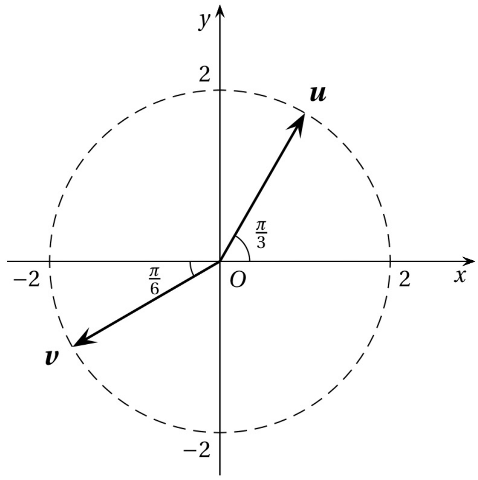
1. Find the vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{u}+\boldsymbol{v}$ and $3\boldsymbol{u}-2\boldsymbol{v}$ in $\mathbb{R}^2$ where $\boldsymbol{u}$ and $\boldsymbol{u}$ and $\boldsymbol{v}$ are shown in the figure below.



$u = (1, \sqrt{3})$

$V = (-\sqrt{3}, -1)$

$u+v = (1-\sqrt{3}, -1+\sqrt{3})$

$3u - 2v = (3+2\sqrt{3}, 2+3\sqrt{3})$.

2. Express each of the following by the set notation in both implicit and explicit form:

(a) The line in $\mathbb{R}^2$ passing through the points $(1,2)$ and $(2,-1)$.

(b) The plane in $\mathbb{R}^3$ containing the points $(0,1,-1)$, $(1,-1,0)$ and $(0,2,0)$.

(c) The line in $\mathbb{R}^3$ through the points $(0,1,-1)$ and $(1,-1,0)$.

a)

$(x, y) = (1,2)$ and $(2,-1)$

$\rightarrow \; ax + by = c.$

$$\begin{cases} a + 2b - c = 0 \\ 2a - b + c = 0. \end{cases}$$

$a = \frac{3}{5}c, \; b = \frac{1}{5}c.$

$\therefore \{(x, y) \mid 3x + y = 5\}$ (implicit).

$\{(\frac{5-t}{3}, t) \mid t \in \mathbb{R}\}$ (explicit)

3. Consider the following subsets of $\mathbb{R}^3$:

$$A = \text{a line passes through the origin and } (9,9,9),$$
$$B = \{(k,k,k) \mid k \in \mathbb{R}\},$$
$$C = \{(x_1, x_2, x_3) \mid x_1 = x_2 = x_3\},$$
$$D = \{(x,y,z) \mid 2x - y - z = 0\},$$
$$E = \{(a,b,c) \mid 2a - b - c = 0 \text{ and } a + b + c = 0\},$$
$$F = \{(u,v,w) \mid 2u - v - w = 0 \text{ and } 3u - 2v - w = 0\}.$$

Which of these subsets are the same?

A = B = C = F

D

E

4. Let U, V, W be three planes in $\mathbb{R}^3$ where

$$U = \{(x,y,z) \mid 2x - y + 3z = 0\}, \quad ✓$$
$$V = \{(x,y,z) \mid 3x + 2y - z = 0\}, \quad ✓$$
$$W = \{(x,y,z) \mid x - 3y - 2z = 1\}. \quad ✗$$

(a) Determine which of U, V, W contain the origin.

U, V, not W.

5. Let $A = \{(1+t, 1+2t, 1+3t) \mid t \in \mathbb{R}\}$ be a subset of $\mathbb{R}^3$.

$x+y-z=1$
$x-2y+z=0$

$\begin{pmatrix} 1 & 1 & -1 \\ 1 & -2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(a) Describe A geometrically.

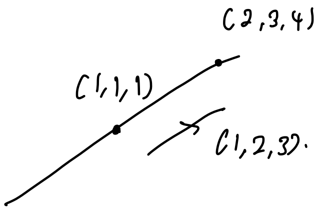
(b) Show that $A = \{(x, y, z) \mid x + y - z = 1 \text{ and } x - 2y + z = 0\}$.

(c) Write down a matrix equation $\boldsymbol{Mx} = \boldsymbol{b}$ where $\boldsymbol{M}$ is a 3×3 matrix and $\boldsymbol{b}$ is a 3×1 matrix such that its solution set is A.

c. $\begin{pmatrix} 1 & 1 & -1 \\ 1 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

5. a. $A : (1,1,1) + t(1,2,3)$

Geometrically:



b. $S = \{(x,y,z) \mid x+y-z=1,\ x-2y+z=0\}$

plane   plane

2 diff plane

$\therefore$ S is a line

$A = S$?

$A \subseteq S \wedge S \subseteq A$

$\Downarrow$

$A = S$. $\cancel{A \neq S}$.

Only possible scenario.

$A: \begin{pmatrix} 1+t \\ 1+2t \\ 1+3t \end{pmatrix} \subseteq S$? ✓

① $x+y-z=1$?

$(1+t) + (1+2t) - (1+3t) = 0$

② $x-2y+z=0$?

$(1+t) - 2(1+2t) + (1+3t) = 0$.

6. Determine whether the following subsets of $\mathbb{R}^4$ are equal to each other.

$$S = \{(p, q, p, q) \mid p, q \in \mathbb{R}\},$$

$$T = \{(x, y, z, w) \mid x + y - z - w = 0\},$$

$$V = \left\{ (a, b, c, d) \left| \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ a & b & c & d \end{vmatrix} = 0 \right. \right\}.$$

Briefly explain why one subset is equal (or not equal) to another subset.

$$\begin{vmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ b & c & d \end{vmatrix} + \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ a & b & d \end{vmatrix}$$

$$= a + b - d - c.$$

$\therefore$ $T = V$.

7. Let P represent a plane in $\mathbb{R}^3$ with equation $x - y + z = 1$ and A, B, C represent three different lines given by the following set notation:

$$A = \{(a, a, 1) \mid a \in \mathbb{R}\}, \quad B = \{(b, 0, 0) \mid b \in \mathbb{R}\}, \quad C = \{(c, 0, -c) \mid c \in \mathbb{R}\}.$$

(a) Express the plane P in explicit set notation.

(b) Does any of the three lines above lie completely on the plane P? Briefly explain your answer.

(c) Find all the points of intersection of the line B with the plane P.

(d) Find the equation of another plane that is parallel to (but not overlapping) the plane P, and contains exactly one of the three lines above.

(e) Can you find a nonzero linear system whose solution set contains all the three lines? Justify your answer.

a) $\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \end{array}\right)$ REF -

$\uparrow\ \uparrow\ \uparrow$

P N N

$y = s$, $z = t$.

$x = y - z + 1$

$= 1 + s - t$

$\therefore\ P = \{(1+s-t, s, t) \mid (s,t) \in \mathbb{R}^2\}$

b) $A \subseteq P$? ✓ $B \subseteq P$? X $C \subseteq P$? X

$(a, a, 1)$ → $a - a + 1 = 1$

$(b, 0, 0)$

$(c, 0, -c)$ → $c - 0 + (-c) = 0 \neq 1$

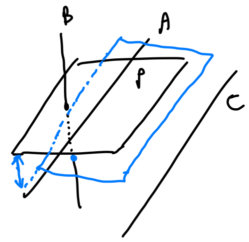
$x - y + z = 1$?

$b - 0 + 0 = \underline{\underline{b = 1}}$?

$(1, 0, 0)$ intersect P.

c)

d)



P: $x - y + z = 1$

P': $x - y + z = 0$ ✓

LHS: $c - 0 + (-c) = 0$

$(c, 0, -c)$

e) linear system (non-zero)

- point
- line
- plane

A: diagonal line between x, y axes $z = 1$

B: x axis

C: diag line between x, z axes

plane AB = xy plane with $\underline{\underline{z = 1}}$.

8. Let $\boldsymbol{u}_1 = (2,1,0,3)$, $\boldsymbol{u}_2 = (3,-1,5,2)$, and $\boldsymbol{u}_3 = (-1,0,2,1)$. Which of the following vectors are linear combinations of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$?

(a) $(2,3,-7,3)$, (b) $(0,0,0,0)$, (c) $(1,1,1,1)$, (d) $(-4,6,-13,4)$.

Is there C_1, C_2, C_3 st

$C_1 U_1 + C_2 U_2 + C_3 U_3 = \text{target}$

$(U_1, U_2, U_3)\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \text{target}$

$\begin{pmatrix} 2 & 3 & -1 \\ 1 & -1 & 0 \\ 0 & 5 & 2 \\ 3 & 2 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \text{target}?$

↳ REF

(find operation)

$\begin{pmatrix} 2 & 3 & -1 \\ 1 & -1 & 0 \\ 0 & 5 & 2 \\ 3 & 2 & 1 \end{pmatrix} \xrightarrow{R_2 - \frac{1}{2}R_1} \begin{pmatrix} 2 & 3 & 1 \\ 0 & -5/2 & 1/2 \\ 0 & 5 & 2 \\ 3 & 2 & 1 \end{pmatrix}$

$\xrightarrow{2R_2} \begin{pmatrix} 2 & 3 & 1 \\ 0 & -5 & 1 \\ 0 & 5 & 2 \\ 3 & 2 & 1 \end{pmatrix}$

$\xrightarrow{R_3 + R_2} \begin{pmatrix} 2 & 3 & 1 \\ 0 & -5 & 1 \\ 0 & 0 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

$\xrightarrow{R_4 - \frac{3}{2}R_1} \begin{pmatrix} 2 & 3 & 1 \\ 0 & -5 & 1 \\ 0 & 0 & 3 \\ 0 & -5/2 & 5/2 \end{pmatrix}$

$\xrightarrow{2R_4} \begin{pmatrix} 2 & 3 & 1 \\ 0 & -5 & 1 \\ 0 & 0 & 3 \\ 0 & -5 & 5 \end{pmatrix}$

$\xrightarrow{R_4 - R_2} \begin{pmatrix} 2 & 3 & 1 \\ 0 & -5 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 4 \end{pmatrix}$

$\xrightarrow{R_4 - \frac{4}{3}R_3} \begin{pmatrix} 2 & 3 & -1 \\ 0 & -5 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$ REF

a) $\begin{pmatrix} 2 \\ 3 \\ -7 \\ 3 \end{pmatrix}$ ... same operation. $\xrightarrow[R_2 - \frac{1}{2}R_1]{} \begin{pmatrix} 2 \\ 4 \\ -3 \\ 3 \end{pmatrix}$

$2R_2$

$\xrightarrow[R_4 - \frac{3}{2}R_1]{R_3 + R_2} \begin{pmatrix} 2 \\ 4 \\ -3 \\ 0 \end{pmatrix}$

$\xrightarrow[R_4 - R_2]{2R_4} \begin{pmatrix} 2 \\ 4 \\ -3 \\ -4 \end{pmatrix}$

$\xrightarrow{R_4 - \frac{4}{3}R_3} \begin{pmatrix} 2 \\ 4 \\ -3 \\ 0 \end{pmatrix}$

(a) (b) (c) (d).

$\left(\begin{array}{ccc|c|c|c|c} 2 & 3 & -1 & 2 & 0 & 1 & -4 \\ 0 & -5 & 1 & 4 & 0 & 1 & 16 \\ 0 & 0 & 3 & -3 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & -\frac{4}{3} & 0 \end{array}\right)$

N

∴ consistent.

P?!

∴ non consistent

consistent.

b) $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{\cdots} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

RHS 0 → system always consistent
since its homogeneous.

c) $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \xrightarrow[\text{all ops}]{\cdots} \begin{pmatrix} 1 \\ 1 \\ 2 \\ \boxed{-14/3} \end{pmatrix}$

d) $\begin{pmatrix} -4 \\ 6 \\ 15 \\ 4 \end{pmatrix} \xrightarrow{\text{all ops}} \begin{pmatrix} -4 \\ 16 \\ 3 \\ 0 \end{pmatrix}$

9. For each of the following sets, determine whether the set spans $\mathbb{R}^3$.

(a) $S_1 = \{(1,1,-1),(-2,2,1)\}$. ✗

(b) $S_2 = \{(1,1,-1),(-2,-2,2)\}$. ✗

(c) $S_3 = \{(1,1,-1),(-2,2,1),(1,5,-2)\}$. ✗

(d) $S_4 = \{(1,1,-1),(-2,2,1),(4,0,3)\}$. ✓

(e) $S_5 = \{(1,1,-1),(-2,2,1),(1,5,-2),(0,8,-2)\}$. ✗

(f) $S_6 = \{(1,1,-1),(-2,2,1),(4,0,3),(2,6,-3)\}$. ✓

10. Let $V = \{(x, y, z) \mid x - y - z = 0\}$ be a subset of $\mathbb{R}^3$.

(a) Let $S = \{(1,1,0), (5,2,3)\}$. Show that $\text{span}(S) = V$.

(b) Let $S' = \{(1,1,0), (5,2,3), (0,0,1)\}$. Show that $\text{span}(S') = \mathbb{R}^3$.

a) $\text{span}(S) \subseteq V \rightarrow c_1\begin{pmatrix}1\\1\\0\end{pmatrix}$ $x-y-z = 1-1-0$ $=0$. ✓

$V \subseteq \text{span}(S)$ $c_2\begin{pmatrix}5\\2\\3\end{pmatrix}$ $x-y-z = 5-2-3$ $=0$. ✓

explicit form : $y = s$

$z = t$

$x = y+z$

$x = s+t$

$V = \{(s+t, s, t) \mid (s,t) \in \mathbb{R}^2\}$

Are there c_1, c_2 s.t.

$c_1\begin{pmatrix}1\\1\\0\end{pmatrix} + c_2\begin{pmatrix}5\\2\\3\end{pmatrix} = \begin{pmatrix}s+t\\s\\t\end{pmatrix}$ $\forall s,t \in \mathbb{R}^2$

$\begin{pmatrix}1 & 5\\1 & 2\\0 & 3\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}s+t\\s\\t\end{pmatrix}$

↳ REF.

$\left(\begin{array}{cc|c}1 & 5 & s+t\\1 & 2 & s\\0 & 3 & t\end{array}\right)$

$\xrightarrow[R_3+R_2]{R_2-R_1} \left(\begin{array}{cc|c}1 & 5 & s+t\\0 & -3 & -t\\0 & 0 & 0\end{array}\right)$

$\therefore$ Always Consistent.

$\therefore \text{span}(S) = V$

b) find c_1, c_2, c_3 s.t

$$c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \forall x, y, z.$$

(to show $\mathbb{R}^3 \subseteq \operatorname{span}(S')$)

$\operatorname{span}(S') \subseteq \mathbb{R}^3$. ✓

$$\left(\begin{array}{ccc|c} 1 & 5 & 0 & \cdot \\ 1 & 2 & 0 & \cdot \\ 0 & 3 & 1 & \cdot \end{array}\right) \xrightarrow[R_3 + R_2]{R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 5 & 0 & \cdot \\ 0 & -3 & 0 & \cdot \\ 0 & 0 & 1 & \cdot \end{array}\right)$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ p & p & p \end{array}$$

$\therefore$ Consistent.

11. Determine whether span$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ = span$\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ if

(a) $\boldsymbol{u}_1 = (2,-2,0)$, $\boldsymbol{u}_2 = (-1,1,-1)$, $\boldsymbol{u}_3 = (0,0,9)$, $\boldsymbol{v}_1 = (1,-1,-5)$, $\boldsymbol{v}_2 = (0,1,1)$.

(b) $\boldsymbol{u}_1 = (1,6,4)$, $\boldsymbol{u}_2 = (2,4,-1)$, $\boldsymbol{u}_3 = (-1,2,5)$, $\boldsymbol{v}_1 = (1,-2,-5)$, $\boldsymbol{v}_2 = (0,8,9)$.

a)

$$\left(\begin{array}{cc|c} 1 & 0 & 2 \\ -1 & 1 & -2 \\ 5 & 1 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 10 \end{array}\right)$$

Since $u_1 \notin$ span$\{V_1, V_2\}$, span$\{U_1, U_2, U_3\} \neq$ span$\{V_1, V_2\}$.

b)

$$\left(\begin{array}{ccc|c|c} 1 & 2 & -1 & 1 & 0 \\ 6 & 4 & 2 & -2 & 8 \\ 4 & -1 & 5 & -5 & 9 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c|c} 1 & 2 & -1 & 1 & 0 \\ 0 & -8 & 8 & -8 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\therefore$ The system are consistent $\therefore$ span$\{V_1, V_2\} \subseteq$ span$\{U_1, U_2, U_3\}$

$$\left(\begin{array}{cc|c|c|c} 1 & 0 & 1 & 2 & -1 \\ -2 & 8 & 6 & 4 & 2 \\ -5 & 9 & 4 & -1 & 5 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cc|c|c|c} 1 & 0 & 1 & 2 & -1 \\ 0 & 8 & 8 & 8 & 0 \\ 0 & 0 & 0 & 10 & 0 \end{array}\right)$$

$\therefore$ The systems are consistent $\therefore$ span$\{U_1, U_2, U_3\} \subseteq$ span$\{V_1, V_2\}$.

$\therefore$ span$\{V_1, V_2\}$ = span$\{U_1, U_2, U_3\}$.

12. Let $\boldsymbol{u}_1 = (2,0,2,-4)$, $\boldsymbol{u}_2 = (1,0,2,5)$, $\boldsymbol{u}_3 = (0,3,6,9)$, $\boldsymbol{u}_4 = (1,1,2,-1)$, $\boldsymbol{v}_1 = (-1,2,1,0)$, $\boldsymbol{v}_2 = (3,1,4,0)$, $\boldsymbol{v}_3 = (0,1,1,3)$, $\boldsymbol{v}_4 = (-4,3,-1,6)$. Determine if the following are true.

(a) $\operatorname{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4\} \subseteq \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$.

(b) $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\} \subseteq \operatorname{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4\}$. → same ✓

(c) $\operatorname{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4\} = \mathbb{R}^4$.

(d) $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\} = \mathbb{R}^4$. → no. If true, then (a) would be true.

a) For each $\underbrace{u_1, u_2, u_3, u_4}_{\text{target vector}}$, we can find

c_1, c_2, c_3, c_4 s.t. $c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = \text{target}$.

$$\begin{pmatrix} -1 & 3 & 0 & -4 \\ 2 & 1 & 1 & 3 \\ 1 & 4 & 1 & -1 \\ 0 & 0 & 3 & 6 \end{pmatrix}$$

1. $R_2 + 2R_1$
2. $R_3 + R_1$
3. $R_3 - R_2$
4. $R_4 \leftrightarrow R_2$.

$$\longrightarrow \left(\begin{array}{cccc|c|c} -1 & 3 & 0 & -4 & 2 & 1 \\ 0 & 7 & 1 & -5 & 4 & 2 \\ 0 & 0 & 3 & 6 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 & -7 \end{array}\right)$$

N ∴ u_1 is consistent.

P ∴ u_2 is non-consistent

$$u_1 \xrightarrow[\text{ops.}]{\text{all}} \begin{pmatrix} 2 \\ 4 \\ -4 \\ 0 \end{pmatrix}$$

$$u_2 \xrightarrow[\text{ops}]{\text{all}} \begin{pmatrix} 1 \\ 2 \\ 7 \\ 2 \end{pmatrix}$$

not in span of V_r

∴ Stop for u_1 and u_2.

13. Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ be vectors in $\mathbb{R}^3$ and let

$$\cancel{S_1 = \{\boldsymbol{u}, \boldsymbol{v}\}}, \quad \cancel{S_2} = \{\boldsymbol{u}-\boldsymbol{v}, \boldsymbol{v}-\boldsymbol{w}, \boldsymbol{w}-\boldsymbol{u}\}, \quad S_3 = \{\boldsymbol{u}-\boldsymbol{v}, \boldsymbol{v}-\boldsymbol{w}, \boldsymbol{u}+\boldsymbol{w}\}, \checkmark$$
$$S_4 = \{\boldsymbol{u}, \boldsymbol{u}+\boldsymbol{v}, \boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w}\}, \checkmark S_5 = \{\boldsymbol{u}+\boldsymbol{v}, \boldsymbol{v}+\boldsymbol{w}, \boldsymbol{u}+\boldsymbol{w}, \boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w}\}. \checkmark$$

Suppose $n = 3$ and $\text{span}\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\} = \mathbb{R}^3$. Determine which of the sets above span $\mathbb{R}^3$.

$$w - u = -(u-v) - (v-w)$$

$\text{span}(S_2) = \text{span}\{u-v, v-w\}$ $\therefore$ does not span $\mathbb{R}^3$.

$$u = \tfrac{1}{2}[(u-v) + (v-w) + (u+w)].$$
$$v = \tfrac{1}{2}[-(u-v) + (v-w) + (u+w)].$$
$$w = \tfrac{1}{2}[-(u-v) - (v-w) + (u+w)].$$

14. Determine which of the following statements are true. Justify your answer.

(a) If $\boldsymbol{u}$ is a nonzero vector in $\mathbb{R}^1$, then $\text{span}\{\boldsymbol{u}\} = \mathbb{R}^1$.

(b) If $\boldsymbol{u}, \boldsymbol{v}$ are nonzero vectors in $\mathbb{R}^2$ such that $\boldsymbol{u} \neq \boldsymbol{v}$, then $\text{span}\{\boldsymbol{u}, \boldsymbol{v}\} = \mathbb{R}^2$.

(c) If S_1 and S_2 are two subsets of $\mathbb{R}^n$, then $\text{span}(S_1 \cap S_2) = \text{span}(S_1) \cap \text{span}(S_2)$.

(d) If S_1 and S_2 are two subsets of $\mathbb{R}^n$, then $\text{span}(S_1 \cup S_2) = \text{span}(S_1) \cup \text{span}(S_2)$.

a) True.

let $\vec{u} = (u)$ for $u \neq 0$,

then for any $(c) \in \mathbb{R}^1$, $(c) = \frac{c}{u} \vec{u}$.

b) False.

let $u = (1,1)$, $v = (2,2)$, $v = ku$

c) False

let $S_1 = \{(1,0), (0,1)\}$, $S_2 = \{(1,0), (0,2)\}$.

d) False.

$S_1 = \{(1,0)\}$ $S_2 = \{(0,1)\}$.

15. Determine which of the following are subspaces of $\mathbb{R}^3$. Justify your answer.

(a) $\{(0,0,0)\}$. Yes.

(b) $\{(1,1,1)\}$. No zero vector

(c) $\{(0,0,0),(1,1,1)\}$. No $\rightarrow (2,2,2)$?

(d) $\{(0,0,c) \mid c$ is an integer$\}$. No $(0,0,\frac{1}{2})$?

(e) $\{(0,0,c) \mid c$ is a real number$\}$. Yes.

(f) $\{(1,1,c) \mid c$ is a real number$\}$. No zero vector.

(g) $\{(a,b,c) \mid a,b,c$ are real numbers and $abc=0\}$. No $(1,1,0)+(0,0,1)=\underline{(1,1,1)} \notin S$.

(h) $\{(a,b,c) \mid a,b,c$ are real numbers and $a \geq b \geq c\}$. No. $-(3,2,1)$ not in set

(i) $\{(a,b,c) \mid a,b$ are real numbers and $4a=3b\}$. yes $\Rightarrow$ similar to homogeneous system.

(j) $\{(a,b,b) \mid a,b$ are real numbers$\}$. Yes

(k) $\{(a,b,ab) \mid a,b$ are real numbers$\}$. No.

$$(1,1,1)+(2,2,4) = \underline{(3,3,5)} \notin S.$$

16. Determine which of the following are subspaces of $\mathbb{R}^4$. Justify your answers.

(a) $\{(w,x,y,z) \mid w+x=y+z\}$. Solution to a homogeneous linear system.

(b) $\{(w,x,y,z) \mid wx=yz\}$. → Counter Examples

(c) $\{(w,x,y,z) \mid w+x+y=z^2\}$.

(d) $\{(w,x,y,z) \mid w=0 \text{ and } y=0\}$.

(e) $\{(w,x,y,z) \mid w=0 \text{ or } y=0\}$. usually is and. ∴ unlikely a subspace ⇒ counter example.

(f) $\{(w,x,y,z) \mid w=1 \text{ and } y=0\}$.

(g) $\{(w,x,y,z) \mid w+z=0 \text{ and } x+y-4z=0 \text{ and } 4w+y-z=0\}$. homogeneous linear system

(h) $\{(w,x,y,z) \mid w+z=0 \text{ or } x+y-4z=0 \text{ or } 4w+y-z=0\}$.

17. Give an example of a 2×3 matrix A, if possible, such that the so

b) Subspace must be closed under addition.

$V_1 \in$ set $= (1,0,1,0)$

$V_2 \in$ set. $= (0,2,0,1)$

$V_1+V_2 \notin$ set $= (1,2,1,1)$

$2 \neq 1$

∴ counter-

e) $V_1 = (0,0,1,0)$

$V_2 = (1,0,0,0)$.

$V_1+V_2 = (1,0,1,0)$

$\Downarrow$

$\notin$ set.

17. Give an example of a 2×3 matrix $\boldsymbol{A}$, if possible, such that the solution space of the linear system $\boldsymbol{Ax} = \mathbf{0}$ is

(a) $\mathbb{R}^3$.

(b) the plane $\{(x, y, z) \mid 2x + 3y - z = 0\}$.

(c) the line $\{(t, 2t, 3t) \mid t \in \mathbb{R}\}$.

(d) the zero subspace.

a) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ✓

b) $\begin{pmatrix} 2 & 3 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ ✓

c) $\begin{pmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & -\frac{2}{3} \end{pmatrix}$ ✓

d) Not possible. ✓

18. Let W be a subspace of $\mathbb{R}^n$ and let $\boldsymbol{v} \in \mathbb{R}^n$. The set

$$W + \boldsymbol{v} = \{\boldsymbol{u} + \boldsymbol{v} \mid \boldsymbol{u} \in W\}$$

is called a *coset* of W containing $\boldsymbol{v}$. For each of the following, give a geometric interpretation for the coset $W + \boldsymbol{v}$.

(a) $W = \{(x, y) \mid x + y = 0\}$ and $\boldsymbol{v} = (1, 1)$.

(b) $W = \{c(1, 1, 1) \mid c \in \mathbb{R}\}$ and $\boldsymbol{v} = (0, 0, 1)$.

(c) $W = \{(x, y, z) \mid x + y + z = 0\}$ and $\boldsymbol{v} = (2, 0, -1)$.

a)

?

$W+V$ is the line $x+y=2$ in $\mathbb{R}^2$.

b) $W+V$ is the line $\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + c \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \mid c \in \mathbb{R} \right\}$ in $\mathbb{R}^3$.

c) $W+V$ is the plane $x+y+z=1$ in $\mathbb{R}^3$?

19. Let U, V and W be the three planes defined in Question 3.4. Is $U \cap V$ a subspace of $\mathbb{R}^3$? Is $V \cap W$ a subspace of $\mathbb{R}^3$? Justify your answers.

$U \cap V$ is a subspace of $\mathbb{R}^3$

since its a line in $\mathbb{R}^3$ passing through the origin

$V \cap W$ is not a subplane since it does not contain origin

20. Let V and W be subspaces of $\mathbb{R}^n$. Define $V+W = \{\boldsymbol{v}+\boldsymbol{v} \mid \boldsymbol{v} \in V \text{ and } \boldsymbol{w} \in W\}$.

(a) Show that $V+W$ is a subspace of $\mathbb{R}^n$.

(b) Write down the subspace $V+W$ explicitly if

(i) $V = \{(t,0) \mid t \in \mathbb{R}\}$ and $W = \{(0,t) \mid t \in \mathbb{R}\}$.

(ii) $V = \{(t,t,t) \mid t \in \mathbb{R}\}$ and $W = \{(t,-t,0) \mid t \in \mathbb{R}\}$.

a) Let $V = \text{span}\{v_1 \ldots v_n\}$

$W = \text{span}\{w_1 \cdots w_n\}$

then

$V+W = \{v+w \mid v \in V \text{ and } w \in W\}$.

$= \{a_1 v_1 + \ldots + a_m v_m + b_1 w_1 + \ldots + b_n w_n \mid a_1 \ldots a_m \in \mathbb{R},\ b_1 \ldots b_n \in \mathbb{R}\}$

$= \text{span}\{v_1 \ldots v_m, w_1 \ldots w_n\}$.

$\therefore$ $V+W$ is subspace of $\mathbb{R}^n$.

b) i) $V+W = \mathbb{R}^2$

ii) $V+W = \{s(1,1,1) + t(1,-1,0) \mid s,t \in \mathbb{R}\}$.

21. (All vectors in this question are written as column vectors.) Let $\boldsymbol{A}$ be an $m \times n$ matrix. Define $V_{\boldsymbol{A}}$ to be the subset $\{\boldsymbol{Au} \mid \boldsymbol{u} \in \mathbb{R}^n\}$ of $\mathbb{R}^m$.

(a) Show that $V_{\boldsymbol{A}}$ is a subspace of $\mathbb{R}^m$.

(b) Write down the subspace $V_{\boldsymbol{A}}$ explicitly if

(i) $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \end{pmatrix}$.

(ii) $\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}$.

a) $V_A = \{ Au \mid u \in \mathbb{R}^n \} \in \mathbb{R}^m.$

let $A = (c_1 \cdots c_n)$ where $c_1 \cdots c_n$ are columns of A.

$\therefore$ For any $u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \in \mathbb{R}^n$, $Au = u_1 c_1 + \cdots + u_n c_n$.

$\therefore$ $V_A = \text{span}\{c_1 \cdots c_n\}$ is a subspace of $\mathbb{R}^m$.

b) i) $V_A = \mathbb{R}^2$

ii) $V_A = \left\{ s\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \mid s, t \in \mathbb{R} \right\}$

22. (All vectors in this question are written in column vectors.) Let $\boldsymbol{A}$ be an $n \times n$ matrix. Define $W_{\boldsymbol{A}}$ to be the subset $\{\boldsymbol{u} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{u} = \boldsymbol{u}\}$ of $\mathbb{R}^n$.

(a) Show that $W_{\boldsymbol{A}}$ is a subspace of $\mathbb{R}^n$.

(b) Let $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Write down the subspace $W_{\boldsymbol{A}}$ explicitly.

a) $Au = u \iff (A - I)u = 0.$

W_A is the soln set to homogeneous system $(A-I)u = 0$.

$\therefore$ W_A is a subspace of $\mathbb{R}^n$.

b) $A - I = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

A general soln of $\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

is $x = s$, $y = t$, $z = 0$.

where $s, t \in \mathbb{R}$, $W_A = \left\{ \begin{pmatrix} s \\ t \\ 0 \end{pmatrix} \middle| \ s, t \in \mathbb{R} \right\}$

23. Determine which of the following statements are true. Justify your answer.

(a) $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$.

(b) The solution set of $x+2y-z=0$ is a subspace of $\mathbb{R}^3$.

(c) The solution set of $x+2y-z=1$ is a subspace of $\mathbb{R}^3$.

(d) If S_1 and S_2 are two subsets of $\mathbb{R}^n$, then $\text{span}(S_1 \cup S_2) = \text{span}(S_1) + \text{span}(S_2)$. (See Question 3.20.)

a) False $\mathbb{R}^2 \notin \mathbb{R}^3$.

b) True, forms a homogeneous system of linear eqn.

c) False $(0,0,0)$ is not a solution

d) True

24. Let V and W be subspaces of $\mathbb{R}^n$.

(a) Show that $V \cap W$ is a subspace of $\mathbb{R}^n$. (*Hint*: Use Remark 3.3.8.)

(b) Give an example of V and W in $\mathbb{R}^2$ such that $V \cup W$ is not a subspace.

(c) Show that $V \cup W$ is a subspace of $\mathbb{R}^n$ if and only if $V \subseteq W$ or $W \subseteq V$.

a) Since both V and W contain zero vector, the zero vector is contained in $V \cap W$.

$\therefore$ $V \cap W$ is non-empty.

Let u and v be any 2 vectors in $V \cap W$,

Let a and b be any real numbers.

Since u and v are contained in V, $au + bv$ is also contained in V.

$\therefore$ $au + bv$ is also contained in W.

$\therefore$ $au + bv$ is contained in $V \cap W$.

$\therefore$ $V \cap W$ is a subspace of $\mathbb{R}^n$

b) $V = \{(x, 0) \mid x \in \mathbb{R}\}$

$W = \{(0, y) \mid y \in \mathbb{R}\}$.

Then both V and W are line through origin $\therefore$ subspaces of $\mathbb{R}^n$.

but $V \cup W$ is a union of 2 lines which is not a subspace of $\mathbb{R}^n$.

c) Suppose $V \not\subseteq W$. $(\Rightarrow)$

Take any vector $x \in W$.

Since $V \not\subseteq W$, $\exists$ a vector $y \in V$ but $y \notin W$

As $V \cup W$ is a subspace of $\mathbb{R}^n$ and $x, y \in V \cup W$,

we have $x + y \in V \cup W$,

either $x + y \in V$ or $x + y \in W$.

Assume $x + y \in W$. As W is a subspace of $\mathbb{R}^n$ and $-x \in W$, we have $y = (x+y) + (-x) \in W$ which contradicts that $y \notin W$ as mentioned above

$\therefore$ we know $x + y \in V$.

As V is a subspace of $\mathbb{R}^n$ and $-y \in V$

we have $x = (x+y) + (-y) \in V$

Since every vector in W is contained in V,

$W \subseteq V$.

$(\Leftarrow)$

If $V \subseteq W$, then $V \cup W = W \in \mathbb{R}^n$

If $W \subseteq V$, then $W \cup V = V \in \mathbb{R}^n$,

(a) $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$.

(b) The solution set of $x+2y-z=0$ is a subspace of $\mathbb{R}^3$.

(c) The solution set of $x+2y-z=1$ is a subspace of $\mathbb{R}^3$.

(d) If S_1 and S_2 are two subsets of $\mathbb{R}^n$, then $\text{span}(S_1 \cup S_2) = \text{span}(S_1) + \text{span}(S_2)$. (See Question 3.20.)

24. Let V and W be subspaces of $\mathbb{R}^n$.

(a) Show that $V \cap W$ is a subspace of $\mathbb{R}^n$. (*Hint*: Use Remark 3.3.8.)

(b) Give an example of V and W in $\mathbb{R}^2$ such that $V \cup W$ is not a subspace.

(c) Show that $V \cup W$ is a subspace of $\mathbb{R}^n$ if and only if $V \subseteq W$ or $W \subseteq V$.

25. For each of the sets S_1 to S_6 in Question 3.9, determine whether the set is linearly independent.

26. (a) Let $\boldsymbol{R} = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

Show that the three nonzero rows of $\boldsymbol{R}$ are linearly independent vectors.

(b) For a nonzero matrix in row-echelon form, is it true that the nonzero rows are always linearly independent?

27. In Question 3.13, suppose $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are linearly independent vectors in $\mathbb{R}^n$. Determine which of the sets S_1 to S_5 are linearly independent.

28. Let $\boldsymbol{u}_1 = (a, 1, -1)$, $\boldsymbol{u}_2 = (-1, a, 1)$, $\boldsymbol{u}_3 = (1, -1, a)$. For what values of a are $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$ linearly independent?

29. Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ be vectors in $\mathbb{R}^3$ such that $V = \text{span}\{\boldsymbol{u}, \boldsymbol{v}\}$ and $W = \text{span}\{\boldsymbol{u}, \boldsymbol{w}\}$ are planes in $\mathbb{R}^3$. Find $V \cap W$ if

(a) $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are linearly independent.

(b) $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are linearly dependent.

30. (All vectors in this question are written as column vectors.) Let $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ be vectors in $\mathbb{R}^n$ and $\boldsymbol{P}$ a square matrix of order n.

(a) Show that if $\boldsymbol{P}\boldsymbol{u}_1, \boldsymbol{P}\boldsymbol{u}_2, \ldots, \boldsymbol{P}\boldsymbol{u}_k$ are linearly independent, then $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ are linearly independent.

(b) Suppose $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ are linearly independent.

(i) Show that if $\boldsymbol{P}$ is invertible, then $\boldsymbol{P}\boldsymbol{u}_1, \boldsymbol{P}\boldsymbol{u}_2, \ldots, \boldsymbol{P}\boldsymbol{u}_k$ are linearly independent.

(ii) If $\boldsymbol{P}$ is not invertible, are $\boldsymbol{P}\boldsymbol{u}_1, \boldsymbol{P}\boldsymbol{u}_2, \ldots, \boldsymbol{P}\boldsymbol{u}_k$ linearly independent?

31. Prove Remark 3.3.8:

Let V be a non-empty subset of $\mathbb{R}^n$. Show that V is a subspace of $\mathbb{R}^n$ if and only if for all $\boldsymbol{u}, \boldsymbol{v} \in V$ and $c, d \in \mathbb{R}$, $c\boldsymbol{u} + d\boldsymbol{v} \in V$. (*Hint*: For the "if" part, you need to find a finite set S of vectors that spans V. By Theorem 3.4.7, there are at most n linearly independent vectors in

25. For each of the sets S_1 to S_6 in Question 3.9, determine whether the set is linearly independent.

27. In Question 3.13, suppose $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are linearly independent vectors in $\mathbb{R}^n$. Determine which of the sets S_1 to S_5 are linearly independent.

29. Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ be vectors in $\mathbb{R}^3$ such that $V = \operatorname{span}\{\boldsymbol{u}, \boldsymbol{v}\}$ and $W = \operatorname{span}\{\boldsymbol{u}, \boldsymbol{w}\}$ are planes in $\mathbb{R}^3$. Find $V \cap W$ if

(a) $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are linearly independent.

(b) $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are linearly dependent.

30. (All vectors in this question are written as column vectors.) Let $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ be vectors in $\mathbb{R}^n$ and $\boldsymbol{P}$ a square matrix of order n.

(a) Show that if $\boldsymbol{P}\boldsymbol{u}_1, \boldsymbol{P}\boldsymbol{u}_2, \ldots, \boldsymbol{P}\boldsymbol{u}_k$ are linearly independent, then $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ are linearly independent.

(b) Suppose $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ are linearly independent.

(i) Show that if $\boldsymbol{P}$ is invertible, then $\boldsymbol{P}\boldsymbol{u}_1, \boldsymbol{P}\boldsymbol{u}_2, \ldots, \boldsymbol{P}\boldsymbol{u}_k$ are linearly independent.

(ii) If $\boldsymbol{P}$ is not invertible, are $\boldsymbol{P}\boldsymbol{u}_1, \boldsymbol{P}\boldsymbol{u}_2, \ldots, \boldsymbol{P}\boldsymbol{u}_k$ linearly independent?

V. When $V \neq \{\mathbf{0}\}$, let S be a largest set of linearly independent vectors in V. Then show that $\operatorname{span}(S) = V$.

Question 3.32 to Question 3.49 are exercises for Sections 3.5 to 3.7.

32. Determine which of the following sets are bases for $\mathbb{R}^3$.

(a) $S_1 = \{(1,0,-1),(-1,2,3)\}$.

(b) $S_2 = \{(1,0,-1),(-1,2,3),(0,3,0)\}$.

(c) $S_3 = \{(1,0,-1),(-1,2,3),(0,3,3)\}$.

(d) $S_4 = \{(1,0,-1),(-1,2,3),(0,3,0),(1,-1,1)\}$.

33. Find a basis for the solution space of each of the following homogeneous systems.

(a) $x_1 + 3x_2 - x_3 + 2x_4 = 0$.

(b) $\begin{cases} x_1 + 3x_2 - x_3 + 2x_4 = 0 \\ -3x_2 + x_3 = 0. \end{cases}$

(c) $\begin{cases} x_1 + 3x_2 - x_3 + 2x_4 = 0 \\ -3x_2 + x_3 = 0 \\ x_1 - x_4 = 0. \end{cases}$

34. For each of the following cases, find the coordinate vector of $\boldsymbol{v}$ relative to the basis $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$.

(a) $\boldsymbol{v} = (1,-2,6)$, $\boldsymbol{u}_1 = (1,1,1)$, $\boldsymbol{u}_2 = (0,2,2)$, $\boldsymbol{u}_3 = (0,0,3)$.

(b) $\boldsymbol{v} = (0,0,1)$, $\boldsymbol{u}_1 = (1,1,2)$, $\boldsymbol{u}_2 = (-1,1,-2)$, $\boldsymbol{u}_3 = (1,3,3)$.

35. Let $V = \{(a+b, a+c, c+d, b+d) \mid a,b,c,d \in \mathbb{R}\}$ and $S = \{(1,1,0,0),(1,0,-1,0),(0,-1,0,1)\}$.

(a) Show that V is a subspace of $\mathbb{R}^4$ and S is a basis for V.

(b) Find the coordinate vector of $\boldsymbol{u} = (1,2,3,2)$ relative to S.

(c) Find a vector $\boldsymbol{v}$ such that $(\boldsymbol{v})_S = (1,3,-1)$.

36. Find a basis for and determine the dimension of each of the following subspaces of $\mathbb{R}^3$:

(a) the plane $x - y + z = 0$,

(b) the plane $x = y$,

(c) the line $x = t$, $y = -t$ and $z = 2t$ for $t \in \mathbb{R}$.

37. Find a basis for and determine the dimension of each of the following subspaces of $\mathbb{R}^4$:

(a) the subspace containing all vectors of the form $(w,0,y,0)$.

(b) the subspace containing all vectors of the form (w,x,x,w).

(c) the subspace containing all vectors of the form (w,x,y,z) with $w = 2x = 3y$.

(d) the solution space of

$$\begin{cases} 2w + 3x + y + z = 0 \\ -3w + x + 4y - 7z = 0 \\ w + 2x + y = 0. \end{cases}$$

(e) the subspace $\{(w,x,y,z) \mid y = w + x \text{ and } z = w - x\}$.

38. Let $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ be a basis for a vector space V. Determine whether $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is a basis for V if

(a) $\boldsymbol{v}_1 = \boldsymbol{u}_1$, $\boldsymbol{v}_2 = \boldsymbol{u}_1 + \boldsymbol{u}_2$, $\boldsymbol{v}_3 = \boldsymbol{u}_1 + \boldsymbol{u}_2 + \boldsymbol{u}_3$.

(b) $\boldsymbol{v}_1 = \boldsymbol{u}_1 - \boldsymbol{u}_2$, $\boldsymbol{v}_2 = \boldsymbol{u}_2 - \boldsymbol{u}_3$, $\boldsymbol{v}_3 = \boldsymbol{u}_3 - \boldsymbol{u}_1$.

39. Give an example of a family of subspaces $V_1, V_2, \ldots, V_n$ of $\mathbb{R}^n$ such that $\dim(V_i) = i$ for $i = 1, 2, \ldots, n$ and

$$V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n.$$

Justify your answer.

40. Let $\boldsymbol{u}_1 = (1,0,1,1)$, $\boldsymbol{u}_2 = (-3,3,7,1)$, $\boldsymbol{u}_3 = (-1,3,9,3)$, $\boldsymbol{u}_4 = (-5,3,5,-1)$ and let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4\}$ and $V = \mathrm{span}(S)$.

(a) Find a non-trivial solution to the equation

$$a\boldsymbol{u}_1 + b\boldsymbol{u}_2 + c\boldsymbol{u}_3 + d\boldsymbol{u}_4 = \mathbf{0}.$$

(b) Express $\boldsymbol{u}_3$ and $\boldsymbol{u}_4$ (separately) as linear combinations of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$.

(c) Find a basis for and determine the dimension of V.

(d) Find a subspace W of $\mathbb{R}^4$ such that $\dim(W) = 3$ and $\dim(W \cap V) = 2$. Justify your answer.

41. Let V be a vector space.

(a) Suppose S is a finite subset of V such that $\mathrm{span}(S) = V$. Show that there exists a subset S' of S such that S' is a basis for V.

(b) Suppose T is a finite subset of V such that T is linearly independent. Show that there exists a basis T^* for V such that $T \subseteq T^*$.

42. Let V be a vector space of dimension n. Show that there exist $n+1$ vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{n+1}$ such that every vector in V can be expressed as a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{n+1}$ with non-negative coefficients.

43. Let V and W be subspaces of $\mathbb{R}^n$. Show that

$$\dim(V + W) = \dim(V) + \dim(W) - \dim(V \cap W).$$

(See Question 3.20 and Question 3.24.)

44. Let $U = \mathrm{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ and $V = \mathrm{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ be subspaces of $\mathbb{R}^5$ such that $\dim(U \cap V) = 2$. Suppose W is the smallest subspace of $\mathbb{R}^5$ that contains both U and V. Determine all possible dimensions of W. Justify your answers.

45. Determine which of the following statements are true. Justify your answer.

(a) If S_1 and S_2 are bases for V and W respectively, where V and W are subspaces of a vector space, then $S_1 \cap S_2$ is a basis for $V \cap W$. (See Question 3.24.)

(b) If S_1 and S_2 are bases for V and W respectively, where V and W are subspaces of a vector space, then $S_1 \cup S_2$ is a basis for $V + W$. (See Question 3.20.)

(c) If V and W are subspaces of a vector space, then there exists a basis S_1 for V and a basis S_2 for W such that $S_1 \cap S_2$ is a basis for $V \cap W$.

(d) If V and W are subspaces of a vector space, then there exists a basis S_1 for V and a basis S_2 for W such that $S_1 \cup S_2$ is a basis for $V + W$.

46. (a) Let $\boldsymbol{u}_1 = (1, 2, -1)$, $\boldsymbol{u}_2 = (0, 2, 1)$, $\boldsymbol{u}_3 = (0, -1, 3)$. Show that $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ forms a basis for $\mathbb{R}^3$.

(b) Suppose $\boldsymbol{w} = (1, 1, 1)$. Find the coordinate vector of $\boldsymbol{w}$ relative to S.

(c) Let $T = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ be another basis for $\mathbb{R}^3$ where $\boldsymbol{v}_1 = (1, 5, 4)$, $\boldsymbol{v}_2 = (-1, 3, 7)$, $\boldsymbol{v}_3 = (2, 2, 4)$. Find the transition matrix from T to S.

(d) Find the transition matrix from S to T.

(e) Use the vector $\boldsymbol{w}$ in Part (b). Find the coordinate vector of $\boldsymbol{w}$ relative to T.

47. Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ where $\boldsymbol{u}_1 = (3, -2, 5)$, $\boldsymbol{u}_2 = (1, -4, 4)$, $\boldsymbol{u}_3 = (0, 3, -2)$.

(a) Show that S is a basis for $\mathbb{R}^3$.

(b) Show that $T = \{\boldsymbol{u}_1 - \boldsymbol{u}_2, \boldsymbol{u}_1 + 2\boldsymbol{u}_2 - \boldsymbol{u}_3, \boldsymbol{u}_2 + 2\boldsymbol{u}_3\}$ is also a basis for $\mathbb{R}^3$.

(c) Find the coordinate vector of $\boldsymbol{v} = (1, 0, 1)$ relative to S.

(d) Find a vector $\boldsymbol{w}$ in $\mathbb{R}^3$ such that $(\boldsymbol{w})_T = (1, 0, 1)$.

(e) Find the transition matrix from T to S and the transition matrix from S to T.

(f) Let $\boldsymbol{x}$ be a vector in $\mathbb{R}^3$ such that $(\boldsymbol{x})_T = (1, 1, 2)$. Find $(\boldsymbol{x})_S$.

48. Consider the vector space $V = \{(x, y, z) \mid 2x - y + z = 0\}$. Let $S = \{(0, 1, 1), (1, 2, 0)\}$ and $T = \{(1, 1, -1), (1, 0, -2)\}$.

(a) Show that both S and T are bases for V.

(b) Find the transition matrix from T to S and the transition matrix from S to T.

(c) Find $(\boldsymbol{w})_S$ and $(\boldsymbol{w})_T$ where $\boldsymbol{w} = (1, -1, -3)$.

49. Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ be a basis for $\mathbb{R}^3$ and $T = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ where

$$\boldsymbol{v}_1 = \boldsymbol{u}_1 + \boldsymbol{u}_2 + \boldsymbol{u}_3, \quad \boldsymbol{v}_2 = \boldsymbol{u}_2 + \boldsymbol{u}_3 \quad \text{and} \quad \boldsymbol{v}_3 = \boldsymbol{u}_2 - \boldsymbol{u}_3.$$

(a) Show that T is a basis for $\mathbb{R}^3$.

(b) Find the transition matrix from S to T.