Section 5.1

Inner Products in $\mathbf{R}^n$

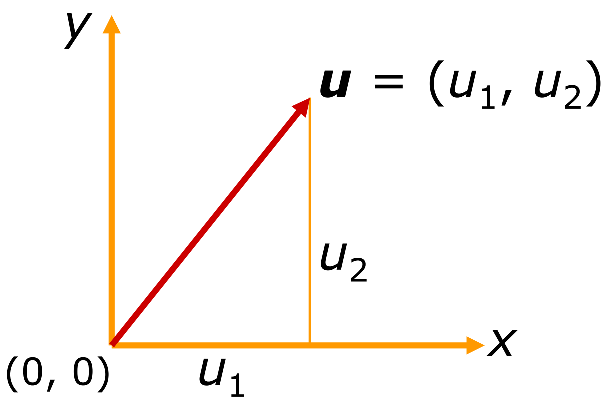
Objectives

- What are the algebraic representation of length, distance and angles in $\mathbf{R}^n$?
- What is the dot product of vectors?

Discussion 5.1.1

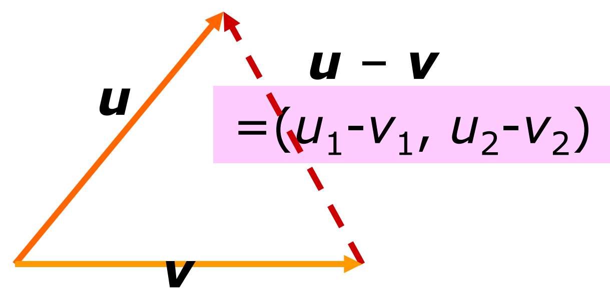
$\boldsymbol{u} = (u_1, u_2)$ and $\boldsymbol{v} = (v_1, v_2)$: vectors in $\mathbf{R}^2$

length of vector



$$\|\boldsymbol{u}\| = \sqrt{u_1^{\,2} + u_2^{\,2}}$$

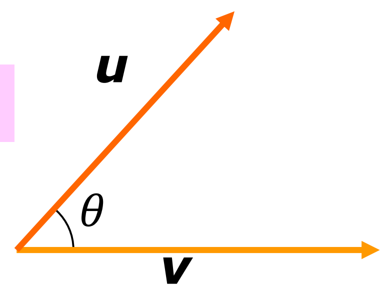
distance between two vectors



$$\|\boldsymbol{u} - \boldsymbol{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2}$$

angle between two vectors

$0 \leq \theta < \pi$



$$\cos\theta = \frac{u_1 v_1 + u_2 v_2}{||\boldsymbol{u}||\,||\boldsymbol{v}||}$$

derived from cosine rule

$$\theta = \cos^{-1}\left(\frac{u_1 v_1 + u_2 v_2}{||\boldsymbol{u}||\,||\boldsymbol{v}||}\right)$$

Similarly for $\mathbf{R}^3$ case

Definition 5.1.2

$\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$, $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ vectors in $\mathbf{R}^n$

	length $\|\|\boldsymbol{u}\|\|$ of vector	distance $\|\|\boldsymbol{u} - \boldsymbol{v}\|\|$ between two vectors	angle θ between two vectors
$\mathbf{R}^2$	$\sqrt{u_1{}^2 + u_2{}^2}$	$\sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2}$	$\cos^{-1}\left(\dfrac{u_1v_1 + u_2v_2}{\|\|\boldsymbol{u}\|\|\,\|\|\boldsymbol{v}\|\|}\right)$
$\mathbf{R}^3$	$\sqrt{u_1{}^2 + u_2{}^2 + u_3{}^2}$	$\sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2}$	$\cos^{-1}\left(\dfrac{u_1v_1 + u_2v_2 + u_3v_3}{\|\|\boldsymbol{u}\|\|\,\|\|\boldsymbol{v}\|\|}\right)$
$\mathbf{R}^n$	$\sqrt{u_1{}^2 + u_2{}^2 + \ldots + u_n{}^2}$	$\sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \ldots + (u_n - v_n)^2}$	$\cos^{-1}\left(\dfrac{u_1v_1 + u_2v_2 + \ldots + u_nv_n}{\|\|\boldsymbol{u}\|\|\,\|\|\boldsymbol{v}\|\|}\right)$

cumbersome

What is dot product?

Definition 5.1.2.1

$\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$, $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ vectors in $\mathbf{R}^n$

The dot product of $\boldsymbol{u}$ and $\boldsymbol{v}$ is defined to be the value (scalar)

scalar product

$$\boxed{\boldsymbol{u} \cdot \boldsymbol{v}} = \boxed{u_1 v_1 + u_2 v_2 + \ldots + u_n v_n}$$

product of two vectors

scalar

inner product

In particular,

$$\boldsymbol{u} \cdot \boldsymbol{u} = u_1^{\,2} + u_2^{\,2} + \ldots + u_n^{\,2}$$

Length, distance and angles in terms of dot product

Definition 5.1.2 ($\mathbf{R}^n$ case)

$\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$, $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ vectors in $\mathbf{R}^n$

What for?

norm length $\|\|\boldsymbol{u}\|\|$ of vector	distance $\|\|\boldsymbol{u} - \boldsymbol{v}\|\|$ between two vectors	angle θ between two vectors
$\sqrt{u_1^{\,2} + u_2^{\,2} + \ldots + u_n^{\,2}}$	$\sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \ldots + (u_n - v_n)^2}$	$\cos^{-1}\left(\dfrac{u_1v_1 + u_2v_2 + \ldots + u_nv_n}{\|\|\boldsymbol{u}\|\|\,\|\|\boldsymbol{v}\|\|}\right)$
$\sqrt{\boldsymbol{u}\cdot\boldsymbol{u}}$	$\sqrt{(\boldsymbol{u}-\boldsymbol{v})\cdot(\boldsymbol{u}-\boldsymbol{v})}$	$\cos^{-1}\left(\boxed{\dfrac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\|\boldsymbol{u}\|\|\,\|\|\boldsymbol{v}\|\|}}\right)$

vectors of norm 1 are called unit vectors

Does this quotient have value between -1 and 1 ?

$\boldsymbol{u}$ is a unit vector $\Leftrightarrow$ $||\boldsymbol{u}|| = 1$

Dot product as matrix multiplication

Remark 5.1.3

$\boldsymbol{u} = (u_1 \ \ u_2 \ \ \ldots \ u_n)$ and $\boldsymbol{v} = (v_1 \ \ v_2 \ \ldots \ \ v_n)$

regarded as row matrix

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n = \begin{pmatrix} u_1 & u_2 & \cdots & u_n \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \boldsymbol{u}\boldsymbol{v}^T$$

$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}\mathbf{v}^T$

1 x 1 matrix

$$\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \qquad \boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$

regarded as column matrix

Properties of dot product

Theorem 5.1.5

Let c be a scalar and $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ vectors in $\mathbf{R}^n$.

1. $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}$ — commutative law
2. $(\boldsymbol{u} + \boldsymbol{v}) \cdot \boldsymbol{w} = \boldsymbol{u} \cdot \boldsymbol{w} + \boldsymbol{v} \cdot \boldsymbol{w}$
   $\boldsymbol{w} \cdot (\boldsymbol{u} + \boldsymbol{v}) = \boldsymbol{w} \cdot \boldsymbol{u} + \boldsymbol{w} \cdot \boldsymbol{v}$ — distributive law
3. $(c\boldsymbol{u}) \cdot \boldsymbol{v} = \boldsymbol{u} \cdot (c\boldsymbol{v}) = c(\boldsymbol{u} \cdot \boldsymbol{v})$ — scalar mult.
4. $\|c\boldsymbol{u}\| = |c|\,\|\boldsymbol{u}\|$ — (not $c\,\|\boldsymbol{u}\|$)
5. (i) $\boldsymbol{u} \cdot \boldsymbol{u} \geq 0$
   (ii) $\boldsymbol{u} \cdot \boldsymbol{u} = 0$ if and only if $\boldsymbol{u} = \mathbf{0}$.

$u_1^2 + u_2^2 + \cdots + u_n^2 = 0$ $\Rightarrow$ $u_1 = 0,\ u_2 = 0,\ \ldots,\ u_n = 0$

Additional example

$\boxed{\boldsymbol{Av} = \mathbf{0} \quad \text{if and only if } \boldsymbol{A}^{\mathrm{T}}\boldsymbol{Av} = \mathbf{0}}$

Proof

$$\boldsymbol{Av} = \mathbf{0} \Rightarrow \boldsymbol{A}^{\mathrm{T}}\boldsymbol{Av} = \boldsymbol{A}^{\mathrm{T}}\mathbf{0} \Rightarrow \boldsymbol{A}^{\mathrm{T}}\boldsymbol{Av} = \mathbf{0}$$

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{Av} = \mathbf{0}$$

$$\boldsymbol{v}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{Av} = \boldsymbol{v}^{\mathrm{T}}\mathbf{0}$$

$$(\boldsymbol{v}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}})\boldsymbol{Av} = \mathbf{0}$$

$$(\boldsymbol{Av})^{\mathrm{T}}\boldsymbol{Av} = \mathbf{0}$$

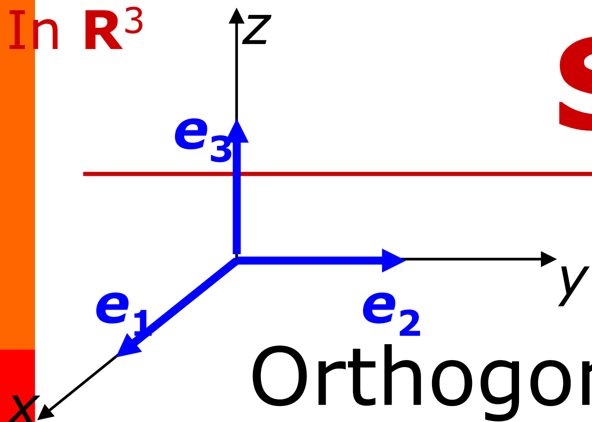
$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T\mathbf{v}$ for column vectors

$$(\boldsymbol{Av})\cdot(\boldsymbol{Av}) = \mathbf{0}$$

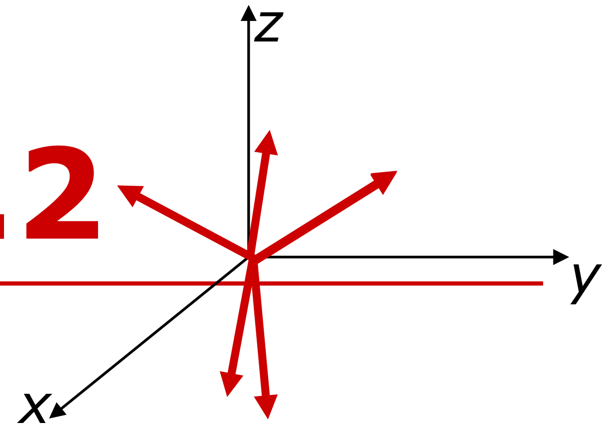
$$\boldsymbol{Av} = \mathbf{0}$$

$\boldsymbol{u} \cdot \boldsymbol{u} = 0$ if and only if $\boldsymbol{u} = \mathbf{0}$

In $\mathbf{R}^3$



Section 5.2



Orthogonal and Orthonormal Basis

Objectives

- What is an orthogonal/orthonormal set?
- How to normalize a vector?
- What are the properties of orthogonal sets?

Ortho- means: *straight, upright, right, correct*

What is an orthogonal/orthonormal set?

Definition 5.2.1

1. Two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbf{R}^n$ are called orthogonal if $\boldsymbol{u} \cdot \boldsymbol{v} = 0$.

   In $\mathbf{R}^2$ and $\mathbf{R}^3$, it means "perpendicular"

2. A set S of vectors in $\mathbf{R}^n$ is called orthogonal if every pair of distinct vectors in S are orthogonal.

   $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$

   $\boldsymbol{u}_1 \cdot \boldsymbol{u}_2 = 0,\ \boldsymbol{u}_1 \cdot \boldsymbol{u}_3 = 0,\ \ldots\ \boldsymbol{u}_{k-1} \cdot \boldsymbol{u}_k = 0$

   $||\boldsymbol{u}_1|| = ||\boldsymbol{u}_2|| = \ldots = ||\boldsymbol{u}_k|| = 1$

3. A set S of vectors in $\mathbf{R}^n$ is called orthonormal if S is orthogonal and every vector in S is a unit vector.

Angle between two orthogonal vectors

Remark 5.2.2

Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two vectors in $\mathbf{R}^n$.

If $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal, $\Rightarrow \boldsymbol{u} \cdot \boldsymbol{v} = 0$

the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$:

$$\cos^{-1}\left(\frac{\boldsymbol{u} \cdot \boldsymbol{v}}{||\boldsymbol{u}||\,||\boldsymbol{v}||}\right) = \cos^{-1}(0) = \frac{\pi}{2}$$

So $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular

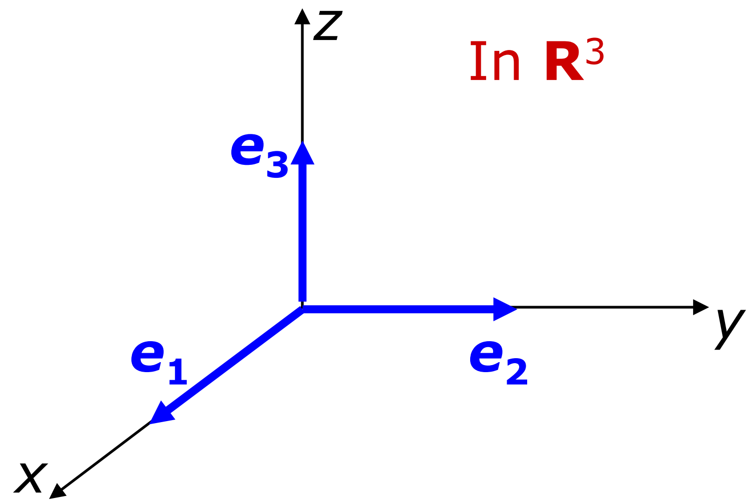
Example 5.2.3.3

Consider the standard basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ for $\mathbf{R}^n$.

$\boldsymbol{e}_1 = (1, 0, \ldots, 0)$

$\boldsymbol{e}_2 = (0, 1, \ldots, 0)$

$\boldsymbol{e}_n = (0, 0, \ldots, 1)$



For $i \neq j$, $\boldsymbol{e}_i \cdot \boldsymbol{e}_j = 0$.

So the standard basis is an orthogonal set

For $i = 1, 2, \ldots, n$, $||\boldsymbol{e}_i|| = 1$.

So the standard basis is also an orthonormal set.

Another example of an orthogonal set

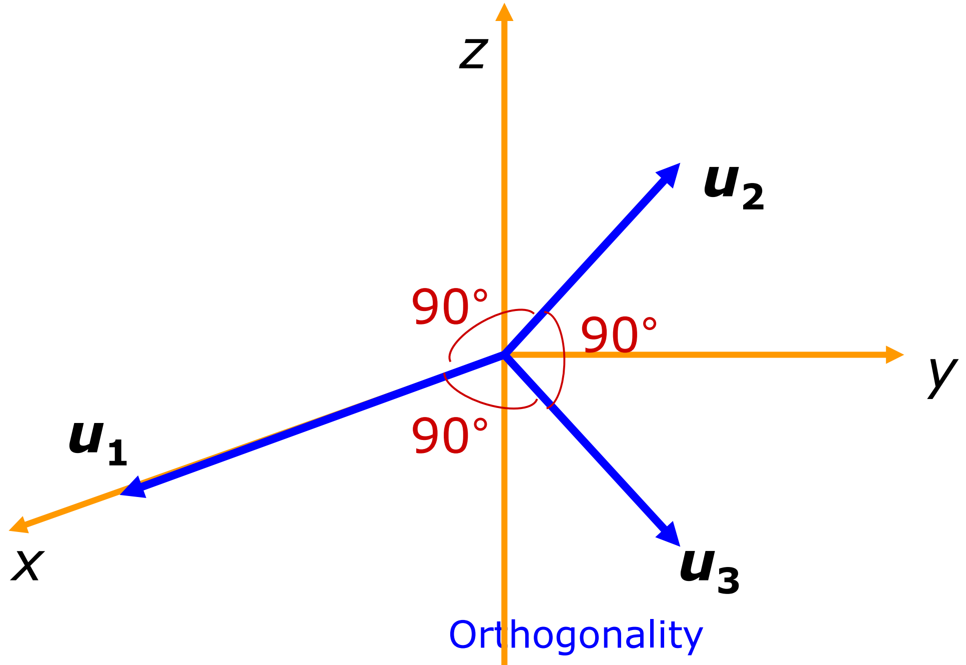
Example 5.2.3.2

$\boldsymbol{u}_1 = (2, 0, 0)$, $\boldsymbol{u}_2 = (0, 1, 1)$ and $\boldsymbol{u}_3 = (0, 1, -1)$.

$$\boldsymbol{u}_1 \cdot \boldsymbol{u}_2 = 0, \quad \boldsymbol{u}_1 \cdot \boldsymbol{u}_3 = 0 \quad \text{and} \quad \boldsymbol{u}_2 \cdot \boldsymbol{u}_3 = 0$$

$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ is an orthogonal set.

It is not an orthonormal set.



Converting orthogonal to orthonormal set

Example 5.2.3.2

$\boldsymbol{u_1} = (2, 0, 0)$ $\boldsymbol{u_2} = (0, 1, 1)$ $\boldsymbol{u_3} = (0, 1, -1)$

$$\left.\begin{aligned}
\boldsymbol{v}_1 &= \frac{1}{\|\boldsymbol{u}_1\|}\boldsymbol{u}_1 = \tfrac{1}{2}(2, 0, 0) = (1, 0, 0)\\
\boldsymbol{v}_2 &= \frac{1}{\|\boldsymbol{u}_2\|}\boldsymbol{u}_2 = \tfrac{1}{\sqrt{2}}(0, 1, 1) = (0, \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}})\\
\boldsymbol{v}_3 &= \frac{1}{\|\boldsymbol{u}_3\|}\boldsymbol{u}_3 = \tfrac{1}{\sqrt{2}}(0, 1, -1) = (0, \tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}})
\end{aligned}\right\}\text{unit vectors}$$

$$\|\boldsymbol{v}_i\| = \left\| \frac{1}{\|\boldsymbol{u}_i\|}\boldsymbol{u}_i \right\| = \frac{1}{\|\mathbf{u}_i\|}\|\mathbf{u}_i\| = 1$$

For $i \neq j$,

$$\boldsymbol{v}_i \cdot \boldsymbol{v}_j = \left(\frac{1}{\|\boldsymbol{u}_i\|}\boldsymbol{u}_i\right)\cdot\left(\frac{1}{\|\boldsymbol{u}_j\|}\boldsymbol{u}_j\right) = \frac{1}{\|\boldsymbol{u}_i\|\|\boldsymbol{u}_j\|}(\boldsymbol{u}_i \cdot \boldsymbol{u}_j) = 0$$

So the set $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ is orthonormal.

Remark on Example 5.2.3.2

$$\boldsymbol{u} \xrightarrow{\text{normalizing}} \frac{1}{||\boldsymbol{u}||}\boldsymbol{u}$$

any non-zero vector → unit vector

Scalar multiple of the original vector

$$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} \xrightarrow{\text{normalizing}} \left\{\frac{1}{||\boldsymbol{u}_1||}\boldsymbol{u}_1, \quad \frac{1}{||\boldsymbol{u}_2||}\boldsymbol{u}_2, \quad \frac{1}{||\boldsymbol{u}_3||}\boldsymbol{u}_3\right\}$$

an orthogonal set → an orthonormal set

orthogonal ⇒ linearly independent

Theorem 5.2.4

Let S be an orthogonal set of nonzero vectors in a vector space.

Then S is linearly independent.

Proof

Let $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_n}\}$ orthogonal set

$c_1 \boldsymbol{u_1} + c_2 \boldsymbol{u_2} + \ldots + c_n \boldsymbol{u_n} = \mathbf{0}$

Want to show:

$c_1 = 0, c_2 = 0, \ldots, c_n = 0$ is the only solution

Take dot product on both sides with $\boldsymbol{u_i}$ for every i.

orthogonal ⇒ linearly independent

$\boldsymbol{u_i} \cdot \boldsymbol{u_i} \neq 0$ for all i

Theorem 5.2.4

$\boldsymbol{u_j} \cdot \boldsymbol{u_i} = 0$ if $j \neq i$

Proof

S = {$\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_n}$} orthogonal set
nonzero vectors

$$c_1 \boldsymbol{u_1} + c_2 \boldsymbol{u_2} + \ldots + c_n \boldsymbol{u_n} = \boldsymbol{0}$$

$$(c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k}) \cdot \boldsymbol{u_1} = \boldsymbol{0} \cdot \boldsymbol{u_1}$$

$$c_1(\boldsymbol{u_1} \cdot \boldsymbol{u_1}) + c_2(\boldsymbol{u_2} \cdot \boldsymbol{u_1}) + \cdots + c_k(\boldsymbol{u_k} \cdot \boldsymbol{u_1}) = 0$$

$$c_1(\boldsymbol{u_1} \cdot \boldsymbol{u_1}) = 0$$

$$c_1 = 0$$

Similarly we can show $c_2 = 0, \ldots, c_n = 0$

What is an orthogonal/orthonormal basis?

Definition 5.2.5

1. A basis S for a vector space is called an orthogonal basis if S is orthogonal.

$\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is an orthogonal basis for $\mathbf{R}^3$

$\{(2,0,0), (0,1,1), (0,1,-1)\}$ is an orthogonal basis for $\mathbf{R}^3$

$\{(1,0,0), (1,1,0), (1,1,1)\}$ is not an orthogonal basis for $\mathbf{R}^3$

2. A basis S for a vector space is called an orthonormal basis if S is orthonormal.

$\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is an orthonormal basis for $\mathbf{R}^3$

$\{(2,0,0), (0,1,1), (0,1,-1)\}$ is not an orthonormal basis for $\mathbf{R}^3$

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$\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is a basis for $\mathbf{R}^3$

$\{(2,0,0), (0,1,1), (0,1,-1)\}$ is a basis for $\mathbf{R}^3$

$\{(1,0,0), (1,1,0), (1,1,1)\}$ is a basis for $\mathbf{R}^3$

How to check a set is an orthogonal basis?

Remark 5.2.6

A set S of nonzero vectors in a vector space V.

To check whether S is an orthonormal basis for V:

S is an orthogonal set

S spans V

S lin. indep

Theorem 5.2.4

Only need to check:

(i) S is orthonormal and

(ii) span(S) = V.

If we know dim V,

Only need to check:

(i) S is orthonormal and

(ii) $|S|$ = dim V.

How to check a set is an orthogonal basis?

Example 5.2.7.2

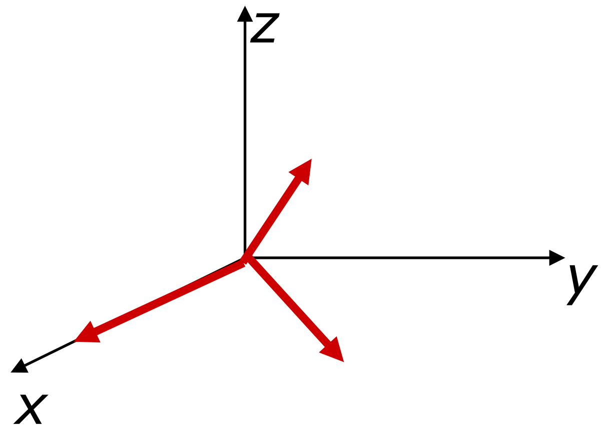
$$\boldsymbol{u}_1 = (2, 0, 0) \quad \boldsymbol{u}_2 = (0, 1, 1) \quad \boldsymbol{u}_3 = (0, 1, -1)$$

$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$

- an orthogonal set

- has three vectors = dim $\mathbf{R}^3$

$\Rightarrow$ an orthogonal basis for $\mathbf{R}^3$.



Quiz Time

True or false

$$\mathbf{u}_1 = (1, -1, 1, -1) \quad \mathbf{u}_2 = (1, 1, 1, 1) \quad \mathbf{u}_3 = (0, 1, 0, -1)$$

$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ is an orthogonal basis for

V = span$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$

Check:

$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ is an orthogonal set

$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ spans V

$\Rightarrow$ $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ is an orthogonal basis for V.

Coordinate vector w.r.t. orthogonal basis

Example 5.2.9.2

$\boldsymbol{u}_1 = (1, 1, 1)$, $\boldsymbol{u}_2 = (1, 0, -1)$ and $\boldsymbol{u}_3 = (1, -2, 1)$.

$S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ is an orthogonal basis for $\mathbf{R}^3$.

Let $\boldsymbol{w} = (1, -1, 0)$. Find $(\boldsymbol{w})_S$

coordinate vector w.r.t. basis S

$$\boldsymbol{w} = c_1 \boldsymbol{u}_1 + c_2 \boldsymbol{u}_2 + c_3 \boldsymbol{u}_3 \quad \Rightarrow (\boldsymbol{w})_S = (c_1, c_2, c_3)$$

standard approach: need to solve linear system

Short cut formula (when S is orthogonal) :

$$(\boldsymbol{w})_S = \left(\frac{\boldsymbol{w} \cdot \boldsymbol{u}_1}{||\boldsymbol{u}_1||^2}, \frac{\boldsymbol{w} \cdot \boldsymbol{u}_2}{||\boldsymbol{u}_2||^2}, \frac{\boldsymbol{w} \cdot \boldsymbol{u}_3}{||\boldsymbol{u}_3||^2} \right) = \left(0, \frac{1}{2}, \frac{1}{2}\right)$$

Coordinate vector w.r.t. orthogonal basis

Theorem 5.2.8.1

$S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}\}$: an orthogonal basis for V

For any vector $\boldsymbol{w}$ in V,

$$\boldsymbol{w} = c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \ldots + c_k\boldsymbol{u_k}$$

$$c_1 = \frac{\boldsymbol{w} \cdot \boldsymbol{u_1}}{||\boldsymbol{u_1}||^2} \qquad c_2 = \frac{\boldsymbol{w} \cdot \boldsymbol{u}_2}{||\boldsymbol{u}_2||^2} \qquad c_k = \frac{\boldsymbol{w} \cdot \boldsymbol{u}_k}{||\boldsymbol{u}_k||^2}$$

$$(\boldsymbol{w})_S = \left(\frac{\boldsymbol{w} \cdot \boldsymbol{u_1}}{||\boldsymbol{u_1}||^2}, \frac{\boldsymbol{w} \cdot \boldsymbol{u_2}}{||\boldsymbol{u_2}||^2}, \cdots, \frac{\boldsymbol{w} \cdot \boldsymbol{u_k}}{||\boldsymbol{u_k}||^2} \right)$$

Theorem 5.2.8.2: orthonormal basis
Special case of part 1, with $||\boldsymbol{u}_i||^2 = 1$ for all i

The proof

$\boldsymbol{u_j} \cdot \boldsymbol{u_i} = 0$ if $j \neq i$

Theorem 5.2.8.1

$\boldsymbol{u_i} \cdot \boldsymbol{u_i} = ||\boldsymbol{u_i}||^2$

Let $\boldsymbol{w} = c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k}$

WTS: $c_i = \dfrac{\mathbf{w} \cdot \mathbf{u}_i}{||\mathbf{u}_i||^2}$

$$\boldsymbol{w} \cdot \boldsymbol{u_1} = (c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k}) \cdot \boldsymbol{u_1}$$

$$= c_1(\boldsymbol{u_1} \cdot \boldsymbol{u_1}) + c_2(\boldsymbol{u_2} \cdot \boldsymbol{u_1}) + \cdots + c_k(\boldsymbol{u_k} \cdot \boldsymbol{u_1})$$

$$= c_1(\boldsymbol{u_1} \cdot \boldsymbol{u_1})$$

$$= c_1 ||\boldsymbol{u_1}||^2$$

So $c_1 = \dfrac{\mathbf{w} \cdot \mathbf{u}_1}{||\mathbf{u}_1||^2}$

Coordinate vector w.r.t. orthonormal basis

Example 5.2.9.1

$$\boldsymbol{v_1} = \left(\frac{3}{5}, \frac{4}{5}\right) \quad \boldsymbol{v_2} = \left(\frac{4}{5}, -\frac{3}{5}\right)$$

$||\boldsymbol{v_i}||^2 = 1 \quad \boldsymbol{v_1}\cdot\boldsymbol{v_2} = 0$

$S = \{\boldsymbol{v_1}, \boldsymbol{v_2}\}$ is an orthonormal basis for $\mathbf{R}^2$.

Let $\boldsymbol{w} = (x, y)$ be any vector in $\mathbf{R}^2$.

Express $(\boldsymbol{w})_s$ in terms of x and y

$$\left.\begin{aligned} \boldsymbol{w}\cdot\boldsymbol{v}_1 &= \frac{3x+4y}{5} \\ \boldsymbol{w}\cdot\boldsymbol{v}_2 &= \frac{4x-3y}{5} \end{aligned}\right\} \Rightarrow$$

$\boldsymbol{w} = (\boldsymbol{w}\cdot\boldsymbol{v_1})\boldsymbol{v_1} + (\boldsymbol{w}\cdot\boldsymbol{v_2})\boldsymbol{v_2}$

$$\boldsymbol{w} = \frac{3x+4y}{5}\boldsymbol{v}_1 + \frac{4x-3y}{5}\boldsymbol{v}_2$$

$$(\boldsymbol{w})_S = \left(\frac{3x+4y}{5}, \frac{4x-3y}{5}\right)$$

Coordinate vector w.r.t. orthogonal basis

Example 5.2.9.2

$\boldsymbol{u}_1 = (1, 1, 1)$, $\boldsymbol{u}_2 = (1, 0, -1)$ and $\boldsymbol{u}_3 = (1, -2, 1)$.

$S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ is an orthogonal basis for $\mathbf{R}^3$.

Let $\boldsymbol{w} = (1, -1, 0)$. Find $(\boldsymbol{w})_S$

coordinate vector w.r.t. basis S

$$\boldsymbol{w} = c_1 \boldsymbol{u}_1 + c_2 \boldsymbol{u}_2 + c_3 \boldsymbol{u}_3 \quad \Rightarrow (\boldsymbol{w})_S = (c_1, c_2, c_3)$$

Theorem 5.2.8 (when S is orthogonal) :

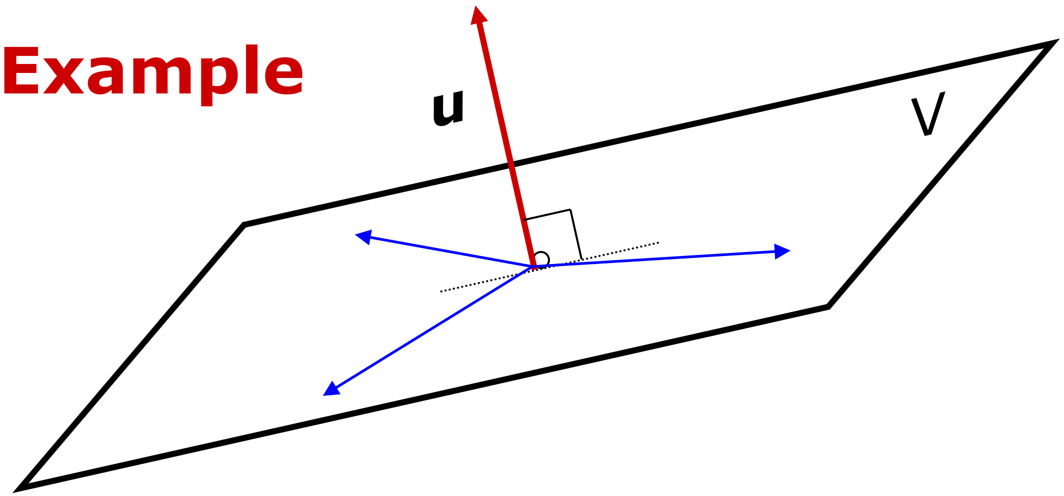
$$(\boldsymbol{w})_S = \left(\frac{\boldsymbol{w} \cdot \boldsymbol{u}_1}{||\boldsymbol{u}_1||^2}, \frac{\boldsymbol{w} \cdot \boldsymbol{u}_2}{||\boldsymbol{u}_2||^2}, \frac{\boldsymbol{w} \cdot \boldsymbol{u}_3}{||\boldsymbol{u}_3||^2} \right) = \left(0, \frac{1}{2}, \frac{1}{2}\right)$$

Definition 5.2.10

Let V be a subspace of $\mathbf{R}^n$.

A vector $\boldsymbol{u}$ is orthogonal to the subspace V if $\boldsymbol{u}$ is orthogonal to all vectors in V.

Example



A vector orthogonal to a plane

Example 5.2.11.1

$3x - 5y + 11z = 0$

V a plane in $\mathbf{R}^3$ with equation $ax + by + cz = 0$.

(3, -5, 11)

$\boldsymbol{n} = (a, b, c)$ Why it works?

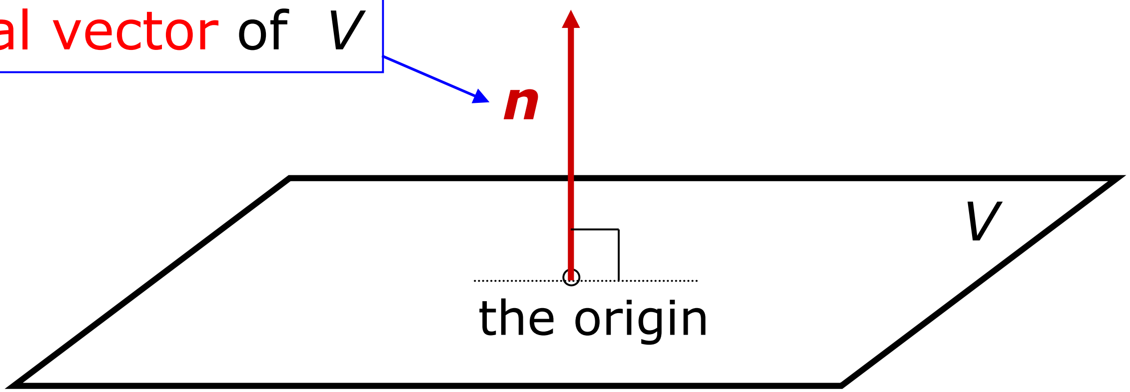
satisfies the equation

Take any $\boldsymbol{u} = (x_0, y_0, z_0)$ in V

Take the dot product $\boldsymbol{n} \cdot \boldsymbol{u} = ax_0 + by_0 + cz_0 = 0$

So $\boldsymbol{n}$ is orthogonal to every vector $\boldsymbol{u}$ in V

a normal vector of V



How to find vectors orthogonal to a subspace?

Example 5.2.11.2

$\boldsymbol{u_1} = (1, 1, 1, 0)$ and $\boldsymbol{u_2} = (0, -1, -1, 1)$

$V = \text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}\}$ a subspace of $\mathbf{R}^4$

Find all vectors that are orthogonal to V.

(w, x, y, z)

Let $\boldsymbol{v}$ be orthogonal to $V = \text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}\}$

$\Leftrightarrow$ $\boldsymbol{v}$ is orthogonal to $a\boldsymbol{u_1} + b\boldsymbol{u_2}$ for all a, b

$\Leftrightarrow$ $\boldsymbol{v} \cdot (a\boldsymbol{u_1} + b\boldsymbol{u_2}) = 0$ for all a, b

$\Leftrightarrow$ $\boldsymbol{v} \cdot \boldsymbol{u_1} = 0$ and $\boldsymbol{v} \cdot \boldsymbol{u_2} = 0$

$$\underbrace{w + x + y = 0 \quad \text{and} \quad -x - y + z = 0}_{\text{solve this homog. system}}$$

$\Rightarrow$ general solution

Section 5.2

Orthogonal and Orthonormal Basis

Objectives

- What is the projection of a vector onto a subspace?
- What is Gram-Schmidt Process?

Usage of the word "Orthogonal"

- A vector ***u*** is orthogonal to another vector ***v*** (same as: two vectors ***u*** and ***v*** are orthogonal)
- A set of vectors is orthogonal (same as: every pair of vectors in the set is orthogonal)
- A vector ***u*** is orthogonal to a subspace *V* (same as: ***u*** is orthogonal to every vector in subspace V)

Remark

To show a vector $\boldsymbol{v}$ is orthogonal to a subspace $V = \text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}\}$ of $\mathbf{R}^n$

Show: $\boldsymbol{v} \cdot \boldsymbol{u_1} = 0, \boldsymbol{v} \cdot \boldsymbol{u_2} = 0, \ldots, \boldsymbol{v} \cdot \boldsymbol{u_k} = 0$

To find a vector $\boldsymbol{v}$ that is orthogonal to a subspace $V = \text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}\}$ of $\mathbf{R}^n$

Let $\boldsymbol{v} = (x_1, x_2, \ldots, x_n)$ (unknowns)

Convert $\boldsymbol{v} \cdot \boldsymbol{u_1} = 0, \boldsymbol{v} \cdot \boldsymbol{u_2} = 0, \ldots, \boldsymbol{v} \cdot \boldsymbol{u_k} = 0$

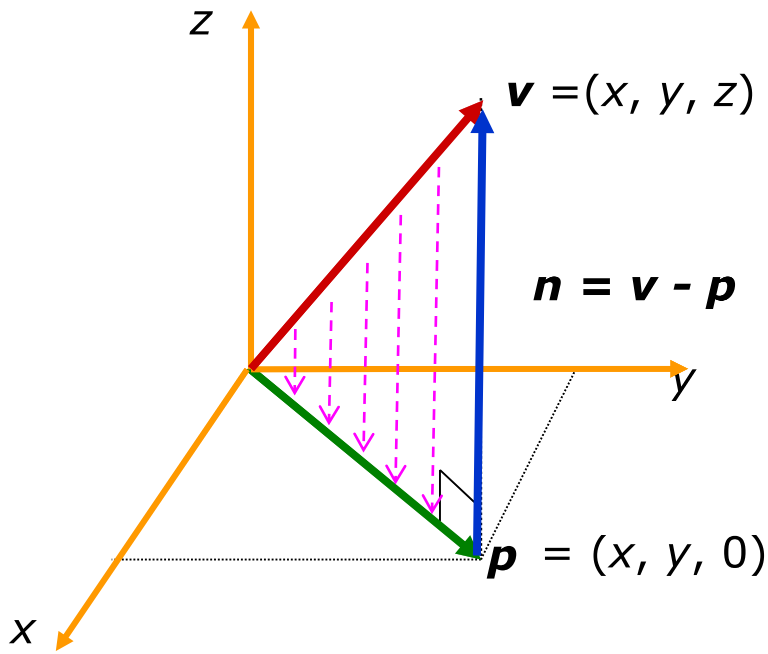
into a homogeneous system.

Solve the system.

Example 5.2.11.2

Example 5.2.14.2

The projection of $\boldsymbol{v} = (x, y, z)$ onto the xy-plane



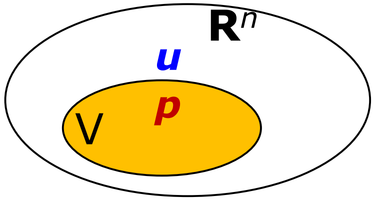
orthogonal to xy-plane

$\boldsymbol{p}$ is a projection of $\boldsymbol{v}$ onto the plane $\Longleftrightarrow$ $\boldsymbol{v} - \boldsymbol{p}$ is orthogonal to the plane

Projection of a vector onto a subspace of $\mathbf{R}^n$

Definition 5.2.13

Let V be a subspace of $\mathbf{R}^n$ and $\boldsymbol{u}$ a vector in $\mathbf{R}^n$.



Let $\boldsymbol{p}$ be a vector in V.

$\boldsymbol{p}$ is called the projection of $\boldsymbol{u}$ onto V

if $\boldsymbol{u} - \boldsymbol{p}$ is a vector orthogonal to V.

good for checking, but not finding projection.

Every vector has exactly one projection onto a given subspace.

see Ex5 Q18

How to find projection in general?

Example 5.2.16

This is the xz-plane

V = span{(1,0,1), (1,0,-1)}   a plane in $\mathbf{R}^3$

Find the projection $\boldsymbol{p}$ of $\boldsymbol{w} = (1, 1, 0)$ onto V

$\boldsymbol{u}_1 = (1, 0, 1)$ and $\boldsymbol{u}_2 = (1, 0, -1)$ orthogonal basis for V

$\boldsymbol{p}$ lies on V

$$\Rightarrow \boldsymbol{p} = c_1 \boldsymbol{u}_1 + c_2 \boldsymbol{u}_2 = \frac{1}{2}(1, 0, 1) + \frac{1}{2}(1, 0, -1)$$

Theorem 5.2.15

$$c_1 = \frac{\boldsymbol{w} \cdot \boldsymbol{u}_1}{||\boldsymbol{u}_1||^2} = \frac{1}{2} \qquad c_2 = \frac{\boldsymbol{w} \cdot \boldsymbol{u}_2}{||\boldsymbol{u}_2||^2} = \frac{1}{2}$$

$= (1, 0, 0)$

This is the projection of $\boldsymbol{w}$ onto V

Check: $\boldsymbol{w} - \boldsymbol{p}$ is orthogonal to V

How to find projection using orthogonal basis?

Theorem 5.2.15

Let V be a subspace of $\mathbf{R}^n$ and $\boldsymbol{w}$ a vector in $\mathbf{R}^n$.

1. $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$: an orthogonal basis for V, the projection $\boldsymbol{p}$ of $\boldsymbol{w}$ onto V is

Look familiar?

$$\mathbf{p} = \frac{\mathbf{w} \cdot \mathbf{u}_1}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 + \frac{\mathbf{w} \cdot \mathbf{u}_2}{\|\mathbf{u}_2\|^2}\mathbf{u}_2 + \cdots + \frac{\mathbf{w} \cdot \mathbf{u}_k}{\|\mathbf{u}_k\|^2}\mathbf{u}_k$$

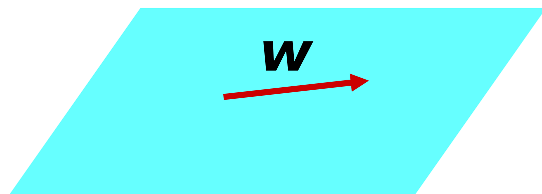
2. $T = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$: an orthonormal basis for V, the projection $\boldsymbol{p}$ of $\boldsymbol{w}$ onto V is

$$\boldsymbol{p} = (\boldsymbol{w} \cdot \boldsymbol{v}_1)\boldsymbol{v}_1 + (\boldsymbol{w} \cdot \boldsymbol{v}_2)\boldsymbol{v}_2 + \cdots + (\boldsymbol{w} \cdot \boldsymbol{v}_k)\boldsymbol{v}_k$$

Theorems 5.2.8 VS 5.2.15

Theorem 5.2.8

w a vector in *V*

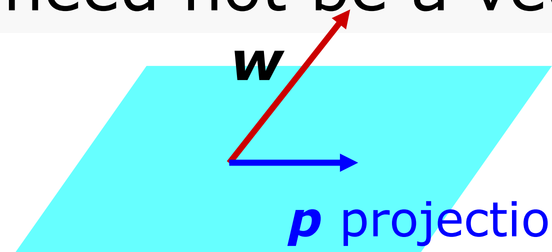


V a subspace

S = {$\mathbf{u}_1$, $\mathbf{u}_2$, …, $\mathbf{u}_k$}
orthogonal basis

Theorem 5.2.15

w need not be a vector in *V*



V a subspace

S = {$\mathbf{u}_1$, $\mathbf{u}_2$, …, $\mathbf{u}_k$}
orthogonal basis

$$\frac{\mathbf{w}\cdot\mathbf{u}_1}{\|\mathbf{u}_1\|^2}\mathbf{u}_1+\frac{\mathbf{w}\cdot\mathbf{u}_2}{\|\mathbf{u}_2\|^2}\mathbf{u}_2+\cdots+\frac{\mathbf{w}\cdot\mathbf{u}_k}{\|\mathbf{u}_k\|^2}\mathbf{u}_k=\begin{cases}\mathbf{w} & \text{if } \mathbf{w}\in V\\ \mathbf{p} & \text{if } \mathbf{w}\notin V\end{cases}$$

The proof

$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$
an orthogonal basis for V

Theorem 5.2.15

Let $\boldsymbol{p} = \frac{\boldsymbol{w} \cdot \boldsymbol{u}_1}{||\boldsymbol{u}_1||^2}\boldsymbol{u}_1 + \frac{\boldsymbol{w} \cdot \boldsymbol{u}_2}{||\boldsymbol{u}_2||^2}\boldsymbol{u}_2 + \cdots + \frac{\boldsymbol{w} \cdot \boldsymbol{u}_k}{||\boldsymbol{u}_k||^2}\boldsymbol{u}_k$

Show $\boldsymbol{p}$ is the projection of $\boldsymbol{w}$ onto V

Just need to show $\boldsymbol{w}$ - $\boldsymbol{p}$ is orthogonal to V

span$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$

Just need to show $\boldsymbol{w}$ - $\boldsymbol{p}$ is orthogonal to $\boldsymbol{u}_i$ for all i.

$$(\boldsymbol{w} - \boldsymbol{p}) \cdot \boldsymbol{u}_1 = \boldsymbol{w} \cdot \boldsymbol{u}_1 - \boldsymbol{p} \cdot \boldsymbol{u}_1$$

$$\boldsymbol{p} = \frac{\boldsymbol{w} \cdot \boldsymbol{u}_1}{||\boldsymbol{u}_1||^2}\boldsymbol{u}_1 + \frac{\boldsymbol{w} \cdot \boldsymbol{u}_2}{||\boldsymbol{u}_2||^2}\boldsymbol{u}_2 + \cdots + + \frac{\boldsymbol{w} \cdot \boldsymbol{u}_k}{||\boldsymbol{u}_k||^2}\boldsymbol{u}_k$$

$$= \mathbf{w} \cdot \mathbf{u}_1 - \frac{\mathbf{w} \cdot \mathbf{u}_1}{||\mathbf{u}_1||^2}\mathbf{u}_1 \cdot \mathbf{u}_1 \qquad = 0$$

How to convert a basis to an orthogonal basis?

Discussion 5.2.18.1

$\boldsymbol{u_1}$ $\boldsymbol{u_2 - p}$

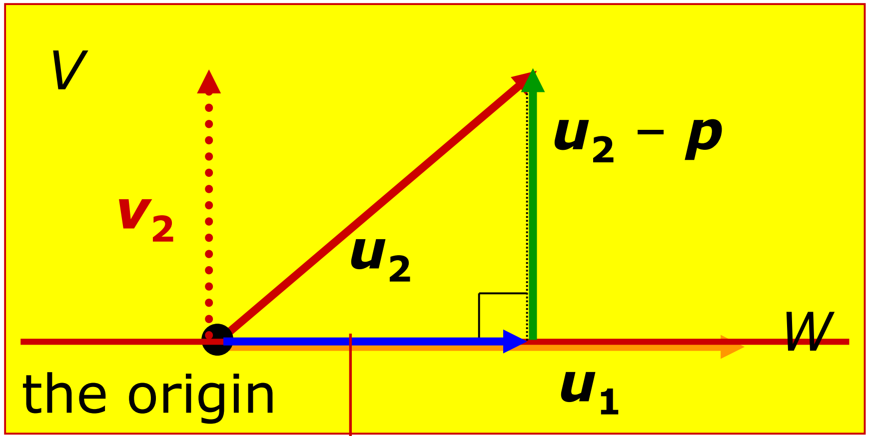
$\boldsymbol{v_1}, \boldsymbol{v_2}$

V = span{$\boldsymbol{u_1}, \boldsymbol{u_2}$} a plane

basis → orthogonal basis

W = span{ $\boldsymbol{u_1}$} a line

projection of $\boldsymbol{u_2}$ onto W



An orthogonal basis for V

$$\mathbf{v}_1 = \mathbf{u}_1$$

$$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\|\mathbf{v}_1\|^2}\mathbf{v}_1$$

$$\boldsymbol{p} = \frac{\boldsymbol{u}_2 \cdot \boldsymbol{u}_1}{\|\boldsymbol{u}_1\|^2}\boldsymbol{u}_1$$

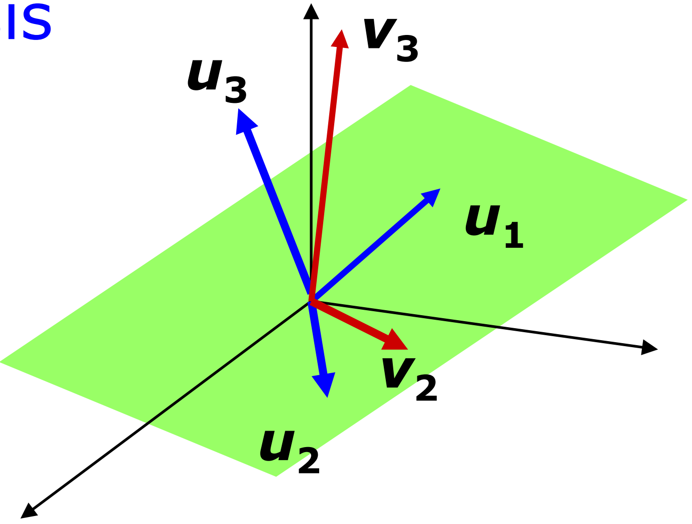
Theorem 5.2.15

How to convert a basis to an orthogonal basis?

Discussion 5.2.18.2

Let $\{\boxed{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3}\}$ be a basis for $\mathbf{R}^3$.

"Convert" to an orthogonal basis



$V = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ a plane

$$\mathbf{v}_1 = \mathbf{u}_1$$

$$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\|\mathbf{v}_1\|^2}\mathbf{v}_1$$

an orthogonal basis for V

How to convert a basis to an orthogonal basis?

Discussion 5.2.18.2

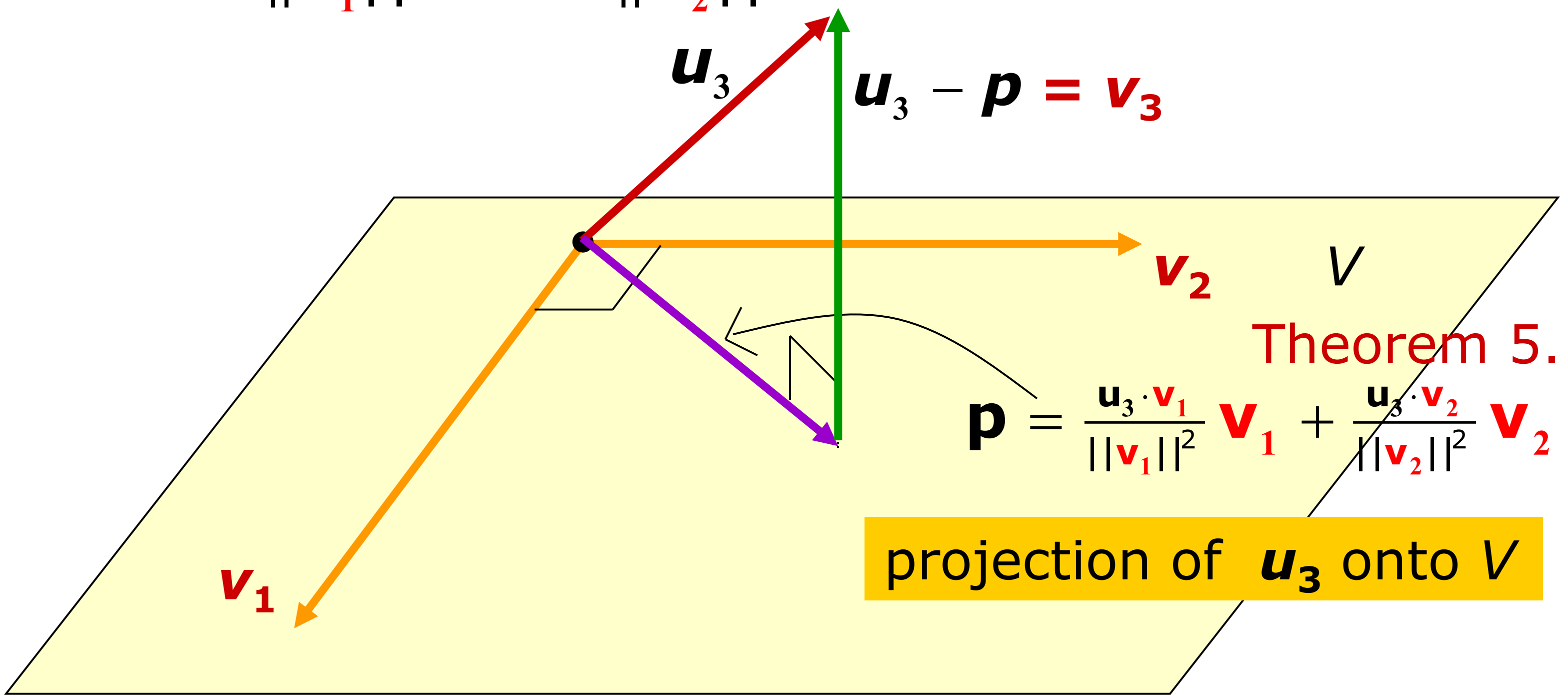
$$\mathbf{v}_1 = \mathbf{u}_1$$

$$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{||\mathbf{v}_1||^2}\mathbf{v}_1$$

orthogonal basis for V

$$\mathbf{v}_3 = \mathbf{u}_3 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{||\mathbf{v}_1||^2}\mathbf{v}_1 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{||\mathbf{v}_2||^2}\mathbf{v}_2$$

orthogonal basis for $\mathbf{R}^3$



Theorem 5.2.15

$$\mathbf{p} = \frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{||\mathbf{v}_1||^2}\mathbf{v}_1 + \frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{||\mathbf{v}_2||^2}\mathbf{v}_2$$

projection of $\boldsymbol{u}_3$ onto V

Gram-Schmidt Process

Theorem 5.2.19

$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$: a basis for a vector space V.

Define $\mathbf{v}_1 = \mathbf{u}_1$

$$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\|\mathbf{v}_1\|^2}\mathbf{v}_1$$

orthogonal to $\boldsymbol{v}_1$

$$\mathbf{v}_3 = \mathbf{u}_3 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\|\mathbf{v}_2\|^2}\mathbf{v}_2$$

orthogonal to $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$

$$\vdots$$

$$\mathbf{v}_k = \mathbf{u}_k - \frac{\mathbf{u}_k \cdot \mathbf{v}_1}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\mathbf{u}_k \cdot \mathbf{v}_2}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 - \cdots - \frac{\mathbf{u}_k \cdot \mathbf{v}_{k-1}}{\|\mathbf{v}_{k-1}\|^2}\mathbf{v}_{k-1}$$

orthogonal to $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k-1}$

$\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ is an orthogonal basis for V.

Gram-Schmidt Process

$\{u_1, u_2, \ldots, u_k\}$
basis for V

$\{v_1, v_2, \ldots, v_k\}$
orthogonal basis for V

Theorem 5.2.19

$\{v_1, v_2, \ldots, v_k\}$ is an orthogonal basis for V.

Normalize this basis:

$$\mathbf{w}_1 = \frac{1}{||\mathbf{v}_1||}\mathbf{v}_1 \quad \mathbf{w}_2 = \frac{1}{||\mathbf{v}_2||}\mathbf{v}_2 \quad \cdots \quad \mathbf{w}_k = \frac{1}{||\mathbf{v}_k||}\mathbf{v}_k$$

$\{w_1, w_2, \ldots, w_k\}$ is an orthonormal basis for V.

Example 5.2.20

$$\boldsymbol{u}_1 = (1, -1, 2) \quad \boldsymbol{u}_2 = (2, 1, 0) \quad \boldsymbol{u}_3 = (0, 0, 1)$$

$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ is a basis for $\mathbf{R}^3$.

Apply the Gram-Schmidt Process to transform this basis into an orthonormal basis.

Gram-Schmidt Process

Example 5.2.20

$$\mathbf{u}_1 = (1, -1, 2)$$
$$\mathbf{u}_2 = (2, 1, 0)$$
$$\mathbf{u}_3 = (0, 0, 1)$$

$$\mathbf{v}_1 = \mathbf{u}_1 = (1, -1, 2)$$

Visualization tool

$$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\|\mathbf{v}_1\|^2}\mathbf{v}_1$$
$$= (2, 1, 0) - \tfrac{1}{6}(1, -1, 2) = \left(\tfrac{11}{6}, \tfrac{7}{6}, -\tfrac{1}{3}\right)$$

$$\mathbf{v}_3 = \mathbf{u}_3 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\|\mathbf{v}_2\|^2}\mathbf{v}_2$$
$$= (0, 0, 1) - \tfrac{2}{6}(1, -1, 2) - \tfrac{-1/3}{29/6}\left(\tfrac{11}{6}, \tfrac{7}{6}, -\tfrac{1}{3}\right)$$
$$= \left(-\tfrac{6}{29}, \tfrac{12}{29}, \tfrac{9}{29}\right)$$

$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis for $\mathbf{R}^3$.

Gram-Schmidt Process

Example 5.2.20

$$\mathbf{w}_1 = \frac{1}{\|\mathbf{v}_1\|}\mathbf{v}_1 = \frac{1}{\sqrt{6}}(1, -1, 2) = \left(\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$$

$$\mathbf{w}_2 = \frac{1}{\|\mathbf{v}_2\|}\mathbf{v}_2 = \frac{1}{\sqrt{29/6}}\left(\frac{11}{6}, \frac{7}{6}, -\frac{1}{3}\right) = \left(\frac{11}{\sqrt{174}}, \frac{7}{\sqrt{174}}, -\frac{2}{\sqrt{174}}\right)$$

$$\mathbf{w}_3 = \frac{1}{\|\mathbf{v}_3\|}\mathbf{v}_3 = \frac{1}{\sqrt{9/29}}\left(-\frac{6}{29}, \frac{12}{29}, \frac{9}{29}\right) = \left(-\frac{2}{\sqrt{29}}, \frac{4}{\sqrt{29}}, \frac{3}{\sqrt{29}}\right)$$

$\{\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3\}$ is an orthonormal basis for $\mathbf{R}^3$.