NATIONAL UNIVERSITY OF SINGAPORE

SEMESTER 1, 2021/2022

MA1521 Calculus for Computing **Tutorial 7**

1. Find the area of the region between the curve $y = 3 - x^2$ and the line $y = -1$ by integrating with respect to (a) x, (b) y.

   (Thomas' Calculus (14^{th} edition), p. 323, Problem 76)

2. Compute the volume of the solid generated by revolving the triangular region bounded by the lines $2y = x + 4, y = x$, and $x = 0$ about (a) the x-axis using the washer method, (b) the y-axis using the cylindrical method.

   (Thomas' Calculus (14^{th} edition), p. 350, Problem 30 (a), (b))

3. Find the length of the curve $y = (1 - x^{2/3})^{3/2}$ where $\sqrt{2}/4 \leq x \leq 1$.

   (Thomas' Calculus (14^{th} edition), p. 355, Problem 24)

4. Find the area of the surface generated by revolving about the x-axis the portion of the astroid (the name of the curve) $x^{2/3} + y^{2/3} = 1$.

   (Thomas' Calculus (14^{th} edition), p. 361, Problem 32)

5. Solve the differential equation $\dfrac{dy}{dx} = 3x^2 e^{-y}$

   (Thomas' Calculus (14^{th} edition), p. 423, Problem 12)

6. Which of the following sequences (a_n) converge and which diverge? Find the limit of each convergent sequence.
   (a) $a_n = 1 + (-1)^n$

   (Thomas' Calculus (14^{th} edition), p. 540, Problem 39)

(b) $a_n = \dfrac{\ln n}{n^{1/n}}$

(Thomas' Calculus (14th edition), p. 540, Problem 63)

(c) $a_n = \ln n - \ln(n+1)$

(Thomas' Calculus (14th edition), p. 540, Problem 64)

(d) $a_n = \dfrac{n!}{n^n}$

(Thomas' Calculus (14th edition), p. 540, Problem 67)

(e) $a_n = \dfrac{1}{n}\displaystyle\int_1^n \frac{1}{t}\,dt$

(Thomas' Calculus (14th edition), p. 541, Problem 99)

7. Let
$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt, \;\; x > 0.$$
(a) Show that $\Gamma(1) = 1$.

(b) Apply integration by parts to the integral for $\Gamma(x+1)$ to show that
$$\Gamma(x+1) = x\Gamma(x).$$

(c) Show by induction that
$$\Gamma(n+1) = n!$$
for every nonnegative integer n.

(Thomas' Calculus (14th edition), p. 529, Problem 43)

8. Which of the following series converge, and which diverge? Give reasons for your answers. If a series converges, find its sum.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$

(Thomas' Calculus (14th edition), p. 551, Problem 66)

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n + 4^n}{3^n + 4^n}$

(Thomas' Calculus (14th edition), p. 551, Problem 68)

(c) $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$

(Thomas' Calculus (14^{th} edition), p. 551, Problem 69)

(d) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{e}{\pi}\right)^n$

(Thomas' Calculus (14^{th} edition), p. 551, Problem 71)

(e) $\displaystyle\sum_{n=1}^{\infty} \left(\cos\left(\frac{\pi}{n}\right) + \sin\left(\frac{\pi}{n}\right)\right)$

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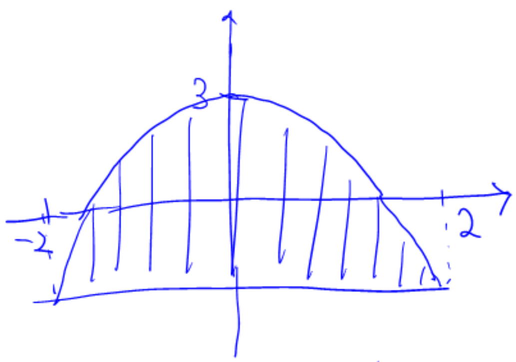
MA1521 Calculus for Computing

Solutions to problems for Tutorial 7

Tutorial 7, Problem 1

Find the area of the region between the curve $y = 3 - x^2$ and the line $y = -1$ by integrating with respect to (a) x, (b) y.



$y = -1$,

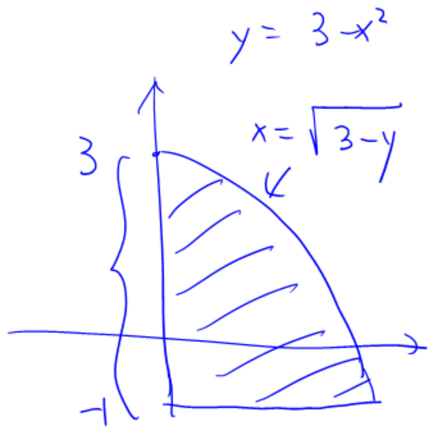
$3 - x^2 = -1 \Rightarrow x^2 = 4$

$\therefore\ x = \pm 2$.

$$\text{Area} = \int_{-2}^{2} 3 - x^2 - (-1)\,dx$$

$$= \int_{-2}^{2} 4 - x^2\,dx = \left(4x - \frac{x^3}{3}\right)\Bigg|_{-2}^{2}$$

$$= 8 - \frac{8}{3} + 8 - \frac{8}{3} = 16 - \frac{16}{3} = \frac{32}{3} \quad \square$$



With respect to y:

$$\text{Area} = 2\int_{-1}^{3} \sqrt{3-y}\, dy$$

$$= -2 \left. \frac{(3-y)^{3/2}}{3/2} \right|_{-1}^{3}$$

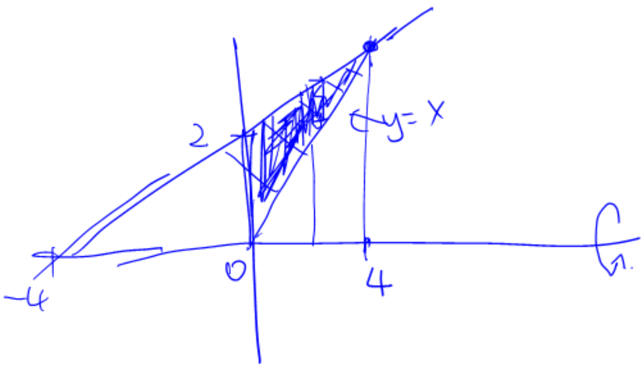
$$= \frac{4}{3}\left(-0 + 4^{3/2}\right)$$

$$= \frac{4}{3} \cdot 8 = \frac{32}{3}$$

$\square$

Tutorial 7, Problem 2

Compute the volume of the solid generated by revolving the triangular region bounded by the lines $2y = x + 4, y = x$, and $x = 0$ about (a) the x-axis using the washer method, (b) the y-axis using the cylindrical method.



$2x = x + 4$

$\Rightarrow x = 4$

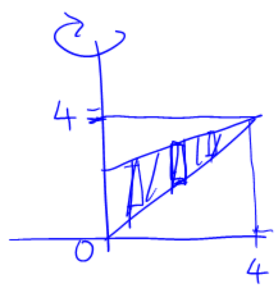
(a) Volume $= \int_0^4 \pi \left(\left(\frac{x+4}{2}\right)^2 - x^2 \right) dx$

$= \frac{\pi}{4} \int_0^4 (x+4)^2 - 4x^2 \, dx$

$= \frac{\pi}{4} \left(\frac{(x+4)^3}{3} - \frac{4x^3}{3} \Big|_0^4 \right)$

$= \frac{\pi}{12} \left(8^3 - 4 \cdot 4^3 - 4^3 \right) = \boxed{16\pi}$

(b)



(b) Volume $= \int_0^4 2\pi x \left(\frac{x+4}{2} - x \right) dx$

$= 2\pi \int_0^4 x \left(2 - \frac{x}{2} \right) dx$

$= 2\pi \left(x^2 - \frac{x^3}{6} \right) \Big|_0^4$

$= 2\pi \left(4^2 - \frac{4^3}{6} \right) = \boxed{\frac{32\pi}{3}}$

Tutorial 7, Problem 3

Find the length of the curve $y = (1 - x^{2/3})^{3/2}$ where $\sqrt{2}/4 \le x \le 1$.

Arc Length $= \displaystyle\int_{\sqrt{2}/4}^{1} \sqrt{1 + (y')^2}\, dx.$

$$y' = \frac{3}{2}\left(1 - x^{2/3}\right)^{1/2} \cdot \left(-\frac{2}{3} x^{-\frac{1}{3}}\right)$$

$$= -x^{-1/3}\left(1 - x^{2/3}\right)^{1/2}.$$

$$1 + (y')^2 = 1 + x^{-2/3}\left(1 - x^{2/3}\right) = 1 + x^{-2/3} - 1 = x^{-2/3}.$$

$$\therefore \int_{\sqrt{2}/4}^{1} \sqrt{1 + (y')^2}\, dx = \int_{\sqrt{2}/4}^{1} x^{-1/3}\, dx = \left.\frac{x^{\frac{2}{3}}}{(2/3)}\right|_{\sqrt{2}/4}^{1}$$

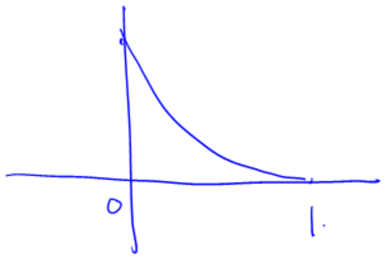
$$= \frac{3}{2}\left(1 - \left(\frac{\sqrt{2}}{4}\right)^{\frac{2}{3}}\right).$$

$$= \frac{3}{2}\left(1 - \frac{1}{2}\right) = \boxed{\frac{3}{4}}$$

$\square$

Tutorial 7, Problem 4

Find the area of the surface generated by revolving about the x-axis the portion of the astroid (the name of the curve) $x^{2/3} + y^{2/3} = 1$.



Surface area

$$= 2\int_0^1 (2\pi y)\sqrt{1+(y')^2}\,dx$$

$$\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}y' = 0. \qquad y' = -x^{-\frac{1}{3}}y^{\frac{1}{3}}.$$

$$1+(y')^2 = 1 + x^{-\frac{2}{3}}y^{\frac{2}{3}} = 1 + x^{-\frac{2}{3}}\left(1-x^{\frac{2}{3}}\right)$$

$$= 1 + x^{-\frac{2}{3}} - 1 = x^{-\frac{2}{3}}$$

$y^{\frac{2}{3}} = 1 - x^{\frac{2}{3}}$.

$$y\sqrt{1+(y')^2} = y\,x^{-\frac{1}{3}}$$

$$y^{2/3} = 1 - x^{\frac{2}{3}} \Rightarrow y = \left(1-x^{\frac{2}{3}}\right)^{\frac{3}{2}}.$$

$$y\,x^{-1/3} = \left(1-x^{\frac{2}{3}}\right)^{\frac{3}{2}}x^{-\frac{1}{3}}$$

$u = x^{\frac{2}{3}}$. $du = \frac{2}{3}x^{-\frac{1}{3}}dx$.

$$\int \left(1-x^{\frac{2}{3}}\right)^{\frac{3}{2}}x^{-\frac{1}{3}}dx = \frac{3}{2}\int (1-u)^{3/2}du = -\frac{3}{2}\frac{(1-u)^{5/2}}{5/2} = -\frac{3}{5}(1-u)^{5/2}.$$

$$\therefore \quad 2\int_0^1 2\pi y\sqrt{1+(y')^2}\,dy = (4\pi)\left(-\frac{3}{5}\right)\left(1-x^{2/3}\right)^{5/2}\Big|_0^1$$

$$= -\frac{12\pi}{5}(0-1) = \boxed{\frac{12\pi}{5}}.$$

$\square$

Tutorial 7, Problem 5

Solve the differential equation $\dfrac{dy}{dx} = 3x^2e^{-y}$

$$\frac{dy}{dx} = 3x^2 e^{-y} \Rightarrow e^y \frac{dy}{dx} = 3x^2$$

$$\int e^y \, dy = \int 3x^2 \, dx$$

$$e^y = 3 \cdot \frac{x^3}{3} + k$$

$$e^y = x^3 + k. \quad (\text{or } y = \ln(x^3 + k).)$$

$\square$

Tutorial 7, Problem 6

Which of the following sequences (a_n) converge and which diverge? Find the limit of each convergent sequence.

(a) $a_n = 1 + (-1)^n$

(Thomas' Calculus (14th edition), p. 540, Problem 39)

(b) $a_n = \dfrac{\ln n}{n^{1/n}}$

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(e) $a_n = \dfrac{1}{n}\displaystyle\int_1^n \frac{1}{t}\,dt$

(Thomas' Calculus (14th edition), p. 541, Problem 99)

(a) $a_1 = 0, \ a_2 = 2, \ a_3 = 0, \ldots$

$(a_n)_{n=1}^{\infty}$ diverges because $\lim_{n\to\infty} a_{2n} = 2$ and

$\lim_{n\to\infty} a_{2n+1} = 0.$

$\therefore$ a_n does not converge to a unique limit.

(b) $a_n = \frac{\ln n}{n^{1/n}}$. Observe that $f(n) = \frac{\ln n}{n^{1/n}} = a_n$

where $f(x) = \frac{\ln x}{x^{1/x}}$.

$$\lim_{x\to\infty} \frac{\ln x}{x^{1/x}} = \lim_{x\to\infty} \frac{\ln x}{e^{(\ln x)/x}}.$$

Now, $\lim_{x\to\infty} \frac{\ln x}{x} \underset{\uparrow}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$ $\left(\begin{array}{l} \lim_{x\to\infty} \ln x = \infty \\ \lim_{x\to\infty} x = \infty \end{array} \right)$

L'Hôpital's Rule

$$\therefore \lim_{x\to\infty} \frac{\ln x}{e^{\ln x/x}} = \infty$$

$\therefore$ a_n diverges

(c) $a_n = \ln n - \ln(n+1) = \ln \frac{n}{(n+1)}$.

Let $f(x) = \ln \frac{x}{x+1}$.

$$\lim_{x\to\infty} \ln \frac{x}{x+1} = \lim_{x\to\infty} \ln \frac{1}{(1+\frac{1}{x})} = \ln 1 = 0.$$

$\therefore$ $(a_n)_{n=1}^{\infty}$ converges & $\lim_{n\to\infty} a_n = 0$.

(d) $$0 \le \frac{n!}{n^n} = \frac{1}{n}\left(\frac{2}{n} \cdots \frac{n-1}{n} \ \frac{n}{n}\right)$$

$$\le \frac{1}{n}(1)(1) \cdots (1) = \frac{1}{n}$$

$\lim_{n\to\infty} \frac{1}{n} = 0$. $\therefore$ By Squeeze Theorem,

$$\frac{n!}{n^n} \to 0 \text{ as } n \to \infty$$

The sequence $\left(\frac{n!}{n^n}\right)_{n=1}^{\infty}$ is convergent and

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0.$$

(e) $a_n = \frac{1}{n}\int_1^n \frac{1}{t}\,dt$. Let $f(x) = \frac{1}{x}\int_1^x \frac{1}{t}\,dt$.

$$\lim_{x\to\infty} \int_1^x \frac{1}{t}\,dt = \lim_{x\to\infty} \ln x = \infty$$

$\lim_{x\to\infty} x = \infty$. By L'Hôpital's Rule,

$$\lim_{x\to\infty} \frac{1}{x}\int_1^x \frac{1}{t}\,dt = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0.$$

$\therefore$ $(a_n)_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} a_n = 0$.

Tutorial 7, Problem 7

Let

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt, \ x > 0.$$

(a) Show that $\Gamma(1) = 1$.

(b) Apply integration by parts to the integral for $\Gamma(x+1)$ to show that

$$\Gamma(x+1) = x\Gamma(x).$$

(c) Show by induction that

$$\Gamma(n+1) = n!$$

for every nonnegative integer n.

(a) $\Gamma(1) = \int_0^\infty e^{-t} dt$. $\lim_{M\to\infty} \int_0^M e^{-t} dt = \lim_{M\to\infty} \left.\frac{e^{-t}}{-1}\right|_0^M$

$= \lim_{M\to\infty} (-e^{-M} + 1) = 1.$

$\therefore \Gamma(1) = 1.$

(b) Let $m \geq 1$.

$e^{-t} dt = du \Rightarrow u = -e^{-t}$

Let $v = t^m$.

Then $\int t^m \, d(-e^{-t}) + \int -e^{-t} \, dt^m = -t^m e^{-t}$

$\int t^m e^{-t} dt - \int e^{-t} m t^{m-1} dt = -t^m e^{-t}$

$$\int_0^M t^m e^{-t} dt = -t^m e^{-t}\Big|_0^M + m\int_0^M e^{-t} t^{m-1} dt$$

Let $M \to \infty$:

$$\Gamma(m+1) = \lim_{M\to\infty}\left(\frac{-t^M}{e^M} + 0\right) + m\,\Gamma(m)$$

$$\left(\because \lim_{M\to\infty} \frac{t^M}{e^M} = 0 \quad (\text{By L'Hôpital's rule.})\right)$$

(c). $\Gamma(2) = 1 \cdot \Gamma(1)$ by (b).

$\Rightarrow \Gamma(2) = 1!$

Suppose $\Gamma(n) = (n-1)!$

Then $\Gamma(n+1) = n\,\Gamma(n)$ by (b).

$= n\,(n-1)! = n!$

Tutorial 7, Problem 8

Which of the following series converge, and which diverge? Give reasons for your answers. If a series converges, find its sum.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$

(Thomas' Calculus (14th edition), p. 551, Problem 66)

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n + 4^n}{3^n + 4^n}$

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(a) $$\frac{n^n}{n!} = \frac{n}{1} \cdot \frac{n}{2} \cdots \frac{n}{n} \geq 1$$

$$\therefore \lim_{n\to\infty} \frac{n^n}{n!} \geq 1 \Rightarrow \lim_{n\to\infty} \frac{n^n}{n!} \neq 0$$

$\therefore$ By n-th term test, $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$ is divergent.

(b) $$\frac{2^n+4^n}{3^n+4^n} = \frac{\left(\frac{2}{4}\right)^n+1}{\left(\frac{3}{4}\right)^n+1} \to 1 \quad \text{as } n\to\infty$$

$\therefore$ By n-th term test, $\sum_{n=1}^{\infty} \frac{2^n+4^n}{3^n+4^n}$ is divergent.

(c). $$\sum_{j=1}^{n} \ln\left(\frac{j}{j+1}\right) = \ln 1 - \ln 2 + \ln 2 - \ln 3 \cdots + \ln N - \ln(N+1)$$

$$= \ln 1 - \ln(n+1)$$

$$\lim_{n\to\infty} \sum_{j=1}^{n} \ln\left(\frac{j}{j+1}\right) = -\infty \quad \text{as} \quad n\to\infty$$

$\therefore$ The series $\sum_{j=1}^{\infty} \ln\left(\frac{j}{j+1}\right)$ diverges.

(d) $\underset{2.718\ldots}{e} < \pi = 3.14\ldots$ $\qquad \frac{e}{\pi} < 1$

$\therefore \sum_{n=1}^{\infty} \left(\frac{e}{\pi}\right)^n$ is convergent

since it is a geometric series with common ratio $\frac{e}{\pi} < 1$.

(e) $$\cos\frac{\pi}{n} + \sin\frac{\pi}{n} \to \cos 0 + \sin 0 = 1 \quad \text{as} \quad n\to\infty$$

$\therefore$ By n-th term test, $\sum_{n=1}^{\infty}\left(\cos\frac{\pi}{n}+\sin\frac{\pi}{n}\right)$ is divergent.