MA2001 LINEAR ALGEBRA

Linear Transformation

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Definition

- Recall that a **linear equation** has the form:
  - $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$,

  $a_1, \ldots, a_n, b$ are **constants**, $x_1, \ldots, x_n$ are **variables**.
- **Definition.** We say the mapping $f : \mathbb{R}^n \to \mathbb{R}$ defined by
  - $f(x_1, x_2, \ldots, x_n) = a_1x_1 + a_2x_2 + \cdots + a_nx_n$

  a **linear transformation** from $\mathbb{R}^n$ to $\mathbb{R}$.

  It can be viewed as the **matrix form**:
  - $f\left(\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}\right) = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$.
- In this chapter, all vectors are viewed as **column vectors**.

Definition

- Recall that a **linear system** has the form:
  - $\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \quad\vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$

  where a_{ij}, b_i are **constants** and $x_1, \ldots, x_n$ are **variables**.
- **Definition.** We say the mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by
  - $T\left(\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}\right) = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix}$

  a **linear transformation** from $\mathbb{R}^n$ to $\mathbb{R}^m$.
  - T is called a **linear operator** on $\mathbb{R}^n$ if $m = n$.

Definition

- Recall that a **linear system** has the form:
  - $$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \quad\vdots \qquad\quad \vdots \qquad \vdots \qquad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

  where a_{ij}, b_i are **constants** and $x_1, \dots, x_n$ are **variables**.
- A **linear transformation** is viewed as the **matrix form**:
  - $$T\left(\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}\right) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$
  - $T : \mathbb{R}^n \to \mathbb{R}^m$ such that $T(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$, for $\boldsymbol{x} \in \mathbb{R}^n$.
    - $\boldsymbol{A} = (a_{ij})_{m\times n}$ is the **standard matrix** for T.

Examples

- **Definition.** Let $I : \mathbb{R}^n \to \mathbb{R}^n$ be the **linear transformation**
  - $I(\boldsymbol{x}) = \boldsymbol{x}$ for $\boldsymbol{x} \in \mathbb{R}^n$.

  It is called the **identity transformation**.
  - It is the **identity operator** on $\mathbb{R}^n$.
  - $I(\boldsymbol{x}) = \boldsymbol{x} = \boldsymbol{I}_n\boldsymbol{x} \Rightarrow \boldsymbol{I}_n$ is the **standard matrix** for I.
- **Definition.** Let $O : \mathbb{R}^n \to \mathbb{R}^m$ be the **linear transformation**
  - $O(\boldsymbol{x}) = \mathbf{0}$ for $\boldsymbol{x} \in \mathbb{R}^n$.

  It is called the **zero transformation**.
  - $O(\boldsymbol{x}) = \mathbf{0} = \mathbf{0}_{m\times n}\mathbf{0} \Rightarrow \mathbf{0}_{m\times n}$ is the **standard matrix**.
- Given a **linear transformation** $T : \mathbb{R}^n \to \mathbb{R}^m$.
  - Is the **standard matrix unique**?

Examples

- Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a **linear transformation** such that
  - $T(\boldsymbol{x}) = \boldsymbol{Ax} = \boldsymbol{Bx}$ for all $\boldsymbol{x} \in \mathbb{R}^n$.
    - For all $\boldsymbol{x} \in \mathbb{R}^n$, $\quad \boldsymbol{0} = \boldsymbol{Ax} - \boldsymbol{Bx} = (\boldsymbol{A} - \boldsymbol{B})\boldsymbol{x}$.
    - Nullspace of $\boldsymbol{A} - \boldsymbol{B}$ is $\mathbb{R}^n$.
    - $\mathrm{nullity}(\boldsymbol{A} - \boldsymbol{B}) = \dim \mathbb{R}^n = n$.
    - $\mathrm{rank}(\boldsymbol{A} - \boldsymbol{B}) = n - \mathrm{nullity}(\boldsymbol{A} - \boldsymbol{B}) = n - n = 0$.

  $\therefore$ $\boldsymbol{A} - \boldsymbol{B} = \boldsymbol{0}$; or equivalently, $\boldsymbol{A} = \boldsymbol{B}$.
    - Alternatively: $\boldsymbol{Ae}_1 = \boldsymbol{Be}_1, \ldots, \boldsymbol{Ae}_n = \boldsymbol{Be}_n$.

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{Ae}_1 & \cdots & \boldsymbol{Ae}_n \end{pmatrix} = \begin{pmatrix} \boldsymbol{Be}_1 & \cdots & \boldsymbol{Be}_n \end{pmatrix} = \boldsymbol{B}.$$

- **Conclusion**:
  - The **standard matrix** of a linear transformation is **unique**.

Examples

- To show that $T : \mathbb{R}^n \to \mathbb{R}^m$ **is** a **linear transformation**,
  - just find a matrix $\boldsymbol{A}$ so that $T(\boldsymbol{x}) = \boldsymbol{Ax}$ for all $\boldsymbol{x} \in \mathbb{R}^n$.
- **Example.** Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined as
  - $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x + y \\ 2x \\ -3y \end{pmatrix}$ for $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^n$.
    - $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x + y \\ 2x + 0y \\ 0x - 3y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

  $\therefore$ T is a **linear transformation**.
    - The **standard matrix** for T is $\begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 0 & -3 \end{pmatrix}$.

Linearity

- Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a **linear transformation**.
  - Let $\boldsymbol{A}$ be the **standard matrix** for T.
    - That is, $T(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$ for all $\boldsymbol{x} \in \mathbb{R}^n$.

1. $T(\mathbf{0}) = \boldsymbol{A}\mathbf{0} = \mathbf{0}$.
2. $T(c\boldsymbol{v}) = \boldsymbol{A}(c\boldsymbol{v}) = c(\boldsymbol{A}\boldsymbol{v}) = cT(\boldsymbol{v})$.
3. $T(\boldsymbol{u}+\boldsymbol{v}) = \boldsymbol{A}(\boldsymbol{u}+\boldsymbol{v}) = \boldsymbol{A}\boldsymbol{u} + \boldsymbol{A}\boldsymbol{v} = T(\boldsymbol{u}) + T(\boldsymbol{v})$.
4. For any $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k \in \mathbb{R}^n$ and $c_1, \ldots, c_k \in \mathbb{R}$,

$$\begin{aligned} T(c_1\boldsymbol{v}_1 + \cdots + c_k\boldsymbol{v}_k) &= \boldsymbol{A}(c_1\boldsymbol{v}_1 + \cdots + c_k\boldsymbol{v}_k) \\ &= \boldsymbol{A}(c_1\boldsymbol{v}_1) + \cdots + \boldsymbol{A}(c_k\boldsymbol{v}_k) \\ &= c_1(\boldsymbol{A}\boldsymbol{v}_1) + \cdots + c_k(\boldsymbol{A}\boldsymbol{v}_k) \\ &= c_1T(\boldsymbol{v}_1) + \cdots + c_kT(\boldsymbol{v}_k). \end{aligned}$$

Linearity

- **Theorem**. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a **linear transformation**.
  - $T(\mathbf{0}) = \mathbf{0}$. More precisely, $T(\mathbf{0}_n) = \mathbf{0}_m$.
  - If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k \in \mathbb{R}^n$ and $c_1, \ldots, c_k \in \mathbb{R}$,
    - $T(c_1\boldsymbol{v}_1 + \cdots + c_k\boldsymbol{v}_k) = c_1T(\boldsymbol{v}_1) + \cdots + c_kT(\boldsymbol{v}_k)$.
- If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a **linear transformation**, then
  - $T(c\boldsymbol{v}) = cT(\boldsymbol{v})$ for all $\boldsymbol{v} \in \mathbb{R}^n$ and $c \in \mathbb{R}$.
  - $T(\boldsymbol{u}+\boldsymbol{v}) = T(\boldsymbol{u}) + T(\boldsymbol{v})$ for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$.
- To show that a mapping T is **not** a **linear transformation**.
  - Show that $T(\mathbf{0}) \neq \mathbf{0}$; or
  - Find $\boldsymbol{v} \in \mathbb{R}^n$, $c \in \mathbb{R}$ such that $T(c\boldsymbol{v}) \neq cT(\boldsymbol{v})$; or
  - Find $\boldsymbol{u} \in \mathbb{R}^n$ such that $T(\boldsymbol{u}+\boldsymbol{v}) \neq T(\boldsymbol{u}) + T(\boldsymbol{v})$.

Examples

- Let $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by
  - $T_1\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+1 \\ y+3 \end{pmatrix}$ for $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

    $T_1\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow T_1$ is **not linear**.
  - Alternatively,
    - $T_1\left(2\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = T_1\left(\begin{pmatrix} 2 \\ 2 \end{pmatrix}\right) = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$,
    - $2\,T_1\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = 2\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$.

    $T_1\left(2\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) \neq 2\,T_1\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) \Rightarrow T_1$ is **not linear**.

Examples

- Let $T_2 : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by
  - $T_2\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x^2 \\ yz \end{pmatrix}$ for $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$.
    - $T_2\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\right) = T_2\left(\begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}\right) = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$.
    - $T_2\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) + T_2\left(\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$.
  - $T_2\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\right) \neq T_2\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) + T_2\left(\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\right)$.
    - T_2 is **not** a **linear transformation**.

Representation

- Recall that for a **linear transformation** $T : \mathbb{R}^n \to \mathbb{R}^m$:
  - $T(\mathbf{0}) = \mathbf{0}$. More precisely, $T(\mathbf{0}_n) = \mathbf{0}_m$.
  - If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k \in \mathbb{R}^n$ and $c_1, \ldots, c_k \in \mathbb{R}$, then
    - $T(c_1\boldsymbol{v}_1 + \cdots + c_k\boldsymbol{v}_k) = c_1T(\boldsymbol{v}_1) + \cdots + c_kT(\boldsymbol{v}_k)$.
- Let $E = \{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ be the **standard basis** for $\mathbb{R}^n$.
  - Every $\boldsymbol{v} \in \mathbb{R}^n$ has the form $v_1\boldsymbol{e}_1 + \cdots + v_n\boldsymbol{e}_n$.

  Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a **linear transformation**.

$$\begin{aligned} T(\boldsymbol{v}) &= T(v_1\boldsymbol{e}_1 + \cdots + v_n\boldsymbol{e}_n) \\ &= v_1T(\boldsymbol{e}_1) + \cdots + v_nT(\boldsymbol{e}_n). \end{aligned}$$

  - $T(\boldsymbol{v})$ is **completely** determined by $T(\boldsymbol{e}_1), \ldots, T(\boldsymbol{e}_n)$.

Representation

- Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a **linear transformation**, and let $\boldsymbol{A}$ be the **standard matrix** for T.
  - $T(\boldsymbol{v}) = \boldsymbol{A}\boldsymbol{v} \quad$ for all $\boldsymbol{v} \in \mathbb{R}^n$.
  - $T(\boldsymbol{e}_1) = \boldsymbol{A}\boldsymbol{e}_1, \ \ldots, \ T(\boldsymbol{e}_n) = \boldsymbol{A}\boldsymbol{e}_n$.

$$\begin{aligned} \boldsymbol{A} = \boldsymbol{A}\boldsymbol{I} = \boldsymbol{A}\begin{pmatrix}\boldsymbol{e}_1 & \cdots & \boldsymbol{e}_n\end{pmatrix} &\\ = \begin{pmatrix}\boldsymbol{A}\boldsymbol{e}_1 & \cdots & \boldsymbol{A}\boldsymbol{e}_n\end{pmatrix} &\\ = \begin{pmatrix}T(\boldsymbol{e}_1) & \cdots & T(\boldsymbol{e}_n)\end{pmatrix} & \end{aligned}$$

- **Example.** If T is a **linear transformation** such that
  - $T\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = \begin{pmatrix}1\\2\\3\end{pmatrix}, T\left(\begin{pmatrix}0\\1\end{pmatrix}\right) = \begin{pmatrix}4\\5\\6\end{pmatrix}$.
  - The **standard matrix** for T is $\begin{pmatrix}1 & 4\\2 & 5\\3 & 6\end{pmatrix}$.

Representation

- Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a mapping satisfying
  - $T(c_1\boldsymbol{v}_1 + \cdots + c_k\boldsymbol{v}_k) = c_1T(\boldsymbol{v}_1) + \cdots + c_kT(\boldsymbol{v}_k)$

    for all $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k \in \mathbb{R}^n$ and $c_1, \ldots, c_k \in \mathbb{R}$.
  - Let $\boldsymbol{A} = \begin{pmatrix} T(\boldsymbol{e}_1) & \cdots & T(\boldsymbol{e}_n) \end{pmatrix}$.
    - Write $\boldsymbol{v} \in \mathbb{R}^n$ as $\boldsymbol{v} = v_1\boldsymbol{e}_1 + \cdots + v_n\boldsymbol{e}_n$.

$$\begin{aligned} T(\boldsymbol{v}) &= v_1T(\boldsymbol{e}_1) + \cdots + v_nT(\boldsymbol{e}_n) \\ &= \begin{pmatrix} T(\boldsymbol{e}_1) & \cdots & T(\boldsymbol{e}_n) \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \\ &= \begin{pmatrix} T(\boldsymbol{e}_1) & \cdots & T(\boldsymbol{e}_n) \end{pmatrix} \boldsymbol{v} \\ &= \boldsymbol{A}\boldsymbol{v}. \end{aligned}$$

$\therefore$ T is a **linear transformation** with **standard matrix** $\boldsymbol{A}$.

Representation

- A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is a **linear transformation**, i.e., T has the form $T(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} \Leftrightarrow$

  $T(c_1\boldsymbol{v}_1 + \cdots + c_k\boldsymbol{v}_k) = c_1T(\boldsymbol{v}_1) + \cdots + c_kT(\boldsymbol{v}_k)$
  - for all $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k \in \mathbb{R}^n$ and $c_1, \ldots, c_k \in \mathbb{R}$.
- **Exercise.** A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is a **linear transformation**, i.e., T has the form $T(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$ $\Leftrightarrow$

  $T(c\boldsymbol{u} + d\boldsymbol{v}) = cT(\boldsymbol{u}) + dT(\boldsymbol{v})$
  - for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$.
- **General Definition.** Let V and W be **vector spaces**.
  - A mapping $T : V \to W$ is a **linear transformation** if

    $T(c\boldsymbol{u} + d\boldsymbol{v}) = cT(\boldsymbol{u}) + dT(\boldsymbol{v})$
    - for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$.

Representation

- Let $S = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ be a **basis** for $\mathbb{R}^n$.
  - For $\boldsymbol{v} \in \mathbb{R}^n$, write $(\boldsymbol{v})_S = (c_1, \ldots, c_n)$;
    - i.e., $\boldsymbol{v} = c_1\boldsymbol{v}_1 + \cdots + c_n\boldsymbol{v}_n$.

  Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a **linear transformation**.

$$\begin{aligned} T(\boldsymbol{v}) &= T(c_1\boldsymbol{v}_1 + \cdots + c_n\boldsymbol{v}_n) \\ &= c_1 T(\boldsymbol{v}_1) + \cdots + c_n T(\boldsymbol{v}_n) \\ &= \begin{pmatrix} T(\boldsymbol{v}_1) & \cdots & T(\boldsymbol{v}_n) \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \end{aligned}$$

  - $T(\boldsymbol{v})$ is **completely** determined by $T(\boldsymbol{v}_1), \ldots, T(\boldsymbol{v}_n)$.

Example

- Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a **linear transformation**:
  - $T\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, T\left(\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} -1 \\ 2 \end{pmatrix},$

    $T\left(\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}\right) = \begin{pmatrix} 4 \\ -1 \end{pmatrix}.$
- Consider the given condition:
  - $\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}\right\}$ is a **basis** for $\mathbb{R}^3$,

    $\because \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$ is invertible.
  - So the given information **completely determines** T.

Example

- $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \right\}$ is a **basis** for $\mathbb{R}^3$.
  - Every vector in $\mathbb{R}^3$ is a **unique linear combination**:
    - $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$
    - $\begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & x \\ 1 & 1 & 0 & y \\ 1 & 1 & -1 & z \end{array}\right) \xrightarrow{\text{G.-J.E.}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & x-2y+2z \\ 0 & 1 & 0 & -x+3y-2z \\ 0 & 0 & 1 & y-z \end{array}\right)$$

Example

- $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \right\}$ is a **basis** for $\mathbb{R}^3$.
  - Every vector in $\mathbb{R}^3$ is a **unique linear combination**:
    - $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$
    - $\begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.
    - $\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} x-2y+2z \\ -x+3y-2z \\ y-z \end{pmatrix}$

Example

- $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}.$

$$\begin{aligned} T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) &= c_1 T\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right) + c_2 T\left(\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right) + c_3 T\left(\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}\right) \\ &= (x - 2y + 2z)\begin{pmatrix} 1 \\ 3 \end{pmatrix} + (-x + 3y - 2z)\begin{pmatrix} -1 \\ 2 \end{pmatrix} \\ &\quad + (y - z)\begin{pmatrix} 4 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} 2x - y \\ x - y + 3z \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 \\ 1 & -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}. \end{aligned}$$

  - The **standard matrix** for T is $\begin{pmatrix} 2 & -1 & 0 \\ 1 & -1 & 3 \end{pmatrix}$.

Change of Bases

- Let $S = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ be a **basis** for $\mathbb{R}^n$.
  - For $\boldsymbol{v} \in \mathbb{R}^n$, write $(\boldsymbol{v})_S = (c_1, \ldots, c_n)$;

    $$\boldsymbol{v} = c_1\boldsymbol{v}_1 + \cdots + c_n\boldsymbol{v}_n = \begin{pmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \end{pmatrix} [\boldsymbol{v}]_S.$$

  - Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a **linear transformation**.

$$\begin{aligned} T(\boldsymbol{v}) &= T(c_1\boldsymbol{v}_1 + \cdots + c_n\boldsymbol{v}_n) \\ &= c_1 T(\boldsymbol{v}_1) + \cdots + c_n T(\boldsymbol{v}_n) \\ &= \begin{pmatrix} T(\boldsymbol{v}_1) & \cdots & T(\boldsymbol{v}_n) \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \\ &= \begin{pmatrix} T(\boldsymbol{v}_1) & \cdots & T(\boldsymbol{v}_n) \end{pmatrix} [\boldsymbol{v}]_S. \end{aligned}$$

Change of Bases

- Let $S = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ be a **basis** for $\mathbb{R}^n$.
  - For $\boldsymbol{v} \in \mathbb{R}^n$, write $(\boldsymbol{v})_S = (c_1, \ldots, c_n)$;

    $\boldsymbol{v} = c_1\boldsymbol{v}_1 + \cdots + c_n\boldsymbol{v}_n = \begin{pmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \end{pmatrix} [\boldsymbol{v}]_S.$
  - Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a **linear transformation**.

$$T(\boldsymbol{v}) = \begin{pmatrix} T(\boldsymbol{v}_1) & \cdots & T(\boldsymbol{v}_n) \end{pmatrix} [\boldsymbol{v}]_S = \boldsymbol{B}[\boldsymbol{v}]_S,$$

    - where $\boldsymbol{B} = \begin{pmatrix} T(\boldsymbol{v}_1) & \cdots & T(\boldsymbol{v}_n) \end{pmatrix}$.

    Let $\boldsymbol{A}$ be the **standard matrix** for T. Then
    - $T(\boldsymbol{v}) = \boldsymbol{A}\boldsymbol{v} = \boldsymbol{A} \begin{pmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \end{pmatrix} [\boldsymbol{v}]_S$.

  $\therefore$ $\boldsymbol{A}\boldsymbol{P} = \boldsymbol{B}$, where $\boldsymbol{P} = \begin{pmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \end{pmatrix}$.
    - Or equivalently, $\boldsymbol{A} = \boldsymbol{B}\boldsymbol{P}^{-1}$.

Example

- Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a **linear transformation**:
  - $T\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, T\left(\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} -1 \\ 2 \end{pmatrix},$
    $T\left(\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}\right) = \begin{pmatrix} 4 \\ -1 \end{pmatrix}.$
  - Let $\boldsymbol{P} = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$: **basis** for $\mathbb{R}^n$.
  - Let $\boldsymbol{B} = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \end{pmatrix}$: the **images**.

  $\therefore$ The **standard matrix** $\boldsymbol{A} = \boldsymbol{B}\boldsymbol{P}^{-1} = \begin{pmatrix} 2 & -1 & 0 \\ 1 & -1 & 3 \end{pmatrix}$.

Change of Bases

- Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a **linear transformation**.
  - If $S = \{\boldsymbol{u}_1, \dots, \boldsymbol{u}_n\}$ is a **basis** for $\mathbb{R}^n$,
    - $T(\boldsymbol{v}) = \boldsymbol{B}[\boldsymbol{v}]_S$, $\boldsymbol{B} = \begin{pmatrix} T(\boldsymbol{u}_1) & \cdots & T(\boldsymbol{u}_n) \end{pmatrix}$
  - If $R = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ is a basis for $\mathbb{R}^n$,
    - $T(\boldsymbol{v}) = \boldsymbol{C}[\boldsymbol{v}]_R$, $\boldsymbol{C} = \begin{pmatrix} T(\boldsymbol{v}_1) & \cdots & T(\boldsymbol{v}_n) \end{pmatrix}$

  We can conclude the **relation** between $\boldsymbol{B}$ and $\boldsymbol{C}$:
  - Let $\boldsymbol{P}$ be the **transition matrix** from S to R:
    - $\boldsymbol{P}[\boldsymbol{v}]_S = [\boldsymbol{v}]_R \Rightarrow \boldsymbol{C}\boldsymbol{P}[\boldsymbol{v}]_S = \boldsymbol{C}[\boldsymbol{v}]_R = T(\boldsymbol{v})$

      $\Rightarrow \boldsymbol{B} = \boldsymbol{C}\boldsymbol{P}$
  - Let $\boldsymbol{Q} = \boldsymbol{P}^{-1}$ be the **transition matrix** from R to S:
    - $\boldsymbol{Q}[\boldsymbol{v}]_R = [\boldsymbol{v}]_S \Rightarrow \boldsymbol{B}\boldsymbol{Q}[\boldsymbol{v}]_R = \boldsymbol{B}[\boldsymbol{v}]_S T(\boldsymbol{v})$
      - $\Rightarrow \boldsymbol{C} = \boldsymbol{B}\boldsymbol{Q}$.

Change of Bases

- Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a **linear operation** on $\mathbb{R}^n$.
  - Let $\boldsymbol{A}$ be the **standard matrix**. Then $\boldsymbol{A}$ is square.
    - $T(\boldsymbol{v}) = \boldsymbol{A}\boldsymbol{v}$ for all $\boldsymbol{v} \in \mathbb{R}^n$.

  Let $S = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ be a **basis** for $\mathbb{R}^n$.
  - Let $\boldsymbol{P} = \begin{pmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \end{pmatrix}$. Then $\boldsymbol{P}$ is invertible.
    - $\boldsymbol{v} = \boldsymbol{P}[\boldsymbol{v}]_S$ for all $\boldsymbol{v} \in \mathbb{R}^n$.

    Then we can write
    - $T(\boldsymbol{v}) = \boldsymbol{P}[T(\boldsymbol{v})]_S$ and $\boldsymbol{A}\boldsymbol{v} = \boldsymbol{A}\boldsymbol{P}[\boldsymbol{v}]_S$.
  - $\boldsymbol{P}[T(\boldsymbol{v})]_S = \boldsymbol{A}\boldsymbol{P}[\boldsymbol{v}]_S \Rightarrow [T(\boldsymbol{v})]_S = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}[\boldsymbol{v}]_S$.

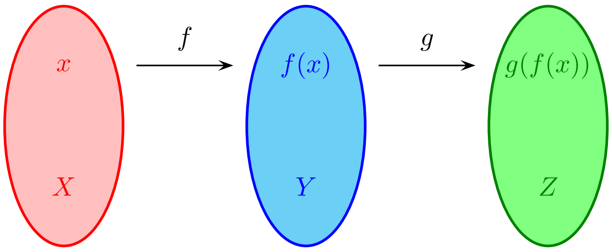
$\therefore$ T can be **represented** by $[\boldsymbol{v}]_S \mapsto \boldsymbol{B}[\boldsymbol{v}]_S$,
  - where $\boldsymbol{B} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$. We say $\boldsymbol{A}$ and $\boldsymbol{B}$ are **similar**.

Example

- Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 0.2x + 0.2y \\ 0.8x + 0.8y \end{pmatrix}$.
  - $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 0.2 & 0.2 \\ 0.8 & 0.8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.
    - $\begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 0.2 & 0.2 \\ 0.8 & 0.8 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.
  - Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ where $\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
    - Then $T(\boldsymbol{u}) = \boldsymbol{v}$, where
      - $[\boldsymbol{u}]_S = \begin{pmatrix} x \\ y \end{pmatrix}$ and $[\boldsymbol{v}]_S = \begin{pmatrix} x \\ 0 \end{pmatrix}$.
    - More precisely, $T(c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2) = c_1\boldsymbol{v}_1$.

Composition

- Consider two **functions** $f : X \to Y$ and $g : Y \to Z$.

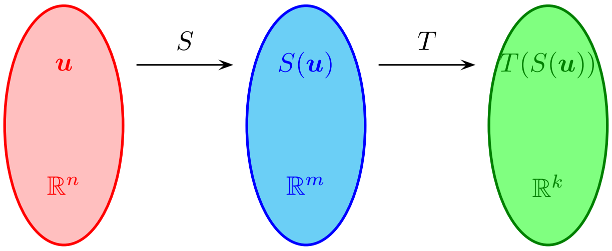


  - Let $g \circ f$ denote the function $X \to Z$ such that
    - $g \circ f(x) = g(f(x)), \quad$ for all $x \in X$.

    This is called the **composition** of g with f.
- **Note**: In general, $g \circ f \neq f \circ g$.

Composition

- **Definition.** Let $S : \mathbb{R}^n \to \mathbb{R}^m$ and $T : \mathbb{R}^m \to \mathbb{R}^k$ be **linear transformations**.



  - Let $T \circ S$ denote the mapping $\mathbb{R}^n \to \mathbb{R}^k$ such that
    - $(T \circ S)(\boldsymbol{u}) = T(S(\boldsymbol{u}))$, for all $\boldsymbol{u} \in \mathbb{R}^n$.

    This is called the **composition** of T with S.

- **Note**. In general, $T \circ S \neq S \circ T$.

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Example

- Let $S : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by
  - $S\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x+y \\ z \end{pmatrix}$, for $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$.

  Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by
  - $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} y \\ y \\ x \end{pmatrix}$, for $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

  Then $T \circ S : \mathbb{R}^3 \to \mathbb{R}^3$ is the mapping given by

$$\begin{aligned}(T \circ S)\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) &= T\left(S\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right)\right) \\ &= T\left(\begin{pmatrix} x+y \\ z \end{pmatrix}\right) = \begin{pmatrix} z \\ z \\ x+y \end{pmatrix}.\end{aligned}$$

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Example

- Let $S : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by
  - $S\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x+y \\ z \end{pmatrix}$, for $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$.

  Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by
  - $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} y \\ y \\ x \end{pmatrix}$, for $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

  Then $T \circ S : \mathbb{R}^3 \to \mathbb{R}^3$ is the mapping given by
  - $(T \circ S)\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} z \\ z \\ x+y \end{pmatrix}$,
    - The **standard matrix** for $T \circ S$ is $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

Example

- Let $S : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by
  - $S\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x+y \\ z \end{pmatrix}$, for $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$.

  Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by
  - $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} y \\ y \\ x \end{pmatrix}$, for $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

  Then $S \circ T : \mathbb{R}^2 \to \mathbb{R}^2$ is the mapping given by

$$\begin{aligned}(S \circ T)\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) &= S\left(T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)\right) \\ &= S\left(\begin{pmatrix} y \\ y \\ x \end{pmatrix}\right) = \begin{pmatrix} 2y \\ x \end{pmatrix}.\end{aligned}$$

Example

- Let $S : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by
  - $S\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x+y \\ z \end{pmatrix}$, for $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$.

  Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by
  - $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} y \\ y \\ x \end{pmatrix}$, for $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

  Then $S \circ T : \mathbb{R}^2 \to \mathbb{R}^2$ is the mapping given by
  - $(S \circ T)\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2y \\ x \end{pmatrix}$,
    - The **standard matrix** for $S \circ T$ is $\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$.

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Example

- Standard matrix for $S : \mathbb{R}^3 \to \mathbb{R}^2$: $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

  Standard matrix for $T : \mathbb{R}^2 \to \mathbb{R}^3$: $\begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$.
  - Standard matrix for $T \circ S : \mathbb{R}^3 \to \mathbb{R}^3$: $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.
    - $\begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

    The **standard matrix** for $T \circ S$ is
    - (Standard matrix for T) $\times$ (Standard matrix for S).

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Example

- Standard matrix for $S : \mathbb{R}^3 \to \mathbb{R}^2$: $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

  Standard matrix for $T : \mathbb{R}^2 \to \mathbb{R}^3$: $\begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$.

  - Standard matrix for $S \circ T : \mathbb{R}^2 \to \mathbb{R}^2$: $\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$.
    - $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$.

    The **standard matrix** for $S \circ T$ is
    - (Standard matrix for S) $\times$ (Standard matrix for T).
  - Moreover, $T \circ S$ and $S \circ T$ are linear transformations.

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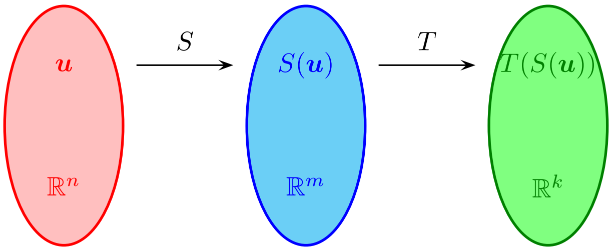
Properties

- Let $S : \mathbb{R}^n \to \mathbb{R}^m$ and $T : \mathbb{R}^m \to \mathbb{R}^k$ be **linear transformations**.



  - Let $\boldsymbol{A}$ be the **standard matrix** for S.
    - $S(\boldsymbol{u}) = \boldsymbol{A}\boldsymbol{u}$ for all $\boldsymbol{u} \in \mathbb{R}^n$.
  - Let $\boldsymbol{B}$ be the **standard matrix** for T.
    - $T(\boldsymbol{v}) = \boldsymbol{B}\boldsymbol{v}$ for all $\boldsymbol{v} \in \mathbb{R}^m$.

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Properties

- Let $S : \mathbb{R}^n \to \mathbb{R}^m$ and $T : \mathbb{R}^m \to \mathbb{R}^k$ be **linear transformations**.
  - Let $\boldsymbol{A}$ be the **standard matrix** for S.
    - $S(\boldsymbol{u}) = \boldsymbol{A}\boldsymbol{u}$ for all $\boldsymbol{u} \in \mathbb{R}^n$.
  - Let $\boldsymbol{B}$ be the **standard matrix** for T.
    - $T(\boldsymbol{v}) = \boldsymbol{B}\boldsymbol{v}$ for all $\boldsymbol{v} \in \mathbb{R}^m$.

  For all $\boldsymbol{u} \in \mathbb{R}^n$,
$$\begin{aligned}(T \circ S)(\boldsymbol{u}) &= T(S(\boldsymbol{u})) = T(\boldsymbol{A}\boldsymbol{u}) \\ &= \boldsymbol{B}(\boldsymbol{A}\boldsymbol{u}) = (\boldsymbol{B}\boldsymbol{A})\boldsymbol{u}.\end{aligned}$$

$\therefore$ $T \circ S : \mathbb{R}^n \to \mathbb{R}^k$ is a **linear transformation** and its **standard matrix** is $\boldsymbol{B}\boldsymbol{A}$.

Composition

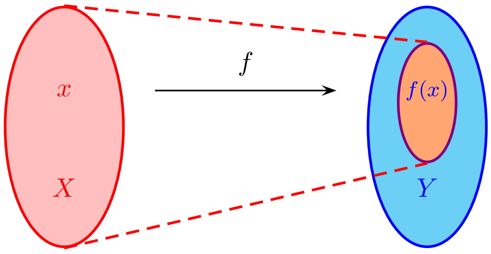
- **Theorem.** If $S : \mathbb{R}^n \to \mathbb{R}^m$ and $T : \mathbb{R}^m \to \mathbb{R}^k$ are **linear transformations**,
  - then $T \circ S : \mathbb{R}^n \to \mathbb{R}^k$ is also a **linear transformation**.

  Moreover, if $\boldsymbol{A}$ is the **standard matrix** for S and $\boldsymbol{B}$ is the **standard matrix** for T,
  - then $\boldsymbol{B}\boldsymbol{A}$ is the **standard matrix** for $T \circ S$.

- **Exercises.**
  - $I \circ S = S \circ I = S; \quad O \circ S = S \circ O = O;$
  - $c(T \circ S) = (cT) \circ S = T \circ (cS);$
  - $U \circ (T \circ S) = (U \circ T) \circ S;$
  - $(T_1 + T_2) \circ S = T_1 \circ S + T_2 \circ S;$
  - $T \circ (S_1 + S_2) = T \circ S_1 + T \circ S_2.$

Range of Function

- Let $f : X \to Y$ be a **function**:



  - The **range** of f is the set of all **images** of f:
    - $\mathrm{R}(f) = \{f(x) \mid x \in X\} \subseteq Y$.
- **Examples**. Let $f(x) = x^2$. Then $\mathrm{R}(f) = [0, \infty)$.

  Let $f(x) = \sin x$. Then $\mathrm{R}(f) = [-1, 1]$.

Range of Linear Transformation

- **Definition.** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a **linear transformation**.
  - The **range** of T is the set of all **images** of T:
    - $\mathrm{R}(T) = \{T(\boldsymbol{v}) \mid \boldsymbol{v} \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$.
- **Examples.** Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by
  - $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ y \\ x \end{pmatrix}$, for $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.
    - $\mathrm{R}(T) = \left\{ \begin{pmatrix} x+y \\ y \\ x \end{pmatrix} \middle| \, x, y \in \mathbb{R} \right\}$.
    - $\begin{pmatrix} x+y \\ y \\ x \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + y \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

Range of Linear Transformation

- **Definition.** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a **linear transformation**.
  - The **range** of T is the set of all **images** of T:
    - $\mathrm{R}(T) = \{T(\boldsymbol{v}) \mid \boldsymbol{v} \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$.
- **Examples.** Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by
  - $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ y \\ x \end{pmatrix}$, for $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.
    - $\mathrm{R}(T) = \mathrm{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right\} =$ vector space.
    - $\begin{pmatrix} x+y \\ y \\ x \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + y\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

Representation of Range

- Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a **linear transformation**.
  - How to determine the **range** of T?

Let $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ be a **basis** for $\mathbb{R}^n$.

- For any $\boldsymbol{v} \in \mathbb{R}^n$, write $\boldsymbol{v} = c_1\boldsymbol{v}_1 + \cdots + c_n\boldsymbol{v}_n$.

$$\begin{aligned} T(\boldsymbol{v}) &= c_1T(\boldsymbol{v}_1) + \cdots + c_nT(\boldsymbol{v}_n) \\ &\in \mathrm{span}\{T(\boldsymbol{v}_1), \ldots, T(\boldsymbol{v}_n)\}. \end{aligned}$$

$$\therefore \quad \begin{aligned} \mathrm{R}(T) &= \{T(\boldsymbol{v}) \mid \boldsymbol{v} \in \mathbb{R}^m\} \\ &\subseteq \mathrm{span}\{T(\boldsymbol{v}_1), \ldots, T(\boldsymbol{v}_n)\} \end{aligned}$$

On the other hand, every linear combination

- $c_1T(\boldsymbol{v}_1) + \cdots + c_nT(\boldsymbol{v}_n) = T(\boldsymbol{v}) \in \mathrm{R}(T)$.

$\therefore \quad \mathrm{span}\{T(\boldsymbol{v}_1), \ldots, T(\boldsymbol{v}_n)\} \subseteq \mathrm{R}(T)$.

Representation of Range

- **Theorem.** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a **linear transformation**.
  - Then the **range** of T is given by
    - $\mathrm{R}(T) = \mathrm{span}\{T(\boldsymbol{v}_1), \ldots, T(\boldsymbol{v}_n)\}$,

    where $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ is any basis for $\mathbb{R}^n$.
  - In particular, $\mathrm{R}(T)$ is a **subspace** of $\mathbb{R}^m$.
- **Example.** $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ y \\ x \end{pmatrix}$.
  - Use the **standard basis**:
    - $T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $T\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.
    - $\mathrm{R}(T) = \mathrm{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right\}$.

Representation of Range

- Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a **linear transformation**.
  - Let $\boldsymbol{A}$ be the **standard matrix** for T.
    - $T(\boldsymbol{v}) = \boldsymbol{A}\boldsymbol{v}$ for all $\boldsymbol{v} \in \mathbb{R}^n$.

    Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ be the **standard basis** for $\mathbb{R}^n$.
    - $T(\boldsymbol{e}_j) = \boldsymbol{A}\boldsymbol{e}_j = j$th column of $\boldsymbol{A}$.
  - Recall that $\mathrm{R}(T) = \mathrm{span}\{T(\boldsymbol{e}_1), T(\boldsymbol{e}_2), \ldots, T(\boldsymbol{e}_n)\}$.
    - $\mathrm{R}(T)$ is the subspace of $\mathbb{R}^m$ spanned by columns of $\boldsymbol{A}$.

  $\therefore$ $\mathrm{R}(T) =$ column space of $\boldsymbol{A}$.
- **Theorem.** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a **linear transformation** and $\boldsymbol{A}$ the **standard matrix** for T.
  - Then $\mathrm{R}(T) =$ column space of $\boldsymbol{A}$.

Representation of Range

- **Definition.** Let T be a **linear transformation**.
  - The **rank** of T is defined as the dimension of $\mathrm{R}(T)$:
    - $\operatorname{rank}(T) = \dim \mathrm{R}(T)$.
- Let $\boldsymbol{A}$ be the standard matrix for a linear transformation T.
  - $\mathrm{R}(T) =$ column space of $\boldsymbol{A}$.
  - $\operatorname{rank}(T) = \dim \mathrm{R}(T) = \dim$ (coln space of $\boldsymbol{A}$) $= \operatorname{rank}(\boldsymbol{A})$.
- **Example.** $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ y \\ x \end{pmatrix}$.
  - Standard matrix: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Representation of Range

- **Definition.** Let T be a **linear transformation**.
  - The **rank** of T is defined as the dimension of $\mathrm{R}(T)$:
    - $\operatorname{rank}(T) = \dim \mathrm{R}(T)$.
- Let $\boldsymbol{A}$ be the standard matrix for a linear transformation T.
  - $\mathrm{R}(T) =$ column space of $\boldsymbol{A}$.
  - $\operatorname{rank}(T) = \dim \mathrm{R}(T) = \dim$ (coln space of $\boldsymbol{A}$) $= \operatorname{rank}(\boldsymbol{A})$.
- **Example.** $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ y \\ x \end{pmatrix}$.
  - $\mathrm{R}(T) = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right\}$; $\operatorname{rank}(T) = 2$.

Example

- Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be defined by
  - $T\left(\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x+2y+z \\ x+3y \\ x+4y-z \\ y-z \end{pmatrix}, \quad \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \in \mathbb{R}^4.$
    - Standard matrix: $\begin{pmatrix} 0 & 1 & 2 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$
    - $\mathrm{R}(T) = \mathrm{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ -1 \end{pmatrix} \right\}.$
  - How to find a **basis** for $\mathrm{R}(T)$?

Example

- Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be defined by
  - $T\left(\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x+2y+z \\ x+3y \\ x+4y-z \\ y-z \end{pmatrix}, \quad \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \in \mathbb{R}^4.$
    - $\begin{pmatrix} 0 & 1 & 2 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \overset{\text{G.E.}}{\cdots\longrightarrow} \begin{pmatrix} 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
    - $\mathrm{R}(T) = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \\ 1 \end{pmatrix} \right\}.$

      $\mathrm{rank}(T) = \dim \mathrm{R}(T) = 2.$

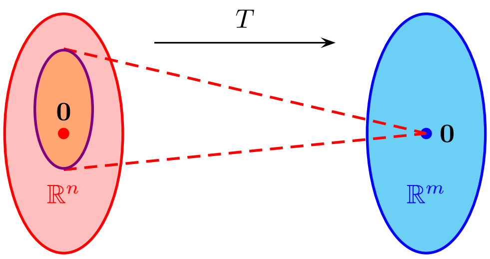
Example

- Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be defined by
  - $T\left(\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x + 2y + z \\ x + 3y \\ x + 4y - z \\ y - z \end{pmatrix}, \quad \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \in \mathbb{R}^4.$
    - $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 3 & 4 & 1 \\ 1 & 0 & -1 & -1 \end{pmatrix} \xrightarrow{\text{G.J.E.}} \cdots \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
    - $\mathrm{R}(T) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix} \right\}.$

      $\operatorname{rank}(T) = \dim \mathrm{R}(T) = 2.$

Kernel of Linear Transformation

- **Definition.** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a **linear transformation**.



  - The **kernel** of T is the set of all vectors in $\mathbb{R}^n$ whose image is the zero vector in $\mathbb{R}^m$.
    - $\mathrm{Ker}(T) = \{\boldsymbol{v} \in \mathbb{R}^n \mid T(\boldsymbol{v}) = \mathbf{0}\} \subseteq \mathbb{R}^n.$
  - Recall that $T(\mathbf{0}) = \mathbf{0}$.
    - $\mathrm{Ker}(T)$ contains the zero vector in $\mathbb{R}^n$.

Examples

- Let $T_1 : \mathbb{R}^3 \to \mathbb{R}^4$ be defined by
  - $T_1\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 2x - y \\ x - y + 3z \\ -5x + y \\ x - z \end{pmatrix}$, for $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$.
  - Find the **kernel** of T_1.
    - Let $T_1\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 2x - y \\ x - y + 3z \\ -5x + y \\ x - z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.
    - $\begin{pmatrix} 2 & -1 & 0 \\ 1 & -1 & 3 \\ -5 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

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Examples

- Let $T_1 : \mathbb{R}^3 \to \mathbb{R}^4$ be defined by
  - $T_1\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 2x - y \\ x - y + 3z \\ -5x + y \\ x - z \end{pmatrix}$, for $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$.
  - Find the **kernel** of T_1.
    - Let $T_1\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 2x - y \\ x - y + 3z \\ -5x + y \\ x - z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.
    - $\begin{pmatrix} 2 & -1 & 0 \\ 1 & -1 & 3 \\ -5 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \cdots \xrightarrow{\text{G.J.E.}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

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Examples

- Let $T_1 : \mathbb{R}^3 \to \mathbb{R}^4$ be defined by
  - $T_1\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 2x - y \\ x - y + 3z \\ -5x + y \\ x - z \end{pmatrix}$, for $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$.
  - Find the **kernel** of T_1.
    - Let $T_1\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 2x - y \\ x - y + 3z \\ -5x + y \\ x - z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.
    - $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \mathrm{Ker}(T_1) = \left\{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right\}$.

Examples

- Let $T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by
  - $T_2\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} z - y \\ 0 \\ x \end{pmatrix}$, for $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$.
  - Find the **kernel** of T_2.
    - Let $T_2\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} z - y \\ 0 \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.
    - $z = y$ and $x = 0 \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = y\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.
    - $\mathrm{Ker}(T_2) = \left\{\begin{pmatrix} 0 \\ y \\ y \end{pmatrix} \middle| \, y \in \mathbb{R}\right\} = \mathrm{span}\left\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$.

Representation of Kernel

- Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a **linear transformation**.
  - Let $\boldsymbol{A}$ be the **standard matrix** for T.
    - $T(\boldsymbol{v}) = \boldsymbol{A}\boldsymbol{v}$ for all $\boldsymbol{v} \in \mathbb{R}^n$.

$$\begin{aligned} \mathrm{Ker}(T) &= \{\boldsymbol{v} \in \mathbb{R}^n \mid T(\boldsymbol{v}) = \boldsymbol{0}\} \\ &= \{\boldsymbol{v} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{v} = \boldsymbol{0}\} \\ &= \text{nullspace of } \boldsymbol{A}. \end{aligned}$$

- **Theorem.** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a **linear transformation** and $\boldsymbol{A}$ the **standard matrix** for T.
  - $\mathrm{Ker}(T) =$ nullspace of $\boldsymbol{A}$.

  In particular, $\mathrm{Ker}(T)$ is always a **subspace** of $\mathbb{R}^n$.

Representation of Kernel

- **Definition.** Let T be a **linear transformation**.
  - The **nullity** of T is defined as the dimension of $\mathrm{Ker}(T)$.
    - $\mathrm{nullity}(T) = \dim \mathrm{Ker}(T)$.
- Recall that if $\boldsymbol{A}$ is the standard matrix for T, then
  - $\mathrm{Ker}(T) =$ nullspace of $\boldsymbol{A}$.

$$\begin{aligned} \mathrm{nullity}(T) = \dim \mathrm{Ker}(T) &= \dim(\text{nullspace of } \boldsymbol{A}) \\ &= \mathrm{nullity}(\boldsymbol{A}). \end{aligned}$$

- **Examples.** $\mathrm{Ker}(T_1) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$.
  - $\mathrm{nullity}(T_1) = 0$.

Representation of Kernel

- **Definition.** Let T be a **linear transformation**.
  - The **nullity** of T is defined as the dimension of $\mathrm{Ker}(T)$.
    - $\mathrm{nullity}(T) = \dim \mathrm{Ker}(T)$.
- Recall that if $\boldsymbol{A}$ is the standard matrix for T, then
  - $\mathrm{Ker}(T) =$ nullspace of $\boldsymbol{A}$.

$$\begin{aligned}\mathrm{nullity}(T) = \dim \mathrm{Ker}(T) &= \dim(\text{nullspace of } \boldsymbol{A}) \\ &= \mathrm{nullity}(\boldsymbol{A}).\end{aligned}$$

- **Examples.** $\mathrm{Ker}(T_2) = \mathrm{span}\left\{\begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$.
  - $\mathrm{nullity}(T_2) = 1$.

Example

- Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be defined by
  - $T\left(\begin{pmatrix}w\\x\\y\\z\end{pmatrix}\right) = \begin{pmatrix}x+2y+z\\x+3y\\x+4y-z\\y-z\end{pmatrix}$, for $\begin{pmatrix}w\\x\\y\\z\end{pmatrix} \in \mathbb{R}^4$.
    - Standard matrix: $\boldsymbol{A} = \begin{pmatrix}0&1&2&1\\0&1&3&0\\0&1&4&-1\\0&0&1&-1\end{pmatrix}$.

$$\begin{pmatrix}0&1&2&1\\0&1&3&0\\0&1&4&-1\\0&0&1&-1\end{pmatrix} \cdots\xrightarrow{\text{G.J.E.}} \begin{pmatrix}0&1&0&3\\0&0&1&-1\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

$w = s$, $z = t$ and $x = -3t$, $y = t$.

Example

- Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be defined by
  - $T\left(\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x+2y+z \\ x+3y \\ x+4y-z \\ y-z \end{pmatrix}$, for $\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \in \mathbb{R}^4$.
    - $\mathrm{Ker}(T) =$ null sp. of $\boldsymbol{A} = \left\{ \begin{pmatrix} s \\ -3t \\ t \\ t \end{pmatrix} \middle| s, t \in \mathbb{R} \right\}$

      $\mathrm{Ker}(T) = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 1 \\ 1 \end{pmatrix} \right\}$,

      $\mathrm{nullity}(T) = \dim \mathrm{Ker}(T) = \mathrm{nullity}(\boldsymbol{A}) = 2$.

Properties

- Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a **linear transformation**.
  - Let $\boldsymbol{A}$ be the **standard matrix** for T.
    - $\boldsymbol{A}$ is $m \times n$ such that $T(\boldsymbol{v}) = \boldsymbol{A}\boldsymbol{v}$ for all $\boldsymbol{v} \in \mathbb{R}^n$.

  We have proved that

  1. $\mathrm{R}(T) =$ column space of $\boldsymbol{A}$.
     - $\mathrm{rank}(T) = \mathrm{rank}(\boldsymbol{A})$.
  2. $\mathrm{Ker}(T) =$ nullspace of $\boldsymbol{A}$.
     - $\mathrm{nullity}(T) = \mathrm{nullity}(\boldsymbol{A})$.

  Recall **Dimension Theorem for Matrices**:

  - $\mathrm{rank}(\boldsymbol{A}) + \mathrm{nullity}(\boldsymbol{A}) =$ number of colns of $\boldsymbol{A} = n$.

  $\therefore$ $\mathrm{rank}(T) + \mathrm{nullity}(T) = n =$ dimension of domain.

Properties

- **Dimension Theorem for Linear Transformations**.

  Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a **linear transformation**. Then

  - $\text{rank}(T) + \text{nullity}(T) = n$.

- Recall that $T : V \to W$ between **vector spaces** is a **linear transformation** if

  - $T(c\boldsymbol{u} + d\boldsymbol{v}) = cT(\boldsymbol{u}) + dT(\boldsymbol{v})$, $\boldsymbol{u}, \boldsymbol{v} \in V$, $c, d \in \mathbb{R}$.

  We can similarly define and prove that

  - $\text{R}(T) = \{T(\boldsymbol{v}) \mid \boldsymbol{v} \in V\}$ is a subspace of W.
    - $\text{rank}(T) = \dim \text{R}(T)$.
  - $\text{Ker}(T) = \{\boldsymbol{v} \in V \mid T(\boldsymbol{v}) = \boldsymbol{0}\}$ is a subspace of V.
    - $\text{nullity}(T) = \dim \text{Ker}(T)$.
  - $\text{rank}(T) + \text{nullity}(T) = \dim V$.

Geometric Linear Transformations

Introduction

- Recall that a linear transformation is uniquely determined by its images on a basis:

  Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and $S = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ a basis for $\mathbb{R}^n$.

  - If $(\boldsymbol{v})_S = (c_1, \ldots, c_n)$, then
    - $T(\boldsymbol{v}) = c_1 T(\boldsymbol{v}_1) + \cdots + c_n T(\boldsymbol{v}_n)$.

  In particular, let $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ be the standard basis for $\mathbb{R}^n$.

  - If $\boldsymbol{v} = (v_1, \ldots, v_n)$, then
    - $T(\boldsymbol{v}) = v_1 T(\boldsymbol{e}_1) + \cdots + v_n T(\boldsymbol{e}_n)$.

- To study the geometric interpretation a linear transformation,

  - it suffices to check the effect of the linear transformation on a basis (in particular, standard basis) for its domain.

Scalings

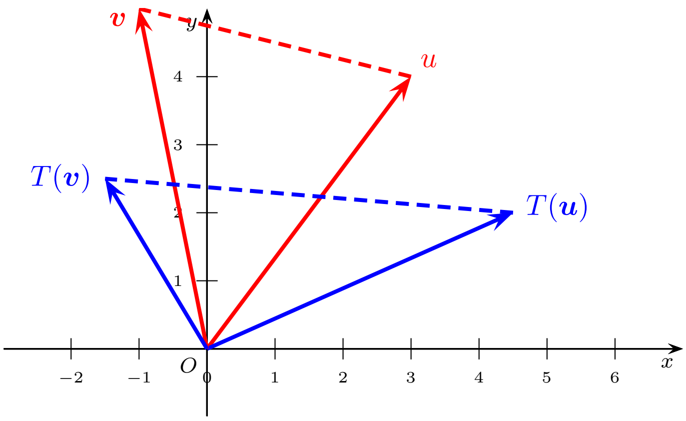
- Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation such that
  - $T\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = \begin{pmatrix}\lambda_1\\0\end{pmatrix}$, $T\left(\begin{pmatrix}0\\1\end{pmatrix}\right) = \begin{pmatrix}0\\\lambda_2\end{pmatrix}$.

  Then the standard matrix for T is $\begin{pmatrix}\lambda_1 & 0\\0 & \lambda_2\end{pmatrix}$.
  - $T\left(\begin{pmatrix}x\\y\end{pmatrix}\right) = \begin{pmatrix}\lambda_1 & 0\\0 & \lambda_2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\lambda_1 x\\\lambda_2 y\end{pmatrix}$.

  Suppose that $\lambda_1 > 0$ and $\lambda_2 > 0$.
  - Then T is a **scaling** in $\mathbb{R}^2$
    - along the x-axis by a factor of λ_1, and
    - along the y-axis by a factor of λ_2.

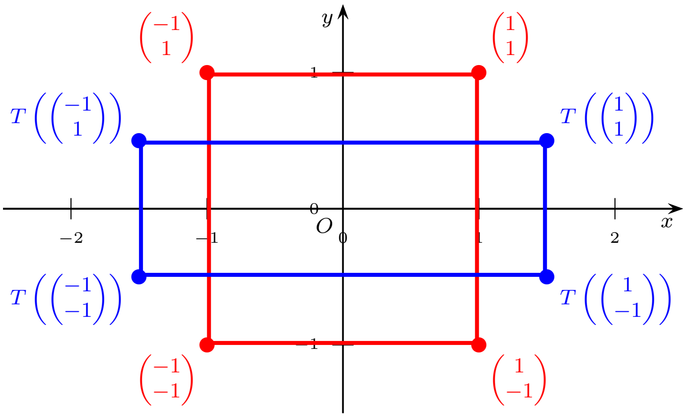
Example

- Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ with standard matrix $\begin{pmatrix}1.5 & 0\\0 & 0.5\end{pmatrix}$.
  - Then T is a **scaling** in $\mathbb{R}^2$
    - along the x-axis by 1.5 & along the y-axis by 0.5.



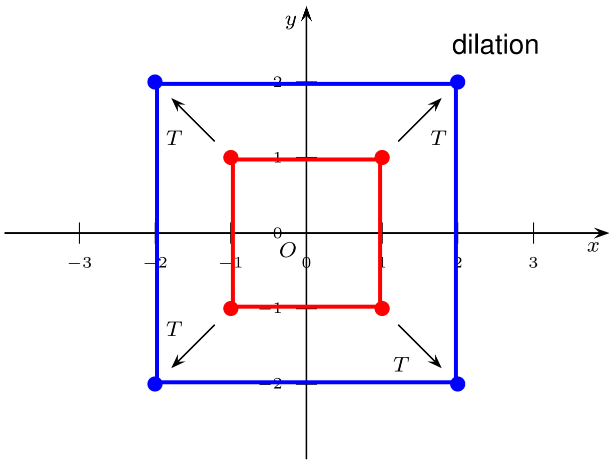
Example

- Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ with standard matrix $\begin{pmatrix} 1.5 & 0 \\ 0 & 0.5 \end{pmatrix}$.
  - Then T is a **scaling** in $\mathbb{R}^2$
    - along the x-axis by 1.5 & along the y-axis by 0.5.



Remark

- Suppose that the scaling T satisfies $\lambda_1 = \lambda_2$.
  - Let $\lambda = \lambda_1 = \lambda_2$. The standard matrix of T is $\lambda \boldsymbol{I}_2$.
    - T is a **dilation** if $\lambda > 1$.
    - T is a **contraction** if $0 < \lambda < 1$.



Remark

- Suppose that the scaling T satisfies $\lambda_1 = \lambda_2$.
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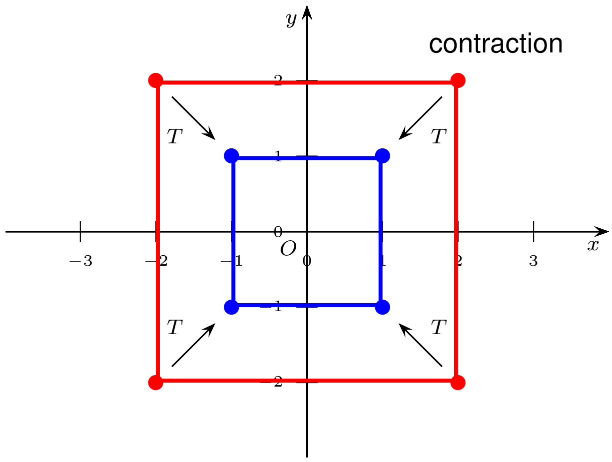


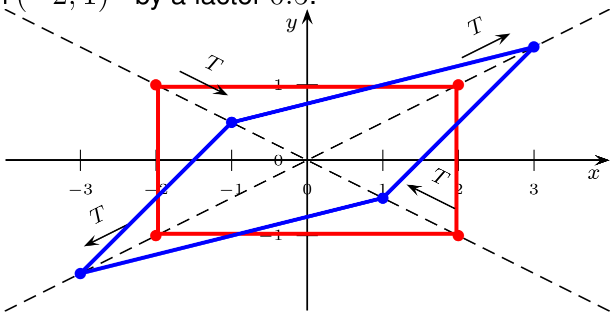
Diagonalization

- Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation.
  - Let $\boldsymbol{A}$ be the standard matrix.

  Assume: $\boldsymbol{A}$ is diagonalizable with positive eigenvalues λ_1, λ_2.
  - There exists invertible $\boldsymbol{P}$ such that
    - $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.
  - Let $\boldsymbol{P} = \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{pmatrix}$. $T(\boldsymbol{v}_1) = \lambda_1 \boldsymbol{v}_1$, $T(\boldsymbol{v}_2) = \lambda \boldsymbol{v}_2$.
    - Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2\}$. Then S is a basis for $\mathbb{R}^2$.
    - $[T(\boldsymbol{v})]_S = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} [\boldsymbol{v}]_S$.
  - T can be viewed as a scaling
    - along the direction of $\boldsymbol{v}_1$ by factor $\lambda_1 > 0$, &
    - along the direction of $\boldsymbol{v}_2$ by factor $\lambda_2 > 0$.

Example

- Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation with
  - Standard matrix $\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 0.25 & 1 \end{pmatrix}$.
  - $\begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ 0.25 & 1 \end{pmatrix} \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1.5 & 0 \\ 0 & 0.5 \end{pmatrix}$.
  - T is a scaling
    - along the direction $(2, 1)^{\mathrm{T}}$ by a factor 1.5, and
    - along the direction $(-2, 1)^{\mathrm{T}}$ by a factor 0.5.



Scaling in $\mathbb{R}^3$

- Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation with
  - Standard matrix $\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$, $\lambda_1, \lambda_2, \lambda_3 > 0$.

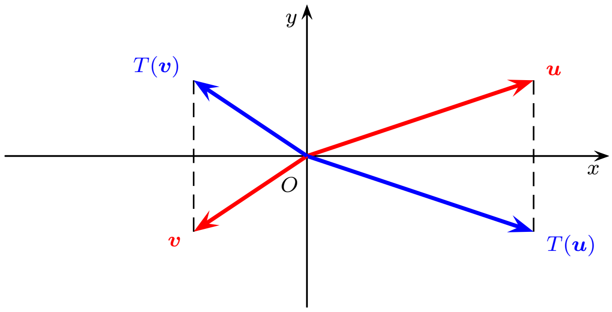
  Then T is a **scaling**
  - along the x-axis by factor λ_1,
  - along the y-axis by factor λ_2,
  - along the z-axis by factor λ_3.

  Suppose that $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$.
  - T is a **dilation** if $\lambda > 1$.
  - T is a **contraction** if $0 < \lambda < 1$.
- Suppose T has standard matrix $\boldsymbol{A}$.
  - Assume $\boldsymbol{A}$ is diagonalizable with positive eigenvalues.
  - Then T can be viewed as a scaling with respect to a basis for $\mathbb{R}^3$. (Exercise.)

Reflection

- Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with
  - Standard matrix: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
  - $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x \\ -y \end{pmatrix}$.

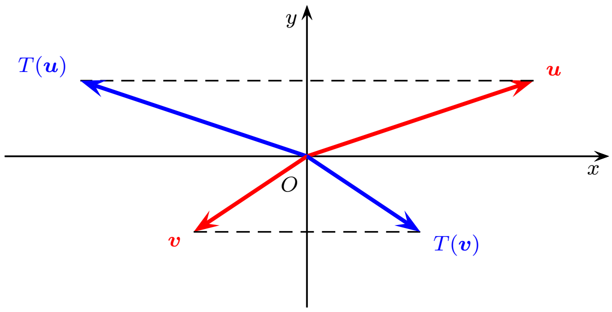
  T is the **reflection** with respect to the x-axis.



Reflection

- Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with
  - Standard matrix: $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.
  - $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} -x \\ y \end{pmatrix}$.

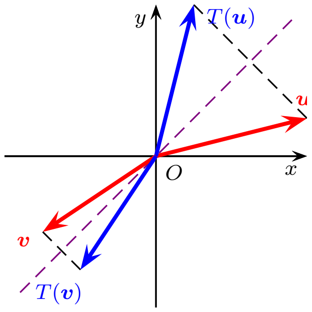
  T is the **reflection** with respect to the y-axis.



Reflection

- Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with
  - Standard matrix: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
  - $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} y \\ x \end{pmatrix}$.

  T is the **reflection** with respect to the line $y = x$.



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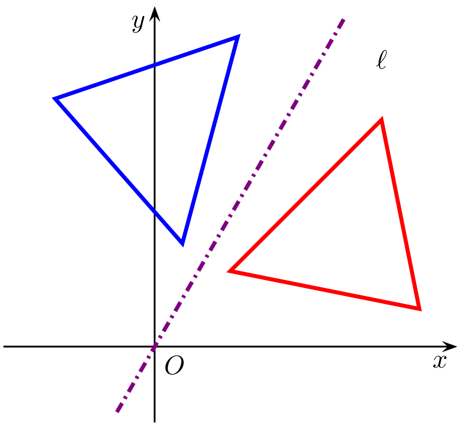
Reflection

- Consider a line ℓ passing through the origin $(0, 0)$.



Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ denote the reflection with respect to ℓ.
  - Then T is a linear transformation (show by geometry).

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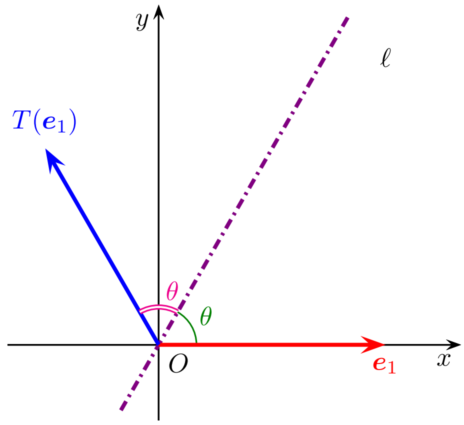
Reflection

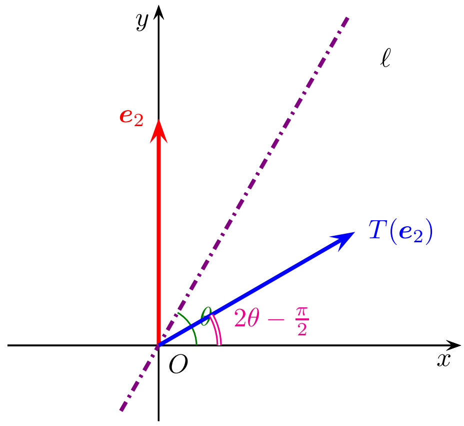
- Let θ be the angle between ℓ and the x-axis.



  - $T\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = \begin{pmatrix}\cos(2\theta)\\ \sin(2\theta)\end{pmatrix}$.

Reflection

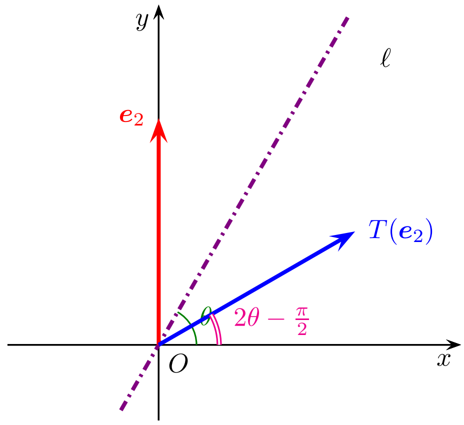
- Let θ be the angle between ℓ and the x-axis.



  - $T\left(\begin{pmatrix}0\\1\end{pmatrix}\right) = \begin{pmatrix}\cos(2\theta - \frac{\pi}{2})\\ \sin(2\theta - \frac{\pi}{2})\end{pmatrix} = \begin{pmatrix}\sin(2\theta)\\ -\cos(2\theta)\end{pmatrix}$.

Reflection

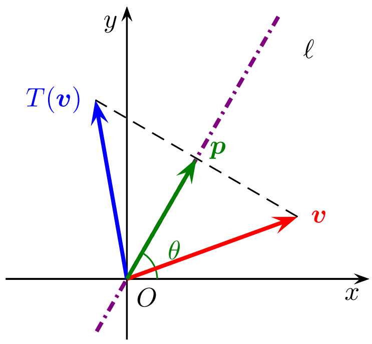
- Let θ be the angle between ℓ and the x-axis.



  - The standard matrix for T is $\begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$.
    - Every orthogonal matrix of $\det = -1$ is in this form.

Remark

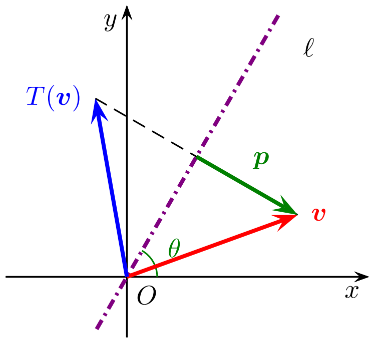
- Let $\boldsymbol{n} = (\cos\theta, \sin\theta)^{\mathrm{T}}$ be a unit vector on ℓ.



  - $\boldsymbol{p}$ is the projection of $\boldsymbol{v}$ onto $\operatorname{span}\{\boldsymbol{n}\}$.
    - $\boldsymbol{p} = (\boldsymbol{v} \cdot \boldsymbol{n})\boldsymbol{n}$.
  - $\boldsymbol{p}$ is the midpoint of $\boldsymbol{v}$ and $T(\boldsymbol{v})$.
    - $T(\boldsymbol{v}) = 2\boldsymbol{p} - \boldsymbol{v} = 2(\boldsymbol{v} \cdot \boldsymbol{n})\boldsymbol{n} - \boldsymbol{v}$.

Remark

- Let $\boldsymbol{n} = (\sin\theta, -\cos\theta)^{\mathrm{T}}$ be a unit vector orthogonal to ℓ.



  - $\boldsymbol{p}$ is the projection of $\boldsymbol{v}$ onto $\mathrm{span}\{\boldsymbol{n}\}$.
    - $\boldsymbol{p} = (\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{n}$.
  - Note that $T(\boldsymbol{v}) + 2\boldsymbol{p} = \boldsymbol{v}$.
    - $T(\boldsymbol{v}) = \boldsymbol{v} - 2\boldsymbol{p} = \boldsymbol{v} - 2(\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{n}$.

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Reflections in $\mathbb{R}^3$

- Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation.
  - If the standard matrix is $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$,
    - then T is the reflection with respect to the xy-plane.
  - If the standard matrix is $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,
    - then T is the reflection with respect to the xy-plane.
  - If the standard matrix is $\boldsymbol{A} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,
    - then T is the reflection with respect to the yz-plane.

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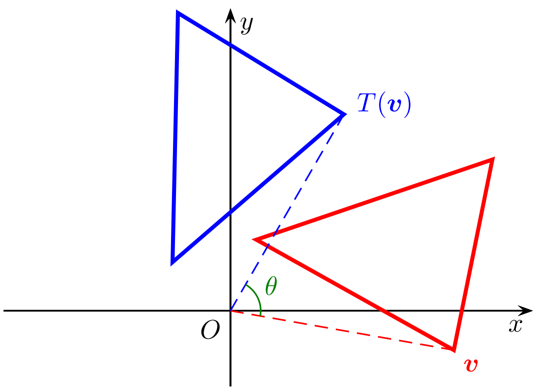
Reflections in $\mathbb{R}^3$

- Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the **reflection** with respect to the plane $ax + by + cz = 0$, where a, b, c not all zero.
  - Then $\boldsymbol{n} = (a, b, c)^{\mathrm{T}}$ is orthogonal to the plane.

  (Exercise) $T(\boldsymbol{v}) = \boldsymbol{v} - \left(2\frac{\boldsymbol{v} \cdot \boldsymbol{n}}{\|\boldsymbol{n}\|^2}\right)\boldsymbol{n}, \quad \boldsymbol{v} \in \mathbb{R}^3.$
  - *Hint*: The midpoint of $\boldsymbol{v}$ and $T(\boldsymbol{v})$ is the projection of $\boldsymbol{v}$ onto the plane $ax + by + cz = 0$.
- **Problem.** Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the reflection with respect to a straight line passing through the origin O.
  - Can you find the formula of T?

Rotations

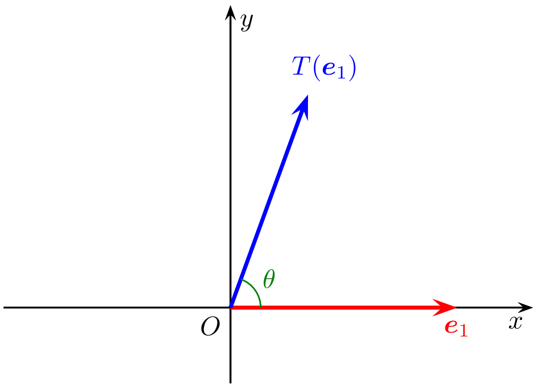
- Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the **rotation** about the origin by θ.
  - Then T is a linear transformation.



  - It suffices to determine $T(\boldsymbol{e}_1)$ and $T(\boldsymbol{e}_2)$.

Rotations

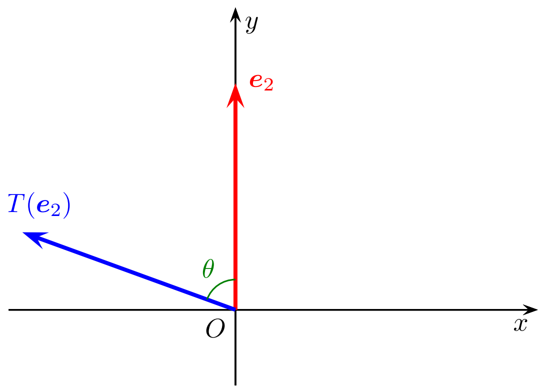
- Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the **rotation** about the origin by θ.
  - Then T is a linear transformation.



  - $T\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$.

Rotations

- Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the **rotation** about the origin by θ.
  - Then T is a linear transformation.



  - $T\left(\begin{pmatrix}0\\1\end{pmatrix}\right) = \begin{pmatrix}\cos(\theta+\frac{\pi}{2})\\ \sin(\theta+\frac{\pi}{2})\end{pmatrix} = \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$.

Rotations

- Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the **rotation** about the origin by θ.
  - Then T is a linear transformation.
  - The standard matrix for T is $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.
    - Every orthogonal matrix of $\det = 1$ is in this form.
- Suppose standard matrix $\boldsymbol{A}$ for $T : \mathbb{R}^2 \to \mathbb{R}^2$ is orthogonal.
  - If $\det(\boldsymbol{A}) = 1$, T represents a rotation about the origin.
  - If $\det(\boldsymbol{A}) = -1$, T represents the reflection with respect to a line passing through the origin.
- $\begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$.
  - Let ℓ denote the line $\mathrm{span}\{(\cos\theta, \sin\theta)^{\mathrm{T}}\}$.
    - Reflection with respect to ℓ
      $\Leftrightarrow$ reflection with respect to the x-axis
      & rotation about the origin anticlockwise by 2θ.

Rotations in $\mathbb{R}^3$

- Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the **rotation** about the z-axis anticlockwise by angle θ.
  - The z-coordinate does not change.
  - On the xy-plane, it is the rotation about the origin on the plane $z = z_0$ anticlockwise by θ.
    - $T\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$.
    - $T\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}$.
    - $T\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

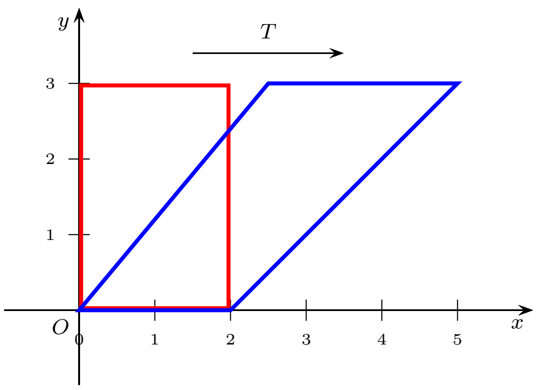
Rotations in $\mathbb{R}^3$

- Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the rotation about the z-axis anticlockwise by angle θ.
  - Standard matrix $\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
- Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the rotation about the x-axis anticlockwise by angle θ.
  - Standard matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$.
- Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the rotation about the y-axis anticlockwise by angle θ.
  - Standard matrix $\begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$.

Shears

- Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by
  - $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x + ky \\ y \end{pmatrix}$.

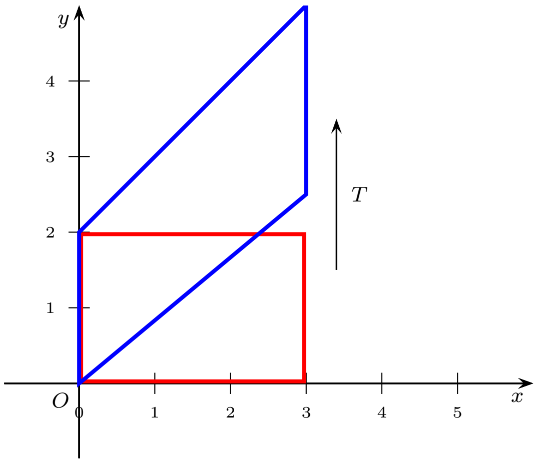
  Then T is a **shear** in the x-direction by a factor k.



Shears

- Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by
  - $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x \\ kx + y \end{pmatrix}$.

  Then T is a **shear** in the y-direction by a factor k.



Shears

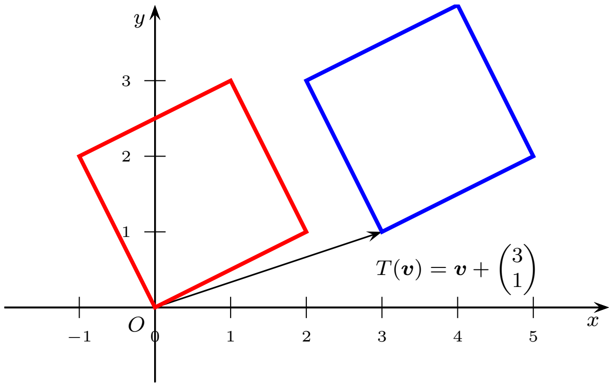
- Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by
  - $T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x + k_1 z \\ y + k_2 z \\ z \end{pmatrix}$.

  Then T is a **shear** in the x-direction by factor k_1, and in the y-direction by factor k_2.

  - On yz-plane $x = 0$, it is a shear in y-direction by k_2.
  - On xz-plane $y = 0$, it is a share in x-direction by k_1.
  - On the plane $z = 1$,
    - $T\left(\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}\right) = \begin{pmatrix} x + k_1 \\ y + k_2 \\ 1 \end{pmatrix}$.

Translations

- Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by
  - $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+a \\ y+b \end{pmatrix}$, a, b are real numbers.
- T is called a **translation** by $(a, b)^{\mathrm{T}}$.
  - T is **not** a linear transformation unless $a = b = 0$.

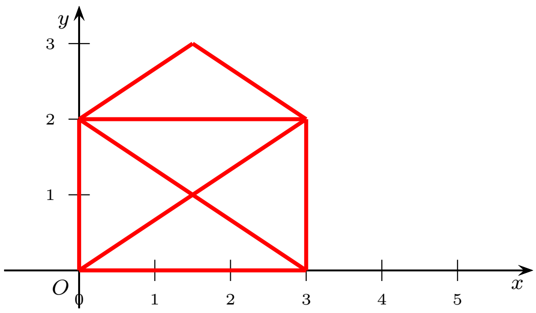


2D Computer Graphic

- In 2D computer graphic, a figure is drawn by connecting
  - points $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$.

  It can be written as an $2 \times n$ matrix:
  - $\boldsymbol{M} = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{pmatrix}$.

  For example, $\boldsymbol{M} = \begin{pmatrix} 0 & 3 & 3 & 0 & 0 & 3 & 1.5 & 0 & 3 \\ 0 & 0 & 2 & 0 & 2 & 2 & 3 & 2 & 0 \end{pmatrix}$.



2D Computer Graphic

- Primary geometric transformations on 2D graphics:
  - Scalings, Reflections, Rotations and Translations.
- Let T be a scaling/reflection/rotation/translation on $\mathbb{R}^2$.

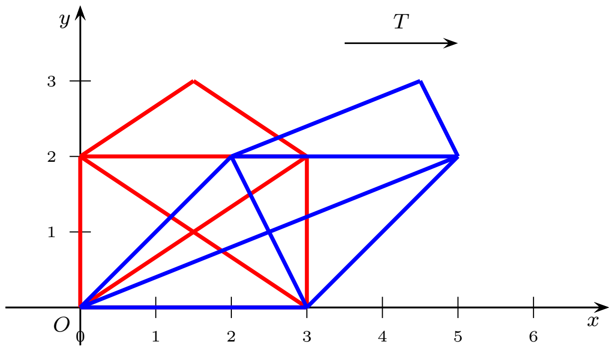
  Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ be a 2D computer graphic.
  - The resulting graphic by T is $T(\boldsymbol{v}_1), \ldots, T(\boldsymbol{v}_n)$.
- Suppose T is a scaling, reflection or rotation.
  - Then T is linear with standard matrix $\boldsymbol{A}$.

  If the 2D computer graphic is $\boldsymbol{M} = \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{pmatrix}$,
  - then the resulting graphic by T is

$$\begin{aligned} \begin{pmatrix} T(\boldsymbol{v}_1) & \cdots & T(\boldsymbol{v}_n) \end{pmatrix} &= \begin{pmatrix} \boldsymbol{A}\boldsymbol{v}_1 & \cdots & \boldsymbol{A}\boldsymbol{v}_n \end{pmatrix} \\ &= \boldsymbol{A} \begin{pmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \end{pmatrix} = \boldsymbol{A}\boldsymbol{M}. \end{aligned}$$

Example

- Let $\boldsymbol{M} = \begin{pmatrix} 0 & 3 & 3 & 0 & 0 & 3 & 1.5 & 0 & 3 \\ 0 & 0 & 2 & 0 & 2 & 2 & 3 & 2 & 0 \end{pmatrix}$.
  - Let $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ y \end{pmatrix}$. $\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.
    - $\boldsymbol{A}\boldsymbol{M} = \begin{pmatrix} 0 & 3 & 5 & 0 & 2 & 5 & 4.5 & 2 & 3 \\ 0 & 0 & 2 & 0 & 2 & 2 & 3 & 2 & 0 \end{pmatrix}$.



Homogeneous Coordinate System

- **Homogeneous coordinate system** is formed by identifying $\mathbb{R}^2$ with plane $z = 1$ in $\mathbb{R}^3$: $\begin{pmatrix} a \\ b \end{pmatrix} \leftrightarrow \begin{pmatrix} a \\ b \\ 1 \end{pmatrix}$.
- A graphic $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$ is identified by
  - $(a_1, b_1, 1), (a_2, b_2, 1), \ldots, (a_n, b_n, 1)$.

  The associated matrix is $\boldsymbol{M} = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \\ 1 & 1 & \cdots & 1 \end{pmatrix}$.

  Let T be the translation $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+a \\ y+b \end{pmatrix}$.
  - The shear $T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x+az \\ y+bz \\ z \end{pmatrix}$ will do the job:

    $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$. $\boldsymbol{AM} = \begin{pmatrix} a_1+a & \cdots & a_n+a \\ b_1+b & \cdots & b_n+b \\ 1 & \cdots & 1 \end{pmatrix}$.

Example

- Let $\boldsymbol{M} = \begin{pmatrix} 0 & 3 & 3 & 0 & 0 & 3 & 1.5 & 0 & 3 \\ 0 & 0 & 2 & 0 & 2 & 2 & 3 & 2 & 0 \end{pmatrix}$.
  - Let $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+2 \\ y+1 \end{pmatrix}$.

  Set $\boldsymbol{M}' = \begin{pmatrix} 0 & 3 & 3 & 0 & 0 & 3 & 1.5 & 0 & 3 \\ 0 & 0 & 2 & 0 & 2 & 2 & 3 & 2 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$.
  - Standard matrix of the shear: $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.
    - $\boldsymbol{AM}' = \begin{pmatrix} 2 & 5 & 5 & 2 & 2 & 5 & 3.5 & 2 & 5 \\ 1 & 1 & 3 & 1 & 3 & 3 & 4 & 3 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$.

  Result graph: $\begin{pmatrix} 2 & 5 & 5 & 2 & 2 & 5 & 3.5 & 2 & 5 \\ 1 & 1 & 3 & 1 & 3 & 3 & 4 & 3 & 1 \end{pmatrix}$.

Example

- Let $M = \begin{pmatrix} 0 & 3 & 3 & 0 & 0 & 3 & 1.5 & 0 & 3 \\ 0 & 0 & 2 & 0 & 2 & 2 & 3 & 2 & 0 \end{pmatrix}$.
  - Let $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+2 \\ y+1 \end{pmatrix}$.
  - Result graph: $\begin{pmatrix} 2 & 5 & 5 & 2 & 2 & 5 & 3.5 & 2 & 5 \\ 1 & 1 & 3 & 1 & 3 & 3 & 4 & 3 & 1 \end{pmatrix}$