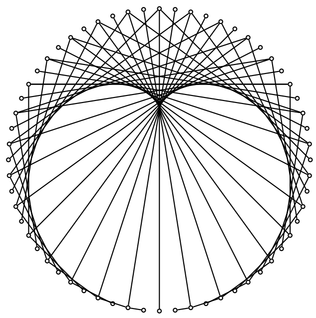
CS1231S Tutorial 5: Functions and partial orders

National University of Singapore

2021/22 Semester 1

Questions for discussion on the LumiNUS Forum

Answers to these questions will not be provided.



Multiplication by 2 modulo 60

D1. Write down all functions $\{a, b, c\} \to \{1, 2\}$.

D2. Is $\varnothing$ antisymmetric as a relation on a nonempty set?

D3. Draw a (pretty) Hasse diagram of the subset relation on $\mathcal{P}(\{1, 2, 3, 4\})$.

D4. Re-label the nodes in the diagram you drew in Question D3 such that it becomes a Hasse diagram of the divisibility relation $\mid$ on $\{d \in \mathbb{Z}^+ : d \mid n\}$ for some $n \in \mathbb{Z}^+$. Which n did you get?

Tutorial questions

1. (a) Define $f\colon \mathbb{Q} \to \mathbb{Q}$ by setting $f(n) = \pm n$ for each $n \in \mathbb{Q}$.

   (b) Define $g\colon \mathbb{Q} \to \mathbb{Q}$ by setting $g(n) = 2\sqrt{n}$ for each $n \in \mathbb{Q}$.

   (c) Define $h\colon \mathbb{Q} \to \mathbb{Q}$ by setting $h(n) = \frac{1}{n^2+1}$ for each $n \in \mathbb{Q}$.

   (d) Define $k\colon \mathbb{Q} \to \mathbb{Q}$ by setting $k(n) = \lfloor \sin n \rfloor$ for each $n \in \mathbb{Q}$.

   Which of f, g, h, k are well defined? Which of them are not?

   Here $\lfloor x \rfloor$ denotes the biggest integer that is less than or equal to x.

2. Let U be a set and $A \subseteq U$ such that $\varnothing \neq A \neq U$. Define the function $\chi\colon U \to \mathbb{Z}$ by setting, for all $x \in U$,

$$\chi(x) = \begin{cases} 0, & \text{if } x \notin A; \\ 1, & \text{if } x \in A. \end{cases}$$

   What are the domain and the codomain of χ? Write down $\{\chi(x) : x \in U\}$ in roster notation.

3. **Motivation.** One can represent a function using three sets: its domain, its codomain, and its graph, as defined below.

   Fix sets A, B. Define the *graph* of a function $f\colon A \to B$ to be

   $$\{(x, f(x)) : x \in A\}.$$

   (a) Suppose $A = \{0, 1, 2\}$ and $B = \mathbb{Z}$. Define $f\colon A \to B$ by setting $f(x) = 3x + 1$ for each $x \in A$. Write down the graph of A in roster notation.

   (b) Assuming $A \neq \varnothing$, find a subset $S \subseteq A \times B$ that cannot be the graph of any function $A \to B$.

   (c) Show that a subset $S \subseteq A \times B$ is the graph of a function $A \to B$ if and only if

   $$\forall x \in A \quad \exists! y \in B \quad (x, y) \in S.$$

   Recall that $\exists!$ means "there exists unique".

4. Consider the relation $\sim$ on $\mathbb{Q}$ defined by setting, for all $x, y \in \mathbb{Q}$,

   $$x \sim y \quad \Leftrightarrow \quad x - y \in \mathbb{Z}.$$

   It was shown in Question 5 of Tutorial 4 that $\sim$ is an equivalence relation. Define addition and multiplication on $\mathbb{Q}/{\sim}$ as follows: whenever $[x], [y] \in \mathbb{Q}/{\sim}$,

   $$[x] + [y] = [x + y] \quad \text{and} \quad [x] \cdot [y] = [x \cdot y].$$

   Is $+$ is well defined on $\mathbb{Q}/{\sim}$? Is $\cdot$ is well defined on $\mathbb{Q}/{\sim}$? Prove that your answers are correct.

5. Prove that the divisibility relation $\mid$ on $\mathbb{Z}^+$ is antisymmetric.

   You may use the following theorem without proof for this question.

   Lemma. For all $a, b \in \mathbb{Z}^+$, if $a \mid b$, then $a \leqslant b$.

6. Consider the "divides" relation on each of the following sets of integers. For each of these, draw a Hasse diagram, find all largest, smallest, maximal and minimal elements.

   (a) $A = \{1, 2, 4, 5, 10, 15, 20\}$.

   (b) $B = \{2, 3, 4, 6, 8, 9, 12, 18\}$.

7. Consider a set A and a total order $\preccurlyeq$ on A. Show that all minimal elements are smallest.

8. **Definition.** Consider a partial order $\preccurlyeq$ on a set A. Let $a, b \in A$.

   - We say a, b are *comparable* if $a \preccurlyeq b$ or $b \preccurlyeq a$.
   - We say a, b are *compatible* if there exists $c \in A$ such that $a \preccurlyeq c$ and $b \preccurlyeq c$.

   (a) Is it true that, in all partially ordered sets, any two comparable elements are compatible? Justify your answer.

   (b) Is it true that, in all partially ordered sets, any two compatible elements are comparable? Justify your answer.

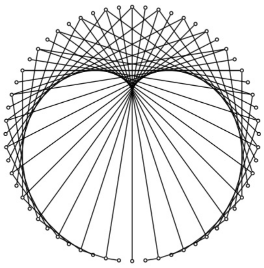
9. Let $A = \{a, b, c, d\}$, where a, b, c, d are mutually distinct. Consider the following partial order on A:

   $$R = \{(a, a), (a, b), (a, c), (a, d), (b, b), (b, c), (c, c), (d, d)\}.$$

   (a) Draw a Hasse diagram of R.

   (b) Draw Hasse diagrams of all the linearizations of R.

Answers to these questions will not be provided.



Multiplication by 2 modulo 60

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D2. Is $\varnothing$ antisymmetric as a relation on a nonempty set?

R is antisymmetric if $\forall x, y \in A\ (xRy \land yRx \to x = y)$

D3. Draw a (pretty) Hasse diagram of the subset relation on $\mathcal{P}(\{1, 2, 3, 4\})$.

D4. Re-label the nodes in the diagram you drew in Question D3 such that it becomes a Hasse diagram of the divisibility relation $|$ on $\{d \in \mathbb{Z}^+ : d \mid n\}$ for some $n \in \mathbb{Z}^+$. Which n did you get?

D1. Define $\underbrace{f : \{a,b,c\}}_{\text{domain}} \to \underbrace{\{1,2\}}_{\text{codomain}}$ by setting,

for each $x \in \{a,b,c\}$,

$f(x) = \begin{cases} 1, & x = a,b \\ 2, & x = c \end{cases}$ $\quad f(x) = \begin{cases} 1, & x = a \\ 2, & x = b,c \end{cases}$

$f(x) = \begin{cases} 1, & x = a,c \\ 2, & x = b \end{cases}$ $\quad f(x) = \begin{cases} 1, & x = b \\ 2, & x = a,c \end{cases}$

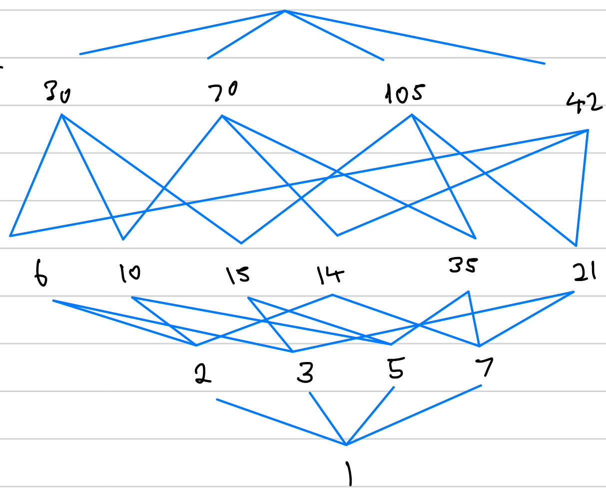
$f(x) = \begin{cases} 1, & x = b,c \\ 2, & x = a. \end{cases}$ $\quad f(x) = \begin{cases} 1, & x = c \\ 2, & x = a,b. \end{cases}$

Then the domain of f is $\{a, b, c\}$ and codomain of f is $\{1, 2\}$.

D4.

2	30	70	105	42
5	15	35	105	21
7	3	7	21	21
3	3	1	3	3

$n = 210$



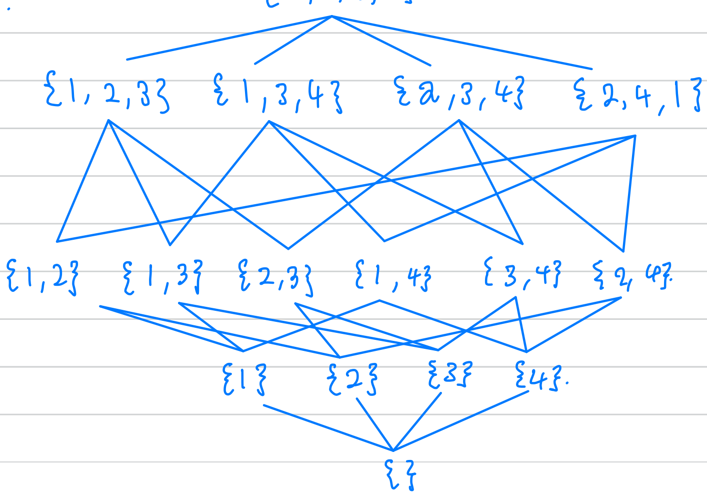
D2. Let A be a non-empty set,

$\therefore\ \forall x \in A\ (\varnothing R x \land x R \varnothing \to \varnothing = x)$

$\varnothing \neq x$

$\therefore$ Is not antisymmetric.

D3.



Tutorial questions

1. (a) Define $f\colon \mathbb{Q} \to \mathbb{Q}$ by setting $f(n) = \pm n$ for each $n \in \mathbb{Q}$.
   (b) Define $g\colon \mathbb{Q} \to \mathbb{Q}$ by setting $g(n) = 2\sqrt{n}$ for each $n \in \mathbb{Q}$.
   (c) Define $h\colon \mathbb{Q} \to \mathbb{Q}$ by setting $h(n) = \frac{1}{n^2+1}$ for each $n \in \mathbb{Q}$.
   (d) Define $k\colon \mathbb{Q} \to \mathbb{Q}$ by setting $k(n) = \lfloor \sin n \rfloor$ for each $n \in \mathbb{Q}$.

   Which of f, g, h, k are well defined? Which of them are not?
   Here $\lfloor x \rfloor$ denotes the biggest integer that is less than or equal to x.

1. a) $f(n) = \pm n$

when $n \in \mathbb{Q}$, $f(n) \in \mathbb{Q}$,
hence it is well-defined.

b) $g(n) = 2\sqrt{n}$

let $a, b \in \mathbb{Z}$ and $r = \mathbb{Q}$ s.t. $r = \frac{a}{b}$ (by definition of rational number)

let $n = r = \frac{a}{b}$, then $n \in \mathbb{Q}$,

then $g(n) = 2\sqrt{\frac{a}{b}}$
$= 2\frac{\sqrt{a}}{\sqrt{b}}$

let $a = 2$ and $b = 5$,

$g(\frac{2}{5}) = 2\frac{\sqrt{2}}{\sqrt{5}}$

since $2\sqrt{2}$ and $\sqrt{5} \notin \mathbb{Z}$,
then by definition of rational numbers, $g(\frac{2}{5})$ is not rational,
hence it is not well defined

c) $h(n) = \frac{1}{n^2+1}$

let $a, b \in \mathbb{Z}$ and $r = \mathbb{Q}$ s.t. $r = \frac{a}{b}$ (by definition of rational number)
let $n = r = \frac{a}{b}$, then $n \in \mathbb{Q}$,
Then, $h(n) = \frac{1}{(\frac{a}{b})^2+1}$

$= \frac{1}{\frac{a^2}{b^2} + 1 \cdot \frac{b^2}{b^2}}$

$= \frac{1}{\frac{a^2+b^2}{b^2}}$

$= \frac{b^2}{a^2+b^2}$ by basic algebra

Since $b^2 = b \cdot b \in \mathbb{Z}$ (by closure of integers under multiplication)
then $a^2 \in \mathbb{Z}$ and $a^2+b^2 \in \mathbb{Z}$ (by closure of integers under addition)
Then by definition of rational numbers, $\frac{b^2}{a^2+b^2} \in \mathbb{Q}$, thus $h(n) \in \mathbb{Q}$ //
∴ function is well-defined

d) $k(n) = \lfloor \sin n \rfloor$

let $a, b \in \mathbb{Z}$ and $r = \mathbb{Q}$ s.t. $r = \frac{a}{b}$ (by definition of rational number)

let $n = r = \frac{a}{b}$, then $n \in \mathbb{Q}$,

then $k(n) = \lfloor \sin(\frac{a}{b}) \rfloor$

By definition of $\lfloor\ \rfloor$, $k(n) \in \mathbb{Z}$.

Since $\mathbb{Z} \in \mathbb{Q}$, then $k(n)$ is well defined.

domain codomain

2. Let U be a set and $A \subseteq U$ such that $\varnothing \neq A \neq U$. Define the function $\chi : U \to \mathbb{Z}$ by setting, for all $x \in U$,

$$\chi(x) = \begin{cases} 0, & \text{if } x \notin A; \\ 1, & \text{if } x \in A. \end{cases}$$

image: values

What are the domain and the codomain of χ? Write down $\{\chi(x) : x \in U\}$ in roster notation.

input.

a) $\chi(x) : U \to \mathbb{Z}$

$\therefore$ domain : $\{x \in U \mid x\}$. ✓

codomain = $\{0, 1\}$. × $\mathbb{Z}$

b) $\{\chi(x) : x \in U\} = \{0, 1\}$. ✓

3. **Motivation.** One can represent a function using three sets: its domain, its codomain, and its graph, as defined below.

Fix sets A, B. Define the *graph* of a function $f \colon A \to B$ to be

$$\{(x, f(x)) : x \in A\}.$$

(a) Suppose $A = \{0, 1, 2\}$ and $B = \mathbb{Z}$. Define $f \colon A \to B$ by setting $f(x) = 3x + 1$ for each $x \in A$. Write down the graph of A in roster notation.

(b) Assuming $A \neq \varnothing$, find a subset $S \subseteq A \times B$ that cannot be the graph of any function $A \to B$. $f: A \to B$ x.

(c) Show that a subset $S \subseteq A \times B$ is the graph of a function $A \to B$ if and only if

$$\forall x \in A \quad \exists! y \in B \quad (x, y) \in S.$$

Recall that $\exists!$ means "there exists unique".

a) $\{(0,1), (1,4), (2,7)\}$. ✓

b) $\{(0,1), (1,4)\}$
$\{(0,1), (2,7)\}$
$\{(1,4), (2,7)\}$
$\{(0,1)\}, \{(1,4)\}, \{(2,7)\}$
$\{\}$
$\{(0,3)\}$ ✗

subset of

Cannot be the graph of $f: A \to B$.

$\varnothing$ cannot be graph of $A \to B$.

c) $\forall x \in A \quad \exists! y \in B \quad (x,y) \in S$.
$A = \{0,1,2\}$ $\quad B \in \mathbb{Z}$ $\quad S \subseteq A \times B$ is graph of $f: A \to B$.

Suppose A is the set $\{0,1,2\}$, B is $\mathbb{Z}$, $S \subseteq A \times B$ — prove if
Then $x \in A$ and $y \in B$, — existence
then $(x,y) \in S \subseteq A \times B$. — uniqueness
From (a), $A \times B = \{(0,1), (1,4), (2,7)\}$. — only if
Hence the subset of $A \times B$ is (x,y).
Let $x = 0$,
then $\exists! y = 1$ for $\{(x,y)\} = \{(0,1)\} \subseteq A \times B$.
let $x = 1$,
then $\exists! y = 4$ for $\{(x,y)\} = \{(1,4)\} \subseteq A \times B$.
let $x = 2$,
then $\exists! y = 7$ for $\{(x,y)\} = \{(2,7)\} \subseteq A \times B$.
Therefore, to all over, $\exists! y$ for the statement $\forall x \in A, \exists! y \in B, (x,y) \in S$ to be true

Recall that ∃! means "there exists unique".

4. Consider the relation $\sim$ on $\mathbb{Q}$ defined by setting, for all $x, y \in \mathbb{Q}$,

$$x \sim y \quad \Leftrightarrow \quad x - y \in \mathbb{Z}.$$

It was shown in Question 5 of Tutorial 4 that $\sim$ is an equivalence relation. Define addition and multiplication on $\mathbb{Q}/\sim$ as follows: whenever $[x], [y] \in \mathbb{Q}/\sim$,

set of all equivalence classes

$$[x] + [y] = [x + y] \quad \text{and} \quad [x] \cdot [y] = [x \cdot y].$$

Is $+$ is well defined on $\mathbb{Q}/\sim$? Is $\cdot$ is well defined on $\mathbb{Q}/\sim$? Prove that your answers are correct.

$[x]_\sim = \{ y \in A : x \sim y \}$.

$[x] = \{ y \in \mathbb{Q} : x - y \in \mathbb{Z} \}$.

$[y] = \{ z \in \mathbb{Q} : y - z \in \mathbb{Z} \}$.

eg: $[\frac{1}{3}] = \{ \dots, -\frac{2}{3}, \frac{1}{3}, \frac{4}{3}, \frac{7}{3}, \dots \}$

Lemma 6.4.4

Define addition and multiplication on $\mathbb{Q}/\sim$ as follows:

Whenever $[x], [y] \in \mathbb{Q}/\sim$,

$$[x] + [y] = [x+y] \quad \text{and} \quad [x] \cdot [y] = [x \cdot y]$$

$+$: let $a, b, c, d \in \mathbb{Z}$, then $r, q \in \mathbb{Q}$,

then $r = \frac{a}{b}$ and $q = \frac{c}{d}$ (by defn of $\mathbb{Q}$)

Then, $r - q = \frac{a}{b} - \frac{c}{d}$

$= \frac{ad - bc}{bd} \in \mathbb{Z}$

then $bd = 1$, $b = \frac{1}{d}$. (by basic algebra).

Then, $r = ad$, $q = \frac{c}{d}$

Thus, $[r] + [q] = [ad] + [\frac{c}{d}]$

$= [ad + \frac{c}{d}]$

$= [\frac{a+c}{d}]$

Since $a + c \in \mathbb{Z}$, $d \in \mathbb{Z}$, (by closure of $\mathbb{Z}$ under $+$)

Then $\frac{a+c}{d} \in \mathbb{Q}$ (by definition of $\mathbb{Q}$)

Hence it is well defined on $\mathbb{Q}/\sim$

$(x_1 + y_1) - (x_2 + y_2) = (x_1 - x_2) + (y_1 - y_2)$

$= k + l \in \mathbb{Z}$ ($\mathbb{Z}$ closed)

$x_1 + y_1 \sim x_2 + y_2$

$\cdot$: let $a, b, c, d \in \mathbb{Z}$, then $r, q \in \mathbb{Q}$,

then $r = \frac{a}{b}$ and $q = \frac{c}{d}$ (by defn of $\mathbb{Q}$)

Then, $r - q = \frac{a}{b} - \frac{c}{d}$

$= \frac{ad - bc}{bd} \in \mathbb{Z}$

then $bd = 1$, $b = \frac{1}{d}$. (by basic algebra

Then, $r = ad$, $q = \frac{c}{d}$

Thus, $[r] \cdot [q] = [ad] \cdot [\frac{c}{d}]$

$= [ad \cdot \frac{c}{d}]$

$= [ac]$

Since $ac \in \mathbb{Z}$, (by closure of $\mathbb{Z}$ under $\cdot$)

Then $ac \in \mathbb{Q}$ (by definition of $\mathbb{Q}$)

Hence it is well defined on $\mathbb{Q}/\sim$

not well defined.

by counterexample.

$[\frac{1}{4}] = [\frac{1}{2}] \cdot [\frac{1}{2}] = [\frac{1}{2} \cdot \frac{1}{2}]$

not same.

$[-\frac{1}{4}] = [\frac{1}{2} \cdot -\frac{1}{2}] = [\frac{1}{2}] \cdot [-\frac{1}{2}]$.

R is antisymmetric if $\forall x, y \in A \ (xRy \wedge yRx \Rightarrow x = y)$

5. Prove that the divisibility relation $|$ on $\mathbb{Z}^+$ is antisymmetric.

You may use the following theorem without proof for this question.

Lemma. For all $a, b \in \mathbb{Z}^+$, if $a \mid b$, then $a \leqslant b$.

$\forall a, b \in \mathbb{Z}^+ \ (a | b \to a \leq b)$

Let $a, b \in \mathbb{Z}^+$,

Suppose $a | b$,

then $a \leq b$ ✓ by lemma

Suppose $b | a$,

then $b \leq a$ ✓

Since $b \leq a$ and $a \leq b$, then $a = b$

$\therefore \forall a, b \in \mathbb{Z}^+, (a | b \wedge b | a \Rightarrow a = b)$ ✓

$\therefore$ divisibility relation on $\mathbb{Z}^+$ is antisymmetric.

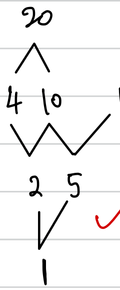
Lemma: For all $a, b \in \mathbb{Z}^+$, if $a \mid b$, then $a \leqslant b$.

6. Consider the "divides" relation on each of the following sets of integers. For each of these, draw a Hasse diagram, find all largest, smallest, maximal and minimal elements.

   (a) $A = \{1, 2, 4, 5, 10, 15, 20\}$.

   (b) $B = \{2, 3, 4, 6, 8, 9, 12, 18\}$.

a)



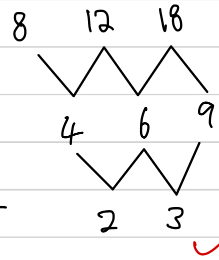
Largest: None.

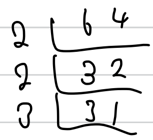
Smallest: 1

Maximal: 20, 15.

Minimal: 1

b)





Largest: None.

Smallest: None

Maximal: 8, 12, 18

Minimal: 2, 3.

7. Consider a set A and a total order $\preccurlyeq$ on A. Show that all minimal elements are smallest.

Let A be a set and R be relation on A.
Since R is a total order, thus R is a partial order
and R is reflexive, transitive and antisymmetric.

$\therefore \forall x, y \in A\ (xRy \wedge yRx \Rightarrow x = y)$

and every pair of element x, y are comparable, $\therefore$

$\forall x, y \in A\ (xRy \vee yRx)$

Let $c \in A$,

c is a minimum element if no $x \in A$ is strictly $\preccurlyeq$ less than c,

$\forall x \in A\ (x \preccurlyeq c \rightarrow c = x)$

c is the smallest element if all $x \in A$ are $\preccurlyeq$ bigger than or equal to c,

$\forall x \in A\ (c \preccurlyeq x)$.

Suppose $c \in A$ is a minimum element, ✓ → pick any $x \in A$ (for sets)

thus $\forall x \in A\ (x \preccurlyeq c \rightarrow c = x)$.

Since A is a total order, then R is a partial order and every pair of elements are comparable

$\therefore \forall x, y \in A\ (x \preccurlyeq y \wedge (xRy \vee yRx))$

Suppose there is more than one minimum element, Total order → $x \leq C$ or $C \leq x$.

Then, let these minimum elements be m.

Then, $\forall m, x \in A\ (m \preccurlyeq x \wedge (mRx \vee xRm))$.

However, m cannot be $m \preccurlyeq x$ since there are multiple m.

$x = C$ $C \leq x$

by minimality of C as $\leq$ is reflexive. $\therefore C \leq x$

all cases

Suppose there is only one minimum element,

Then, $\forall x \in A\ (c \preccurlyeq x \wedge (cRx \vee xRc))$

Since c is the only minimum element, then c is the smallest element (by defn of smallest)

and $c \preccurlyeq x$ is true.

In any case, there must be one minimum to satisfy the defn of total order,

hence all minimum element are smallest (by defn of smallest).

8. **Definition.** Consider a partial order $\preccurlyeq$ on a set A. Let $a, b \in A$.

- We say a, b are *comparable* if $a \preccurlyeq b$ or $b \preccurlyeq a$.
- We say a, b are *compatible* if there exists $c \in A$ such that $a \preccurlyeq c$ and $b \preccurlyeq c$.

(a) Is it true that, in all partially ordered sets, any two comparable elements are compatible? Justify your answer.

(b) Is it true that, in all partially ordered sets, any two compatible elements are comparable? Justify your answer.

a) $\forall a, b \in A\, ((a \preccurlyeq b \lor b \preccurlyeq a) \to \exists c \in A\, (a \preccurlyeq c \land b \preccurlyeq c))$

Suppose $a, b, c \in A$; and $a \preccurlyeq b \lor b \preccurlyeq a$,

if $a \preccurlyeq c$,

then either $a \preccurlyeq c \preccurlyeq b$ or $a \preccurlyeq b \preccurlyeq c$. or $b \preccurlyeq a \preccurlyeq c$

$\therefore \exists c$ s.t. $b \preccurlyeq c$.

if $b \preccurlyeq c$,

then either $b \preccurlyeq a \preccurlyeq c$ or $a \preccurlyeq b \preccurlyeq c$. or $b \preccurlyeq c \preccurlyeq a$.

$\therefore \exists c$ s.t. $a \preccurlyeq c$.

$\therefore$ Statement is true.

$a \leq b = c$ by assumption

use reflexivity

b) $\forall a, b \in A\, (\exists c \in A\, (a \preccurlyeq c \land b \preccurlyeq c) \to (a \preccurlyeq b \lor b \preccurlyeq a))$

Not true.

Suppose $a, b, c \in A$ and $a \preccurlyeq c \land b \preccurlyeq c$,

then, $a \preccurlyeq b \preccurlyeq c$ or $b \preccurlyeq a \preccurlyeq c$,

$\therefore a \preccurlyeq b$ or $b \preccurlyeq a$.

$\therefore$ Statement is true.

divides relation

2 and 3 are compatible but not comparable

$2 \mid 6$ and $3 \mid 6$

$2 \nmid 3$ and $3 \nmid 2$.

9. Let $A = \{a, b, c, d\}$, where a, b, c, d are mutually distinct. Consider the following partial order on A:

$$R = \{(a,a), (a,b), (a,c), (a,d), (b,b), (b,c), (c,c), (d,d)\}.$$

(a) Draw a Hasse diagram of R.

(b) Draw Hasse diagrams of all the linearizations of R.

R is relation on A.

$\forall x, y \in A \, (xRy \wedge yRx \rightarrow x = y)$

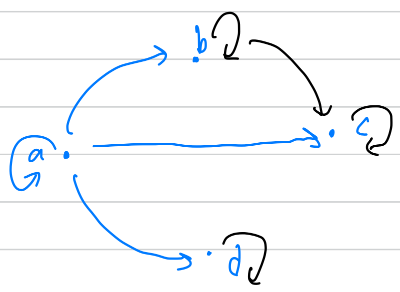
R is reflexive, transitive,

antisymmetric.

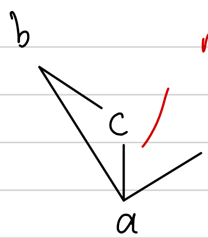
x, y is comparable:

xRy or yRx.

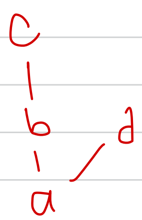
$(A, R) \Rightarrow$ partially ordered set



a)



related a/c by indirectly



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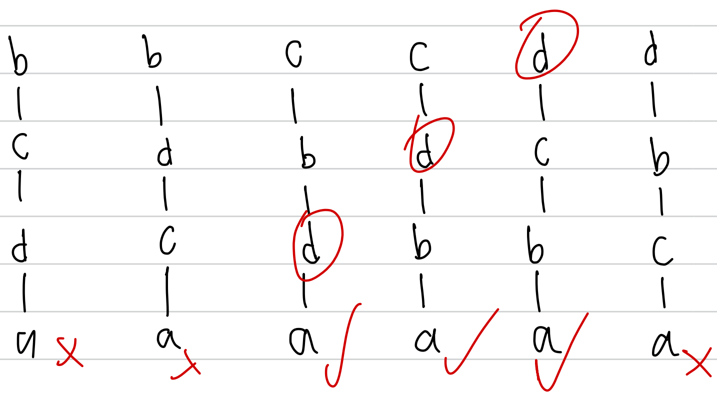


CS1231S Tutorial 5: Functions and partial orders Solutions

National University of Singapore

2021/22 Semester 1

1. (a) Define $f\colon \mathbb{Q} \to \mathbb{Q}$ by setting $f(n) = \pm n$ for each $n \in \mathbb{Q}$.
   (b) Define $g\colon \mathbb{Q} \to \mathbb{Q}$ by setting $g(n) = 2\sqrt{n}$ for each $n \in \mathbb{Q}$.
   (c) Define $h\colon \mathbb{Q} \to \mathbb{Q}$ by setting $h(n) = \frac{1}{n^2+1}$ for each $n \in \mathbb{Q}$.
   (d) Define $k\colon \mathbb{Q} \to \mathbb{Q}$ by setting $k(n) = \lfloor \sin n \rfloor$ for each $n \in \mathbb{Q}$.

   Which of f, g, h, k are well defined? Which of them are not?

   Solution. The functions h, k are well defined, but f, g are not.

   Additional information. The function f is not well defined because it is not clear from the definition whether $f(1)$ is 1, -1 or both. The function g is not well defined because $1/2 \in \mathbb{Q}$ but according to the definition,

   $$g\Big(\frac{1}{2}\Big) = 2\sqrt{\frac{1}{2}} = \frac{2}{\sqrt{2}} = \sqrt{2} \notin \mathbb{Q}.$$

2. Let U be a set and $A \subseteq U$ such that $\varnothing \neq A \neq U$. Define the function $\chi\colon U \to \mathbb{Z}$ by setting, for all $x \in U$,

   $$\chi(x) = \begin{cases} 0, & \text{if } x \notin A; \\ 1, & \text{if } x \in A. \end{cases}$$

   What are the domain and the codomain of χ? Write down $\{\chi(x) : x \in U\}$ in roster notation.

   Solution. The domain is U. The codomain is $\mathbb{Z}$. One can show that $\{\chi(x) : x \in U\} = \{0, 1\}$.

   Additional information. One can rewrite the displayed part of the definition of χ as

   $$(x \notin A \Rightarrow \chi(x) = 0) \wedge (x \in A \Rightarrow \chi(x) = 1).$$

3. Fix sets A, B. Define the *graph* of a function $f\colon A \to B$ to be

   $$\{(x, f(x)) : x \in A\}.$$

   (a) Suppose $A = \{0, 1, 2\}$ and $B = \mathbb{Z}$. Define $f\colon A \to B$ by setting $f(x) = 3x + 1$ for each $x \in A$. Write down the graph of A in roster notation.
   (b) Assuming $A \neq \varnothing$, find a subset $S \subseteq A \times B$ that cannot be the graph of any function $A \to B$.
   (c) Show that a subset $S \subseteq A \times B$ is the graph of a function $A \to B$ if and only if

   $$\forall x \in A \quad \exists! y \in B \quad (x, y) \in S.$$

   Recall that $\exists!$ means “there exists unique”.

Solution.

(a) The graph of f is $\{(0,1),(1,4),(2,7)\}$.

(b) We claim that $S = \varnothing$ has the required property.

1. We prove this by contradiction.
   1.1. Suppose $\varnothing$ is the graph of the function $f\colon A \to B$.
   1.2. Since $A \neq \varnothing$, it has an element, say x.
   1.3. Then $(x, f(x)) \in \varnothing$ by the definition of graphs.
   1.4. This contradicts the fact that $\varnothing$ has no element.
2. So $\varnothing$ cannot be the graph of any function $A \to B$. □

(c) 1. ("Only if")
   1.1. Suppose S is the graph of a function $f\colon A \to B$.
   1.2. Pick any $x \in A$.
   1.3. ("Existence part")
      1.3.1. Then $f(x) \in B$ as B is the codomain of f.
      1.3.2. As S is the graph of f, we know $(x, f(x)) \in S$.
      1.3.3. So $(x, y) \in S$ for some $y \in B$.
   1.4. ("Uniqueness part")
   It suffices to show that $\forall y \in B\ \big((x,y) \in S \Rightarrow y = f(x)\big)$.
      1.4.1. Let $y \in B$ such that $(x, y) \in S$.
      1.4.2. As S is the graph of f, we know $y = f(x)$.
   1.5. So there is a unique $y \in B$ such that $(x, y) \in S$.
2. ("If")
   2.1. Suppose $\forall x \in A\ \exists! y \in B\ \ (x, y) \in S$.
   2.2. Define $f\colon A \to B$ by setting $f(x)$ to be the unique $y \in B$ such that $(x, y) \in S$, for every $x \in A$.
   2.3. This f is well defined by line 2.1.
   2.4. By the definition of f, for all $(x, y) \in A \times B$,

$$(x, y) \in S \quad \Leftrightarrow \quad y = f(x).$$

   2.5. So S is indeed the graph of f. □

4. Consider the relation $\sim$ on $\mathbb{Q}$ defined by setting, for all $x, y \in \mathbb{Q}$,

$$x \sim y \quad \Leftrightarrow \quad x - y \in \mathbb{Z}.$$

It was shown in Question 5 of Tutorial 4 that $\sim$ is an equivalence relation. Define addition and multiplication on $\mathbb{Q}/{\sim}$ as follows: whenever $[x], [y] \in \mathbb{Q}/{\sim}$,

$$[x] + [y] = [x + y] \quad \text{and} \quad [x] \cdot [y] = [x \cdot y].$$

Is $+$ is well defined on $\mathbb{Q}/{\sim}$? Is $\cdot$ is well defined on $\mathbb{Q}/{\sim}$? Prove that your answers are correct.

Solution. We claim that $+$ is well defined on $\mathbb{Q}/{\sim}$.

1. Let $[x_1], [y_1], [x_2], [y_2] \in \mathbb{Q}/{\sim}$ such that $[x_1] = [x_2]$ and $[y_1] = [y_2]$.
2. Then Lemma 6.4.4 implies $x_1 \sim x_2$ and $y_1 \sim y_2$.
3. Use the definition of $\sim$ to find $k, \ell \in \mathbb{Z}$ such that $x_1 - x_2 = k$ and $y_1 - y_2 = \ell$.
4. Note $(x_1 + y_1) - (x_2 + y_2) = (x_1 - x_2) + (y_1 - y_2) = k + \ell \in \mathbb{Z}$ as $\mathbb{Z}$ is closed under $+$.
5. So the definition of $\sim$ tells us $x_1 + y_1 \sim x_2 + y_2$.
6. Hence $[x_1] + [y_1] = [x_1 + y_1] = [x_2 + y_2] = [x_2] + [y_2]$ by Lemma 6.4.4. □

We claim that $\cdot$ is not well defined on $\mathbb{Q}/{\sim}$.

1. Note that $\frac{1}{2} - \frac{-1}{2} = 1 \in \mathbb{Z}$ and $\frac{1}{4} - \frac{-1}{4} = \frac{1}{2} \notin \mathbb{Z}$.
2. So $[\frac{1}{2}] = [\frac{-1}{2}]$ and $[\frac{1}{4}] \neq [\frac{-1}{4}]$ by Lemma 6.4.4.

3. Therefore, according to the definition of $\cdot$ on $\mathbb{Q}/{\sim}$,

$$\left[\frac{1}{2}\right] \cdot \left[\frac{1}{2}\right] = \left[\frac{1}{2} \cdot \frac{1}{2}\right] = \left[\frac{1}{4}\right] \neq \left[\frac{-1}{4}\right] = \left[\frac{1}{2} \cdot \frac{-1}{2}\right] = \left[\frac{1}{2}\right] \cdot \left[\frac{-1}{2}\right]. \qquad \square$$

5. Prove that the divisibility relation $\mid$ on $\mathbb{Z}^+$ is antisymmetric.

   You may use the following theorem without proof for this question.

   Lemma. For all $a, b \in \mathbb{Z}^+$, if $a \mid b$, then $a \leqslant b$.

   Solution.

   1. Let $a, b \in \mathbb{Z}^+$ such that $a \mid b$ and $b \mid a$.
   2. Then the lemma given in the question tells us $a \leqslant b$ and $b \leqslant a$.
   3. So $a = b$. $\square$

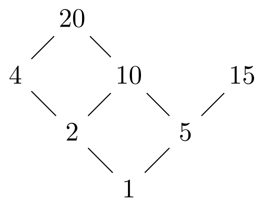
6. Consider the "divides" relation on each of the following sets of integers. For each of these, draw a Hasse diagram, find all largest, smallest, maximal and minimal elements.

   (a) $A = \{1, 2, 4, 5, 10, 15, 20\}$.

   (b) $B = \{2, 3, 4, 6, 8, 9, 12, 18\}$.

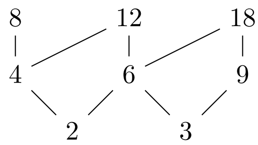
   Solution.

   (a)



   1 is the only minimal element and is the smallest element. 15 and 20 are maximal elements. There is no largest element.

   (b)



   2 and 3 are minimal elements. 8, 12 and 18 are maximal elements. There is no largest element. There is no smallest element.

7. Consider a set A and a total order $\preccurlyeq$ on A. Show that all minimal elements are smallest.

   Solution.

   1. Let $c \in A$ that is minimal with respect to $\preccurlyeq$.
   2. Pick any $x \in A$.
   3. As $\preccurlyeq$ is a total order, either $x \preccurlyeq c$ or $c \preccurlyeq x$.
   4. Case 1: suppose $x \preccurlyeq c$.
      4.1. Then $x = c$ by the minimality of c.
      4.2. So the reflexivity of $\preccurlyeq$ tells us $c \preccurlyeq x$.
   5. Case 2: suppose $c \preccurlyeq x$.
      5.1. Then clearly $c \preccurlyeq x$.
   6. So $c \preccurlyeq x$ in all cases. $\square$

8. **Definition.** Consider a partial order $\preccurlyeq$ on a set A. Let $a, b \in A$.

   - We say a, b are *comparable* if $a \preccurlyeq b$ or $b \preccurlyeq a$.
   - We say a, b are *compatible* if there exists $c \in A$ such that $a \preccurlyeq c$ and $b \preccurlyeq c$.

   (a) Is it true that, in all partially ordered sets, any two comparable elements are compatible? Justify your answer.

(b) Is it true that, in all partially ordered sets, any two compatible elements are comparable? Justify your answer.

Solution.

(a) Yes. If a and b are comparable, then either $a \preccurlyeq b$ or $b \preccurlyeq a$. In the former case, we have $a \preccurlyeq b$ and $b \preccurlyeq b$ by the reflexivity of $\preccurlyeq$, and so a and b are compatible. In the latter case, we have $a \preccurlyeq a$ and $b \preccurlyeq a$ by the reflexivity of $\preccurlyeq$, and so a and b are compatible.

(b) No. Consider the "divides" relation $|$ on $\mathbb{Z}^+$. This is a partial order on $\mathbb{Z}^+$. We know $2 \mid 6$ and $3 \mid 6$. So 2 and 3 are compatible. However, we also know that $2 \nmid 3$ and $3 \nmid 2$. So 2 and 3 are not comparable.

9. Let $A = \{a, b, c, d\}$, where a, b, c, d are mutually distinct. Consider the following partial order on A:

$$R = \{(a,a), (a,b), (a,c), (a,d), (b,b), (b,c), (c,c), (d,d)\}.$$

(a) Draw a Hasse diagram of R.

(b) Draw Hasse diagrams of all the linearizations of R.

Solution.

(a)

$$\begin{array}{ccc} c & & \\ | & & \\ b & & d \\ | & / & \\ a & & \end{array}$$

Note that Hasse diagrams have no loop and no arrowhead. There is no edge between a and c because there is b in between.

(b)

$$\begin{array}{c} d \\ | \\ c \\ | \\ b \\ | \\ a \end{array} \qquad\qquad \begin{array}{c} c \\ | \\ d \\ | \\ b \\ | \\ a \end{array} \qquad\qquad \begin{array}{c} c \\ | \\ b \\ | \\ d \\ | \\ a \end{array}$$