CS1231S: Discrete Structures

Tutorial #11: Graphs and Trees

(Week 13: 8 – 12 November 2021)

I. Discussion Questions

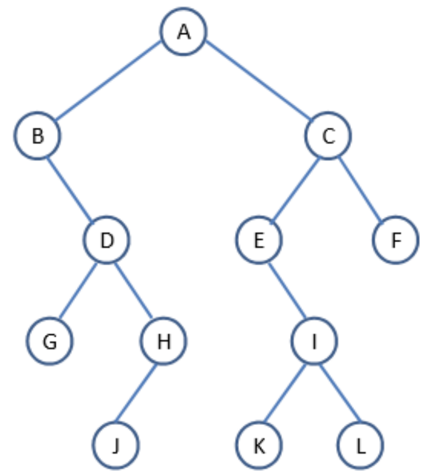
These are meant for you to discuss on the LumiNUS Forum. No answers will be provided.

D1. For any simple connected graph with n $(n > 0)$ vertices, what is the minimum and maximum number of edges the graph may have?

D2. (AY2016/17 Semester 1 Exam Question)

How many simple graphs on 3 vertices are there? In general, how many simple graphs on n $(n > 1)$ vertices are there?

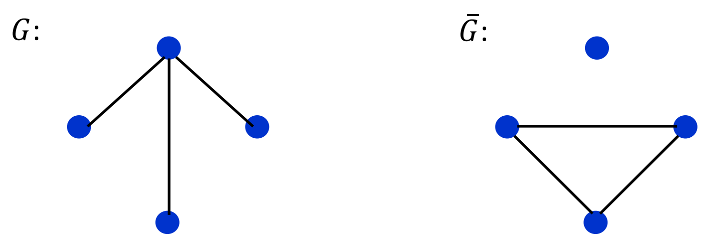
D3. Given the binary tree on the right, write the pre-order, in-order, and post-order traversals of its vertices.



II. Definitions

Definition 1. If G is a simple graph, the *complement* of G, denoted $\bar{G}$, is obtained as follows: the vertex set of $\bar{G}$ is identical to the vertex set of G. However, two distinct vertices v and w of $\bar{G}$ are connected by an edge if and only if v and w are not connected by an edge in G.

The figure below shows a graph G and its complement $\bar{G}$.



A graph G and its complement $\bar{G}$.

Definition 2. A *self-complementary* graph is isomorphic with its complement.

Definition 3. A simple circuit (cycle) of length three is called a *triangle*.

III. Tutorial Questions

1. Draw all self-complementary graphs with (a) four vertices; (b) five vertices.

2. (AY2016/17 Semester 1 Exam Question)

   Let G be a simple graph with n vertices where every vertex has degree at least $\left\lfloor \frac{n}{2} \right\rfloor$. Prove that G is connected.

3. (AY2020/21 Semester 2 Exam Question)

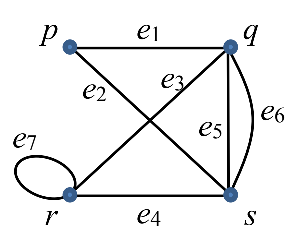
   A *height-balanced binary tree* (or simply *balanced binary tree*) is a binary tree in which the heights of the left and right subtrees under any vertex differ by not more than one. Draw all balanced full binary trees with 9 vertices

4. Show that every simple graph with at least two vertices has two vertices of the same degree.

   (This is similar to the popular puzzle: "Prove that at a party with at least two persons, there are two people who know the same number of people".)

5. Prove that for any simple graph G with six vertices, G or its complementary graph $\bar{G}$ contains a triangle.

6. Given the graph shown below:



   (a) Write the adjacency matrix A for the graph. Let the rows and columns be p, q, r and s.

   (b) Find A^2 and A^3.

   (c) How many walks of length 2 are there from p to q? From s to itself? List out all the walks.

   (d) How many walks of length 3 are there from r to s? From s to p? List out all the walks.

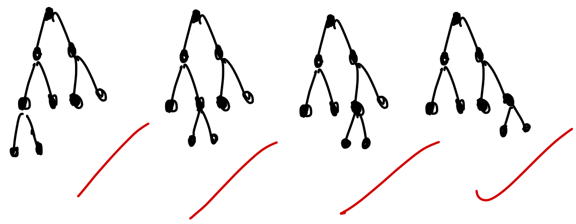
1. Draw all self-complementary graphs with (a) four vertices; (b) five vertices.

2. (AY2016/17 Semester 1 Exam Question)
Let G be a simple graph with n vertices where every vertex has degree at least $\left\lfloor \frac{n}{2} \right\rfloor$. Prove that G is connected.

3. (AY2020/21 Semester 2 Exam Question)
A *height-balanced binary tree* (or simply *balanced binary tree*) is a binary tree in which the heights of the left and right subtrees under any vertex differ by not more than one. Draw all balanced full binary trees with 9 vertices

no child
2 children.

3.



1. a)

degrees:
1 2 2 1

max degree for each vertex
= 4 − 1
= 3.

Complement ⇒ 3 - degree

{a, b, c, d}

⇒ {1, 2, 1, 2}

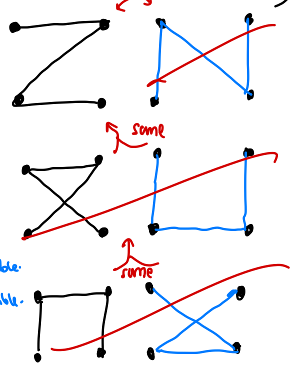
~~{3, 0, 3, 0}~~ : impossible.

~~{1, 2, 3, 0}~~ : impossible.

G same $\bar{G}$

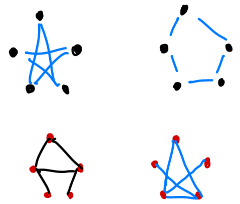
same

same



b)

G $\bar{G}$



(4 - deg)

⇒ {2, 2, 2, 2, 2}

{1, 1, 2, 3, 3}

~~{1, 2, 2, 2, 3}~~ impossible

Simple graph property.

2. (AY2016/17 Semester 1 Exam Question)
Let G be a simple graph with n vertices where every vertex has degree at least $\left\lfloor \frac{n}{2} \right\rfloor$. Prove that G is connected. Contradiction

2. n vertices → degree at least $\lfloor \frac{n}{2} \rfloor$ ⇒ G connected.

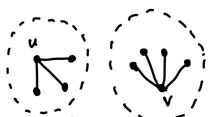
1. Since G is a simple graph, G does not have any loops or parallel edges.

2. Then for all vertexes, if an edge is originating from one vertex, it must have another different vertex as endpoint.

3. Suppose we split all the vertices of G into 2 sets.

   3.1 If every vertices in set A is connected to every vertices in set B, there is at least $\lfloor \frac{n}{2} \rfloor$ degree on each vertices.

   3.2 By definition of a complete bipartite graph, G is hence connected.



u
v

u connects to $\lfloor \frac{n}{2} \rfloor$ vertices

v connects to $\lfloor \frac{n}{2} \rfloor$ vertices

1. Suppose G is not connected.

2. Let u and v be the vertices in 2 separate connected components.

3. Then the number of vertices in the union of their neighbourhoods, including u and v, is at least $\lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{2} \rfloor + 2 \geq n+1$

   3.1 Case: n is even, then $n = 2k$, $k \in \mathbb{Z}$,
   $$\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + 2 = \left\lfloor \frac{2k}{2} \right\rfloor + \left\lfloor \frac{2k}{2} \right\rfloor + 2 = k + k + 2 = n + 2.$$

   3.2 Case: n is odd, then $n = 2k+1$, $k \in \mathbb{Z}$,
   $$\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + 2 = \left\lfloor \frac{2k+1}{2} \right\rfloor + \left\lfloor \frac{2k+1}{2} \right\rfloor + 2 = k + k + 2 = 2k + 2 = n + 1$$

4. This contradicts that there are n vertices

5. Hence G is connected.

4. Show that every simple graph with at least two vertices has two vertices of the same degree.

existence

pigeon

hole.

(This is similar to the popular puzzle: “Prove that at a party with at least two persons, there are two people who know the same number of people”.)

if 1 vertex has $n-1$ degrees
⇒ all other vertex has at least 1 degree.

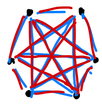
4.

1. Let G be a simple graph of 2 vertices without loops or parallel edges.
2. Case 1: G is connected.
   2.1 There must be a edge from one vertices to another
   2.2. Since the number of edges incident on both vertices is 1, thus both vertices has the same degree of 1.
3. Case 2: G is not connected
   3.1. There is no edge in the graph.
   3.2. Since there is no edge, the degree on both vertices are 0.
4. In any case, the degree on both vertices is the same.

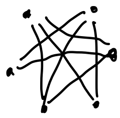
1. Let G be a simple graph with at least 2 vertices
2. Case 1: If G has no vertex with degree $n-1$
   2.1 Then all vertices in G have degree lying in the range 0 to $n-2$ inclusive. ($n-1$ holes)
3. Case 2: If G has a vertex v with degree $n-1$
   3.1 Firstly, v does not have degree 0 (since $n-1>0$)
   3.2 Also, as v has degree $n-1$, it is connected to every other vertex, hence no other vertex can have degree 0 too.
   3.3. Hence, all vertices in G have degree in range 1 to $n-1$ inclusive. ($n-1$ holes)
4. In all cases, there are at most $n-1$ possible vertex degrees among the n vertices.
5. Therefore, at least 2 vertices must have the same degree (by PHP).

vertices overflow degrees.

5. Prove that for any simple graph G with six vertices, G or its complementary graph $\bar{G}$ contains a triangle.



6 vertices
15 edges



5. 1. Let G be a simple graph with 6 vertices.
2. Case 1: G has 0 edges.
   2-1 Then $\bar{G}$ is a complete graph with a triangle
3. Case 2: G has 15 edges.
   3.1 Then G is a complete graph with a triangle.
4. Case 3: G has 1 to 14 edges

1. Take any simple graph G with 6 vertices
2. Draw a black edge between adjacent vertices in G and draw a red edge between non-adjacent vertices in G

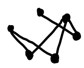


Figure 4a: Graph G

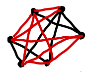


Figure 4b: Graph G'

3. Call this graph G' (figure 4b). Now G' is a complete graph with every edge either black or red, and we want to prove that it has a black triangle or a red triangle.

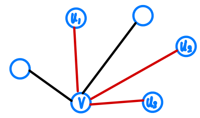
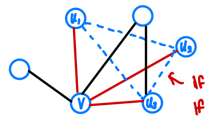


Figure 4c.

4. Let v be an arbitrary vertex of G'.
   4.1 There are 5 edges incident to v, which are either black or red.
   4.2. Therefore, at least 3 of these 5 edges are same colour c (by Generalized PHP)
   4.3. For these 3 edges of colour c, name the vertices at the other end of these edges u_1, u_2, u_3 (figure 4c.)

5 edges
2 colours.
∴ at least 3 edges of the same colour.



If any of dotted line is red = red triangle.
If all not red = black triangle.

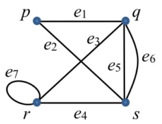
Figure 4d.

4.4. Case 1: If there is an edge of colour c between any two of u_1, u_2, u_3, then that edge forms a triangle of colour c with the 2 edges coming from v
(In Figure 4d, the 3 dashed lines are the edges btw u_1, u_2, u_3.
In this example, two of them – $\{u_1, u_3\}$ and $\{u_2, u_3\}$ are of colour c (red). Let's pick $\{u_1, u_3\}$.
We have a triangle of colour c with v, u_1, u_3)

4.5. Case 2: If there are no edges of colour c between any pair of u_1, u_2, u_3. Then the edges between these 3 vertices form a triangle of colour opposite to c.

5. In all cases, there is a triangle of the same colour.

6. Given the graph shown below:



(a) Write the adjacency matrix A for the graph. Let the rows and columns be p, q, r and s.

(b) Find A^2 and A^3.

(c) How many walks of length 2 are there from p to q? From s to itself? List out all the walks.

(d) How many walks of length 3 are there from r to s? From s to p? List out all the walks.

6. a)

$$\begin{array}{c|c} & \begin{matrix} p & q & r & s \end{matrix} \\ \begin{matrix} p \\ q \\ r \\ s \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix} \end{array}$$

symmetric
⇒ no directions.

b) $A^2 = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 2 & 6 & 3 & 2 \\ 2 & 3 & 3 & 3 \\ 2 & 2 & 3 & 6 \end{bmatrix}$

$A^3 = \begin{bmatrix} 4 & 8 & 6 & 8 \\ 8 & 9 & 11 & 17 \\ 6 & 11 & 9 & 11 \\ 8 & 17 & 11 & 9 \end{bmatrix}$

c) p to q : 2

$e_2 e_5$
$e_2 e_6$

s to s : 6

$e_5 e_6$ $e_4 e_4$ $e_6 e_6$
$e_6 e_5$ $e_5 e_5$ $e_2 e_2$.

d) r to s : 11

$e_7 e_7 e_4$ $e_3 e_1 e_2$ $e_4 e_4 e_4$ $e_4 e_2 e_2$
$e_7 e_3 e_5$ $e_4 e_5 e_6$ $e_4 e_5 e_5$ ~~$e_4 e_6 e_5$~~.
$e_7 e_3 e_6$ $e_4 e_6 e_5$ $e_4 e_6 e_6$ $e_3 e_3 e_4$.

s to p : 8

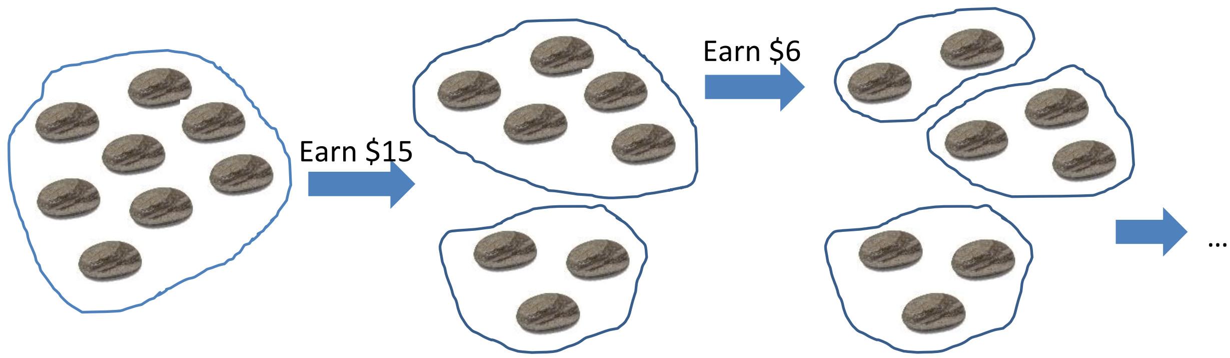
$e_1 e_2 e_7$ $e_5 e_6 e_2$
$e_5 e_5 e_2$ $e_6 e_5 e_2$ $e_2 e_1 e_1$
$e_6 e_6 e_2$ $e_4 e_7 e_1$ $e_4 e_4 e_2$

7. (AY2017/18 Semester 1 Exam Question)
Suppose you are given a pile of stones. At each step, you can separate a pile of k stones into two piles of k_1 and k_2 stones. (Obviously, $k_1 + k_2 = k$.) On doing this, you earn $\$(k_1 \times k_2)$.

What is the maximum amount of money you can earn at the end if you start with a pile of n stones? Explain your answer.

The diagram below illustrates the (incomplete) process of separating a pile of 8 stones.

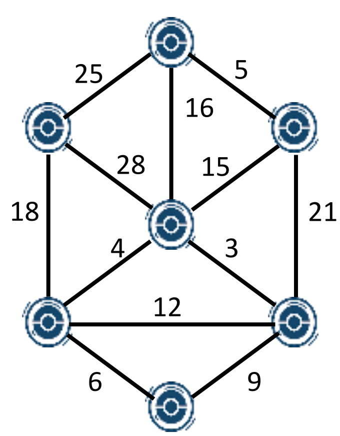
(This problem may be solved without using graph theory, but here we want to model it as a graph problem.)



8. How many possible binary trees with 4 vertices *A*, *B*, *C* and *D* have this in-order traversal: *A B C D*? Draw them.

9. (AY2016/17 Semester 1 Exam Question)
The figure below shows a graph where the vertices are Pokestops. Using either Kruskal's algorithm or Prim's algorithm, find its minimum spanning tree (MST). If you use Prim's algorithm, you must start with the top-most vertex.

Indicate the order of the edges inserted into the MST in your answer.

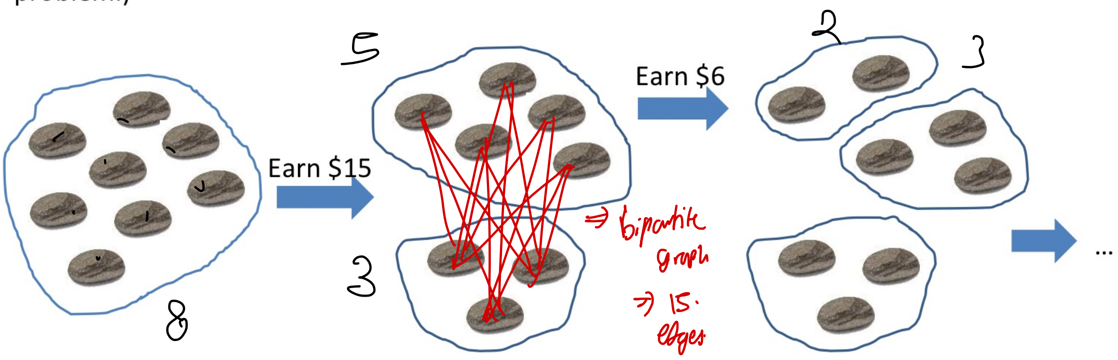


7. (AY2017/18 Semester 1 Exam Question)
Suppose you are given a pile of stones. At each step, you can separate a pile of k stones into two piles of k_1 and k_2 stones. (Obviously, $k_1 + k_2 = k$.) On doing this, you earn $\$(k_1 \times k_2)$.

What is the maximum amount of money you can earn at the end if you start with a pile of n stones? Explain your answer.

The diagram below illustrates the (incomplete) process of separating a pile of 8 stones.

(This problem may be solved without using graph theory, but here we want to model it as a graph problem.)

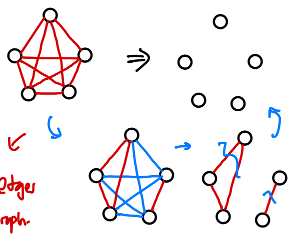


Separating a pile of k stones into 2 piles of k_1 and k_2 stones corresponds to removing $k_1 \times k_2$ edges from K_k.

Model this as a graph problem.
Let vertices represent the stones.
Draw an edge between 2 vertices if the stones they represent are in the same pile

In the beginning, we have a K_n graph.
eg. K_5

In the end, we have a graph of n isolated vertices.



removing $k_1 \times k_2$ edges from graph

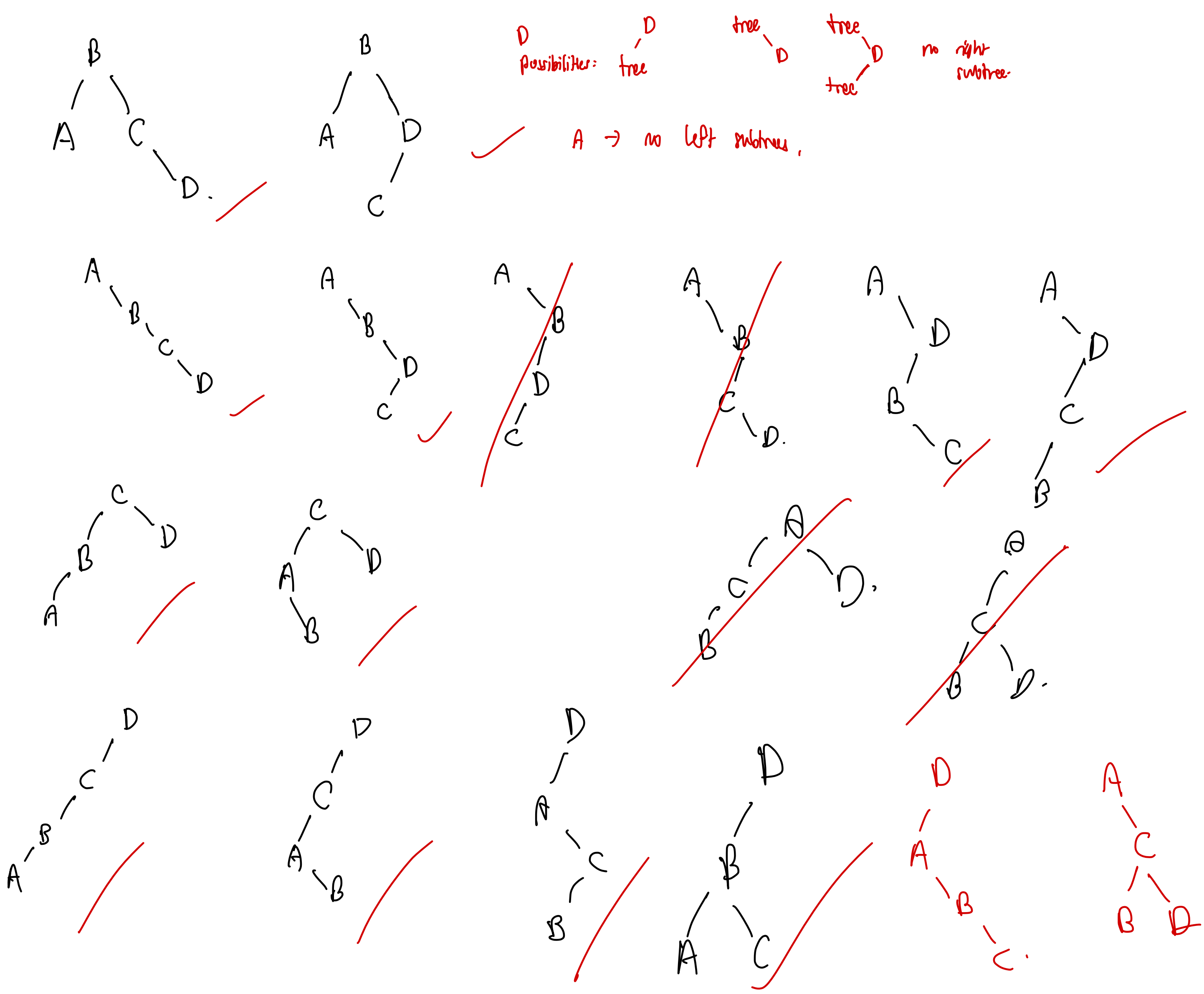
"breaking the edges"

A K_n graph has $\binom{n}{2} = \frac{n(n-1)}{2}$ edges

Therefore, the maximum amount one can earn is $\$ \frac{n(n-1)}{2}$.

8. How many possible binary trees with 4 vertices *A*, *B*, *C* and *D* have this in-order traversal: *A B C D*? Draw them.

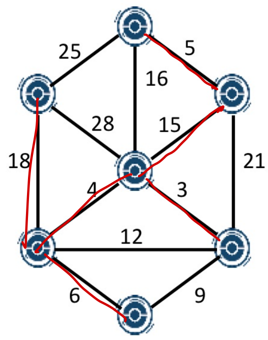
16.



12 | 14.

9. (AY2016/17 Semester 1 Exam Question)
The figure below shows a graph where the vertices are Pokestops. Using either Kruskal's algorithm or Prim's algorithm, find its minimum spanning tree (MST). If you use Prim's algorithm, you must start with the top-most vertex.

Indicate the order of the edges inserted into the MST in your answer.



5, 15, 3, 4, 6, 18.

= 51

10. Construct the binary tree given the following in-order and pre-order traversals of the tree:

In-order: I A D J N H B E K O F L G C M

Pre-order: H N A I J D O B K E C L F G M

Draw diagrams to trace the steps of your construction.

10. Construct the binary tree given the following in-order and pre-order traversals of the tree:

In-order: I A D J N H B E K O F L G C M

Pre-order: H N A I J D O B K E C L F G M

Draw diagrams to trace the steps of your construction.