Background

Definition. Let R be a relation from a set A to a set B. Define the relation R^{-1} from B to A by setting

$$y \, R^{-1} \, x \quad \Leftrightarrow \quad x \, R \, y$$

for each $y \in B$ and each $x \in A$.

Definition. A positive integer is *prime* if it has exactly two positive divisors.

Definition: Divisibility

If n and d are integers and $d \neq 0$, then

n is divisible by d iff n equals d times some integer.

We use the notation $d \mid n$ to mean "d divides n". Symbolically, if $n, d \in \mathbb{Z}$ and $d \neq 0$:

$$d \mid n \Leftrightarrow \exists k \in \mathbb{Z} \text{ such that } n = dk.$$

$dk = n$, k is some integers.

1. Let $A = \{1, 2, \ldots, 10\}$ and $B = \{2, 4, 6, 8, 10, 12, 14\}$. Define a relation R from A to B by setting

$$x \, R \, y \quad \Leftrightarrow \quad x \text{ is prime and } x \mid y$$

for each $x \in A$ and each $y \in B$. Write down the sets R and R^{-1} in roster notation. Do not use ellipses $(\ldots)$ in your answers.

1. $R = \{(2,2), (2,4), (2,6), (2,8), (2,10), (2,12), (2,14), (3,6), (3,12), (5,10), (7,14)\}$.

$R^{-1} = \{(2,2), (2,4), (2,6), (2,8), (2,10)\}$. $\{(2,2), (4,2), (6,2), (8,2), (10,2), (12,2), (14,2), (6,3), (12,3), (10,5), (14,7)\}$

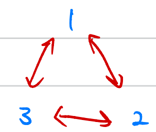
$yR^{-1}x \Leftrightarrow xRy$

2. Let R be a relation on a set A. Show that the following are equivalent.

(i) R is symmetric, i.e., $\forall x, y \in A \; (x \, R \, y \Rightarrow y \, R \, x)$.

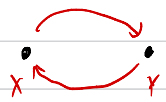
(ii) $\forall x, y \in A \; (x \, R \, y \Leftrightarrow y \, R \, x)$.

(iii) $R = R^{-1}$.



Directly prove $1 \to 3$, $3 \to 2$, $2 \to 1$

2. i) S_1 : $\forall x, y \in A \; (xRy \Rightarrow yRx)$



ii) S_2 : $\forall x, y \in A \; (xRy \Leftrightarrow yRx)$

iii) S_3 : $R = R^{-1} \Rightarrow xRy = yRx$

1. For every statement, every statement is related to itself. Hence the relationship between S_1, S_2 and S_3 is reflexive
2. $S_1 = S_2$, since $xRy \Leftrightarrow yRx \Rightarrow xRy \Rightarrow yRx \wedge yRx \Rightarrow xRy$ and $xRy \Rightarrow yRx$ is same by the definition of symmetry.
   $S_2 = S_3$, since $xRy \Leftrightarrow yRx$ is same as $xRy = yRx$.
   $S_1 = S_3$, by transitivity.
   Thus, the relation between S_1, S_2 and S_3 are symmetric.
3. Since $S_1 = S_2$ and $S_2 = S_3$ and $S_1 = S_3$, thus the relations between S_1, S_2 and S_3 are transitive.
4. Thus, by definition of equivalence, S_1, S_2 and S_3 are equivalent.

1. (i) → (ii)
   - 1.1 Suppose R is symmetric
   - 1.2 Let $x, y \in A$.
   - 1.3 ($\Rightarrow$) Then $xRy \Rightarrow yRx$ as R is symmetric.
   - 1.4 ($\Leftarrow$) Similarly $yRx \Rightarrow xRy$ as R is symmetric
   - 1.5 So $xRy \Leftrightarrow yRx$ by 1.3 and 1.4.
2. (ii) → (iii)
   - 2.1 Assume $\forall x, y \in A \; (xRy \Leftrightarrow yRx)$
   - 2.2 R^{-1} is $yR^{-1}x \Leftrightarrow xRy$, for all x, y. $R \in A \times A$.
     - 2.2.1 $(x,y) \in R \Leftrightarrow xRy$ by defn. of xRy.
     - 2.2.2 $\Leftrightarrow yRx$ by 2.1
     - 2.2.3 $\Leftrightarrow xR^{-1}y$ by def of R^{-1}
     - 2.2.4 $\Leftrightarrow (x,y) \in R^{-1}$ by def of $xR^{-1}y$

     $R \subseteq R^{-1}$ — note: need to prove the other way also.

     $R^{-1} \subseteq R$
   - 2.3 This shows $R = R^{-1}$
3. (iii) → (i)
   - 3.1 Suppose $R = R^{-1}$
   - 3.2
     - 3.2.1 Let $x, y \in A$ s.t. xRy
     - 3.2.2 Then $xR^{-1}y$ as $R = R^{-1}$.
     - 3.2.3 So yRx. by defn of R^{-1}
   - 3.3 $\therefore R$ is symmetric

3. For each of the relations defined below, determine whether it is (i) reflexive, (ii) symmetric, (iii) transitive, and (iv) an equivalence relation.

(a) Define $A = \{1, 2, 3\}$ and $Q = \{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$. Consider Q as a relation on A.

(b) Define the relation R on $\mathbb{Q}$ by setting, for all $x, y \in \mathbb{Q}$,

$$x \, R \, y \quad \Leftrightarrow \quad xy \geqslant 0.$$

(c) Define the relation S on $\mathbb{Q}$ by setting, for all $x, y \in \mathbb{Q}$,

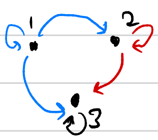
$$x \, S \, y \quad \Leftrightarrow \quad xy > 0.$$

(d) Define the relation T on $\mathbb{Z}$ by setting, for all $x, y \in \mathbb{Z}$,

$$x \, S \, y \quad \Leftrightarrow \quad -2 \leqslant x - y \leqslant 2.$$

Note that $0 \in \mathbb{Q}$.

a)



Q is reflexive

Q is not symmetric.

Q is transitive.

Q is not equivalence

$1 \to 2$
$2 \to 3$
$1 \to 3$

b) i) R is reflexive iff the following statement is true,

$\forall x \in \mathbb{Q}, \ x R x.$

This is the same as saying,

$\forall x \in \mathbb{Q}, \ x \cdot x \geq 0$

This is true for $x < 0$, $x > 0$ and $x = 0$,

Thus R is reflexive

ii) R is symmetric iff the following is true,

$\forall x, y \in \mathbb{Q}$, if xRy then yRx.

this is the same as,

$\forall x, y \in \mathbb{Q}$, if $xy \geq 0$ then $yx \geq 0$.

This is true as the value of xy and yx is the same by commutative law.

iii) R is transitive iff the following is true,

$\forall x, y, z \in \mathbb{Q}$, if xRy, yRz, then xRz.

This is same as,

$\forall x, y, z \in \mathbb{Q}$, if $xy \geq 0$, $yz \geq 0$, then $xz \geq 0$.

This is false

Counter-example: $x > 0$, $y = 0$, $z < 0$, then $xy \geq 0$, $yz \geq 0$ but $xz < 0 \neq xz \geq 0$.

iv) Since R is not transitive, R is not an equivalence relation.

3. (c) Define the relation S on $\mathbb{Q}$ by setting, for all $x, y \in \mathbb{Q}$,

$$x \, S \, y \quad \Leftrightarrow \quad xy > 0.$$

(d) Define the relation T on $\mathbb{Z}$ by setting, for all $x, y \in \mathbb{Z}$,

$$x \, T \, y \quad \Leftrightarrow \quad -2 \leqslant x - y \leqslant 2.$$

c) i) S is reflexive iff the following statement is true,

$\forall x \in \mathbb{Q},\ x S x.$

This is the same as saying,

$\forall x \in \mathbb{Q},\ x \cdot x > 0$

This is false when $x = 0$.

Thus S is not reflexive.

ii) S is symmetric iff the following is true,

$\forall x, y \in \mathbb{Q}$, if xSy then ySx.

this is the same as,

$\forall x, y \in \mathbb{Q}$, if $xy > 0$ then $yx > 0$

This is true as the value of xy and yx is the same by commutative law.

iii) S is transitive iff the following is true,

$\forall x, y, z \in \mathbb{Q}$, if xSy, ySz, then xSz.

this is same as,

$\forall x, y, z \in \mathbb{Q}$, if $xy > 0$, $yz > 0$, then $xz > 0$.

This is true. When $x > 0$, $y > 0$, $xy > 0$ and $z > 0$, $yz > 0$, then $xz > 0$.

When $x < 0$, $y < 0$, $xy > 0$ and $z < 0$, $yz > 0$, then $xz > 0$.

iv) Since S is not reflexive, S is not an equivalence relation.

d) i) $-2 \leq x - x \leq 2$

$-2 \leq 0 \leq 2$ is true.

$\therefore$ reflexive

ii) $-2 \leq x - y \leq 2$, $-2 \leq y - x \leq 2$.

True as the difference is the same but only the sign change.

Since $|x-y| = |y-x|$ but

if $x - y > 0$, $y - x < 0$

and if $x - y < 0$, $y - x > 0$.

iii) if $-2 \leq x - y \leq 2$, $-2 \leq y - z \leq 2$,

then $-2 \leq x - z \leq 2$

False. Counter: $x = 6$, $y = 4$, $z = 2$.

Then $x - y = 2$

$y - z = 2$

$x - z = 4 > 2$

iv) Since T is not transitive, T is not a equivalence relation.

4. (2020/21 Semester 1 exam question 18) Define an equivalence relation $\sim$ on $\mathbb{Z}^+ \times \mathbb{Z}^+$ by setting, for all $a, b, c, d \in \mathbb{Z}^+$,

$$(a, b) \sim (c, d) \quad \Leftrightarrow \quad ab = cd.$$

all the elements inside A that are related to

Write down the equivalence classes $[(1,1)]$ and $[(4,3)]$ in roster notation.

set of all non-empty subsets of $\mathbb{Z}^+ \times \mathbb{Z}^+$ where $ab = cd$.

$[(1,1)] = \{(a,b) \in (\mathbb{Z}^+ \times \mathbb{Z}^+) \mid (a,b) \sim (1,1)\}$.
$= \{(1,1)\}$

$[(4,3)] = \{(a,b) \in (\mathbb{Z}^+ \times \mathbb{Z}^+) \mid (a,b) \sim (4,3)\}$.
$= \{(4,3), (3,4), (2,6), (6,2), (1,12), (12,1)\}$

5. Define a relation R on $\mathbb{Q}$ as follows: for all $x, y \in \mathbb{Q}$,

$$x \, R \, y \quad \Leftrightarrow \quad x - y \in \mathbb{Z}.$$

(a) Show that R is an equivalence relation.

(b) Find an element a in the equivalence class $[\frac{37}{7}]$ that satisfies $0 \leqslant a < 1$.

(c) Devise a general method to find, for each given equivalence class $[x]$, where $x \in \mathbb{Q}$, an element $a \in [x]$ such that $0 \leqslant a < 1$. Justify your answer.

a) R is reflexive iff the following statement is true,
$\forall x \in \mathbb{Q},\ xRx$.
This is the same as saying,
$\forall x \in \mathbb{Q},\ x - x \in \mathbb{Z}$.
This is true since $x - x = 0 \in \mathbb{Z}$.
Thus R is reflexive.

R is symmetric iff the following is true,
$\forall x, y \in \mathbb{Q}$, if xRy then yRx.
This is the same as,
$\forall x, y \in \mathbb{Q}$, if $x - y \in \mathbb{Z}$ then $y - x \in \mathbb{Z}$. (by closure of integer under multiplication)
This is true as the difference between x and y is an integer, then difference between y and x is also an integer.

R is transitive iff the following is true,
$\forall x, y, z \in \mathbb{Q}$, if xRy, yRz, then xRz.
This is same as,
$\forall x, y, z \in \mathbb{Q}$, if $x - y \in \mathbb{Z}$, $y - z \in \mathbb{Z}$, then $x - z \in \mathbb{Z}$.
This is true. If $|x - y| \in \mathbb{Z}$ and $|y - z| \in \mathbb{Z}$,
then $|x - z| \in \mathbb{Z}$ by closure of integer under subtraction

$x - y = a \wedge y - z = b$
$(x - y) + (y - z) = a + b$
$x - z = a + b \in \mathbb{Z}$
(by closure under addition)

Since R reflexive, symmetric and transitive, R is a equivalence relation.

5. Define a relation R on $\mathbb{Q}$ as follows: for all $x, y \in \mathbb{Q}$,

$$x \, R \, y \quad \Leftrightarrow \quad x - y \in \mathbb{Z}.$$

(a) Show that R is an equivalence relation.

(b) Find an element a in the equivalence class $[\frac{37}{7}]$ that satisfies $0 \leqslant a < 1$.

(c) Devise a general method to find, for each given equivalence class $[x]$, where $x \in \mathbb{Q}$, an element $a \in [x]$ such that $0 \leqslant a < 1$. Justify your answer.

all elems that are related to $\frac{37}{7}$ $\{ \ldots \frac{37}{7} - 1, \frac{37}{7}, \frac{37}{7} + 1 \ldots \}$

b) $[\frac{37}{7}] = \{ a \in \mathbb{Q} \mid a - \frac{37}{7} \in \mathbb{Z} \}$

$\therefore a - \frac{37}{7} \in \mathbb{Z}$.

$a - 5\frac{2}{7}$

$0 \leq \therefore a = \frac{2}{7} \cancel{<1}$

c) $[x] = \{ a \in \mathbb{Q} \mid a - x \in \mathbb{Z} \}$

let $a = x - \lfloor x \rfloor$.

In order for $0 \leq a < 1$, a need to make up the difference between $\lfloor x \rfloor$ and x for $a - x \in \mathbb{Z}$.

Thus, the only viable solution is $x - \lfloor x \rfloor$.

Using the definition

1. Input: $x \in \mathbb{Q}$
2. Use defn of $\mathbb{Q}$, find $m, n \in \mathbb{Z}$ s.t. $x = \frac{m}{n}$ and $n \neq 0$
3. Replace m, n with $-m, -n$ if needed, assume $n > 0$
4. Use division theorem, find $q, r \in \mathbb{Z}$ s.t. $m = nq + r$ and $0 \leq r < n$.
5. output $= \frac{r}{n}$.

· Note $0 \leq a < 1$ as $0 \leq r < n$

· Also $x - a = \frac{m}{n} - \frac{r}{n} = \frac{\overbrace{m-r}^{=nq}}{n} = \frac{\cancel{n}q}{\cancel{n}} = q \in \mathbb{Z}$ as $m = nq + r$

You may use the following theorem without proof for this question.

Division Theorem. For all $n \in \mathbb{Z}$ and $d \in \mathbb{Z}^+$, there exist unique $q, r \in \mathbb{Z}$ such that

$$n = dq + r \quad \text{and} \quad 0 \leqslant r < d.$$

6. Fix $m, n \in \mathbb{Z}^+$. Let $\sim_m$ and $\sim_n$ denote respectively the congruence-mod-m and the congruence-mod-n relations on $\mathbb{Z}$. Prove that $[x]_{\sim_m} \subseteq [y]_{\sim_n}$ for some $x, y \in \mathbb{Z}$ if and only if $n \mid m$.

(Hint: Example 6.4.3 may be helpful.)

m apart   must be inside y   no need all.

$x_1 \cdots \underbrace{y_1 \cdots y_2}_{n} \cdots x_2$, m

$m, n \in \mathbb{Z}^+$.

$a \sim_m b \Leftrightarrow m \mid (a-b) \Leftrightarrow a \equiv b \pmod m$

$a \sim_n b \Leftrightarrow n \mid (a-b) \Leftrightarrow a \equiv b \pmod n$.

prove $[x]_{\sim m} \subseteq [y]_{\sim n}$, $x, y \in \mathbb{Z}$ iff $n \mid m$. $\Leftrightarrow m = nk, k \in \mathbb{Z}$. ($n$ is smaller than m)

(m is of multiples of n)

(The smaller the mod, the more elements in the set).

elements in $[x]$ are m apart

$[x]_{\sim m} = \{a \in \mathbb{Z} \mid m \mid (a-x)\}. = \{a \in \mathbb{Z} \mid a - x = mk, k \in \mathbb{Z}\}$.

$[y]_{\sim n} = \{b \in \mathbb{Z} \mid n \mid (b-y)\}. = \{b \in \mathbb{Z} \mid b - y = nk, k \in \mathbb{Z}\}$.

elements in $[y]$ are n apart.

1. "$\Leftarrow$"

1.1 Suppose $n \mid m$

1.2 Use the defn of divisibility to find $a \in \mathbb{Z}$ s.t. $m = an$

1.3 Then $[0]_{\sim m} = \{mk + 0 : k \in \mathbb{Z}\}$ by example 6.4.3.

$= \{ank + 0 : k \in \mathbb{Z}\}$ by choice of a

$\subseteq \{nl + 0 : l \in \mathbb{Z}\}$ $\mathbb{Z}$ is closed under $\times$.

$= [0]_{\sim n}$ by example 6.4.3.

$\therefore$ Only if $m = nk$,

then $[x]_{\sim m} = \{a \in \mathbb{Z} \mid a - x = (nk_n)k_m, k_m \in \mathbb{Z}\}$.

and $[y]_{\sim n} = \{b \in \mathbb{Z} \mid b - y = nk_n, k_n \in \mathbb{Z}\}$.

$\therefore [x]_{\sim m} = \{a \in \mathbb{Z} \mid a - x = nk, k \in \mathbb{Z}\}$.

$[y]_{\sim n} = \{b \in \mathbb{Z} \mid b - y = nk_n, k_n \in \mathbb{Z}\}$.

$\therefore a - x = b - y$ for $k = k_n$.

$[x]_{\sim m} = [y]_{\sim n}$ for $k_m = 1$

$\therefore$ Set of $[x]_{\sim m}$ is smaller than set of $[y]_{\sim n}$ if $k_m \neq 1$

For example, if $\{a \in \mathbb{Z} \mid a - x = nk_nk_m\} \supset \{a \in \mathbb{Z} \mid a - y = nk_n\}$.

but $[x]_{\sim m}$ needs to satisfy more conditions than $[y]_{\sim n}$, hence $[x]_{\sim m} \supset [y]_{\sim n}$ cannot be true, by contradiction, statement is true.

2. "$\Rightarrow$"

2.1 Suppose $x, y \in \mathbb{Z}$ s.t. $[x]_{\sim m} \subseteq [y]_{\sim n}$

2.2 Then Example 6.4.3, $\{\underbrace{(mk+x)}_{\text{this form}} : k \in \mathbb{Z}\} \subseteq \{(nl + y) : l \in \mathbb{Z}\}$

2.3. Find $l_1, l_2 \in \mathbb{Z}$ s.t. $x = m \cdot 0 + x = nl_1 + y$ and $x + m = m \cdot 1 + x = nl_2 + y$.

2.4. Then $m = (x+m) - x = (nl_2 + y) - (nl_1 + y) = n(l_2 - l_1)$.

2.5. As $l_1, l_2 \in \mathbb{Z}$, we know $l_2 - l_1 \in \mathbb{Z}$. $\mathbb{Z}$ closed under $-$

2.6. So $n \mid m$ by definition of divisibility.

(If) $[x]_{\sim m} = \{a \in \mathbb{Z} \mid a - x = nk_nk_m\}$.

$\subseteq [y]_{\sim n} = \{a \in \mathbb{Z} \mid a - y = nk_n\}$.

Then $m = nk$.

Since p is true, then q is true.

C is a set of all subset of A that are non-empty. every element of A must be in one element of C

7. Let $\mathscr{C}$ be a partition of a set A. Denote by $\sim$ the same-component relation with respect to $\mathscr{C}$, i.e., for all $x, y \in A$,

$$x \sim y \quad \Leftrightarrow \quad x \text{ is in the same component of } \mathscr{C} \text{ as } y$$
$$\Leftrightarrow \quad x, y \in S \text{ for some } S \in \mathscr{C}.$$

Recall from Proposition 6.2.16 that $\sim$ is an equivalence relation on A.

(a) Prove that if $x \in S \in \mathscr{C}$, then $[x] = S$. → If inside a group. that all things that are related to an element in the group then it is in

(b) Prove that $A/\sim = \mathscr{C}$.

↳ set of all equivalence classes of elements in A.

a) $x \in S \in C \rightarrow [x] = S$

Suppose $x \notin S \in C \rightsquigarrow [x] = S$.

and $[x] = \{a \in A \mid a \sim x\}$

$\therefore$ a is in same component as x.

$a, x \in S$ for some $S \in C$.

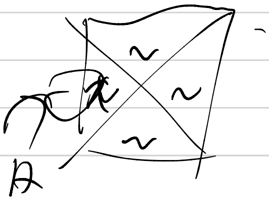
$\therefore$ By definition,

$x \notin S \in C$ cannot be true.

$\therefore$ By contradiction, $x \in S \in C$ must be true for $[x] = S$.

a)
1. Let $x \in S \in C$
2. "$\supseteq$" If $y \in S$, then
   2.1 $x \sim y$ by defn of $\sim$ as $x, y \in S \in C$
   2.2 $\therefore y \in [x]$ by defn of $[x]$. (any y is related to x)
3. "$\subseteq$"
   3.1 Let $y \in [x]$.
   3.2 Then $x \sim y$ by defn of $[x]$. ↗ to show that same group $= S$.
   3.3 Use defn of $\sim$ to find $\hat{S} \in C$ such that $x, y \in \hat{S}$
   3.4 Since $x \in S \cap \hat{S}$, we deduce that $S = \hat{S}$ as $S, \hat{S}$ are components in partition C.
   iff different $\therefore \Rightarrow S \cap \hat{S} = \emptyset$.
   3.5 Hence $y \in \hat{S} = S$.
4. Thus $[x] = S$.

b) $A/\sim = C$.



$A/\sim$: $x \not\sim y$: x is not in the same component of C as y,

$x, y \notin S$ for some $S \in C$.

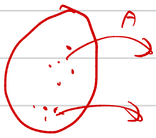
$\therefore$ Set A w/o $x \sim y$ is still the partition of set A.

$\therefore$

$A/\sim$ = quotient of A by $\sim$.

$\Rightarrow$ set of all the equivalence class of A.

$= \{ \quad \}$



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$\{\}$

Saying same thing as first qns.

$\Rightarrow$ All equivalence class form the entire partition

CS1231S Tutorial 4: Equivalence relations
Solutions

National University of Singapore

2021/22 Semester 1

1. Let $A = \{1, 2, \ldots, 10\}$ and $B = \{2, 4, 6, 8, 10, 12, 14\}$. Define a relation R from A to B by setting
$$x \; R \; y \quad \Leftrightarrow \quad x \text{ is prime and } x \mid y$$
for each $x \in A$ and each $y \in B$. Write down the sets R and R^{-1} in roster notation. Do not use ellipses $(\ldots)$ in your answers.

Solution.

$$R = \{(2,2),(2,4),(2,6),(2,8),(2,10),(2,12),(2,14),(3,6),(3,12),(5,10),(7,14)\}.$$
$$R^{-1} = \{(2,2),(4,2),(6,2),(8,2),(10,2),(12,2),(14,2),(6,3),(12,3),(10,5),(14,7)\}.$$

2. Let R be a relation on a set A. Show that the following are equivalent.

(i) R is symmetric, i.e., $\forall x, y \in A \; (x \; R \; y \Rightarrow y \; R \; x)$.

(ii) $\forall x, y \in A \; (x \; R \; y \Leftrightarrow y \; R \; x)$.

(iii) $R = R^{-1}$.

Solution.

1. ((i) $\Rightarrow$ (ii))
   1.1. Suppose R is symmetric.
   1.2. Let $x, y \in A$.
   1.3. $(\Rightarrow)$ If $x \; R \; y$, then $y \; R \; x$ by the symmetry of R.
   1.4. $(\Leftarrow)$ If $y \; R \; x$, then $x \; R \; y$ by the symmetry of R.
   1.5. Combining the two, we have $x \; R \; y \Leftrightarrow y \; R \; x$, as required.
2. ((ii) $\Rightarrow$ (iii))
   2.1. Assume $\forall x, y \in A \; (x \; R \; y \Leftrightarrow y \; R \; x)$.
   2.2. Then for all x, y,

2.2.1.	$(x, y) \in R$	$\Leftrightarrow$	$x \; R \; y$	by the definition of $x \; R \; y$;
2.2.2.		$\Leftrightarrow$	$y \; R \; x$	by our assumption on line 2.1;
2.2.3.		$\Leftrightarrow$	$x \; R^{-1} \; y$	by the definition of R^{-1};
2.2.4.		$\Leftrightarrow$	$(x, y) \in R^{-1}$	by the definition of $x \; R^{-1} \; y$.

   2.3. This shows $R = R^{-1}$.
3. ((iii) $\Rightarrow$ (i))
   3.1. Suppose $R = R^{-1}$.
   3.2. 3.2.1. Let $x, y \in A$ such that $x \; R \; y$.
        3.2.2. Then $x \; R^{-1} \; y$ as $R = R^{-1}$.
        3.2.3. $\therefore$ $y \; R \; x$ by the definition of R^{-1}.
   3.3. So R is symmetric. $\square$

3. For each of the relations defined below, determine whether it is (i) reflexive, (ii) symmetric, (iii) transitive, and (iv) an equivalence relation.

   (a) Define $A = \{1, 2, 3\}$ and $Q = \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\}$. Consider Q as a relation on A.

   (b) Define the relation R on $\mathbb{Q}$ by setting, for all $x, y \in \mathbb{Q}$,
   $$x \mathrel{R} y \quad\Leftrightarrow\quad xy \geqslant 0.$$

   (c) Define the relation S on $\mathbb{Q}$ by setting, for all $x, y \in \mathbb{Q}$,
   $$x \mathrel{S} y \quad\Leftrightarrow\quad xy > 0.$$

   (d) Define the relation T on $\mathbb{Z}$ by setting, for all $x, y \in \mathbb{Z}$,
   $$x \mathrel{S} y \quad\Leftrightarrow\quad -2 \leqslant x - y \leqslant 2.$$

   Solution.

   (a) Q is reflexive and transitive. It is not symmetric because $1 \mathrel{Q} 2$ but $2 \not\mathrel{Q} 1$.

   (b) R is reflexive and symmetric. It is not transitive because $1 \mathrel{R} 0$ and $0 \mathrel{R} -1$ but $1 \not\mathrel{R} -1$. Since it is not transitive, it is not an equivalence relation.

   (c) S is symmetric and transitive. It is not reflexive because $0 \not\mathrel{S} 0$. Since it is not reflexive, it is not an equivalence relation.

   (d) T is reflexive and symmetric. It is not transitive because $-2 \mathrel{T} 0$ and $0 \mathrel{T} 2$ but $-2 \not\mathrel{T} 2$. Since it is not transitive, it is not an equivalence relation.

4. (2020/21 Semester 1 exam question 18) Define an equivalence relation $\sim$ on $\mathbb{Z}^+ \times \mathbb{Z}^+$ by setting, for all $a, b, c, d \in \mathbb{Z}^+$,
   $$(a, b) \sim (c, d) \quad\Leftrightarrow\quad ab = cd.$$
   Write down the equivalence classes $[(1, 1)]$ and $[(4, 3)]$ in roster notation.

   Solution. (No working is required for this question.)
   $$\begin{aligned}
   [(1,1)] &= \{(a,b) \in \mathbb{Z}^+ \times \mathbb{Z}^+ : (1,1) \sim (a,b)\} \\
   &= \{(a,b) \in \mathbb{Z}^+ \times \mathbb{Z}^+ : ab = 1 \times 1 = 1\} \\
   &= \{(1,1)\}. \\
   [(4,3)] &= \{(a,b) \in \mathbb{Z}^+ \times \mathbb{Z}^+ : (4,3) \sim (a,b)\} \\
   &= \{(a,b) \in \mathbb{Z}^+ \times \mathbb{Z}^+ : ab = 4 \times 3 = 12\} \\
   &= \{(1,12), (2,6), (3,4), (4,3), (6,2), (12,1)\}.
   \end{aligned}$$

5. Define a relation R on $\mathbb{Q}$ as follows: for all $x, y \in \mathbb{Q}$,
   $$x \mathrel{R} y \quad\Leftrightarrow\quad x - y \in \mathbb{Z}.$$

   (a) Show that R is an equivalence relation.

   (b) Find an element a in the equivalence class $[\frac{37}{7}]$ that satisfies $0 \leqslant a < 1$.

   (c) Devise a general method to find, for each given equivalence class $[x]$, where $x \in \mathbb{Q}$, an element $a \in [x]$ such that $0 \leqslant a < 1$. Justify your answer.

   Solution.

   (a) 1. ("Reflexivity")

1.1. Let $x \in \mathbb{Q}$.
1.2. Then $x - x = 0 \in \mathbb{Z}$.
1.3. So $x \mathrel{R} x$.

2. ("Symmetry")
   2.1. Let $x, y \in \mathbb{Q}$ such that $x \mathrel{R} y$.
   2.2. Then $x - y \in \mathbb{Z}$ by the definition of R.
   2.3. So $y - x = (-1)(x - y) \in \mathbb{Z}$ as $\mathbb{Z}$ is closed under multiplication.
   2.4. This implies $y \mathrel{R} x$ by the definition of R.
3. ("Transitivity")
   3.1. Let $x, y, z \in \mathbb{Q}$ such that $x \mathrel{R} y$ and $y \mathrel{R} z$.
   3.2. Then $x - y \in \mathbb{Z}$ and $y - z \in \mathbb{Z}$ by the definition of R.
   3.3. So $x - z = (x - y) + (y - z) \in \mathbb{Z}$ as $\mathbb{Z}$ is closed under addition.
   3.4. This implies $x \mathrel{R} z$ by the definition of R.
4. Since R is reflexive, symmetric and transitive, it is an equivalence relation. □

(b) Note that $\frac{37}{7} = 5\frac{2}{7}$. Thus $\frac{37}{7} - \frac{2}{7} = 5 \in \mathbb{Z}$. This implies $\frac{37}{7} \mathrel{R} \frac{2}{7}$ and hence $\frac{2}{7} \in [\frac{37}{7}]$.

(c) Let $x \in \mathbb{Q}$. Take $m \in \mathbb{Z}$ and $n \in \mathbb{Z} \setminus \{0\}$ such that $x = m/n$. Replacing m, n with $-m, -n$ if needed, we may assume $n > 0$. Apply the Division Theorem to find $q, r \in \mathbb{Z}$ such that

$$m = nq + r \quad \text{and} \quad 0 \leqslant r < n.$$

Define $a = r/n$. Then we know $0 \leqslant a < 1$ because $0 \leqslant r < n$. In addition,

$$x - a = \frac{m}{n} - \frac{r}{n} = \frac{m - r}{n} = \frac{nq}{n} = q \in \mathbb{Z}$$

by the choice of q and r. Thus $x \mathrel{R} a$ and so $a \in [x]$.

6. Fix $m, n \in \mathbb{Z}^+$. Let $\sim_m$ and $\sim_n$ denote respectively the congruence-mod-m and the congruence-mod-n relations on $\mathbb{Z}$. Prove that $[x]_{\sim_m} \subseteq [y]_{\sim_n}$ for some $x, y \in \mathbb{Z}$ if and only if $n \mid m$.

Solution.

1. ("If")
   1.1. Suppose $n \mid m$.
   1.2. Use the definition of divisibility to find $a \in \mathbb{Z}$ such that $m = an$.
   1.3. Then $[0]_{\sim_m} = \{mk + 0 : k \in \mathbb{Z}\}$ by Example 6.4.3;
   1.4. $= \{(an)k + 0 : k \in \mathbb{Z}\}$ by the choice of a;
   1.5. $\subseteq \{n\ell + 0 : \ell \in \mathbb{Z}\}$ as $\mathbb{Z}$ is closed under multiplication;
   1.6. $= [0]_{\sim_n}$ by Example 6.4.3.
2. ("Only if")
   2.1. Suppose $x, y \in \mathbb{Z}$ such that $[x]_{\sim_m} \subseteq [y]_{\sim_n}$.
   2.2. Then Example 6.4.3 tells us $\{mk + x : k \in \mathbb{Z}\} \subseteq \{n\ell + y : \ell \in \mathbb{Z}\}$.
   2.3. Find $\ell_1, \ell_2 \in \mathbb{Z}$ such that $x = m \cdot 0 + x = n\ell_1 + y$ and $x + m = m \cdot 1 + x = n\ell_2 + y$.
   2.4. Then $m = (x + m) - x = (n\ell_2 + y) - (n\ell_1 + y) = n(\ell_2 - \ell_1)$.
   2.5. As $\ell_1, \ell_2 \in \mathbb{Z}$, we know $\ell_2 - \ell_1 \in \mathbb{Z}$ because $\mathbb{Z}$ is closed under subtraction.
   2.6. So $n \mid m$ by the definition of divisibility. □

7. Let $\mathscr{C}$ be a partition of a set A. Denote by $\sim$ the same-component relation with respect to $\mathscr{C}$, i.e., for all $x, y \in A$,

$$\begin{aligned} x \sim y \quad &\Leftrightarrow \quad x \text{ is in the same component of } \mathscr{C} \text{ as } y \\ &\Leftrightarrow \quad x, y \in S \text{ for some } S \in \mathscr{C}. \end{aligned}$$

Recall from Proposition 6.2.16 that $\sim$ is an equivalence relation on A. By Theorem 6.4.9, we know $A/{\sim}$ is a partition of A. The aim of this question is to show that this partition is the same as the partition $\mathscr{C}$ that we started off with.

(a) Prove that if $x \in S \in \mathscr{C}$, then $[x] = S$.

(b) Prove that $A/{\sim} = \mathscr{C}$.

Solution.

(a) 1. Let $x \in S \in \mathscr{C}$.

2. ($\supseteq$) If $y \in S$, then

   2.1. $x \sim y$ by the definition of $\sim$, as $x, y \in S \in \mathscr{C}$;

   2.2. $\therefore$ $y \in [x]$ by the definition of $[x]$.

3. ($\subseteq$)
   3.1. Let $y \in [x]$.
   3.2. Then $x \sim y$ by the definition of $[x]$.
   3.3. Use the definition of $\sim$ to find $\hat{S} \in \mathscr{C}$ such that $x, y \in \hat{S}$.
   3.4. Since $x \in S \cap \hat{S}$, we deduce that $S = \hat{S}$, because S and $\hat{S}$ are components in the partition $\mathscr{C}$.
   3.5. Hence $y \in \hat{S} = S$.

4. Thus $[x] = S$. □

(b) 1. ($\subseteq$)
   1.1. Let $[x] \in A/{\sim}$.
   1.2. Use the assumption that $\mathscr{C}$ is a partition of A to find $S \in \mathscr{C}$ such that $x \in S$.
   1.3. Then part (a) implies $[x] = S \in \mathscr{C}$.

2. ($\supseteq$)
   2.1. Let $S \in \mathscr{C}$.
   2.2. Then $S \neq \varnothing$ as S is a component in a partition.
   2.3. Take $x \in S$.
   2.4. Then part (a) implies $S = [x] \in A/{\sim}$.

3. Hence $A/{\sim} = \mathscr{C}$. □