EXERCISES FOR CHAPTER 7: LINEAR TRANSFORMATIONS

Question 7.1 to Question 7.17 are exercises for Sections 7.1 and 7.2.

1. Determine whether the following are linear transformations. Write down the standard matrix for each of the linear transformations.

(a) $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T_1\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ y-x \end{pmatrix}$ for $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

(b) $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T_2\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2^x \\ 0 \end{pmatrix}$ for $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

(c) $T_3 : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T_1\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ 0 \\ 0 \end{pmatrix}$ for $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

(d) $T_4 : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T_4\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 1 \\ y-x \\ y-z \end{pmatrix}$ for $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$.

(e) $T_5 : \mathbb{R}^n \to \mathbb{R}$ such that $T_5(\boldsymbol{x}) = \boldsymbol{x} \cdot \boldsymbol{y}$ for $\boldsymbol{x} \in \mathbb{R}^n$ where $\boldsymbol{y} = (y_1, \ldots, y_n)^{\mathrm{T}}$ is a fixed vector in $\mathbb{R}^n$.

(f) $T_6 : \mathbb{R}^n \to \mathbb{R}$ such that $T_6(\boldsymbol{x}) = \boldsymbol{x} \cdot \boldsymbol{x}$ for $x \in \mathbb{R}^n$.

(In Parts (e) and (f), $\mathbb{R}$ is regarded as $\mathbb{R}^1$.)

2. For each of the following linear transformations,

(i) determine whether there is enough information for us to find the formula of T; and

(ii) find the formula and the standard matrix for T if possible.

(a) $T : \mathbb{R}^3 \to \mathbb{R}^4$ such that

$$T\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 3 \\ 0 \\ 1 \end{pmatrix}, \quad T\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ 2 \\ -1 \\ 4 \end{pmatrix} \quad \text{and} \quad T\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 4 \\ 1 \\ 6 \end{pmatrix}.$$

(b) $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that

$$T\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad T\left(\begin{pmatrix} 2 \\ 2 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

(c) $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$T\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad T\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad \text{and} \quad T\left(\begin{pmatrix} 2 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

(d) $T:\mathbb{R}^3 \to \mathbb{R}^2$ such that

$$T\left(\begin{pmatrix}1\\1\\1\end{pmatrix}\right)=\begin{pmatrix}2\\3\end{pmatrix},\quad T\left(\begin{pmatrix}2\\1\\3\end{pmatrix}\right)=\begin{pmatrix}-1\\0\end{pmatrix}\quad\text{and}\quad T\left(\begin{pmatrix}-1\\1\\2\end{pmatrix}\right)=\begin{pmatrix}0\\1\end{pmatrix}.$$

(e) $T:\mathbb{R}^3 \to \mathbb{R}^3$ such that

$$T\left(\begin{pmatrix}1\\0\\1\end{pmatrix}\right)=\begin{pmatrix}1\\1\\2\end{pmatrix},\quad T\left(\begin{pmatrix}1\\1\\0\end{pmatrix}\right)=\begin{pmatrix}1\\0\\1\end{pmatrix}\quad\text{and}\quad T\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\right)=\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

(f) $T:\mathbb{R}^3 \to \mathbb{R}$ such that

$$T\left(\begin{pmatrix}1\\-1\\0\end{pmatrix}\right)=-1,\quad T\left(\begin{pmatrix}0\\1\\-1\end{pmatrix}\right)=1\quad\text{and}\quad T\left(\begin{pmatrix}-1\\0\\1\end{pmatrix}\right)=0.$$

3. Let S and T be linear transformations as defined below. Determine the formulae of the compositions $S\circ T$ and $T\circ S$ whenever they are defined.

(a) $S:\mathbb{R}^2 \to \mathbb{R}^3$ such that $S\left(\begin{pmatrix}x\\y\end{pmatrix}\right)=\begin{pmatrix}x+y\\x-y\\x\end{pmatrix}$;

$T:\mathbb{R}^2 \to \mathbb{R}^2$ such that $T\left(\begin{pmatrix}x\\y\end{pmatrix}\right)=\begin{pmatrix}x+y\\x-y\end{pmatrix}$.

(b) $S:\mathbb{R}^2 \to \mathbb{R}^3$ such that $T\left(\begin{pmatrix}x\\y\end{pmatrix}\right)=\begin{pmatrix}y\\x\\x+y\end{pmatrix}$;

$T:\mathbb{R}^3 \to \mathbb{R}^2$ such that $T\left(\begin{pmatrix}x\\y\\z\end{pmatrix}\right)=\begin{pmatrix}-x-y+3z\\-2x-y+3z\end{pmatrix}$.

4. Prove Remark 7.1.3:

Show that a mapping $T:\mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if and only if

$$T(c\boldsymbol{u}+d\boldsymbol{v})=cT(\boldsymbol{u})+dT(\boldsymbol{v})\quad\text{for all } \boldsymbol{u},\boldsymbol{v}\in\mathbb{R}^n \text{ and } c,d\in\mathbb{R}.$$

5. (a) Let $T_1:\mathbb{R}^n \to \mathbb{R}^m$ and $T_2:\mathbb{R}^n \to \mathbb{R}^m$ be linear transformations with standard matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ respectively. Define a mapping $T_1+T_2:\mathbb{R}^n \to \mathbb{R}^m$ such that

$$(T_1+T_2)(\boldsymbol{u})=T_1(\boldsymbol{u})+T_2(\boldsymbol{u})\quad\text{for } \boldsymbol{u}\in\mathbb{R}^n.$$

Show that T_1+T_2 is a linear transformation and find the standard matrix.

(b) Let $T:\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with the standard matrix $\boldsymbol{A}$ and let λ be a scalar. Define a mapping $\lambda T:\mathbb{R}^n \to \mathbb{R}^m$ such that

$$(\lambda T)(\boldsymbol{u})=\lambda T(\boldsymbol{u})\quad\text{for } \boldsymbol{u}\in\mathbb{R}^n.$$

Show that λT is a linear transformation and find the standard matrix.

6. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. If there exists a linear operator $S : \mathbb{R}^n \to \mathbb{R}^n$ such that $S \circ T$ is the identity transformation, i.e.

$$(S \circ T)(\boldsymbol{u}) = \boldsymbol{u} \quad \text{for all } \boldsymbol{u} \in \mathbb{R}^n,$$

then T is said to be *invertible* and S is called the *inverse* of T.

(a) For each of the following, determine whether T is invertible and find the inverse of T if possible.

(i) $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} y \\ x \end{pmatrix}$ for $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

(ii) $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ 0 \end{pmatrix}$ for $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$.

(b) Suppose T is invertible and $\boldsymbol{A}$ is the standard matrix for T. Find the standard matrix for the inverse of T.

7. Let $\boldsymbol{n}$ be a unit vector in $\mathbb{R}^n$. Define $P : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$P(\boldsymbol{x}) = \boldsymbol{x} - (\boldsymbol{n} \cdot \boldsymbol{x})\boldsymbol{n} \quad \text{for } \boldsymbol{x} \in \mathbb{R}^n.$$

(a) Show that P is a linear transformation and find the standard matrix for P.

(b) Prove that $P \circ P = P$.

8. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation such that $T \circ T = T$.

(a) If T is not the zero transformation, show that there exists a nonzero vector $\boldsymbol{u} \in \mathbb{R}^n$ such that $T(\boldsymbol{u}) = \boldsymbol{u}$.

(b) If T is not the identity transformation, show that there exists a nonzero vector $\boldsymbol{v} \in \mathbb{R}^n$ such that $T(\boldsymbol{v}) = \boldsymbol{0}$.

(c) Find all linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T \circ T = T$. (*Hint*: See Question 6.4.)

9. Let $\boldsymbol{n}$ be a unit vector in $\mathbb{R}^n$. Define $F : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$F(\boldsymbol{x}) = \boldsymbol{x} - 2(\boldsymbol{n} \cdot \boldsymbol{x})\boldsymbol{n} \quad \text{for } \boldsymbol{x} \in \mathbb{R}^n.$$

(See also Remark 7.3.7 and Question 7.26.)

(a) Show that F is a linear transformation and find the standard matrix for F.

(b) Prove that $F \circ F$ is the identity transformation.

(c) Show that the standard matrix for F is an orthogonal matrix.

10. A linear operator T on $\mathbb{R}^n$ is called an *isometry* if $\|T(\boldsymbol{u})\| = \|\boldsymbol{u}\|$ for all $\boldsymbol{u} \in \mathbb{R}^n$.

(a) If T is an isometry on $\mathbb{R}^n$, show that $T(\boldsymbol{u}) \cdot T(\boldsymbol{v}) = \boldsymbol{u} \cdot \boldsymbol{v}$ for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$. (*Hint*: Compute $T(\boldsymbol{u}+\boldsymbol{v}) \cdot T(\boldsymbol{u}+\boldsymbol{v})$ in two different ways.)

(b) Let $\boldsymbol{A}$ be the standard matrix for a linear operator T. Show that T is an isometry if and only if $\boldsymbol{A}$ is an orthogonal matrix. (See also Question 5.32.)

(c) Find all isometries on $\mathbb{R}^2$. (*Hint*: See Question 2.57.)

11. Let $T:\mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation defined by

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 2x+y \\ x-y+z \end{pmatrix} \quad \text{for } \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3.$$

(a) Find a basis for the range of T.

(b) Find a basis for the kernel of T.

(c) Use this example to verify the Dimension Theorem for Linear Transformation.

(d) Extend the basis found in Part (b) to a basis for $\mathbb{R}^3$.

12. Let $T:\mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation with the standard matrix

$$\begin{pmatrix} 3 & -1 & 2 & 7 \\ 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & -1 \end{pmatrix}.$$

(a) Find a basis for the range of T.

(b) Find a basis for the kernel of T.

(c) Use this example to verify the Dimension Theorem for Linear Transformation.

13. In each of the following parts, use the given information to find the nullity of the linearly transformation T.

(a) $T:\mathbb{R}^4 \to \mathbb{R}^6$ has rank 2.

(b) The range of $T:\mathbb{R}^6 \to \mathbb{R}^4$ is $\mathbb{R}^4$.

(c) The reduced row-echelon form of the standard matrix for $T:\mathbb{R}^6 \to \mathbb{R}^6$ has 4 nonzero rows.

14. Let $T:\mathbb{R}^n \to \mathbb{R}^n$ be a linear operator defined by $T(\boldsymbol{v}) = 2\boldsymbol{v}$.

(a) What is the kernel of T?

(b) What is the range of T?

15. Let V be a subspace of $\mathbb{R}^n$. Define a mapping $P:\mathbb{R}^n \to \mathbb{R}^n$ such that for $\boldsymbol{u} \in \mathbb{R}^n$, $P(\boldsymbol{u})$ is the projection of $\boldsymbol{u}$ onto V.

(a) Show that P is a linear transformation.

(b) Suppose $n = 3$ and V is the plane $ax+by+cz = 0$ where a, b, c are not all zero. Find $\mathrm{Ker}(P)$ and $\mathrm{R}(P)$.

16. Let $T:\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Show that $\mathrm{Ker}(T) = \{\mathbf{0}\}$ if and only if T is one-to-one, i.e. for any two vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$, if $\boldsymbol{u} \neq \boldsymbol{v}$, then $T(\boldsymbol{u}) \neq T(\boldsymbol{v})$.

17. Let $S:\mathbb{R}^n \to \mathbb{R}^m$ and $T:\mathbb{R}^m \to \mathbb{R}^k$ be linear transformations.

(a) Show that $\mathrm{Ker}(S) \subseteq \mathrm{Ker}(T \circ S)$.

(b) Show that $\mathrm{R}(T \circ S) \subseteq \mathrm{R}(T)$.

Question 7.18 to Question 7.28 are exercises for Section 7.3.

18. Describe geometrically the effect of each of the following linear operators $T:\mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$.

(a) $\boldsymbol{A} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$,

(b) $\boldsymbol{A} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}$,

(c) $\boldsymbol{A} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$,

(d) $\boldsymbol{A} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$,

(e) $\boldsymbol{A} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$,

(f) $\boldsymbol{A} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$,

(g) $\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ -11 & \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1}$.

19. Let $F_1 : \mathbb{R}^2 \to \mathbb{R}^2$ and $F_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be two reflections about lines $y = x\tan(\theta)$ and $y = x\tan(\phi)$ respectively. Show that $F_2 \circ F_1$ is an anti-clockwise rotation about the origin through an angle $2(\phi - \theta)$. (*Hint*: Compute the standard matrix for $F_2 \circ F_1$.)

20. For each of the following linear operators on $\mathbb{R}^2$,

(i) find the standard matrices for $T_1 \circ T_2$ and $T_2 \circ T_1$; and

(ii) determine whether $T_1 \circ T_2 = T_2 \circ T_1$.

(a) T_1 is the reflection about the line $x - y = 0$ and T_2 is the dilation by a factor of 2.

(b) T_1 is the reflection about the line $x - y = 0$ and T_2 is the scaling about the x and y-axes by factors of 1 and 2 respectively.

(c) T_1 is the reflection about the line $x - y = 0$ and T_2 is the anti-clockwise rotation about the origin through an angle $\frac{\pi}{2}$.

21. Determine which of the following statements are true. Justify your answer.

(a) If R_1 and R_2 are two rotations about the origin in $\mathbb{R}^2$, then $R_2 \circ R_1$ is a rotation about the origin in $\mathbb{R}^2$.

(b) If R_1 and R_2 are two rotations about the origin in $\mathbb{R}^2$, then $R_1 \circ R_2 = R_2 \circ R_1$.

(c) If R is a rotation about the origin and F is a reflection about a line through the origin in $\mathbb{R}^2$, then $F \circ R$ is a reflection about a line through the origin in $\mathbb{R}^2$.

(d) If R is a rotation about the origin and F is a reflection about a line through the origin in $\mathbb{R}^2$, then $R \circ F = F \circ R$.

(e) If F_1 and F_2 are two reflections about lines through the origin in $\mathbb{R}^2$, then $F_2 \circ F_1$ is a reflection about a line through the origin in $\mathbb{R}^2$.

(f) If F_1 and F_2 are two reflections about lines through the origin in $\mathbb{R}^2$, then $F_1 \circ F_2 = F_2 \circ F_1$.

22. Describe geometrically the effect of each of the following linear operators $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$.

(a) $\boldsymbol{A} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$,

(b) $\boldsymbol{A} = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$,

(c) $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{4}{5} & -\frac{3}{5} \\ 0 & \frac{3}{5} & \frac{4}{5} \end{pmatrix}$, (d) $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{4}{5} \\ 0 & \frac{4}{5} & -\frac{3}{5} \end{pmatrix}$,

(e) $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{4}{5} & -\frac{3}{5} \\ 0 & \frac{3}{5} & \frac{4}{5} \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{4}{5} & -\frac{3}{5} \\ 0 & \frac{3}{5} & \frac{4}{5} \end{pmatrix}^{-1}$.

23. For each of the following linear operators on $\mathbb{R}^3$,

(i) find the standard matrices for $T_1 \circ T_2$ and $T_2 \circ T_1$; and

(ii) determine whether $T_1 \circ T_2 = T_2 \circ T_1$.

(a) T_1 is the anti-clockwise rotation about the z-axis through an angle $\frac{\pi}{2}$ and T_2 is the contraction by a factor of 0.5.

(b) T_1 is the anti-clockwise rotation about the z-axis through an angle $\frac{\pi}{2}$ and T_2 is the reflection about the xy-plane.

(c) T_1 is the anti-clockwise rotation about the z-axis through an angle $\frac{\pi}{2}$ and T_2 is the reflection about the yz-plane.

24. Let $R : \mathbb{R}^3 \to \mathbb{R}^3$ be the (anti-clockwise) rotation about the axis in the direction $\boldsymbol{n} = (0,1,1)^{\mathrm{T}}$ through an angle π. Find the formula and the standard matrix for R. (*Hint*: Check the images of $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ under R geometrically.

25. Let $R : \mathbb{R}^3 \to \mathbb{R}^3$ be the anti-clockwise rotation about the axis in the direction of $\boldsymbol{n} = (1,1,-1)^{\mathrm{T}}$ through an angle θ. Let $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ be the basis for $\mathbb{R}^3$ obtained in Question 5.14(c). Find $R(\boldsymbol{v}_1), R(\boldsymbol{v}_2)$ and $R(\boldsymbol{v}_3)$. (*Hint*: Check the effect of the rotation on the vectors in the plane $x + y - z = 0$.)

26. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a mapping such that

$$T(\boldsymbol{u}) = \boldsymbol{u} - 2\left(\frac{\boldsymbol{u}\cdot\boldsymbol{n}}{\boldsymbol{n}\cdot\boldsymbol{n}}\right)\boldsymbol{u} \quad \text{for } \boldsymbol{n} \in \mathbb{R}^3$$

where $\boldsymbol{n} = (a,b,c)^{\mathrm{T}}$ is a nonzero vector. Show that T is the reflection about the plane $ax + by + cz = 0$ in $\mathbb{R}^3$ and write down the standard matrix for T.

27. Let $\boldsymbol{B} = \begin{pmatrix} 0 & 0 & 2 & 2 \\ 0 & 2 & 0 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}$. The column vectors of $\boldsymbol{B}$ represent the homogeneous coordinates of points in the xy-plane (see Discussion 7.3.13).

(a) Sketch the figure represented by $\boldsymbol{B}$.

(b) For each of the following, sketch the figure represented by $\boldsymbol{PB}$ and describe geometrically the effect of the corresponding transformation.

(i) $\boldsymbol{P} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$, (ii) $\boldsymbol{P} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

(iii) $\boldsymbol{P} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, (iv) $\boldsymbol{P} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

28. Consider the homogeneous coordinates discussed in Discussion 7.3.13.

(a) Write down the matrix that corresponds to the translation that moves $(x, y)^{\mathrm{T}}$ to $(x-1, y+2)^{\mathrm{T}}$.

(b) Write down the inverse of the matrix obtained in Part (a) and describe geometrically the effect of the corresponding transformation.

(c) Write down the matrix that corresponds to the anti-clockwise rotation about the origin through an angle $\frac{\pi}{4}$.

(d) Hence, or otherwise, find the matrix that corresponds to the anti-clockwise rotation about the point $(-1, 2)^{\mathrm{T}}$ through an angle $\frac{\pi}{4}$.