

CS1231S: Discrete Structures
Tutorial #10: Counting and Probability II
(Week 12: 1 – 5 November 2021)

I. Discussion Questions

You are strongly encouraged to discuss D1 – D3 on LumiNUS forum. No answers will be provided.

- D1. Suppose a random sample of 2 lightbulbs is selected from a group of 8 bulbs in which 3 are defective, what is the expected value of the number of defective bulbs in the sample? Let X represent the number of defective bulbs that occur on a given trial, where $X = 0, 1, 2$. Find $E[X]$.
- D2. How many **injective functions** are there from a set A with m elements to a set B with n elements, where $m \leq n$?
- D3. How many **surjective functions** are there from a 5-element set A to a 3-element set B ?

II. Tutorial Questions

1. [CS1231 Past Year's Exam Question]

You wish to select five persons from seven men and six women to form a committee that includes at least three men.

- (a) In how many ways can you form the committee?
(b) If you randomly choose five persons to form the committee, what is the probability that you will get a committee with at least three men? Give your answer correct to 4 significant figures.

2. Think of a set with $m + n$ elements as composed of two parts, one with m elements and the other with n elements. Give a **combinatorial argument** to show that

$$\binom{m+n}{r} = \binom{m}{0} \binom{n}{r} + \binom{m}{1} \binom{n}{r-1} + \cdots + \binom{m}{r} \binom{n}{0}$$

where $m, n \in \mathbb{Z}^+$, $r \leq m$ and $r \leq n$.

Call the above equation (A). Using equation (A), prove that for all integers $n \geq 0$,

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2.$$

II. Tutorial Questions

7M 6W

1. [CS1231 Past Year's Exam Question]

You wish to select five persons from seven men and six women to form a committee that includes at least three men.

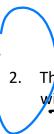
- (a) In how many ways can you form the committee?
- (b) If you randomly choose five persons to form the committee, what is the probability that you will get a committee with at least three men? Give your answer correct to 4 significant figures.

a) ${}^7C_3 \cdot {}^{10}C_2$ ${}^7M{}^2W + {}^4M{}^1W + {}^5M$

interesting results.
Choose items from specific gp \rightarrow general gp. \Rightarrow ends up having extra terms.

b) $\frac{{}^7C_5}{{}^{13}C_5}$

58.74%



2. Think of a set with $m+n$ elements as composed of two parts, one with m elements and the other with n elements. Give a combinatorial argument to show that

$$\binom{m+n}{r} = \binom{m}{0} \binom{n}{r} + \binom{m}{1} \binom{n}{r-1} + \dots + \binom{m}{r} \binom{n}{0}$$

where $m, n \in \mathbb{Z}^+, r \leq m$ and $r \leq n$.

Call the above equation (A). Using equation (A), prove that for all integers $n \geq 0$,

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \dots + \binom{n}{n}^2. \quad \text{--- (B)}$$

Q2. Think of a set with $m+n$ elements as composed of 2 parts, one with m elements and the other with n elements. Give a combinatorial argument to show that:

$$\binom{m+n}{r} = \binom{m}{0} \binom{n}{r} + \binom{m}{1} \binom{n}{r-1} + \dots + \binom{m}{r} \binom{n}{0} \quad \dots (A)$$

where $m, n \in \mathbb{Z}^+, r \leq m$ and $r \leq n$.

Selecting r elements from $m+n$ elements can be viewed as:
dividing into the cases of selecting k elements from the m elements and the remaining $(r-k)$ elements from the n elements, for $0 \leq k \leq r$.

$$\binom{m+n}{r} = \binom{m}{0} \binom{n}{r} + \binom{m}{1} \binom{n}{r-1} + \dots + \binom{m}{r} \binom{n}{0} - \textcircled{A}$$

Let $m=n, r=n$.

Then, from \textcircled{A} ,

$$\binom{2n}{n} = \binom{n}{0} \binom{n}{n} + \binom{n}{1} \binom{n}{n-1} + \dots + \binom{n}{n-1} \binom{n}{1} + \binom{n}{n} \binom{n}{0}$$

$$\binom{n}{r} = \binom{n}{n-r} \quad (\text{Lecture 12 pg. 8})$$

$$= \textcircled{B}.$$

3. [AY2020/21 Semester 2 Exam Question]

How many integer solutions for x_1, x_2, x_3 and x_4 does the following equation have, given that $x_i \geq 2^i + i$, for $1 \leq i \leq 4$?

$$x_1 + x_2 + x_3 + x_4 = 56.$$

4. Find the term independent of x in the expansion of

$$\left(2x^2 + \frac{1}{x}\right)^9$$

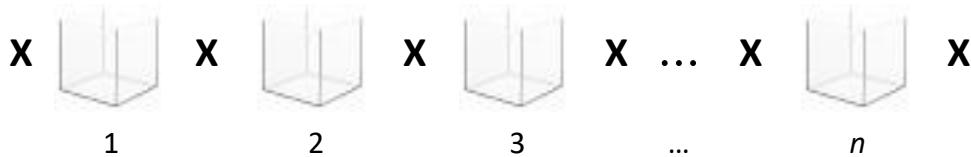
5. Let's revisit Tutorial #9 Question 5:

Given n boxes numbered 1 to n , each box is to be filled with either a white ball or a blue ball such that at least one box contains a white ball and boxes containing white balls must be consecutively numbered. What is the total number of ways this can be done?

Last week, the answer given was

For k ($1 \leq k \leq n$) consecutively numbered boxes that contain white balls, there are $n - k + 1$ ways. Therefore, total number of ways is $\sum_{k=1}^n (n - k + 1) = \sum_{k=1}^n k = \frac{n(n+1)}{2}$.

Now, let's use another approach to solve this problem. Draw crosses on the side of the boxes as shown below. How do you use these crosses?



6. [AY2020/21 Semester 2 Exam Question]

On a die there are 6 numbers. We call 4, 5, 6 the big numbers and 1, 2, 3 the small numbers. Given a loaded die in which the probability of rolling any fixed big number is twice the probability of rolling any fixed small number, answer the following questions.

- What is the probability of rolling a 6? Write your answer as a single fraction.
- If two such loaded dice are rolled, what is the expected value of the maximum of the two dice? Write your answer as a single fraction.

7. An urn contains five balls numbered 1, 2, 2, 8 and 8. If a person selects a set of two balls at random, what is the expected value of the sum of the numbers on the balls?

3. [AY2020/21 Semester 2 Exam Question]

How many integer solutions for x_1, x_2, x_3 and x_4 does the following equation have, given that $x_i \geq 2^i + i$, for $1 \leq i \leq 4$?

Multiset An

$$x_1 + x_2 + x_3 + x_4 = 56.$$

$$\begin{aligned} x_1 &\geq 2^1 + 1 = 3 \\ x_2 &\geq 2^2 + 2 = 6 \\ x_3 &\geq 2^3 + 3 = 11 \\ x_4 &\geq 2^4 + 4 = 20 \end{aligned}$$

$$\begin{aligned} \therefore x_1 &\geq 3 \\ x_2 &\geq 6 \\ x_3 &\geq 11 \\ x_4 &\geq 20. \end{aligned}$$

$$\binom{4+16-1}{16}$$

$r \leftarrow$ combi
 \nearrow repetition
allowed

$$3 + 6 + 11 + 20 = 40$$

$$\therefore 56 - 40 = 16.$$

$x_i \geq 0$ every constant can have 0 balls

Multiset : $n=4, r=16$.

$$\underbrace{x_1' + x_2' + x_3' + x_4'}_{\binom{r+n-1}{r}} = 16.$$

$$\binom{4+16-1}{16}$$

4. Find the term independent of x in the expansion of

$$\left(2x^2 + \frac{1}{x}\right)^9$$

$$(a+b)^n$$

$$\binom{n}{k} a^{n-k} b^k$$

$$= \binom{9}{k} (2x^2)^{9-k} \left(\frac{1}{x}\right)^k$$

$$\text{Let } \underline{k=6}$$

$$\binom{9}{6} (2x^2)^{9-6} \left(\frac{1}{x}\right)^6$$

$$= \binom{9}{6} 2^3 x^6 \cdot \frac{1}{x^6}$$

$$= 2^3 \binom{9}{6}$$

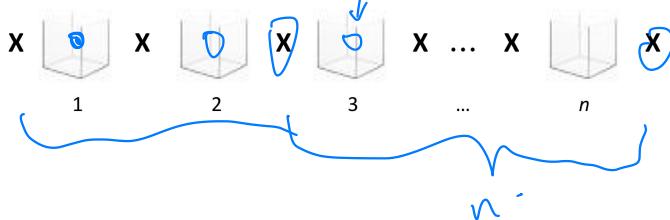
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Now, let's use another approach to solve this problem. Draw crosses on the side of the boxes as shown below. How do you use these crosses?



At 1 cross, change my 2 draw for white ball to be in between

$$n! C_2$$

6. [AY2020/21 Semester 2 Exam Question 12]

On a die there are 6 numbers. We call 4, 5, 6 the big numbers and 1, 2, 3 the small numbers. Given a loaded die in which the probability of rolling any fixed big number is twice the probability of rolling any fixed small number, answer the following questions.

- (i) What is the probability of rolling a 6? Write your answer as a single fraction.
- (ii) If two such loaded dice are rolled, what is the expected value of the maximum of the two dice? Write your answer as a single fraction.

i) $\frac{2}{9}$

W down all probability!

ii) Expected value:

$$\begin{aligned} \sum_{k=1}^6 a_k p_k &= a_1 p_1 + a_2 p_2 + a_3 p_3 + \dots + a_6 p_6 \\ &= 1 \cdot \frac{1}{9} + 2 \cdot 2 \cdot \frac{1}{9} + 1 \cdot 1 \cdot \frac{1}{9} \\ &\quad + 3 \cdot 3 \cdot \frac{1}{9} + 1 \cdot 4 \cdot \frac{2}{9} \\ &\quad + 3 \cdot 5 \cdot \frac{2}{9} + 2 \cdot 5 \cdot \frac{2}{9} \\ &\quad + 3 \cdot 6 \cdot \frac{1}{9} + 3 \cdot 6 \cdot \frac{2}{9} \end{aligned}$$

Maximum	Cases	Probability
1	(1,1)	$\frac{1}{36}$
2	(2,1), (1,2)	$\frac{2}{36}$
3	(3,1), (2,2), (1,3)	$\frac{3}{36}$
4	(4,1), (3,2), (2,3), (1,4)	$\frac{4}{36}$
5	(5,1), (4,2), (3,3), (2,4), (1,5)	$\frac{5}{36}$
6	(6,1), (5,2), (4,3), (3,4), (2,5), (1,6)	$\frac{6}{36}$

$$\begin{aligned} EV &= \frac{1}{6} \cdot 1 + \frac{2}{6} \cdot 2 + \frac{16}{36} \cdot 3 + \frac{24}{36} \cdot 4 + \frac{72}{36} \cdot 5 \\ &= \frac{398}{36} \end{aligned}$$

Different ball

7. An urn contains five balls numbered 1, 2, 2, 8 and 8. If a person selects a set of two balls at random, what is the expected value of the sum of the numbers on the balls? *Set w/o permutation*

$$\text{Expected value} = \sum_{n=1}^r a_n p_n = a_1 p_1 + a_2 p_2 + a_3 p_3 + \dots + a_n p_n$$

$$1+2 = 3$$

$$2+2 = 4$$

$$2+8 = 10$$

$$8+8 = 16$$

$$8+1 = 9.$$

- Q7. An urn contains five balls numbered 1, 2, 2, 8 and 8. If a person selects a set of two balls at random, what is the expected value of the sum of the numbers on the balls?

Let 2_a and 2_b denote the two balls with the number 2, and 8_a and 8_b the two balls with the number 8.

Cases for sums of the numbers on the balls:

- Sum of 3: $\{1, 2_a\}, \{1, 2_b\}$
- Sum of 4: $\{2_a, 2_b\}$
- Sum of 9: $\{1, 8_a\}, \{1, 8_b\}$
- Sum of 10: $\{2_a, 8_a\}, \{2_a, 8_b\}, \{2_b, 8_a\}, \{2_b, 8_b\}$
- Sum of 16: $\{8_a, 8_b\}$

$$\text{Expected sum} = 3 \cdot \left(\frac{2}{10}\right) + 4 \cdot \left(\frac{1}{10}\right) + 9 \cdot \left(\frac{2}{10}\right) + 10 \cdot \left(\frac{4}{10}\right) + 16 \cdot \left(\frac{1}{10}\right)$$

$$\begin{aligned}
 &= 3 \cdot \left(\frac{1}{5} \cdot \frac{2}{4} + \frac{2}{5} \cdot \frac{1}{4}\right) \\
 &+ 4 \cdot \left(\frac{2}{5} \cdot \frac{1}{4}\right) \cancel{\times 2} \rightarrow \text{no need for permutation, it's the same!} \\
 &+ 10 \cdot \left(\frac{2}{5} \cdot \frac{2}{4} + \frac{2}{5} \cdot \frac{1}{4}\right) \\
 &+ 16 \cdot \left(\frac{2}{5} \cdot \frac{2}{4}\right) \cancel{\times 2} \\
 &+ 9 \cdot \left(\frac{1}{5} \cdot \frac{2}{4} + \frac{2}{5} \cdot \frac{1}{4}\right) \\
 \\
 &= 8 \cdot 4
 \end{aligned}$$

8. One urn contains 10 red balls and 25 green balls, and a second urn contains 22 red balls and 15 green balls. A ball is chosen as follows: First an urn is selected by tossing a loaded coin with probability 0.4 of landing heads up and probability of 0.6 of landing tails up. If the coin lands heads up, the first urn is chosen; otherwise, the second urn is chosen. Then a ball is picked at random from the chosen urn.

Write your answers correct to three significant figures.

- (a) What is the probability that the chosen ball is green? *what we want to find.*
- (b) If the chosen ball is green, what is the probability that it was picked from the first urn?

given that $10R \quad 25G \quad 22R \quad 15G$

$0.4H$ $0.6T$

$$\begin{aligned}
 \text{a)} \quad & 0.4 \times \frac{25}{35} + 0.6 \times \frac{15}{37} \\
 & = 0.129 \quad \text{or} \quad \frac{137}{259}.
 \end{aligned}$$

$$\begin{aligned}
 \text{b)} \quad & 0.4 \times (0.129) = 0.540 \\
 & \times \frac{25}{35}
 \end{aligned}$$

$$P(G|I_1) = \frac{P(I_1 \cap G)}{P(I_1)} < \frac{0.4 \times \frac{25}{35}}{(0.129)}$$

Q8. First urn chosen by tossing a loaded coin with probability of 0.4 of landing heads (H) up and probability of 0.6 of landing tails (T) up.
H → first urn is chosen; *T* → second urn is chosen.
Then a ball is picked at random from the chosen urn.

Let G be the event that the chosen ball is green, I_1 the event that the ball came from the first urn, and I_2 the event that the ball came from the second urn.

$P(I_1) = 0.4, \quad P(I_2) = 0.6, \quad P(G|I_1) = \frac{25}{35}, \quad P(G|I_2) = \frac{15}{37}$

By Bayes' Theorem, $P(I_1|G) = \frac{P(G|I_1) \cdot P(I_1)}{P(G|I_1) \cdot P(I_1) + P(G|I_2) \cdot P(I_2)}$

$$\begin{aligned}
 & = \frac{\left(\frac{25}{35}\right) \cdot 0.4}{\left(\frac{25}{35}\right) \cdot 0.4 + \left(\frac{15}{37}\right) \cdot 0.6} = \frac{74}{137} = 54.0\%
 \end{aligned}$$


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Write your answers correct to three significant figures.

- (a) What is the probability that the chosen ball is green?
- (b) If the chosen ball is green, what is the probability that it was picked from the first urn?

9. [AY2015/16 Semester 1 Exam Question]

Let $A = \{1, 2, 3, 4\}$. Since each element of $P(A \times A)$ is a subset of $A \times A$, it is a binary relation on A . ($P(S)$ denotes the powerset of S .)

Assuming each relation in $P(A \times A)$ is equally likely to be chosen, what is the probability that a randomly chosen relation is (a) reflexive? (b) symmetric?

Can you generalize your answer to any set A with n elements?

10. Let's revisit Tutorial #9 Question 1:

In a certain tournament, the first team to win four games wins the tournament. Suppose there are two teams A and B , and team A wins the first two games. How many ways can the tournament be completed?

The solution given last week uses a possibility tree to depict the 15 ways. Now, let's approach this problem using combination.

Let us define a function $W(a, b)$ to be the number of ways the tournament can be completed if team A has to win a more games to win, while team B has to win b more games to win. Hence,

$$W(a, b) = \begin{cases} 1, & \text{if } a = 0 \text{ or } b = 0; \\ W(a, b - 1) + W(a - 1, b), & \text{if } a > 0 \text{ and } b > 0. \end{cases}$$

We may express $W(a, b)$ as a simple combination formula as follows:

$$W(a, b) = \binom{a+b}{a}.$$

Verify the above.

Now, we denote the function $T(n, k)$ to be the number of ways the tournament can be completed, given that the first team to win n games wins the tournament, and team A wins the first k ($k \leq n$) games.

Derive a simple combination formula for $T(n, k)$ (hint: relate function T to function W), and hence solve $T(4, 2)$ which is the problem in Tutorial #9 Question 1.

9. [AY2015/16 Semester 1 Exam Question]

subset of $A \times A \in P(A \times A)$.

Let $A = \{1, 2, 3, 4\}$. Since each element of $P(A \times A)$ is a subset of $A \times A$, it is a binary relation on A . ($P(S)$ denotes the powerset of S .)

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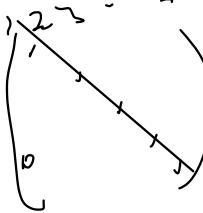
$$A \times A = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), (4,4)\}$$

$\frac{2^{16}-2}{2^{16}}$
 $\frac{n^2-n}{2^{n^2}}$

$$\begin{aligned} |\mathcal{P}(A \times A)| &= 2^{16} \\ &= \{ \emptyset, \{(1,1)\}, \dots, \{(1,1), (4,4)\}, \dots \}. \end{aligned}$$

a) Reflexive: (a, a)

$$P = \frac{2^{12}}{2^{16}}.$$



b) Symmetric: $xRy \rightarrow yRx$

$$C = \frac{n^2-n+n}{2}$$

n is triangular = 4.

General form:

$$\frac{1}{2^{n^2-n}}.$$

Q9. (b) Symmetric relations

1. For a set A with n elements, there are 2^{n^2} possible relations on A .

2. For a relation to be symmetric, for every entry $a_{i,j}$ (where $i < j$), i.e. in the upper triangular region (red triangle), its corresponding mirror image along the main diagonal, $a_{j,i}$ in the lower triangular region (blue triangle) must follow with the same value.

3. There are $\frac{n(n+1)}{2}$ entries in the upper triangle. There are n entries along the main diagonal.

Therefore, there are $\frac{n(n+1)}{2} + n$, or $\frac{n(n+1)}{2} + n$ entries to be filled with 0 or 1.

4. Therefore, there are $2^{\frac{n(n+1)}{2} + n}$ symmetric relations on A with n elements.

5. Hence, the probability that a randomly chosen relation on set A is symmetric is: $\frac{2^{\frac{n(n+1)}{2} + n}}{2^{n^2}} = \frac{1}{2^{n^2-n}}$. In particular, when $n = 4$, the probability is $\frac{1}{2^4}$ or $\frac{1}{16}$.

$$\begin{aligned} &\frac{(1+n-1)(n-1)}{2} + 4 \\ &= \frac{4^2-4}{2} + 4 \\ &= \frac{2^{16}}{2} \\ &= 2^{10} \end{aligned}$$

Q9. Let $A = \{1, 2, 3, 4\}$. Since each element of $\mathcal{P}(A \times A)$ is a subset of $A \times A$, it is a binary relation on A . ($\mathcal{P}(S)$ denotes the power set of S .)

Assuming each relation in $\mathcal{P}(A \times A)$ is equally likely to be chosen, what is the probability that a randomly chosen relation is (a) reflexive? (b) symmetric?

We will solve the general case. Let $A = \{a_1, a_2, \dots, a_n\}$ and so $|A| = n$. A relation R on A can be represented by an $n \times n$ matrix where the entry $a_{i,j} = 1$ if $a_i R a_j$, or $a_{i,j} = 0$ if $a_i \not R a_j$.

Example:

a_1	a_2	a_3	a_4
0	1	0	1
0	1	0	0
1	0	0	0
0	0	1	1

This matrix represents this relation R on A : $R = \{(a_1, a_2), (a_1, a_4), (a_2, a_3), (a_3, a_1), (a_4, a_3)\}$

Q9. (a) Reflexive relations

1. For a set A with n elements, there are 2^{n^2} possible relations on A . (why?)

2. For a relation to be reflexive, $a_i R a_i \forall a_i \in A$. Hence, the main diagonal entries $a_{i,i}$ must be filled with 1, as shown below.

$$\begin{bmatrix} 1 & ? & ? & \dots & ? \\ ? & 1 & ? & \dots & ? \\ ? & ? & 1 & \dots & ? \\ \vdots & \vdots & \vdots & \ddots & ? \\ ? & ? & ? & ? & 1 \end{bmatrix}$$

3. The remaining $n^2 - n$ entries may be filled with 0 or 1 (two choices).

4. Therefore, there are 2^{n^2-n} reflexive relations on A with n elements.

5. Hence, the probability that a randomly chosen relation on set A is reflexive is:

$$\frac{2^{n^2-n}}{2^{n^2}} = \frac{1}{2^n}$$

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We may express $W(a, b)$ as a simple combination formula as follows:

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Verify the above.

Now, we denote the function $T(n, k)$ to be the number of ways the tournament can be completed, given that the first team to win n games wins the tournament, and team A wins the first k ($k \leq n$) games.

Derive a simple combination formula for $T(n, k)$ (hint: relate function T to function W), and hence solve $T(4, 2)$ which is the problem in Tutorial #9 Question 1.

$$\begin{aligned} W(a, b) &= \begin{cases} 1 & \text{if } a=0 \text{ or } b=0 \\ W(a, b-1) + W(a-1, b) & \text{if } a>0 \text{ and } b>0 \end{cases} \\ &= \binom{a+b}{a} \quad \xrightarrow{\substack{W(a, b-1) \\ b-1-1-1}} \quad \begin{matrix} & \text{if } a=0 \text{ or } b=0 \\ & W(a, b-1-1) \\ & b-1-1-1 \end{matrix} \\ \text{if } a=0, &= \binom{b}{0} = 1 \quad \checkmark \\ \text{if } b=0, &= \binom{a}{a} = 1 \quad \checkmark \quad \text{commutative} \\ \text{else} &= \binom{a+(b-1)}{a} + \binom{(a-1)+b}{a} = \binom{a+b}{a} \quad \text{pascal's formula} \\ &= \binom{a+b-1}{a-1} + \binom{a+b-1}{a} \end{aligned}$$

$$T(n, k) = W(n-k, n)$$

$$\begin{aligned} &= \binom{n-k+n}{n-k} \\ &= \binom{2n-k}{n-k} \quad \cancel{=} \quad \binom{2n-k}{n} \end{aligned}$$

$$\therefore T(4, 2) = \binom{2+4-2}{4-2}$$

$$= \binom{4}{2}$$

$$= 6 C_2$$