Section 7.1

Linear Transformations from $\mathbf{R}^n$ to $\mathbf{R}^m$

Objective

- What is a linear transformation?
- How are linear transformations related to matrices?
- What are the conditions of a linear transformation?
- How to use basis to determine linear transformation?

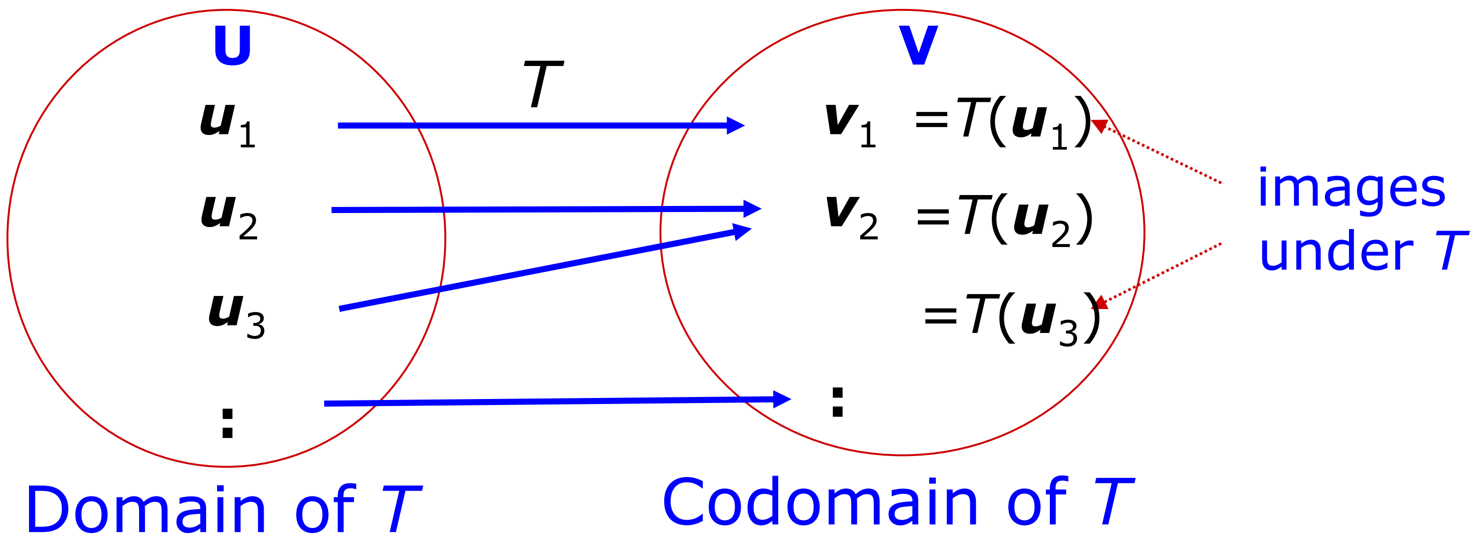
In this chapter, we shall always write vectors in $\mathbf{R}^n$ as column vectors.

Mapping $\boxed{T : \mathbf{U} \to \mathbf{V}}$

Let **U** and **V** be two sets

A mapping from **U** to **V**

assigns every element of **U** with an element of **V**

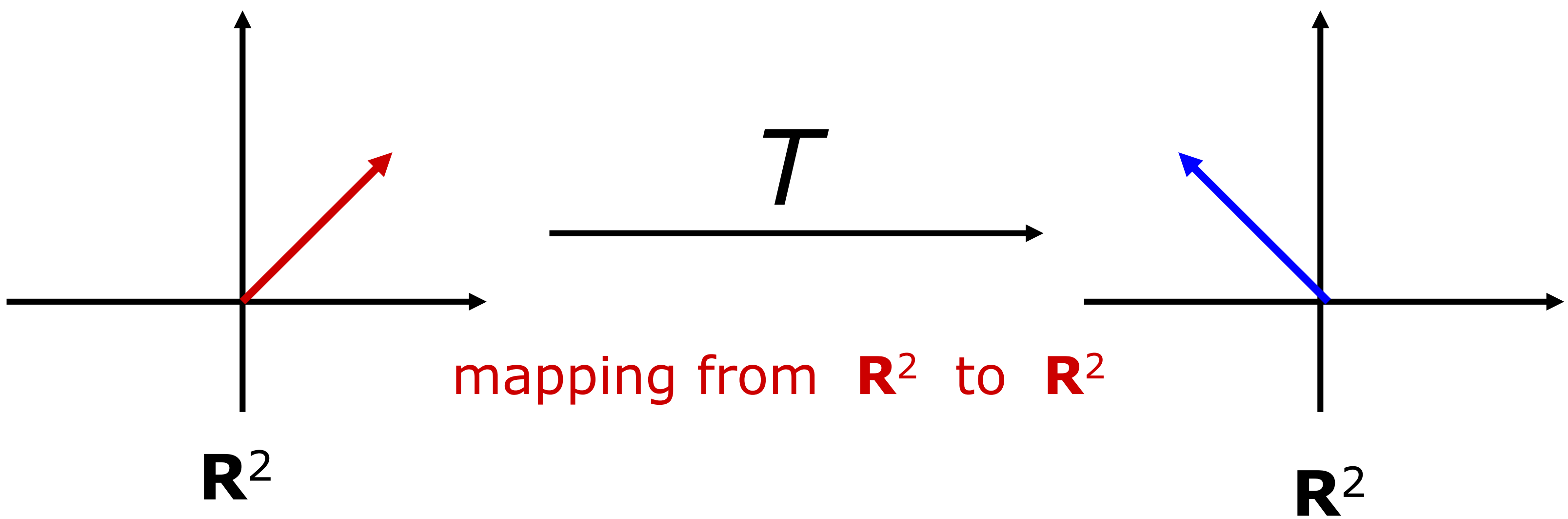


We call a mapping defined this way a linear transformation.

Matrix as a mapping

$$\mathbf{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \longrightarrow \boldsymbol{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \longrightarrow \mathbf{Au} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

input output



Notation: $T : \mathbf{R}^2 \to \mathbf{R}^2$

defined by $T(\boldsymbol{u}) = \boldsymbol{Au}$ for all $\boldsymbol{u}$ in $\mathbf{R}^2$

Matrix as a mapping

$T : \mathbf{R}^2 \to \mathbf{R}^2$

defined by $T(\boldsymbol{u}) = \boldsymbol{A}\boldsymbol{u}$

$$\boldsymbol{u} = \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \boldsymbol{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \boldsymbol{A}\boldsymbol{u} = \begin{pmatrix} -y \\ x \end{pmatrix}$$

input output

Formula of $T : \mathbf{R}^2 \to \mathbf{R}^2$ is given by

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} -y \\ x \end{pmatrix} \quad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{R}^2$$

Geometrical meaning

Rotation anticlockwise 90°

What is a linear transformation?

Definition 7.1.1

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$\mathbf{u} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \Rightarrow \boldsymbol{A} = m \times n \text{ matrix} \Rightarrow \mathbf{Au} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$$

input
Domain $\mathbf{R}^n$

$T : \mathbf{R}^n \to \mathbf{R}^m$

defined by $T(\boldsymbol{u}) = \boldsymbol{Au}$ for all $\boldsymbol{u} \in \mathbf{R}^n$

output
Codomain $\mathbf{R}^m$

T is called a linear transformation from $\mathbf{R}^n$ to $\mathbf{R}^m$

$\boldsymbol{A}$ is called the standard matrix of the linear transformation

Formula of T

$$T\left(\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}\right) = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix}$$

An example of linear transformation

Example 7.1.2.3

$T : \mathbf{R}^2 \to \mathbf{R}^3$ defined by formula

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ 2x \\ -3y \end{pmatrix} = x\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + y\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 0 & -3 \end{pmatrix}}_{\boldsymbol{A}}\begin{pmatrix} x \\ y \end{pmatrix}$$

Is T a linear transformation?

$T(\boldsymbol{u}) = \boldsymbol{A}\boldsymbol{u}$ for some $\boldsymbol{A}$?

So T is a linear transformation with standard matrix $\boldsymbol{A}$

An example of non-linear transformation

Example 7.1.5.1

$T_1 : \mathbf{R}^2 \to \mathbf{R}^2$ defined by formula

$$T_1\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x + 1 \\ y + 3 \end{pmatrix}$$

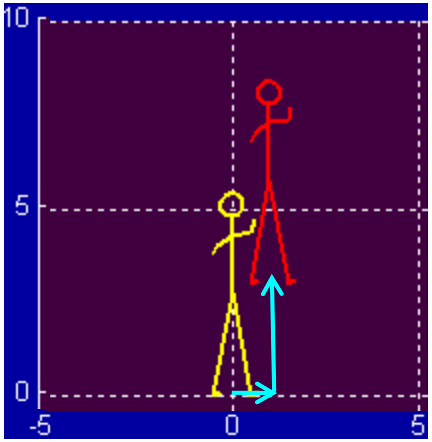
Can't have constant terms in the formula

Why?

There is no 2 x 2 matrix **A** such that $T_1\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix}$

T_1 is not a linear transformation.

T_1 represent a translation in xy-plane



Example 7.1.5.2

$T_2 : \mathbf{R}^3 \to \mathbf{R}^2$ defined by formula

$$T_2\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x^2 \\ yz \end{pmatrix}$$

Can't have non-linear terms in the formula

This is not a linear transformation. Why?

Identity transformation

Example 7.1.2.1

$I : \mathbf{R}^n \to \mathbf{R}^n$: the identity transformation

$I(\boldsymbol{u}) = \boldsymbol{u}$ for all $\boldsymbol{u}$ in $\mathbf{R}^n$. Do-nothing mapping

Is I a linear transformation?

$I(\boldsymbol{u}) = \boldsymbol{A}\boldsymbol{u}$ for some $\boldsymbol{A}$?

$$I(\boldsymbol{u}) = \boldsymbol{I}_{\text{n}}\boldsymbol{u} \qquad \mathbf{I}_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \text{ identity matrix}$$

Formula of I

So I is a linear transformation with standard matrix $\boldsymbol{I}_{\text{n}}$

$$\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \qquad T_1\left(\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}\right) = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$$

Zero transformation

Example 7.1.2.2

$O : \mathbf{R}^n \to \mathbf{R}^m$: the zero transformation

$O(\boldsymbol{u}) = \mathbf{0}$ for all $\boldsymbol{u}$ in $\mathbf{R}^n$. Kill-everything mapping

Is O a linear transformation?

$O(\boldsymbol{u}) = \boldsymbol{A}\boldsymbol{u}$ for some $\boldsymbol{A}$?

$$O(\boldsymbol{u}) = \boldsymbol{0}_{m\times n}\boldsymbol{u} \qquad \mathbf{0}_{m\times n} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$ zero matrix

Formula of O

$$\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \qquad O\left(\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

So O is a linear transformation with standard matrix $\boldsymbol{0}_{m\times n}$

scalar multiplication $2\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix}(2)$ matrix multiplication

Ex 7 Q7 (Tutorial 11)

P: $\mathbf{R}^n \to \mathbf{R}^n$ defined by $P(\mathbf{x}) = \mathbf{x} - (\mathbf{n} \cdot \mathbf{x})\,\mathbf{n}$

n is some fixed vector

Show that P is a linear transformation.

Hint: Show $P(\mathbf{x}) = \mathbf{A}\mathbf{x}$ for some matrix **A**

$$(\mathbf{n} \cdot \mathbf{x})\,\mathbf{n} = \mathbf{n}\,(\mathbf{n} \cdot \mathbf{x}) = \mathbf{n}\,(\mathbf{n}^{\mathrm{T}}\mathbf{x}) = (\mathbf{n}\mathbf{n}^{\mathrm{T}})\,\mathbf{x}$$

Properties of linear transformation

Theorem 7.1.4

If $T : \mathbf{R}^n \to \mathbf{R}^m$ is a linear transformation, then

1. $T(\mathbf{0}) = \mathbf{0}$ $\quad \boldsymbol{A}\mathbf{0} = \mathbf{0}$ $\quad T$ preserves zero vector

2. $T(\,c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k}\,)$ $\quad T$ preserves linear combinations

a linear combination in $\mathbf{R}^n$

$= c_1T(\boldsymbol{u_1}) + c_2T(\boldsymbol{u_2}) + \cdots + c_kT(\boldsymbol{u_k})$

a linear combination in $\mathbf{R}^m$

$\boldsymbol{A}(\,c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k}\,)$

$= c_1\boldsymbol{Au_1} + c_2\boldsymbol{Au_2} + \cdots + c_k\boldsymbol{Au_k}$

Linearity condition

Remark 7.1.3

Formal definition of Linear Transformation

A linear transformation $T : \mathbf{R}^n \to \mathbf{R}^m$

is a mapping from $\mathbf{R}^n$ to $\mathbf{R}^m$

that satisfies the following condition:

For all vectors $\boldsymbol{u}, \boldsymbol{v}$ in $\mathbf{R}^n$ and scalars a, b

$$T(a\boldsymbol{u} + b\boldsymbol{v}) = aT(\boldsymbol{u}) + bT(\boldsymbol{v})$$

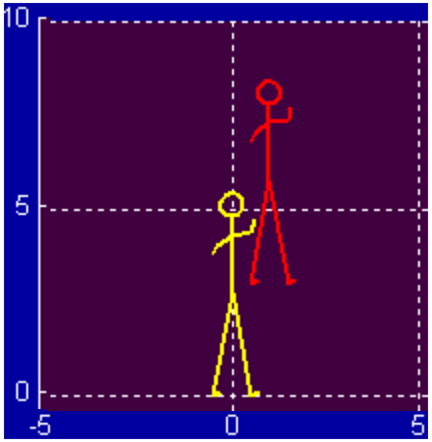
Linearity conditions of T

1. $T(\boldsymbol{u} + \boldsymbol{v}) = T(\boldsymbol{u}) + T(\boldsymbol{v})$ — T preserves addition
2. $T(c\boldsymbol{u}) = cT(\boldsymbol{u})$ — T preserves scalar multiplication

How to show a mapping is not linear transformation?

Example 7.1.5.1 revisited

$T_1 : \mathbf{R}^2 \to \mathbf{R}^2 \quad T_1\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+1 \\ y+3 \end{pmatrix}$



Check the image of zero vector ***0***:

$$T_1\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The property $T(\mathbf{0}) = \mathbf{0}$ is violated

Thus T_1 is not a linear transformation.

How to show a mapping is not linear transformation?

Example 7.1.5.2 revisited

$$T_2 : \mathbf{R}^3 \to \mathbf{R}^2 \qquad T_2\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x^2 \\ yz \end{pmatrix}$$

The linearity condition
$T(\boldsymbol{u} + \boldsymbol{v}) = T(\boldsymbol{u}) + T(\boldsymbol{v})$
is violated

Check the image of zero vector ***0***:

$$T_2\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Does not violate $T(\mathbf{0}) = \mathbf{0}$

$$T_2\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$T_2\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$T_2\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) = T_2\left(\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$T_2\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) + T_2\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Thus T_2 is not a linear transformation.

What is a linear operator?

Definition 7.1.1

If a linear transformation $T : \mathbf{R}^n \to \mathbf{R}^n$

maps from $\mathbf{R}^n$ to itself,

we say T is a linear operator on $\mathbf{R}^n$

Domain of T = Codomain of T

In this case, the standard matrix for T is a square matrix.

In example 7.1.2,
I is a linear operator;
O is a linear operator if domain = codomain;
T is not a linear operator.

LT without formula

$T : \mathbf{R}^3 \to \mathbf{R}^2$: the linear transformation such that

$$T\left(\begin{pmatrix}1\\1\\1\end{pmatrix}\right)=\begin{pmatrix}1\\3\end{pmatrix} \quad T\left(\begin{pmatrix}0\\1\\1\end{pmatrix}\right)=\begin{pmatrix}-1\\2\end{pmatrix} \quad T\left(\begin{pmatrix}2\\0\\-1\end{pmatrix}\right)=\begin{pmatrix}4\\-1\end{pmatrix}$$

basis for $\mathbf{R}^3$

If the formula /standard matrix of T is NOT given, can we find the image of every vector in $\mathbf{R}^3$ under T ?

YES ! Provided …

How to determine LT from basis?

Example 7.1.7

$T : \mathbf{R}^3 \to \mathbf{R}^2$: the linear transformation such that

$$T\left(\begin{pmatrix}1\\1\\1\end{pmatrix}\right) = \begin{pmatrix}1\\3\end{pmatrix} \quad T\left(\begin{pmatrix}0\\1\\1\end{pmatrix}\right) = \begin{pmatrix}-1\\2\end{pmatrix} \quad T\left(\begin{pmatrix}2\\0\\-1\end{pmatrix}\right) = \begin{pmatrix}4\\-1\end{pmatrix}$$

basis for $\mathbf{R}^3$

(a) Find the image of $\begin{pmatrix}-1\\4\\6\end{pmatrix}$ under T.

(b) Find the formula of T.

How to determine LT from basis?

Example 7.1.7.1

$$T\left(\begin{pmatrix}1\\1\\1\end{pmatrix}\right)=\begin{pmatrix}1\\3\end{pmatrix}\quad T\left(\begin{pmatrix}0\\1\\1\end{pmatrix}\right)=\begin{pmatrix}-1\\2\end{pmatrix}\quad T\left(\begin{pmatrix}2\\0\\-1\end{pmatrix}\right)=\begin{pmatrix}4\\-1\end{pmatrix}$$

$$\begin{pmatrix}-1\\4\\6\end{pmatrix}=3\begin{pmatrix}1\\1\\1\end{pmatrix}+1\begin{pmatrix}0\\1\\1\end{pmatrix}+-2\begin{pmatrix}2\\0\\-1\end{pmatrix}$$

use Gaussian elimination to find the coefficients

$$T\left(\begin{pmatrix}-1\\4\\6\end{pmatrix}\right)=\boxed{T\left(3\begin{pmatrix}1\\1\\1\end{pmatrix}+\begin{pmatrix}0\\1\\1\end{pmatrix}-2\begin{pmatrix}2\\0\\-1\end{pmatrix}\right)}\rightarrow$$

this step can be skipped

Linearity condition

$$=3T\left(\begin{pmatrix}1\\1\\1\end{pmatrix}\right)+T\left(\begin{pmatrix}0\\1\\1\end{pmatrix}\right)-2T\left(\begin{pmatrix}2\\0\\-1\end{pmatrix}\right)=3\begin{pmatrix}1\\3\end{pmatrix}+\begin{pmatrix}-1\\2\end{pmatrix}-2\begin{pmatrix}4\\-1\end{pmatrix}=\begin{pmatrix}-6\\13\end{pmatrix}$$

Images under LT in terms of basis

Discussion 7.1.6

$\{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_n}\}$: a basis for $\mathbf{R}^n$

Any $\boldsymbol{v}$ in $\mathbf{R}^n$

$$\boldsymbol{v} = c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_n\boldsymbol{u_n}$$

for some scalar $c_1, c_2, \ldots, c_n$

Suppose $T : \mathbf{R}^n \to \mathbf{R}^m$ is a linear transformation.

Linearity condition

$$\boxed{T(\boldsymbol{v})} = T(\, c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_n\boldsymbol{u_n}\,)$$

image of a general vector $\boldsymbol{v}$

$$= c_1\boxed{T(\boldsymbol{u_1})} + c_2\boxed{T(\boldsymbol{u_2})} + \cdots + c_n\boxed{T(\boldsymbol{u_n})}$$

images of the basis vectors

Images under LT in terms of basis

Discussion 7.1.6

$\{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_n}\}$: a basis for $\mathbf{R}^n$

Any $\boldsymbol{v}$ in $\mathbf{R}^n$

$$T(\boldsymbol{v}) = c_1 T(\boldsymbol{u_1}) + c_2 T(\boldsymbol{u_2}) + \cdots + c_n T(\boldsymbol{u_n})$$

Knowing the images $T(\boldsymbol{u_1}), T(\boldsymbol{u_2}), \ldots, T(\boldsymbol{u_n})$
is enough to determine
the image $T(\boldsymbol{v})$ of any vector $\boldsymbol{v}$ in the domain $\mathbf{R}^n$.

The linear transformation T
is completely determined by the images
$T(\boldsymbol{u_1}), T(\boldsymbol{u_2}), \ldots, T(\boldsymbol{u_n})$ of the basis.

How to determine LT from basis?

Example 7.1.7

$T : \mathbf{R}^3 \to \mathbf{R}^2$: the linear transformation such that

$$T\left(\begin{pmatrix}1\\1\\1\end{pmatrix}\right)=\begin{pmatrix}1\\3\end{pmatrix}\quad T\left(\begin{pmatrix}0\\1\\1\end{pmatrix}\right)=\begin{pmatrix}-1\\2\end{pmatrix}\quad T\left(\begin{pmatrix}2\\0\\-1\end{pmatrix}\right)=\begin{pmatrix}4\\-1\end{pmatrix}$$

Find the formula of T.

Method 1: Direct Gaussian elimination

Method 2: Find $T(\mathbf{e}_1)$, $T(\mathbf{e}_2)$, $T(\mathbf{e}_3)$

Method 3: Stacking matrices

How to determine LT from basis?

Method 1

Example 7.1.7.2

$$T\left(\begin{pmatrix}1\\1\\1\end{pmatrix}\right)=\begin{pmatrix}1\\3\end{pmatrix}\quad T\left(\begin{pmatrix}0\\1\\1\end{pmatrix}\right)=\begin{pmatrix}-1\\2\end{pmatrix}\quad T\left(\begin{pmatrix}2\\0\\-1\end{pmatrix}\right)=\begin{pmatrix}4\\-1\end{pmatrix}$$

Find the formula of T

$$T\left(\begin{pmatrix}x\\y\\z\end{pmatrix}\right)=\begin{pmatrix}2x-y\\x-y+3z\end{pmatrix}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=c_1\begin{pmatrix}1\\1\\1\end{pmatrix}+c_2\begin{pmatrix}0\\1\\1\end{pmatrix}+c_3\begin{pmatrix}2\\0\\-1\end{pmatrix}$$

$$\boxed{(x-2y+2z)}\quad\boxed{(-x+3y-2z)}\quad\boxed{(y-z)}$$

use Gaussian elimination to find the coefficients

$$\left(\begin{array}{ccc|c}1&0&2&x\\1&1&0&y\\1&1&-1&z\end{array}\right)$$

$$T\left(\begin{pmatrix}x\\y\\z\end{pmatrix}\right)=c_1T\left(\begin{pmatrix}1\\1\\1\end{pmatrix}\right)+c_2T\left(\begin{pmatrix}0\\1\\1\end{pmatrix}\right)+c_3T\left(\begin{pmatrix}2\\0\\-1\end{pmatrix}\right)=c_1\begin{pmatrix}1\\3\end{pmatrix}+c_2\begin{pmatrix}-1\\2\end{pmatrix}+c_3\begin{pmatrix}4\\-1\end{pmatrix}$$

in terms of x, y, z

Images of standard basis and standard matrix

Discussion 7.1.8

$T : \mathbf{R}^n \to \mathbf{R}^m$: any linear transformation

$\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$: the standard basis for $\mathbf{R}^n$

$$\boldsymbol{A} = \begin{pmatrix} \boxed{\begin{matrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{matrix}} & \begin{matrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{matrix} & \begin{matrix} \cdots \\ \cdots \\ \\ \cdots \end{matrix} & \begin{matrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{matrix} \end{pmatrix}$$

the standard matrix of T

$$\boxed{T(\mathbf{e}_1) = \mathbf{A}\mathbf{e}_1 = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}}$$

The image T($\boldsymbol{e}_j$) = the jth column of $\boldsymbol{A}$

$$\boldsymbol{A} = (T(\mathbf{e}_1) \;\; T(\mathbf{e}_2) \;\; \cdots \;\; T(\mathbf{e}_n))$$

Images of standard basis and standard matrix

Example 7.1.9

Method 2

$$T\left(\begin{pmatrix}1\\1\\1\end{pmatrix}\right)=\begin{pmatrix}1\\3\end{pmatrix}\quad T\left(\begin{pmatrix}0\\1\\1\end{pmatrix}\right)=\begin{pmatrix}-1\\2\end{pmatrix}\quad T\left(\begin{pmatrix}2\\0\\-1\end{pmatrix}\right)=\begin{pmatrix}4\\-1\end{pmatrix}$$

$$\mathbf{A} = (\ T(\mathbf{e_1})\ T(\mathbf{e_2})\ T(\mathbf{e_3})\ )$$

$$\mathbf{A} = \begin{pmatrix}2 & -1 & 0\\1 & -1 & 3\end{pmatrix}$$

Find $T(\mathbf{e_1})$, $T(\mathbf{e_2})$, $T(\mathbf{e_3})$

Find $\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$ in terms of $\begin{pmatrix}1\\1\\1\end{pmatrix}$ $\begin{pmatrix}0\\1\\1\end{pmatrix}$ $\begin{pmatrix}2\\0\\-1\end{pmatrix}$

$$\left(\begin{array}{ccc|c|c|c}1 & 0 & 2 & 1 & 0 & 0\\1 & 1 & 0 & 0 & 1 & 0\\1 & 1 & -1 & 0 & 0 & 1\end{array}\right)\xrightarrow[\text{elimination}]{\text{Gauss-Jordan}}\left(\begin{array}{ccc|c|c|c}1 & 0 & 0 & 1 & -2 & 2\\0 & 1 & 0 & -1 & 3 & -2\\0 & 0 & 1 & 0 & 1 & -1\end{array}\right)$$

$$\mathbf{e}_1=\begin{pmatrix}1\\1\\1\end{pmatrix}-\begin{pmatrix}0\\1\\1\end{pmatrix}\qquad \mathbf{e}_2=-2\begin{pmatrix}1\\1\\1\end{pmatrix}+3\begin{pmatrix}0\\1\\1\end{pmatrix}+\begin{pmatrix}2\\0\\1\end{pmatrix}\qquad \mathbf{e}_3=2\begin{pmatrix}1\\1\\1\end{pmatrix}-2\begin{pmatrix}0\\1\\1\end{pmatrix}-\begin{pmatrix}2\\0\\1\end{pmatrix}$$

$$T(\mathbf{e}_1)=T\begin{pmatrix}1\\1\\1\end{pmatrix}-T\begin{pmatrix}0\\1\\1\end{pmatrix}\qquad T(\mathbf{e}_2)=-2T\begin{pmatrix}1\\1\\1\end{pmatrix}+3T\begin{pmatrix}0\\1\\1\end{pmatrix}+T\begin{pmatrix}2\\0\\1\end{pmatrix}\qquad T(\mathbf{e}_3)=2T\begin{pmatrix}1\\1\\1\end{pmatrix}-2T\begin{pmatrix}0\\1\\1\end{pmatrix}-T\begin{pmatrix}2\\0\\1\end{pmatrix}$$

Stacking the matrix

Method 3

$$T\left(\begin{pmatrix}1\\1\\1\end{pmatrix}\right)=\begin{pmatrix}1\\3\end{pmatrix}\quad T\left(\begin{pmatrix}0\\1\\1\end{pmatrix}\right)=\begin{pmatrix}-1\\2\end{pmatrix}\quad T\left(\begin{pmatrix}2\\0\\-1\end{pmatrix}\right)=\begin{pmatrix}4\\-1\end{pmatrix}$$

$$\mathbf{A}=\begin{pmatrix}2 & -1 & 0\\1 & -1 & 3\end{pmatrix}$$

$$\left(\begin{array}{ccc|c|c|c}1 & 0 & 2 & 1 & 0 & 0\\1 & 1 & 0 & 0 & 1 & 0\\1 & 1 & -1 & 0 & 0 & 1\end{array}\right)\xrightarrow[\text{elimination}]{\text{Gauss-Jordan}}\left(\begin{array}{ccc|c|c|c}1 & 0 & 0 & 1 & -2 & 2\\0 & 1 & 0 & -1 & 3 & -2\\0 & 0 & 1 & 0 & 1 & -1\end{array}\right)$$

$$\mathbf{A}\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}1\\3\end{pmatrix},\ \mathbf{A}\begin{pmatrix}0\\1\\1\end{pmatrix}=\begin{pmatrix}-1\\2\end{pmatrix},\ \mathbf{A}\begin{pmatrix}2\\0\\-1\end{pmatrix}=\begin{pmatrix}4\\-1\end{pmatrix}$$

$$\mathbf{A}\begin{pmatrix}1 & 0 & 2\\1 & 1 & 0\\1 & 1 & -1\end{pmatrix}=\begin{pmatrix}1 & -1 & 4\\3 & 2 & -1\end{pmatrix}$$

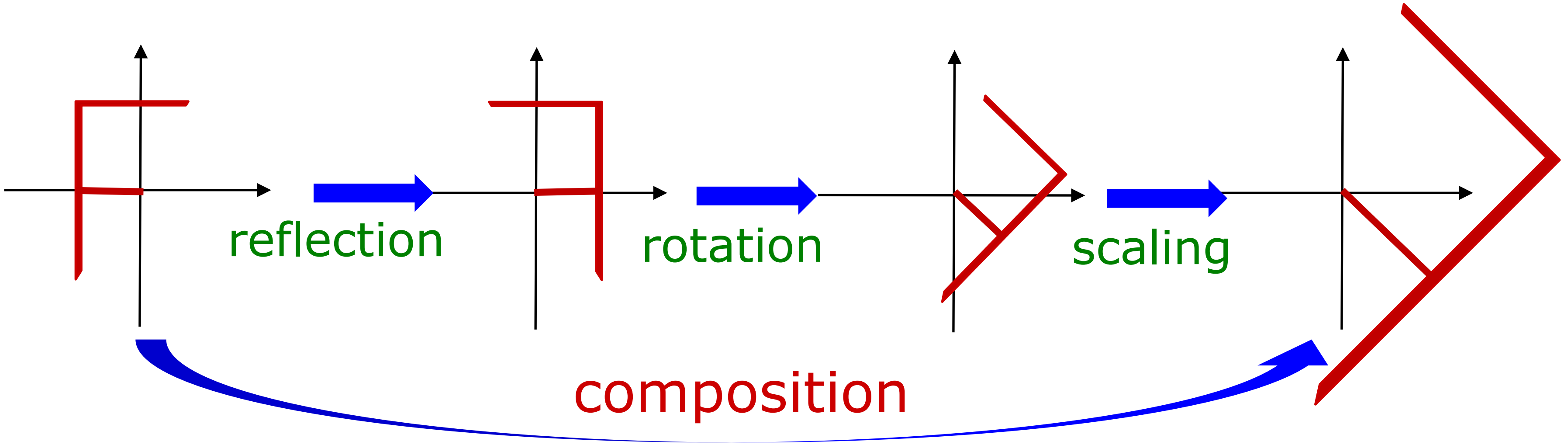
$$\mathbf{A}=\begin{pmatrix}1 & -1 & 4\\3 & 2 & -1\end{pmatrix}\begin{pmatrix}1 & 0 & 2\\1 & 1 & 0\\1 & 1 & -1\end{pmatrix}^{-1}=\begin{pmatrix}1 & -1 & 4\\3 & 2 & -1\end{pmatrix}\begin{pmatrix}1 & -2 & 2\\-1 & 3 & -2\\0 & 1 & -1\end{pmatrix}$$

Section 7.1

Linear Transformations from $\mathbf{R}^n$ to $\mathbf{R}^m$

Objective

- What is the composition of linear transformations?



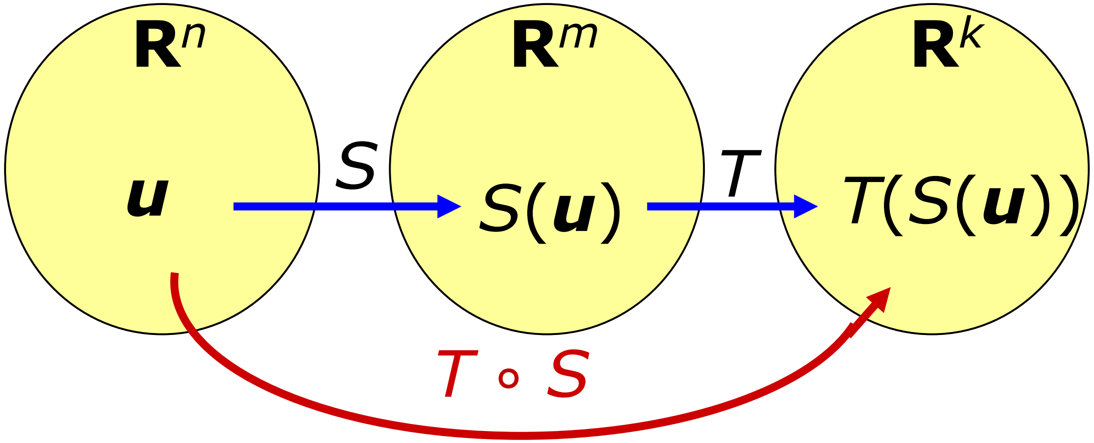
Composition of LT's

Definition 7.1.10

Let $S : \mathbf{R}^n \to \mathbf{R}^m$ and $T : \mathbf{R}^m \to \mathbf{R}^k$ be linear transformations.

The composition of T with S, denoted by $T \circ S$ First S, then T is a mapping from $\mathbf{R}^n$ to $\mathbf{R}^k$ such that

$$(T \circ S)(\boldsymbol{u}) = T(S(\boldsymbol{u})) \quad \text{for all } \boldsymbol{u} \text{ in } \mathbf{R}^n.$$



Example 7.1.12

$S\colon \mathbf{R}^3 \to \mathbf{R}^2$: the linear transformation defined by

$$S\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x+y \\ z \end{pmatrix} \quad \text{for all} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{in} \quad \mathbf{R}^3.$$

$T : \mathbf{R}^2 \to \mathbf{R}^3$: the linear transformation defined by

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} y \\ y \\ x \end{pmatrix} \quad \text{for all} \quad \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{in} \quad \mathbf{R}^2.$$

Find the composition of *T* with *S*.

Composition of LT's

Example 7.1.12

$$S\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x+y \\ z \end{pmatrix}$$

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} y \\ y \\ x \end{pmatrix}$$

$T \circ S$ is a mapping from $\mathbf{R}^3$ to $\mathbf{R}^3$:

$$(T \circ S)\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = T\left(S\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right)\right) = T\left(\begin{pmatrix} x+y \\ z \end{pmatrix}\right) = \begin{pmatrix} z \\ z \\ x+y \end{pmatrix}$$

Not recommended; alternative approach later

Is $T \circ S$ a linear transformation ?

Standard matrix of composition of LT's

Theorem 7.1.11

If $S : \mathbf{R}^n \to \mathbf{R}^m$ and $T : \mathbf{R}^m \to \mathbf{R}^k$
are linear transformations

S, T have standard matrices $\boldsymbol{A}$, $\boldsymbol{B}$ respectively

then $T \circ S : \mathbf{R}^n \to \mathbf{R}^k$
is again a linear transformation.

$T \circ S$ has standard matrix $\boldsymbol{BA}$

The proof

Theorem 7.1.11

linear transformation	standard matrix
$S : \mathbf{R}^n \to \mathbf{R}^m$	$\boldsymbol{A}$
$T : \mathbf{R}^m \to \mathbf{R}^k$	$\boldsymbol{B}$
$T \circ S : \mathbf{R}^n \to \mathbf{R}^k$	$\boldsymbol{BA}$

For all $\boldsymbol{u}$ in $\mathbf{R}^n$,

$$(T \circ S)(\boldsymbol{u}) = T(S(\boldsymbol{u})) = T(\boldsymbol{Au}) = \boldsymbol{B}(\boldsymbol{Au}) = (\boldsymbol{BA})\boldsymbol{u}$$

$T \circ S$ is a linear transformation

Standard matrix of composition of LT's

Example 7.1.12

$$S\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x+y \\ z \end{pmatrix} \qquad T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} y \\ y \\ x \end{pmatrix} \qquad (T \circ S)\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} z \\ z \\ x+y \end{pmatrix}$$

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\boldsymbol{BA} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

standard matrix of $T \circ S$

$$(T \circ S)\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \mathbf{BA}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ z \\ x+y \end{pmatrix}$$

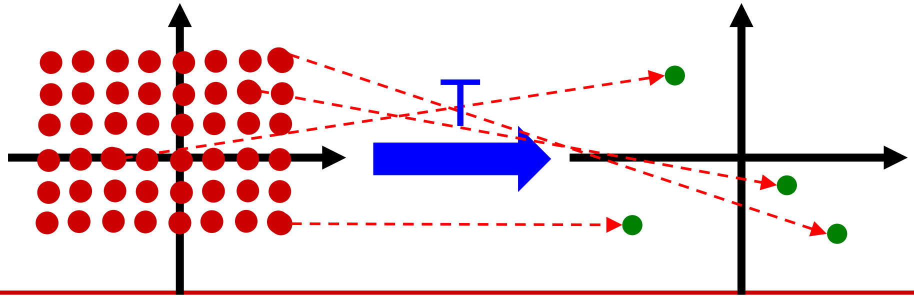
Section 7.2

Ranges and Kernel

Objective

- What are the range and kernel of a linear transformation?
- What are the rank and nullity of a linear transformation?
- What is the Dimension Theorem of linear transformation?

Visualization



$T : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ linear transformation

Three possibilities:

- Images under T fill up the whole xy-plane ($\mathbf{R}^2$)
- Images under T all lie on a line
- Images under T all are the same point

range of T

$S : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ linear transformation

Four possibilities:

- Images under S fill up the whole xyz-space ($\mathbf{R}^3$)
- Images under S all lie on a plane
- Images under S all lie on a line
- Images under S all are the same point

range of S

What is the range of a LT?

Definition 7.2.1

Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation.

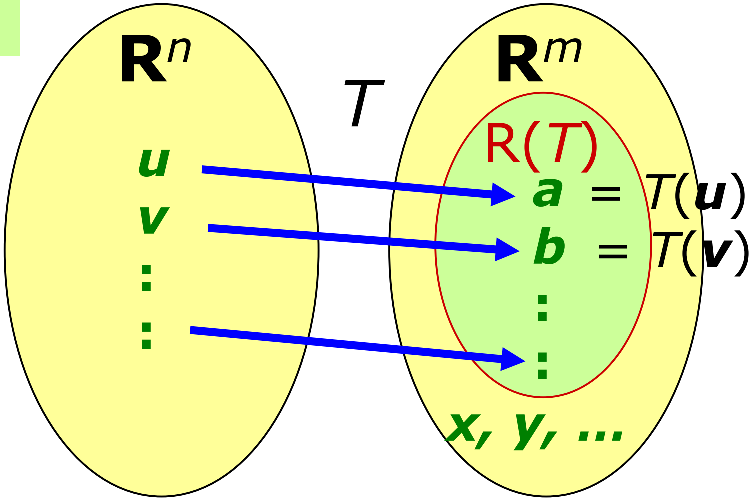
The range of T, denoted by $R(T)$,

is the set of images of T.

$R(T)$ = {images of T }

$R(T) = \{ T(\boldsymbol{u}) \mid \boldsymbol{u} \in \mathbf{R}^n \}$

explicit set notation



$R(T)$ is a subset of $\mathbf{R}^m$

$R(T)$ may not be equal to $\mathbf{R}^m$

range of T ⊆ codomain of T

What is the range of a LT?

Example 7.2.2

$\mathrm{R}(T) = \{T(\boldsymbol{u}) \mid \boldsymbol{u} \in \mathbf{R}^n\}$

$T : \mathbf{R}^2 \to \mathbf{R}^3$: the linear transformation defined by

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ y \\ x \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + y\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \text{ in } \mathbf{R}^2.$$

What is $\mathrm{R}(T)$?

$$\mathrm{R}(T) = \left\{ \begin{pmatrix} x+y \\ y \\ x \end{pmatrix} \middle|\; x, y \in \boldsymbol{R} \right\} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

explicit set notation

linear span form

a plane in $\mathbf{R}^3$

Example 7.2.2

$T : \mathbf{R}^2 \to \mathbf{R}^3$: the linear transformation defined by

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ y \\ x \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + y\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{ for all } \begin{pmatrix} x \\ y \end{pmatrix} \text{ in } \mathbf{R}^2.$$

What is R(T)?

standard matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\mathrm{R}(T) = \left\{ \boxed{\begin{pmatrix} x+y \\ y \\ x \end{pmatrix}} \,\middle|\, \boxed{x, y \in \boldsymbol{R}} \right\} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

explicit set notation

linear span form

column space of $\boldsymbol{A}$

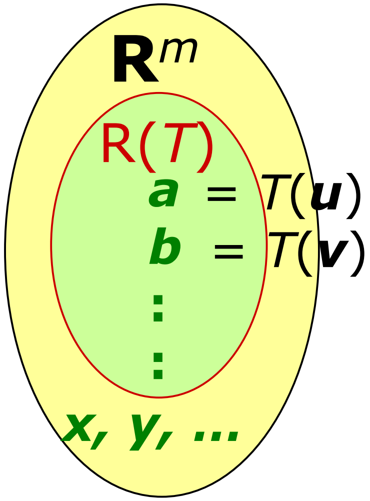
R(T) is the column space of standard matrix

Theorem 7.2.4

$T : \mathbf{R}^n \to \mathbf{R}^m$: a linear transformation

$\boldsymbol{A}$ the standard matrix for T

Then R(T) = span{ columns of $\boldsymbol{A}$ }
= the column space of $\boldsymbol{A}$



R(T) is a subspace of $\mathbf{R}^m$

R(T) is a subset of $\mathbf{R}^m$

What is the rank of a LT?

Definition 7.2.5

Let T be a linear transformation.

The dimension of $R(T)$ = dimension of column space of ***A***

called the rank of T denoted by rank(T)

A the standard matrix for T rank(T) = rank(***A***)

Example 7.2.2:

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ y \\ x \end{pmatrix} \quad R(T) = \text{span}\left\{\underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}_{\text{basis}}\right\} \quad \text{rank}(T) = 2$$

How to find a basis for R(T)?

Example 7.2.6

$T : \mathbf{R}^4 \to \mathbf{R}^4$: a linear transformation defined by

$$T\left(\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x+2y+z \\ x+3y \\ x+4y-z \\ y-z \end{pmatrix} \quad \text{for all} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \in \boldsymbol{R}^4$$

Find a basis for the range of T and determine the rank of T.

Let $\boldsymbol{A}$ be the standard matrix for T

Same as to find:
a basis for column space of $\boldsymbol{A}$ and rank($\boldsymbol{A}$).

Discussion 7.2.3

$T : \mathbf{R}^n \to \mathbf{R}^m$ a linear transformation

$\mathrm{R}(T) = \mathrm{span}\{ \text{ columns of } \boldsymbol{A} \}$

$= \mathrm{span}\{ T(\boldsymbol{e_1}), T(\boldsymbol{e_2}), \ldots, T(\boldsymbol{e_n}) \}$

If $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_n}\}$ is any basis for $\mathbf{R}^n$

then $\boxed{\mathrm{R}(T) = \mathrm{span}\{ T(\boldsymbol{u_1}), T(\boldsymbol{u_2}), \ldots, T(\boldsymbol{u_n}) \}}$ **?**

$$\Downarrow \qquad\qquad\qquad \Uparrow$$

$$T(\boldsymbol{v}) \qquad c_1T(\boldsymbol{u_1}) + c_2T(\boldsymbol{u_2}) + \cdots + c_nT(\boldsymbol{u_n})$$

$$\Downarrow \qquad\qquad\qquad \Uparrow$$

$$T(\, c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_n\boldsymbol{u_n} \,)$$

We can write $\boldsymbol{v} = c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_n\boldsymbol{u_n}$

Finding range R(T) and $\boxed{\text{its basis}}$

T: $\mathbf{R}^n \to \mathbf{R}^m$

I. if formula of T is given

- $R(T) = \{\text{formula in } x_1, x_2, \ldots, x_n \mid x_1, x_2, \ldots, x_n \in \mathbf{R}\}$

II. if standard matrix **A** is given

- $R(T) = \text{span}\{\text{ columns of } \mathbf{A} \}$

  or part I above

Find basis for column space of **A**

III. if image of a basis $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ for $\mathbf{R}^n$ is given

- $R(T) = \text{span}\{T(\mathbf{u}_1), T(\mathbf{u}_2), \ldots, T(\mathbf{u}_n)\}$

Throw out the redundant vectors in the span (use column space method if necessary)

Visualization

$T : \mathbf{R}^2 \to \mathbf{R}^2$ linear transformation

- Images under T fill up the whole xy-plane ($\mathbf{R}^2$)
- Images under T all lie on a line
- Images under T all are the same point

range of T

Some information is lost

kernel of T (or S)

$S : \mathbf{R}^3 \to \mathbf{R}^3$ linear transformation

- Images under S fill up the whole xyz-space ($\mathbf{R}^3$)
- Images under S all lie on a plane
- Images under S all lie on a line
- Images under S all are the same point

range of S

What is the kernel of a LT?

Definition 7.2.7

Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation.

The kernel of T, denoted by $\ker(T)$, is the set of vectors in $\mathbf{R}^n$ whose image is the zero vector in $\mathbf{R}^m$.

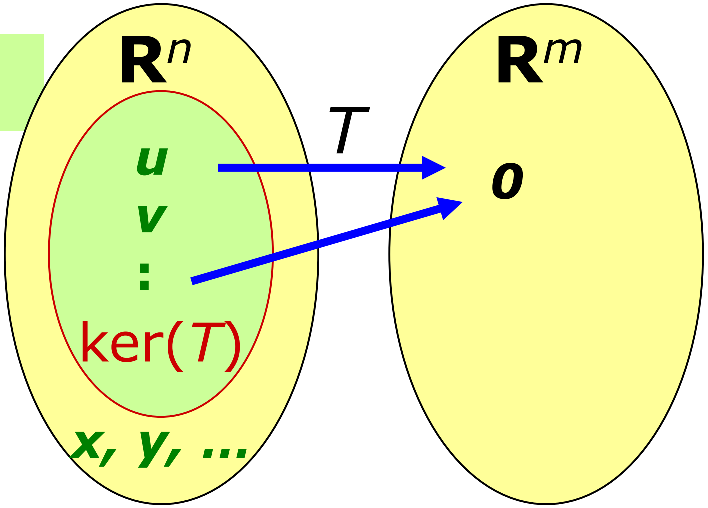
$\ker(T)$ = {vectors that map to $\mathbf{0}$ under T }

$$\ker(T) = \{\, \boldsymbol{u} \in \mathbf{R}^n \mid T(\boldsymbol{u}) = \mathbf{0} \,\}$$

implicit set notation

$\ker(T)$ is a subset of $\mathbf{R}^n$

$\ker(T)$ may not be equal to $\mathbf{R}^n$



How to find kernel of a LT?

Example 7.2.8.1

$\ker(T) = \{\boldsymbol{u} \in \mathbf{R}^3 \mid \boxed{T(\boldsymbol{u}) = \mathbf{0}}\}$

$T : \mathbf{R}^3 \to \mathbf{R}^4$: a linear transformation defined by

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 2x - y \\ x - y + 3z \\ -5x + y \\ x - z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ for all } \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \boldsymbol{R}^3$$

homog. system $\rightarrow$ only trivial solution

What is the kernel of T ?

$$\ker(T) = \left\{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right\}$$

Find all $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \boldsymbol{R}^3$ that satisfy this hom. system.

the zero space

How to find kernel of a LT?

Example 7.2.8.2

$\ker(T) = \{\boldsymbol{u} \in \mathbf{R}^3 \mid \boxed{T(\boldsymbol{u}) = \mathbf{0}}\}$

Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be a linear transformation defined by

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} z-y \\ 0 \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{for all } \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \boldsymbol{R}^3$$

solve for x, y, z

we get $z = y$ and $x = 0$

$$\ker(T) = \left\{ \begin{pmatrix} 0 \\ y \\ y \end{pmatrix} \;\middle|\; y \in \mathbf{R} \right\} = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

a subspace of dimension 1

Ker(T) is the nullspace of standard matrix

Theorem 7.2.9

$T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation

$\boldsymbol{A}$ the standard matrix for T

$T(\boldsymbol{u}) = \boldsymbol{A}\boldsymbol{u}$

ker(T) = all $\boldsymbol{u}$ such that $T(\boldsymbol{u}) = \mathbf{0}$

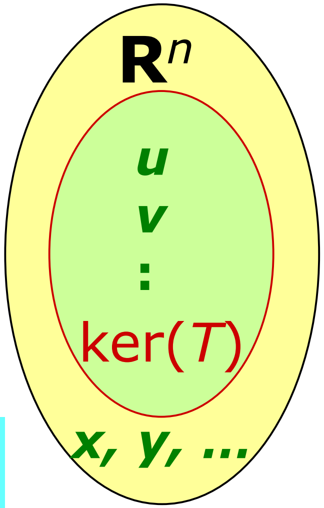
= all $\boldsymbol{u}$ such that $\boldsymbol{A}\boldsymbol{u} = \mathbf{0}$

= the solution space of $\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$

= the nullspace of $\boldsymbol{A}$

ker(T) is a subspace of $\mathbf{R}^n$

ker(T) is a subset of $\mathbf{R}^n$



What is the nullity of a LT?

Definition 7.2.10

Let T be a linear transformation.

The dimension of $\ker(T)$

called the nullity of T

denoted by $\text{nullity}(T)$

$\ker(T)$ = the nullspace of standard matrix $\boldsymbol{A}$

$$\boxed{\text{nullity}(T) = \text{nullity}(\boldsymbol{A})}$$

How to find a basis for ker(T)?

Example 7.2.11.1

In example 7.2.8.1,

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 2x - y \\ x - y + 3z \\ -5x + y \\ x - z \end{pmatrix}$$

$$\ker(T) = \left\{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right\}$$

the nullity of T is 0

In example 7.2.8.2,

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x - z \\ 0 \\ y \end{pmatrix}$$

$$\ker(T) = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\}$$

the nullity of T is 1

Dimension Theorem for LT

Theorem 7.2.12

Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be any linear transformation.

$$\text{rank}(T) + \text{nullity}(T) = n$$

By Thm 4.3.4. $\text{rank}(\boldsymbol{A}) + \text{nullity}(\boldsymbol{A}) = n$ (number of columns)

Proof

The standard matrix $\boldsymbol{A}$ of T is of size m x n

Range and kernel in proof

Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation.

$\mathrm{Ker}(T) = \{ \mathbf{v} \in \mathbf{R}^n \mid T(\mathbf{v}) = \mathbf{0} \}$

if you want to show:

In a proof, if you start with: $\mathbf{v} \in \ker(T)$,

try to show:

you should follow by: $T(\mathbf{v}) = \mathbf{0}$.

$R(T) = \{ T(\mathbf{v}) \mid \mathbf{v} \in \mathbf{R}^n \}$

if you want to show:

In a proof, if you start with: $\mathbf{v} \in R(T)$,

try to show:

you should follow by: $\mathbf{v} = T(\mathbf{u})$ for some $\mathbf{u} \in \mathbf{R}^n$.

Ex 7 Q17

S: $\mathbf{R}^n \to \mathbf{R}^m$ and T: $\mathbf{R}^m \to \mathbf{R}^k$ linear transformations

$\mathrm{Ker}(S) \subseteq \mathrm{Ker}(T \circ S)$

Hint: Take $\mathbf{u} \in \ker(S)$. Show that $\mathbf{u} \in \ker(T \circ S)$.

$S(\mathbf{u}) = \mathbf{0}$

$(T \circ S)(\mathbf{u}) = \mathbf{0}$

$R(T \circ S) \subseteq R(T)$

Hint: Take $\mathbf{u} \in R(T \circ S)$. Show that $\mathbf{u} \in R(T)$.

$\mathbf{u} = (T \circ S)(\mathbf{v})$ for some $\mathbf{v} \in \mathbf{R}^n$

$\mathbf{u} = T(\mathbf{w})$ for some $\mathbf{w} \in \mathbf{R}^m$