CS1231S: Discrete Structures

Tutorial #9: Counting and Probability I

(Week 11: 25 – 29 October 2021)

I. Discussion Questions

These are meant for you to discuss on the LumiNUS Forum. No answers will be provided.

D1. A box contains three blue balls and seven white balls. One ball is drawn, its colour recorded, and it is returned to the box. Then another ball is drawn and its colour is recorded as well.

(a) What is the probability that the first ball is blue and the second is white?
(b) What is the probability that both balls drawn are white?
(c) What is the probability that the second ball drawn is blue?

D2. Calculate
(a) the probability that a randomly chosen positive three-digit integer is a multiple of 6.
(b) the probability that a randomly chosen positive four-digit integer is a multiple of 7.

D3. In Tutorial 7 D3, you are asked to write down all possible functions $\{1,2,3\} \rightarrow \{4,5\}$. How many possible functions $f: A \rightarrow B$ are there if $|A| = n$ and $|B| = k$?

D4. Assuming that all years have 365 days and all birthdays occur with equal probability. What is the smallest value for n so that in any randomly chosen group of n people, the probability that two or more persons having the same birthday is at least 50%?

Write out the equation to solve for n and write a program to compute n.
(This is the well-known *birthday problem*, whose solution is counter-intuitive but true.)

II. Tutorial Questions

1. In a certain tournament, the first team to win four games wins the tournament. Suppose there are two teams A and B, and team A wins the first two games. How many ways can the tournament be completed?
(We will use possibility tree to solve this problem for now. In the next tutorial, we will approach this problem using combination.)

2. A pack of cards consists of 52 cards with 4 suits: spades (♠), hearts (♥), diamonds (♦) and clubs (♣). Each suit has 13 cards: 2, 3, 4, 5, 6, 7, 8, 9, Ten, Jack, Queen, King and Ace.

You draw a sequence of 5 cards from a pack of cards with replacement.

(a) How many sequences will have at least one Queen?

(b) How many sequences will have at least one Queen or one King?

Interpretation is 'important

1. In a certain tournament, the first team to win four games wins the tournament. Suppose there are two teams A and B, and team A wins the first two games. How many ways can the tournament be completed?
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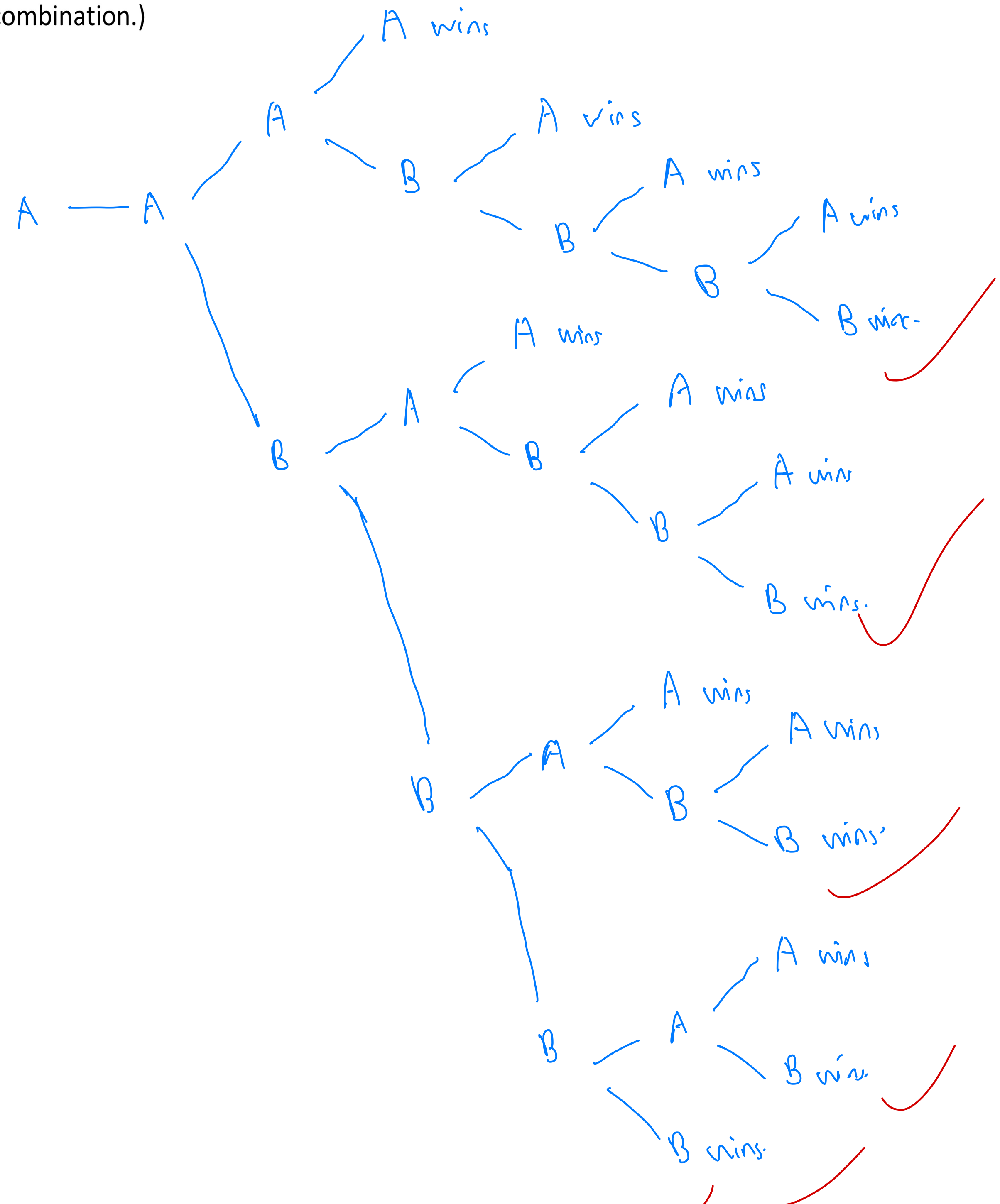

Total ways: 15

2. A pack of cards consists of 52 cards with 4 suits: spades (♠), hearts (♥), diamonds (♦) and clubs (♣). Each suit has 13 cards: 2, 3, 4, 5, 6, 7, 8, 9, Ten, Jack, Queen, King and Ace.

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(a) How many sequences will have at least one Queen?

(b) How many sequences will have at least one Queen or one King?

a) Total sequence − Sequences w/ no Queen

$52^5 - (52-4)^5$

$= 125400064$

Difference rule.

b) Total sequences − Sequences w/ no Queen or King

$52^5 - (52-8)^5$

$= 215287808$

Difference rule.

3. There are 789 CS students in SoC. Among them, 672 are taking CS1231S, 629 are taking CS1101S, 153 are taking MA1101R, 608 are taking CS1231S and CS1101S, 87 are taking CS1231S and MA1101R, 53 are taking CS1101S and MA1101R, and 46 are taking all three modules.

   How many CS students are not taking any of these three modules?

4. Among all permutations of n positive integers from 1 through n, where $n \geq 3$, how many of them have integers 1, 2 or 3 in the correct position?

   An integer k is in the correct position if it is at the k^{th} position in the permutation. For example, the permutation 3, $\underline{2}$, 4, 1, $\underline{5}$ has integers 2 and 5 in their correct positions, and the permutation 12, 1, $\underline{3}$, 9, 10, 8, $\underline{7}$, 6, 2, 4, $\underline{11}$, 5 has integers 3, 7, and 11 in their correct positions. Integers that are in their correct positions are underlined for illustration.

5. Given n boxes numbered 1 to n, each box is to be filled with either a white ball or a blue ball such that at least one box contains a white ball and boxes containing white balls must be consecutively numbered. What is the total number of ways this can be done?

   (For this tutorial, use sum of a sequence to solve this problem. In the next tutorial, we will revisit this problem using a different approach.)

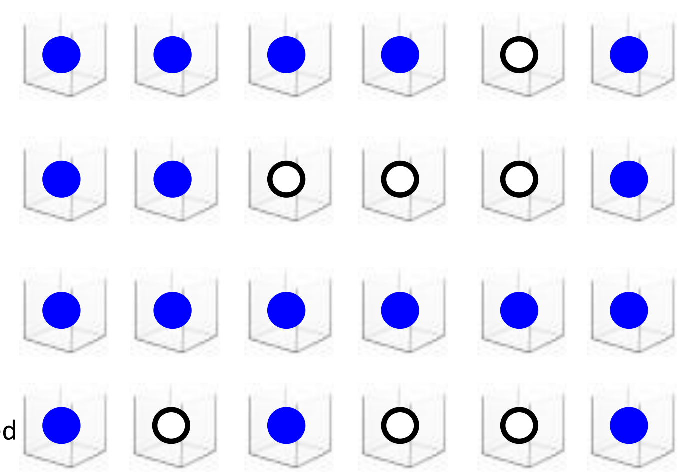
   Some examples for $n = 6$ are shown below for your reference.

Valid:

Valid:

Invalid:
(no white ball)

Invalid:
(white balls not numbered consecutively)



1 2 3 4 5 6

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How many CS students are not taking any of these three modules?

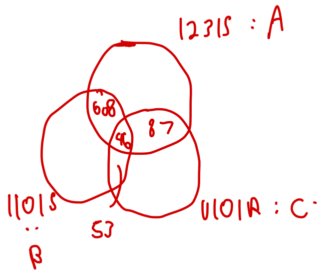
Inclusion exclusion principle.

1101R 1231S 1101S 583 1101R

~~153~~ 107 ~~672~~ 626 ~~629~~ ~~153~~ 107

~~87~~ 41 ~~608~~ 562. ~~53~~. 7

46



Not taking all 3 : 789 − 46
= 743.
−7
−41
−562
−59
−23
−14
= 37 ✓

1101R 1231S 1101S 1101R

107 626 583 107

41 562 7

100 626 576 100

59 585 576 59.

59 23 14

$|A\cup B\cup C| = |A|+|B|+|C| - |A\cap B| - |A\cap C| - |B\cap C| + |A\cap B\cap C|$

$= 752$

$|\bar{A}\cup\bar{B}\cup\bar{C}| = |\overline{(A\cup B\cup C)}| = |U| - |A\cup B\cup C|$

$= 37.$

tricky ∵ IEP.

4. Among all permutations of n positive integers from 1 through n, where $n \geq 3$, how many of them have integers 1, 2 $\boxed{\text{or}}$ 3 in the correct position?

An integer k is in the correct position if it is at the k^{th} position in the permutation. For example, the permutation 3, 2, 4, 1, 5 has integers 2 and 5 in their correct positions, and the permutation 12, 1, 3, 9, 10, 8, 7, 6, 2, 4, 11, 5 has integers 3, 7, and 11 in their correct positions. Integers that are in their correct positions are underlined for illustration.

Inclusion - Exclusion. rule.

$\underline{1}\ \underline{2}\ \underline{3}\ \underbrace{a\ b\ c}_{\text{any.}}\ d \dots$

len = n.

∴ $\underline{(n-3)!}$

Let P_k be permutation, with integer k in the correct position.

$|P_1| = |P_2| = |P_3| = (n-1)!$

$|P_1 \cap P_2| = |P_2 \cap P_3| = |P_1 \cap P_3| = (n-2)!$

$|P_1 \cap P_2 \cap P_3| = (n-3)!$

∴ $|P_1 \cup P_2 \cup P_3| = 3(n-1)! - 3(n-2)! + (n-3)!$
$= (3n^2 - 12n + 18)(n-3)!$

5. Given n boxes numbered 1 to n, each box is to be filled with either a white ball $\boxed{\text{or}}$ a blue ball such that at least one box contains a white ball and boxes containing white balls must be consecutively numbered. What is the total number of ways this can be done?

(For this tutorial, use sum of a sequence to solve this problem. In the next tutorial, we will revisit this problem using a different approach.)

Some examples for $n = 6$ are shown below for your reference.

$\frac{6!}{5!} \times 1! = 6.$

Valid:

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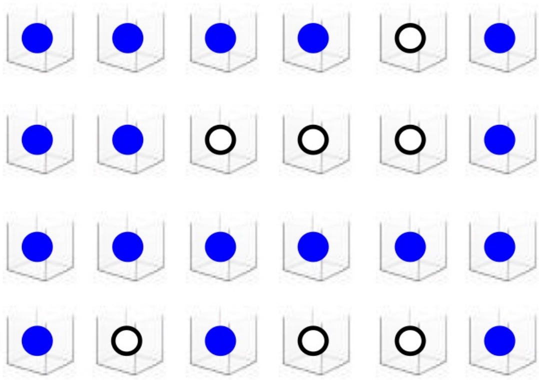
Invalid:
(no white ball)

Invalid:
(white balls not numbered consecutively)



1 2 3 4 5 6

balls are same.

$\dots + 6 + 5 + 4 + 3 + 2 + 1$

$= \sum_{k=1}^{n} (n-(k+1))$

Addition rule.

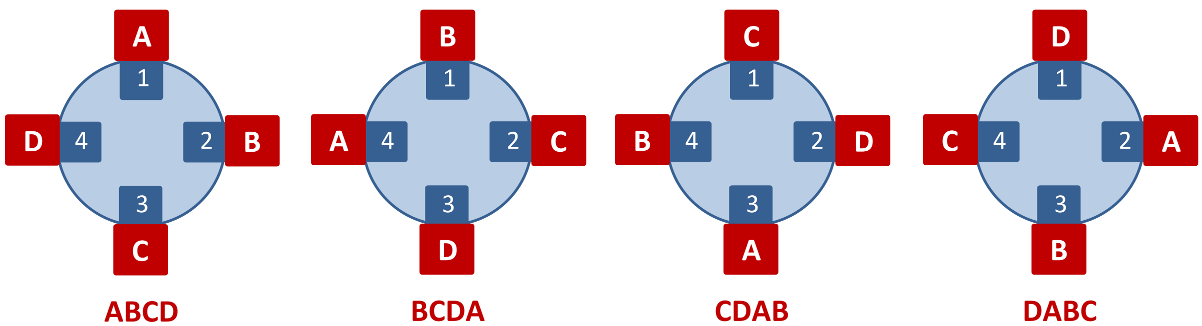
$= \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

6. [AY2020/21 Semester 2 exam question.]
Aaron loves Hawaii and is always looking forward to spending his holiday there. He is intrigued to hear that new bridges will be constructed on the islands of Maui, Moloka'i and Lana'i, making a total of 13 bridges between these islands. Each bridge connects two islands.

   If there are 11 bridges with one end on Maui and 6 with one end on Lana'i, (i) how many bridges are there connecting Maui and Moloka'i, and (ii) how many bridges are there connecting Moloka'i and Lana'i?

7. We have learned that the number of permutations of n distinct objects is $n!$, but that is on a straight line. If we seat four guests Anna, Barbie, Chris and Dorcas on chairs on a straight line they can be seated in 4! or 24 ways.

   What if we seat them around a circular table? Examine the figure below.



   The four seating arrangements (clockwise from top) *ABCD*, *BCDA*, *CDAB* and *DABC* are just a single permutation, as in each arrangement the persons on the left and on the right of each guest are still the same persons. Hence, these four arrangements are considered as one permutation.

   This is known as *circular permutation*. The number of linear permutations of 4 persons is four times its number of circular permutations. Hence, there are $\frac{4!}{4}$ or 3! ways of circular permutations for 4 persons. In general, the number of circular permutations of n objects is $(n-1)!$

   Answer the following questions:

   a. In how many ways can 8 boys and 4 girls sit around a circular table, so that no two girls sit together?

   b. In how many ways can 6 people sit around a circular table, but Eric would not sit next to Freddy?

   c. In how many ways can $n-1$ people sit around a circular table with n chairs?

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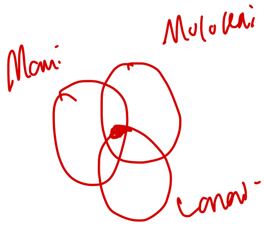

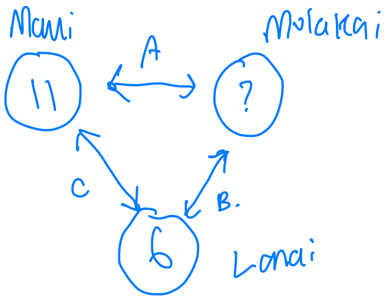

Sets

$|\text{Maui} \cap \text{Molokai}|$
$= |\text{Maui} \cup \text{Molokai} \cup \text{Lanai}| - |\text{Lanai}|$
$= 7.$

$|\text{Maui} \cap \text{Lanai}|$
$= |\text{Maui} \cup \text{Molokai} \cup \text{Lanai}| - |\text{Maui}|$
$= 2.$

$A + C = 11 \rightarrow A + 6 - B = 11$

$C + B = 6. \rightarrow C = 6 - B.$

$A + B + C = 13.$

$A + \cancel{B} + 6 - \cancel{B} = 13.$

$A + 6 = 13.$

$A = 13 - 6$
$= 7$

$C = 11 - A = 4.$

$\therefore B = 6 - C.$
$= 2$

Ans: i) $A = 7.$
ii) $B = 2$
$C = 4.$ //

7. We have learned that the number of permutations of n distinct objects is $n!$, but that is on a straight line. If we seat four guests Anna, Barbie, Chris and Dorcas on chairs on a straight line they can be seated in 4! or 24 ways.

Permutate separators first!

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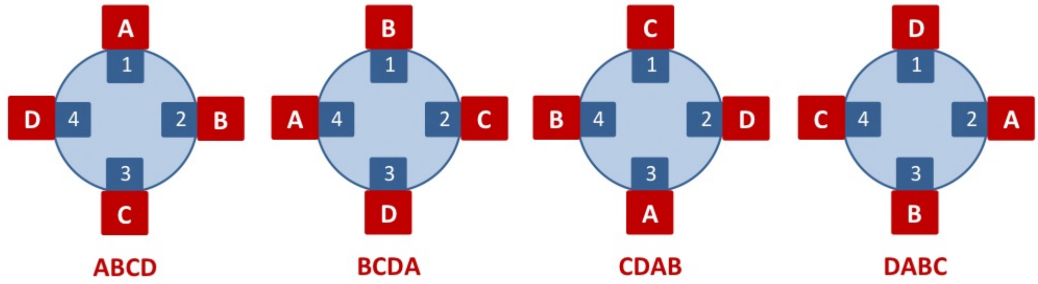

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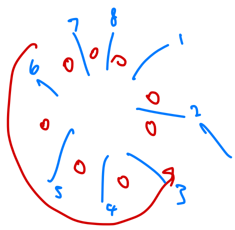
gaps are numbered cause 6 are distinct



a. No. of ways to permutate b = 7!

8 gaps. ∴ total no. of ways = $7! \times P(8,4)$ → $C(8,4) \times 4!$

$= 5040 \times 1680$

$= 8467200.$

Can permutate

b. $5! - (2 \times 4!)$

ordered btw E and F.

$= 72$

c. Treat empty chair as just another person: $\frac{(n-1)!}{1!}$

8. [Past year's exam question.]
   Prove that if you randomly put 51 points inside a unit square, there are always three points that can be covered by a circle of radius 1/7.

9. Let $S = \{3,4,5,6,7,8,9,10,11,12\}$. What is the smallest number of integers you must choose from S such that two of them sum to 15? In other words, what is the smallest $n \in \mathbb{Z}_{n\geq 2}$ such that for all subsets X of S where $|X| = n$, there exists two distinct elements $x, y \in X$ such that $x + y = 15$.

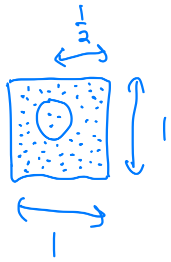
10. This is the famous chess master problem to illustrate the use of the Pigeonhole Principle. Try it out yourself before googling for the answer.

    A chess master who has 11 weeks to prepare for a tournament decides to play at least one game every day, but in order not to tire herself, she decides not to play more than 12 games during any one week. Show that there exists a succession of consecutive days during which the chess master will have played exactly 21 games.

Redo.

PHP.

8. [Past year's exam question.]
Prove that if you randomly put 51 points inside a unit square, there are always three points that can be covered by a circle of radius 1/7.



area is infinite

Pigeonhole Principle (PHP)

A function from one finite set to a smaller finite set cannot be one-to-one: There must be at least 2 elements in the domain that have the same image in the co-domain.

Generalized Pigeonhole Principle

For any function f from a finite set X with n elements to a finite set Y with m elements and for any positive integer k, if $k < n/m$, then there is some $y \in Y$ such that y is the image of at least $k + 1$ distinct elements of X.

Generalized Pigeonhole Principle (Contrapositive Form)

For any function f from a finite set X with n elements to a finite set Y with m elements and for any positive integer k,
if for each $y \in Y$, $f^{-1}(\{y\})$ has at most k elements,
then X has at most km elements; in other words, $n \leq km$.

1. Let area of circle be $\pi\left(\frac{1}{7}\right)^2 = \frac{\pi}{49}$. (by defn of area of circle).
Let area of unit square be $1 \cdot 1 = 1$ (by defn of unit square).

1.1 Then the set of all things in the circle is a smaller finite set than the set of all things in the unit square as area of circle < area of unit square.

1.2. By the PHP, the function from unit square to circle cannot be one-to-one.

1.3. Therefore there must be at least 2 elements in the domain that have the same image in the codomain.

1.4. Therefore, since $\frac{\pi}{49} / 1 = \frac{\pi}{49}$, there must exist at least $\frac{\pi}{49} \times 51$ points in the circle. (by basic algebra).

1.5. Hence, $\lfloor \frac{\pi}{49} \times 51 \rfloor = 3$ (by defn of floor function)

2. ∴ there are always 3 points in the circle

10 pigeons.

9. Let $S = \{3,4,5,6,7,8,9,10,11,12\}$. What is the smallest number of integers you must choose from S such that two of them sum to 15? In other words, what is the smallest $n \in \mathbb{Z}_{n\geq 2}$ such that for all subsets X of S where $|X| = n$, there exists two distinct elements $x, y \in X$ such that $x + y = 15$.

keyword

$X \subseteq S.$ $|X| = n$: smallest $n \in \mathbb{Z}_{\geq 2}$.

$\exists\, x, y \in X$ s.t. $x + y = 15$.

let $X = \{3, 12\} \rightarrow 3 + 12 = 15$.

$\therefore |X| = 2$

let $X = \{4, 11\} \rightarrow 4 + 11 = 15$.

$\therefore |X| = 2$

let $X = \{5, 10\} \rightarrow 5 + 10 = 15$.

$\therefore |X| = 2$

let $X = \{6, 9\} \rightarrow 6 + 9 = 15$.

$\therefore |X| = 2$.

let $X = \{7, 8\} \rightarrow 7 + 8 = 15$.

$\therefore |X| = 2$

holes.

Hence, in all cases, $n = 2$.

$n \leq 5$: no 2 elem sum to 15.

$n > 5$: 2 of them must be from same subset.

smallest n is 6.

10. This is the famous chess master problem to illustrate the use of the Pigeonhole Principle. Try it out yourself before googling for the answer.

A chess master who has 11 weeks to prepare for a tournament decides to play at least one game every day, but in order not to tire herself, she decides not to play more than 12 games during any one week. Show that there exists a succession of consecutive days during which the chess master will have played exactly 21 games.

PHP

11 weeks. → 77 days.

at least 1 game per day.

at most 12 games per week. → at most 132 games.

∃ succession of consecutive days (play exactly 21 games).

$D_1\ D_2\ D_3\ D_4\ \dots$ succession where 21 games were played. $\dots\ D_{69}\ D_{70}\ \dots\ D_{77}$

Pigeonhole Principle (PHP)

A function from one finite set to a smaller finite set cannot be one-to-one: There must be at least 2 elements in the domain that have the same image in the co-domain.

Generalized Pigeonhole Principle

For any function f from a finite set X with n elements to a finite set Y with m elements and for any positive integer k, if $k < n/m$, then there is some $y \in Y$ such that y is the image of at least $k + 1$ distinct elements of X.

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For any function f from a finite set X with n elements to a finite set Y with m elements and for any positive integer k, if for each $y \in Y$, $f^{-1}(\{y\})$ has at most k elements, then X has at most km elements; in other words, $n \leq km$.

7 1 1 1 1 1 1 1 a week.

12 2 2 2 2 2 1 1 at most 5 2s.

12 3 3 2 1 1 1 1 at most 2 3s.

12 4 2 2 1 1 1 1 at most 1 4.

12 4 3 1 1 1 1 1

12 5 2 1 1 1 1 1 at most 1 5.

12. 6 1 1 1 1 1 1 at most 1 6.

no bigger than 6.

min

max.

1. Since at most 12 games per week can be played, and while at least 1 game must be played per day;
at most 6 games can be played per day in one week.

2. In 11 weeks, at most 132 games can be played.

2.1 Suppose only 1 game is played per day for 77 days.

2.1.1: In any 21 consecutive days, the game is played exactly 21 times.

2.2. By PHP, since the finite set of at most 132 games can be played is larger than the finite set of exactly 21 games played.

2.3 the function from the larger set to the smaller set cannot be one-to-one.

2.4 ∴ There must be at least 2 image in the domain that have the same image in the codomain.

3. ∴ In any permutation of days, there exist a consecutive day for 21 games to be played.