi(1 - x^2)}_{\text{base area}} \cdot \underbrace{\Delta x}_{\text{width}}.$$

In general, suppose an object S is put along the x-axis from a to b. Let $A(x)$ be the area of the cross-section of S perpendicular to the x-axis and passing through the point x. Then the volume of the disk near x is $\Delta V = A(x)\,\Delta x$. Then $\dfrac{dV}{dx} = \lim\limits_{\Delta x \to 0} \dfrac{\Delta V}{\Delta x} = A(x)$. Therefore, we can define the **volume** of S to be

$$V = \int_a^b A(x)\,dx.$$

Hence the volume of the unit sphere is

$$2\int_0^1 \pi(1 - x^2)\,dx = 2\pi\left[x - \frac{x^3}{3}\right]_{x=0}^{x=1} = \frac{4}{3}\pi.$$

9.2.1 Solids of revolution

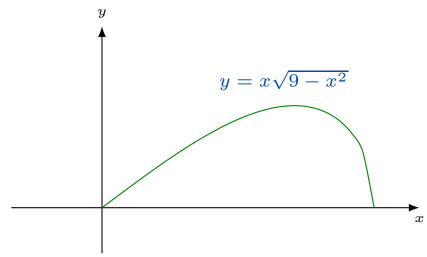
In particular, if the solid if obtained by rotating the region bounded by the curve $y = f(x)$ and the x-axis from a to b about the x-axis, then the area of the cross-section at x is $A(x) = \pi(f(x))^2$. Therefore, its volume can be computed by the integral

$$V = \int_a^b \pi(f(x))^2\,dx.$$

This is known as the **washer method**.

EXAMPLE 9.2

Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \dfrac{x}{4}\sqrt{9-x^2}$ from 0 to 3.



Solution

The volume of revolution generated by the curve about the x-axis is

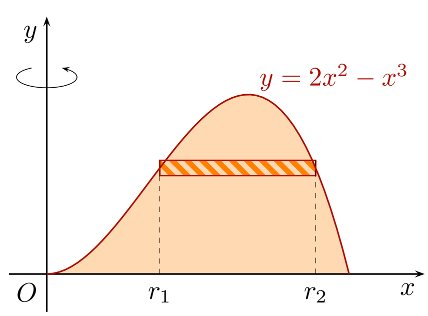
$$\int_0^2 \pi \left(\frac{x}{4}\sqrt{9-x^2}\right)^2 dx = \int_0^3 \pi \frac{x^2}{16}(9-x^2)\,dx = F(3) - F(0) = \frac{81\pi}{40},$$

where

$$F(x) = \pi\left(\frac{3x^3}{16} - \frac{x^5}{80}\right).$$

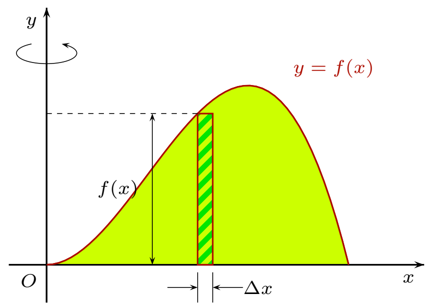
9.3 Volumes by cylindrical shells

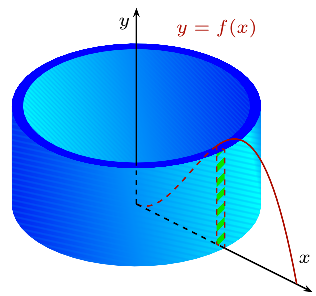
There are solids of revolution where the volume are difficult to compute using the washer method. Let us consider the problem of finding the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.



Slicing in the direction perpendicular to the y-axis, we obtain a washer. The the cross-section at y is an annulus and its area is $A(y) = \pi(r_2^2 - r_1^2)$. But to compute the inner radius r_1 and the outer radius r_2 of the washer we would have to solve the cubic equation $y = 2x^2 - x^3$ for x in terms of y. This could lead to integrals which involve cubic root of a function and we may be faced with the difficult task of finding indefinite integrals for such functions.

We now discuss an alternative method. Instead of *washers*, we use *cylinders* to approximate the volume. We consider the vertical strip and rotate it about the y-axis:





The volume of the little strip near x when rotated about the y-axis is thus

$$\Delta V = \underbrace{\pi[(x+\Delta x)^2 - x^2]}_{\text{base area}} \cdot \underbrace{f(x)}_{\text{height}} = \pi(2x + \Delta x) f(x) \cdot \Delta x.$$

Then

$$\frac{dV}{dx} = \lim_{\Delta x \to 0} \frac{\Delta V}{\Delta x} = \lim_{\Delta x \to 0} \pi(2x + \Delta x) f(x) = 2\pi x\, f(x).$$

This gives the **method of cylindrical shells**:

THEOREM 9.2 Let f be a continuous function such that $f(x) \geq 0$ for all $x \in [a, b]$, $(0 \leq a < b)$. The volume of the solid obtained by rotating about the y-axis the region under the curve $y = f(x)$ from a to b is given by

$$V = 2\pi \int_a^b x\, f(x)\, dx.$$

EXAMPLE 9.3

Find the volume of the solid obtained by rotating about the y-axis the region between $y = x$ and $y = x^2$.

Solution

The volume is given by

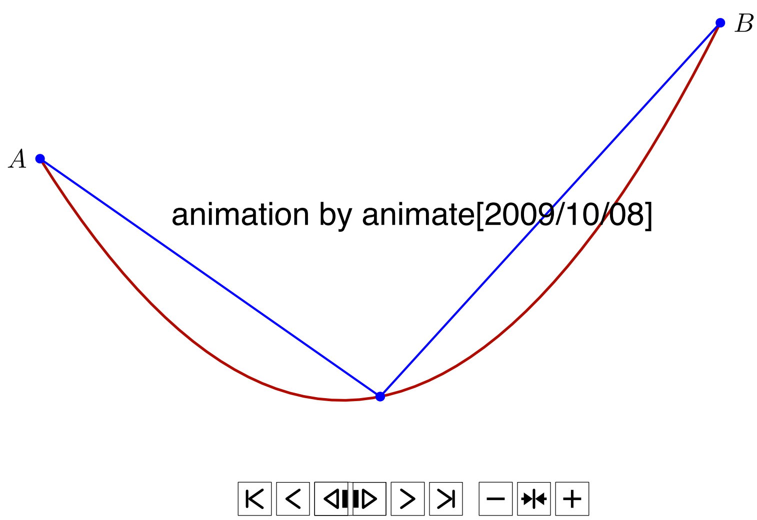
$$\int_0^1 2\pi x(x - x^2)\,dx = \int_0^1 2\pi x^2 - 2\pi x^3\,dx = F(1) - F(0) = \frac{\pi}{6},$$

where

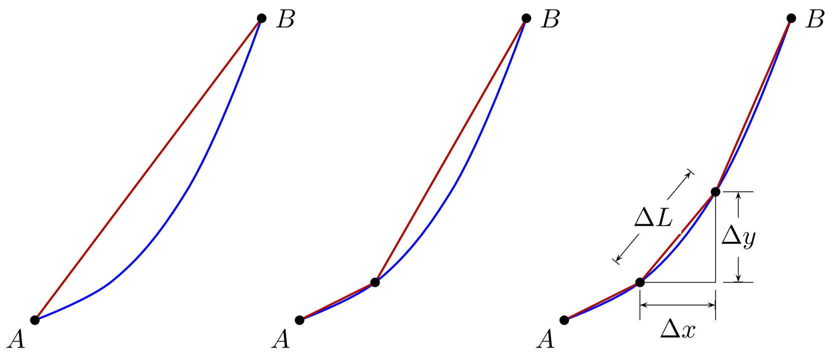
$$F(x) = \frac{2\pi x^3}{3} - \frac{2\pi x^4}{4}.$$

9.4 Arc length

Let f be a differentiable function. It is called **smooth** if f' is continuous. How do we measure the length of a smooth curve $y = f(x)$? Consider the following diagram:



We see that we can approximate the length of a curve by line segments. Using the same idea as in the case of area and volume, we proceed as follow:



Consider a small part of the curve. We estimate its length using the length of the line connecting the initial and terminal points. Then $\Delta L = \sqrt{(\Delta x)^2 + (\Delta y)^2}$. Therefore,

$$\frac{dL}{dx} = \lim_{\Delta x \to 0} \frac{\Delta L}{\Delta x} = \lim_{\Delta x \to 0} \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

This gives a formula to evaluate the arc length:

THEOREM 9.3 Let f be a smooth function. Then the length of the curve $y = f(x)$, $a \le x \le b$, is

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx.$$

EXAMPLE 9.4

Find the length of the curve $y = \dfrac{x^3}{12} + \dfrac{1}{x}$ from $x = 1$ to 4.

Solution

Note that

$$\frac{dy}{dx} = \frac{x^2}{4} - \frac{1}{x^2} = \frac{x^4 - 4}{4x^2}.$$

Therefore,

$$1 + \left(\frac{dy}{dx}\right)^2 = \frac{16x^4 + x^8 + 16 - 8x^4}{16x^4}.$$

Hence, the arc length is given by

$$\int_1^4 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_1^4 \frac{x^4+4}{4x^2}\,dx = F(4)-F(1) = 6,$$

where

$$F(x) = \frac{x^3}{12} - \frac{1}{x}.$$

9.5 Surface area of revolution

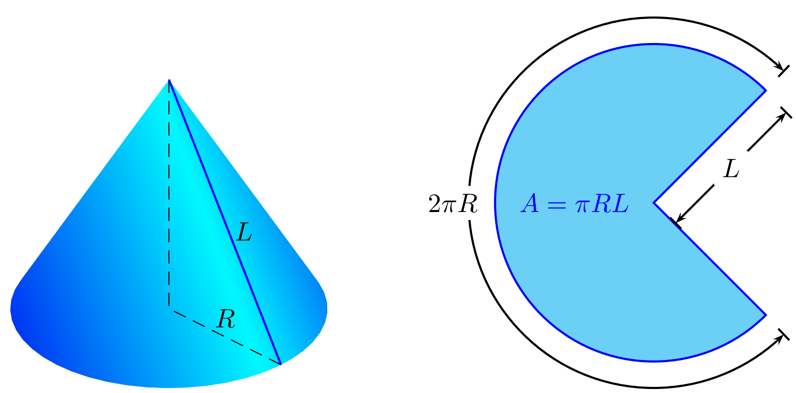
We know that the surface area of the unit sphere is 4π. In order to compute the area, we view the sphere as the rotation of $y=\sqrt{1-x^2}$, $-1 \le x \le 1$, about the x-axis.

In general, we would like to ask: *How can we compute the surface area of revolution?*

Let's consider some simple cases:

A **cone** is the revolution of a segment whose initial point is on the axes. Its surface area is $A = \pi RL$. To see this, recall that the area of a sector of a circle is given by $\frac{1}{2}s^2\theta$ where s is the radius of the circle and θ is the angle in radians subtended by the arc at the centre of the circle. In the above case, the angle θ is found by the ratio $2\pi R/L$. This implies that the surface area of the cone is

$$\frac{1}{2}L^2\frac{2\pi R}{L} = \pi RL.$$



A **frustum of a cone** is the revolution of a segment. (We may view circular cylinder and cone as special cases.) It is also the difference of two cones. By using the formula for the surface area of cone, we observe that the area of the frustum is given by

$$\pi RL - \pi r(L - \Delta L) = \pi L(R - r) + \pi r \Delta L.$$

Note that

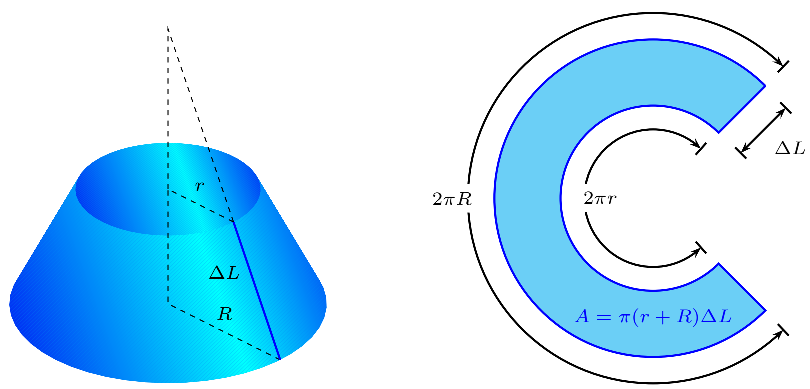
$$\frac{R}{L} = \frac{r}{L - \Delta L}$$

or

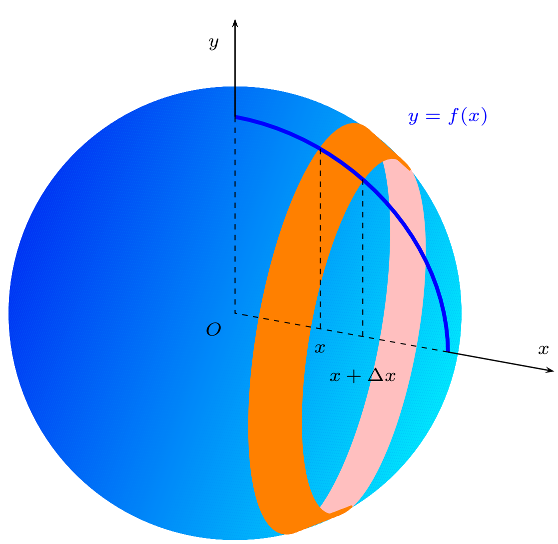
$$L(R - r) = R\Delta L,$$

Therefore, the area of the frustum is given by

$$\pi(R + r)\Delta L.$$



We now consider the general situation: Let f be a smooth function. Then the surface area of revolution of $y = f(x)$ about the x-axis can be approximated by that of frustums of a cone.



We see that $\Delta A = \pi(f(x) + f(x+\Delta x))\,\Delta L$, where $\Delta L = \sqrt{(\Delta x)^2 + (\Delta y)^2}$. Then

$$\begin{aligned} \frac{dA}{dx} = \lim_{\Delta x \to 0} \frac{\Delta A}{\Delta x} &= \pi \lim_{\Delta x \to 0} (f(x) + f(x+\Delta x)) \cdot \lim_{\Delta x \to 0} \frac{\Delta L}{\Delta x} \\ &= \pi \cdot 2f(x) \cdot \sqrt{1 + (f'(x))^2}. \end{aligned}$$

Therefore we have

THEOREM 9.4 Let f be a smooth function. The surface area of the surface obtained by rotating the curve $y = f(x)$, $a \le x \le b$, about the x-axis is

$$A = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\,dx.$$

EXAMPLE 9.5

Find the area of the surface generated by rotating the curve $y = 2\sqrt{x}$, $1 \le x \le 2$, about the x-axis.

Solution

We first evaluate the indefinite integral:

$$\begin{aligned} \int 4\pi\sqrt{x}\sqrt{1 + (y')^2}\,dx &= 4\pi \int \sqrt{x}\sqrt{1 + \frac{1}{x}}\,dx \\ &= 4\pi \int \sqrt{x+1}\,dx \\ &= 4\pi \frac{(1+x)^{3/2}}{3/2}. \end{aligned}$$

Therefore, the surface area is given by

$$\int_1^2 4\pi\sqrt{x}\sqrt{1 + (y')^2}\,dx = \frac{8\pi}{3}\left(3^{3/2} - 2^{3/2}\right).$$

9.6 Applications of integrals: Solving certain differential equations

DEFINITION 9.1 A first-order differential equation is an equation

$$\frac{dy}{dx} = f(x, y)$$

in which $f(x, y)$ is a function of of two variables defined on a region in the xy-plane. The equation is of first-order because it involves only the first derivative dy/dx and not higher derivatives.

A solution of a first order differential equation is a differential equation $y = y(x)$ such that

$$\frac{dy(x)}{dx} = f(x, y(x)).$$

In general, it is not easy to find solutions for a first order differential equation. We can, however, find solutions for special classes of differential equations.

9.6.1 Separable equations

The equation $y' = f(x, y)$ is **separable** if f can be expressed as a product of a functions of x and a function of y. The differential equation then has the form

$$\frac{dy}{dx} = g(x)h(y).$$

We can rewrite this as

$$\frac{1}{h(y)}\frac{dy}{dx} = g(x).$$

This yields, after using integration by substitution, we find that

$$\int \frac{1}{h(y)}dy = \int g(x)dx.$$

After completing the integrations, we obtain the solution y defined implicitly as a function of x.

EXAMPLE 9.6

Solve the differential equation $\dfrac{dy}{dx} = (1 + y)e^x$.

Solution

From the differential equation, we deduce that

$$\frac{1}{1+y}\frac{dy}{dx} = e^x.$$

This implies that

$$\int \frac{1}{1+y}\,dy = \int e^x dx.$$

Therefore,

$$\ln(1+y) = e^x + k.$$

EXAMPLE 9.7

One model for the way diseases die out when properly treated assumes that the rate dy/dt at which the number of infected people changes is proportional to the number y. Suppose that in the course of any given year the number of cases of a disease is reduced by 20%. If there 10 000 cases today, how many years will it take to reduce the number to 1000?

Solution

Let y be the number of people infected with the disease at time t. Then

$$\frac{dy}{dt} = ky.$$

This implies that

$$\ln y = kt + C.$$

Therefore,

$$y = y_0 e^{kt}.$$

At time $t = 0$, $y(0) = 10000$. At time $t = 1$, $y(1) = 8000$, because 2000 are cured. Therefore, $8000 = 10000e^k$ implies that $k = \ln 0.8 < 0$.

Let t^* be the number of years needed for y to hit 1000. This implies that $1000 = 10000e^{(\ln 0.8)t^*}$. Therefore,

$$t^* = \ln 0.1/\ln 0.8 = 10.3188\cdots.$$

So it takes about 10 years for the number to hit 1000.

10 Series

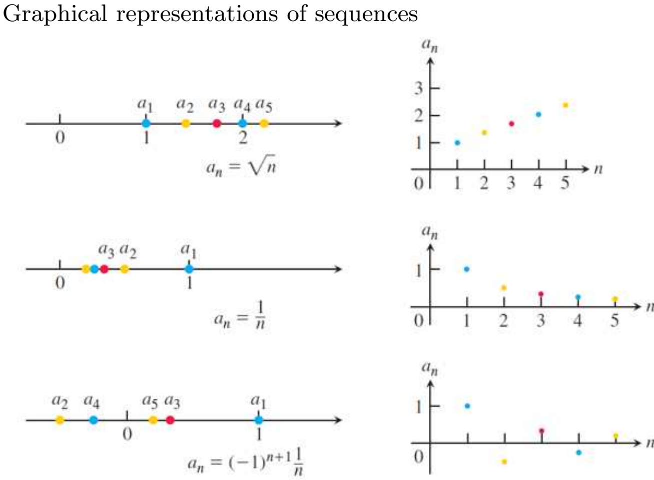
10.1 Sequences

DEFINITION 10.1 An **infinite sequence** of numbers is a function whose domain is the set of positive integers.

We usually denote a sequence by $(a_n)_{n=1}^{\infty}$ (or simply (a_n) when the reference to n is clear).

Sequences can be represented as points on the real line or as points on the Cartesian plane where the horizontal axis n is the index number of the term and the vertical axis a_n is its value. The following are examples of such representations:

Figure 10.1 Graphical representations of sequences



DEFINITION 10.2 The sequence (a_n) **converges** to the number L if for every positive number ϵ, there corresponds an integer N (which depends on ϵ) such

that

$$|a_n - L| < \epsilon \text{ whenever } n\, N.$$

If no such number L exists, we say that (a_n) **diverges**. If (a_n) converges to L, we write

$$\lim_{n\to\infty} a_n = L$$

or simply $a_n \to L$ and call L the limit of the sequence.

If for every $M > 0$, there exists an integer N (depending on M) such that

$$a_n > M \text{ whenever } n > N,$$

then we say that the sequence diverges to ∞ and write

$$\lim_{n\to\infty} a_n = \infty.$$

Similarly, if for every $M > 0$, there exists an integer N (depending on M) such that

$$a_n < -M \text{ whenever } n > N,$$

then we say that the sequence diverges to $-\infty$ and write

$$\lim_{n\to\infty} a_n = -\infty.$$

EXAMPLE 10.1

Show, using the definition of the limit of a sequence, that

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Solution

Let $\epsilon > 0$. Let $N = \left\lfloor \frac{1}{\epsilon} \right\rfloor + 1$. If $n > N$, then $n \geq \left\lfloor \frac{1}{\epsilon} \right\rfloor + 2 > \frac{1}{\epsilon}$. This implies that

$$\frac{1}{n} < \epsilon \text{ whenever } n > N$$

and thus,

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Not all sequences converge. The sequence

$$((-1)^n)$$

is an example of a divergent sequence because the sequence takes on the values 1 and -1 infinitely often.

10.1.1 Calculating Limits of Sequences

This following theorem gives a shortcut to evaluate the limit of some sequences using the limit of functions.

THEOREM 10.1 Let f be function, and (a_n) be a sequence such that $f(n) = a_n$ for all n. If $\lim\limits_{x\to\infty} f(x) = L$, then $\lim\limits_{n\to\infty} a_n = L$.

Here is a sketch of the proof of the above theorem.

Proof
Since $f(x) \to L$ as $x \to \infty$, we know that for every $\epsilon > 0$, there exists M such that if $x > M$, then $|f(x) - L| < \epsilon$. Suppose n is a positive integer greater than $N = \lfloor M \rfloor + 1$. Then $n > N$ implies that $n > M$ and this implies that $|f(n) - L| < \epsilon$. Since $a_n = f(n)$, we conclude that $a_n \to L$ as $n \to \infty$. □

EXAMPLE 10.2

Evaluate $\lim\limits_{n\to\infty} \dfrac{\ln n}{n}$.

Solution

It suffices to compute $\lim\limits_{x\to\infty} \dfrac{\ln x}{x}$. But by L'ôpital's Rule,

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0.$$

EXAMPLE 10.3

Evaluate $\lim_{n\to\infty}\left(\frac{n+1}{n-1}\right)^n$.

Solution

It suffices to compute $\lim_{x\to\infty}\left(\frac{x+1}{x-1}\right)^x$. Let $y=\left(\frac{x+1}{x-1}\right)^x$. Then $\ln y = x\ln((x+1)/(x-1))$. Write

$$\ln y = \frac{\ln((x+1)/(x-1))}{1/x}.$$

Since the numerator and denominator both tend to 0 as $x\to\infty$, we may apply L'Hôpital's rule and deduce that

$$\begin{aligned}\lim_{x\to\infty}\ln y &= \lim_{x\to\infty}\frac{\ln(x+1)-\ln(x-1)}{1/x}\\ &= \lim_{x\to\infty}\frac{1/(x+1)-1/(x-1)}{-1/x^2}\\ &= \lim_{x\to\infty}(-x^2)\frac{-2}{x^2(1-(1/x))(1+1/x)} = 2.\end{aligned}$$

Therefore $\lim_{x\to\infty} y = e^2$ and hence, $\lim_{n\to\infty}\left(\frac{n+1}{n-1}\right)^n = e^2$.

EXAMPLE 10.4

Show that $\lim_{n\to\infty}\left(1+\frac{\nu}{n}\right)^n = e^\nu$.

Solution

It suffices to compute $\lim_{x\to\infty}\left(1+\frac{\nu}{x}\right)^x$. Let $y=\left(1+\frac{\nu}{x}\right)^x$. Then $\ln y = x\ln(1+\nu/x)$. This implies that

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty}\frac{\ln(1+\nu/x)}{1/x} = \lim_{x\to\infty}(-x^2)\frac{1}{1+\nu/x}\frac{\nu}{x^2} = \nu.$$

Therefore, $\lim_{x\to\infty} y = e^\nu$ and this implies that $\lim_{n\to\infty}\left(1+\frac{\nu}{n}\right)^n = e^\nu$.

EXAMPLE 10.5

Let $0 < \nu < 1$. Show that $\lim\limits_{n\to\infty} \nu^n = 0$.

Solution

It suffices to evaluate $\lim\limits_{x\to\infty} \nu^x$. Let $y = \nu^x$. Then $\ln y = x \ln \nu$. Since $\nu < 1$ and ν is positive, we deduce that $\ln \nu < 0$. Therefore $\lim\limits_{x\to\infty} \ln y = -\infty$. Hence, $\lim\limits_{x\to\infty} y = 0$.

10.2 Limit laws for sequences

If (a_n) and (b_n) are convergent sequences and c is a constant, then similarly as functions, we have

$$\begin{aligned}
\lim_{n\to\infty} ca_n &= c \lim_{n\to\infty} a_n,\\
\lim_{n\to\infty} (a_n \pm b_n) &= \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n,\\
\lim_{n\to\infty} a_n b_n &= \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n,\\
\lim_{n\to\infty} \frac{a_n}{b_n} &= \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}, \quad \text{if } \lim_{n\to\infty} b_n \neq 0.
\end{aligned}$$

THEOREM 10.2 (Squeeze Theorem for Sequence) If $a_n \le b_n \le c_n$ for all n and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$, then $\lim\limits_{n\to\infty} b_n = L$.

EXAMPLE 10.6

Show that $\lim\limits_{n\to\infty} \dfrac{\cos n}{n} = 0$.

Solution

We observe that

$$-1 \le \cos n \le 1$$

and that

$$\frac{-1}{n} \le \frac{\cos n}{n} \le \frac{1}{n}.$$

Since $\lim_{n\to\infty} \pm\frac{1}{n} = 0$, we conclude, by the squeeze theorem for sequences, that

$$\lim_{n\to\infty} \frac{\cos n}{n} = 0.$$

EXAMPLE 10.7

Let ν be a positive real number. Show that $\lim_{n\to\infty} \frac{\nu^n}{n!} = 0$.

Solution

There exists an integer N such that $\nu < N$ (take for example, $N = \lfloor \nu \rfloor + 1$). Suppose $n > N$ is an integer. Write $n = N + m$, $m > 1$. Observe that

$$\frac{\nu^n}{n!} = \frac{\nu^N}{N!}\frac{\nu}{N+1} \cdot \frac{\nu}{N+m}.$$

Since $1/(N+j) < 1/N$ when $j > 1$, we conclude that

$$0 \leq \frac{\nu^n}{n!} \leq \frac{\nu^N}{N!}\left(\frac{\nu}{N}\right)^m. \tag{10.1}$$

By Example 10.5, since $\nu/N < 1$, we deduce that $\left(\frac{\nu}{N}\right)^m \to 0$ as $m \to \infty$. Since $n \to \infty$ implies that $m \to \infty$, we conclude that $\lim_{m\to\infty} \frac{\nu^N}{N!}\left(\frac{\nu}{N}\right)^m = 0$. By (10.1) and the squeeze theorem for sequences, we conclude that $\lim_{n\to\infty} \frac{\nu^n}{n!} = 0$.

10.2.1 Bounded Monotonic Sequences

DEFINITION 10.3 A sequence (a_n) is **nondecreasing** if $a_n \leq a_{n+1}$ for all positive integers n. A sequence (a_n) is **nonincreasing** if $a_n \geq a_{n+1}$ for all positive integers n. The sequence (a_n) is **monotonic** if it is either nondecreasing or nonincreasing.

EXAMPLE 10.8

The sequence $\left(\dfrac{n}{n+1}\right)_{n=1}^{\infty}$ is nondecreasing. The sequence $(1/n)_{n=1}^{\infty}$ is nonincreasing.

DEFINITION 10.4 A sequence (a_n) is **bounded from above** if there exists a number M such that $a_n \leq M$ for all positive integers n. The number M is an **upper bound** for (a_n). If M is an upper bound for (a_n) but no number less than M is an upper bound for (a_n), then M is called the **least upper bound** for (a_n). Similarly, A sequence (a_n) is **bounded from below** if there exists a number m such that $m \leq a_n$ for all positive integers n. The number M is an **lower bound** for (a_n). If m is a lower bound for (a_n) but no number greater than m is a lower bound for (a_n), then m is called the **greatest lower bound** for (a_n).

EXAMPLE 10.9

The sequence $(n)_{n=1}^{\infty}$ has no upper bound. The sequence $(n/(n+1))_{n=1}^{\infty}$, on the other hand, has an upper 1. It can be shown, using the definition of the limit of sequence, that the least upper bound for $(n/(n+1))_{n=1}^{\infty}$ is 1.

The following theorem holds for nondecreasing sequences. A similar result holds for non-increasing sequences.

THEOREM 10.3 (The Monotonic Sequence Theorem) A nondecreasing sequence of real numbers converges if and only if it is bounded from above. If a nondecreasing sequence converges, it converges to its least upper bound.

REMARK 10.1 A proof of the above theorem can be found on page 55 of Rudin's Princples of Mathematical Analysis.

EXAMPLE 10.10

The sequence $(n/(n+1))_{n=1}^{\infty}$ is nondecreasing and bounded above by 1. By the Monotonic Sequence Theorem, the sequence converges. Indeed, we can compute directly using the fact that $f(n) = a_n$ where $f(x) = x/(x+1)$ that $a_n \to 1$ as $n \to \infty$.

10.3 Series

Given a sequence $(a_n)_{n=1}^{\infty}$ we can construct a new sequence defined by

$$S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n.$$

(S_n) is called the sequence of **partial sums**. We write its limit as

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n. \tag{$\dagger$}$$

The quantity $(\dagger)$ is called an **infinite series** or in short a **series**.

A series is called **convergent** if its corresponding sequence of partial sums (S_n) is convergent, and it is called **divergent** otherwise.

EXAMPLE 10.11

Show that the geometric series $\sum_{j=1}^{\infty} ar^{j-1}$ converges to $a/(1-r)$ when $|r| < 1$ and diverges otherwise.

Solution

The n-th of the sequence of partial sum (S_n) where the term is (ar^{n-1}) takes the form

$$S_n = a\frac{1-r^n}{1-r}.$$

If $r > 1$ then $r^n = e^{(\ln r)n} \to \infty$ as $n \to \infty$ since $\ln r > 0$. If $0 < r < 1$ then $r^n = e^{(\ln r)n} \to 0$ as $n \to \infty$ since $\ln r < 0$. Therefore S_n diverges if $r > 1$ and converges to $a/(1-r)$ if $0 < r < 1$. Equivalently we say that the geometric series diverges for $r > 1$ and converges to $a/(1-r)$ for $0 < r < 1$. For $r = 0$, the series contains only one term and it converges to a.

For $r < -1$, $|r| > 1$. This means that $r = -|r|$ and $r^n = (-1)^n |r|^n$. Therefore $r^n \to \infty$ if n is even and $-\infty$ when n is odd. Thus, the limit of S_n does not exists.

For $-1 < r < 0$, we use the inequality

$$-|r|^n \le r^n \le |r|^n$$

to conclude that $r_n \to 0$ as $n \to \infty$ by the squeeze theorem for sequences. Therefore, $S_n \to a/(1-r)$ as $n \in \infty$. This completes our study of the geometric series with common ratio r.

REMARK 10.2 The proof that $r^n \to 0$ if $|r|^n \to 0$ as $n \to \infty$ may be generalized to $a_n \to 0$ if $|a_n| \to 0$ as $n \to \infty$. The proof is the same as for the case of $(r^n)_{n=1}^{\infty}$ for $|r| < 1$.

Here are some basic properties of series:

EXAMPLE 10.12

Show that the series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n(n+1)}$ is convergent and find its sum.

Solution

We find that

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Therefore,

$$S_n = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1}.$$

Since $S_n \to 1$ as $n \to \infty$, we conclude that the series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n(n+1)}$ converges to 1.

The next result lists the properties of series.

THEOREM 10.4 If $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$ are convergent series, then so are the series $\sum\limits_{n=1}^{\infty} ca_n$ (where c is a constant) and $\sum\limits_{n=1}^{\infty} (a_n + b_n)$. Moreover,

(a) $\sum\limits_{n=1}^{\infty} ca_n = c \sum\limits_{n=1}^{\infty} a_n$, and

(b) $\sum\limits_{n=1}^{\infty} (a_n + b_n) = \sum\limits_{n=1}^{\infty} a_n + \sum\limits_{n=1}^{\infty} b_n$.

THEOREM 10.5 (If the series $\sum_{n=1}^{\infty} a_n$ is convergent then $\lim_{n\to\infty} a_n = 0$.)

Proof Suppose $\sum_{n=1}^{\infty} a_n = L$. Let $S_n = a_1 + \cdots + a_n$. Then $\lim_{n\to\infty} S_n = L$. Note that $S_n - S_{n-1} = a_n$ for all $n \geq 2$. Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = L - L = 0. \qquad \square$$

As a corollary of the above theorem, we deduce the following test for divergence, which is called the n-th term test.

THEOREM 10.6 (The Test for Divergence) If $\lim_{n\to\infty} a_n$ does not exists or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

EXAMPLE 10.13

Show that the series $\sum_{n=1}^{\infty} \frac{n}{n+1}$ is divergent using the n-th term test.

Solution

The term $n/(n+1) \to 1$ as $n \to \infty$. So $\lim_{n\to\infty} n/(n+1) \neq 0$ and therefore the series $\sum_{n=1}^{\infty} \frac{n}{n+1}$ is divergent.

REMARK 10.3 The test for divergence is **inconclusive** if $\lim_{n\to\infty} a_n = 0$. For example, there are series for which $a_n \to 0$ but $\sum_{n=1}^{\infty} a_n$ is divergent.

10.4 Integral test

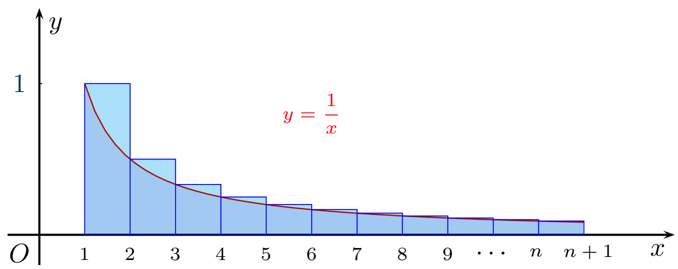
In general, it is difficult to determine if a given series in convergent. In the next few sections, we will discuss some methods that will enable us to test the convergence of certain series. We begin with the integral test.

EXAMPLE 10.14

The **harmonic series** $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Solution

Consider the following diagram:



Note that

$$\sum_{j=1}^{n} \frac{1}{j} \geq \int_{1}^{n+1} \frac{1}{t}\, dt = \ln(n+1).$$

Hence the series diverges since $\ln(n+1) \to \infty$ as $n \to \infty$.

THEOREM 10.7 (The Integral Test) [1] Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and that $a_n = f(n)$ for all n. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_{1}^{\infty} f(x)\, dx$ is convergent. In other words:

(i) If $\int_{1}^{\infty} f(x)\, dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\int_{1}^{\infty} f(x)\, dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Proof For each $i \in \mathbf{Z}^{+}$, $a_{i+1} \leq f(x) \leq a_i$ for all $x \in [i, i+1]$. Then

$$a_{i+1} \leq \int_{i}^{i+1} f(x)\, dx \leq a_i.$$

Let $S_n = a_1 + \cdots + a_n$. Adding the inequalities for $i = 1, \ldots, n$, we have

$$S_{n+1} - a_1 \leq \int_1^{n+1} f(x)\,dx \leq S_n.$$

If $\displaystyle\int_1^{\infty} f(x)\,dx$ is convergent, say, to L, then for each n,

$$S_n \leq \int_1^{n+1} f(x)\,dx + a_1 \leq \int_1^{\infty} f(x)\,dx + a_1 = L + a_1.$$

It follows that the sequence (S_n) is bounded above. Also note that $S_{n+1} - S_n = a_n > 0$ for all n. Then (S_n) is increasing.

By Theorem 10.3, (S_n) is convergent. In other words, $\sum\limits_{n=1}^{\infty} a_n$ is convergent.

If $\displaystyle\int_1^{\infty} f(x)\,dx$ is divergent, then $\displaystyle\int_1^{\infty} f(x)\,dx = \infty$.

Since $S_n > \displaystyle\int_1^{n+1} f(x)\,dx$ and $\displaystyle\lim_{n\to\infty} \int_1^{n+1} f(x)\,dx = \int_1^{\infty} f(x)\,dx = \infty$, we have $\displaystyle\lim_{n\to\infty} S_n = \infty$. Therefore, $\sum\limits_{n=1}^{\infty} a_n$ is divergent. □

EXAMPLE 10.15

Test the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$.

Solution

The integral

$$\int_1^M \frac{1}{x^2+1}\,dx = \tan^{-1} M - \tan^{-1} 1 \to \frac{\pi}{2} - \frac{\pi}{4} \text{ as } M \to \infty.$$

Therefore the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges.

THEOREM 10.8 (The p-series) The ***p*-series** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if and only if $p > 1$.

Proof
If $p > 1$, then

$$\int_1^{\infty} \frac{1}{t^p}\,dt = \lim_{M\to\infty} \frac{M^{1-p}}{1-p} - \frac{1}{1-p} = \frac{1}{p-1}.$$

If $p < 1$, then the integral $\int_1^\infty \frac{1}{t^p}\,dt$ diverges since $M^{1-p} \to \infty$ as $M \to \infty$. $\square$

By finding the limit directly, we can check the convergence of a geometric series; by integration, we could know whether a p-series is convergent or divergent. However, not all the series are easy in computation. So we shall search for more techniques.

In this section we are going to learn how to determine the convergence by comparing the terms of the series in question with those of a series whose convergence properties are known.

10.4.1 The comparison test (Reference: Section 9.4)

THEOREM 10.9 (The Comparison Test) Suppose $\sum a_n$ and $\sum b_n$ are series with **positive terms** such that $a_n \le b_n$ for all n.

(i) If $\sum b_n$ is convergent, then $\sum a_n$ is convergent;
(ii) If $\sum a_n$ is divergent, then $\sum b_n$ is divergent.

Proof Let $S_n = a_1 + \cdots + a_n$ and $T_n = b_1 + \cdots + b_n$.

Suppose $\sum b_n$ is convergent, i.e., (T_n) is convergent, say, to L. Then $S_n \le T_n \le \lim\limits_{n\to\infty} T_n = L$ is bounded above.

Since $S_{n+1} - S_n = a_n > 0$, (S_n) is increasing.

Therefore, by Monotonic Sequence Theorem (S_n) is convergent; that is, $\sum a_n$ is convergent. $\square$

EXAMPLE 10.16

Determine the convergence of the series $\sum\limits_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$.

Solution

Since

$$\frac{5}{2n^2+4n+3} \le \frac{5}{2n^2}$$

and the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, we deduce by comparison test that the series $\frac{5}{2n^2+4n+3} \le \frac{5}{2n^2}$ is convergent.

EXAMPLE 10.17

Determine the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$.

Solution

Observe that $2^{n-1} \geq 1$ for all $n \geq 1$. Therefore $2^n - 2^{n-1} \leq 2^n - 1$ for $n \geq 1$. Thus,

$$\frac{1}{2^n - 1} \leq \frac{1}{2^{n-1}}.$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ is convergent geometric series with common ratio $1/2$, we conclude by comparison test that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ is convergent.

10.4.2 The limit comparison test

THEOREM 10.10 (The Limit Comparison Test) Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, a positive constant, then either both series converge or both diverge.

REMARK 10.4 There are three parts to the limit comparison test but we will study only the most interesting part.

Proof Suppose $\lim_{n\to\infty} \frac{a_n}{b_n} = c \neq 0$. Then there is a number N such that

$$n > N \Rightarrow \left| \frac{a_n}{b_n} - c \right| < \frac{c}{2} \Rightarrow \frac{c}{2} b_n < a_n < \frac{3c}{2} b_n.$$

If $\sum a_n$ is convergent, then $\sum \frac{c}{2} b_n$ is convergent, so is $\sum b_n$.

If $\sum b_n$ is divergent, then $\sum \frac{3c}{2} b_n$ is divergent, so is $\sum b_n$.

$\square$

EXAMPLE 10.18

Show that $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ is convergent using limit comparison test.

Solution

Note that

$$\lim_{n\to\infty} \frac{\frac{1}{2^n}}{\frac{1}{2^n-1}} = 1,$$

and $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent, we conclude, by limit comparison test, that $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ is convergent.

EXAMPLE 10.19

Show that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(\sqrt{n}+1)}$ is convergent using limit comparison test.

Solution

The series is divergent because

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}(\sqrt{n}+1)}}{\frac{1}{n}} = 1$$

and $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

10.5 Absolute convergence and conditional convergence

DEFINITION 10.5 A series $\sum_{n=1}^{\infty} a_n$ is **absolutely convergent** if the series $\sum_{n=1}^{\infty} |a_n|$ converges.

EXAMPLE 10.20

The geometric series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n}$ is absolutely convergent since $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent.

DEFINITION 10.6 A series $\sum_{n=1}^{\infty} a_n$ that converges but does not converge absolutely is said to **converge conditionally**.

In other words, if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges, then we say that the series $\sum_{n=1}^{\infty} a_n$ converges conditionally.

EXAMPLE 10.21

The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is conditionally convergent.

Solution

First, if $\lim_{m\to\infty} S_{2m} = L$ and $\lim_{m\to\infty} S_{2m+1} = L$ then $\lim_{n\to\infty} S_n = L$. The above limits imply that for any $\epsilon > 0$, there exists a positive integer N such that

$$|S_{2m} - L| < \epsilon$$

and

$$|S_{2m+1} - L| < \epsilon$$

for $m > N$. This implies that

$$|S_n - L| < \epsilon$$

for all $n > 2N$. Therefore $\lim_{n\to\infty} S_n = L$.

Now, let

$$(S_n)_{n=1}^{\infty} = (1, 1-\frac{1}{2}, 1-\frac{1}{2}+\frac{1}{3}, 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}, \cdots).$$

$$0 < S_{2m} = \left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\frac{1}{2m-1}-\frac{1}{2m}.$$

From the above arrangement of S_{2m}, we note that S_{2m} is an increasing sequence. Next, we arrange S_{2m} in the following manner:

$$S_{2m} = 1-\left(\frac{1}{2}-\frac{1}{3}\right)-\left(\frac{1}{4}-\frac{1}{5}\right)+\cdots-\left(\frac{1}{2m-2}-\frac{1}{2m-1}\right)-\frac{1}{2m}.$$

The above shows that

$$S_{2m} \leq 1.$$

In other words, the sequence $(S_{2m})_{m=1}^{\infty}$ is a convergent sequence by Theorem 10.3. Let $L = \lim_{m\to\infty} S_{2m}$. Now,

$$S_{2m+1} = S_{2m} + \frac{1}{2m+1}.$$

Therefore,

$$\lim_{m\to\infty} S_{2m+1} = L + 0 = L.$$

By the remark in the beginning of the solution, we deduce that $\lim_{n\to\infty} S_n$ exists and therefore, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is convergent. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, we conclude that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is conditionally convergent.

Following exactly the steps in Example 10.21, we can provide a proof for the following theorem:

THEOREM 10.11 (The Alternating Series Test) If (a_n) is a sequence of positive numbers such that

(a) $a_{n+1} \leq a_n$ and
(b) $\lim_{n\to\infty} a_n = 0$,

then the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

is convergent.

EXAMPLE 10.22

Investigate the convergence of the series $\sum_{n=3}^{\infty}(-1)^n \frac{1}{n \ln n}$.

Solution

Note that $0 < n \ln n \leq (n+1)\ln(n+1)$ and therefore

$$\frac{1}{(n+1)\ln(n+1)} \leq \frac{1}{n \ln n}.$$

Furthermore,

$$\lim_{n\to\infty} \frac{1}{n \ln n} = 0.$$

Therefore, by alternating series test, $\sum_{n=3}^{\infty}(-1)^n \frac{1}{n \ln n}$ is convergent.

We have seen that a convergent series may not be absolutely convergent. It turns out that if a series is absolutely convergent, it is a convergent series.

THEOREM 10.12 If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Proof
The inequality

$$0 \leq a_n + |a_n| \leq 2|a_n|$$

and the comparison test implies that $\sum_{n=1}^{\infty}(a_n + |a_n|)$ is convergent. Now

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}((a_n + |a_n|) - |a_n|)$$

and therefore the series converges. □

EXAMPLE 10.23

The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ converges since its "absolute series" $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by integral test.

EXAMPLE 10.24

To show that the series $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges, we observe that

$$\left|\frac{\sin n}{n^2}\right| \leq \frac{1}{n^2}.$$

The series $\sum_{n=1}^{\infty} \left|\frac{\sin n}{n^2}\right|$ converges by the comparison test. Therefore, by Theorem 10.12, the series $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges.

10.6 The Ratio Test and The Root Test (Section 9.5)

THEOREM 10.13 (The Ratio Test) Suppose $\sum a_n$ is a series such that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$ (L is a finite number or ∞).

(a) If $0 \leq L < 1$, then $\sum a_n$ is absolutely convergent.
(b) If $L > 1$, then $\sum a_n$ is divergent.

Proof Let $L < 1$. Then there exists $\epsilon > 0$ such that $\ell = L + \epsilon < 1$. Since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, we deduce that for this $\epsilon > 0$, there exists a positive integer $N = N_\epsilon$ such that if $n \geq N$, then $||a_{n+1}/a_n| - L| < \epsilon$ or

$$|a_{n+1}| < (L+\epsilon)|a_n| = \ell|a_n|.$$

This implies that

$$|a_{N+1}| < \ell|a_N|, |a_{N+2}| < \ell|a_{N+1} < \ell^2|a_N|, \cdots .$$

In general, for integer $m \geq 1$, we find that

$$|a_{N+m}| < \ell^m|a_N|.$$

Since the series $\sum_{m=1}^{\infty} \ell^2$ for $\ell < 1$, we conclude by comparison test that the series $\sum_{j=N+m}^{\infty} |a_j|$ is convergent and therefore, $\sum_{j=1}^{\infty} |a_j|$ is convergent. This completes the proof of (a).

Suppose $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$. Let $\nu = L - \epsilon > 1$ for some $\epsilon > 0$. Then for this $\epsilon > 0$, there exists a positive integer $N = N_\epsilon$ such that if $n \geq N$, then $||a_{n+1}/a_n| - L| < \epsilon$, or $|a_{n+1}| > (L-\epsilon)|a_n| = \nu|a_n|$. Then

$$|a_{N+1}| > \nu|a_N|, |a_{N+2}| > \nu^2|a_N|, \cdots .$$

Therefore, for any integer $m \geq 1$,

$$|a_{N+m}| \geq \nu^m |a_N|.$$

This implies that

$$\lim_{m\to\infty} |a_{N+m}| \geq \lim_{m\to\infty} \nu^m |a_N|.$$

Since $\nu > 1$, we conclude that

$$\lim_{m\to\infty} \nu^m = \infty$$

and therefore, $\lim_{k\to\infty} |a_k| \neq 0$, which implies that $a_n \not\to 0$ as $n \to \infty$. By n-th term test, we conclude that $\sum_{j=1}^{\infty} a_j$ diverges and the proof of (b) is complete.

□

EXAMPLE 10.25

Investigate the convergence of the series $\sum_{n=1}^{\infty} \frac{2^n+5}{3^n}$.

Solution

Let $a_n = \frac{2^n+5}{3^n}$. Then

$$\frac{a_{n+1}}{a_n} = \frac{2+5/2^n}{1+5/2^n}\frac{1}{3} \to \frac{1}{3} < 1$$

as $n \to \infty$. By the Ratio Test, the series $\sum_{j=1}^{\infty} a_j$ converges absolutely.

EXAMPLE 10.26

Investigate the convergence of the series $\sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$.

Solution

Let $a_n = \dfrac{(2n)!}{n!n!}$. Then

$$\frac{a_{n+1}}{a_n} = \frac{(2n+2)(2n+1)}{(n+1)^2} \to 4 > 1$$

as $n \to \infty$. By the Ratio Test, the series $\sum_{j=1}^{\infty} a_j$ diverges.

THEOREM 10.14 (The Root Test) Suppose $\sum a_n$ is a series such that $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$ (L is a finite number or ∞).

(i) If $0 \le L < 1$, then $\sum a_n$ is absolutely convergent.
(ii) If $L > 1$, then $\sum a_n$ is divergent.

Proof Suppose $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$. Then there exists $\epsilon > 0$ such that $\ell = L+\epsilon$. For this $\epsilon > 0$, there is an integer $N = N_\epsilon$ such that if $n > N$, then

$$|\sqrt[n]{|a_n|} - L| < \epsilon.$$

This implies that

$$\sqrt[n]{|a_n|} < L + \epsilon = \ell.$$

Therefore

$$|a_n| < \ell^n.$$

Since $\sum_{j=1}^{\infty} \ell^j$ is convergent as $\ell < 1$, by comparison test, $\sum_{j=1}^{\infty} |a_j|$ is convergent and the proof of (a) is complete.

Suppose $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$. Let $\epsilon > 0$ be such that $\nu = L - \epsilon > 1$. For this $\epsilon > 0$, there exists an integer $N = N_\epsilon$ such that if $n \ge N$, then

$$|\sqrt[n]{|a_n|} - L| < \epsilon,$$

which implies that

$$\sqrt[n]{|a_n|} > \nu = L - \epsilon > 1.$$

Therefore

$$|a_n| > \nu^n.$$

Since $\nu > 1$ and $\lim_{n\to\infty} \nu^n = \infty$, we deduce that $|a_n| \to \infty$ as $n \to \infty$. In other words, $a_n \not\to 0$ as $n \to \infty$ and by the n-th term test, the series $\sum_{j=1}^{\infty} a_j$ is divergent. $\square$

EXAMPLE 10.27

Test the convergence of the series

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n}.$$

Solution

Let $a_n = \dfrac{n^2}{2^n}$. Then

$$a_n^{1/n} = \frac{n^{1/n}}{2} \to \frac{1}{2} < 1$$

as $n \to \infty$. Therefore, by the root test, the series $\sum_{j=1}^{\infty} a_j$ is absolutely convergent.

EXAMPLE 10.28

Test the convergence of the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{n+1}\right)^n.$$

Solution

Let $a_n = \left(\dfrac{1}{n+1}\right)^n$. Then

$$a_n^{1/n} = \frac{1}{(n+1)} \to 0 < 1$$

as $n \to \infty$. Therefore, by the root test, the series $\sum_{j=1}^{\infty} a_j$ is absolutely convergent.

10.7 Power series

A **power series** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots ,$$

where x is a variable and the c_n's are constants called the **coefficients** of the series. For each fixed x, the power series is a series of numbers that we can test for convergence or divergence.

More generally, a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

is called a **power series centered at a** or a **power series about a**. We adopt the convention that $(x-a)^0 = 1$ even when $x = a$.

EXAMPLE 10.29

The series $\sum_{j=0}^{\infty} x^j$ is a geometric series and it is convergent when $|x| < 1$ and converges to $1/(1-x)$. An important point to note here is that the power series $1 + x + x^2 + \cdots$ when convergent actually defines the function $\dfrac{1}{1-x}$. We say that the power series coincides with $\dfrac{1}{1-x}$ when $|x| < 1$.

EXAMPLE 10.30

The series $\sum_{k=0}^{\infty} \left(\dfrac{-(x-2)}{2}\right)^k$ is convergent for $|x-2| < 2$. It converges to $\dfrac{2}{x}$. We say that $\sum_{k=0}^{\infty} \left(\dfrac{-(x-2)}{2}\right)^k$ is the **power series representation** about the center 2 of $\dfrac{2}{x}$ and the identification of $2/x$ with the power series holds when $|x-2| < 2$.

Solution

Let $b_k = \left(\frac{-(x-2)}{2}\right)^k$. Then if

$$\lim_{k\to\infty} \frac{|b_{k+1}|}{|b_k|} < 1,$$

then $\sum_{k=0}^{\infty} b_k$ converges absolutely. The above condition is equivalent that

$$|x-2| < 2.$$

THEOREM 10.15 For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, exactly one of the following possibilities holds:

(a) The series converges at $x = a$ only.
(b) The series converges for all x.
(c) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

The number R in case (c) is called the **radius of convergence** of the power series. By convention, the radius of convergence in is $R = 0$ in case (a) and $R = \infty$ in case (b). The **interval of convergence** of a power series is the interval consists of all values of x for which the series converges.

In some cases, we can compute R. For example, if $\lim_{n\to\infty} \sqrt[n]{|c_n|} = L$ or $\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = L$, where L is a real number or ∞, then $R = 1/L$.[2]

EXAMPLE 10.31

Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}.$$

Solution

[2] By convention, if $L = 0$ then $R = \infty$, and if $L = \infty$ then $R = 0$.

Let $c_k = \dfrac{(-3)^k}{\sqrt{l+1}}$. The radius of convergence

$$R = \lim_{k\to\infty} \frac{|c_k|}{|c_{k+1}|} = \frac{1}{3}.$$

The interval of convergence is $(-1/3, 1/3)$.

Suppose $\displaystyle\sum_{k=0}^{\infty} c_k(x-a)^k$ is convergent for $|x-a| < R$. Then we may define

$$P(x) = \sum_{k=0}^{\infty} c_k(x-a)^k, \quad |x-a| < R.$$

For example, the series $I(x) = \displaystyle\sum_{k=0}^{\infty} x^k$ is convergent for $|x| < 1$. This power series $I(x)$ has a closed form and it is given by $\dfrac{1}{1-x}$.

THEOREM 10.16 If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable on the interval $|x-a| < R$ and

(i) $f'(x) = \sum_{n=1}^{\infty} n\, c_n(x-a)^n$,

(ii) $\displaystyle\int f(x)\, dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C.$

EXAMPLE 10.32

Find the power series expansion of $1/(1-x)^2$ about 0.

Solution

The series

$$\frac{1}{1-x} = \sum_{j=0}^{\infty} x^j.$$

This implies that

$$\frac{1}{(1-x)^2} = \sum_{j=1}^{\infty} j x^{j-1}.$$

EXAMPLE 10.33

Show that

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k} \quad \text{for } |x| < 1.$$

Solution

Note that

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots = \sum_{j=0}^{\infty} (-1)^j x^j.$$

Therefore,

$$\int_0^x \frac{1}{1+t}\, dt = \sum_{j=0}^{\infty} \frac{(-1)^j}{j+1} x^{j+1}.$$

10.8 Taylor and Maclaurin series

By using Theorem 10.16 we can deduce the following:

Let $f(x)$ be a known function and suppose it has a power series expansion about a of the form $\sum_{k=0}^{\infty} c_k (x-a)^k$. Write $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$. Let $x = a$. This gives $f(a) = c_0$. So we know the value of c_0. Next, $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$. Again, let $x = a$, we conclude that $f'(a) = c_1$. From these few cases, we can guess that $f^{(k)}(x) = k!c_k + (k+1)k \cdot \cdots \cdot 2(x-a) + (k+2)(k+1) \cdot \cdots \cdot 3(x-a)^2 + \cdots$. This yields $f^{(k)}(a) = k!c_k$ or $c_k = \dfrac{f^{(k)}(a)}{k!}$

Let $f(x)$ be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the **Taylor series generated by f at $x = a$** is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k.$$

The **Maclaurin series generated by $f(x)$** is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k.$$

It can be shown that the Taylor series and Maclaurin series converges to $f(x)$ in their respective regions of convergence. The proof is done by using Mean Value Theorem. For more details, see See Section 9.9, page 599.

EXAMPLE 10.34

Find the Taylor series generated by $f(x) = 1/x$ at $a = 2$.

Solution

We observe that

$$\frac{1}{x} = \frac{1}{2+(x-2)} = \frac{1}{2}\frac{1}{1+(x-2)/2} = \frac{1}{2}\sum_{k=0}^{\infty} \frac{(-1)^k (x-2)^k}{2^k}.$$

If we know the Taylor series of known functions, we can use them to evaluate certain limits.

EXAMPLE 10.35

Use Power series to evaluate

$$\lim_{x\to 1} \frac{\ln x}{x-1}.$$

Solution

The function $\ln x$ has the following series about 1:

$$\ln x = \ln(1+(x-1)) = (x-1) - \frac{(x-1)^2}{2} + \cdots.$$

Then

$$\lim_{x\to 1} \frac{\ln x}{x-1} = \lim_{x\to 1} \left(1 - \frac{x-1}{2} + \cdots\right) = 1.$$