CS1231S Tutorial 3: Sets

National University of Singapore

2021/22 Semester 1

When asked to 'find' a set in the following, the answer should involve a list of all of the elements in the set.

Questions for discussion on the LumiNUS Forum

Answers to these questions will not be provided.

Theorem. I will get A+ in CS1231S.

Proof due to Curry (1942). 1. Let $C = \{x : \text{if } x \in x, \text{ then I will get A+ in CS1231S}\}$.

2. By the definition of C, for every z,

$$z \in C \quad \Leftrightarrow \quad \text{"if } z \in z, \text{ then I will get A+ in CS1231S"}.$$

3. Instantiating z to C here gives

$$C \in C \quad \Leftrightarrow \quad \text{"if } C \in C, \text{ then I will get A+ in CS1231S"}.$$

4. 4.1. Assume $C \in C$.
   4.2. Then, by the $\Rightarrow$ part of line 3, we know if $C \in C$, then I will get A+ in CS1231S.
   4.3. Applying modus ponens to lines 4.1 and 4.2, we deduce that I will get A+ in CS1231S.
5. From lines 4.1–4.3, we see that if $C \in C$, then I will get A+ in CS1231S.
6. Thus $C \in C$ by the $\Leftarrow$ part of line 3.
7. Combining lines 5 and 6 using modus ponens, we conclude that I will get A+ in CS1231S. ⁕

D1. Let $A = \{a, \varnothing\}$, where a is a letter. Which of the following are true? Which of them are false?

(a) $a \in A$.
(b) $\{a\} \in A$.
(c) $a \subseteq A$.
(d) $\{a\} \subseteq A$.
(e) $\varnothing \subseteq A$.
(f) $\varnothing \in A$.
(g) $\{\varnothing\} \subseteq A$.
(h) $\{\varnothing\} \in A$.

D2. Find two finite sets A, B such that $A \in B$ and $A \subseteq B$.

D3. Find the power set of each of the following sets:

(a) $\{x, y, z, w\}$;
(b) $\{a, \{a, b\}\}$;
(c) $\varnothing$;
(d) $\{\varnothing\}$;
(e) $\{\{a\}, \varnothing\}$.

Tutorial questions

1. Which of the following are true? Which of them are false?

   (a) $\varnothing \in \varnothing$.
   (b) $\varnothing \subseteq \varnothing$.
   (c) $\varnothing \in \{\varnothing\}$.
   (d) $\varnothing \subseteq \{\varnothing\}$.
   (e) $\{\varnothing, 1\} = \{1\}$.
   (f) $1 \in \{\{1,2\},\{2,3\},4\}$.
   (g) $\{1,2\} \subseteq \{3,2,1\}$.
   (h) $\{3,3,2\} \subsetneq \{3,2,1\}$.

2. Let $A = \{1, \{1,2\}, 2, \{1,2\}\}$. Find $|A|$.

3. Let $A = \{0,1,4,5,6,9\}$ and $B = \{0,2,4,6,8\}$. Find $|A|$, $|B|$, $|A \cap B|$, and $|A \cup B|$.

4. Let $A = \{2n+1 : n \in \mathbb{Z}\}$ and $B = \{2n-1 : n \in \mathbb{Z}\}$. Is $A = B$? Prove that your answer is correct.

5. Let $A = \{x \in \mathbb{Z} : 2 \leqslant x \leqslant 5\}$ and $B = \{x \in \mathbb{Q} : 2 \leqslant x \leqslant 5\}$. Is $A = B$? Prove that your answer is correct.

6. Let $U = \{5, 6, 7, \ldots, 12\}$. Find:

   (a) $\{n \in U : n \text{ is even}\}$;
   (b) $\{n \in U : n = m^2 \text{ for some } m \in \mathbb{Z}\}$;
   (c) $\{-5, -4, -3, \ldots, 5\} \setminus \{1, 2, 3, \ldots, 10\}$;
   (d) $\overline{\{5,7,9\} \cup \{9,11\}}$, where U is considered the universal set;
   (e) $\big\{(x,y) \in \{1,3,5\} \times \{2,4\} : x + y \geqslant 6\big\}$;
   (f) $\mathcal{P}(\{2,4\})$.

7. Show that for all sets A, B, C,
$$A \cap (B \setminus C) = (A \cap B) \setminus C.$$

8. (2009/10 Semester 2 exam question B) Prove that for all sets A and B,
$$(A \cup \overline{B}) \cap (\overline{A} \cup B) = (A \cap B) \cup (\overline{A} \cap \overline{B}).$$

9. Let A, B be sets. Show that $A \subseteq B$ if and only if $A \cup B = B$.

10. For sets A and B, define $A \oplus B = (A \setminus B) \cup (B \setminus A)$.

    (a) Let $A = \{1, 4, 9, 16\}$ and $B = \{2, 4, 6, 8, 10, 12, 14, 16\}$. Find $A \oplus B$.
    (b) Show that for all sets A, B,
$$A \oplus B = (A \cup B) \setminus (A \cap B).$$

11. (2015/16 Semester 1 exam question 16(a)) Denote by $|x|$ the absolute value of the integer x, i.e.,
$$|x| = \begin{cases} x, & \text{if } x \geqslant 0; \\ -x, & \text{if } x < 0. \end{cases}$$
    Given the set $S = \{-9, -6, -1, 3, 5, 8\}$, for each of the following statements, state whether it is true or false, with explanation.

    (a) $\exists z \in S \ \forall x, y \in S \ \ z > |x - y|$.
    (b) $\exists z \in S \ \forall x, y \in S \ \ z < |x - y|$.

12. For sets $A_m, A_{m+1}, \ldots, A_n$, define

$$\bigcup_{i=m}^{n} A_i = A_m \cup A_{m+1} \cup \cdots \cup A_n \quad \text{and} \quad \bigcap_{i=m}^{n} A_i = A_m \cap A_{m+1} \cap \cdots \cap A_n.$$

(a) Let $A_i = \{x \in \mathbb{Z} : x \geqslant i\}$ for each $i \in \mathbb{Z}$. Write down $\bigcup_{i=2}^{5} A_i$ and $\bigcap_{i=2}^{5} A_i$ in roster notation.

(b) Let $B_1, B_2, \ldots, B_k, C_1, C_2, \ldots, C_\ell$ be sets such that

$$\bigcup_{i=1}^{k} B_i \subseteq \bigcap_{j=1}^{\ell} C_j.$$

Show that $B_i \subseteq C_j$ for all $i \in \{1, 2, \ldots, k\}$ and all $j \in \{1, 2, \ldots, \ell\}$.

D1 (a) T
(b) F
(c) F
(d) T
(e) F
(f) T
(g) T
(h) F

4. yes, $2n+1 = 2(n+1)-1$ since $n \in \mathbb{Z}$, there exist a integer $n+1$ such that it satisfied equation B, an since $A(n)$ and $B(n+1)$ yield the same solutions, we can conclude that $A = B$.

As $n \in \mathbb{Z}$, $n+1$ also $\in \mathbb{Z}$

5. No, there exist some rational numbers x in range $2 \le x \le 5$ which is not an integer. For example, $x = \frac{5}{2}$ is an element of B but not A. Hence, $A \neq B$

D2 $A = \{a, \emptyset\}$
$B = \{a, \emptyset, \{a, \emptyset\}\}$

$A = B \Leftrightarrow \forall z\,(z \in A \Leftrightarrow z \in B)$

D3. (a) $\{\{x\}, \{y\}, \{z\}, \{w\}, \{x,y\}, \{x,z\}, \{x,w\}, \{y,z\}, \{y,w\}, \{z,w\}, \{x,y,z\}, \{x,y,w\}, \{x,z,w\}, \{y,z,w\}, \{w,x,y,z\}, \emptyset\}$

(b) $\{\{a\}, \{\{a,b\}\}, \{a, \{a,b\}\}, \emptyset\}$

(c) $\{\emptyset\}$

(d) $\{\emptyset, \{\emptyset\}\}$

(e) $\{\{a\}, \emptyset, \{\{a\}, \emptyset\}, \{\emptyset\}\}$

Tutorial Questions

1 (a) F
(b) T
(c) T
(d) T
(e) F
(f) F
(g) T
(h) T

2. $|A| = 3$

3. $|A| = 6$
$|B| = 5$
$|A \cap B| = 3$
$|A \cup B| = 8$

6 (a) $n = 6, 8, 10, 12$
(b) $n = 9$
(c) $\emptyset$
(d) $\{5, 7, 9, 11\}$
(e) $\{\{3,2\}, \{3,4\}, \{5,2\}, \{5,4\}\}$
(f) $\{2, 4, \{2,4\}, \{0\}\}$

7 $A \cap (B \setminus C) = (A \cap B) \setminus C$

Diagram A

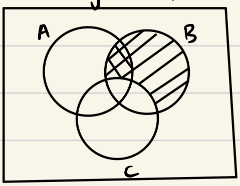
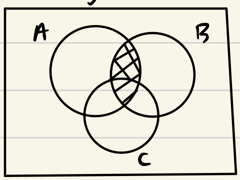


Diagram B



Prove 1

In diagram A, represents $(B \setminus C)$ and represents $A \cap (B \setminus C)$
In diagram B, represents $(A \cap B)$ and represents $(A \cap B) \setminus C$
the diagrams clearly show that both areas are the same.
Hence, we conclude that $A \cap (B \setminus C) = (A \cap B) \setminus C$

$A \cap (B \setminus C)$

$= A \cap (B \cap \bar{C})$ set difference laws

$= (A \cap B) \cap \bar{C}$ associative laws

$= (A \cap B) \setminus C$ set difference laws

Prove 2

8. $(A \cup \bar{B}) \cap (\bar{A} \cup B)$

$= ((A \cup \bar{B}) \cap \bar{A}) \cup ((A \cup \bar{B}) \cap B)$ distributive laws

$= (\bar{A} \cap (A \cup \bar{B})) \cup (B \cap (A \cup \bar{B}))$ commutative laws

$= ((\bar{A} \cap A) \cup (\bar{A} \cap \bar{B})) \cup ((B \cap A) \cup (B \cap \bar{B}))$ distributive laws

$= (\emptyset \cup (\bar{A} \cap \bar{B})) \cup ((B \cap A) \cup \emptyset)$ complement laws

$= (\bar{A} \cap \bar{B}) \cup (B \cap A)$ Identity laws

$= (\bar{A} \cap \bar{B}) \cup (A \cap B)$ commutative laws

$= (A \cap B) \cup (\bar{A} \cap \bar{B})$ commutative laws

10. (a) $A \setminus B = \{1, 9\}$

$B \setminus A = \{2, 6, 8, 10, 12, 14\}$

$(A \setminus B) \cup (B \setminus A) = \{1, 9, 2, 6, 8, 10, 12, 14\}$

(b) $A \oplus B$

$= (A \setminus B) \cup (B \setminus A)$ defined in question of $\oplus$

$= (A \cap \bar{B}) \cup (B \cap \bar{A})$ set difference laws

$= ((A \cap \bar{B}) \cup B) \cap ((A \cap \bar{B}) \cup \bar{A})$ distributive laws

$= (B \cup (A \cap \bar{B})) \cap (\bar{A} \cup (A \cap \bar{B}))$ commutative laws

$= ((B \cup A) \cap (B \cup \bar{B})) \cap ((\bar{A} \cup A) \cap (\bar{A} \cup \bar{B}))$ distributive laws

$= ((B \cup A) \cap U) \cap (U \cap (\bar{A} \cup \bar{B}))$ complement laws

$= (B \cup A) \cap (\bar{A} \cup \bar{B})$ Identity laws

$= (A \cup B) \cap (\bar{A} \cup \bar{B})$ commutative laws

$= (A \cup B) \cap (\overline{A \cap B})$ De Morgan's laws

$= (A \cup B) \setminus (A \cap B)$ set difference laws

11. (a) false, when $x = -9$ & $y = 8$, $|x - y| = 17 > 8 = \max$ of $S \geq Z$.

(b) true, $|x-y|$ always ≥ 0. There exist some value of z which is < 0

12. (a) $\bigcup_{i=2}^{5} A_i = A_2 \cup A_3 \cup A_4 \cup A_5$

$= \{2, 3, 4, \ldots\} \cup \{3, 4, 5, \ldots\} \cup \ldots \cup \{5, 6, 7, \ldots\}$

$= \{2, 3, 4, 5, \ldots\}$

$\bigcap_{i=2}^{5} A_i = A_2 \cap A_3 \cap A_4 \cap A_5$

$= \{5, 6, 7, 8, \ldots\}$

(b)

1. ($\Rightarrow$)

1.1 suppose $A \subseteq B$

1.2 ($A \cup B \subseteq B$)

1.2.1 let $z \in A \cup B$

1.2.2 Then $z \in A$ or $z \in B$

1.2.3 case 1 : $z \in A$

1.2.3.1 Then $z \in B$

1.2.4 Case 2 : $z \in B$

1.2.4.1 Then $z \in B$

1.2.5 In all cases, $z \in B$

1.3 ($A \cup B \supseteq B$)

1.3.1 let $z \in B$

1.3.2 Then $z \in A \vee z \in B$

1.3.3 So $z \in A \cup B$

1.4 Lines 1.2 & 1.3 imply $A \cup B = B$

2 ($\Leftarrow$)

2.1 Suppose $A \cup B = B$

2.2 We show that $A \subseteq B$ as follows

2.2.1 let $z \in A$

2.2.2 Then $z \in A \vee z \in B$

2.2.3 So $z \in A \cup B$

2.2.4 Thus $z \in B$

Q12b. 1 let $B_1, B_2, \ldots B_k, C_1, C_2 \ldots C_l$ be sets such that $\bigcup_{i=1}^{k} B_i \subseteq \bigcap_{j=1}^{l} C_j$.

2 2.1 let $r \in \{1, 2, \ldots k\}$ and $s \in \{1, 2, \ldots l\}$

2.2 Take any $z \in B_r$

2.3 Then $z \in B_1$ or $z \in B_2$ ... or $z \in B_k$ by defn of "or", as $r \in \{1, 2, \ldots, k\}$

2.4 So $z \in B_1 \cup B_2 \cdots \cup B_k = \bigcup_{i=1}^{k} B_i$ by defn of $\cup$, $\bigcup$

2.5 Hence $z \in \bigcap_{j=1}^{l} C_j = C_1 \cap C_2 \cap \ldots \cap C_l$ (by line 1) as $\bigcup_{i=1}^{k} B_i \subseteq \bigcap_{j=1}^{l} C_j$ by subset defn $\bigcup_{i=1}^{k} B_i \subseteq \bigcap_{j=1}^{l} C_j$

2.6 Thus $z \in C_1$ and $z \in C_2$ and ... and $z \in C_l$ by defn of $\bigcap$.

2.7 In particular, we know that $z \in C_s$ as $s \in \{1, 2, \ldots, l\}$.

3 So $B_i \subseteq C_j$ for any $i \{1, 2, \ldots, k\}$ and any $j = \{1, 2, \ldots, l\}$

CS1231S Tutorial 3: Sets
Solutions

National University of Singapore

2021/22 Semester 1

Sometimes there is more than one correct answer.

1. Which of the following are true? Which of them are false?

   (a) $\varnothing \in \varnothing$.

   (b) $\varnothing \subseteq \varnothing$.

   (c) $\varnothing \in \{\varnothing\}$.

   (d) $\varnothing \subseteq \{\varnothing\}$.

   (e) $\{\varnothing, 1\} = \{1\}$.

   (f) $1 \in \{\{1, 2\}, \{2, 3\}, 4\}$.

   (g) $\{1, 2\} \subseteq \{3, 2, 1\}$.

   (h) $\{3, 3, 2\} \subsetneq \{3, 2, 1\}$.

   Solution. F, T, T, T, F, F, T, T.

2. Let $A = \{1, \{1, 2\}, 2, \{1, 2\}\}$. Find $|A|$.

   Solution. $|A| = 3$.

3. Let $A = \{0, 1, 4, 5, 6, 9\}$ and $B = \{0, 2, 4, 6, 8\}$. Find $|A|$, $|B|$, $|A \cap B|$, and $|A \cup B|$.

   Solution. Note that $A \cap B = \{0, 4, 6\}$ and $A \cup B = \{0, 1, 2, 4, 5, 6, 8, 9\}$. So

$$|A| = 6, \quad |B| = 5, \quad |A \cap B| = 3, \quad \text{and} \quad |A \cup B| = 8.$$

4. Let $A = \{2n + 1 : n \in \mathbb{Z}\}$ and $B = \{2n - 1 : n \in \mathbb{Z}\}$. Is $A = B$? Prove that your answer is correct.

   Solution. Yes, as shown below.

   1. ($\subseteq$)
      1.1. Let $a \in A$.
      1.2. Use the definition of A to find $n \in \mathbb{Z}$ such that $a = 2n + 1$.
      1.3. Then $a = 2(n + 1) - 1$.
      1.4. As $n \in \mathbb{Z}$, we know $n + 1 \in \mathbb{Z}$.
      1.5. So $a \in B$ by the definition of B.
   2. ($\supseteq$)
      2.1. Let $b \in B$.
      2.2. Use the definition of B to find $n \in \mathbb{Z}$ such that $a = 2n - 1$.
      2.3. Then $b = 2(n - 1) + 1$.
      2.4. As $n \in \mathbb{Z}$, we know $n - 1 \in \mathbb{Z}$.
      2.5. So $b \in A$ by the definition of A.
   3. Hence $A = B$ by the definition of set equality. $\square$

5. Let $A = \{x \in \mathbb{Z} : 2 \leqslant x \leqslant 5\}$ and $B = \{x \in \mathbb{Q} : 2 \leqslant x \leqslant 5\}$. Is $A = B$? Prove that your answer is correct.

   Solution. No, as shown below.

   1. $3.14 \in \mathbb{Q}$ and $2 \leqslant 3.14 \leqslant 5$.

2. So $3.14 \in B$ by the definition of B.
3. $3.14 \notin \mathbb{Z}$.
4. So $3.14 \notin A$ by the definition of A.
5. Lines 2 and 4 imply $A \neq B$ by the definition of set equality. $\square$

6. Let $U = \{5, 6, 7, \ldots, 12\}$. Find:

 (a) $\{n \in U : n \text{ is even}\}$;

 (b) $\{n \in U : n = m^2 \text{ for some } m \in \mathbb{Z}\}$;

 (c) $\{-5, -4, -3, \ldots, 5\} \setminus \{1, 2, 3, \ldots, 10\}$;

 (d) $\overline{\{5, 7, 9\} \cup \{9, 11\}}$, where U is considered the universal set;

 (e) $\big\{(x, y) \in \{1, 3, 5\} \times \{2, 4\} : x + y \geqslant 6\big\}$;

 (f) $\mathcal{P}(\{2, 4\})$.

 Solution.

 (a) $\{6, 8, 10, 12\}$.

 (b) $\{9\}$.

 (c) $\{-5, -4, -3, -2, -1, 0\}$.

 (d) $\overline{\{5, 7, 9\} \cup \{9, 11\}} = \overline{\{5, 7, 9, 11\}} = \{6, 8, 10, 12\}$ when U is considered the universal set.

 (e) $\{(3, 4), (5, 2), (5, 4)\}$.

 (f) $\{\varnothing, \{2\}, \{4\}, \{2, 4\}\}$.

7. Show that for all sets A, B, C,

$$A \cap (B \setminus C) = (A \cap B) \setminus C.$$

 Solution.

1.	$A \cap (B \setminus C) = \{x : x \in A \text{ and } x \in B \setminus C\}$	by the definition of $\cap$;
2.	$= \{x : x \in A \text{ and } (x \in B \text{ and } x \notin C)\}$	by the definition of $\setminus$;
3.	$= \{x : (x \in A \text{ and } x \in B) \text{ and } x \notin C\}$	as "and" is associative;
4.	$= \{x : x \in A \cap B \text{ and } x \notin C\}$	by the definition of $\cap$;
5.	$= (A \cap B) \setminus C$	by the definition of $\setminus$. $\square$

8. (2009/10 Semester 2 exam question B) Prove that for all sets A and B,

$$(A \cup \overline{B}) \cap (\overline{A} \cup B) = (A \cap B) \cup (\overline{A} \cap \overline{B}).$$

 Solution. (Note that we no longer need to apply the set identities as strictly as we did in the logic part of the module.)

1.	$(A \cup \overline{B}) \cap (\overline{A} \cup B)$	
2.	$= \big((A \cup \overline{B}) \cap \overline{A}\big) \cup \big((A \cup \overline{B}) \cap B\big)$	as $\cap$ distributes over $\cup$;
3.	$= \big((A \cap \overline{A}) \cup (\overline{B} \cap \overline{A})\big) \cup \big((A \cap B) \cup (\overline{B} \cap B)\big)$	as $\cap$ distributes over $\cup$;
4.	$= (\varnothing \cup (\overline{B} \cap \overline{A})) \cup ((A \cap B) \cup \varnothing)$	by the Complement Law;
5.	$= (\overline{B} \cap \overline{A}) \cup (A \cap B)$	by the Identity Law;
6.	$= (A \cap B) \cup (\overline{A} \cap \overline{B})$	by the Commutative Laws. $\square$

 One may alternatively use the element method or the truth-table method.

9. Let A, B be sets. Show that $A \subseteq B$ if and only if $A \cup B = B$.

 Solution.

 1. ("Only if")

1.1. Suppose $A \subseteq B$.
1.2. ("$A \cup B \subseteq B$")
    1.2.1. Let $z \in A \cup B$.
    1.2.2. Then $z \in A$ or $z \in B$ by the definition of $\cup$.
    1.2.3. Case 1: suppose $z \in A$.
        1.2.3.1. Then $z \in B$ as $A \subseteq B$ from line 1.1.
    1.2.4. Case 2: suppose $z \in B$.
        1.2.4.1. Then $z \in B$.
    1.2.5. In either case, we have $z \in B$.
1.3. ("$A \cup B \supseteq B$")
    1.3.1. Let $z \in B$.
    1.3.2. Then $z \in A$ or $z \in B$ by the definition of "or".
    1.3.3. So $z \in A \cup B$ by the definition of $\cup$.
1.4. Lines 1.3 and 1.2 imply $A \cup B = B$ by the definition of set equality.

2. ("If")
2.1. Suppose $A \cup B = B$.
2.2. We prove $A \subseteq B$ as follows.
    2.2.1. Let $z \in A$.
    2.2.2. Then $z \in A$ or $z \in B$ by the definition of "or".
    2.2.3. So $z \in A \cup B$ by the definition of $\cup$.
    2.2.4. This implies $z \in B$ as $A \cup B = B$ by line 2.1. $\square$

10. For sets A and B, define $A \oplus B = (A \setminus B) \cup (B \setminus A)$.

(a) Let $A = \{1, 4, 9, 16\}$ and $B = \{2, 4, 6, 8, 10, 12, 14, 16\}$. Find $A \oplus B$.

(b) Show that for all sets A, B,

$$A \oplus B = (A \cup B) \setminus (A \cap B).$$

Solution.

(a) $A \setminus B = \{1, 9\}$ and $B \setminus A = \{2, 6, 8, 10, 12, 14\}$. So $A \oplus B = \{1, 2, 6, 8, 9, 10, 12, 14\}$.

(b) Compare the following truth tables.

$z \in A$	$z \in B$	$z \in A \setminus B$	$z \in B \setminus A$	$z \in A \oplus B$
T	T	F	F	F
T	F	T	F	T
F	T	F	T	T
F	F	F	F	F

$z \in A$	$z \in B$	$z \in A \cup B$	$z \in A \cap B$	$z \in (A \cup B) \setminus (A \cap B)$
T	T	T	T	F
T	F	T	F	T
F	T	T	F	T
F	F	F	F	F

Since the last columns of the two tables are the same, we conclude that $A \oplus B = (A \cup B) \setminus (A \cap B)$. $\square$

Instead of the truth tables above, one may prove this using the set identities. Here U denotes the universal set.

1.	$A \oplus B$	
2.	$= (A \setminus B) \cup (B \setminus A)$	by the definition of $\oplus$;
3.	$= (A \cap \overline{B}) \cup (B \cap \overline{A})$	by the Set Difference Law;
4.	$= \big((A \cap \overline{B}) \cup B\big) \cap \big((A \cap \overline{B}) \cup \overline{A}\big)$	by the Distributive Law;
5.	$= (A \cup B) \cap (\overline{B} \cup B) \cap (A \cup \overline{A}) \cap (\overline{B} \cup \overline{A})$	by the Distributive Law;
6.	$= (A \cup B) \cap U \cap U \cap (\overline{B} \cup \overline{A})$	by the Complement Law;
7.	$= (A \cup B) \cap U \cap U \cap (\overline{B \cap A})$	by De Morgan's Law;
8.	$= (A \cup B) \cap (\overline{B \cap A})$	by the Identity Law;
9.	$= (A \cup B) \cap (\overline{A \cap B})$	by the Commutative Law;
10.	$= (A \cup B) \setminus (A \cap B)$	by the Set Difference Law. $\square$

Alternatively, one may also use the element method.

11. (2015/16 Semester 1 exam question 16(a)) Denote by $|x|$ the absolute value of the integer x, i.e.,

$$|x| = \begin{cases} x, & \text{if } x \geqslant 0; \\ -x, & \text{if } x < 0. \end{cases}$$

Given the set $S = \{-9, -6, -1, 3, 5, 8\}$, for each of the following statements, state whether it is true or false, with explanation.

(a) $\exists z \in S \ \ \forall x, y \in S \ \ z > |x - y|$.

(b) $\exists z \in S \ \ \forall x, y \in S \ \ z < |x - y|$.

Solution.

(a) This statement is false, as shown below.
1. It suffices to show $\forall z \in S \ \ \exists x, y \in S \ \ z \leqslant |x - y|$.
2. Take any $z \in S$.
3. Let $x = 8$ and $y = -9$.
4. Then $x, y \in S$ and $|x - y| = |8 - (-9)| = 17 > 8 = \max S \geqslant z$. $\square$

(b) This statement is true, as shown below.
1. $-1 \in S$.
2. $|x - y| \geqslant 0 > -1$ for all $x, y \in S$. $\square$

12. For sets $A_m, A_{m+1}, \ldots, A_n$, define

$$\bigcup_{i=m}^{n} A_i = A_m \cup A_{m+1} \cup \cdots \cup A_n \quad \text{and} \quad \bigcap_{i=m}^{n} A_i = A_m \cap A_{m+1} \cap \cdots \cap A_n.$$

(a) Let $A_i = \{x \in \mathbb{Z} : x \geqslant i\}$ for each $i \in \mathbb{Z}$. Write down $\bigcup_{i=2}^{5} A_i$ and $\bigcap_{i=2}^{5} A_i$ in roster notation.

(b) Let $B_1, B_2, \ldots, B_k, C_1, C_2, \ldots, C_\ell$ be sets such that

$$\bigcup_{i=1}^{k} B_i \subseteq \bigcap_{j=1}^{\ell} C_j.$$

Show that $B_i \subseteq C_j$ for all $i \in \{1, 2, \ldots, k\}$ and all $j \in \{1, 2, \ldots, \ell\}$.

Solution.

(a) $\bigcup_{i=2}^{5} A_i = \{2, 3, 4, \ldots\}$ and $\bigcap_{i=2}^{5} A_i = \{5, 6, 7, \ldots\}$.

(b) 1. Let $B_1, B_2, \ldots, B_k, C_1, C_2, \ldots, C_\ell$ be sets such that $\bigcup_{i=1}^{k} B_i \subseteq \bigcap_{j=1}^{\ell} C_j$.
2. 2.1. Let $r \in \{1, 2, \ldots, k\}$ and $s \in \{1, 2, \ldots, \ell\}$.
   2.2. Take any $z \in B_r$.

2.3. Then $z \in B_1$ or $z \in B_2$ or $\ldots$ or $z \in B_k$ by the definition of "or", as $r \in \{1, 2, \ldots, k\}$.

2.4. So $z \in B_1 \cup B_2 \cup \cdots \cup B_k = \bigcup_{i=1}^{k} B_i$ by the definition of $\cup$ and $\bigcup$.

2.5. Hence $z \in \bigcap_{j=1}^{\ell} C_j = C_1 \cap C_2 \cap \cdots \cap C_\ell$ as $\bigcup_{i=1}^{k} B_i \subseteq \bigcap_{j=1}^{\ell} C_j$ by line 1.

2.6. Thus $z \in C_1$ and $z \in C_2$ and $\ldots$ and $z \in C_\ell$ by the definition of $\cap$.

2.7. In particular, we know $z \in C_s$ as $s \in \{1, 2, \ldots, \ell\}$.

3. So $B_i \subseteq C_j$ for any $i \in \{1, 2, \ldots, k\}$ and any $j \in \{1, 2, \ldots, \ell\}$. $\square$