NATIONAL UNIVERSITY OF SINGAPORE

SCHOOL OF COMPUTING

MID-TERM TEST
AY2021/22 Semester 1

CS1231S — DISCRETE STRUCTURES

7 October 2021 Time Allowed: 1 hour 30 minutes

INSTRUCTIONS

1. This assessment paper contains **SIXTEEN (16)** questions in **THREE (3)** parts and comprises **SIX (6)** printed pages.
2. This is an **OPEN BOOK** assessment.
3. Printed/written materials are allowed. Apart from calculators, electronic devices are not allowed.
4. Answer **ALL** questions and write your answers only on the **ANSWER SHEETS**. You may write in pen or pencil.
5. The maximum mark of this assessment is 50.
6. Do not start writing or flip over this page until you are told to do so.

——— END OF INSTRUCTIONS ———

Part A: Multiple Choice Questions (Total: 12 marks)

Each multiple choice question (MCQ) is worth <u>two marks</u> and has exactly **one** correct answer.

1. Given this statement: $C(A) \vee C(D) \rightarrow \forall x \in S\,(C(x))$

   "If Aiken or Dueet can do it, then all CS1231S students can do it."

   Suppose the above is true, which of the following is always true?

   A. "Aiken or Dueet are CS1231S students." $A \vee D \in S$? x.
   B. "If Aiken can do it, then Dueet can do it." $C(A) \rightarrow C(D)$? x
   C. "If Aiken can do it, then all CS1231S students can do it." $C(A) \rightarrow \forall x \in S\,(C(x))$ ✓
   D. "If all CS1231S students can do it, then Aiken or Dueet can do it."
   E. None of (A), (B), (C), (D) is correct. $\forall x \in S\,(C(x)) \rightarrow C(A) \vee C(D)$? x

2. Consider the predicate $P(x, y, z) \equiv$ "$xyz = 1$" for $x, y, z \in \mathbb{Q}^+$.

   Which of the following statements is/are true?

   no 0.

   ~~(I)~~ $\forall x \in \mathbb{Q}^+\ \forall y \in \mathbb{Q}^+\ \forall z \in \mathbb{Q}^+\ P(x, y, z)$. $1 \cdot 1 \cdot 2 = 2 \neq 1$

   ✓(II) $\forall x \in \mathbb{Q}^+\ \forall y \in \mathbb{Q}^+\ \exists z \in \mathbb{Q}^+\ P(x, y, z)$. $1 \cdot 2 \cdot \frac{1}{2} = 1$

   ✓(III) $\exists x \in \mathbb{Q}^+\ \forall y \in \mathbb{Q}^+\ \forall z \in \mathbb{Q}^+\ P(x, y, z)$.

   doesn't matter here.

   A. (I) only.
   B. (II) only.
   C. (III) only.
   D. (II) and (III) only.
   E. None of (A), (B), (C), (D) is correct.

   D.

3. Which of the following statements is/are true?

   ×(I) $\mathcal{P}(\{\emptyset\}) = \mathcal{P}(\{\{\emptyset\}\})$. $\{\emptyset, \{\emptyset\}\} \neq \{\emptyset, \{\{\emptyset\}\}\}$.

   ✓(II) $|\mathcal{P}(\{\emptyset\})| = |\mathcal{P}(\{\{\emptyset\}\})|$. $2^1 = 2^1$ ✓

   A. Both (I) and (II) are true.
   B. (I) is true but (II) is not.
   C. (II) is true but (I) is not.
   D. Both (I) and (II) are not true.

   C.

4. Consider the congruence-mod-5 relation as an equivalence relation on $\mathbb{Z}$. Of which of the following sets is 1231 an element?

   A. $[0]$.
   B. $[1]$. $1231 = 5x + 1$
   C. $[2]$.
   D. $[3]$.
   E. $[4]$.

   B.

5. Define $f\colon \mathbb{Z} \to \mathbb{Z}_{\geqslant 0}$ and $g\colon \mathbb{Q} \to \mathbb{Q}_{\geqslant 0}$ by setting, for all $a \in \mathbb{Z}$ and all $x \in \mathbb{Q}$,

$$✓\ f(a) = \{a^2n^2 : n \in \mathbb{Z}\} \quad \text{and} \quad ✗\ g(x) = x^2\sqrt{2}.$$

Which of the following is true? $(-1)^2(-1)^2 \to \mathbb{Z}_{\geq 0}$ $\quad (\frac{1}{2})^2\sqrt{2} \not\in \mathbb{Q}_{\geq 0}$

   A. f and g are both well defined.
   B. f is well defined but g is not.
   C. g is well defined but f is not. B.
   D. f and g are both not well defined.

6. Consider the equivalence relation $\sim$ on $\mathbb{Z}$ satisfying, for all $x, y \in \mathbb{Z}$,

$$x \sim y \quad \Leftrightarrow \quad x = y \text{ or } x = -y.$$

Define two functions $f, g\colon \mathbb{Z}/\!\sim\ \to \mathbb{Z}/\!\sim$ by setting, for all $x \in \mathbb{Z}$,

$$✗\ f([x]) = [3x + 1] \quad \text{and} \quad g([x]) = [x^4].$$

-1 $\quad 3(-1)+1 = -3+1 = -2.$ $\quad -1 \quad (-1)^4 = 1.$

Which of the following is true? $\quad -100 \quad (-100)^4 = 100^4$

   A. f and g are both well defined.
   B. f is well defined but g is not.
   C. g is well defined but f is not.
   D. f and g are both not well defined. D.

Part B: Multiple Response Questions [Total: 21 marks]

Each multiple response question (MRQ) is worth <u>three marks</u> and may have one answer or multiple answers. Write out **all** correct answers. For example, if you think that A, B, C are the correct answers, write A, B, C. Only if you get all the answers correct will you be awarded three marks. **No partial credit will be given for partially correct answers.**

7. The floor and the ceiling of a real number x, denoted as $\lfloor x \rfloor$ and $\lceil x \rceil$ respectively, are defined as follows:

   $\lfloor x \rfloor$ = the largest integer n such that $n \leq x$. $\quad \lfloor 1.5 \rfloor = 1 \quad \lfloor 1 \rfloor = 1$

   $\lceil x \rceil$ = the smallest integer n such that $n \geq x$. $\quad \lceil 1.5 \rceil = 2 \quad \lceil 1 \rceil = 1$

   Which of the following statements is/are true?

   A. $\forall x \in \mathbb{R}, \lceil \lfloor x \rfloor \rceil = \lfloor x \rfloor$. $\quad \lceil \lfloor 1.5 \rfloor \rceil = 1.$
   ~~B.~~ $\forall x \in \mathbb{R}, \lceil x \rceil = \lfloor x \rfloor + 1$. $\quad \lceil 1.5 \rceil = 2 = \lfloor 1.5 \rfloor + 1 \mid \lceil 1 \rceil = 1 \neq \lfloor 1 \rfloor + 1$
   ~~C.~~ $\forall x \in \mathbb{R}, \lfloor 2x \rfloor = 2\lfloor x \rfloor$. $\quad \lfloor 2 \cdot 1.5 \rfloor = \lfloor 3 \rfloor \neq 2\lfloor 1.5 \rfloor = 2(1) = 2$
   D. $\forall x \in \mathbb{R}, x - 1 < \lfloor x \rfloor \leq \lceil x \rceil < x + 1$. $\quad 1.5 - 1 = 0.5 < \lfloor 1.5 \rfloor = 1 \leq \lceil 1.5 \rceil = 2 < 1.5 + 1 = 2.5.$
   ~~E.~~ $\forall x, y \in \mathbb{R}, \lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$.

   $\lceil 1.5 + 1.5 \rceil = \lceil 1.5 \rceil + \lceil 1.5 \rceil$
   $\lceil 3 \rceil = 2 + 2.$

   $1 - 1 = 0 < \lfloor 1 \rfloor = 1 \leq \lceil 1 \rceil = 1 < 1 + 1 = 2$
   $0 - 1 = -1 < \lfloor 0 \rfloor = 0 \leq \lceil 0 \rceil = 0 < 0 + 1 = 1$

8. Which of the following statements is/are equivalent to $(p \wedge q) \rightarrow q$?

   A. $p \rightarrow p$
   B. $(p \wedge q) \rightarrow p$
   C. $(p \vee q) \rightarrow q$
   D. $p \rightarrow (p \vee q)$
   E. $p \rightarrow (p \wedge q)$

$p \rightarrow (p \wedge q)$
$\neg p \vee (p \wedge q).$
$(\neg p \vee p) \wedge (\neg p \vee q)$
$T \wedge (\neg p \vee q)$
$\neg p \wedge q.$

$\neg p \vee p \equiv T$
$\neg(p \wedge q) \vee p \quad \neg p \vee \neg q \vee p.$
$\neg(p \vee q) \vee p \quad \neg p \wedge \neg q \vee p.$
$T \wedge \neg q.$
$\neg q.$
$\neg p \vee (p \vee q).$
$\neg p \vee p \vee q.$
$T \vee q$
T

$p \wedge q \rightarrow q.$
$\neg(p \wedge q) \vee q.$
$\neg p \vee \neg q \vee q.$
$\neg p \vee T$
$T.$

9. To prove the statement $\forall x \in D\ (P(x) \rightarrow Q(x))$, it is enough to prove that ______

   A. $\exists x \in D\ (P(x) \wedge \sim Q(x)) \rightarrow \exists y \in D\ (P(y) \wedge \sim P(y))$
   B. $\forall x \in D\ (\sim Q(x) \rightarrow \sim P(x))$ Contrapositive
   C. $\forall x \in D\ ((P(x) \wedge \sim Q(x)) \rightarrow (P(x) \wedge \sim P(x)))$
   D. $\exists\, x \in D\ (\sim Q(x) \rightarrow \sim P(x))$

$\forall x \in D(\neg P(x) \vee Q(x))$

F

$\neg(P \wedge \neg q) \vee F \rightarrow \neg P \vee q.$

10. Let $A = \{x \in \mathbb{Q} : 0 \leqslant x \leqslant 1\}$ and $B = \{x \in \mathbb{Q} : 1 \leqslant x \leqslant 2\}$ and $C = \{x \in \mathbb{Q} : 2 \leqslant x \leqslant 3\}$. Which of the following is a partition (or are partitions) of $\mathbb{Q}$?

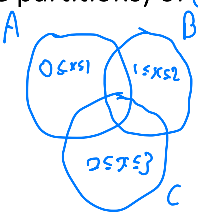
   A. $\{B, \mathbb{Q} \setminus B\}$.
   B. $\{A \cap C, \mathbb{Q} \setminus (A \cap C)\}$.
   C. $\{A, \mathbb{Q} \setminus A, B, \mathbb{Q} \setminus B\}$.
   D. $\{A, C, (\mathbb{Q} \setminus A) \cap (\mathbb{Q} \setminus C)\}$.
   E. $\{A, B, C\}$.

mutually disjoint
non-empty subset
of $\mathbb{Q}$

$A \cap C$ is empty.

$\mathbb{Q} \setminus A \cap \mathbb{Q} \setminus B$

doesn't add up to $\mathbb{Q}$.



11. Let $A = \{3,4,5,6,7,8\}$. Which of the following is/are equal to $A/\sim$ for some equivalence relation $\sim$ on A?

   A. $\{\{1,2,3\}, \{4,5,6\}\}$.
   B. $\{\{3,4,5\}, \{6,7,8\}\}$.
   C. $\{\{\{3,4\}, \{5\}\}, \{6,7,8\}\}$.
   D. $\{\{3,4,5\}, \{6\}, \{7,8\}\}$.
   E. $\{3,4,5,6,7,8\}$.

not a set of sets.

12. Let $A = \{3,4,5,6,7,8\}$. Partially order A by the divisibility relation, i.e., consider the partial order $\preccurlyeq$ on A defined by setting, for all $a, b \in A$,

$$a \preccurlyeq b \quad \Leftrightarrow \quad \exists k \in \mathbb{Z}\ (b = ka).$$

Which of the following is/are equal to the set of all minimal elements in this partially ordered set?

not smallest

6 8 3 4 5 7?

odd

A. $\{x \in A : \exists k \in \mathbb{Z}\ (x = 2k + 1)\}$.

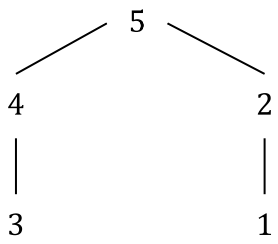
B. $\{3\}$.

C. $A \setminus \{x + x : x \in A\}$. 5 is not minimal

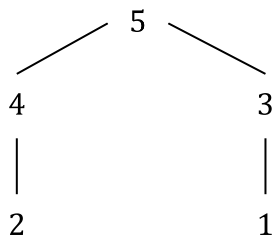
D. $\{x \in \mathbb{Z} : \exists k \in \mathbb{Z}\ (420 = kx)\}$.

E. $\{x \in A : x + x \in A\}$.

13. Which of the following is a Hasse diagram (or are Hasse diagrams) for a partial order of which the usual non-strict order $\leqslant$ on $\{1,2,3,4,5\}$ is a linearization?



A.



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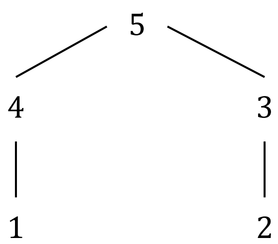
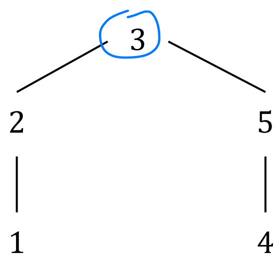


C.



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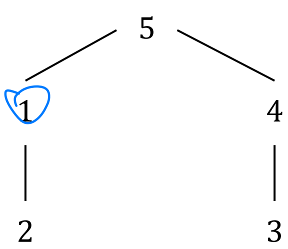

E.

Part C: There are 3 questions in this part [Total: 17 marks]

~~14.~~ Theorem 2.1.1 is given as follows:

1	Commutative laws	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2	Associative laws	$p \wedge q \wedge r$ $\equiv (p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$p \vee q \vee r$ $\equiv (p \vee q) \vee r \equiv p \vee (q \vee r)$
3	Distributive laws	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4	Identity laws	$p \wedge$ **true** $\equiv p$	$p \vee$ **false** $\equiv p$
5	Negation laws	$p \vee {\sim}p \equiv$ **true**	$p \wedge {\sim}p \equiv$ **false**
6	Double negative law	${\sim}({\sim}p) \equiv p$	
7	Idempotent laws	$p \wedge p \equiv p$	$p \vee p \equiv p$
8	Universal bound laws	$p \vee$ **true** $\equiv$ **true**	$p \wedge$ **false** $\equiv$ **false**
9	De Morgan's laws	${\sim}(p \wedge q) \equiv {\sim}p \vee {\sim}q$	${\sim}(p \vee q) \equiv {\sim}p \wedge {\sim}q$
10	Absorption laws	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11	Negation of **true** and **false**	${\sim}$**true** $\equiv$ **false**	${\sim}$**false** $\equiv$ **true**

Simplify the following expression using the laws above, justifying every step. You may combine consecutive steps using the same law in one step. [3 marks]

$$(p \wedge q) \vee (q \wedge r) \vee ({\sim}p \wedge r)$$

15. Prove that $(n^3 - n^2)$ is even for any positive integer n. [4 marks]

(You may quote the claim without proof that an integer is either odd or even but not both.)

$\{(1,1), (1,2), (1,3), \ldots\}$.

16. Let $A = \{1,2,3,4,5,6\}^2$. Define a relation R on A by setting, for all $(a_1, a_2), (b_1, b_2) \in A$,

$$(a_1, a_2)\ R\ (b_1, b_2) \quad \Leftrightarrow \quad \left|\{(i,j) \in \{1,2\}^2 : a_i \leqslant b_j\}\right| \geqslant 2$$

at least 2 that satisfies the eqn. $\{1,2\} \times \{1,2\}$ $\{(1,1), (1,2), (2,1), (2,2)\}$ 4.

all b ≥ all a

(Hint: the number 2 on the right-hand side of the inequality above is equal to $|\{1,2\}^2|/2$.)

(a) Is R reflexive? xRx? ✓ $(1,1)\ R\ (1,1)$ ✓ [3 marks]

~~(b)~~ Is R symmetric? $xRy \rightarrow yRx$?, $(1,2)\ R\ (6,6) \rightarrow (6,6)\ R\ (1,2)$ [2 marks]

~~(c)~~ Is R antisymmetric? $xRy \wedge yRx \rightarrow x = y$? [2 marks]

~~(d)~~ Is R transitive? $xRy \wedge yRz \rightarrow xRz$? [3 marks]

For each part, if your answer is yes, then give a proof; if your answer is no, then give a counterexample.

$(1,1)\ R\ (1,1) \leftrightarrow |\{(i,j) \in \{1,2\}^2 : a_1, a_2 \leq b_1, b_2\}| \geq 2$.

=== END OF PAPER ===

16.

c) $(1,2) R (2,1) \wedge (2,1) R (1,2) \rightarrow (1,2) = (2,1)$?, X.

d) $(6,6) R (6,1) \wedge (6,1) R (2,2) \rightarrow (6,6) R (2,2)$

15. $(n^3 - n^2)$ is even for any $n \in \mathbb{Z}^+$

1. let $a \in \mathbb{Z}^+$.
2. Suppose a is odd,
   2.1 Then $a = 2k+1$, $k \in \mathbb{Z}^+$. by definition of odd and 1.
   2.2. Then $a^3 - a^2 = (2k+1)^3 - (2k+1)^2$.
   $= (2k+1)^2(2k+1-1)$
   $= 2k(2k+1)^2$. by basic algebra.
   2.3. Then $2k(2k+1)^2 = 2b$, $b = k(2k+1)^2 \in \mathbb{Z}^+$ by closure of $\mathbb{Z}$ under $+$ and $\times$.
   2.4. $\therefore$ By definition of even, $(a^3 - a^2)$ is even.
3. Suppose a is even;
   3.1 then $a = 2j$, $j \in \mathbb{Z}^+$. by definition of even and 1.
   3.2 Then $a^3 - a^2 = (2j)^3 - (2j)^2$
   $= (2j)^2(2j-1)$
   $= 4j^2(2j-1)$
   $= 2 \cdot (2j^2(2j-1))$ by basic algebra.
   3.3. Then $2(2j^2(2j-1)) = 2c$, $c = 2j^2(2j-1) \in \mathbb{Z}^+$ by closure of $\mathbb{Z}$ under $+$ and $\times$.
   3.4. $\therefore$ By definition of even, $a^3 - a^2$ is even.
4. In either case, $a^3 - a^2$ is even. (an integer is either odd or even but not both)
   Hence $(n^3 - n^2)$ is even for any $n \in \mathbb{Z}^+$.

14. Consensus Theorem

$(p \wedge q) \vee (q \wedge r) \vee (\neg p \wedge r)$

$\equiv (p \wedge q) \vee (\neg p \wedge r) \vee (q \wedge r)$ by commutative

$\equiv (p \wedge q) \vee (\neg p \wedge r) \vee (T \wedge (q \wedge r))$ by identity law.

$\equiv (p \wedge q) \vee (\neg p \wedge r) \vee ((p \vee \neg p) \wedge (q \wedge r))$ by negation

$\equiv (p \wedge q) \vee (\neg p \wedge r) \vee ((q \wedge r) \wedge (p \vee \neg p))$ by commutative

$\equiv (p \wedge q) \vee (\neg p \wedge r) \vee [(q \wedge r) \wedge p] \vee [(q \wedge r) \wedge \neg p]$ by distributive.

$\equiv [(p \wedge q) \vee ((p \wedge q) \wedge r)] \vee [(\neg p \wedge r) \vee ((\neg p \wedge q) \wedge r)]$ by commutative & associative.

$\equiv (p \wedge q) \vee (\neg p \wedge r)$ by absorption law.