Section 2.5

Determinants

Objectives

- How do matrix operations affect determinants?
- What is the relation between invertibility and determinant?
- What is the adjoint of a matrix?
- What is Cramer's rule?

Two main results:

$\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$

$\mathbf{A}$ is invertible if and only if $\det(\mathbf{A}) \neq 0$

Matrices with two identical rows (columns)

Theorem 2.5.12 & Example 2.5.13

1. The determinant of a square matrix with two identical rows is zero.

2. The determinant of a square matrix with two identical columns is zero.

$$\begin{pmatrix} 4 & -2 \\ 4 & -2 \end{pmatrix}$$

det = 0

$$\begin{pmatrix} 1 & 2 & 4 \\ -1 & 10 & 4 \\ 1 & 2 & 4 \end{pmatrix}$$

det = 0

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & -3 & -3 & 9 \\ 2 & 4 & 4 & 0 \\ 0 & -2 & -2 & -1 \end{pmatrix}$$

det = 0

Theorem 2.5.12 (Exercise 2.58)

Prove by mathematical induction.

The determinant of a square matrix with two identical rows is zero

Base case

2×2: $$\overset{\mathbf{0}}{\begin{vmatrix} a & b \\ a & b \end{vmatrix}} = ab - ab$$

Inductive step $k \times k \Rightarrow (k+1) \times (k+1)$

3×3: $$\overset{\mathbf{0}}{\begin{vmatrix} a & b & c \\ * & * & * \\ a & b & c \end{vmatrix}} = - * \overset{\mathbf{0}}{\begin{vmatrix} b & c \\ b & c \end{vmatrix}} + * \overset{\mathbf{0}}{\begin{vmatrix} a & c \\ a & c \end{vmatrix}} - * \overset{\mathbf{0}}{\begin{vmatrix} a & b \\ a & b \end{vmatrix}}$$

cofactor expansion along row 2

How does e.r.o affect determinants?

Discussion 2.5.14 & Theorem 2.5.15

$$\boldsymbol{A} \xrightarrow{E.R.O.} \boldsymbol{B}$$

What is the relation between det($\boldsymbol{A}$) and det($\boldsymbol{B}$)?

E.R.O	Determinant
$\boldsymbol{A} \xrightarrow{kR_i} \boldsymbol{B}$	det($\boldsymbol{B}$) = k det($\boldsymbol{A}$)
$\boldsymbol{A} \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{B}$	det($\boldsymbol{B}$) = −det($\boldsymbol{A}$)
$\boldsymbol{A} \xrightarrow{R_i + kR_j} \boldsymbol{B}$	det($\boldsymbol{B}$) = det($\boldsymbol{A}$)

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{3R_2} \mathbf{B} = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 0 & 3 \\ 0 & 1 & 1 \end{pmatrix}$$

det(B) = 3 det(A)

$$\mathbf{A} = \begin{pmatrix} 3 & 3 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \mathbf{B} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix}$$

det(B) = −det(A)

Using e.r.o. to find determinants

Example 2.5.17.1

$$\begin{vmatrix} 3 & -1 & 1 & 1 \\ 3 & -1 & 2 & 1 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 2 & -1 \end{vmatrix} \overset{R_2 - R_1}{=} \begin{vmatrix} 3 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 2 & -1 \end{vmatrix}$$

$$\overset{R_2 \leftrightarrow R_3}{=} - \begin{vmatrix} 3 & -1 & 1 & 1 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 \end{vmatrix} \overset{R_4 - 2R_3}{=} - \begin{vmatrix} 3 & -1 & 1 & 1 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}$$

$$= -3 \times 2 \times 1 \times (-1) = 6$$

Gaussian Elimination

Using e.r.o. to find determinants

Example 2.5.17.2

$$\boldsymbol{A} \xrightarrow{R_1 + \frac{2}{9}R_2} \boldsymbol{C} \xrightarrow{R_2 \leftrightarrow R_3} \boldsymbol{D} \xrightarrow{4R_2} \boldsymbol{B} = \begin{pmatrix} 5 & 0 & 8 & -1 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & \frac{1}{3} \end{pmatrix}$$

$\det(\boldsymbol{B}) = 4\det(\boldsymbol{D})$

Find $\det(\boldsymbol{A})$.

$$\det(\boldsymbol{B}) = 5 \times (-2) \times 1 \times \frac{1}{3} = -\frac{10}{3}$$

$$\det(\boldsymbol{A}) = \det(\boldsymbol{C}) = -\det(\boldsymbol{D}) = -\boxed{\frac{1}{4}}\det(\mathbf{B}) = \frac{5}{6}$$

Proof of part 3

To prove: det(***B***) = det(***A***)

Theorem 2.5.15

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \xrightarrow{R_2 + kR_1} \mathbf{B} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} + ka_{11} & a_{22} + ka_{12} & a_{23} + ka_{13} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

the (2, j)-cofactor of ***A*** = the (2, j)-cofactor of ***B***

$$A_{21} = B_{21} \qquad A_{22} = B_{22} \qquad A_{23} = B_{23}$$

Cofactor expansion along row 2 of ***B*** :

$$\det(\mathbf{B}) = (a_{21} + ka_{11})B_{21} + (a_{22} + ka_{12})B_{22} + (a_{23} + ka_{13})B_{23}$$

$$= (a_{21} + ka_{11})A_{21} + (a_{22} + ka_{12})A_{22} + (a_{23} + ka_{13})A_{23}$$

$$= \underbrace{(a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23})}_{= \det(\mathbf{A})} + k\underbrace{(a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23})}_{\neq \det(\mathbf{A})}$$

Proof of part 3

To prove: det(***B***) = det(***A***)

Theorem 2.5.15

shown = 0

entries on row 1

$$\boxed{\det(\mathbf{B}) = \det(\mathbf{A})} + k\boxed{(a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23})}$$

cofactors for row 2

replace row 2 of ***A*** by row 1

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

cofactor expansion (***A***′) along row 2

$A_{21} = A'_{21}$
$A_{22} = A'_{22}$
$A_{23} = A'_{23}$

$$\det(\mathbf{A}') = a_{11}A'_{21} + a_{12}A'_{22} + a_{13}A'_{23}$$

$$= a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23}$$

$$\det(\mathbf{A}') = 0$$

$$\mathbf{A}' = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

two identical rows

Theorem 2.5.15 (part 4)

A : nxn square matrix

	det(***E***)	e.r.o.	Determinant
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{pmatrix}$	k	$\boldsymbol{A} \xrightarrow{kR_i} \boldsymbol{B}$	$\det(\boldsymbol{B}) = k\det(\boldsymbol{A})$ $= \det(\boldsymbol{E})\det(\boldsymbol{A})$
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	-1	$\boldsymbol{A} \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{B}$	$\det(\boldsymbol{B}) = -\det(\boldsymbol{A})$ $= \det(\boldsymbol{E})\det(\boldsymbol{A})$
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{pmatrix}$	1	$\boldsymbol{A} \xrightarrow{R_i + kR_j} \boldsymbol{B}$	$\det(\boldsymbol{B}) = \det(\boldsymbol{A})$ $= \det(\boldsymbol{E})\det(\boldsymbol{A})$

E : nxn elementary matrix

$$\boldsymbol{EA} = \boldsymbol{B} \Rightarrow \boxed{\det(\boldsymbol{EA})} = \det(\boldsymbol{B}) = \boxed{\det(\boldsymbol{E})\det(\boldsymbol{A})}$$

Pre-multiplying a matrix with elementary matrices

Remark

For any square matrix ***A*** and elementary matrix ***E***:

$$\det(\boldsymbol{EA}) = \det(\boldsymbol{E})\det(\boldsymbol{A})$$

$$\begin{aligned}\det(\boldsymbol{E_2E_1A}) &= \det(\boldsymbol{E_2})\det(\boldsymbol{E_1A}) \\ &= \det(\boldsymbol{E_2})\det(\boldsymbol{E_1})\det(\boldsymbol{A})\end{aligned}$$

$$\det(\boldsymbol{E_k}\cdots\boldsymbol{E_2E_1A}) = \det(\boldsymbol{E_k})\cdots\det(\boldsymbol{E_2})\det(\boldsymbol{E_1})\det(\boldsymbol{A})$$

In particular

$$\det(\boldsymbol{E_k}\cdots\boldsymbol{E_2E_1}) = \det(\boldsymbol{E_k})\cdots\det(\boldsymbol{E_2})\det(\boldsymbol{E_1})$$

Note: We have not yet proved that

$$\boxed{\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B})}$$

Remark 2.5.18

Since $\det(\boldsymbol{A}) = \det(\boldsymbol{A}^T)$ for any square matrix $\boldsymbol{A}$, Theorem 2.5.15 is still true if we change "rows" to "columns".

- We have 3 corresponding elementary column operations.
- Column operations have same effect as post-multiplying an elementary matrix.

$$\boldsymbol{A} \xrightarrow[\text{operation}]{C:\text{ column}} \boldsymbol{B} \qquad \boldsymbol{B} = \boldsymbol{A}\boldsymbol{E}$$

Column operations

Remark 2.5.18 = (Theorem 2.5.15)^T

A : nxn square matrix ***E*** : nxn elementary matrix

Elementary column operation	Determinant
$\boldsymbol{A} \xrightarrow{kC_i} \boldsymbol{B}$	$\det(\boldsymbol{B}) = k\det(\boldsymbol{A})$
$\boldsymbol{A} \xrightarrow{C_i \leftrightarrow C_j} \boldsymbol{B}$	$\det(\boldsymbol{B}) = -\det(\boldsymbol{A})$
$\boldsymbol{A} \xrightarrow{C_i + kC_j} \boldsymbol{B}$	$\det(\boldsymbol{B}) = \det(\boldsymbol{A})$

$$\det(\boldsymbol{AE}) = \det(\boldsymbol{A})\det(\boldsymbol{E})$$

Determinant and invertibility

Theorem 2.5.19

A square matrix $\boldsymbol{A}$ is invertible

if and only if

$\det(\boldsymbol{A}) \neq 0$.

contrapositive

converse

$\boldsymbol{A}$ is invertible $\Rightarrow \det(\boldsymbol{A}) \neq 0$

$\boldsymbol{A}$ is invertible $\Leftarrow \det(\boldsymbol{A}) \neq 0$

$\boldsymbol{A}$ is not invertible $\Rightarrow \det(\boldsymbol{A}) = 0$

$\boldsymbol{A}$ is not invertible $\Leftarrow \det(\boldsymbol{A}) = 0$

same meaning

different meaning

different meaning

The proof

Theorem 2.5.19

A square matrix $\boldsymbol{A}$ is invertible
if and only if
$\det(\boldsymbol{A}) \neq 0$.

$$\mathbf{A} \xrightarrow{R_1} \quad \xrightarrow{R_2} \quad \cdots \quad \xrightarrow{R_k} \mathbf{B} \text{ (RREF)}$$

$$\boldsymbol{E_k \cdots E_2 E_1 A} = \boldsymbol{B}$$

$$\det(\boldsymbol{E_k \cdots E_2 E_1 A}) = \det(\boldsymbol{B})$$

$$\underbrace{\det(\boldsymbol{E_k})\cdots\det(\boldsymbol{E_2})\det(\boldsymbol{E_1})}_{\text{non-zero}}\det(\boldsymbol{A}) = \det(\boldsymbol{B})$$

$\boldsymbol{A}$ invertible $\Rightarrow$ RREF $\boldsymbol{B} = \boldsymbol{I} \Rightarrow \det(\boldsymbol{B}) = 1 \Rightarrow \det(\boldsymbol{A}) \neq 0$

$\boldsymbol{A}$ not invertible $\Rightarrow$ RREF $\boldsymbol{B}$ has zero row

$\Rightarrow \det(\boldsymbol{B}) = 0 \Rightarrow \det(\boldsymbol{A}) = 0$

Remark 2.5.21

check invertibility

Theorem 2.5.19

Remark 2.4.10

determinant

row echelon form

- When the determinant is easy to get
- Connecting concepts

In practice

What is the actual value of determinant for?

Theorem 2.5.22

$\boldsymbol{A}$ and $\boldsymbol{B}$: square matrices of order n
c a scalar

$\det(c\boldsymbol{A}) \neq c\det(\boldsymbol{A}$

1. $\det(c\boldsymbol{A}) = c^n \det(\boldsymbol{A})$

2. $\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B})$ Multiplicative property

3. if $\boldsymbol{A}$ is invertible, then

$$\det(\boldsymbol{A}^{-1}) = \frac{1}{\det(\boldsymbol{A})}$$

4. $\det(\boldsymbol{A} + \boldsymbol{B}) \neq \det(\boldsymbol{A}) + \det(\boldsymbol{B})$

Proof of part 2

$\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B})$

$\det(\boldsymbol{EB}) = \det(\boldsymbol{E})\det(\boldsymbol{B})$

Theorem 2.5.22

Case 1: $\boldsymbol{A}$ is singular

By Theorem 2.4.14

$\boldsymbol{AB}$ is singular

$\Rightarrow \det(\boldsymbol{A}) = 0$

$\Rightarrow \det(\boldsymbol{A})\det(\boldsymbol{B}) = 0$

$\Rightarrow \det(\boldsymbol{AB}) = 0$

Case 2: $\boldsymbol{A}$ is invertible

Theorem 2.4.7: (1) implies (4)

$\Rightarrow \boldsymbol{A} = \boldsymbol{E}_1\boldsymbol{E}_2\cdots\boldsymbol{E}_k$ (product of elementary matrices)

$$\begin{aligned}\det(\boldsymbol{AB}) &= \det(\boldsymbol{E}_1\boldsymbol{E}_2\cdots\boldsymbol{E}_k\boldsymbol{B})\\ &= \det(\boldsymbol{E}_1)\det(\boldsymbol{E}_2)\cdots\det(\boldsymbol{E}_k)\det(\boldsymbol{B})\\ &= \det(\boldsymbol{E}_1\boldsymbol{E}_2\cdots\boldsymbol{E}_k)\det(\boldsymbol{B})\\ &= \det(\boldsymbol{A})\det(\boldsymbol{B})\end{aligned}$$

What is adjoint?

Definition 2.5.24

Let $\boldsymbol{A}$ be a square matrix of order n.

The adjoint of $\boldsymbol{A}$ is the $n \times n$ matrix

$$\mathrm{adj}(\boldsymbol{A}) = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}^{T} = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1\mathrm{n}} & A_{2\mathrm{n}} & \cdots & A_{nn} \end{pmatrix}$$

where A_{ij} is the (i, j)-cofactor of $\boldsymbol{A}$.

$(-1)^{i+j} \det(\boldsymbol{M_{ij}})$

What is adjoint?

Example 2.5.26.2

$$\boldsymbol{B}=\begin{pmatrix}1 & -1 & 1\\ 0 & -1 & 0\\ 1 & 0 & 3\end{pmatrix}$$

$$\operatorname{adj}(\boldsymbol{B})=\begin{pmatrix}\begin{vmatrix}-1 & 0\\ 0 & 3\end{vmatrix} & -\begin{vmatrix}0 & 0\\ 1 & 3\end{vmatrix} & \begin{vmatrix}0 & -1\\ 1 & 0\end{vmatrix}\\ -\begin{vmatrix}-1 & 1\\ 0 & 3\end{vmatrix} & \begin{vmatrix}1 & 1\\ 1 & 3\end{vmatrix} & -\begin{vmatrix}1 & -1\\ 1 & 0\end{vmatrix}\\ \begin{vmatrix}-1 & 1\\ -1 & 0\end{vmatrix} & -\begin{vmatrix}1 & 1\\ 0 & 0\end{vmatrix} & \begin{vmatrix}1 & -1\\ 0 & -1\end{vmatrix}\end{pmatrix}^{T}=\begin{pmatrix}-3 & 0 & 1\\ 3 & 2 & -1\\ 1 & 0 & -1\end{pmatrix}^{T}=\begin{pmatrix}-3 & 3 & 1\\ 0 & 2 & 0\\ 1 & -1 & -1\end{pmatrix}$$

What is adjoint for?

Theorem 2.5.25

Let $\boldsymbol{A}$ be a square matrix.

If $\boldsymbol{A}$ is invertible, then

$$\boxed{\boldsymbol{A}^{-1} = \frac{1}{\det(\boldsymbol{A})}\operatorname{adj}(\boldsymbol{A})}$$

Example 2.5.26.2

$$\boldsymbol{B} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 3 \end{pmatrix} \qquad \det(\boldsymbol{B}) = -2 \qquad \operatorname{adj}(\mathbf{B}) = \begin{pmatrix} -3 & 3 & 1 \\ 0 & 2 & 0 \\ 1 & -1 & -1 \end{pmatrix}$$

$$\boldsymbol{B}^{-1} = \frac{1}{\det(\boldsymbol{B})}\operatorname{adj}(\boldsymbol{B}) = -\frac{1}{2}\begin{pmatrix} -3 & 3 & 1 \\ 0 & 2 & 0 \\ 1 & -1 & -1 \end{pmatrix}$$

The proof

Theorem 2.5.25

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \qquad \operatorname{adj}(\boldsymbol{A}) = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}$$

the (i, i)-entry of $\boldsymbol{A}[\operatorname{adj}(\boldsymbol{A})]$ diagonal entries

$$= a_{i1}A_{i1} + a_{i2}A_{i2} + \ldots + a_{in}A_{in} = \det(\boldsymbol{A})$$

cofactor expansion along row i

the (i, j)-entry of $\boldsymbol{A}[\operatorname{adj}(\boldsymbol{A})]$ with $i \neq j$,

non-diagonal entries

$$= a_{i1}A_{j1} + a_{i2}A_{j2} + \ldots + a_{in}A_{jn}$$

$$= 0$$

see the proof of Theorem 2.5.15.3.

The proof

Theorem 2.5.25

$$\boldsymbol{A}\,\mathrm{adj}(\boldsymbol{A}) = \begin{pmatrix} \det(\boldsymbol{A}) & 0 & \dots & 0 \\ 0 & \det(\boldsymbol{A}) & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \det(\boldsymbol{A}) \end{pmatrix} = \det(\boldsymbol{A}) \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

$$= \det(\boldsymbol{A})\boldsymbol{I}$$

$$\Rightarrow \quad \boldsymbol{A}\left[\frac{1}{\det(\boldsymbol{A})}\mathrm{adj}(\boldsymbol{A})\right] = \boldsymbol{I}$$

$$\Rightarrow \mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathrm{adj}(\mathbf{A})$$

Using adjoint to find inverse

Remark

find inverse

Theorem 2.5.25

Discussion 2.4.8

adjoint

reduced row echelon form

for smaller size

for larger size

give explicit formula
for inverse

Example 2.5.28 (Cramer's rule)

Use Cramer's rule to solve the system of linear equations

$$\begin{cases} x + y + 3z = 0 \\ 2x - 2y + 2z = 4 \\ 3x + 9y = 3 \end{cases}$$

Rewrite the linear system as $\begin{pmatrix} 1 & 1 & 3 \\ 2 & -2 & 2 \\ 3 & 9 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix}$

$$\boldsymbol{A}_1 = \begin{pmatrix} 0 & 1 & 3 \\ 4 & -2 & 2 \\ 3 & 9 & 0 \end{pmatrix} \quad \boldsymbol{A}_2 = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 4 & 2 \\ 3 & 3 & 0 \end{pmatrix} \quad \boldsymbol{A}_3 = \begin{pmatrix} 1 & 1 & 0 \\ 2 & -2 & 4 \\ 3 & 9 & 3 \end{pmatrix}$$

What is Cramer's rule?

Example 2.5.28

$$\begin{cases} x + y + 3z = 0 \\ 2x - 2y + 2z = 4 \\ 3x + 9y = 3 \end{cases}$$

Cramer's rule says:

$$x = \frac{\det(A_1)}{\det(A)} = \frac{\begin{vmatrix} 0 & 1 & 3 \\ 4 & -2 & 2 \\ 3 & 9 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 3 \\ 2 & -2 & 2 \\ 3 & 9 & 0 \end{vmatrix}} = \frac{132}{60} = 2.2$$

$$y = \frac{\det(A_2)}{\det(A)} = \frac{\begin{vmatrix} 1 & 0 & 3 \\ 2 & 4 & 2 \\ 3 & 3 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 3 \\ 2 & -2 & 2 \\ 3 & 9 & 0 \end{vmatrix}} = \frac{-24}{60} = -0.4$$

$$z = \frac{\det(A_3)}{\det(A)} = \frac{\begin{vmatrix} 1 & 1 & 0 \\ 2 & -2 & 4 \\ 3 & 9 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 3 \\ 2 & -2 & 2 \\ 3 & 9 & 0 \end{vmatrix}} = \frac{-36}{60} = -0.6$$

this gives the unique solution of the system

What is Cramer's rule?

Theorem 2.5.27 (Cramer's Rule)

Suppose $\boldsymbol{Ax} = \boldsymbol{b}$ is a linear system where $\boldsymbol{A}$ is an n x n invertible matrix.

terms and conditions

Let $\boldsymbol{A_i}$ be the matrix obtained from $\boldsymbol{A}$ by replacing the ith column of $\boldsymbol{A}$ by $\boldsymbol{b}$.

Then the system has a unique solution

$$\boldsymbol{x} = \frac{1}{\det(\boldsymbol{A})}\begin{pmatrix} \det(\boldsymbol{A}_1) \\ \det(\boldsymbol{A}_2) \\ \vdots \\ \det(\boldsymbol{A}_n) \end{pmatrix}$$

$$x_1 = \frac{\det(\boldsymbol{A_1})}{\det(\boldsymbol{A})}$$

$$x_2 = \frac{\det(\boldsymbol{A_2})}{\det(\boldsymbol{A})}$$

$$x_n = \frac{\det(\boldsymbol{A}_n)}{\det(\boldsymbol{A})}$$

The proof

Proof of Theorem 2.5.27

To prove:

$$\boxed{x_i = \frac{\det(\boldsymbol{A}_i)}{\det(\boldsymbol{A})}}$$

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \Leftrightarrow \boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b} = \frac{1}{\det(\boldsymbol{A})}\boxed{\operatorname{adj}(\boldsymbol{A})}\boldsymbol{b}$$

$$\Leftrightarrow \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \frac{1}{\det(A)} \begin{pmatrix} A_{11} & A_{21} & \ldots & A_{n1} \\ A_{12} & A_{22} & \ldots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \ldots & A_{nn} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

To show

$$x_1 = \frac{1}{\det(\mathbf{A})}(b_1A_{11} + b_2A_{21} + \ldots + b_nA_{n1})$$

$$x_i = \frac{1}{\det(\mathbf{A})}(b_1A_{1i} + b_2A_{2i} + \ldots + b_nA_{ni})$$

for $i = 1, 2, \ldots, n$

The proof

To prove:

$$x_i = \frac{\det(\boldsymbol{A}_i)}{\det(\boldsymbol{A})}$$

Proof of Theorem 2.5.27

entries on column i

$$x_i = \frac{1}{\det(\boldsymbol{A})}\underbrace{\boxed{(b_1 A_{1i} + b_2 A_{2i} + \ldots + b_n A_{ni})}}_{\det(\boldsymbol{A_i})} = \frac{\det(\boldsymbol{A_i})}{\det(\boldsymbol{A})}$$

cofactors for column i

cofactor expansion of $\boldsymbol{A}_i$ along column i

$$\det(\boldsymbol{A_i}) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,i-1} & b_1 & a_{1,i+1} & \ldots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2,i-1} & b_2 & a_{2,i+1} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{\mathrm{n},i-1} & b_n & a_{\mathrm{n},i+1} & \ldots & a_{nn} \end{vmatrix}$$

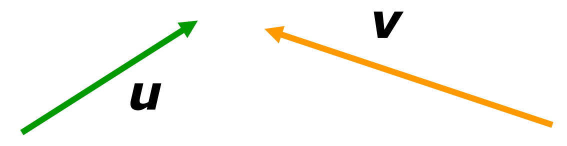
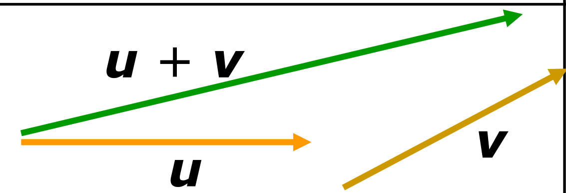
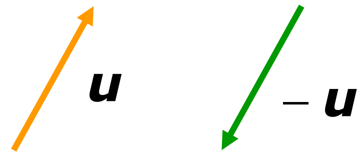
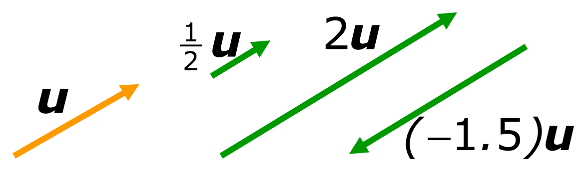
Section 3.1

Euclidean n-Spaces

Objectives

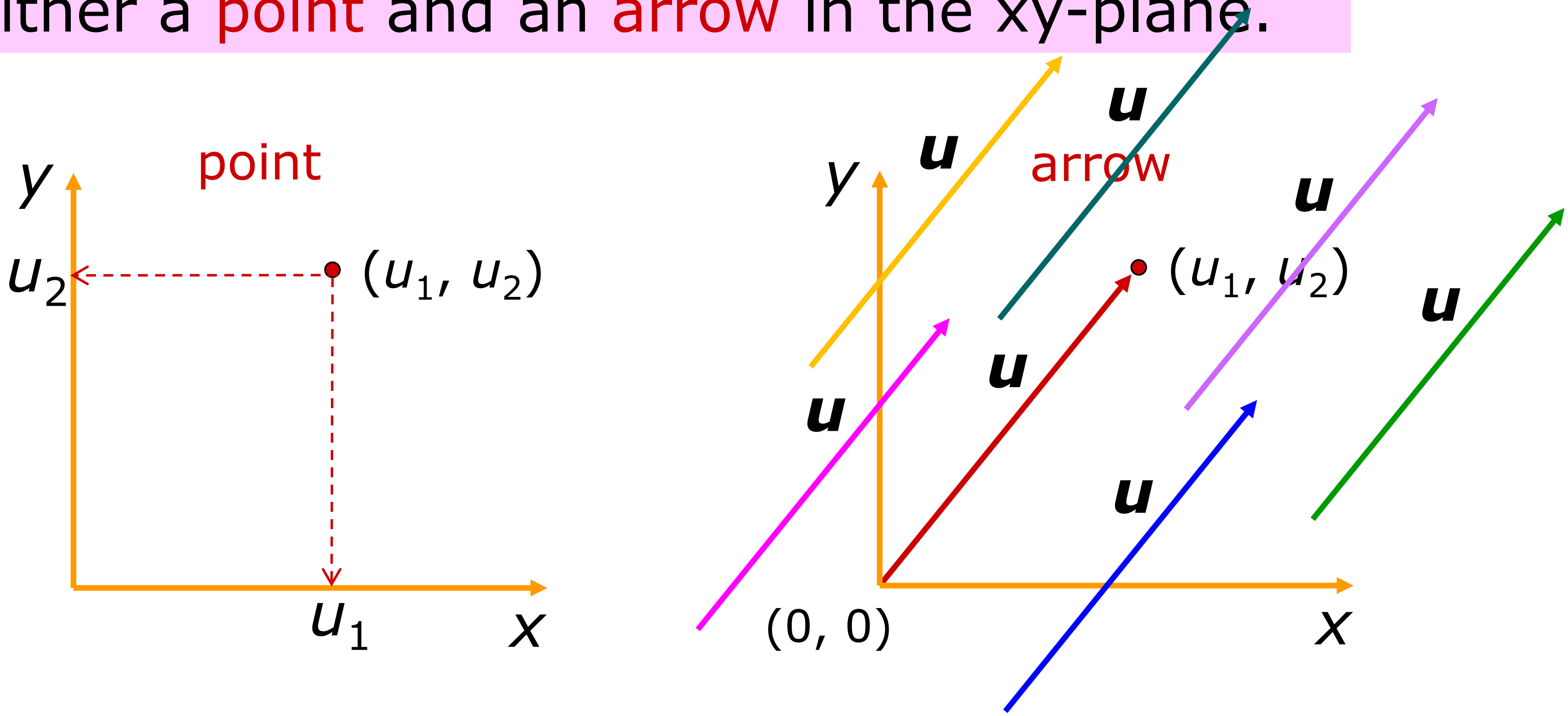
- What is an n-vector?
- What are some operations on n-vectors?
- What is a Euclidean n-space $\mathbf{R}^n$?
- How to express subsets of $\mathbf{R}^n$?

Discussion 3.1.1 - 3.1.2 (Vectors)

Notation	**geometric**	**algebraic** (2-dimension) (3-dimension)
Vector ***u*** Vector ***v***		(u_1, u_2) (v_1, v_2) (u_1, u_2, u_3) (v_1, v_2, v_3)
Addition ***u+v***		(u_1+v_1, u_2+v_2) $(u_1+v_1, u_2+v_2, u_3+v_3)$
Negative ***-u***		$(-u_1, -u_2)$ $(-u_1, -u_2, -u_3)$
Scalar multiple a***u***		(au_1, au_2) (au_1, au_2, au_3)

Discussion 3.1.2.1

Geometrically, (u_1, u_2) can represent either a point and an arrow in the xy-plane.



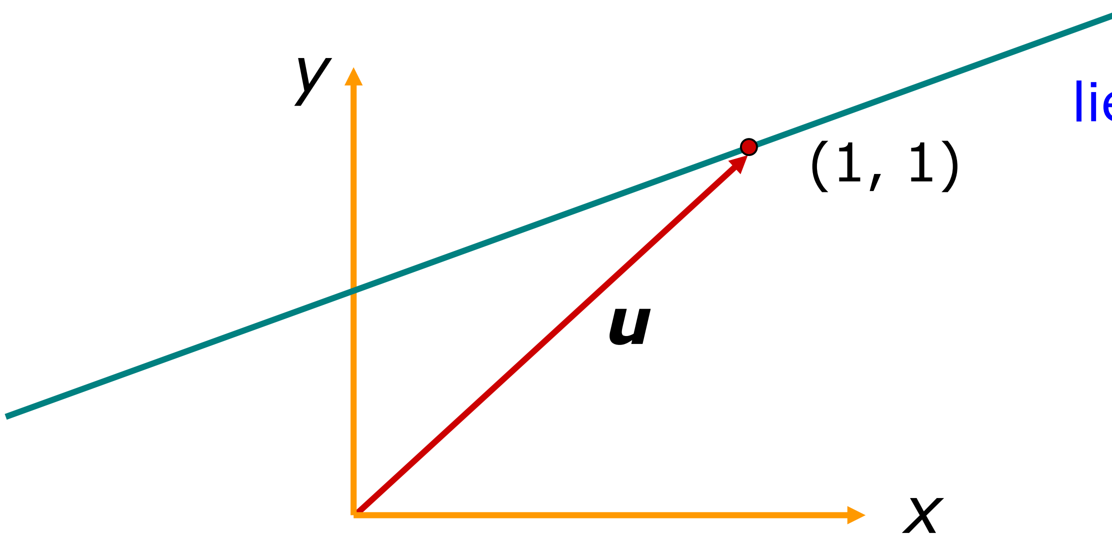
Similarly for (u_1, u_2, u_3) in the xyz-space.

Point or Arrow?

Geometrically, (u_1, u_2) can represent either a point and an arrow in the xy-plane.

Linear equation: $2y - x = 1$

A solution: $x = 1, y = 1$



Does (1, 1) lie on the line?

The point (1, 1) lies on the line,
but the arrow (1, 1) does not lie on the line

What is an n-vector?

(u_1, u_2) 2-vector

(u_1, u_2, u_3) 3-vector

Definition 3.1.3

n-vector $(u_1, u_2, \ldots, u_i, \ldots, u_n) \neq \{u_1, u_2, \ldots, u_i, \ldots, u_n\}$

where $u_1, u_2, \ldots, u_n$ are real numbers

i^{th} component (or i^{th} coordinate) of the n-vector

Always think/view an n-vector
as a SINGLE object
and not n numbers

n-vectors as matrices

Notation 3.1.5

We can identify an n-vector $(u_1, u_2, \ldots, u_n)$ with a $1 \times n$ matrix $(u_1\ u_2\ \ldots\ u_n)$ (row vector)

or an $n \times 1$ matrix $\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$ (column vector).

Which form to use depends on the context.

Definitions and properties of vector operations are similar to matrix operations (see 3.1.3 to 3.1.6)

What is a Euclidean n-space ?

Definition 3.1.7

The set of all n-vectors of real numbers is called the Euclidean n-space and is denoted by $\mathbf{R}^n$.

$\mathbf{R}^4$

(1,2,3,1)

(0,2,1,5)

(-1,2,3,19)

:

$\boldsymbol{u} \in \mathbf{R}^n \longleftrightarrow \boldsymbol{u}$ is an n-vector $\longleftrightarrow \boldsymbol{u} = (u_1, u_2, \ldots, u_n)$

Euclidean 2-space $\mathbf{R}^2$

all the 2-vectors (as points) in xy-plane

Euclidean 3-space $\mathbf{R}^3$

all the 3-vectors (as points) in xyz-space

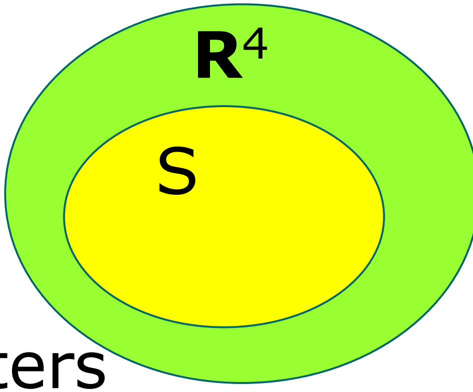
How to express subsets of $\mathbf{R}^n$?

Example 3.1.8.1

Set notation

$$S = \{ \underbrace{(u_1, u_2, u_3, u_4)}_{\text{type of elements is 4-vector}} \mid \underbrace{u_1 = 0 \ \text{ and } \ u_2 = u_4}_{\text{conditions on the components}} \}$$ implicit form

$(0, 0, 0, 0), (0, 1, 5, 1), (0, \pi, -3, \pi) \in S$



$(0, 2, 2, 3), (1, 1, 1, 1) \notin S$

general form $(0, a, b, a)$ $\quad a, b$: parameters

$$S = \{ (0, a, b, a) \mid a, b \in \mathbf{R} \}$$ explicit form

Set notation for subsets of $\mathbf{R}^n$

Implicit form

$$\left\{ \begin{array}{l} \text{general n-tuple} \\ (u_1, u_2, \ldots, u_n) \end{array} \middle| \begin{array}{l} \text{conditions satisfied} \\ \text{by } u_1, u_2, \ldots, u_n \end{array} \right\}$$

$$S = \{ (u_1, u_2, u_3, u_4) \mid u_1 = 0 \text{ and } u_2 = u_4 \}$$

Explicit form   Not always possible to express in explicit form

$$\left\{ \begin{array}{l} \text{n-tuples with} \\ \text{explicit form} \end{array} \middle| \begin{array}{l} \text{range of parameters} \\ \text{appearing on the left} \end{array} \right\}$$

$$S = \{ (0, a, b, a) \mid a, b \in \mathbf{R} \}$$

Don't write $\{a, b \in \mathbf{R} \mid (0, a, b, a) \}$

Solution set as a subset of $\mathbf{R}^n$

Example 3.1.8.2

$$\begin{cases} x + y + z = 0 \\ x - y + 2z = 1 \end{cases}$$

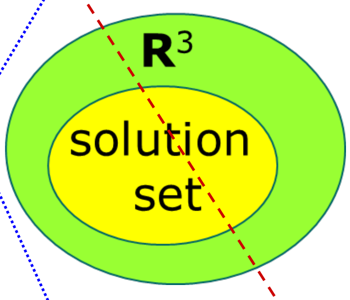
x = 2, y = -1, z = -1

(2, -1, -1)

a 3-vector

general solution: $\begin{cases} x = 0.5 - 1.5t \\ y = -0.5 + 0.5t \\ z = t \end{cases}$

solution set subset of $\mathbf{R}^3$



Explicit form

$$\{\, (0.5 - 1.5t,\ -0.5 + 0.5t,\ t) \mid t \in \mathbf{R} \,\}$$

Implicit form

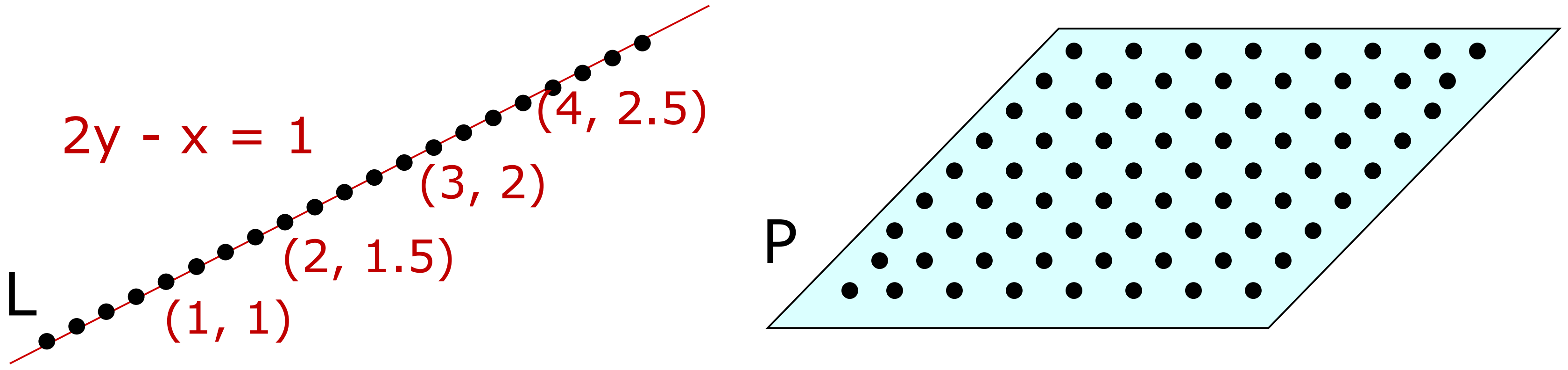
$$\{\, (x, y, z) \mid x + y + z = 0 \ \text{and}\ x - y + 2z = 1 \,\}$$

Line and plane as subsets of $\mathbf{R}^2$ and $\mathbf{R}^3$

Example 3.1.8.3

A line/plane in the xy-plane/ xyz-space can be regarded as a collection of points.

a collection of vectors



L is a subset of $\mathbf{R}^2$

P is a subset of $\mathbf{R}^3$

Set notations of lines and planes

Example 3.1.8.3

Lines in xy-plane

Implicit form: $\{ (x, y) \mid ax + by = c \}$

Explicit form: $\left\{ \left(\dfrac{c - bt}{a}, \ t \right) \mid t \in \mathbf{R} \right\}$

Planes in xyz-space

Implicit form: $\{ (x, y, z) \mid ax + by + cz = d \}$

Explicit form: $\left\{ \left(\dfrac{d - bs - ct}{a}, \ s, \ t \right) \mid s, t \in \mathbf{R} \right\}$

Lines in xyz-space

Implicit form: { (x, y, z) | eqn of the line }**?**

Explicit form: { (general solution) | 1 parameter }**?**

Example 3.1.8.2 (revisited)

$$\begin{cases} x + y + z = 0 \\ x - y + 2z = 1 \end{cases}$$ two planes

solution set (explicit form)

$$\{ (0.5 - 1.5t, -0.5 + 0.5t, t) \mid t \in \mathbf{R} \}$$

This represents a line in the xyz-space

$$(0.5, -0.5, 0) + (-1.5t, 0.5t, t)$$

$$(0.5, -0.5, 0) + t(-1.5, 0.5, 1)$$

a point on the line

the direction of the line

Line as a subset of $\mathbf{R}^3$

Example 3.1.8.3(c)

Set notation (explicit)

$\{\, (a_0 + at,\ b_0 + bt,\ c_0 + ct) \mid t \text{ in } \mathbf{R} \,\}$ t: parameter

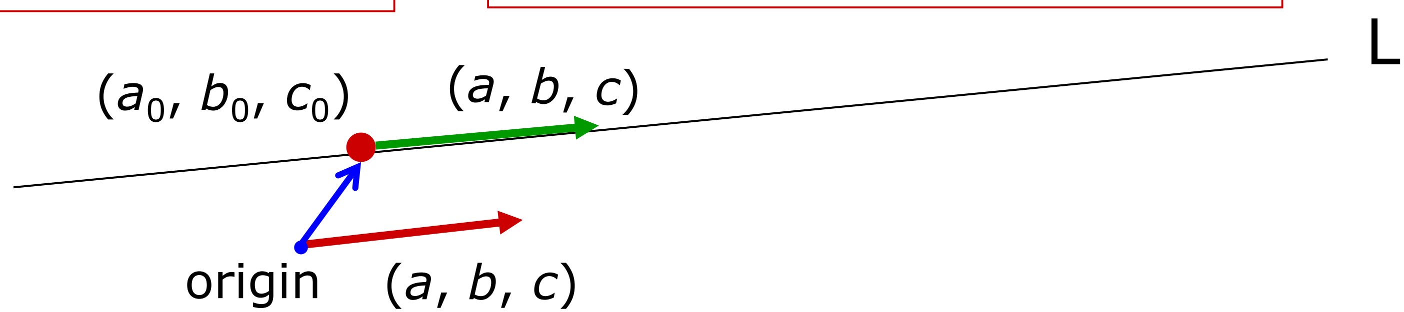
a_0, b_0, c_0, a, b, c are fixed real numbers

a, b, c not all zero

$\{\, (a_0, b_0, c_0) + t(a, b, c) \mid t \text{ in } \mathbf{R} \,\}$

a point on the line

the direction of the line



Line as a subset of $\mathbf{R}^3$

Example 3.1.8.3(c)

Set notation (Implicit)

$\{ (x, y, z) \mid \text{eqn of the line} \}$**?**

A line in $\mathbf{R}^3$ cannot be represented by a single linear equation.

But it can be regarded as the intersection of two planes P_1 and P_2.

Suppose the equations of the two planes are given by

P_1: $a_1x + b_1y + c_1z = d_1$ $\quad$ P_2: $a_2x + b_2y + c_2z = d_2$

Set notation (implicit)

$\{ (x, y, z) \mid a_1x + b_1y + c_1z = d_1 \text{ and } a_2x + b_2y + c_2z = d_2 \}$

refer to 3.1.8.2

Notation 3.1.9 & Example 3.1.10

For a finite set S, we denote the number of elements of S by $|S|$

$S_1 = \{ 1, 2, 3, 4 \}$ $\quad |S_1| = 4$

$S_2 = \{ (1, 2, 3, 4) \}$ $\quad |S_2| = 1$

$S_3 = \{ (1,2,3), (2,3,4) \}$ $\quad |S_3| = 2$