NATIONAL UNIVERSITY OF SINGAPORE

SCHOOL OF COMPUTING

MID-TERM TEST
AY2020/21 Semester 1

CS1231/CS1231S — DISCRETE STRUCTURES

3 October 2020 Time Allowed: 1 hour 30 minutes

INSTRUCTIONS

1. This assessment paper contains **TWENTY TWO (22)** questions (excluding question 0) in **THREE (3)** parts and comprises **EIGHT (8)** printed pages.
2. Answer **ALL** questions.
3. This is an **OPEN BOOK** assessment.
4. The maximum mark of this assessment is 50.
5. You are to submit a **single pdf file** (size $\leqslant$ 20MB) to your submission folder on LumiNUS.
6. Your submitted file should be named after your **Student Number** (eg: A1234567X.pdf) and all pages of your file should contain your Student Number as well.
7. Limit your answers to **TWO pages** if possible, or at most THREE pages.
8. Do not write your name in your submitted file.

——— END OF INSTRUCTIONS ———

1	B	2	D	3	C	4	C	5	C	6	D	7	E
8	B,C	9	A,D	10	A,B,C,D	11	A,D	12	C	13	B		
14	C	15	A	16	B,C	17	D	18	B,C	19	B,C,D		

You do not need to answer question 0. Your tutor will check its correctness.

0. (a) Does my submission folder consist of a **single pdf file**? [1 mark]
   (b) Is my file named correctly, i.e. with my **Student Number** (eg: A1234567X.pdf)? [½ mark]
   (c) Have I written my Student Number on EVERY PAGE of my submitted file? [½ mark]

Part A: Multiple Choice Questions (Total: 14 marks)

Each multiple choice question (MCQ) is worth two marks and has exactly **one** correct answer. You are advised to write your answers on a **single line** to conserve space. For example:

1 A   2 B   3 C   4 D   ...

Please write in **CAPITAL LETTERS**.

1. Given the following statements:

   "I" is a necessary condition for "can".
   "can" is a sufficient condition for "do".
   "can" only if "it".

   Which of the following is logically equivalent to the conjunction of the above three statements?

   A. $"I" \rightarrow "can" \wedge "do" \wedge "it"$
   B. $\boldsymbol{"can" \rightarrow "I" \wedge "do" \wedge "it"}$
   C. $("can" \wedge "I") \rightarrow ("do" \vee "it")$
   D. $("I" \vee "can") \rightarrow ("do" \wedge "it")$
   E. None of (A), (B), (C), (D) is true.

   "I" is a necessary condition for "can": "can" → "I" ≡ $\sim$"can" ∨ "I".
   "can" is a sufficient condition for "do": "can" → "do" ≡ $\sim$"can" ∨ "do".
   "can" only if "it": "can" → "it" ≡ $\sim$"can" ∨ "it".
   $(\sim"can" \vee "I") \wedge (\sim"can" \vee "do") \wedge (\sim"can" \vee "it") \equiv \sim"can" \vee ("I" \wedge "do" \wedge "it")$
   $\therefore "can" \rightarrow "I" \wedge "do" \wedge "it"$.

2. Given the statement:

   The product of two negative real numbers is positive.

   Which of the following is the correct logical statement for the above statement?

   A. $\exists x \in \mathbb{R}\ \forall y \in \mathbb{R}\ (x < 0 \wedge y < 0 \rightarrow xy > 0)$
   B. $\exists x \in \mathbb{R}\ \exists y \in \mathbb{R}\ (x < 0 \wedge y < 0 \wedge xy > 0)$
   C. $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}\ (x < 0 \wedge y < 0 \wedge xy > 0)$
   D. $\boldsymbol{\forall x \in \mathbb{R}\ \forall y \in \mathbb{R}\ (x < 0 \wedge y < 0 \rightarrow xy > 0)}$

3. Given the statement:

$$(\sim p \wedge \sim r) \vee \big(p \wedge \sim(\sim r \vee s)\big) \vee p \vee (\sim p \wedge q \wedge \sim r \wedge s) \vee \sim(q \vee \sim p)$$

Which of the following is logically equivalent to the above?

A. p

B. $\sim r$

C. $\boldsymbol{p} \vee \sim\boldsymbol{r}$

D. $\sim p \vee \sim r$

E. None of (A), (B), (C) or (D).

$(\sim p \wedge \sim r) \vee \big(p \wedge \sim(\sim r \vee s)\big) \vee p \vee (\sim p \wedge q \wedge \sim r \wedge s) \vee \sim(q \vee \sim p)$
$\equiv (\sim p \wedge \sim r) \vee p \vee (\sim p \wedge q \wedge \sim r \wedge s) \vee \sim(q \vee \sim p)$ (absorption law)
$\equiv (\sim p \wedge \sim r) \vee p \vee (\sim p \wedge \sim r \wedge q \wedge s) \vee \sim(q \vee \sim p)$ (associative law, commutative law)
$\equiv (\sim p \wedge \sim r) \vee (\sim p \wedge \sim r \wedge q \wedge s) \vee p \vee \sim(q \vee \sim p)$ (commutative law)
$\equiv (\sim p \wedge \sim r) \vee p \vee \sim(q \vee \sim p)$ (absorption law)
$\equiv (\sim p \wedge \sim r) \vee p \vee (\sim(\sim p) \wedge \sim q)$ (De Morgan's law, commutative law)
$\equiv (\sim p \wedge \sim r) \vee p \vee (p \wedge \sim q)$ (double negative law)
$\equiv (\sim p \wedge \sim r) \vee p$ (absorption law)
$\equiv p \vee (\sim p \wedge \sim r)$ (commutative law)
$\equiv p \vee (p \wedge \sim r) \vee (\sim p \wedge \sim r)$ (absorption law: $p \equiv p \vee (p \wedge \sim r)$)
$\equiv p \vee \big((\sim r \wedge p) \vee (\sim r \wedge \sim p)\big)$ (commutative law)
$\equiv p \vee \big(\sim r \wedge (p \vee \sim p)\big)$ (distributive law)
$\equiv p \vee (\sim r \wedge \mathbf{true})$ (negation law)
$\equiv p \vee \sim r$ (identity law)

4. Let $A = \{1,2,3\}$ and define $f\colon A \to A$ by setting, for all $x \in A$,

$$f(x) = \begin{cases} 2, & \text{if } x = 1; \\ 3, & \text{if } x = 2; \\ 1, & \text{if } x = 3. \end{cases}$$

How many (distinct) elements are there in

$$\{\underbrace{f \circ f \circ \cdots \circ f}_{n\text{-many } f\text{'s}} : n \in \mathbb{Z}^+\}?$$

A. 1.

B. 2.

C. 3.

D. 4.

E. Infinitely many.

Observe that $f = g \circ h$, where g, h are as defined in Question 6 of Tutorial 4. We know from that question that f has order 3, which means f and $f \circ f$ are not equal to id_A but $f \circ f \circ f$ is. Notice $f(1) = 2 \neq 3 = f(2) = f\big(f(1)\big) = (f \circ f)(1)$. So $f \neq f \circ f$. It follows that

$$\{\underbrace{f \circ f \circ \cdots \circ f}_{n\text{-many } f\text{'s}} : n \in \mathbb{Z}^+\}$$

$$= \{f, f\circ f, f\circ f\circ f, f\circ f\circ f\circ f, f\circ f\circ f\circ f\circ f, f\circ f\circ f\circ f\circ f\circ f, \ldots\}$$
$$= \{f, f\circ f, \mathrm{id}_A, \mathrm{id}_A\circ f, \mathrm{id}_A\circ f\circ f, \mathrm{id}_A\circ \mathrm{id}_A, \ldots\}$$
$$= \{f, f\circ f, \mathrm{id}_A, f, f\circ f, \mathrm{id}_A, \ldots\}$$
$$= \{f, f\circ f, \mathrm{id}_A\}$$

has exactly 3 distinct elements.

5. Let $A = \{1,2,3\}$ and define $g, h\colon A \to A$ by setting, for all $x \in A$,

$$g(x) = \begin{cases} 1, & \text{if } x = 2; \\ 2, & \text{if } x = 1; \\ x, & \text{otherwise,} \end{cases} \qquad h(x) = \begin{cases} 2, & \text{if } x = 3; \\ 3, & \text{if } x = 2; \\ x, & \text{otherwise.} \end{cases}$$

What is $(h \circ g)^{-1}$?

A. g.

B. h.

C. $\boldsymbol{g \circ h}$.

D. $h \circ g$.

E. None of the above.

These functions are from Tutorial 4 Question 6. As one can readily verify,

$$g^{-1} = g \quad \text{and} \quad h^{-1} = h.$$

Tutorial 4 Question 7 implies $(h \circ g)^{-1} = g^{-1} \circ h^{-1} = g \circ h$. One can also check that the other answers are not correct.

6. Define the function $f\colon \mathrm{Bool}^2 \to \mathrm{Bool}$ by setting, for all $p, q \in \mathrm{Bool}$,

$$f(p,q) \quad = \quad (p \vee q) \wedge {\sim}(p \wedge q).$$

(Here $\mathrm{Bool} = \{\mathbf{true}, \mathbf{false}\}$.) Which of the following statements is true?

A. f is not well defined, i.e., f does not satisfy the definition of a function.

B. f is a function that is neither injective nor s **Common mistakes:**

- Some students tried to prove by contradiction, but use ONE example to show that $\exists x \in \mathbb{R}\,\big((x^2 > x) \wedge (x \geq 0) \wedge (x \leq 1)\big)$ is false.
- Some students attempted to prove by contraposition:

  $$\forall x \in \mathbb{R}\,\big((0 \leq x \leq 1) \to (x^2 \leq x)\big)$$

  However, they merely wrote, "since $0 \leq x \leq 1$, therefore $x^2 \leq x$" and justified it with "we all know this". This is not acceptable.

  Some simply quoted ONE example to show that the above for all statement is true ("let $x = \frac{1}{2}$, then $x^2 = \frac{1}{4} < \frac{1}{2} < x$. Therefore, the statement is true."). This is not acceptable too.
- Despite our constant reminder that you should justify your steps, many students still did not do so. For example, when students used the given T25, they didn't write (by

T25). Only one student who used theorem T20 in Epp's book explicitly quoted it, while many who used it didn't.

- I am quite lenient here as I did not deduct mark for such lack of justification. I gave the benefit of doubt whenever possible, unless in cases where it just takes too much leap of faith.
- Quite a number of students wrote "let $x = m/n, n \neq 0$". As x here is a real number (which is not a rational number), this is not acceptable.
- I'm surprised that many students couldn't negate $(x < 0)$ correctly, writing $(x > 0)$ instead of $(x \geq 0)$!
- Students mixed up on how to do contrapositive. The contrapositive of $\forall x\ \big(P(x) \rightarrow Q(x)\big)$ is $\forall x\ \big({\sim}Q(x) \rightarrow {\sim}P(x)\big)$, not $\exists x\ \big({\sim}Q(x) \rightarrow {\sim}P(x)\big)$.
- Some students also did the negation wrongly. The negation of $\forall x\ \big(P(x) \rightarrow Q(x)\big)$ is $\exists x\ \big(P(x) \wedge {\sim}Q(x)\big)$, not $\exists x\ \big({\sim}P(x) \rightarrow {\sim}Q(x)\big)$.
- Many students committed converse error when they quoted T25 this way:

  "$x > 0$ and $(x - 1) > 0$, therefore $x(x - 1) > 0$ (by T25)".

  That that is the converse of T25, not T25, and we know that in general a conditional statement is not logically equivalent to its converse.

Comments:

- In programming, we don't just assess a program by correctness alone; its efficiency and elegance are also considered. Likewise, in assessing a proof, we need to uncover the writer's thought process as we read the proof.
- If the proof is cumbersome to read, it reflects the thought process behind it.
- A good proof is one with a smooth flow, instead of making the reader go back and forth to understand the logic.
- In general, when you are asked to prove the statement $\forall x\ \big(P(x) \rightarrow Q(x)\big)$, these are some possible ways:
  - Direct proof: Start from $P(x)$ and arrive at $Q(x)$.
  - Proof by contraposition: Convert to $\forall x\ \big({\sim}Q(x) \rightarrow {\sim}P(x)\big)$, then start from ${\sim}Q(x)$ and arrive at ${\sim}P(x)$.
  - Proof by contradiction: Negate the statement to $\exists x\ \big(P(x) \wedge {\sim}Q(x)\big)$, then arrive at a contradiction.
- In answering this question, many students started with the cases for $x > 0$, $x = 0$ and $x < 0$. This doesn't have connection with the $P(x)$ in this question, which is $x^2 > x$. This makes the proof likely wrong and even if correct, difficult to read and understand. This is the place when I graded it incorrect or I deducted marks even though it is correct.

urjective.

C. f is an injection but not a surjection.

D. f is a surjection but not an injection.

E. f is a bijection.

p	q	$p \vee q$	$p \wedge q$	$\sim(p \wedge q)$	$(p \vee q) \wedge \sim(p \wedge q)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

7. Let A, B, C be sets. Which of the following may not be a subset of $(A \setminus C) \cup (B \setminus C)$?

   A. $(A \cap B) \setminus C$.

   B. $(A \cup B) \setminus C$.

   C. $(A \setminus B) \setminus C$.

   D. $(A \setminus C) \setminus B$.

   E. $\boldsymbol{A \cap B \cap C}$.

- Let $A = B = C = \{1\}$. Then $A \cap B \cap C = \{1\} \nsubseteq \emptyset = (A \setminus C) \cup (B \setminus C)$.
- $(A \cap B) \setminus C \subseteq (A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$.
- $(A \setminus C) \setminus B = (A \setminus B) \setminus C \subseteq (A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$.

Part B: Multiple Response Questions [Total: 24 marks]

Each multiple response question (MRQ) is worth two marks and may have one answer or multiple answers. Write out **all** correct answers. For example, if you think that A, B, C are the correct answers, write A, B, C. Only if you get all the answers correct will you be awarded two marks. **No partial credit will be given for partially correct answers.**

You are advised to write your answers on **at most two lines** to conserve space. For example:

8 A,B   9 B,D   10 C   11 A,B,C,D   …

Please write in **CAPITAL LETTERS**.

8. Let the domain of discourse be the set of real numbers and define $P(x, y)$ and $Q(x, y)$ as follows:

   $$P(x, y) : xy = 0$$
   $$Q(x, y) : x/y = 1$$

   Which of the following is/are FALSE?

   A. $\forall x\, \forall y\, P(x, y) \equiv \forall x\, \forall y\, Q(x, y)$

   B. $\boldsymbol{\forall x\, \exists y\, P(x, y) \equiv \forall x\, \exists y\, Q(x, y)}$

   C. $\boldsymbol{\exists x\, \forall y\, P(x, y) \equiv \exists x\, \forall y\, Q(x, y)}$

   D. $\exists x\, \exists y\, P(x, y) \equiv \exists x\, \exists y\, Q(x, y)$

9. Which of the following is/are FALSE for some predicates $P(x)$ and $Q(x)$?

   A. $\boldsymbol{\forall x\, (P(x) \lor Q(x)) \Leftrightarrow \forall x\, P(x) \lor \forall x\, Q(x)}$

   B. $\forall x\, (P(x) \land Q(x)) \Leftrightarrow \forall x\, P(x) \land \forall x\, Q(x)$

   C. $\exists x\, (P(x) \lor Q(x)) \Leftrightarrow \exists x\, P(x) \lor \exists x\, Q(x)$

   D. $\boldsymbol{\exists x\, (P(x) \land Q(x)) \Leftrightarrow \exists x\, P(x) \land \exists x\, Q(x)}$

10. Let the domain of discourse be the set of real numbers. Which of the following is/are FALSE?

    A. $\forall x\ (x^2 > 0)$

    B. $\exists!\, x\ (x^2 + 2x - 3 = 0)$

    C. $\exists x\ (x^2 + x + 1 = 0)$

    D. $\forall x\ (x \neq 0 \rightarrow x^2 \geq 1)$

    For (B), $x = 1$ or $x = -3$. For (C), there are no real roots.

11. Let $A = \{-2, -1, 0, 1, 2\}$ and $B = \{-1, 0, 1\}$. Which of the following is/are FALSE?

    A. $\forall x, y \in A\ (x + y \in A)$

    B. $\forall x, y \in B\ (xy \in B)$

    C. $\exists x \in A\ \forall y \in A\ \exists z \in \mathbb{Z}\ (y = xz)$

    D. $\forall x \in A\ \forall y \in B\ \exists z \in \mathbb{Z}\ (x + y < z < xy)$

    For (C), let $x = 1$, then $\forall y \in A$ there exists an integer z such that $z = y$.

    For (D), let $x = -2$ and $y = 1$, then there is no z for $-1 < z < -2$.

12. Which of the following set(s) is/are equal to

    $$\{3n + 1 : n \in \mathbb{Z}\}?$$

    A. $\{3n : n \in \mathbb{Z}\} \cup \{1\}$.

    B. $\{3n - 1 : n \in \mathbb{Z}\}$.

    C. $\{3n - 2 : n \in \mathbb{Z}\}$.

    D. The domain of the function $f\colon \mathbb{Z} \to \mathbb{Z}$ satisfying $f(n) = 3n + 1$ for all $n \in \mathbb{Z}$.

    E. The codomain of the function $f\colon \mathbb{Z} \to \mathbb{Z}$ satisfying $f(n) = 3n + 1$ for all $n \in \mathbb{Z}$.

- 0 is an element of the set in (A) but not in the given set.
- -1 is an element of the set in (B) but not in the given set.
- The domain and the codomain of f above are both $\mathbb{Z}$. So 0 is in it but not in the given set.
- One can show that the set in (C) is correct by following the proof in Tutorial 3 Question 4.

13. Let $S = \{\diamondsuit, \clubsuit, \heartsuit, \spadesuit\}$ and $R = \{\mathrm{A}, 2,3,4,5,6,7,8,9,10, \mathrm{J}, \mathrm{Q}, \mathrm{K}\}$. Which of the following **set(s)** contain

    $$\{(\spadesuit, \mathrm{A}), (\spadesuit, \mathrm{K}), (\spadesuit, \mathrm{Q}), (\spadesuit, \mathrm{J}), (\spadesuit, 10)\}$$

    as an **element**?

    A. $S \times R$.

    B. $\mathcal{P}(S \times R)$.

    C. $S \cup R$.

    D. $\mathcal{P}(S \cup R)$.

- All the elements of $S \times R$ are ordered pairs, not sets. So the set displayed cannot be an element of $S \times R$. Similarly, no element of S is a set, and no element of R is a set. So the set displayed cannot be an element of $S \cup R$.
- Every element of the set displayed is an ordered pair in which the first coordinate is an element of S and the second coordinate is an element of R. Thus it is an element of $S \times R$. This shows the set displayed is a subset of $S \times R$ and hence an element of $\mathcal{P}(S \times R)$.
- $(\spadesuit, \mathrm{A})$ is an element of the set displayed above. However, it is neither in S nor in R. So the set displayed above is not a subset of $S \cup R$. This means it is not an element of $\mathcal{P}(S \cup R)$.

14. Let $M_k = \{n \in \mathbb{Z} : n = km \text{ for some } m \in \mathbb{Z}\}$ for each $k \in \mathbb{Z}$. Which of the following contain(s) the number 100 as an element?

A. $M_2 \times M_3 \times M_5$.

B. $(M_2 \setminus M_3) \setminus M_5$.

C. $\boldsymbol{M_2 \cup M_3 \cup M_5}$.

D. $M_2 \cap M_3 \cap M_5$.

- All elements of $M_2 \times M_3 \times M_5$ are ordered 3-tuples. So $100 \notin M_2 \times M_3 \times M_5$ because 100 is not an ordered 3-tuple.
- Note $100 = 5 \times 20 \in M_5$. So $100 \notin (M_2 \setminus M_3) \setminus M_5$ but $100 \in M_2 \cup M_3 \cup M_5$.
- Note $100 \neq 3m$ for any $m \in \mathbb{Z}$. So $100 \notin M_3$ and thus $100 \notin M_2 \cap M_3 \cap M_5$.

15. Which of the following **cannot** be $|A \times (B \cup \mathcal{P}(C))|$ for any choice of mutually disjoint nonempty finite sets A, B, C?

A. 1.

B. 2.

C. 3.

D. 4.

E. 5.

- $|A \times (B \cup \mathcal{P}(C))| \geqslant |A \times \mathcal{P}(C)| = |A| \times 2^{|C|} \geqslant 1 \times 2^1 = 2 > 1$. So A is not possible.
- Let $A = \{3\}$, $B = \{\emptyset\}$ and $C = \{8\}$. Then
$$\begin{aligned} A \times \big(B \cup \mathcal{P}(C)\big) &= \{3\} \times \big(\{\emptyset\} \cup \mathcal{P}(\{8\})\big) \\ &= \{3\} \times \big(\{\emptyset\} \cup \{\emptyset, \{8\}\}\big) \\ &= \{3\} \times \{\emptyset, \{8\}\} \\ &= \{(3, \emptyset), (3, \{8\})\} \end{aligned}$$
has exactly 2 elements. So B is possible.
- Let $A = \{3\}$, $B = \{4\}$, $C = \{8\}$. Then
$$\begin{aligned} A \times \big(B \cup \mathcal{P}(C)\big) &= \{3\} \times \big(\{4\} \cup \mathcal{P}(\{8\})\big) \\ &= \{3\} \times \big(\{4\} \cup \{\emptyset, \{8\}\}\big) \\ &= \{3\} \times \{4, \emptyset, \{8\}\} \\ &= \{(3,4), (3, \emptyset), (3, \{8\})\} \end{aligned}$$
has exactly 3 elements. So C is possible.

- Similarly, one sees that D and E are possible by replacing the set B in the previous bullet point with $\{4,5\}$ and $\{4,5,6\}$ respectively.

16. Which of the following function(s) $f: \mathbb{Z} \to \mathbb{Z}$ defined below satisfies/satisfy

$$\forall X \subseteq \mathbb{Z} \quad f^{-1}\big(f(X)\big) \subseteq X?$$

A. $f: x \mapsto x^2$.

B. $\boldsymbol{f: x \mapsto 3x + 1.}$

C. $\boldsymbol{f: x \mapsto \begin{cases} x, & \text{if } x \text{ is even;} \\ 2x - 1, & \text{if } x \text{ is odd.} \end{cases}}$

D. $f: x \mapsto \left\lfloor \frac{x}{2} \right\rfloor$.

- For A, one counterexample is $X = \{1\}$, so that $f^{-1}\big(f(X)\big) = f^{-1}(1) = \{1, -1\}$.
- For D, one counterexample is $X = \{0\}$, so that $f^{-1}\big(f(X)\big) = f^{-1}(0) = \{0,1\}$.
- As one can show, the required condition is satisfied if f is injective. Since the functions in B and C are injective (see Tutorial 4 Question 4), we know that they satisfy the required condition.

17. Which of the following function(s) $f: \mathbb{Z} \to \mathbb{Z}$ defined below satisfies/satisfy

$$\forall Y \subseteq \mathbb{Z} \quad Y \subseteq f(f^{-1}(Y))?$$

A. $f: x \mapsto x^2$.

B. $f: x \mapsto 3x + 1$.

C. $f: x \mapsto \begin{cases} x, & \text{if } x \text{ is even;} \\ 2x - 1, & \text{if } x \text{ is odd.} \end{cases}$

D. $\boldsymbol{f: x \mapsto \left\lfloor \frac{x}{2} \right\rfloor.}$

- For A, one counterexample is $Y = \{-1\}$, so that $f\big(f^{-1}(Y)\big) = f(\emptyset) = \emptyset$.
- For B, one counterexample is $Y = \{0\}$, so that $f\big(f^{-1}(Y)\big) = f(\emptyset) = \emptyset$.
- For C, one counterexample is $Y = \{3\}$, so that $f(f^{-1}(Y)) = f(\emptyset) = \emptyset$.
- As one can show, the required condition is satisfied if f is surjective. Since the function in D is surjective, we know that it satisfies the required condition.

18. Define a set S recursively as follows.

- $1 \in S$. (base clause)
- If $x \in S$, then $2x \in S$ and $x - 3 \in S$. (recursion clause)
- Membership for S can always be demonstrated by (finitely many) successive applications of the clauses above. (minimality clause)

Which of the following is/are in S?

A. 3

B. 5

C. 7

D. 9

$1 \xrightarrow{\times 2} 2 \xrightarrow{\times 2} 4 \xrightarrow{\times 2} 8 \xrightarrow{-3} 5$ $\qquad$ $1 \xrightarrow{\times 2} 2 \xrightarrow{\times 2} 4 \xrightarrow{\times 2} 8 \xrightarrow{\times 2} 16 \xrightarrow{-3} 13 \xrightarrow{-3} 10 \xrightarrow{-3} 7$

One can show by structural induction over S that 3 cannot divide any element of S.

Thus $3, 9 \notin S$.

19. Let $A_0, A_1, A_2, \ldots$ be the sequence of sets satisfying

$$A_0 = \emptyset \quad \text{and} \quad A_{n+1} = A_n \cup \{A_n\}.$$

Which of the following is/are true?

A. $\{\{\emptyset\}\} \in A_{100}$.

B. $\boldsymbol{\{\{\emptyset\}\} \subseteq A_{100}}$.

C. $\boldsymbol{\{\emptyset, \{\emptyset\}\} \in A_{100}}$.

D. $\boldsymbol{\{\emptyset, \{\emptyset\}\} \subseteq A_{100}}$.

Note that $A_1 = \emptyset \cup \{\emptyset\} = \{\emptyset\}$ and $A_2 = \{\emptyset\} \cup \{\{\emptyset\}\} = \{\emptyset, \{\emptyset\}\}$.

One can show by induction that $A_n = \{A_0, A_1, \ldots, A_{n-1}\}$ for all $n \in \mathbb{Z}_{\geqslant 0}$. $\qquad (*)$

We know from $(*)$ that $A_2 \in A_{100}$ and $A_2 = \{A_0, A_1\} \subseteq \{A_0, A_1, \ldots, A_{99}\} = A_{100}$.

More generally, one can deduce from $(*)$ that if $S = A_n$ for some $n \in \mathbb{Z}_{\geqslant 0}$, then every element of S is a subset of S. Now $\{\emptyset\}$ is an element but not a subset of $\{\{\emptyset\}\}$. So $\{\{\emptyset\}\}$ is not one of the A_n's and thus cannot be an element of A_{100}.

Since $A_1 \in A_{100}$ by $(*)$, we know $\{A_1\} \subseteq A_{100}$.

Part C: There are 3 questions in this part [Total: 10 marks]

20. Given statement variables p, q and r, is the following statement a tautology?

$$\big((p \to q) \wedge (q \to r)\big) \vee (p \to r) \to (r \to p)$$

You need to show your working. Merely stating that it is a tautology or it is not, or giving a counterexample (in the case it is not) without showing that it leads to false is not sufficient and no mark will be awarded. No justification in your steps are needed if you use the definitions of conjunction, disjunction and conditional statement. Do not use truth table. [3 marks]

Answer: **It is not a tautology.** Counterexample: (1) $p = false, q = true, r = true$, or (2) $p = false, q = false, r = true$.

(1) $\big((false \to true) \wedge (true \to true)\big) \vee (false \to true) \to (true \to false)$

$\equiv (true \wedge true) \vee true \to false \equiv true \vee true \to false$

$\equiv true \to false \equiv false$

(2) $\big((false \to false) \wedge (false \to true)\big) \vee (false \to true) \to (true \to false)$

$\equiv (true \wedge true) \vee true \to false \equiv true \vee true \to false$

$\equiv true \to false \equiv false$

Common mistakes:

(Ken) When simplifying $(p \to q) \wedge (q \to r)$, many students wrote $(p \to r)$. This is incorrect. $(p \to q) \wedge (q \to r) \to (p \to r)$ (transitivity) is true, but $(p \to q) \wedge (q \to r) \not\equiv (p \to r)$.

(Aaron) I notice some students simplified the statement to $\sim r \vee p$ and stopped there, claiming that it is not a tautology. You need to justify, by citing a counterexample.

Comments (by Aaron):

Not knowing whether the given statement is true or false, we will attempt to find a counterexample since that is easier. Failing which, we can then conclude that it is true.

For a conditional statement to be false, you can apply some cleverness here. Knowing that $true \to false$ would be false, we aim to find values of the variables such that the antecedent is true and the consequent false. The antecedent in this question is long: $\big((p \to q) \wedge (q \to r)\big) \vee (p \to r)$, so we shall work on the shorter consequent $(r \to p)$ first instead.

For $(r \to p)$ to be false, the only possibility is $r = true$ and $p = false$. That leaves only variable q for you to try. In this case, q can be either true or false. Substituting these values of p, q, r into the given statement, it will evaluate to false.

And viola! There you get the counterexample. There is no need to simplify this complicated statement which takes a lot of time, as many of you did in the test! Always go for the most effective method.

This is the kind of tricks we hope students can derive themselves. In fact, in Assignment #1 Question 4, I have given you the hint in the way the solution is presented.

21. In Appendix A (Properties of the Real Numbers) of Epp's book, the following theorem is given and you may quote it without proof:

    T25. Suppose a and b are real numbers, if $ab > 0$, then both a and b are positive or both are negative.

    Prove: $\forall x \in \mathbb{R}\left((x^2 > x) \rightarrow (x < 0) \vee (x > 1)\right)$

    You are to give justification wherever appropriate. [3 marks]

    Answer: Direct proof.

    1. Let $x \in \mathbb{R}$ and $x^2 > x$.
    2. Then $x^2 - x > 0$, or $x(x-1) > 0$. (by basic algebra)
    3. Either both x and $(x-1)$ are positive or both are negative. (by T25)
    4. Case 1: Both x and $(x-1)$ are positive.
       4.1 Then $x > 0$ and $(x-1) > 0$, hence $x > 1$. (by basic algebra)
    5. Case 2: Both x and $(x-1)$ are negative.
       5.1 Then $x < 0$ and $(x-1) < 0$, hence $x < 0$. (by basic algebra)
    6. From 4.1 and 5.1, $x > 1$ or $x < 0$.

    Alternative answer: Proof by contradiction.

    1. Suppose not, i.e., $\sim\left(\forall x \in \mathbb{R}\left((x^2 > x) \rightarrow (x < 0) \vee (x > 1)\right)\right)$, or $\exists x \in \mathbb{R}\left((x^2 > x) \wedge (x \geq 0) \wedge (x \leq 1)\right)$
       1.1 $(x^2 > x) \Rightarrow x(x-1) > 0$. (by basic algebra)
       1.2 Hence, both x and $(x-1)$ are positive or both are negative. (by T25)
       1.3 Case 1: Both x and $(x-1)$ are positive.
           1.3.1 Then $x > 0$ and $(x-1) > 0$, so $x > 1$. (by basic algebra)
           1.3.2 But this contradicts $x \leq 1$ in line 1.
       1.4 Case 2: Both x and $(x-1)$ are negative.
           1.4.1 Then $x < 0$ and $(x-1) < 0$, so $x < 0$. (by basic algebra)
           1.4.2 But this contradicts $x \geq 0$ in line 1.
       1.5 In all cases there is a contradiction.
    2. Therefore, the supposition is false and the original statement $\forall x \in \mathbb{R}\left((x^2 > x) \rightarrow (x < 0) \vee (x > 1)\right)$ is true.

    Alternative answer: Proof by contraposition (not using T25 but T20)

    1. Contrapositive statement: $\forall x \in \mathbb{R}\left((x \geq 0) \wedge (x \leq 1) \rightarrow (x^2 \leq x)\right)$
    2. $(x \geq 0) \wedge (x \leq 1) \Rightarrow 0 \leq x \leq 1$.
    3. Multiplying by x, we have $0 \leq x^2 \leq x$ since x is non-negative (by T20*)
    4. Therefore, $x^2 \leq x$.

    (Note: T20 in Epp's book actually says "If $a < b$ and $c > 0$, then $ac < bc$". Here, we adapt it to include equality, so: "If $a \leq b$ and $c \geq 0$, then $ac \leq bc$". By right, you are to prove this but I will accept it without proof.)

Alternative answer: Proof by contradiction (not using T25 but T20)

1. Suppose not, i.e., $\sim\left(\forall x \in \mathbb{R}\left((x^2 > x) \rightarrow (x < 0) \vee (x > 1)\right)\right)$, or $\exists x \in \mathbb{R}\left((x^2 > x) \wedge (x \geq 0) \wedge (x \leq 1)\right)$
   1.1 $(x \geq 0) \wedge (x \leq 1) \Rightarrow 0 \leq x \leq 1$.
   1.2 Multiplying by x, we have $0 \leq x^2 \leq x$ since x is non-negative (by T20*).
   1.3 But this contradicts with $x^2 > x$ in line 1.
2. Therefore, the supposition is false and the original statement $\forall x \in \mathbb{R}\left((x^2 > x) \rightarrow (x < 0) \vee (x > 1)\right)$ is true.

(Note: T20 in Epp's book actually says "If $a < b$ and $c > 0$, then $ac < bc$". Here, we adapt it to include equality, so: "If $a \leq b$ and $c \geq 0$, then $ac \leq bc$". By right, you are to prove this but I will accept it without proof.))

Common mistakes:

- Some students tried to prove by contradiction, but use ONE example to show that $\exists x \in \mathbb{R}\left((x^2 > x) \wedge (x \geq 0) \wedge (x \leq 1)\right)$ is false.
- Some students attempted to prove by contraposition:

  $$\forall x \in \mathbb{R}\left((0 \leq x \leq 1) \rightarrow (x^2 \leq x)\right)$$

  However, they merely wrote, "since $0 \leq x \leq 1$, therefore $x^2 \leq x$" and justified it with "we all know this". This is not acceptable.

  Some simply quoted ONE example to show that the above for all statement is true ("let $x = \frac{1}{2}$, then $x^2 = \frac{1}{4} < \frac{1}{2} < x$. Therefore, the statement is true."). This is not acceptable too.
- Despite our constant reminder that you should justify your steps, many students still did not do so. For example, when students used the given T25, they didn't write (by T25). Only one student who used theorem T20 in Epp's book explicitly quoted it, while many who used it didn't.
- I am quite lenient here as I did not deduct mark for such lack of justification. I gave the benefit of doubt whenever possible, unless in cases where it just takes too much leap of faith.
- Quite a number of students wrote "let $x = m/n, n \neq 0$". As x here is a real number (which is not a rational number), this is not acceptable.
- I'm surprised that many students couldn't negate $(x < 0)$ correctly, writing $(x > 0)$ instead of $(x \geq 0)$!

- Students mixed up on how to do contrapositive. The contrapositive of $\forall x\, \big(P(x) \rightarrow Q(x)\big)$ is $\forall x\, \big({\sim}Q(x) \rightarrow {\sim}P(x)\big)$, not $\exists x\, \big({\sim}Q(x) \rightarrow {\sim}P(x)\big)$.
- Some students also did the negation wrongly. The negation of $\forall x\, \big(P(x) \rightarrow Q(x)\big)$ is $\exists x\, \big(P(x) \wedge {\sim}Q(x)\big)$, not $\exists x\, \big({\sim}P(x) \rightarrow {\sim}Q(x)\big)$.
- Many students committed converse error when they quoted T25 this way:

  "$x > 0$ and $(x-1) > 0$, therefore $x(x-1) > 0$ (by T25)".

  That that is the converse of T25, not T25, and we know that in general a conditional statement is not logically equivalent to its converse.

Comments:

- In programming, we don't just assess a program by correctness alone; its efficiency and elegance are also considered. Likewise, in assessing a proof, we need to uncover the writer's thought process as we read the proof.
- If the proof is cumbersome to read, it reflects the thought process behind it.
- A good proof is one with a smooth flow, instead of making the reader go back and forth to understand the logic.
- In general, when you are asked to prove the statement $\forall x\, \big(P(x) \rightarrow Q(x)\big)$, these are some possible ways:
  - Direct proof: Start from $P(x)$ and arrive at $Q(x)$.
  - Proof by contraposition: Convert to $\forall x\, \big({\sim}Q(x) \rightarrow {\sim}P(x)\big)$, then start from ${\sim}Q(x)$ and arrive at ${\sim}P(x)$.
  - Proof by contradiction: Negate the statement to $\exists x\, \big(P(x) \wedge {\sim}Q(x)\big)$, then arrive at a contradiction.
- In answering this question, many students started with the cases for $x > 0$, $x = 0$ and $x < 0$. This doesn't have connection with the $P(x)$ in this question, which is $x^2 > x$. This makes the proof likely wrong and even if correct, difficult to read and understand. This is the place where I graded it incorrect or deducted marks even though it is correct.

(Updated on: 11 Oct 2020)

22. Let $a_0, a_1, a_2, \ldots$ be the sequence satisfying

$$a_0 = 1 \quad \text{and} \quad a_{n+1} = 2a_n + 2$$

for all $n \in \mathbb{Z}_{\geqslant 0}$. Show by induction that

$$a_n = 2^{n+1} + 2^n - 2$$

for all $n \in \mathbb{Z}_{\geqslant 0}$. [4 marks]

Answer:

1. For each $n \in \mathbb{Z}_{\geqslant 0}$, let $P(n)$ be the proposition " $a_n = 2^{n+1} + 2^n - 2$ ".
2. (Base step) $P(0)$ is true because $a_0 = 1 = 2 + 1 - 2 = 2^{0+1} + 2^0 - 2$.
3. (Induction step)

3.1. Let $k \in \mathbb{Z}_{\geqslant 0}$ such that $P(k)$ is true, i.e., such that $a_k = 2^{k+1} + 2^k - 2$.

3.2. Then $a_{k+1} = 2a_k + 2$ by the definition of a_{k+1};

3.3. $= 2(2^{k+1} + 2^k - 2) + 2$ by the induction hypothesis;

3.4. $= 2^{k+2} + 2^{k+1} - 2$

3.5. $= 2^{(k+1)+1} + 2^{k+1} - 2$.

3.6. So $P(k+1)$ is true.

4. Hence $\forall n \in \mathbb{Z}_{\geqslant 0}$ $P(n)$ is true by MI.

Comments (by Lawrence):

- Many did not show $P(0)$ in the base step. We do need $P(0)$ shown as we want $P(n)$ to be true for all integers $n \geqslant 0$. Watch out on where the induction should start.
- Many wrote in the induction step

  $$\text{“}a_{k+1} = 2^{(k+1)+1} + 2^{k+1} - 2 = \cdots = 2a_k + 2.\text{”}$$

  This presents the argument **in the wrong order**. Present the proof in the logical order, not in the order you come up with it. The first equality here is already not justified.
- Quite a number of students used strong induction, which is not necessary but is also not wrong.

=== END OF PAPER ===