Important Theorems:

1. Squeeze Theorem

$g(x) \le f(x) \le h(x)$

$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$

Then $\lim_{x \to c} f(x) = L$.

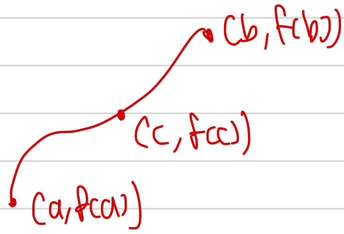
$\lim_{x \to 0} \frac{\sin x}{x} = 1.$

2. Intermediate Value Theorem

If f is continuous on a closed interval $[a, b]$

and y_0 is any value between $f(a)$ and $f(b)$

Then $f(c) = y_0$ for $c \in [a, b]$.



3. Extreme Value Theorem.

If f is continuous on $[a, b]$, then

f has both a absolute max M and a

absolute min m. in $[a, b]$.

$\exists\, x_1$ and x_2 in $[a, b]$ s.t. $f(x_1) = M$, $f(x_2) = m$

and $m \le f(x) \le M$ for all other x in $[a, b]$.

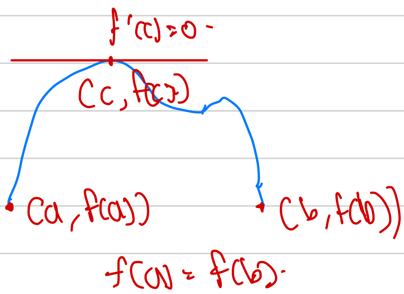
4. Rolle's Theorem.

$y = f(x)$ is continuous in $[a, b]$,

and differentiable at every point of its interior (a, b).

If $f(a) = f(b)$, then there is at least one number c

at which $f'(c) = 0$.



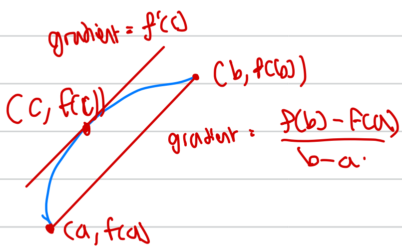
5. Mean Value Theorem.

$y = f(x)$ is continuous in $[a, b]$ and

differentiable on (a, b).

Then there is a point c in (a, b) s.t.

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

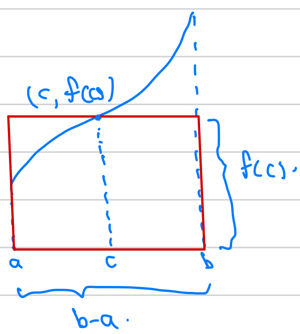


6. MVT for definite Integrals.

If f is continuous on $[a, b]$, then

at some point c in $[a, b]$,

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\,dx$$



$\therefore$ Area $= \int_a^b f(x)\,dx = f(c) \cdot (b-a)$.

1. $y = 2x^2 + 8x + 9.$

$\frac{dy}{dx} = 4x + 8 = 0$

$4x = 8$

$x = 2.$

$\therefore$ when $x=2$, $y = 2(2)^2 + 8(2) + 9$

$= 8 + 16 + 9.$

$= 17 + 16 = 33.$

2. $y = x^3 - 2x + 4.$

$\frac{dy}{dx} = 3x^2 - 2 = 0.$

$3x^2 = 2$

$x^2 = \frac{2}{3}$

$x = \pm\frac{2}{3}.$

when $x = \frac{2}{3}$,

$y = (\frac{2}{3})^3 - 2(\frac{2}{3}) + 4.$

$= \frac{80}{27}$

when $x = -\frac{2}{3}$,

$y = \frac{136}{27}$

3. $y = x^3 + x^2 - 8x + 5.$

$y' = 3x^2 + 2x - 8 = 0.$

$(x - \frac{4}{3})(x+2) = 0.$

$\therefore x = \frac{4}{3}$ or $x = -2$

$y = -\frac{41}{27}$ or $y = 17$ //

4. $y = x^3(x-5)^2$

$\frac{dy}{dx} = x^3(x-5)2(1) + 3x^2(x-5)^2.$

$\frac{dy}{dx} = 2x^3(x-5) + 3x^2(x-5)^2 = 0.$

$2x^4 - 10x^3 + 3x^2(x^2 - 10x + 5^2) = 0.$

$2x^4 - 10x^3 + 3x^4 - 30x^3 + 75x^2 = 0.$

$5x^4 - 40x^3 + 75x^2 = 0.$

$x^2(x^2 - 8x + 15) = 0.$

$x = 0$ or $x = 3$ or $x = 5.$

$y = 0$ or $y = -216$ or $y = 0.$

5. $y = \sqrt{x^2 - 1}$

$y' = \frac{2x}{2\sqrt{x^2-1}}$

$= \frac{x}{\sqrt{x^2-1}} = 0.$

$x = 0.$

$y = \sqrt{-1}.$

$y = i$?

6. $y = x - 4\sqrt{x}.$

$y' = 1 - 4\frac{1}{2\sqrt{x}}. = 0$

$2/\sqrt{x} = 1$

$\sqrt{x} = 2$

$x = 4.$

$y = 4 - 4\sqrt{4}$

$= 4 - 4(2)$

$= -4.$

7. $y = \frac{1}{\sqrt[3]{1-x^2}}.$

$y = (1-x^2)^{-\frac{1}{3}}.$

$= -\frac{1}{3}(1-x^2)^{-\frac{1}{3}-1}.(-2x)$

$= \frac{2}{3}\cdot(1-x^2)^{-\frac{4}{3}} x$ // $= 0.$

$x = 0.$

$y = 1.$

8. $y = \sqrt{3 + 2x - x^2}$

$y' = \frac{2 - 2x}{2\sqrt{3+2x-x^2}}.$

$= \frac{1-x}{\sqrt{3+2x-x^2}} = 0.$

$1 - x = 0.$

$-x = -1$

$x = 1.$

$y = \sqrt{4} = 2.$

9. $y = \dfrac{x}{x^2+1}$

$y' = \dfrac{(x^2+1)(1) - (2x)x}{(x^2+1)^2} = 0$

$x^2+1-2x^2 = 0$

$-x^2 = -1$

$x^2 = 1$

$x = \pm 1$

$y = \frac{1}{2}$ or $-\frac{1}{2}$.

10. $y = \dfrac{x+1}{x^2+2x+2}$

$= \dfrac{(x^2+2x+2)(1) - (2x+2)(x+1)}{(x^2+2x+2)^2} = 0$.

$x^2 + \cancel{2x} + \cancel{2} - (2x^2 + \cancel{2x} + 2x + \cancel{2}) = 0$

$-x^2 + 2x = 0$.

$x(-x+2) = 0$.

$x = 0$ or $-x+2 = 0$

$x = 2$.

$y = \frac{1}{2}$ or $y = \frac{3}{10}$

11. $y = x^3 + 2x + \tan x$

$y' = 3x^2 + 2 + \sec^2 x = 0$.

$3x^2 + \sec^2 x = -2$

There is no value of x such that

$3x^2 + \sec^2 x = -2$.

12. $y = \csc x + 2\cot x$.

No turning points from the graph

13. $y = (x+7)(11-3x)^{\frac{1}{3}}$

$y' = \frac{1}{3}(x+7)(11-3x)^{\frac{1}{3}-1}(-3) + (1)(11-3x)^{\frac{1}{3}}$

$= -(x+7)(11-3x)^{-\frac{2}{3}} + (11-3x)^{\frac{1}{3}} = 0$

$(11-3x)^{\frac{1}{3}}\left(-(x+7)(11-3x)^{-2}+1\right) = 0$

$(11-3x)^{\frac{1}{3}} = 0$ or $\dfrac{(x+7)}{(11-3x)^2} = 1$

$11-3x = 0$.

$-3x = -11$

$x = \frac{11}{3}$

$y =$

$x+7 = (11-3x)^2$

$x+7 = 11^2 - 2(11)(3x) + (3x)^2$

$x+7 = 11^2 - 22 \cdot 3x + 9x^2$

$0 = 11^2 - 7 - 67x + 9x^2$

$0 = 114 - 67x + 9x^2$

$x = 4.812$

or $x = 2.632$.

14. $y = \dfrac{ax+b}{x^2-1}$

when $x=3$,

$y=1$ extreme val.

$1 = \dfrac{a3+b}{3^2-1}$

$1 = \dfrac{3a+b}{8}$

$8 = 3a+b$. — ①

$y' = \dfrac{(x^2-1)(a) - 2x(ax+b)}{(x^2-1)^2} = 0$

$ax^2 - a - 2x^2a - 2xb = 0$.

$9a - a - 18a - 6b = 0$.

$-10a - 6b = 0$.

$10a + 6b = 0$. — ②

$a = \dfrac{32}{7}$

$b = -\dfrac{40}{7}$,

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9. $\int_{-2}^{2} 3f(x)\,dx = 12$

$\int_{-2}^{5} f(x)\,dx = 6.$

$\overbrace{\underbrace{-2 \quad 2}_{4} \quad 5}^{6}$

$\int_{-2}^{5} g(x)\,dx = 2.$

a) $\int_{-2}^{2} f(x)\,dx = \frac{12}{3} = 4.$ ✓

b) $\int_{2}^{5} f(x)\,dx = \int_{-2}^{5} f(x)\,dx - \int_{-2}^{2} f(x)\,dx.$
$= 6 - 4$
$= 2.$ ✓

c) $\int_{5}^{-2} g(x)\,dx = -2$ ✓

d) $\int_{-2}^{5} -\pi g(x)\,dx = -2\pi.$ ✓

e) $\int_{-2}^{5} \left(\frac{f(x)+g(x)}{5}\right)$

$= \frac{1}{5}\left(\int_{-2}^{5} f(x)\,dx + \int_{-2}^{5} g(x)\,dx\right)$

$= \frac{1}{5}(6+2)$

$= \cancel{4}.\ \frac{8}{5}$ ✓

10. $\int_{0}^{2} f(x)\,dx = \pi$
$\int_{0}^{2} 7g(x)\,dx = 7.$
$\int_{0}^{1} g(x)\,dx = 2$

a) $\int_{0}^{2} g(x)\,dx = 1$ ✓

b) $\int_{1}^{2} g(x)\,dx = 1 - 2 = -1$ ✓

c) $\int_{2}^{0} f(x)\,dx = -\pi.$ ✓

d) $\int_{0}^{2} \sqrt{2}\,f(x)\,dx = \pi\sqrt{2}.$ ✓

e) $\int_{0}^{2} (g(x) - 3f(x))\,dx.$
$= \int_{0}^{2} g(x)\,dx - 3\int_{0}^{2} f(x)\,dx$
$= 1 - 3\pi.$ ✓

11. $\int_{0}^{3} x^2 - 4x + 3\,dx$
$= \frac{x^3}{3} - \frac{4x^2}{2} + 3x \Big|_{0}^{3}$ ✓
$= \left(\frac{3^3}{3} - 2(3)^2 + 3(3)\right) - (0 - 0 + 0).$
$= 3^2 - 18 + 9.$
$= 0.$

Area $= \int_{0}^{1} x^2 - 4x + 3\,dx - \int_{1}^{3} (x^2 - 4x + 3)\,dx$
$= \frac{8}{3}.$

12. $\int_{-2}^{3} 1 - \frac{1}{4}\cdot x^2\,dx$

Area $= \int_{-2}^{2} \left(1 - \frac{x^2}{4}\right) dx - \int_{2}^{3} \left(1 - \frac{x^2}{4}\right) dx$
$= \frac{13}{4}.$

$= x - \frac{1}{4}\cdot\frac{x^3}{3} \Big|_{-2}^{3}$ ✓
$= \left(3 - \frac{1}{4}\cdot\frac{3^3}{3}\right) - \left(-2 - \frac{1}{4}\cdot\frac{(-2)^3}{3}\right)$
$= \frac{25}{12}$

13. $\int_{-1}^{8} 5 - 5x^{2/3}\,dx.$ ✓
$= 5x - 5\cdot\frac{x^{2/3+1}}{2/3+1} \Big|_{-1}^{8}.$
$= \left(5(8) - 5\cdot\frac{8^{2/3+1}}{2/3+1}\right) - \left(5(-1) - 5\cdot\frac{(-1)^{2/3+1}}{2/3+1}\right)$
$= -54.$

Area: $\int_{-1}^{1} (5 - 5x^{2/3})\,dx - \int_{1}^{8} (5 - 5x^{2/3})\,dx.$
$= 62.$

14. $\int_{0}^{4} 1 - \sqrt{x}.$
$= x - \frac{x^{1/2+1}}{1/2+1} \Big|_{0}^{4}.$ ✓
$= \left(4 - \frac{4^{1/2+1}}{1/2+1}\right) - 0.$
$= -\frac{4}{3}.$

$\int_{0}^{1} 1 - \sqrt{x}\,dx - \int_{1}^{4} (1 - \sqrt{x})\,dx$

15. $\int_1^2 x \, dx - \int_1^2 \frac{1}{x^2} \, dx.$

$= \frac{x^2}{2} \Big|_1^2 - \frac{x^{-2+1}}{-2+1} \Big|_1^2$

$= \left(\frac{2^2}{2} - \frac{1^2}{2}\right) - \left(\frac{2^{-1}}{-1} - \frac{1^{-1}}{-1}\right).$

$= \frac{3}{2} - \frac{1}{2}$

$= 1$ //

16. $\int_1^2 x \, dx - \int_1^2 \frac{1}{\sqrt{x}} \, dx.$

$= \frac{x^2}{2} \Big|_1^2 - \frac{x^{-1/2+1}}{-1/2+1} \Big|_1^2$

$= \left(\frac{2^2}{2} - \frac{1^2}{2}\right) - \left(\frac{2^{1/2}}{1/2} - \frac{1^{1/2}}{1/2}\right).$

$= \frac{3}{2} - \left(2\sqrt{2} - 2\right)$

$= \frac{7}{2} - 2\sqrt{2}$ //

17. $\sqrt{y} = 1 - \sqrt{x}$

$y = (1-\sqrt{x})^2.$

$\int_0^1 (1-\sqrt{x})^2 \, dx.$

$= \int_0^1 (1 - 2\sqrt{x} + x) \, dx.$

$= x - 2\frac{x^{1/2+1}}{1/2+1} + \frac{x^2}{2} \Big|_0^1$

$= \left(1 - \frac{2}{3/2} + \frac{1}{2}\right) - 0$

$= \frac{1}{6}.$

18. $x^3 + \sqrt{y} = 1.$

$\sqrt{y} = 1 - x^3.$

$y = (1-x^3)^2.$

$\int_0^1 (1-x^3)^2 \, dx$

$= \int_0^1 (1 - 2x^3 + (x^3)^2) \, dx.$

$= \int_0^1 (1 - 2x^3 + x^6) \, dx$

$= x - 2\frac{x^4}{4} + \frac{x^7}{7} \Big|_0^1$

$= \left(1 - 2 \cdot \frac{1}{4} + \frac{1}{7}\right) - 0$

$= \frac{9}{14}$ //

19. $\int_0^3 2y^2 \, dy.$

$= 2 \cdot \frac{y^3}{3} \Big|_0^3$

$= 2 \cdot \frac{3^3}{3} - 0.$

$= 18.$

20. $\int_{-2}^2 4 - y^2 \, dy.$

$= 4y - \frac{y^3}{3} \Big|_{-2}^2.$

$= \left(4(2) - \frac{2^3}{3}\right) - \left(4(-2) - \frac{(-2)^3}{3}\right).$

$= \frac{32}{3}$

21. $y^2 = 4x \rightarrow x = \frac{y^2}{4}.$

$y = 4x - 2. \rightarrow x = \frac{y+2}{4}.$

$\therefore \int_{-1}^2 \frac{y+2}{4} \, dy - \int_{-1}^2 \frac{y^2}{4} \, dy$

$= \frac{1}{4}\left(\frac{y^2}{2} + 2y\right)\Big|_{-1}^2 - \frac{1}{4}\left(\frac{y^3}{3}\right)\Big|_{-1}^2$

$= \frac{1}{4}\left(\frac{2^2}{2} + 2 \cdot 2 - \frac{(-1)^2}{2} - 2 \cdot (-1)\right) - \frac{1}{4}\left(\frac{2^3}{3} - \frac{(-1)^3}{3}\right)$

$= \frac{9}{8}.$

33. $\frac{d^2y}{dx^2} = 2 - \frac{1}{x^2}$ $y'(1) = 3$.

$y(1) = 1$.

$\frac{dy}{dx} = \int 2 - \frac{1}{x^2}\, dx$.

$= 2x - \frac{x^{-2+1}}{-2+1} + C$

$= 2x - \frac{x^{-1}}{-1} + C$.

$= 2x + \frac{1}{x} + C$

$2(1) + \frac{1}{1} + C = 3$. $\rightarrow$ $C = 3 - 2 - 1 = 0$.

$\frac{dy}{dx} = 2x + \frac{1}{x}$

$y = \int 2x + \frac{1}{x}\, dx$.

$y = \int 2x\, dx + \int \frac{1}{x}\, dx$.

$y = 2\frac{x^2}{2} + \int_1^x \frac{1}{t}\, dt + C$

$y = x^2 + \int_1^x \frac{1}{t}\, dt + C$

$(1)^2 + \int_1^1 \frac{1}{t}\, dt + C = 1$.

$C = 1 - 1 = 0$.

$\therefore y = x^2 + \int_1^x \frac{1}{t}\, dt$. //

34. $\frac{d^2y}{dx^2} = \sqrt{\sec x}\ \tan x$ $y'(0) = 3$

$y(0) = 0$.

$y = \int_0^x (1 + 2\sqrt{\sec t})\, dt$.

$\frac{dy}{dx} = \frac{dy}{dx} \int_0^x (1 + 2\sqrt{\sec t})\, dt$

$= 1 + 2\sqrt{\sec x}$

$\frac{d^2y}{dx^2} = \cancel{2}\ \frac{\sec x \tan x}{\cancel{2}\sqrt{\sec x}}$.

$= \frac{\sec x \tan x}{\sqrt{\sec x}}$.

$= \sqrt{\sec x}\ \tan x$.

35. $\frac{dy}{dx} = \frac{\sin x}{x}$ $y(5) = -3$.

$y = \int \frac{\sin x}{x}\, dx + C$

$y = \int_5^x \frac{\sin t}{t}\, dt - 3$

36. $\frac{dy}{dx} = \sqrt{2 - \sin^2 x}$

$y = \int_{-1}^x \sqrt{2 - \sin^2 t}\, dt + 2$

37. $\int 2 (\cos x)^{-1/2} \sin x\, dx$

$= 2 \int \frac{\sin x}{\sqrt{\cos x}}\, dx$.

$u = \sqrt{\cos x}$

$du = \frac{-\sin x}{2\sqrt{\cos x}}\, dx$

$\therefore -2 \cdot 2 \cdot \int -\frac{\sin x}{2\sqrt{\cos x}}\, dx$.

$= -2 \cdot 2 \cdot \int 1\, du$.

$= -2 \cdot 2 \cdot u$

$= -4 \cdot \sqrt{\cos x}$.

$= -4\sqrt{\cos x} + C$ //

38. $\int (\tan x)^{-3/2} \sec^2 x\, dx$.

$u = \tan x$.

$\frac{du}{dx} = \sec^2 x$.

$= \int (u)^{-3/2}\, du$

$= \frac{u^{-3/2+1}}{-3/2+1} + C$

$= -2(\tan x)^{-\frac{1}{2}} + C$.

39. $\int 2\theta + 1 + 2\cos(2\theta + 1)\, d\theta$

$= \frac{2\theta^2}{2} + \theta + 2\int \cos(2\theta + 1)\, d\theta$ $u = 2\theta + 1$, $du = 2$

$= \theta^2 + \theta + (\sin u) + C$

$= \theta^2 + \theta + \sin(2\theta + 1) + C$.

40. $\int \frac{1}{\sqrt{2\theta - \pi}} + 2\sec^2(2\theta - \pi)\, d\theta$.

$u = 2\theta - \pi$.

$du = 2\, dx$.

$= \frac{1}{2} \int \frac{1}{\sqrt{u}} + 2\sec^2 u\, du$.

$= \frac{1}{2} \int \frac{1}{\sqrt{u}}\, du + \int \sec^2 u\, du$.

$= \cancel{\frac{1}{2}} \cdot \frac{u^{-\frac{1}{2}+1}}{\cancel{2}\frac{1}{2}} + \tan u + C$.

$= \sqrt{u} + \tan u + C$

$= \sqrt{2\theta - \pi} + \tan(2\theta - \pi) + C$ //

41. $\int \left(t-\frac{2}{t}\right)\left(t+\frac{2}{t}\right) dt$

$= \int \left(t^2-\left(\frac{2}{t}\right)^2\right) dt$

$= \int t^2 dt - \int \left(\frac{2}{t}\right)^2 dt$

$= \frac{t^3}{3} - 4 \cdot \frac{t^{-2+1}}{-2+1} + C$

$= \frac{t^3}{3} + 4t^{-1} + C$

42. $\int \frac{(t+1)^2-1}{t^4} dt$

$= \int \frac{1}{t^4} \cdot [(t+1-1)(t+1+1)]$

$= \int \frac{1}{t^{4}} [\cancel{t}(t+2)]$

$= \int \frac{1}{t^3}(t+2)$

$= \int \frac{1}{t^2} dt + 2\int \frac{1}{t^3} dt$

$= \frac{t^{-2+1}}{-2+1} + 2 \cdot \frac{t^{-3+1}}{-3+1} + C$

$= -\frac{1}{t} - \frac{1}{t^2} + C$

43. $\int \sqrt{t} \sin(2t^{3/2}) dt$

$u = 2t^{3/2}$

$du = 2 \cdot \frac{3}{2} \cdot t^{\frac{1}{2}} dt$

$= 3\sqrt{t}\, dt$

$\therefore \frac{1}{3}\int \sin(u)\, du$

$= \frac{1}{3} \cdot -\cos u + C$

$= -\frac{1}{3}\cos(2t^{3/2}) + C$

44. $\int (\sec\theta \tan\theta)\sqrt{1+\sec\theta}\, d\theta$

$u = 1+\sec\theta$

$du = \sec\theta \tan\theta\, d\theta$

$= \int \sqrt{u}\, du$

$= \frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$

$= \frac{2}{3} \cdot u^{\frac{3}{2}} + C$

$= \frac{2}{3}(1+\sec\theta)^{\frac{3}{2}} + C$

45. $\int \frac{\sin 2\theta - \cos 2\theta}{(\sin 2\theta + \cos 2\theta)^3} d\theta$

$u = \sin 2\theta + \cos 2\theta$

$du = 2\cos 2\theta - 2\sin 2\theta\, d\theta$

$\therefore = -2(\sin 2\theta - \cos 2\theta)\, d\theta$

$-\frac{1}{2}\int \frac{1}{u^3} du$

$= -\frac{1}{2}\frac{u^{-3+1}}{-3+1} + C$

$= \frac{1}{4}u^{-2} + C$

$= \frac{1}{4}(\sin 2\theta + \cos 2\theta)^{-2} + C$

46. $\int \cos\theta \cdot \sin(\sin\theta)\, d\theta$

$u = \sin\theta$

$du = \cos\theta\, d\theta$

$= \int \sin u\, du$

$= -\cos u + C$

$= -\cos(\sin\theta) + C$

52. $\int_1^4 \frac{(1+\sqrt{u})^{1/2}}{\sqrt{u}} du$

$u=1, v=2$

$u=4, v=3$

$v = 1+\sqrt{u}$

$dv = \frac{1}{2\sqrt{u}} du$

change

$= 2\int_1^4 v^{1/2}\, dv$

$= 2 \cdot \frac{v^{1/2+1}}{1/2+1}\Big|_1^4$

$= \frac{4}{3}(4^{3/2} - 1^{3/2})$

$= \frac{28}{3}$ X

Tutorial Exercises

6: 1. a) $\int x\sqrt{4-x}\,dx$

$\cancel{u = \sqrt{4-x} \qquad dv = x}$

$\cancel{du = \frac{-1}{2\sqrt{4-x}} \qquad v = \frac{1}{2}x^2}$

$\cancel{= \frac{1}{2}x^2\sqrt{4-x} - \int \frac{1}{2}x^2 \cdot \frac{-1}{2\sqrt{4-x}}}$

$\cancel{+ \frac{1}{4}\int x^2 \cdot \frac{1}{\sqrt{4-x}}\,dx}$

$u = 4-x \qquad x = 4-u$

$du = -1\,dx$

$= -\int (4-u)\sqrt{u}\,du$

$= \int u\sqrt{u} - 4\sqrt{u}\,du$

$= \int u\sqrt{u}\,du - 4\int\sqrt{u}\,du$

$= \int u \cdot u^{\frac{1}{2}}\,du - 4\,\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$

$= \int u^{\frac{3}{2}}\,du - \frac{4}{\frac{3}{2}} \cdot u^{\frac{3}{2}} + C$

$= \frac{u^{\frac{3}{2}+1}}{\frac{3}{2}+1} - \frac{8}{3}u^{\frac{3}{2}} + C$

$= \frac{2}{5} \cdot u^{\frac{5}{2}} - \frac{8}{3}u^{\frac{3}{2}} + C$

$= \frac{2}{5}(4-x)^{\frac{5}{2}} - \frac{8}{3}(4-x)^{\frac{3}{2}} + C$

b) $\int \frac{1}{x^2} \sin\frac{1}{x} \cos\frac{1}{x}\,dx$

$u = \frac{1}{x}$

$du = -x^{-2}$

$= -\frac{1}{x^2}\,dx$

$= -\int \sin u \cos u\,du$

$= -\frac{1}{2}\int 2\sin u\cos u\,du$

$= -\frac{1}{2}\int \sin 2u\,du \qquad v = 2u,\ dv = 2\,du$

$= -\frac{1}{2} \cdot \frac{1}{2}\int \sin v\,dv$

$= +\frac{1}{4}\cos v + C$

$= \frac{1}{4}\cos 2u + C$

$= \frac{1}{4}\cos\frac{2}{x} + C$

2. $I = \int_0^a \frac{f(x)}{f(x) + f(a-x)}\,dx$

$u = a-x \qquad x = a-u$

$du = -dx$

$= \int_0^a \frac{f(a-u)}{f(a-u)+f(u)}\,du$

$= \int_0^a \frac{f(a-x)}{f(a-x)+f(x)}\,dx$

$I + I = \int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx + \int_0^a \frac{f(a-x)}{f(x)+f(a-x)}\,dx$

$= \int_0^a \frac{f(x) - f(a-x)}{f(x)+f(a-x)}\,dx$

$= \int_0^a 1\,dx$

$= a$

$I = \frac{a}{2}$

3. a) $\int x(\ln x)^2\,dx$

$u=(\ln x)^2 \qquad dv = x$

$du = 2\ln x\left(\frac{1}{x}\right) \qquad v=\frac{1}{2}x^2$

$= \frac{1}{2}x^2(\ln x)^2 - \int \cancel{\frac{1}{2}}x^{\cancel{2}} \cdot \cancel{2}\ln x \frac{1}{\cancel{x}}\,dx$

$= \frac{1}{2}x^2(\ln x)^2 - \int x\ln x\,dx$

$u=\ln x \qquad dv = x$

$du=\frac{1}{x} \qquad v=\frac{1}{2}x^2$

$= \frac{1}{2}x^2(\ln x)^2 - \left(\ln x\cdot\frac{1}{2}x^2 - \int \frac{1}{2}x^2\cdot\frac{1}{x}\,dx\right)$

$= \frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2\ln x + \frac{1}{2}\int x\,dx$

$= \frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2\ln x + \frac{1}{2}\cdot\frac{x^2}{2}+C$

$= \frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2\ln x + \frac{1}{4}x^2 + C$

b) $\int \tan^{-1}x\,dx$

$u=\tan^{-1}x \qquad dv=1$

$du=\frac{1}{1+x^2}\,dx \qquad v=x$

$= x\tan^{-1}x - \int\frac{x}{1+x^2}\,dx$

$u=1+x^2$

$du=2x\,dx$

$= x\tan^{-1}x - \frac{1}{2}\int\frac{1}{u}\,du$

$= x\tan^{-1}x - \frac{1}{2}\ln u + C$

$= x\tan^{-1}x - \frac{1}{2}\ln|1+x^2| + C$

6. $x=\int_0^y \frac{1}{\sqrt{1+4t^2}}\,dt$

$\frac{dx}{dy}=\frac{d}{dy}\int_0^y\frac{1}{\sqrt{1+4t^2}}\,dt$

$=\frac{1}{\sqrt{1+4y^2}}$

$\frac{dy}{dx}=\sqrt{1+4y^2}$

$\frac{d^2y}{dx^2}=\frac{1}{2\sqrt{1+4y^2}}\cdot 8y\cdot\frac{dy}{dx}$

$=\frac{4y}{\cancel{\sqrt{1+4y^2}}}\cdot\cancel{\sqrt{1+4y^2}}$

$\frac{d^2y}{dx^2}=4y$

$\frac{1}{y}\cdot\frac{d^2y}{dx^2}=4 \in$ constant

4. $\int\frac{1}{x(x^4+1)}\,dx$

$=\int\frac{x^3}{x^4(x^4+1)}\,dx$

$u=x^4$

$du=4x^3\,dx$

$=\frac{1}{4}\int\frac{1}{u(u+1)}\,du$

$=\frac{1}{4}\int\frac{A}{u}+\frac{B}{u+1}\,du$

$=\frac{1}{4}\int\frac{1}{u}-\frac{1}{u+1}\,du$

$=\frac{1}{4}\left(\int\frac{1}{u}\,du - \int\frac{1}{u+1}\,du\right)$

$=\frac{1}{4}\left(\ln u - \ln(u+1)\right)$

$=\frac{1}{4}\ln x^4 - \frac{1}{4}\ln(x^4+1)+C$

$=\ln x - \frac{1}{4}\ln(x^4+1)+C$

5. $\int\sec x\,dx$

$=\int\frac{1}{\cos x}\,dx$

$=\int\frac{\cos x}{\cos^2 x}\,dx$

$=\int\frac{\cos x}{1-\sin^2 x}\,dx$

$u=\sin x$

$du=\cos x\,dx$

$=\int\frac{1}{1-u^2}\,du$

$=\int\frac{1}{(1-u)(1+u)}\,du$

$=\int -\frac{1}{2(u-1)}+\frac{1}{2(u+1)}\,du$

$=-\frac{1}{2}\int\frac{1}{u-1}\,du+\frac{1}{2}\int\frac{1}{u+1}\,du$

$=-\frac{1}{2}\ln(u-1)+\frac{1}{2}\ln(u+1)+C$

$=\frac{1}{2}\left(\ln(\sin x+1)-\ln(\sin x-1)\right)+C$

$=\frac{1}{2}\left(\ln\left(\frac{\sin x+1}{\sin x-1}\right)\right)+C$

7. a) $-1\le\frac{2x}{1+x^2}\le 1$

let $y=\frac{2x}{1+x^2}$

$\therefore y'=\frac{(1+x^2)\cdot 2-2x\cdot(2x)}{(1+x)^2}=0$

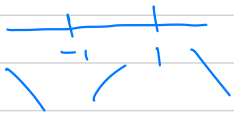
$2(1+x^2)-4x^2=0$

$2+2x^2-4x^2=0$

$-2x^2=-2$

$x^2=1$

$x=\pm 1$



$\therefore x=1$ is abs max,

$x=-1$ is abs min,

$-1\le\frac{2x}{1+x^2}\le 1$.

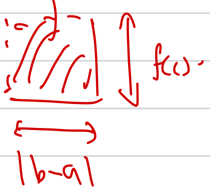
b) 6. MVT for definite Integrals.

If f is continuous on $[a,b]$, then at some point c in $[a,b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

And $\left|\frac{2x}{1+x^2}\right| \leq 1$

$\int_a^b \frac{2x}{1+x^2}\,dx$



$\int \frac{2x}{1+x^2}\,dx.$

$u = 1+x^2.$

$du = 2x.$

$= \int \frac{1}{u}\,du$

$= \ln u + C$

$= \ln(1+x^2) + C.$

$\therefore \int_a^b \frac{2x}{1+x^2}\,dx = \ln(1+x^2)\Big|_a^b.$

$= (\ln(1+b^2) - \ln(1+a^2))$ $\rightarrow f(c) = \frac{1}{b-a}(\ln(1+b^2) - \ln(1+a^2))$

$\therefore |f(c)| = \left|\frac{2c}{1+c^2}\right| \leq 1.$

$|\ln(1+b^2) - \ln(1+a^2)| \leq |b-a|.$

8. a) $\displaystyle\int \frac{1}{x\sqrt{x+9}}\,dx.$

b) $\displaystyle\int \frac{1}{x^6(x^5+4)}\,dx.$

a)

$u = x+9$

$x = u-9$

$du = 1\,dx.$

$$= \int \frac{1}{(u-9)\sqrt{u}}\,du.$$

let $v = \sqrt{u}$,

$dv = \frac{1}{2\sqrt{u}}\,du.$

$u = v^2.$

$$= 2\int \frac{1}{(v^2-9)}\,dv.$$

$$= 2\int \frac{1}{(v-3)(v+3)}\,dv.$$

$$= 2\int \frac{1}{6(v-3)} - \frac{1}{6(v+3)}\,dv.$$

$$= \frac{1}{3}\ln(v-3) - \frac{1}{3}\ln(v+3) + C$$

$$= \frac{1}{3}\ln(\sqrt{u}-3) - \frac{1}{3}\ln(\sqrt{u}+3) + C$$

$$= \frac{1}{3}\ln(\sqrt{x+9}-3) - \frac{1}{3}\ln(\sqrt{x+9}+3) + C$$

b)

$u = \frac{1}{x^5}$, $x^5 = \frac{1}{u}.$

$du = -5\frac{1}{x^6}\,dx.$

$$= \frac{1}{-5}\int \frac{1}{\frac{1}{u}+4}\,du.$$

$$= -\frac{1}{5}\int \frac{1}{\frac{4u+1}{u}}\,du.$$

$$= -\frac{1}{5}\int \frac{u}{4u+1}\,du.$$

$$= -\frac{1}{5\cdot 4}\int \frac{4u+1-1}{4u+1}\,du.$$

$$= -\frac{1}{20}\int \frac{4u+1}{4u+1} - \frac{1}{4u+1}\,du.$$

$$= -\frac{1}{20}\left(\int 1\,du - \int \frac{1}{4u+1}\,du\right)$$

$$= -\frac{1}{20}\left(u - \frac{\ln(4u+1)}{4}\right)$$

$$= -\frac{1}{20}x^{-5} + \frac{1}{80}\ln(4x^{-5}+1) + C.$$

9. a) $\displaystyle\int_2^{\infty} \frac{2}{t^2-1}\,dt.$

$$= \lim_{k\to\infty}\int_2^{k} \frac{2}{t^2-1}\,dt$$

$$= 2\int \frac{1}{(t-1)(t+1)}\,dt$$

$$= \int \frac{1}{\cancel{2}(t-1)} - \frac{1}{\cancel{2}(t+1)}\,dt$$

$$= \int_2^{k} \frac{1}{(t-1)}\,dt - \int_2^{k} \frac{1}{(t+1)}\,dt.$$

$$= \ln(t-1)\Big|_2^{k} - \ln(t+1)\Big|_2^{k}$$

$$= \left|\ln(k-1) - \overset{0}{\cancel{\ln(2-1)}}\right| - \left(\ln(k+1) - \ln(2+1)\right)$$

$$= \ln(k-1) - \ln(k+1) + \ln 3.$$

As $k\to\infty$,

$\to \ln 3.$ Since $\ln\left(\frac{k-1}{k+1}\right) \to 0.$

b) $\int_0^1 (-\ln x)\,dx$

$\lim_{k\to 0} \int_k^1 -\ln x\,dx$

$u = \ln x \quad dv = -1$

$du = \frac{1}{x} \quad v = -x$

$= -x\ln x - \int -\cancel{x}\frac{1}{\cancel{x}}\,dx$

$= -x\ln x + \int 1\,dx$

$= -x\ln x + x \Big|_k^1$

$= -(1)(\ln(1)) + (1) - (-k\ln k + 0)$

$= 1 + k\ln k.$

As $k \to 0^+$,

$k\ln k \to 0.$

$\therefore = 1.$

5: 1. $f(x) = \cos x$

By MVT,

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$-\sin c = \frac{\cos b - \cos a}{b-a}$$

Suppose the interval $[0, x]$.

Then,

$$f'(c) = \frac{f(x)-f(0)}{x-0}$$

$$f'(c) = \frac{f(x)-f(0)}{x}$$

$$f'(c) = \frac{\cos x - 1}{x}$$

$$\sin c = \frac{\cos x - 1}{x}$$

Since $|\sin c| \le 1$,

$$\left|\frac{\cos x - 1}{x}\right| \le 1$$

$|\cos x - 1| \le |x|$ shown

Suppose the interval $[-x, 0]$

Then,

$$f'(c) = \frac{f(0)-f(-x)}{0-(-x)}$$

$$= \frac{\cos(0) - \cos(-x)}{x}$$

$$= \frac{1-\cos(-x)}{x}$$

$$\sin c = \frac{1-\cos(x)}{x}$$

Since, $|\sin c| \le 1$,

then, $\left|\frac{1-\cos x}{x}\right| \le 1$

$|1-\cos x| \le |x|$

$|\cos x - 1| \le |x|$ shown

$\therefore$ For $[-x, x]$, $|\cos x - 1| \le |x|$

2. $f(x) = \frac{1}{x}$ $\quad b > a > 0$.

$c \in (a, b)$.

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$= \frac{\frac{1}{b}-\frac{1}{a}}{b-a}$$

$$= \left(\frac{1}{b}-\frac{1}{a}\right) \cdot \frac{1}{b-a}$$

$$= \frac{a-b}{ab} \cdot \frac{-1}{a-b}$$

$$-\frac{1}{c^2} = -\frac{1}{ab}$$

$$c^2 = ab$$

$$c = \sqrt{ab}$$

3. $f(x) = \frac{ax+b}{x^2-1}$ $\quad (3, 1) \rightarrow$ extreme value

$$f(3) = \frac{3a+b}{9-1} = 1$$

$$3a+b = 8 \quad - (1)$$

$$f'(x) = \frac{(x^2-1)(a) - (ax+b)(2x)}{(x^2-1)^2} = 0$$

$$a(x^2-1) - 2x(ax+b) = 0$$

$x = 3$.

$$a(8) - 2(3)(a3+b) = 0$$

$$8a - 6(3a+b) = 0$$

$$8a - 6(8) = 0$$

$8a = 48 \rightarrow a = 6, b = -10$

4. a) $\lim_{x\to\infty} \left(\frac{x+2}{x-1}\right)^x$.

let $y = \left(\frac{x+2}{x-1}\right)^x$.

$\ln y = x \ln\left(\frac{x+2}{x-1}\right)$.

$= \dfrac{\ln\left(\frac{x+2}{x-1}\right)}{\frac{1}{x}}$

$\lim_{x\to\infty}\left[\dfrac{\ln\left(\frac{x+2}{x-1}\right)}{\frac{1}{x}}\right]$

$= \lim_{x\to\infty}\left(\dfrac{\ln\left(\frac{1+\frac{2}{x}}{1-\frac{1}{x}}\right)}{\frac{1}{x}\to 0}\right)$

$\therefore$ By l'Hôpital's rule

$= \lim_{x\to\infty}\left(\dfrac{\frac{1}{x+2}-\frac{1}{x-1}}{-\frac{1}{x^2}}\right)$

$\frac{1}{x+2}-\frac{1}{x-1}\cdot(-x^2)$

$= (-x^2)\dfrac{x-1-x-2}{(x+2)(x-1)}$

$= \dfrac{3x^2}{(x+2)(x-1)}$

$= \lim_{x\to\infty} \dfrac{3x^2}{x^2-x+2x-2}$

x^2+x-2

$= \lim_{x\to\infty} \dfrac{3x^2}{x^2+x-2}\cdot\dfrac{\frac{1}{x^2}}{\frac{1}{x^2}}$

$= \lim_{x\to\infty} \dfrac{3}{1+\frac{1}{x}\to 0-\frac{2}{x^2}\to 0}$

$= 3$.

$\therefore \lim_{x\to\infty} \ln y = 3$.

$\ln\left(\lim_{x\to\infty} y\right) = 3$.

$\lim_{x\to\infty} y = e^3$.

b) $\lim_{x\to\infty} \dfrac{2^x-3^x}{3^x+4^x}\cdot\dfrac{\frac{1}{3^x}}{\frac{1}{3^x}}$

$= \lim_{x\to\infty}\dfrac{\frac{2^x}{3^x}-1}{1+\frac{4^x}{3^x}}$

$= \lim_{x\to\infty}\dfrac{\left(\frac{2}{3}\right)^x\to 0-1}{1+\left(\frac{4}{3}\right)^x\to\infty}$

$= 0$.

If $0<r<1$,

then $\lim_{x\to\infty} r^x = 0$

$\lim_{x\to\infty} r^x = \lim_{x\to\infty} e^{x\ln r}$

Since $r<1$, $\ln r < \ln 1 = 0$.

$\therefore \lim_{x\to\infty} e^{x\ln r} = 0$.

If $r>1$, $\ln r > \ln 1 = 0$

$\therefore \lim_{x\to\infty} r^x = \lim_{x\to\infty} e^{x\ln r} = \infty$.

5. a) Riemann integral.

$0 \le \sin x \le 1$. for $[0,1]$

$\therefore \int_0^1 \sin x \, dx = \int_0^1 1 \, dx = 1$

b) $\int_0^1 \sin x \, dx \le \int_0^1 x \, dx$.

$= \frac{x^2}{2} \Big|_0^1 = \frac{1}{2}$

6. a) $\int_{-\pi/3}^{\pi/3} \sin^2 t \, dt$.

$= \int 1 - \cos^2 t \, dt$ $\qquad 2\cos^2\theta - 1 = \cos 2\theta$

$= \int 1 - \frac{\cos 2t + 1}{2} dt$ $\qquad \cos^2\theta = \frac{\cos 2\theta + 1}{2}$

$= \int 1 \, dt - \frac{1}{2}\left(\int \cos 2t \, dt + \int 1 \, dt \right)$.

$= \frac{1}{2} t - \frac{1}{4} \sin 2t + C$.

$\therefore \int_{-\pi/3}^{\pi/3} \sin^2 t \, dt$

$= \frac{1}{2} t - \frac{1}{4} \sin 2t \Big|_{-\pi/3}^{\pi/3}$

$= \left(\frac{1}{2}\left(\frac{\pi}{3}\right) - \frac{1}{4} \sin\left(\frac{2}{3}\pi\right) \right) - \left(\frac{1}{2}\left(-\frac{\pi}{3}\right) - \frac{1}{4} \sin\left(-\frac{2}{3}\pi\right) \right)$

$= \frac{\pi}{6} - \frac{\sqrt{3}}{2} \cdot \frac{1}{4} + \frac{\pi}{6} - \frac{\sqrt{3}}{2} \cdot \frac{1}{4}$

$= \frac{\pi}{3} - \frac{\sqrt{3}}{4}$ //

b) $\int_{-3}^{-1} \frac{y^5 - 2y}{y^3} dy$.

$\int \frac{y^5 - 2y}{y^3} dy$.

$= \int \frac{y^5}{y^3} - 2\frac{y}{y^3} dy$.

$= \int y^2 \, dy - 2 \int \frac{1}{y^2} dy$.

$= \frac{y^3}{3} - 2 \cdot \frac{y^{-2+1}}{-2+1}$

$= \frac{y^3}{3} + \frac{2}{y} + C$.

$\therefore \frac{y^3}{3} + \frac{2}{y} \Big|_{-3}^{-1}$

$= \frac{(-1)^3}{3} + \frac{2}{-1} - \frac{(-3)^3}{3} - \frac{2}{-3}$.

$= \frac{22}{3}$ //

7. a) $f(x) = x \int_a^{x^2} \sin(t^3) dt$.

let $H(x) = \int_a^x \sin(t^3) dt$

Then $H'(x) = \sin x^3$

$f(x) = x \cdot H(x^2)$

$f'(x) = \frac{dy}{dy}\left(x \cdot H(x^2) \right)$

$= x \cdot H'(x^2) + H(x^2)$

$= x \cdot \sin x^6 \cdot (2x) + \int_a^{x^2} \sin(t^3) dt$.

$= 2x^2 \sin x^6 + \int_a^{x^2} \sin(t^3) dt$.

b) $g(x) = \int_0^{\sin x} \frac{1}{\sqrt{1-t^2}} dt$, $|x| < \frac{\pi}{2}$.

let $H(x) = \int_0^x \frac{1}{\sqrt{1-t^2}} dt$

then $H'(x) = \frac{1}{\sqrt{1-x^2}}$

$\therefore g(x) = H(\sin x)$

$g'(x) = H'(\sin x) \cdot \cos x$

$= \frac{1}{\sqrt{1-\sin^2 x}} \cdot \cos x$.

$= \frac{\cos x}{\cos x}$

$= 1$

8. a) $f(x) = ax^2 + 2bx + c$

$f'(x) = 2ax + 2b = 0.$

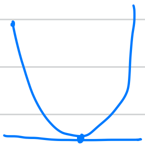
$2ax = -2b.$

$x = -\frac{b}{a}.$

$f''(x) = 2a.$

when $x = -\frac{b}{a}$, $f''(x) > 0.$

$\therefore$ $f(x)$ has absolute minimum at $x = -\frac{b}{a}.$



$x = -\frac{b}{a}$

$\therefore$ Since abs min is when $x = -\frac{b}{a}$,

for $f(x) \geq 0$, then.

$$a \cdot \left(-\frac{b}{a}\right)^2 + 2b\left(-\frac{b}{a}\right) + c \geq 0.$$

$$\frac{b^2}{a} - \frac{2b^2}{a} + c \geq 0.$$

$$-\frac{b^2}{a} + c \geq 0.$$

$$-b^2 + ac \geq 0.$$

or $b^2 - ac \leq 0$ //

(other side) * Suppose $b^2 - ac \leq 0.$

Then, $c - \frac{b^2}{a} \geq 0$

Since $f(x)$ has abs max at $x = -\frac{b}{a}.$

Then $f(x) \geq f\left(-\frac{b}{a}\right) = c - \frac{b^2}{a} \geq 0.$

$f(x) \geq 0 \iff b^2 - ac \leq 0.$

b) $f(x) = (a_1x + b_1)^2 + (a_2x + b_2)^2 + \dots + (a_nx + b_n)^2$

$f(x) = ((a_1x)^2 + 2a_1xb_1 + b_1^2) + \dots$

$= \underbrace{(a_1^2 + \dots + a_n^2)}_{A}x^2 + 2x\underbrace{(a_1b_1 + \dots + a_nb_n)}_{B} + \underbrace{(b_1^2 + \dots + b_n^2)}_{C}$

$= Ax^2 + 2Bx + C.$

$f(x) \geq 0$, Since $(a_ix + b_i)^2 \geq 0$ for all $x \in \mathbb{R}$.

From (a),

$B^2 - AC \leq 0.$

$B^2 \leq AC$

or $(a_1b_1 + \dots + a_nb_n)^2 \leq (a_1^2 + \dots + a_n^2)(b_1^2 + \dots + b_n^2)$

4 :   1.