NATIONAL UNIVERSITY OF SINGAPORE

SEMESTER 1, 2021/2022

MA1521 Calculus for Computing **Tutorial 8**

1. Let p be a positive real number.

   (a) Use integral test to show that if $p > 1$ then $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ is convergent.

   (b) Use integral test to show that if $p \leq 1$ then $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ is divergent.

   (Thomas' Calculus (14^{th} edition), p. 559, Problem 61 (Modified))

2. Which of the following series converge and which diverge? Give your reasons for your answers. (You may use any test of convergence or divergence.)

   (a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{5n + 10\sqrt{n}}$

   (Thomas' Calculus (14^{th} edition), p. 557, Problem 12)

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^2}$

   (Thomas' Calculus (14^{th} edition), p. 563, Problem 19)

   (c) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^n$

   (Thomas' Calculus (14^{th} edition), p. 563, Problem 25)

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+2)!}$

(Thomas' Calculus (14$^{\text{th}}$ edition), p. 563, Problem 44)

(e) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n - \ln n}$

(Thomas' Calculus (14$^{\text{th}}$ edition), p. 576, Problem 30)

3. Find the radius and interval of convergence of the following power series.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{n}{5^n}(x+3)^n$

(Thomas' Calculus (14$^{\text{th}}$ edition), p. 587, Problem 17)

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2 \cdot 4 \cdot 6 \cdots (2n)} x^n$

(Thomas' Calculus (14$^{\text{th}}$ edition), p. 587, Problem 33)

(c) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{n}{n+1}\right)^{n^2} x^n$

(Thomas' Calculus (14$^{\text{th}}$ edition), p. 587, Problem 40)

4. Let a_n be nonnegative numbers and suppose $\displaystyle\sum_{n=1}^{\infty} a_n$ converges. Show that $\displaystyle\sum_{n=1}^{\infty} a_n^2$ converges.

Hint: Use the comparison test.

(Thomas' Calculus (14$^{\text{th}}$ edition), p. 564, Problem 60)

5. Let a_n be nonnegative numbers and suppose $\displaystyle\sum_{n=1}^{\infty} a_n$ converges. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges.

Hint: Use the comparison test.

(Thomas' Calculus (14$^{\text{th}}$ edition), p. 564, Problem 58)

6. Use power series operations to find the Taylor's series at $x = 0$ for the following functions:
   (a) xe^x

   (Thomas' Calculus (14^{th} edition), p. 600, Problem 13)

   (b) $\ln(1+x) - \ln(1-x)$

   (Thomas' Calculus (14^{th} edition), p. 600, Problem 30)

7. Use series to evaluate the following limits:
   (a) $\lim_{x\to 0} \frac{1}{x^2}\left(e^{x^2} - 1\right)$

   (Thomas' Calculus (14^{th} edition), p. 608, Problem 35 (modified))

   (b) $\lim_{x\to 0} \frac{\ln(1+x^2)}{1-\cos x}$

   (Thomas' Calculus (14^{th} edition), p. 608, Problem 37)

1. Let p be a positive real number.

   (a) Use integral test to show that if $p > 1$ then $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ is convergent.

   (b) Use integral test to show that if $p \leq 1$ then $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ is divergent.

(Thomas' Calculus (14th edition), p. 559, Problem 61 (Modified))

1. a) By integral test,

$$\int_2^{\infty} \frac{1}{n(\ln n)^p}\,dn$$

$$= \lim_{a\to\infty} \int_2^{a} \frac{1}{n(\ln n)^p}\,dn$$

$u = \ln n$

$du = \frac{1}{n}$

$$= \lim_{a\to\infty} \int_{\ln 2}^{\ln a} \frac{1}{u^p}\,du$$

$$= \lim_{a\to\infty} \left(\frac{u^{1-p}}{1-p} \Big|_{\ln 2}^{\ln a} \right)$$

$$= \lim_{a\to\infty} \left(\frac{(\ln a)^{1-p}}{1-p} - \frac{(\ln 2)^{1-p}}{1-p} \right)$$

$$= 0 - \frac{(\ln 2)^{1-p}}{1-p}$$

$$= \frac{1}{p-1}(\ln 2)^{1-p}$$

$\therefore$ For series to be convergent, $p > 1$.

b) By integral test,

$$\int_2^{\infty} \frac{1}{n(\ln n)^p}\,dn.$$

$$= \lim_{a\to\infty} \left(\frac{(\ln a)^{1-p}}{1-p} - \frac{(\ln 2)^{1-p}}{1-p} \right)$$

$\therefore$ If $p \leq 1$,

$(\ln a)^{1-p} \to \infty$ as $a \to \infty$.

$\therefore$ Series diverges

No limit for sequence (test 3)

2. Which of the following series converge and which diverge? Give your reasons for your answers. (You may use any test of convergence or divergence.)

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{5n+10\sqrt{n}}$

(Thomas' Calculus (14th edition), p. 557, Problem 12)

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^2}$

(Thomas' Calculus (14th edition), p. 563, Problem 19)

(c) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^n$

(Thomas' Calculus (14th edition), p. 563, Problem 25)

2.a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{5n+10\sqrt{n}}$.

By integral test,

$$\int_2^\infty \frac{1}{5n+10\sqrt{n}}\,dn$$

$$= \lim_{a\to\infty} \cdot \frac{1}{5}\int_2^a \frac{1}{n+2\sqrt{n}}\,dn.$$

$u = \sqrt{n}$

$du = \frac{1}{2\sqrt{n}}$.

$$= \lim_{a\to\infty} \cdot \frac{1}{5}\int_{\sqrt{2}}^{\sqrt{a}} \frac{1}{2\sqrt{n}\left(\frac{1}{2}\sqrt{n}+1\right)}\,dn.$$

$$= \lim_{a\to\infty} \frac{1}{5}\int_{\sqrt{2}}^{\sqrt{a}} \frac{1}{\frac{1}{2}u+1}\,du.$$

$$= \lim_{a\to\infty} \frac{1}{5}\left(2\ln\left(\frac{u}{2}+1\right)\Big|_{\sqrt{2}}^{\sqrt{a}}\right)$$

$$= \lim_{a\to\infty} \frac{1}{5}\left(2\ln\left(\frac{\sqrt{a}}{2}+1\right) - 2\ln\left(\frac{\sqrt{2}}{2}+1\right)\right)$$

$$= \lim_{a\to\infty} \left(\frac{2}{5}\ln\left(\frac{\sqrt{a}}{2}+1\right) - \frac{2}{5}\ln\left(\frac{\sqrt{2}}{2}+1\right)\right)$$

Since $\ln\left(\frac{\sqrt{a}}{2}+1\right) \to \infty$ as $a \to \infty$,
the series is divergent.

Limit comparison

$$\lim_{n\to\infty}\left(\frac{\frac{1}{5n+10\sqrt{n}}}{\frac{1}{n}}\right)$$

$$= \lim_{n\to\infty}\left(\frac{n}{5n+10\sqrt{n}}\right)$$

$$= \lim_{n\to\infty}\left(\frac{\cancel{n}}{\cancel{n}} \cdot \frac{1}{5+\frac{10}{\sqrt{n}}}\right)$$

$$= \frac{1}{5} \neq 0.$$

Since $\sum \frac{1}{n}$ diverges,

$\therefore \sum \frac{1}{5n+10\sqrt{n}}$ diverges

b) By Comparison test,

Since $\frac{(\sin n)^2}{n^2} \leq \frac{1}{n^2}$

Then the series $\displaystyle\sum_{n=1}^{\infty}\frac{\sin^2 n}{n^2}$ converges as $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges.

c) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n}{3n+1}\right)^n$.

By ~~limit~~ comparison test,

$$\cancel{\lim_{n\to\infty} \frac{\left(\frac{n}{3n+1}\right)^n}{\left(\frac{1}{n}\right)^n}}$$

$$\cancel{= \lim_{n\to\infty}\left(\frac{n}{3n+1}\cdot\frac{1}{n}\right)^n}$$

By root test

$$\sum_{n=1}^{\infty}\left(\frac{n}{3n+1}\right)^n$$

$$\left(\left(\frac{n}{3n+1}\right)^n\right)^{\frac{1}{n}} = \left(\frac{n}{3n+1}\right) \to \frac{1}{3} \text{ as } n\to\infty$$

$\therefore \sum\left(\frac{n}{3n+1}\right)^n$ converges.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+2)!}$

(Thomas' Calculus (14th edition), p. 563, Problem 44)

(e) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n - \ln n}$ — or telescopic sum.

(Thomas' Calculus (14th edition), p. 576, Problem 30)

d) By ratio test,

$$\lim_{n\to\infty} \left(\frac{\frac{n!}{(n+3)!}}{\frac{(n-1)!}{(n+2)!}} \right) \quad u_{n+1},\ u_n.$$

$$= \lim_{n\to\infty} \left[\frac{n!}{(n+3)!} \cdot \frac{(n+2)!}{(n-1)!} \right]$$

$$= \lim_{n\to\infty} \left(\frac{n \times \cancel{(n-1)!}}{(n+3) \times \cancel{(n+2)!}} \cdot \frac{\cancel{(n+2)!}}{\cancel{(n-1)!}} \right)$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+3} \right)$$

By l'hôpital rule,

$$= \lim_{n\to\infty} \left(\frac{1}{3} \right).$$

$$= \frac{1}{3} < 1$$

$\therefore$ series converges.

Comparison test.

$$\frac{(n-1)!}{(n+2)!} = \frac{\cancel{(n-1)!}}{(n+2)(n+1)(n)\cancel{(n-1)!}}$$

$$= \frac{1}{(n+2)(n+1)n}.$$

Comparison

$$\lim_{n\to\infty} \frac{\frac{1}{n(n+1)(n+2)}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{\cancel{\frac{1}{n^3}}\frac{1}{(1)(1+\cancel{\frac{1}{n}}^{0})(1+\cancel{\frac{2}{n}}^{0})}}{\cancel{\frac{1}{n^3}}}$$

$$= 1.$$

Since $\sum \frac{1}{n^3}$ converges, series converges

e) By alternating series test,

$$\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n - \ln n}. \quad > 0 \text{ if } n > 1 \quad ③$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \cdot (-1)^{-1} \frac{\ln n}{n - \ln n}$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\ln n}{\ln n - n}$$

$\therefore$ let $a_n = \frac{\ln n}{\ln n - n}$

$a_{n+1} = \frac{\ln(n+1)}{\ln(n+1) - (n+1)}$

① Since $\ln(n+1) - (n+1) > \ln n - n$.

$\therefore a_{n+1} \leq a_n$.

$$f'(x) = \frac{(x - \ln x)\frac{1}{x} - \ln x(1 - \frac{1}{x})}{(x - \ln x)^2}.$$

$$= \frac{1 - \frac{\ln x}{x} - \ln x + \frac{\ln x}{x}}{(x - \ln x)^2} < 0$$

if $x \geq 3$.

show

$\therefore$ for $n \geq 3$,

$$\frac{\ln(n+1)}{(n+1) - \ln(n+1)} < \frac{\ln n}{n - \ln n}.$$

② $\lim_{n\to\infty} \frac{\ln n}{\ln n - n}$

$$= \lim_{n\to\infty} \frac{\cancel{\frac{1}{n}}^{0}}{\cancel{\frac{1}{n}}^{0} - 1}$$

$= 0$

$\therefore$ series is convergent

$\therefore$ Converge.

3. Find the radius and interval of convergence of the following power series.

(a) $\displaystyle\sum_{n=0}^{\infty} \underbrace{\frac{n}{5^n}}_{a_n}(x+3)^n$

(Thomas' Calculus (14th edition), p. 587, Problem 17)

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2\cdot 4\cdot 6\cdots(2n)}x^n$

(Thomas' Calculus (14th edition), p. 587, Problem 33)

(c) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{n}{n+1}\right)^{n^2} x^n$

(Thomas' Calculus (14th edition), p. 587, Problem 40)

3. a) $\displaystyle\sum_{n=0}^{\infty} \frac{n}{5^n}(x+3)^n$

By ratio test,

$$\lim_{n\to\infty}\left|\frac{\frac{n+1}{5^{n+1}}(x+3)^{n+1}}{\frac{n}{5^n}(x+3)^n}\right|$$

$$= \left|\frac{\frac{n+1}{5^{n+1}}\cancel{(x+3)^n}\cdot(x+3)}{\frac{n}{5^n}\cancel{(x+3)^n}}\right|$$

$$= \left|\frac{n+1}{5^{n+1}}(x+3)\cdot\frac{5^n}{n}\right|$$

$$= \left|\frac{(n+1)(x+3)}{5n}\right|$$

$$= \left|\frac{x+3}{5}+\frac{x+3}{5n}\right|$$

$$\therefore \lim_{n\to\infty}\left(\left|\frac{x+3}{5}+\cancel{\frac{x+3}{5n}}\right|\right) < 1 \quad 0$$

$$\left|\frac{x+3}{5}\right| < 1$$

$$|x+3| < 5.$$

$x+3<5$ or $x+3>-5$.

$x<2$ or $x>-8$

Radius: 10.

Interval: $-8<x<2$

If $|x-3| < \displaystyle\lim_{n\to\infty}\frac{5(n+1)}{n} = 5.$

then $\sum$ converges.

endpoints $\left\{ \begin{array}{l} |x+3|=5, \quad x+3=-5. \\ \displaystyle\sum_{n=1}^{\infty}\frac{n}{5^n}5^n \text{ diverges.} \quad \sum\frac{(-1)^n 5^n}{5^n}n \text{ diverges.} \end{array}\right.$

$\therefore |x-3|<5.$

b) $\displaystyle\sum_{n=0}^{\infty}\frac{1}{2\cdot4\cdot6\cdots(2n)}x^n$

$a_n = \frac{1}{2}\cdot\frac{1}{2\cdot2}\cdot\frac{1}{2\cdot3}\cdots\frac{1}{2n}$

$= \frac{1}{2^n}\cdot\frac{1}{n!}$

By Ratio test,

$$\lim_{n\to\infty}\left|\frac{\frac{1}{2\cdot4\cdot6\cdots(2(n+1))}x^{n+1}}{\frac{1}{2\cdot4\cdot6\cdots(2n)}x^n}\right|$$

$x^{n+1}=x^n\cdot x.$

$$=\lim_{n\to\infty}\left|\frac{\frac{1}{\cancel{(2\cdot4\cdot6\cdots2n)}\cdot(2(n+1))}\cancel{x^n}x}{\frac{1}{\cancel{2\cdot4\cdot6\cdots2n}}\cancel{x^n}}\right|$$

$\dfrac{a_n}{a_{n+1}}$

$$=\lim_{n\to\infty}\left|\frac{x}{2(n+1)}\right| < 1 \quad \to\infty$$

$$= 0 < 1$$

$\therefore$ radius $=0$.

interval : 0

converges at 0.

$\therefore$ converge for all $x\in\mathbb{R}$.

$\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}=e^x.$

c) $\sum_{n=0}^{\infty} \left(\frac{n}{n+1}\right)^{n^2} x^n$

$$\lim_{n\to\infty} \left| \frac{\left(\frac{n+1}{n+2}\right)^{(n+1)^2} \cancel{x^{n+1}}}{\left(\frac{n}{n+1}\right)^{n^2} \cancel{x^n}} \right|$$

$(n+1)^2 = n^2 + 2n + 1$

$$= \lim_{n\to\infty} \left| x \cdot \frac{\left(\frac{n+1}{n+2}\right)^{(n+1)^2}}{\left(\frac{n}{n+1}\right)^{n^2}} \right|$$

$$= \lim_{n\to\infty} \left| x \cdot \frac{(n+1)^{(n+1)^2}}{(n+2)^{(n+1)^2}} \cdot \frac{(n+1)^{n^2}}{n^{n^2}} \right| < 1$$

?

root test.

$$\sum_{n=0}^{\infty} \underbrace{\left(\frac{n}{n+1}\right)^{n^2}}_{a_n} x^n.$$

$$R = \lim_{n\to\infty} \frac{1}{\left|\left(\frac{n}{n+1}\right)^{n^2}\right|^{\frac{1}{n}}}$$

$$= \lim_{n\to\infty} \frac{1}{\left(\frac{n}{n+1}\right)^n}$$

$$R = \frac{1}{(1/e)} = e.$$

$$y = \left(\frac{x}{x+1}\right)^x$$

$$\ln y = x \ln\left(\frac{x}{x+1}\right)$$

$$= \frac{\ln x - \ln(x+1)}{\frac{1}{x}}$$

Converges for $|x| < e$

l'hopital rule $\lim_{x\to\infty} \frac{\ln x - \ln(x+1)}{\frac{1}{x}}$

$$= \lim_{n\to\infty} \frac{\frac{1}{x} - \frac{1}{x+1}}{\frac{1}{x^2}}.$$

$= -1.$

$y \to e^{-1}$

$$\left(\frac{n+1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n \to e.$$

$$\left(\frac{n+1}{n}\right)^{-1} \to \frac{1}{e}$$

4. Let a_n be nonnegative numbers and suppose $\sum_{n=1}^{\infty} a_n$ converges. Show that $\sum_{n=1}^{\infty} a_n^2$ converges.

Hint: Use the comparison test.

(Thomas' Calculus (14th edition), p. 564, Problem 60)

By limit comparison test,

$$\lim_{n \to \infty} \frac{a_n^2}{a_n}$$

$$= \lim_{n \to \infty} a_n$$

Since $\sum_{n=0}^{\infty} a_n$ converges,

$\lim_{n \to \infty} a_n = L$,

mean $\sum_{n=0}^{\infty} a_n^2$ also converges.

5. Let a_n be nonnegative numbers and suppose $\sum_{n=1}^{\infty} a_n$ converges. Show that $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges.

Hint: Use the comparison test.

(Thomas' Calculus (14th edition), p. 564, Problem 58)

By comparison test,

Since $0 \leq \frac{a_n}{n} \leq a_n$ since $n \in \mathbb{Z}^+$,

and since $\int_1^{\infty} a_n$ converges,

$\int_1^{\infty} \frac{a_n}{n}$ also converges.

6. Use power series operations to find the Taylor's series at $x = 0$ for the following functions:
(a) xe^x

(Thomas' Calculus (14th edition), p. 600, Problem 13)

(b) $\ln(1 + x) - \ln(1 - x)$

(Thomas' Calculus (14th edition), p. 600, Problem 30)

DEFINITIONS Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the **Taylor series generated by f at $x = a$** is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.$$

The **Maclaurin series of f** is the Taylor series generated by f at $x = 0$, or

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots.$$

a) $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$

when $a=0$,
$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$

$= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$

$1 + (xe^x + e^x) + (xe^x + e^x + e^x)$

$= \frac{(xe^x + 2e^x)}{2!} x^2$

$\therefore$ nth term: $\frac{(xe^x + ne^x)}{n!} x^n$.

$\therefore$ Taylor series : $\sum_{k=0}^{\infty} \frac{k}{k!} x^k = \sum_{k=0}^{\infty} \frac{1}{(k-1)!} x^k$ X $\sum_{0} \frac{x^{n+1}}{n!}$

b) when $a=0$

$= \ln(1) - \ln(1-1) + \left(\cancel{\frac{1}{1+x}} + \frac{1}{1-x}\right)x + \frac{\left(\cancel{-\frac{1}{(1+x)^2}} + \frac{1}{(1-x)^2}\right)}{2!}x^2$ 2

$+ \dots \cancel{\frac{2}{(1+x)^3} - \frac{2}{(1-x)^3}}$ 0

$\therefore$ nth term: $\frac{0}{n!}x^n$.

Taylor series : $\sum_{k=0}^{\infty} \frac{0}{k!} x^k$.

$= \ln(1) - \ln(0) + 2x$. X

this series may not be tested

7. Use series to evaluate the following limits:

(a) $\lim_{x\to 0} \frac{1}{x^2}\left(e^{x^2}-1\right)$ 1

(Thomas' Calculus (14th edition), p. 608, Problem 35 (modified))

(b) $\lim_{x\to 0} \frac{\ln(1+x^2)}{1-\cos x}$ 2

(Thomas' Calculus (14th edition), p. 608, Problem 37)

DEFINITIONS Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the **Taylor series generated by f at $x = a$** is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.$$

DEFINITIONS **A power series about $x = 0$** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots. \quad (1)$$

A power series about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots \quad (2)$$

in which the **center** a and the **coefficients** $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants.

a) $\lim_{x\to 0} \frac{1}{x^2}(e^{x^2}-1)$

$$e^{x^2} = \sum_{m=0}^{\infty} \frac{x^{2m}}{m!} = 1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \ldots$$

$$\frac{e^{x^2}-1}{x^2} = \frac{1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \ldots - 1}{x^2} \to 1 \text{ as } x \to 0$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n.$$

$f(x) = e^x$ $f(0) = 1$

$f'(x) = e^x$ $f'(0) = 1$

$\vdots$

$f^n(x) = e^x$, $f^n(0) = 1$.

Maclaurin → centers at 0.

Taylor → centers at anything.

$$e\left[\sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}\right]$$

b) $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots$

$\int (1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots$

$\int (1+x^2) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} + \dots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \dots$

$1 - \cos x = \frac{x^2}{2!} - \frac{x^4}{4!} \dots$

$\lim \frac{\cancel{x^2}(1 - \frac{x^2}{2} + \dots)}{\cancel{x^2}(\frac{1}{2!} - \frac{x^2}{4!} \dots)}$

$= 2$