CS1231S Tutorial 8: Cardinality

National University of Singapore

2021/22 Semester 1

More challenging questions are indicated by an asterisk. You may use an earlier question in a later question.

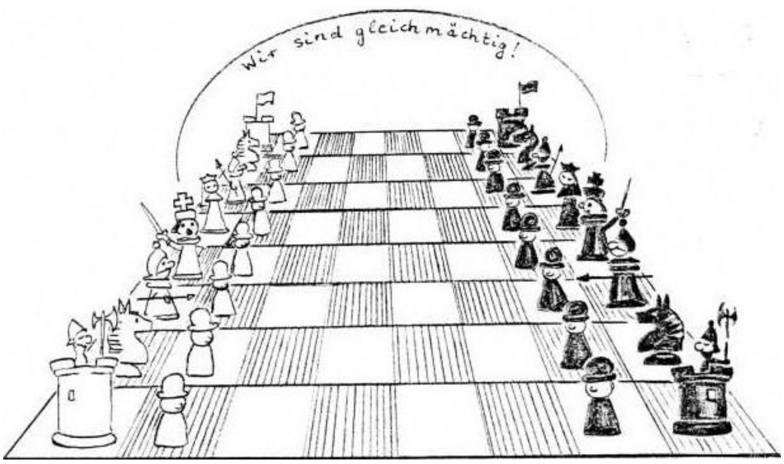
Background

Definition. Let S_i be a set for each $i \in \mathbb{Z}_{\geqslant 0}$. Then

$$\bigcup_{i \in \mathbb{Z}_{\geqslant 0}} S_i = \{x : x \in S_i \text{ for some } i \in \mathbb{Z}_{\geqslant 0}\}.$$

Questions for discussion on the LumiNUS Forum

Answers to these questions will not be provided.



Marcel Erné

D1. We know $\mathbb{Z}^+ \setminus \{1, 2, \ldots, n\} \neq \varnothing$ for each $n \in \mathbb{Z}^+$. Does this imply that $\mathbb{Z}^+ \setminus \{1, 2, 3, \ldots\} \neq \varnothing$? Why?

D2. Let A, B be finite sets such that $|A| \geqslant |B| > 0$. Prove that there is a surjection $A \to B$.

D3. Let A, B be sets. Prove that if A is countable and $|A| = |B|$, then B is countable.

D4.* (Cantor) Prove that the function $f \colon \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ defined by setting

$$f(x, y) = \frac{(x + y)(x + y + 1)}{2} + y$$

for all $x, y \in \mathbb{Z}_{\geqslant 0}$ is a bijection.

Tutorial questions

1. Let $n \in \mathbb{Z}^+$. Recall from Definition 7.1.4 that $\mathbb{Z}_n$ denotes the quotient of $\mathbb{Z}$ by the congruence-mod-n relation. Let $[a], [b] \in \mathbb{Z}_n$. Prove that $[a]$ has the same cardinality as $[b]$. (Hint: You may find Example 6.4.3 helpful.)

2. Let B be a countable infinite set and C be a finite set. Show that $B \cup C$ is countable.

3. Let B be a (not necessarily countable) infinite set and C be a finite set. Define a bijection $B \cup C \to B$.

4. Prove that a set B is infinite if and only if there is $A \subsetneq B$ such that $|A| = |B|$.

5. Let S_i be a countable infinite set for each $i \in \mathbb{Z}_{\geqslant 0}$. Prove that $\bigcup_{i \in \mathbb{Z}_{\geqslant 0}} S_i$ is countable.

6. Let S_i be a countable (not necessarily infinite) set for each $i \in \mathbb{Z}_{\geqslant 0}$. Prove that $\bigcup_{i \in \mathbb{Z}_{\geqslant 0}} S_i$ is countable.

7. For each $n \in \mathbb{Z}_{\geqslant 0}$, define $F_n = \{X \in \mathcal{P}(\mathbb{Z}_{\geqslant 0}) : |X| = n\}$. Let $F = \bigcup_{n \in \mathbb{Z}_{\geqslant 0}} F_n$.

   (a) Prove that F_n is countable for every $n \in \mathbb{Z}_{\geqslant 1}$ by induction on n.

   (b) Deduce that F is countable.

8. In the answer to Exercise 10.4.4 in the notes, it is proved that $\mathbb{Q}_{\geqslant 0}$ is countable. Use this fact to show that $\mathbb{Q}$ is countable.

9. Let $A = \{x \in \mathbb{R} : 0 \leqslant x < 1\}$. Using a diagonal argument, or otherwise, prove that A is uncountable.

   You may use without proof the fact that the elements of A are precisely those real numbers that have a decimal representation

   $$0.d_0 d_1 d_2 d_3 d_4 \ldots$$

   without a tail of 9's; moreover, such a representation is unique.

10. Prove that $\mathbb{R}$ and $\mathbb{C}$ are uncountable.

Cardinality - size of set.

- size of 2 sets equal iff same no. of elements.

For infinite set.

- if bijection from A to B- then set has same cardinality
- Countable if finite or has same cardinality as $\mathbb{Z}_{\geq 0}$
- Countable iff sequence $b_0, b_1, b_2 \ldots$ in which every element appears exactly once.

$S: \mathbb{N} \rightarrow R$.

- Subset of countable set is countable
- infinite set has a countable infinite subset
- cartesian product for $\mathbb{N}$ numbers is countable.

$[a] = \{nk+a : k \in \mathbb{Z}\}. = \{\dots a-3n, a-2n, a-n, a, a+n, \dots\}.$

$[b] \cdot \{\dots b-3n, b-2n, b-n, b, b+n, \dots\}.$

$a \bmod n$?

result,

$f(x) = x+b-a.$

1. Let $n \in \mathbb{Z}^+$. Recall from Definition 7.1.4 that $\mathbb{Z}_n$ denotes the quotient of $\mathbb{Z}$ by the congruence-mod-n relation. Let $[a], [b] \in \mathbb{Z}_n$. Prove that $[a]$ has the same cardinality as $[b]$. (Hint: You may find Example 6.4.3 helpful.)

$|[a]| = |[b]|$ & $|[b]| = |[a]|$

1. Define $f: \mathbb{Z} \to \mathbb{Z}_n$ by setting.

$f(x) = [x].$

2. (injective) Suppose $f(x_1) = f(x_2)$ for some $x_1, x_2 \in \mathbb{Z}$.

2.1 Then $[x_1] = [x_2]$ by definition of f.

2.2 Hence f is injective

3. (surjective) Suppose $[x]$ is any element of $\mathbb{Z}_n$,

3.1 Then it follows that $f(x) = [x]$.

3.2. Thus, there exist an x in $\mathbb{Z}$ such that $[x]$.

3.3. Hence, f is surjective.

4. therefore, by definition of cardinality, $[a]$ and $[b]$ have the same cardinality.

Pattern of proof $\to$ Find the bijection, proof well-defined, proof bijective

1. Define $f: [a] \to [b]$ by setting $f(x) = x+b-a$.

2. (f is well defined)

2.1 let $x \in [a]$.

2.2. $x = nk+a$, $k \in \mathbb{Z}$.

2.3. $f(x) = x+b-a = nk+a+b-a = nk+b$.

2.4 $f(x) \in [b]$.

3. (Injective)

3.1 $x_1, x_2 \in [a]$ st $f(x_1) = f(x_2)$.

3.2 $x_1+b-a = x_2+b-a$. (by def. of f)

3.3. $x_1 = x_2$.

4. (surjective).

4.1 Let $y \in [b]$.

4.2 Find $l \in \mathbb{Z}$ s.t. $y = nl+b$.

4.3. Define $x = y+a-b$.

4.4 Then $x = nl+b+a-b = nl+a$.

4.5. $x \in [a]$.

$\mathbb{Z}/n$

$a \equiv x \bmod n \Rightarrow a - x = kn$

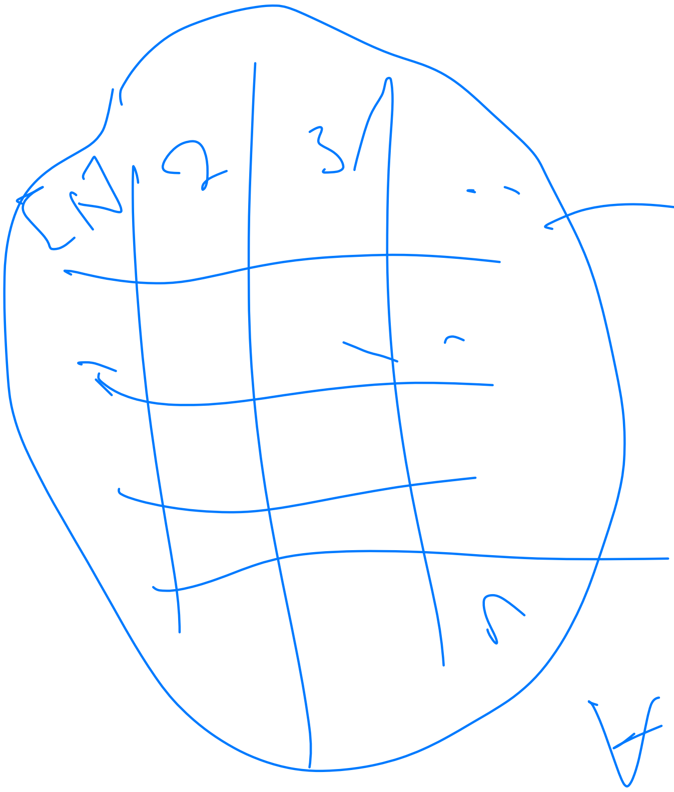
$a = \boxed{kn} + \underline{x}.$

$[a] = \{\ldots, -2n+x, -n+x, x, n+x,$
$2n+x, \ldots\}$

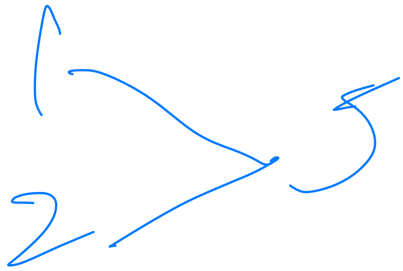
$\mathbb{Z}/\sim_3$

$[1] = \{-2, 1, 4, \cdots\}$

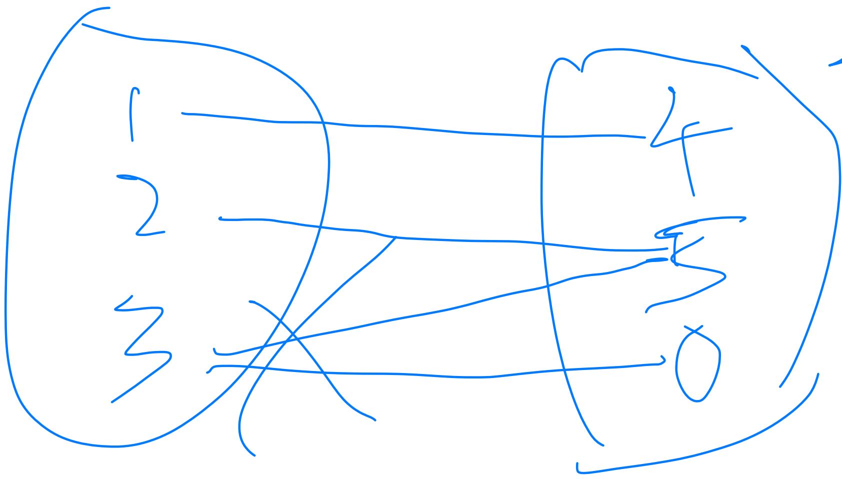
$[2] = \{-4, -1, 2, 5, \cdots\}$



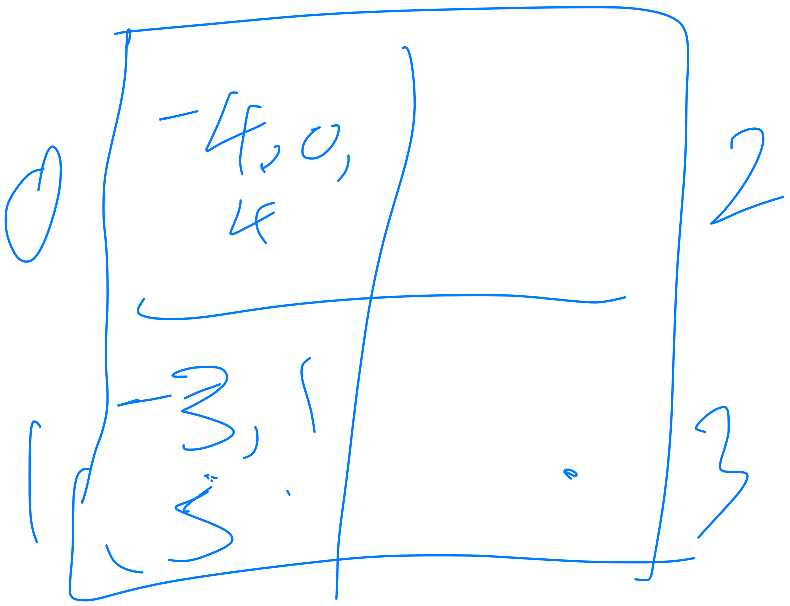
$c = \mathbb{N}$



$\forall x_1, x_2$
$f(x_1) = f(x_2)$



$\Rightarrow x_1 = x_2$



mod-4

$$[a] = \{ \ldots, -2n+a, -n+a, a, n+a, 2n+a, \ldots \}$$

$$[b] = \{ \ldots, -2n+b, -n+b, b, n+b, 2n+b, \ldots \}$$

$$x \in [a]$$

$$x = kn + a.$$

$$f(x) = kn + a + b - a$$

$$= \underline{kn + b} \in [b].$$

$$f(x_1) = f(x_2)$$

$$x_1 + b - a = x_2 + b - a$$

$$\Rightarrow \underline{\underline{x_1 = x_2}}$$

$\forall y \in [b], \ \underline{\exists x \in [a]}$

s.t. $f(x) = y$

$f(x) = x + \cancel{a+b}$ $b-a$

$y \in [b]$

$y = ln + b.$

$\cancel{x = (ln + b)}$

$ln \cancel{+b} = x \cancel{+b} - a$

$x = ln + a.$

$\therefore x \in [a]$

2

$a_1, a_2, a_3, \ldots$

φ

$|A| =$

$\exists f$, f is a bijection,

$$B_0 \cup C_0 \to B_0$$

1. B is infinite $\to \exists A \subsetneq B\ (|A| = |B|)$

1.1 Let $b \in B$

1.2 $A = B \setminus \{b\}$

1.3 $B = \underline{\underline{A}} \cup \underline{\underline{\{b\}}}$

1.4 $|A \cup \{b\}| = \underline{\underline{|A| = |B|}}$

2. $\exists A \subsetneq B\ (|A| = |B|) \to B$ is infinite

2.1 B is finite $\to \forall A \subsetneq B\ (|A| \neq |B|)$

2.2

$S_0 \cup S_1 \cup S_2 \cup \ldots$ is countable.

$$|\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}| = \underline{|\mathbb{Z}_{\geq 0}|}$$

$$f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$$

$$S_0 = b_{0,0}, b_{0,1}, b_{0,2} \ldots$$

$$S_1 = b_{1,0}, b_{1,1}, b_{1,2} \ldots$$

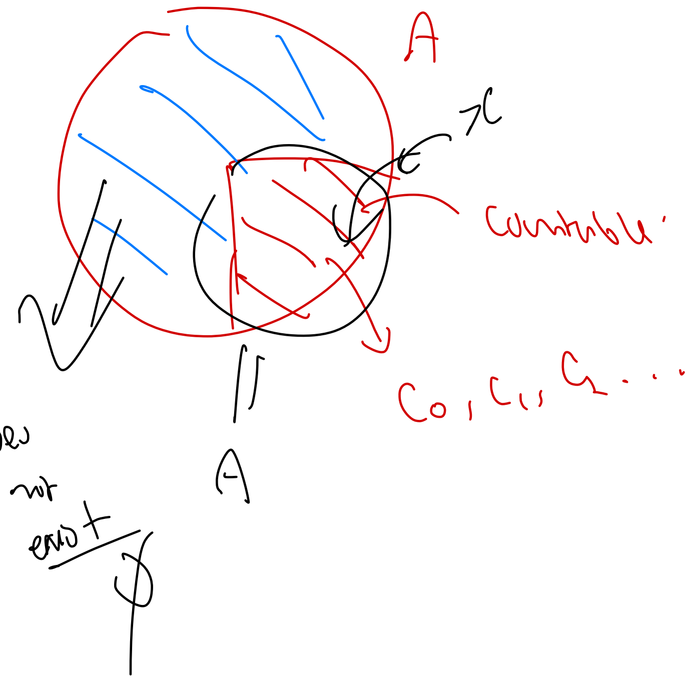
$$f(n) = (i, j)$$

$$\bigcup_{i \in \mathbb{Z}_{\geq 0}} S_i = \{c_0, c_1, c_2, \ldots\}.$$

infinite A

✓

countable?



$x \in A$

$x \in S_i$

$x \in \{b_{i,0}, b_{i,1}, b_{i,2} \ldots\}.$

$\therefore x = b_{i,j}$

1. $\boxed{R_i} = S_i \cup \mathbb{Z}_{\geq 0}$

R_i is always ctble

$$\bigcup_{i \in \mathbb{Z}_{\geq}} S_i \subseteq \bigcup_{i \in \mathbb{Z}_{\geq 0}} R_i$$

c c

$F_n = \{ \underline{X \in P(\mathbb{Z}_{\geq 0})} : \underline{|X| = n} \}$

$n=1 \quad \{1\}, \{2\}, \{3\}, \ldots$

$n=2, \{1,1\} \ldots$

$n=3, \{1,1,1\} \ldots$

$F_{k,0} = \{ B_0 \cup \{0\}, B_1 \cup \{0\}, \ldots \}$

$F_{k,1} = \{ B_0 \cup \{1\}, B_1 \cup \{1\}, \ldots \}$

$\vdots \qquad \vdots$

$F_{3,0} = \{ \{1,1,1\} \cup \{0\} \ldots$

$\underline{\bigcup_{j \in \mathbb{Z}_{\geq 0}} F_{k,j}}$

$\{a, b, c\} \cup \{d\}$

$\underline{\{a, b, c, d\}}$

$F_n = \{X \in P(\mathbb{Z}_{\geq 0}) : |X| = n\}$

$P(k)$: $B_0, B_1, B_2 \ldots \in F_k$

$F_{k,j} = \{ B_0 \cup \{j\},$
$B_1 \cup \{j\},$
$B_2 \cup \{j\},$
$\vdots \},$

$\therefore \bigcup_{j \in \mathbb{Z}_{\geq 0}} F_{k,j}$ countable

$F_{k+1} \subseteq \bigcup_{j \in \mathbb{Z}_{\geq 0}} F_{k,j}$.

$C \in F_{k+1}$

$C \subseteq \mathbb{Z}_{\geq 0}$ $\quad |k+1| \geq 1 + 1 = 2$,

$m \in C$.

$|C \setminus \{m\}| = |C| - 1 = k$.

$\in F_k$: B_0, B_1, B_2

$B_i = C \setminus \{m\}$.

$C = B_i \cup \{m\} \in F_{k,m} \subseteq \bigcup \ldots$

$$\bigcup F_{k,j} \begin{cases} \text{some } \underline{k+1} \\ \text{some } \underline{k} \end{cases}$$

F_{k+1} — $k+1$

$$F_{k+1} \subseteq \bigcup F_{k,j}$$

$C \in F_{k+1}$

$m \in C \Rightarrow |C \setminus \{m\}| = k$

$C \setminus \{m\} = B_i$

$\Downarrow$

$C = B_i \cup \{m\}$

$$\mathbb{Q} = \mathbb{Q}_{>0} \cup \{0\} \cup \mathbb{Q}_{<0}$$

$$= \boxed{\mathbb{Q}_{>0} \cup \mathbb{Q}_{<0}} \cup \boxed{\{0\}}$$

$$f: \quad f(x) = -x$$

$$f\big(\mathbb{Q}_{>0}\big) \to \big|\mathbb{Q}_{<0}\big| : \quad f(x) = -x$$

x in binary

$0.11010110\ldots$

$x_1 = 0.11010\ldots$

$x_2 = 0.00011\ldots$

$x_3 = 0.10111\ldots$

$\vdots$

$x_n = 0.010$

$x_n \neq x_{n-1} \neq x_{n-2} \neq \ldots$

x_1

$x_1 = 0.101010...$

$x_2 = 0.101011...$

$x_{1.5} = 0.11111$

$\mathbb{Q} \to$ countable:

$x = \frac{1}{1} \quad \frac{2}{1} \quad \cdots \quad \frac{7}{1} \quad \cdots$

$\frac{1}{2}$

$\vdots$

$x_1 \in A$

x_2

$\mathbb{Q}$

$\frac{1}{2}$ $\frac{5}{2}$ $\frac{3}{2}$ $\frac{4}{3}$

$\boxed{\frac{1}{1} \quad \frac{2}{1} \quad \frac{3}{1}}$

$A = \{x_1, x_2, \ldots x_n\}$

in an order.

$x_i < x_j$

$\cancel{\emptyset}\ y = \dfrac{x_i + x_j}{2} \neq x_i \neq x_j$

2. Let B be a countable infinite set and C be a finite set. Show that $B \cup C$ is countable.

1 Let B be a countable infinite set, C be a finite set.

2. Then by definition of countable, C is countable.

3. Since both sets are countable, $B \cup C$ is countable by prop. 10.4.1.

$B = b_0, b_1, b_2, \ldots$ sequence

$C = \{c_0, c_1, \ldots c_{n-1}$

Then $c_0, c_1, \ldots, c_{n-1}, b_0, b_1, b_2, \ldots$. sequence

$\therefore$ every elem of $B \cup C$ appears.

$\therefore B \cup C$ is countable.

~~set A is :~~

~~$A \subseteq B \wedge B$ countable~~

~~$\Rightarrow A$ is countable.~~

~~A is uncountable~~

~~$\Rightarrow$~~

$\mathbb{R}$ uncountable, $\mathbb{C}$

$A \subseteq B \wedge B$ countable $\to A$ countable.

A uncountable $\to \neg(A \subseteq B \wedge B$ count$)$

$A \not\subseteq B \vee B$ uncountable

$A \subseteq B$

$A = \{x \in \mathbb{R} : 0 \leq x < 1\}$. $\to$ uncount

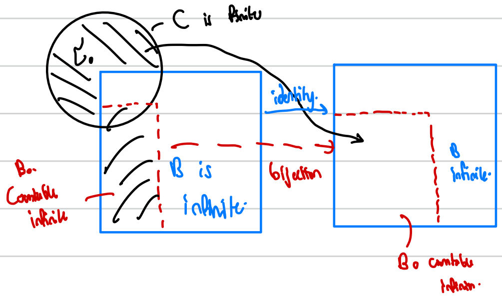
$A \subseteq \mathbb{R} \wedge A \subseteq \mathbb{C}$.

By contrapositive: Let B be either $\mathbb{R}$ or $\mathbb{C}$:

Since $\underbrace{A \text{ uncount}}_{T} \Rightarrow \underbrace{B \text{ uncount}}_{F} \vee \underbrace{A \not\subseteq \mathbb{C}}_{F}$.

3. Let B be a (not necessarily countable) infinite set and C be a finite set. Define a bijection $B \cup C \to B$.

countable $\therefore$ Find bijection.



$|B_0| = |B_0 \cup C_0|$

1. Find countable infinite subset $B_0 \subseteq B$.
2. $C_0 = C \setminus B$ so C_0 is finite.
3. Then $B_0 \cup C_0$ is countable by Q2.
4. $|B_0 \cup C_0| = |\mathbb{Z}_{\geq 0}| = |B_0|$
5. Bijection $f : B_0 \cup C_0 \to B_0$.
6. Then $g : B \cup C \to B$.

$$g(x) = \begin{cases} f(x) & \text{if } x \in B_0 \cup C_0 \\ x & \text{otherwise} \end{cases}$$

is a bijection.

4. Prove that a set B is infinite if and only if there is $A \subsetneq B$ such that $|A| = |B|$.

$p \longrightarrow (q \rightarrow r)$.

1. (→) Suppose B is not infinite.
   1.1 Let A be a set such that $A \subsetneq B$.
   1.2. Then by definition of cardinality, $|A| < |B|$
   1.3. This contradicts $|A| = |B|$, B is infinite.
2. (←) Suppose there is no $A \subsetneq B$ such that $|A| = |B|$.
   2.1. Since B is infinite, $\exists A$ s.t. $|A| = |B|$.
   2.2. Hence by contradiction, there is $A \subsetneq B$ such that $|A| = |B|$.
3. Hence B is infinite iff $A \subsetneq B$ s.t. $|A| = |B|$.

1. (→)
   1.1 To prove B is infinite, $\exists A \subsetneq B\, (|A| = |B|)$
   1.2. Let $b \in B$ and we remove b from B. s.t. $A = B \setminus \{b\}$.
   1.3. $A \cup \{b\} = B$.
      1.3.1 Show that A is infinite.
      1.3.2 Suppose A is finite
      1.3.3 $A \cup \{b\} = \{b, a_1, a_2, \ldots a_n\}$.
      1.3.4. ∴ B is finite (contradiction).
   1.4. Since A is infinite, by Q3, $A \cup \{b\}$ is a bijection onto A
      1.4.1 $|A \cup \{b\}| = |A|$
   1.5. ∴ $|A| = |B|$.

1.1 Suppose B is infinite
1.2 any $b \in B$.
1.3 Define $A = B \setminus \{b\}$.
1.4 Then $A \subsetneq B$.
1.5 $A \to B$ is infinite, A is infinite
1.6 ∴ $|B| = |A \cup \{b\}| = |A|$ by Q3.

2. (←)
   2.1. Suppose B is finite
   2.2. Case 1: $B = \emptyset$
      2.2.1 If $A \subsetneq B$, then $A = B$.
      2.2.2. So there is no $A \subsetneq B$
   2.3. Case 2: $B \neq \emptyset$.
      2.3.1 Take any $b \in B$.
      2.3.2 Let $A = B \setminus \{b\}$, so that $A \subsetneq B$ and A is finite
      2.3.3. Strictly fewer elements in A than B.
      2.3.4. ∴ no bijection $A \to B$.
      2.3.5. $|A| \neq |B|$.

Q4. Prove that a set B is infinite if and only if there is $A \subsetneq B$ such that $|A| = |B|$.

1. ("Only if")
   1.1. Suppose B is infinite.
   1.2. Take any $b \in B$.
   1.3. Define $A = B \setminus \{b\}$.
   1.4. Then $A \subsetneq B$.
   1.5. As B is infinite, we know A is infinite too.
   1.6. So $|B| = |A \cup \{b\}| = |A|$ by Q3.
2. ("If") We prove the contrapositive.
   2.1. Suppose B is finite.
   2.2. Case 1: $B = \emptyset$.
      2.2.1. If $A \subseteq B$, then $A = B$ by Remark 5.1.22(3).
      2.2.2. So there is no $A \subsetneq B$.
   2.3. Case 2: $B \neq \emptyset$.
      2.3.1. Take any $b \in B$.
      2.3.2. Let $A = B \setminus \{b\}$, so that $A \subsetneq B$ and A is finite.
      2.3.3. There are strictly fewer elements in A than in B.
      2.3.4. So Theorem 10.1.3 tells us there is no bijection $A \to B$.
      2.3.5. This means $|A| \neq |B|$.

Q3. Let B be an infinite set and C be a finite set. Then there is a bijection $B \cup C \to B$. So $|B \cup C| = |B|$.

Theorem 10.1.3. Let A and B be finite sets. Then $|A| = |B|$ if and only if there is a bijection $A \to B$.

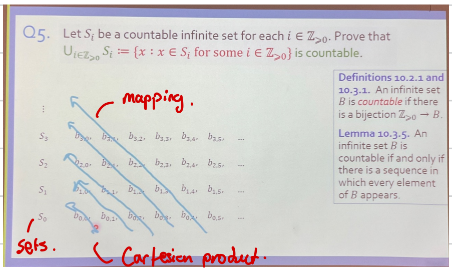
Definition 5.1.23. $A \subsetneq B$ means $A \subseteq B$ and $A \neq B$.

infinitely many union.

5. Let S_i be a countable infinite set for each $i \in \mathbb{Z}_{\geqslant 0}$. Prove that $\bigcup_{i \in \mathbb{Z}_{\geqslant 0}} S_i$ is countable.

Definition. Let S_i be a set for each $i \in \mathbb{Z}_{\geqslant 0}$. Then

$$\bigcup_{i \in \mathbb{Z}_{\geqslant 0}} S_i = \{x : x \in S_i \text{ for some } i \in \mathbb{Z}_{\geqslant 0}\}.$$



Q5. Let S_i be a countable infinite set for each $i \in \mathbb{Z}_{>0}$. Prove that $\bigcup_{i\in\mathbb{Z}_{>0}} S_i := \{x : x \in S_i \text{ for some } i \in \mathbb{Z}_{>0}\}$ is countable.

1. Theorem 10.4.2 tells us $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ is countable.
2. Since this set is also infinite, there is a bijection $\mathbb{Z}_{>0} \to \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$, say f.
3. For each $i \in \mathbb{Z}_{>0}$, apply Lemma 10.3.5 to find a sequence $b_{i,0}, b_{i,1}, b_{i,2}, \ldots$ in which every element of S_i appears.
4. Define a sequence $c_0, c_1, c_2, \ldots$ by setting each $c_k = b_{i,j}$, where $(i,j) = f(k)$. coord
5. In view of Lemma 10.3.5, it suffices to show that every element of $\bigcup_{i\in\mathbb{Z}_{>0}} S_i$ appears in the sequence $c_0, c_1, c_2, \ldots$.
   - 5.1. Take $x \in \bigcup_{i\in\mathbb{Z}_{>0}} S_i$.
   - 5.2. The definition of $\bigcup_{i\in\mathbb{Z}_{>0}} S_i$ gives $i \in \mathbb{Z}_{>0}$ such that $x \in S_i$.
   - 5.3. So line 3 tells us x appears in the sequence $b_{i,0}, b_{i,1}, b_{i,2}, \ldots$.
   - 5.4. Let $j \in \mathbb{Z}_{>0}$ such that $x = b_{i,j}$.
   - 5.5. From the surjectivity of f, we obtain $k \in \mathbb{Z}_{>0}$ such that $f(k) = (i,j)$. coord
   - 5.6. Then $x = b_{i,j} = c_k$ by the definition of c_k.

Definitions 10.2.1 and 10.3.1. An infinite set B is *countable* if there is a bijection $\mathbb{Z}_{>0} \to B$.

Lemma 10.3.5. An infinite set B is countable if and only if there is a sequence in which every element of B appears.

x. S_i

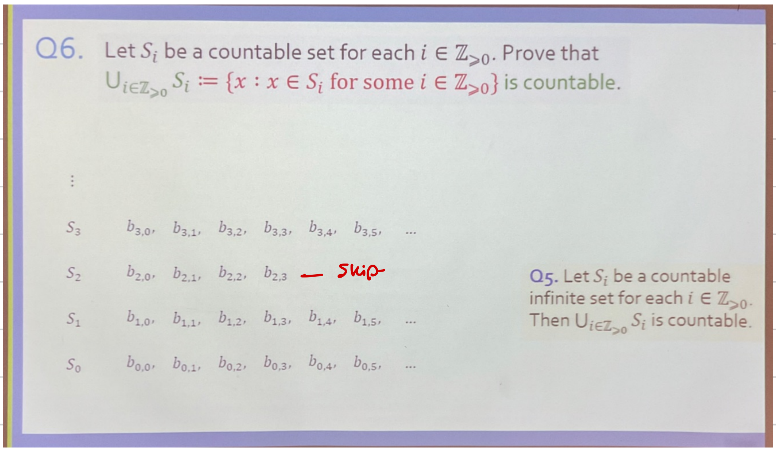
Theorem 10.4.2. $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ is countable.

Definition 9.3.6(1). A function $f : A \to B$ is *surjective* if $\forall y \in B\ \exists x \in A\ (y = f(x))$.

6. Let S_i be a countable (not necessarily infinite) set for each $i \in \mathbb{Z}_{\geqslant 0}$. Prove that $\bigcup_{i\in\mathbb{Z}_{\geqslant 0}} S_i$ is countable.

Definition. Let S_i be a set for each $i \in \mathbb{Z}_{\geqslant 0}$. Then

$$\bigcup_{i\in\mathbb{Z}_{\geqslant 0}} S_i = \{x : x \in S_i \text{ for some } i \in \mathbb{Z}_{\geqslant 0}\}.$$



Q6. Let S_i be a countable set for each $i \in \mathbb{Z}_{\geq 0}$. Prove that $\bigcup_{i\in\mathbb{Z}_{\geq 0}} S_i := \{x : x \in S_i \text{ for some } i \in \mathbb{Z}_{\geq 0}\}$ is countable.

$\vdots$

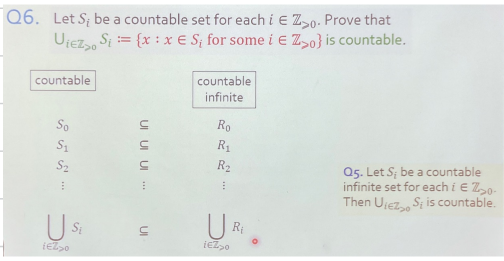
S_3 $\quad b_{3,0}, b_{3,1}, b_{3,2}, b_{3,3}, b_{3,4}, b_{3,5}, \ldots$

S_2 $\quad b_{2,0}, b_{2,1}, b_{2,2}, b_{2,3}$ — skip

S_1 $\quad b_{1,0}, b_{1,1}, b_{1,2}, b_{1,3}, b_{1,4}, b_{1,5}, \ldots$

S_0 $\quad b_{0,0}, b_{0,1}, b_{0,2}, b_{0,3}, b_{0,4}, b_{0,5}, \ldots$

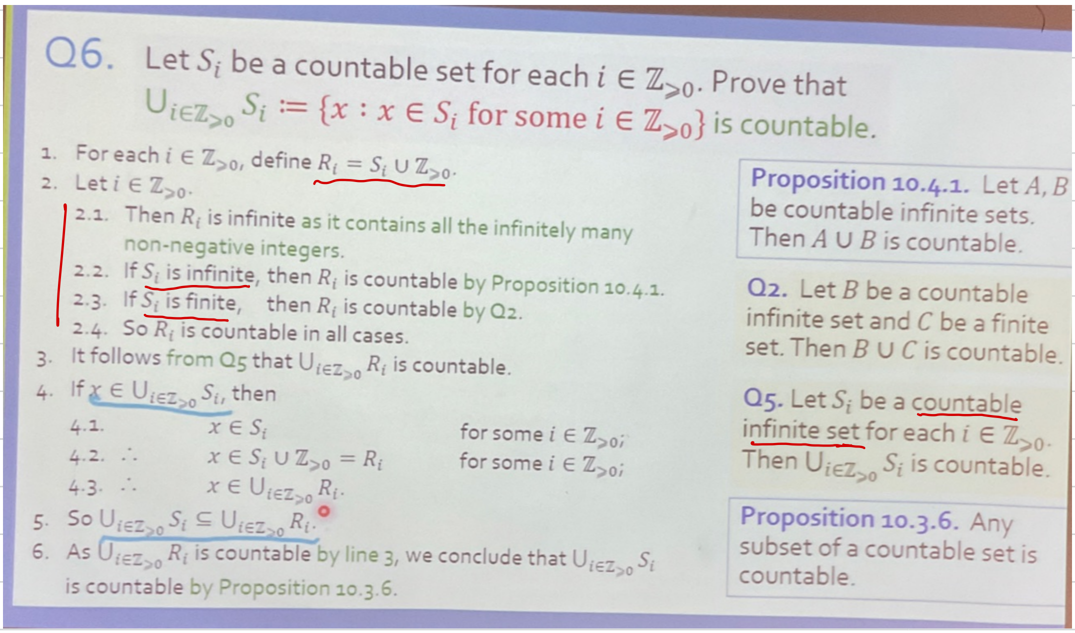
Q5. Let S_i be a countable infinite set for each $i \in \mathbb{Z}_{\geq 0}$. Then $\bigcup_{i\in\mathbb{Z}_{\geq 0}} S_i$ is countable.



Q6. Let S_i be a countable set for each $i \in \mathbb{Z}_{\geq 0}$. Prove that $\bigcup_{i\in\mathbb{Z}_{\geq 0}} S_i := \{x : x \in S_i \text{ for some } i \in \mathbb{Z}_{\geq 0}\}$ is countable.

countable		countable infinite
S_0	$\subseteq$	R_0
S_1	$\subseteq$	R_1
S_2	$\subseteq$	R_2
$\vdots$	$\vdots$	$\vdots$
$\bigcup_{i\in\mathbb{Z}_{\geq 0}} S_i$	$\subseteq$	$\bigcup_{i\in\mathbb{Z}_{\geq 0}} R_i$

Q5. Let S_i be a countable infinite set for each $i \in \mathbb{Z}_{\geq 0}$. Then $\bigcup_{i\in\mathbb{Z}_{\geq 0}} S_i$ is countable.



Q6. Let S_i be a countable set for each $i \in \mathbb{Z}_{\geq 0}$. Prove that $\bigcup_{i\in\mathbb{Z}_{\geq 0}} S_i := \{x : x \in S_i \text{ for some } i \in \mathbb{Z}_{\geq 0}\}$ is countable.

1. For each $i \in \mathbb{Z}_{\geq 0}$, define $R_i = S_i \cup \mathbb{Z}_{\geq 0}$.
2. Let $i \in \mathbb{Z}_{\geq 0}$.
   2.1. Then R_i is infinite as it contains all the infinitely many non-negative integers.
   2.2. If S_i is infinite, then R_i is countable by Proposition 10.4.1.
   2.3. If S_i is finite, then R_i is countable by Q2.
   2.4. So R_i is countable in all cases.
3. It follows from Q5 that $\bigcup_{i\in\mathbb{Z}_{\geq 0}} R_i$ is countable.
4. If $x \in \bigcup_{i\in\mathbb{Z}_{\geq 0}} S_i$, then
   4.1. $\quad x \in S_i \quad$ for some $i \in \mathbb{Z}_{\geq 0}$;
   4.2. $\therefore \quad x \in S_i \cup \mathbb{Z}_{\geq 0} = R_i \quad$ for some $i \in \mathbb{Z}_{\geq 0}$;
   4.3. $\therefore \quad x \in \bigcup_{i\in\mathbb{Z}_{\geq 0}} R_i$.
5. So $\bigcup_{i\in\mathbb{Z}_{\geq 0}} S_i \subseteq \bigcup_{i\in\mathbb{Z}_{\geq 0}} R_i$.
6. As $\bigcup_{i\in\mathbb{Z}_{\geq 0}} R_i$ is countable by line 3, we conclude that $\bigcup_{i\in\mathbb{Z}_{\geq 0}} S_i$ is countable by Proposition 10.3.6.

Proposition 10.4.1. Let A, B be countable infinite sets. Then $A \cup B$ is countable.

Q2. Let B be a countable infinite set and C be a finite set. Then $B \cup C$ is countable.

Q5. Let S_i be a countable infinite set for each $i \in \mathbb{Z}_{\geq 0}$. Then $\bigcup_{i\in\mathbb{Z}_{\geq 0}} S_i$ is countable.

Proposition 10.3.6. Any subset of a countable set is countable.

$F_1 = \{\{0\}, \{1\}, \{2\}, \ldots\}$.
$F_2 = \{\{0,1\}, \{0,2\}, \{1,1\}, \ldots\}$
$F_n = ?$

7. For each $n \in \mathbb{Z}_{\geqslant 0}$, define $F_n = \{X \in \mathcal{P}(\mathbb{Z}_{\geqslant 0}) : |X| = n\}$. Let $F = \bigcup_{n \in \mathbb{Z}_{\geqslant 0}} F_n$.

(a) Prove that F_n is countable for every $n \in \mathbb{Z}_{\geqslant 1}$ by induction on n.

(b) Deduce that F is countable.

F_1 is countable
F_k is countable $\to$ F_{k+1} is countable.

Q7(a). For each $n \in \mathbb{Z}_{\geq 0}$, define $F_n = \{X \in \mathcal{P}(\mathbb{Z}_{\geq 0}) : |X| = n\}$. Prove that F_n is countable for every $n \in \mathbb{Z}_{\geq 1}$ by induction on n.

1. For each $n \in \mathbb{Z}_{\geq 1}$, let $P(n)$ be "F_n is countable".
2. (Base step)
   - 2.1. $\{0\}, \{1\}, \{2\}, \ldots$ is a sequence in which every element of F_1 appears.
   - 2.2. So F_1 is countable by Lemma 10.3.5.
   - 2.3. This shows $P(1)$.
3. (Induction step)
   - 3.1. Let $k \in \mathbb{Z}_{\geq 1}$ such that $P(k)$ is true, i.e., F_k is countable.
   - 3.2. Use Note 10.3.4 to find a sequence $B_0, B_1, B_2, \ldots \in F_k$ in which every element of F_k appears.
   - c elems of set F_{k+1}
   - 3.3. For each $j \in \mathbb{Z}_{\geq 0}$, the set $F_{k,j} := \{B_0 \cup \{j\}, B_1 \cup \{j\}, B_2 \cup \{j\}, \ldots\}$ is countable by Lemma 10.3.5.
   - 3.4. So $\bigcup_{j \in \mathbb{Z}_{\geq 0}} F_{k,j}$ is countable by Q6.
   - 3.5. We claim that $F_{k+1} \subseteq \bigcup_{j \in \mathbb{Z}_{\geq 0}} F_{k,j}$. ...
   - 3.6. Thus F_{k+1} is countable by Proposition 10.3.6.
   - 3.7. This shows $P(k+1)$.

Hence $\forall n \in \mathbb{Z}_{\geq 1}$ $P(n)$ is true by MI.

why? Trying to get a set that contains k+1 (k+1) set.

Note 10.3.4 and Lemma 10.3.5. An infinite set B is countable if and only if there is a sequence $b_0, b_1, b_2, \ldots$ (of elements of B) in which every element of B appears.

Q6. Let S_i be a countable set for each $i \in \mathbb{Z}_{\geq 0}$. Then $\bigcup_{i \in \mathbb{Z}_{\geq 0}} S_i$ is countable.

Proposition 10.3.6. Any subset of a countable set is countable.

- 3.5.1. Take any $C \in F_{k+1}$.
- 3.5.2. This means C is a subset of $\mathbb{Z}_{\geq 0}$ of cardinality $k + 1 \geq 1 + 1 = 2$.
- 3.5.3. Pick $m \in C$.
- 3.5.4. Then $|C \setminus \{m\}| = |C| - 1 = (k+1) - 1 = k$.
- 3.5.5. So $C \setminus \{m\} \in F_k$ by the definition of F_k.
- 3.5.6. Then $C \setminus \{m\} \in F_k$ must appear in the sequence $B_0, B_1, B_2, \ldots$ as every element of F_k appears in this sequence.
- 3.5.7. Let $i \in \mathbb{Z}_{\geq 0}$ such that $B_i = C \setminus \{m\}$.
- 3.5.8. Then $C = B_i \cup \{m\} \in F_{k,m} \subseteq \bigcup_{j \in \mathbb{Z}_{\geq 0}} F_{k,j}$.

$F_{k,j} = \{B_0 \cup \{j\}, \ldots\}$

$\bigcup_{j \in \mathbb{Z}_{\geq 0}} F_{k,j}$: $F_{k,0} = \{B_0 \cup \{0\}, B_1 \cup \{0\}, \ldots\}$.

$F_{k,1} = \{B_0 \cup \{1\}, \ldots\}$

$F_{k,2} = \{B_0 \cup \{2\}, \ldots\}$.

$\vdots$

$F_{k,m}$, $m \in C$, $C \subseteq \mathbb{Z}_{\geq 0}$

$\therefore m \in \mathbb{Z}_{\geq 0}$ $\quad B_i = C \setminus \{m\}$.

$\therefore F_{k,m} = \{B_0 \cup \{m\}, B_1 \cup \{m\}, \ldots \underline{B_i \cup \{m\}}\}$.

$\Downarrow$

C.

Q7(b). For each $n \in \mathbb{Z}_{\geq 0}$, define $F_n = \{X \in \mathcal{P}(\mathbb{Z}_{\geq 0}) : |X| = n\}$. Prove that $F := \bigcup_{n \in \mathbb{Z}_{\geq 0}} F_n$ is countable.

1. Note that $F_0 = \{X \in \mathcal{P}(\mathbb{Z}_{\geq 0}) : |X| = 0\} = \{\emptyset\}$.
2. So F_0 is finite and hence countable.
3. It follows from (a) that F_n is countable for all $n \in \mathbb{Z}_{\geq 0}$.
4. Hence F is countable by Q6.

Q7(a). F_n is countable for all $n \in \mathbb{Z}_{\geq 1}$.

Q6. Let S_i be a countable set for each $i \in \mathbb{Z}_{\geq 0}$. Then $\bigcup_{i \in \mathbb{Z}_{\geq 0}} S_i$ is countable.

Definition 10.3.1. A set is *countable* if it is finite or it has the same cardinality as $\mathbb{Z}_{\geq 0}$.

8. In the answer to Exercise 10.4.4 in the notes, it is proved that $\mathbb{Q}_{\geqslant 0}$ is countable. Use this fact to show that $\mathbb{Q}$ is countable.

$= \mathbb{Q}^+ \cup \{0\} \cup \mathbb{Q}^-$

prof bijective

$f(q) = -q.$

$(1,0), (1,1)$

$(0,0), (0,1).$ $\to \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$

$f(m,n) = \frac{m}{n}$

$f(1,1) = f(2,2) = 1$

Q8. Show that $\mathbb{Q}$ is countable. $= \mathbb{Q}^+ \cup \{0\} \cup \mathbb{Q}$

1. Use Note 10.3.4 to find a sequence $b_0, b_1, b_2, \ldots \in \mathbb{Q}_{\geqslant 0}$ in which every element of $\mathbb{Q}_{\geqslant 0}$ appears.
2. Consider the sequence $c_0, c_1, c_2, \ldots$ defined by setting $c_{2i} = b_i$ and $c_{2i+1} = -b_i$ for all $i \in \mathbb{Z}_{\geqslant 0}$.
3. In view of Lemma 10.3.5, it suffices to show that every element of $\mathbb{Q}$ appears in it.
   - 3.1. Take $y \in \mathbb{Q}$.
   - 3.2. Case 1: suppose $y \geqslant 0$.
     - 3.2.1. Then y must appear in the sequence $b_0, b_1, b_2, \ldots$ as every element of $\mathbb{Q}_{\geqslant 0}$ appears in this sequence.
     - 3.2.2. Let $i \in \mathbb{Z}_{\geqslant 0}$ such that $b_i = y$.
     - 3.2.3. Then $y = b_i = c_{2i}$.
   - 3.3. Case 2: suppose $y < 0$.
     - 3.3.1. Then $-y > 0$.
     - 3.3.2. Then $-y$ must appear in the sequence $b_0, b_1, b_2, \ldots$ as every element of $\mathbb{Q}_{\geqslant 0}$ appears in this sequence.
     - 3.3.3. Let $i \in \mathbb{Z}_{\geqslant 0}$ such that $b_i = -y$.
     - 3.3.4. Then $y = -b_i = c_{2i+1}$.
   - 3.4. In either case, we see that y appears in the sequence $c_0, c_1, c_2, \ldots$.

$f(q) = -q$

Exercise 10.4.4. $\mathbb{Q}_{\geqslant 0}$ is countable.

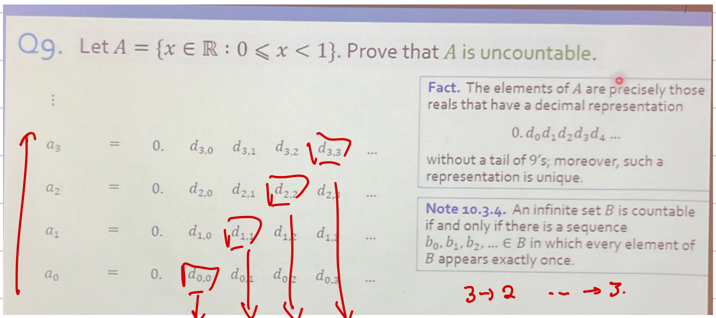
Note 10.3.4 and Lemma 10.3.5. An infinite set B is countable if and only if there is a sequence $b_0, b_1, b_2, \ldots$ (of elements of B) in which every element of B appears.

9. Let $A = \{x \in \mathbb{R} : 0 \leqslant x < 1\}$. Using a diagonal argument, or otherwise, prove that A is uncountable.

You may use without proof the fact that the elements of A are precisely those real numbers that have a decimal representation

$$0.d_0d_1d_2d_3d_4\ldots$$

without a tail of 9's; moreover, such a representation is unique.



$c = 0.$

$0.12\ 4\ 3\ 0$

$0.10223\ldots$

$0.32135\ldots$

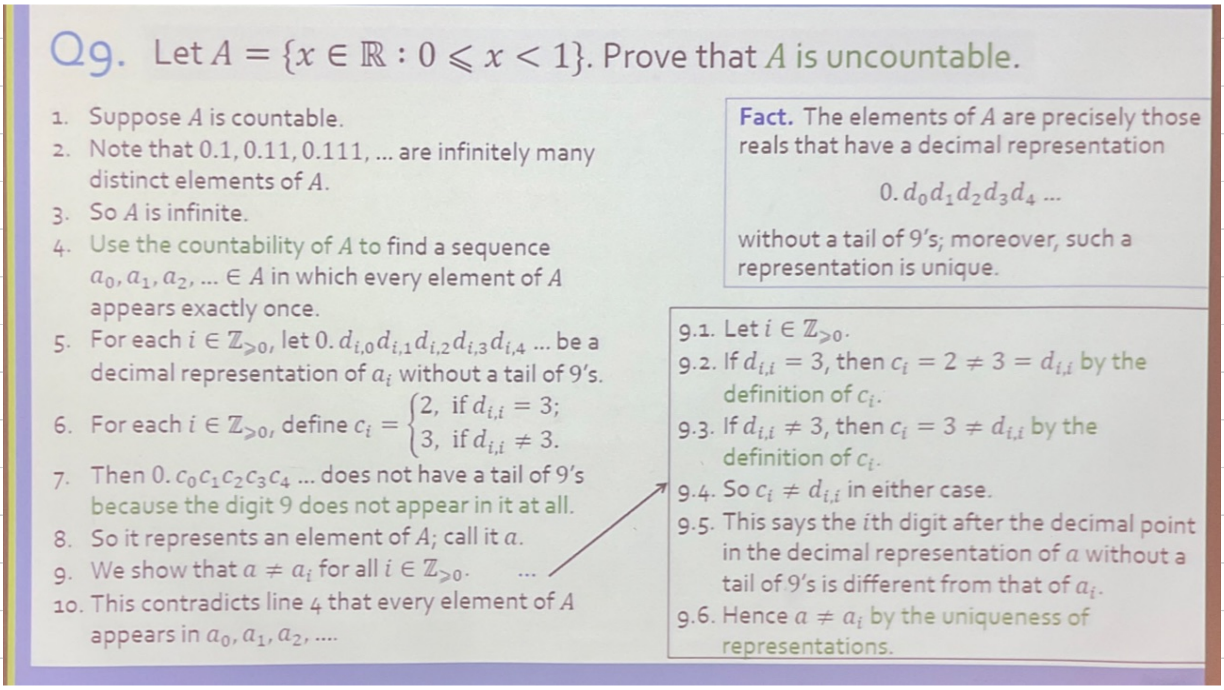
$0.11274\ldots$

$0.3332\ldots$

$3 \to 2$

$\ldots \to 3$

always different, by 1 decimal place



10. Prove that $\mathbb{R}$ and $\mathbb{C}$ are uncountable.