

Answers/Solutions of Exercise 4 (Version: August 18, 2014)

Remark: Please note that bases for vector spaces are not unique. In the following, if a question asks for a basis, the answer given is only one of the possible answers.

1. In order to answer (iv), we obtain the reduced row-echelon form of each of the matrices. (To answer (i)-(iii), we only need a row-echelon form.)

- (a) Performing Gauss-Jordan elimination, we obtain the reduced row-echelon form of \mathbf{A} :

$$\begin{pmatrix} 1 & 0 & 0 & 1 & -\frac{2}{7} \\ 0 & 1 & 0 & 1 & \frac{4}{7} \\ 0 & 0 & 1 & -1 & \frac{13}{7} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

- (i) $\{(1, 0, 0, 1, -\frac{2}{7}), (0, 1, 0, 1, \frac{4}{7}), (0, 0, 1, -1, \frac{13}{7})\}$ is a basis for the row space.

$\{(1, 2, -1, 1)^T, (4, 1, 3, -1)^T, (0, 0, 0, 1)^T\}$ is a basis for the column space.

- (ii) $\{(1, 0, 0, 1, -\frac{2}{7}), (0, 1, 0, 1, \frac{4}{7}), (0, 0, 1, -1, \frac{13}{7}), (0, 0, 0, 1, 0), (0, 0, 0, 0, 1)\}$ is a basis for \mathbb{R}^5 .

(iii) $\begin{pmatrix} 1 & 2 & -1 & 1 \\ 4 & 1 & 3 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & -7 & 7 & -6 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$

So $\{(1, 2, -1, 1)^T, (4, 1, 3, -1)^T, (0, 0, 0, 1)^T, (0, 0, 1, 0)^T\}$ is a basis for \mathbb{R}^4 .

- (iv) $\{(-1, -1, 1, 1, 0)^T, (\frac{2}{7}, -\frac{4}{7}, -\frac{13}{7}, 0, 1)^T\}$ is a basis for the nullspace.

- (v) $\text{rank}(\mathbf{A}) = 3$ and $\text{nullity}(\mathbf{A}) = 2$.

Hence $\text{rank}(\mathbf{A}) + \text{nullity}(\mathbf{A}) = 3 + 2 = 5 = \text{the number of column in } \mathbf{A}$.

- (vi) No.

- (b) Performing Gauss-Jordan elimination, we obtain the reduced row-echelon form of \mathbf{B} :

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

- (i) $\{1, 0, 0\}, \{0, 1, 0\}, \{0, 0, 1\}$ is a basis for the row space.

$\{(1, 0, -1, 2, 3)^T, (2, 1, 3, 1, 1)^T, (0, 1, 6, 0, -1)^T\}$ is a basis for the column space.

- (ii) $\{1, 0, 0\}, \{0, 1, 0\}, \{0, 0, 1\}$ is already a basis for \mathbb{R}^3 .

$$(iii) \begin{pmatrix} 1 & 0 & -1 & 2 & 3 \\ 2 & 1 & 3 & 1 & 1 \\ 0 & 1 & 6 & 0 & -1 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 1 & 0 & -1 & 2 & 3 \\ 0 & 1 & 5 & -3 & -5 \\ 0 & 0 & 1 & 3 & 4 \end{pmatrix}.$$

So $\{(1, 0, -1, 2, 3)^T, (2, 1, 3, 1, 1)^T, (0, 1, 6, 0, -1)^T, (0, 0, 0, 1, 0)^T, (0, 0, 0, 0, 1)^T\}$ is a basis for \mathbb{R}^5 .

(iv) \emptyset is the basis for the nullspace.

(v) $\text{rank}(\mathbf{B}) = 3$ and $\text{nullity}(\mathbf{B}) = 0$.

Hence $\text{rank}(\mathbf{B}) + \text{nullity}(\mathbf{B}) = 3 + 0 = 3 = \text{the number of column in } \mathbf{B}$.

(vi) Yes.

(c) Performing Gauss-Jordan elimination, we obtain the reduced row-echelon form of \mathbf{C} :

$$\begin{pmatrix} 1 & \frac{1}{2} & 0 & \frac{5}{6} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{6} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(i) $\{(1, \frac{1}{2}, 0, \frac{5}{6}, \frac{1}{3}), (0, 0, 1, -\frac{1}{6}, \frac{1}{3})\}$ is a basis for the row space.

$\{(2, 4, 2, 6)^T, (4, 2, -2, 6)^T\}$ is a basis for the column space.

(ii) $\{(1, \frac{1}{2}, 0, \frac{5}{6}, \frac{1}{3}), (0, 0, 1, -\frac{1}{6}, \frac{1}{3}), (0, 1, 0, 0, 0), (0, 0, 0, 1, 0), (0, 0, 0, 0, 1)\}$ is a basis for \mathbb{R}^5 .

$$(iii) \begin{pmatrix} 2 & 4 & 2 & 6 \\ 4 & 2 & -2 & 6 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 2 & 4 & 2 & 6 \\ 0 & -6 & -6 & -6 \end{pmatrix}.$$

So $\{(2, 4, 2, 6)^T, (4, 2, -2, 6)^T, (0, 0, 1, 0)^T, (0, 0, 0, 1)^T\}$ is a basis for \mathbb{R}^4 .

(iv) $\{(-\frac{1}{2}, 1, 0, 0, 0)^T, (-\frac{5}{6}, 0, \frac{1}{6}, 1, 0)^T, (-\frac{1}{3}, 0, -\frac{1}{3}, 0, 1)^T\}$ is the basis for the nullspace.

(v) $\text{rank}(\mathbf{C}) = 2$ and $\text{nullity}(\mathbf{C}) = 3$.

Hence $\text{rank}(\mathbf{C}) + \text{nullity}(\mathbf{C}) = 2 + 3 = 5 = \text{the number of column in } \mathbf{C}$.

(vi) No.

(d) Performing Gauss-Jordan elimination, we obtain the reduced row-echelon form of \mathbf{D} :

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(i) $\{(1, 0, 1, 0), (0, 1, 1, 0), (0, 0, 0, 1)\}$ is a basis for the row space.

$\{(1, -1, 2)^T, (4, 4, 0)^T, (8, 0, 1)^T\}$ is a basis for the column space.

(ii) $\{(1, 0, 1, 0), (0, 1, 1, 0), (0, 0, 0, 1), (0, 0, 1, 0)\}$ is a basis for \mathbb{R}^4 .

(iii) $\{(1, -1, 2)^T, (4, 4, 0)^T, (8, 0, 1)^T\}$ is already a basis for \mathbb{R}^3 .

(iv) $\{(-1, -1, 1, 0)^T\}$ is a basis for the nullspace.

(v) $\text{rank}(\mathbf{D}) = 3$ and $\text{nullity}(\mathbf{D}) = 1$.

Hence $\text{rank}(\mathbf{D}) + \text{nullity}(\mathbf{D}) = 3 + 1 = 4 =$ the number of column in \mathbf{D} .

(vi) Yes.

$$2. \quad (a) \quad \begin{pmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 1 & 15 & 8 & 6 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & -1 & -3 & -2 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$\{(1, -2, 0, 0, 3), (0, -1, -3, -2, 0), (0, 0, 0, -2, 0)\}$ is a basis for W .

(b) $\dim(W) = 3$

(c) $\{(1, -2, 0, 0, 3), (0, -1, -3, -2, 0), (0, 0, 0, -2, 0), (0, 0, 1, 0, 0), (0, 0, 0, 0, 1)\}$ is a basis for \mathbb{R}^5 .

$$3. \quad (a) \quad \begin{pmatrix} 1 & 2 & -1 & 0 & 3 \\ 0 & -1 & 3 & 1 & -1 \\ 1 & 0 & 5 & 2 & 1 \\ 3 & 1 & 12 & 5 & 4 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 1 & 2 & -1 & 0 & 3 \\ 0 & -1 & 3 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

So $S' = \{(1, 0, 1, 3), (2, -1, 0, 1)\}$ is a basis for V .

$$(b) \quad \begin{pmatrix} 1 & 2 & -3 & 0 & 0 \\ 0 & 1 & -2 & 5 & 4 \\ 1 & -2 & 5 & 2 & 6 \\ 3 & 1 & 1 & 1 & 6 \\ 4 & 0 & 4 & 1 & 9 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 1 & 2 & -3 & 0 & 0 \\ 0 & 1 & -2 & 5 & 4 \\ 0 & 0 & 0 & 22 & 22 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

So $S' = \{(1, 0, 1, 3, 4), (2, 1, -2, 1, 0), (0, 5, 2, 1, 1)\}$ is a basis for V .

4. Since

$$(a + b + 3c + 3d, b + 2c + d, a + c + 2d, -a - b - 3c - 3d, a + c + 2d) \\ = a(1, 0, 1, -1, 1) + b(1, 1, 0, -1, 0) + c(3, 2, 1, -3, 1) + d(3, 1, 2, -3, 2),$$

$V = \text{span}\{(1, 0, 1, -1, 1), (1, 1, 0, -1, 0), (3, 2, 1, -3, 1), (3, 1, 2, -3, 2)\}$. By

$$\begin{pmatrix} 1 & 0 & 1 & -1 & 1 \\ 1 & 1 & 0 & -1 & 0 \\ 3 & 2 & 1 & -3 & 1 \\ 3 & 1 & 2 & -3 & 2 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 1 & 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

So $\{(1, 0, 1, -1, 1), (0, 1, -1, 0, -1)\}$ is a basis for V .

$$5. \quad (a) \quad \mathbf{A} \quad \xrightarrow{R_2 - R_1} \quad \xrightarrow{R_3 + R_1} \quad \xrightarrow{R_1 + R_3} \quad \xrightarrow{R_2 - 3R_3} \quad \mathbf{R}$$

- (b) (i) Note that $\text{span}(S)$ and $\text{span}(T)$ are the row spaces of \mathbf{B} and \mathbf{R} respectively. Since \mathbf{B} and \mathbf{R} are row equivalent, $\text{span}(S) = \text{span}(T)$. Also, $\dim(\text{span}(T)) = \text{rank}(R) = 3$. So by Theorem 3.6.7, S is a basis for $\text{span}(T)$.

$$(ii) \quad \left(\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 & 2 \\ 1 & -1 & 0 & 1 & 0 & -1 \\ 2 & -1 & 1 & 1 & 3 & 0 \end{array} \right) \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left(\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right).$$

(You do not need to really do any computations to claim the result on the RHS. Why?)

$$\text{So the transition matrix from } S \text{ to } T \text{ is } \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ -1 & 2 & 2 \end{pmatrix}.$$

$$6. \quad \left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 1 & a & a & a & a \\ 1 & a & a^2 & a & a^2 \\ 1 & a^3 & a & 2a - a^3 & a \end{array} \right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 0 & a - 1 & a - 1 & a - 1 & a - 1 \\ 0 & 0 & a^2 - a & 0 & a^2 - a \\ 0 & 0 & 0 & 2a - 2a^3 & 0 \end{array} \right)$$

- If $a = 1$, then $\{(1, 1, 1, 1, 1)\}$ is a basis for V and $\dim(V) = 1$.
- If $a = 0$, then $\{(1, 1, 1, 1, 1), (0, 1, 1, 1, 1)\}$ is a basis for V and $\dim(V) = 2$.
- If $a = -1$, then $\{(1, 1, 1, 1, 1), (0, -2, -2, -2, -2), (0, 0, 2, 0, 2)\}$ is a basis for V and $\dim(V) = 3$.
- If $a \notin \{1, 0, -1\}$, then $\{(1, 1, 1, 1, 1), (0, a - 1, a - 1, a - 1, a - 1), (0, 0, a^2 - a, 0, a^2 - a), (0, 0, 0, 2a - 2a^3, 0)\}$ is a basis for V and $\dim(V) = 4$.

$$7. \quad V + W = \text{span}\{(1, 1, 0, 0), (-1, 0, 1, 0), (-1, 2, 3, 0), (2, -1, 2, -1)\}$$

$$\left(\begin{array}{cccc} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 2 & 3 & 0 \\ 2 & -1 & 2 & -1 \end{array} \right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 5 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right) = \mathbf{R}$$

$\{(1, 1, 0, 0), (0, 1, 1, 0), (0, 0, 5, -1)\}$ is a basis for $V + W$.

$$8. \quad (a) \quad \text{We can choose } \mathbf{A} = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(b) We can choose $\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

(c) V is the solution space of the linear equation $2x_1 - x_2 - x_3 + 0x_4 = 0$.

We can choose $\mathbf{C} = \begin{pmatrix} 2 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

9. (a) Since $(x_1, x_2, x_3, x_4)^T = (t - 2s, s + t, s, t)^T = s(-2, 1, 1, 0)^T + t(1, 1, 0, 1)^T$, $\{(-2, 1, 1, 0)^T, (1, 1, 0, 1)^T\}$ is a basis for the nullspace of \mathbf{A} . The nullity of \mathbf{A} is 2.

(b) A general solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$ is $x_1 = t - 2s + 1$, $x_2 = s + t$, $x_3 = s - 1$, $x_4 = t$ where s, t are arbitrary parameters.

(c) The reduced row-echelon form of \mathbf{A} is $\begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

(d) $\{(1, 0, 2, -1), (0, 1, -1, -1)\}$ is a basis for the row space of \mathbf{A} . The rank of \mathbf{A} is 2.

(e) No, we cannot find the column space of \mathbf{A} with the given information.

10. (a) Let \mathbf{R} be the reduced row-echelon form of \mathbf{A} . Since $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are linearly independent, the first three columns of \mathbf{R} are linearly independent. Thus

the first three columns of \mathbf{R} must be $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Together with the infor-

mation given for the fourth and fifth columns, $\mathbf{R} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

(b) $\{(1, 0, 0, 1, 0), (0, 1, 0, -2, 1), (0, 0, 1, 1, 1)\}$ is a basis for the row space of \mathbf{A} ; and $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is a basis for the column space of \mathbf{A} .

11. (a) $\mathbf{x} = (2, -1, 3)$ is the solution to the linear system.

Thus $\begin{pmatrix} 16 \\ 13 \\ -4 \\ 7 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 1 \\ 1 \end{pmatrix} + 3 \begin{pmatrix} 5 \\ 4 \\ -1 \\ 2 \end{pmatrix}$.

- (b) $\mathbf{x} = (-3 + s + t, 13 - 3s - 2t, 1 - t, s, t)$, where $s, t \in \mathbb{R}$, is a general solution for the linear system.

In particular,
$$\begin{pmatrix} -1 \\ 9 \\ 4 \end{pmatrix} = -3 \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + 13 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

12. (a) For example, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$

- (b) No. By Theorem 4.2.1, the dimensions of the row space and column space of a matrix must be the same.

(c) For example, $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}.$

(d) For example, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$

13.

	largest possible rank	smallest possible nullity
(a)	5	0
(b)	4	2
(c)	3	0

14. (a) $a = b = c = d = 0$.

(b) $ad - bc \neq 0$.

(c) $ad - bc = 0$ but not all a, b, c, d are zero.

15. (a) If $a = 1$, $\text{rank}(\mathbf{A}) = 1$.

If $a = -2$, $\text{rank}(\mathbf{A}) = 2$.

If $a \neq 1$ and $a \neq -2$, $\text{rank}(\mathbf{A}) = 3$.

(b) If $b = c = d = e = f = 0$, $\text{rank}(\mathbf{B}) = 0$.

If either (i) $b = c = 0$ and not all d, e, f are zero or (ii) $d = e = 0$ and not all b, c, f are zero, $\text{rank}(\mathbf{B}) = 1$.

If not all b, c are zero and not all d, e are zero, $\text{rank}(\mathbf{B}) = 2$.

16. (a) $\mathbf{X}_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \xrightarrow{R_3 - R_1} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

So $\text{rank}(\mathbf{X}_1) = 2$ and $\text{nullity}(\mathbf{X}_1) = 3 - 2 = 1$.

$$\mathbf{X}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_5 - R_1} \xrightarrow{R_4 - R_2} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

So $\text{rank}(\mathbf{X}_2) = 3$ and $\text{nullity}(\mathbf{X}_2) = 5 - 3 = 2$.

$$(b) \quad \mathbf{X}_n \xrightarrow{R_{2n+1} - R_1} \xrightarrow{R_{2n} - R_2} \cdots \xrightarrow{R_{n+2} - R_n} \begin{pmatrix} 1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 \\ 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 & 1 & 0 \\ & & \cdot & & & & \cdot & & \\ & & & \cdot & & \cdot & & & \\ 0 & \cdot & \cdot & 0 & 1 & 0 & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & 0 & 0 & 0 & \cdot & \cdot & 0 \\ & & \cdot & & & & \cdot & & \\ & & \cdot & & & & & \cdot & \\ 0 & \cdot & \cdot & 0 & 0 & 0 & \cdot & \cdot & 0 \end{pmatrix}$$

So $\text{rank}(\mathbf{X}_n) = n + 1$ and $\text{nullity}(\mathbf{X}_n) = (2n + 1) - (n + 1) = n$.

17. When the rank is 0, the solution set is the entire \mathbb{R}^3 .

When the rank is 1, the solution set is a plane in \mathbb{R}^3 that passes through the origin.

When the rank is 2, the solution set is a line in \mathbb{R}^3 that passes through the origin.

When the rank is 3, the solution set is $\{\mathbf{0}\}$.

18. Let $\mathbf{A} = (\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n)$ and \mathbf{B} be $m \times n$ matrices where \mathbf{a}_i is the i th column of \mathbf{A} . Suppose \mathbf{A} and \mathbf{B} are row equivalent, i.e. there exists elementary matrices $\mathbf{E}_1, \mathbf{E}_2, \dots, \mathbf{E}_k$ such that

$$\mathbf{B} = \mathbf{E}_k \cdots \mathbf{E}_2 \mathbf{E}_1 \mathbf{A}.$$

Define $\mathbf{P} = \mathbf{E}_k \cdots \mathbf{E}_2 \mathbf{E}_1$. Then $\mathbf{B} = \mathbf{P}\mathbf{A} = (\mathbf{P}\mathbf{a}_1 \ \mathbf{P}\mathbf{a}_2 \ \cdots \ \mathbf{P}\mathbf{a}_n)$ where $\mathbf{P}\mathbf{a}_i$ is the i th column of \mathbf{B} . By Theorem 2.4.7, \mathbf{P} is invertible.

Let $S_1 = \{\mathbf{a}_{i_1}, \mathbf{a}_{i_2}, \dots, \mathbf{a}_{i_r}\}$ be a set of columns of \mathbf{A} . Note that $S_2 = \{\mathbf{P}\mathbf{a}_{i_1}, \mathbf{P}\mathbf{a}_{i_2}, \dots, \mathbf{P}\mathbf{a}_{i_r}\}$ is the set of corresponding columns of \mathbf{B} .

- (a) Since \mathbf{P} is invertible, by Question 3.30, S_1 is linearly independent if and only if S_2 is linearly independent.

- (b) Suppose S_1 is a basis for the column space of \mathbf{A} . We want to show that S_2 is a basis for the column space of \mathbf{B} :

- (i) By (a), S_2 is linearly independent.
(ii) It is obvious that $\text{span}(S_2) \subseteq \text{column space of } \mathbf{B}$.

Take any $\mathbf{u} \in \text{column space of } \mathbf{B}$, i.e. for some $c_1, c_2, \dots, c_n \in \mathbb{R}$,

$$\mathbf{u} = c_1 \mathbf{P}\mathbf{a}_1 + c_2 \mathbf{P}\mathbf{a}_2 + \dots + c_n \mathbf{P}\mathbf{a}_n.$$

Since $\text{span}(S_1) = \text{column space of } \mathbf{A}$,

$$\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n \in \text{span}(S_1) = \text{span}\{\mathbf{a}_{i_1}, \mathbf{a}_{i_2}, \dots, \mathbf{a}_{i_r}\}$$

and hence

$$\mathbf{P}\mathbf{a}_1, \mathbf{P}\mathbf{a}_2, \dots, \mathbf{P}\mathbf{a}_n \in \text{span}\{\mathbf{P}\mathbf{a}_{i_1}, \mathbf{P}\mathbf{a}_{i_2}, \dots, \mathbf{P}\mathbf{a}_{i_r}\} = \text{span}(S_2).$$

By Theorem 3.2.9.2, $\mathbf{u} \in \text{span}(S_2)$. So the column space of $\mathbf{B} \subseteq \text{span}(S_2)$.

We have shown that $\text{span}(S_2) = \text{column space of } \mathbf{B}$.

By (i) and (ii), S_2 is a basis for the column space of \mathbf{B} .

Similarly, follow the arguments above by replacing \mathbf{a}_i by $\mathbf{P}\mathbf{a}_i$ and \mathbf{P} by \mathbf{P}^{-1} . We conclude that if S_2 is a basis for the column space of \mathbf{B} , then S_1 is a basis for the column space of \mathbf{A} .

$$19. \quad (a) \quad \left(\begin{array}{cccc|c|c|c} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{array} \right) \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left(\begin{array}{cccc|c|c|c} 1 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 1 \end{array} \right)$$

$$(i) \quad \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ t \\ 1-t \\ -1 \end{pmatrix} \text{ where } t \in \mathbb{R}.$$

$$(ii) \quad \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ t \\ -1-t \\ 1 \end{pmatrix} \text{ where } t \in \mathbb{R}.$$

$$(iii) \quad \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ t \\ -t \\ 1 \end{pmatrix} \text{ where } t \in \mathbb{R}.$$

If \mathbf{u}_1 is a solution of (i), \mathbf{u}_2 a solution of (ii) and \mathbf{u}_3 a solution of (iii), then $(\mathbf{u}_1 \ \mathbf{u}_2 \ \mathbf{u}_3)$ is a right inverse of \mathbf{B} . The answer is certainly not unique.

For example, $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$ is a right inverse of \mathbf{B} .

(b) For example, $\mathbf{B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ has no right inverse.

(c) Let $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_m\}$ be the standard basis for \mathbb{R}^m .

\mathbf{B} has a right inverse

$$\Leftrightarrow \mathbf{B} \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_m \end{pmatrix} = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_m \end{pmatrix} \text{ for some } \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m \in \mathbb{R}^n$$

$$\Leftrightarrow \text{All linear systems } \mathbf{B}\mathbf{x} = \mathbf{e}_1, \mathbf{B}\mathbf{x} = \mathbf{e}_2, \dots, \mathbf{B}\mathbf{x} = \mathbf{e}_m \text{ are consistent.}$$

$$\Leftrightarrow \mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_m \in \text{the column space of } \mathbf{B}$$

$$\Leftrightarrow \text{the column space of } \mathbf{B} = \mathbb{R}^m$$

$$\Leftrightarrow \dim(\text{the column space of } \mathbf{B}) = m$$

$$\Leftrightarrow \text{rank}(\mathbf{B}) = m.$$

20. Let $\mathbf{B} = (\mathbf{b}_1 \ \cdots \ \mathbf{b}_n)$ where \mathbf{b}_j is the j th column of \mathbf{B} .

$$\mathbf{A}\mathbf{B} = \mathbf{0} \Rightarrow (\mathbf{A}\mathbf{b}_1 \ \cdots \ \mathbf{A}\mathbf{b}_n) = \mathbf{0} \Rightarrow \mathbf{A}\mathbf{b}_j = \mathbf{0} \text{ for all } j,$$

i.e. $\mathbf{b}_1, \dots, \mathbf{b}_n$ are contained in the nullspace of \mathbf{A} .

So the column space of $\mathbf{B} = \text{span}\{\mathbf{b}_1, \dots, \mathbf{b}_n\} \subseteq \text{the nullspace of } \mathbf{A}$.

21. Let $\mathbf{A} = \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_n \end{pmatrix}$ be a matrix where \mathbf{a}_i is the i th row of \mathbf{A} .

Let $\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{R}^n$ such that \mathbf{u}^T is a vector in the nullspace of \mathbf{A} . Then

$$\mathbf{A}\mathbf{u}^T = \mathbf{0} \Rightarrow \begin{pmatrix} \mathbf{a}_1\mathbf{u}^T \\ \vdots \\ \mathbf{a}_n\mathbf{u}^T \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow \mathbf{a}_i\mathbf{u}^T = 0 \text{ for all } i.$$

Assume that \mathbf{u} is also contained in the row space of \mathbf{A} , i.e. $\mathbf{u} = c_1\mathbf{a}_1 + \cdots + c_n\mathbf{a}_n$ for some $c_1, \dots, c_n \in \mathbb{R}$. We have

$$\mathbf{u}\mathbf{u}^T = c_1\mathbf{a}_1\mathbf{u}^T + \cdots + c_n\mathbf{a}_n\mathbf{u}^T = 0.$$

On the other hand, $\mathbf{u}\mathbf{u}^T = u_1^2 + \cdots + u_n^2$. So $u_1^2 + \cdots + u_n^2 = 0$ which implies $u_1 = 0, \dots, u_n = 0$, i.e. \mathbf{u} is the zero vector.

22. (a) Since \mathbf{P} is invertible, we can write $\mathbf{P} = \mathbf{E}_n \cdots \mathbf{E}_1$ where \mathbf{E}_i are elementary matrices. So $\mathbf{PA} = \mathbf{E}_n \cdots \mathbf{E}_1 \mathbf{A}$ and \mathbf{A} are row-equivalent matrices. They have the same row space. Thus

$$\begin{aligned}\text{rank}(\mathbf{PA}) &= \dim(\text{the row space of } \mathbf{PA}) \\ &= \dim(\text{the row space of } \mathbf{A}) = \text{rank}(\mathbf{A}).\end{aligned}$$

(b) For example, $\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(c) No. For example, let $\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

23. Let $\mathbf{A} = (\mathbf{a}_1 \cdots \mathbf{a}_n)$ and $\mathbf{B} = (\mathbf{b}_1 \cdots \mathbf{b}_n)$ where \mathbf{a}_j is the j th column of \mathbf{A} and \mathbf{b}_j is the j th column of \mathbf{B} . Let $\{\mathbf{a}'_1, \dots, \mathbf{a}'_r\}$ be a basis for the column space of \mathbf{A} and let $\{\mathbf{b}'_1, \dots, \mathbf{b}'_s\}$ be a basis for the column space of \mathbf{B} . Then

$$\begin{aligned}\text{the column space of } \mathbf{A} + \mathbf{B} &= \text{span}\{\mathbf{a}_1 + \mathbf{b}_1, \dots, \mathbf{a}_n + \mathbf{b}_n\} \\ &\subseteq \text{span}\{\mathbf{a}'_1, \dots, \mathbf{a}'_r, \mathbf{b}'_1, \dots, \mathbf{b}'_s\}.\end{aligned}$$

So

$$\text{rank}(\mathbf{A} + \mathbf{B}) = \dim(\text{the column space of } \mathbf{A} + \mathbf{B}) \leq r + s = \text{rank}(\mathbf{A}) + \text{rank}(\mathbf{B}).$$

24. Since $\mathbf{Ax} = \mathbf{b}$ is consistent for all $\mathbf{b} \in \mathbb{R}^m$, by Theorem 4.1.16, the column space of \mathbf{A} is \mathbb{R}^m , i.e. $\text{rank}(\mathbf{A}) = m$. Hence

$$\text{nullity}(\mathbf{A}^T) = m - \text{rank}(\mathbf{A}^T) = m - \text{rank}(\mathbf{A}) = 0.$$

It means that the linear system $\mathbf{A}^T \mathbf{y} = \mathbf{0}$ has only the trivial solution.

Alternative Solution: Let $\mathbf{e}_1, \dots, \mathbf{e}_m$ be the standard basis for \mathbb{R}^m and let $\mathbf{u}_1, \dots, \mathbf{u}_m$ be vectors in \mathbb{R}^n such that $\mathbf{Au}_i = \mathbf{e}_i$ for each i . (In here, all the vectors are column vectors.) Suppose $\mathbf{v} = (v_1, \dots, v_m)^T$ is a solution to the system $\mathbf{A}^T \mathbf{v} = \mathbf{0}$. Then for $i = 1, \dots, m$,

$$v_i = \mathbf{e}_i^T \mathbf{v} = (\mathbf{Au}_i)^T \mathbf{v} = \mathbf{u}_i^T \mathbf{A}^T \mathbf{v} = \mathbf{u}_i^T \mathbf{0} = 0.$$

So $\mathbf{v} = \mathbf{0}$. That is, the system $\mathbf{A}^T \mathbf{y} = \mathbf{0}$ has only the trivial solution.

25. (a) Let \mathbf{u} be any vector in the nullspace of \mathbf{A} , i.e. $\mathbf{Au} = \mathbf{0}$. Then $\mathbf{A}^T \mathbf{Au} = \mathbf{A}^T \mathbf{0} = \mathbf{0}$. So \mathbf{u} is also a vector in the nullspace of $\mathbf{A}^T \mathbf{A}$. We have shown that the nullspace of \mathbf{A} is a subspace of the nullspace of $\mathbf{A}^T \mathbf{A}$.

Let \mathbf{v} be any vector in the nullspace of $\mathbf{A}^T \mathbf{A}$, i.e. $\mathbf{A}^T \mathbf{A} \mathbf{v} = \mathbf{0}$. Suppose $\mathbf{A} \mathbf{v} = (b_1, b_2, \dots, b_m)^T$. Then

$$\begin{aligned} (\mathbf{A} \mathbf{v})^T (\mathbf{A} \mathbf{v}) &= \mathbf{v}^T \mathbf{A}^T \mathbf{A} \mathbf{v} = \mathbf{v}^T \mathbf{0} = 0 \\ \Rightarrow b_1^2 + b_2^2 + \dots + b_m^2 &= 0 \\ \Rightarrow b_1 = b_2 = \dots = b_m &= 0. \end{aligned}$$

That is, $\mathbf{A} \mathbf{v} = \mathbf{0}$. So \mathbf{v} is also a vector in the nullspace of \mathbf{A} .

We have shown that the nullspace of $\mathbf{A}^T \mathbf{A}$ is a subspace of the nullspace of \mathbf{A} .

Hence the nullspace of \mathbf{A} is equal to the nullspace of $\mathbf{A}^T \mathbf{A}$.

(b) By (a), $\text{nullity}(\mathbf{A}) = \text{nullity}(\mathbf{A}^T \mathbf{A})$.

Since \mathbf{A} is an $m \times n$ matrix, $\mathbf{A}^T \mathbf{A}$ is an $n \times n$ matrix. By the Dimension Theorem for Matrices (Theorem 4.3.4),

$$\text{rank}(\mathbf{A}) = n - \text{nullity}(\mathbf{A}) = n - \text{nullity}(\mathbf{A}^T \mathbf{A}) = \text{rank}(\mathbf{A}^T \mathbf{A}).$$

(c) No. For example, let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

(d) Yes. By (b) and Remark 4.2.5.3, $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}^T) = \text{rank}((\mathbf{A}^T)^T \mathbf{A}^T) = \text{rank}(\mathbf{A} \mathbf{A}^T)$.

26. (a) False. For example, let $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

(b) True. By Theorem 4.1.7, the row space of \mathbf{A} and the row space of \mathbf{B} are the same. Hence the column space of \mathbf{A}^T and the column space of \mathbf{B}^T are the same.

(c) False. For example, let $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

(d) False. For example, let $\mathbf{A} = \mathbf{B} = \mathbf{I}_2$.

(e) False. For example, let $\mathbf{A} = \mathbf{B} = \mathbf{0}_{2 \times 2}$.

(f) False. For example, let $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

(g) False. For example, let $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.