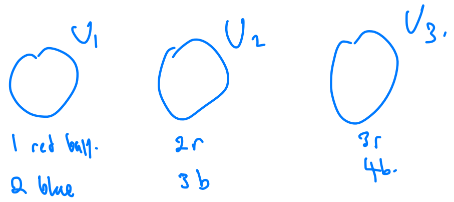
(c) There are three urns U_1, U_2 and U_3. Urn U_k $(1 \le k \le 3)$ contains k red balls and $k+1$ blue balls.

(i) If you draw 2 balls at random from U_2 without replacement, what is the probability of drawing **at least one red ball**? Write your answer as a single fraction. [2 marks]

(ii) Four words "I", "CAN", "DO" and "IT" are separately written on 4 slips of paper and concealed, each having an equal chance of being selected.

You select a slip of paper at random and reveal the word written on it. The length of the word, k, directs you to urn U_k to pick a ball. If the ball picked is **blue**, what is the probability that it comes from U_2? Write your answer as a single fraction or a percentage rounded to 4 significant figures. [3 marks]



i) $P(\text{at least one red ball}) = P(\text{choose } 2) - P(\text{no red ball})$

$= \left(\frac{5}{5}\right)\left(\frac{4}{4}\right) - \left(\frac{3}{5}\right)\left(\frac{2}{4}\right)$

$= \frac{7}{10}$.

ii)

Word	length		probability
I	1	$\rightarrow U_1$	$\frac{1}{4}$
do	2	$\rightarrow U_2$	$\frac{1}{4}$ } $\frac{2}{4}$.
it	2	$\rightarrow U_2$	$\frac{1}{4}$
Can	3	$\rightarrow U_3$.	$\frac{1}{4}$

$\therefore P(U_2 \mid \text{blue}) = \dfrac{P(U_2 \cap \text{blue})}{P(\text{blue})}$ $P(\text{blue}) = P(U_1 \cap \text{Blue}) + P(U_2 \cap \text{Blue}) + P(U_3 \cap \text{Blue})$.

$= \dfrac{\frac{2}{4} \times \frac{3}{5}}{\frac{1}{4} \times \frac{2}{3} + \frac{2}{4} \times \frac{3}{5} + \frac{1}{4} \times \frac{4}{7}}$

$= \frac{63}{128}$.

(e) You are to pick 14 numbers from 1 through 20. Is it true that no matter how you pick the 14 numbers, there is always a pair of numbers such that one is three times the other? Explain your answer. [4 marks]

1 2 3 4 5 6 7 8 9 10
11 12 13 14 15 16 17 18 19 20

No.

Counter-example : {2, 3, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 20}.
There exist a set that contains no pair of numbers such that one is 3 times the other.

PHP:

14 sets: {1, 3, 9}, {2, 6, 18}, {4, 12}, {5, 15}, {7}, {8}, {10}, {11}, {13}, {14}, {16}, {17}, {19}, {20}

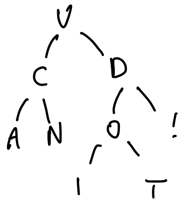
$\therefore$ Can pick one number from each of the 14 sets and do not get any pair of numbers such that one is 3 times the other.

(a) The **pre-order traversal** and **post-order traversal** of a full binary tree with 9 vertices are given below:

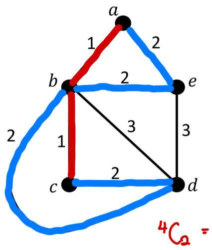
Pre-order: **U C A N D O I T !** 9 elems.

Post-order: **A N C I T O ! D U**

Draw the full binary tree (clearly!). [3 marks]



(b) In the following weighted graph, how many minimum spanning trees are there? You do not need to provide working or diagrams. [3 marks]



abced 2x2

abced = 4

doesn't matter where start.

$^4C_2 = 6$

but 2 of them result in a circuit.

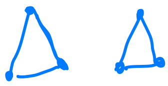
$\therefore 6-2=4$ //

(d) A *regular graph* is a simple undirected graph where every vertex has the same degree.
A *2-regular graph* is a regular graph where every vertex has degree 2.

Prove or disprove the following statement:

All 2-regular graphs are connected graphs. [3 marks]

No.



$\therefore$ not connected but have 2 degree

(e) Assume that all graphs in this question are simple undirected graphs.

Given two simple undirected graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ where $V_1 \cap V_2 = \emptyset$ and $E_1 \cap E_2 = \emptyset$, the *graph union*, *graph join* and *graph product* are defined as follows:

> Graph union:
> The union $G_\cup = G_1 \cup G_2$ has vertex set $V_\cup = V_1 \cup V_2$ and edge set $E_\cup = E_1 \cup E_2$.

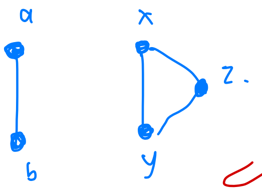
> Graph join:
> The join $G_+ = G_1 + G_2$ has vertex set $V_+ = V_1 \cup V_2$ and edge set $E_+ = E_1 \cup E_2 \cup$ {all edges connecting every vertex in V_1 with every vertex in V_2}.

> Graph product:
> The product $G_\times = G_1 \times G_2$ has vertex set $V_\times = V_1 \times V_2$ (Cartesian product of V_1 and V_2) and two vertices $(\alpha, \beta), (\gamma, \delta) \in V_\times$ are connected by an edge if and only if the vertices $\alpha, \beta, \gamma, \delta$ satisfy the following (with $\sim$ denoting "is adjacent to"):
> $$(\alpha = \gamma \wedge \beta \sim \delta) \vee (\beta = \delta \wedge \alpha \sim \gamma)$$

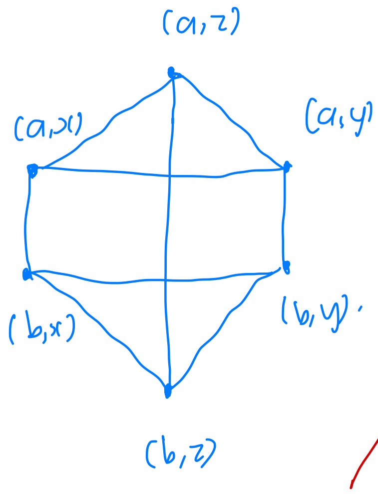
{ (a,x) (a,z), (a,y)
(b,x) (b,z), (b,y) }

Given the following graphs G_1 and G_2, draw their union graph (1 mark), join graph (2 marks) and product graph (4 marks). You should label the vertices clearly on your graphs. [7 marks]

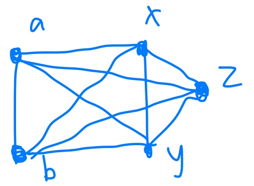
G_u :



$G_\times$:



G_+ :



(b) A simple graph is a graph with no loops and parallel edges. Given a simple undirected graph $G = (V, E)$, the **complement graph** of G is a simple undirected graph $\bar{G} = (V, \bar{E})$

where $\bar{E} = \{\{u, v\}: u, v \in V \wedge u \neq v \wedge \{u, v\} \notin E\}$.

A graph G is **self-complementary** if and only if G and $\bar{G}$ are isomorphic.

bijection $\therefore$ edges must half. $\Rightarrow$ same no. of edges.

Suppose a self-complementary simple undirected graph G has n vertices. How many edges does G have? Working is not required for this question. [2 marks]

By the definition of complementary graph, the union graph of G and $\bar{G}$ is a complete graph K_n, where $n = |V|$, which has $\binom{n}{2} = \frac{n(n-1)}{2}$ edges.

For self-complementary graph G, G and $\bar{G}$ must have half this number of edges, therefore it has $\frac{n(n-1)}{4}$ edges.

(c) Given the definition of a self-complementary graph in part (b) above, is there a self-complementary simple undirected graph with $4k + 2$ vertices, where $k \in \mathbb{Z}^+$? Explain your answer. [4 marks]

can verify by drawing simple graphs

From previous question, a self-complementary graph with n vertices has $\frac{n(n-1)}{4}$ edges

If $n = 4k+2$, then there are $\frac{(4k+2)(4k+1)}{4} = \frac{(2k+1)(4k+1)}{2}$ edges

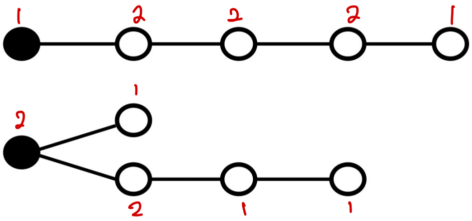
Since both $(2k+1)$ and $(4k+1)$ are odd, their product is odd (?).

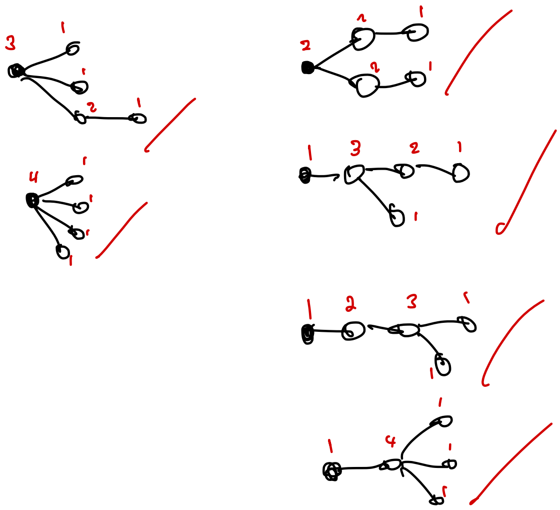
As odd number is not divisible by 2, the number of edges would not be an integer

Therefore, there are no self-complementary simple undirected graph with $4k+2$ vertices.

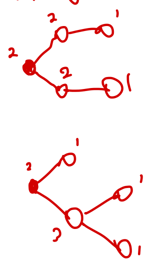
(e) For two rooted trees to be isomorphic under a permutation π, the image $\pi(r)$ of a root r must also be a root.

Draw (clearly!) a list of rooted trees on 5 vertices such that every rooted tree on 5 vertices is isomorphic to exactly one in your list. Draw the roots as solid black circles and the rest of the vertices as hollow circles. Two such trees are already shown below and you do not need to re~~-draw them~~ in your answers. [5 marks]





Missing:



Be systematic!

11. In how many ways can the letters of the word "CREATE" be arranged such that the vowels always stay together? Vowels are A, E, I, O and U.

A. 72

B. 120

C. 144

D. 240

E. 360

CREATE

Vowels non-vowels

EAE CRT

$\frac{3!}{2!}$

$\therefore 4! \times \frac{3!}{2!} = 72.$ $\therefore$ A

no replacement

12. Suppose a random sample of 2 lightbulbs is selected from a group of 6 bulbs in which 2 are defective, what is the expected value of the number of defective bulbs in the sample?

A. 1/18

B. 1/3

C. 8/9

D. 2/3

E. 14/15

D D

$\therefore$ no. of defective

$P(X=0) = \frac{4}{6} \times \frac{3}{5} = \frac{2}{5}.$

$P(X=1) = \left(\frac{4}{6} \times \frac{2}{5}\right) + \left(\frac{2}{6} \times \frac{4}{5}\right) = \frac{8}{15}.$

$P(X=2) = \frac{2}{6} \times \frac{1}{5} = \frac{1}{15}.$

$\therefore E(X) = \frac{2}{3}.$ $\therefore$ D

13. What is the coefficient of x^7 in the expansion of $(x + 3)^9$?

A. 7

B. 36

C. 108

D. 320

E. None of the above.

$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b^n + \dots + \binom{n}{r}a^{n-r}b^r + \dots + b^n$

$\binom{9}{2}x^{9-2}3^2$

$= 36 \times 9x^7$

$= 324x^7.$

$\therefore$ E.

(a) How many non-negative integer solutions for a, b and c does the following equation have?

Multiset.

$$a + b + c = 100.$$ [2 marks]

${}^{r+n-1}C_r = {}^{100+3-1}C_{100} = {}^{102}C_{100}$

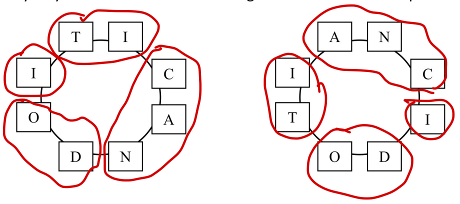
$= 5151.$

(b) Four children are to be selected from 6 boys and 4 girls to participate in a training camp. There must be at least one boy among the selected children. In how many ways can the children be selected? [3 marks]

$N(4 \text{ children}) - N(4 \text{ girls}) = {}^{10}C_4 - {}^4C_4 = 210 - 1$

$= 209.$

(c) You want to lay the letter tiles of these four words "I", "CAN", "DO", "IT" in a circular arrangement. The letters in the groups "CAN", "DO" and "IT" must be kept together in each group, but the letters within each group may be arranged in any order within that group. Also, no two similar letters should be placed next to each other. In how many ways can this be done? The diagram below shows two possible arrangements. [4 marks]



Do not permutate the circle.

Case 1: "I" and "IT" are opposite of each other.

$2 \times 3! \times 2! \times 2! = 48.$

swap I and IT, perm can, perm do, perm IT.

Case 2: "I" and "IT" are next to each other "ITI"

$2 \times 3! \times 2! = 24.$

swap can and do, perm can, perm do

$+ \Rightarrow 72$

(d) Each of the 9 cells in the 3×3 grid below is to be filled with the number -1, 0 or 1. Prove that among the 3 row-sums, 3 column-sums and 2 diagonal-sums, there are two sums that are equal in value. [3 marks]

$\therefore 3+3+2 = 8$ pigeons.

possible sums:

$1+1+1 = 3$

$1+1+0 = 2$

$1+0+0 = 1$

$0+0+0 = 0$

$-1+0+0 = -1$

$-1+-1+0 = -2$

$-1+-1+-1 = -3$.

$\therefore$ 9 possible pigeon holes.

$\therefore$ By PHP, 2 sum with same value.

(e) It is known that 0.3% of the population are sufferers of a certain disease. The probability of a sufferer tested positive by a diagnostic test is 98%, while the probability of a non-sufferer tested negative by the test is 95%.

Suppose the test is administered to a person randomly chosen from the population, answer the following parts, writing your answer correct to 3 significant figures.

(i) What is the probability that the test result of the person will be positive?

(ii) If the test result is positive, what is the probability that the person is a sufferer?

(iii) What is the probability that the person will be misclassified? A person is misclassified if he has the disease but is tested negative, or he does not have the disease but is tested positive.

[8 marks]

i) Let T = "tested positive", S = "sufferer".

$P(S) = 0.003$;

$P(T|S) = 0.98$;

$P(T|\bar{S}) = 0.05$;

note!

$\therefore P(T) = P(T|S)\cdot P(S) + P(T|\bar{S})\cdot P(\bar{S})$

$= (0.98 \times 0.003) + (0.05 \times 0.997)$

$\approx 0.05279 = 5.28\%$

ii) $P(S|T) = \dfrac{P(T|S)\cdot P(S)}{P(T)}$

$= \dfrac{0.98 \times 0.003}{0.05279} = 0.05569. \approx 5.57\%$

iii) $P(\text{misclassified}) = P(T \cap \bar{S}) + P(\bar{T} \cap S)$

$= P(T|\bar{S})\cdot P(\bar{S}) + P(\bar{T}|S)\cdot P(S)$

$= (0.05 \times 0.997) + (0.02 \times 0.003)$

$= 0.04991 = 4.99\%$

Given a fair coin, what is the expected number of coin tosses to get 5 consecutive heads?

Approach 1: 1. Let E be the expected number of tosses.

2. If we get a T (with probability $\frac{1}{2}$), then the expected value is $E+1$.

3. If we get a HT (with probability $\frac{1}{4}$), then the expected value is $E+2$.

4. If HHT (probability $\frac{1}{8}$), then the expected value is $E+3$.

5. If HHHT (probability $\frac{1}{16}$), then $E+4$.

6. If HHHHT (probability $\frac{1}{32}$), then $E+5$.

7. If HHHHH (probability $\frac{1}{32}$), then $E = 5$.

8. $\therefore E = \frac{1}{2}(E+1) + \frac{1}{4}(E+2) + \frac{1}{8}(E+3) + \frac{1}{16}(E+4) + \frac{1}{32}(E+5) + \frac{1}{32}(5)$

9. $\therefore E = 62$.

Generalizing, $E_n = 2(2^n - 1)$ where n is the number of consecutive head.

Approach 2: Find the recurrence relation for E_n, the expected number of tosses to get n consecutive heads.

After getting $n-1$ consecutive heads, there are 2 scenarios:

- If we get a head (with probability $\frac{1}{2}$), in which case we have got n consecutive heads and the expected value is $E_{n-1}+1$.
- If we get a tail (with probability $\frac{1}{2}$), in which case we have to start all over, and hence the expected value is $E_{n-1}+1+E_n$

$E_0 = 0$

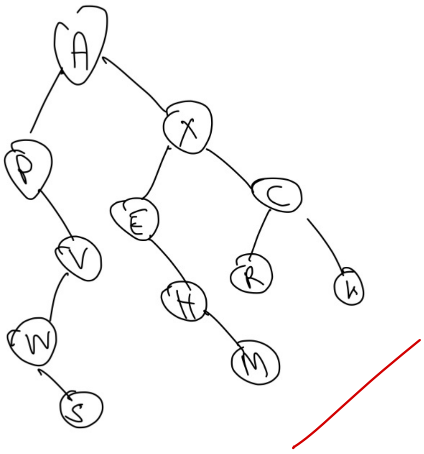
$E_n = \frac{1}{2}(E_{n-1}+1) + \frac{1}{2}(E_{n-1}+1+E_n)$ or $E_n = 2E_{n-1}+2$

(c) The post-order traversal and in-order traversal of a binary tree with 12 vertices are given below:

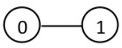
Post-order: S W V P M H E R K C X A

In-order: P W S V A E H M X R C K

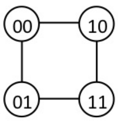
Draw the binary tree. The root has been drawn for you. [4 marks]



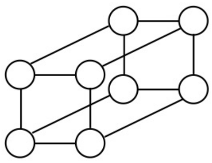
(d) A hypercube graph Q_n may be constructed by creating a vertex for each n-digit binary number (leading zeroes are added if necessary to form n digits), with two vertices adjacent when their binary representations differ in a single digit. The following figures show Q_1, Q_2, Q_3 and Q_4. $N_v(G)$ and $N_e(G)$ denote the number of vertices and number of edges, respectively, in graph G. The vertices in Q_1 and Q_2 are also labelled as shown in the figures.



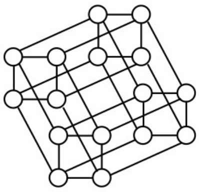
Graph Q_1
$N_v(Q_1) = 2$
$N_e(Q_1) = 1$



Graph Q_2
$N_v(Q_2) = 4$
$N_e(Q_2) = 4$



Graph Q_3
$N_v(Q_3) = 8$
$N_e(Q_3) = 12$



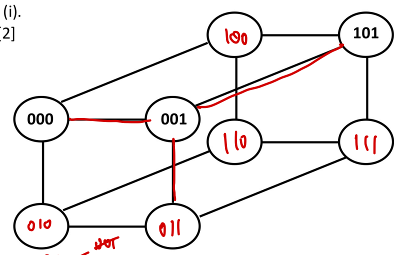
Graph Q_4
$N_v(Q_4) = 16$
$N_e(Q_4) = 32$

(i) Fill in the vertex labels for Q_3 on the **Answer Booklet**. Note that three of the vertices have been labelled 000, 001 and 101 for you. [2 marks]

(ii) What is the closed-form formula for $N_v(Q_n)$? [3 marks]

(iii) What is the closed-form formula for $N_e(Q_n)$? [5 marks]

d. (i).
[2]



_01 < 001 / 101
or _10 < 010 / 110

ii) $Nv(Q_n) = 2^n$

iii) $N_e(Q_n) = \frac{n2^n}{2}$.

1. n-bit number can differ by 1 bit in n ways. ⇒ (1011 ⇒ 0111, 1111, 1001, 1010)
2. Each vertex has degree n.
3. There are 2^n vertices.
4. Total degree of $Q_n = n2^n$.
5. ∴ By handshake theorem, total degree of graph = 2x number of edges
6. ∴ $Ne(Q_n) = \frac{n2^n}{2}$.

Q17. Graphs (14 marks)

The *lazy caterer's sequence* describes the number of maximum pieces of a pancake (or pizza) that can be made with a given number of straight cuts.

For example, with three straight cuts, you get seven pieces as shown in Figure 2 below.

Figure 2: Pancake (Photo credit: Wikipedia).

Figure 3 below shows the first few values in the lazy caterer's sequence starting with $n = 0$ where n is the number of straight cuts. The sequence is 1, 2, 4, 7, 11, 16, ...

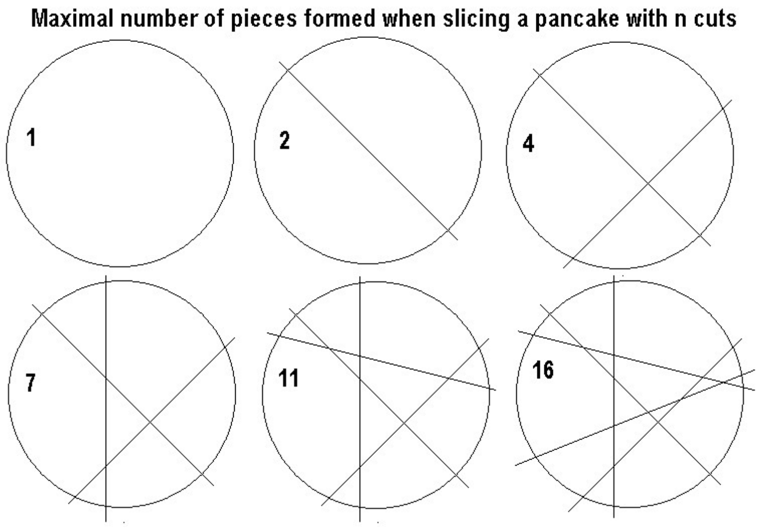


Figure 3: Lazy Caterer's Sequence.

We may model this problem by using a graph. Figure 4 shows the graph corresponding to $n = 3$, where the vertices are the intersections among the cuts and the boundary of the pancake.

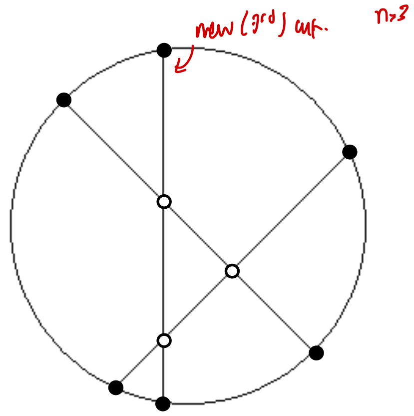
We may define the following functions:

$P(n)$: number of pieces of pancakes with n cuts

$V(n)$: number of vertices of a graph corresponding to a pancake with n cuts

$E(n)$: number of edges of a graph corresponding to a pancake with n cuts

In Figure 4, $P(3) = 7, V(3) = 9$ and $E(3) = 15$.



$P(3) = 7$
$V(3) = 9$
$V_3(3) = 6$
$V_4(3) = 3.$
$E(3) = 15.$

Figure 4: Graph representation.

The vertices of the graph are of two types: those with degree of three (which lie on the boundary of the circle, shown as black dots) and those with degree of four (which lie inside the circle, shown as white dots). Let's define two more functions:

$V(n)$ {
- $V_3(n)$: number of vertices with degree three $= 2n$ (every cut creates 2 new vertices on the boundary).
- $V_4(n)$: number of vertices with degree four $= V_4(n-1) + \underbrace{n-1}_{\text{new vertices}}$ (n^{th} cut intersects the existing $n-1$ cuts) $= \frac{n(n-1)}{2}$ for $n>0$.

In Figure 4, $V_3(3) = 6$ and $V_4(3) = 3$.

Answer the following parts. Working is not required.

(a) (2 marks) Express $E(n)$ in terms of $V_3(n)$ and $V_4(n)$. $E(n) = \frac{3V_3(n) + 4(V_4(n))}{2}$ (Handshake Theorem)

(b) (2 marks) Write the recurrence relation for $V(n)$. The base case is $V(0) = 0$. $V(n) = V(n-1) + n+1$ ← n^{th} cut cut thru existing $n-1$ cuts and it also cuts the boundary at 2 places.

(c) (2 marks) Write the closed form formula for $V(n)$. $\therefore V(n) = V_3(n) + V_4(n) = 2n + \frac{n(n-1)}{2} = \frac{n^2+3n}{2}.$

(d) (2 marks) Write the recurrence relation for $E(n)$. The base case is $E(0) = 0$.

(e) (2 marks) Write the closed form formula for $E(n)$.

(f) (2 marks) Euler's formula is given as $v - e + f = 2$. Relate v, e and f with the functions defined in this question.

(g) (2 marks) From part (f), or otherwise, derive the closed form formula for $P(n)$.

d) $E(n) = E(n-1) + 2n+1$

(The n^{th} cut adds $n+1$ vertices as it cuts through the existing $n-1$ cuts and opposite sides of the circle, splitting each of the $n+1$ edges it cuts through into 2, and also introducing n new edges).

e) $E(n) = \frac{3V_3(n) + 4V_4(n)}{2}$

$= \frac{3(2n) + 2n(n-1)}{2}$

$= n^2 + 2n$

f) $P(n) = E(n) - V(n) + 1$ (using euler's formula $v - e + f = 2$, where $f = P(n) + 1$ since the region outside constitute 1 face).

g) $P(n) = P(n-1) + n$

$P(n) = \frac{n^2+n+2}{2}.$

Q18. Functions (12 marks)

Private cars in Singapore have license plates (see Figure 5) in the format: $S\alpha_1\alpha_2\ x_1x_2x_3x_4\ c$, where each α_1 and α_2 is a single letter taken from the usual English alphabet (excluding I and O), and each $x_1, \ldots, x_4$ is a single digit taken from $\{0, 1, \ldots, 9\}$. The last letter c is a checksum letter, ie. a function of the preceding letters and numbers. Its purpose is to serve as a quick check on the validity of the license plate.

Figure 5: A typical Singapore car license plate. (Photo Credit: Wikipedia)

Let $\mathcal{L}$ denote the set of all possible strings of the form: $\alpha_1\alpha_2x_1x_2x_3x_4$. The possible values of α_i and x_j are as described above. Also, let $\mathcal{K} = \{A, Z, Y, X, U, T, S, R, P, M, L, K, J, H, G, E, D, C, B\}$. Then the checksum function may be defined as $f : \mathcal{L} \to \mathcal{K}$, where $f(\alpha_1\alpha_2x_1x_2x_3x_4)$ is calculated in three steps:

F1. Let n_1 be the positional value of α_1 in the English alphabet, ie. $A = 1,\ B = 2,\ C = 3, \ldots,\ Z = 26$. And let n_2 be the positional value of α_2. (Note that since I and O are not allowed, $n_1, n_2 \notin \{9, 15\}$.)

F2. Compute $t = 9n_1 + 4n_2 + 5x_1 + 4x_2 + 3x_3 + 2x_4$, and $r = t\%19$. That is, r is the remainder of t modulo 19, which means $0 \leq r < 19$.

F3. The checksum letter $c =$ the letter in $\mathcal{K}$ indexed by r, where $0 = A, 1 = Z, 2 = Y, \ldots, 18 = B$. (Here, we are treating $\mathcal{K}$ as an ordered set, in which its elements are indexed by position, starting from 0.)

Using the example in Figure 5:

F1. $\alpha_1 = D,\ \alpha_2 = N,\ x_1 = 7,\ x_2 = 4,\ x_3 = 8,\ x_4 = 4$; and so $n_1 = 4,\ n_2 = 14$.

F2. Then $t = 9 \cdot 4 + 4 \cdot 14 + 5 \cdot 7 + 4 \cdot 4 + 3 \cdot 8 + 2 \cdot 4 = 175$, and so $r = 175\%19 = 4$.

F3. Hence $c = U$.

(a) (2 marks) Determine the checksum letter for $CS1231$. (No working needed.)

(b) (2 marks) Show that f is not one-to-one by finding another $y \in \mathcal{L}$ such that $f(y)$ is the same checksum letter as that in Figure 5. (No working needed. Just state a suitable y.)

note: must be same value.

(c) (8 marks) (Difficult) Is f onto? Prove or disprove.

END OF PAPER

a) F1: $\alpha_1 = C$, $\alpha_2 = S$, so $n_1 = 3$, $n_2 = 19$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, $x_4 = 1$

F2: $t = 9\cdot 3 + 4\cdot 19 + 5\cdot 1 + 4\cdot 2 + 3\cdot 3 + 2\cdot 1 = 127$

$\therefore r = 127 \% 19 = 13$.

F3: $c = H$.

b) Weights of n_2 and x_2 are the same, which allow us to increase n_2 by the same amount as we decrease x_2. yielding the same value for t.

$\therefore y$: DP 7284 is one possible answer.

c) Is f onto? Prove.

f is onto.

construct the table which shows the value of a, b such that $3a + 2b \equiv r \pmod{19}$.

for $r = 0, 1, 2 \ldots, 18$.

Note that this table is a bijection between a, b and r.

a\b	0	1	2	3	4
0	0	2	4		
1	3	5	7		
2	6	8	10	12	
3	9	11	13	15	17
4		14	16	18	1.

Proof (by construction): or list out 19. car plate number.

1. Take any $w \in K$
2. From step F3, there is a unique r which is the positional index of w. Note that $0 \le r \le 19$.
3. (we will construct a string: $y =$ BE00ab , where a, b will be derived from r, such that $f(y) = w$.)
4. Given r from line 2 , look up table for unique pair a, b.
5. let $y =$ BE00ab , Clearly $y \in L$.
6. Now $f(y)$ may be calculated as follows:

    6.1 F2: $t = 9\cdot 2 + 4\cdot 5 + \cancel{5\cdot 0 + 4\cdot} 0 + 3a + 2b$
    $= 38 + 3a + 2b$.

    6.2. Since $38 + 3a + 2b \equiv 3a + 2b \equiv r \pmod{19}$, we have $f(y) = w$.
7. Thus, $\exists y \in L$ such that $f(y) = w$.
8. Hence, f is onto.

> **Definition 1.** *A **Group**, denoted $(G, *)$, is a set G along with a binary operator $*$ that satisfies these four axioms:*
>
> | | | |
> |---|---|---|
> | *(A1)* | *Closure:* | $\forall a, b \in G,\ a * b \in G.$ |
> | *(A2)* | *Associativity:* | $\forall a, b, c \in G,\ (a * b) * c = a * (b * c).$ |
> | *(A3)* | *Identity:* | $\exists e \in G$ *such that* $\forall a \in G,\ a * e = e * a = a.$ |
> | *(A4)* | *Inverse:* | $\forall a \in G,\ \exists b \in G$ *(called the **inverse** of a) such that* $a * b = b * a = e.$ |

Remarks:

1. It is usual to write ab to mean $a * b$.
2. Because of Associativity, there is no ambiguity in writing abc, since the result is the same whichever way it is evaluated.
3. The element $e \in G$ is called the **identity element**, or simply **identity**.
4. The inverse of a is usually denoted a^{-1}.

> **Definition 2.** *If G is finite with $m \in \mathbb{Z}^+$ elements, then the **group order** of G is said to be m. For small m, it is often helpful to see the group as a **Cayley table**. Let $a_1, a_2, \ldots, a_m$ be the elements of G, then each entry, c_{ij} in the $m \times m$ Cayley table is defined as $c_{ij} = a_i a_j$, for $i, j = 1, 2, \ldots, m$ (see Table 1(a) on page 6.)*

An example of a group is $(\{\mathbf{true}, \mathbf{false}\}, \oplus)$. Its Cayley table is shown in Table 1(b). As you may recall, $\oplus$ is the logical *exclusive-OR* binary operator, which evaluates to **true** when both its inputs are different, and **false** otherwise.

To check whether this is a group, let's see if it satisfies the four axioms. Closure is obvious from the table, and Associativity is guaranteed by the $\oplus$ operator. From the table, the *identity* is **false**, because **true** $\oplus$ **false** $=$ **true** , and **false** $\oplus$ **false** $=$ **false**, thus satisfying Axiom (A3). The last equality also means that **false** is its own inverse; while **true** $\oplus$ **true** $=$ **false** means that **true** is its own inverse too. Thus, $(\{\mathbf{true}, \mathbf{false}\}, \oplus)$ is a group since it satisfies all 4 axioms.

$*$	a_1	$\ldots$	a_j	$\ldots$	a_m
a_1					
$\vdots$			$\vdots$		
a_i		$\ldots$	c_{ij}	$\ldots$	
$\vdots$			$\vdots$		
a_m					

(a) General Cayley table

$\oplus$	**true**	**false**
true	**false**	**true**
false	**true**	**false**

(b) Cayley table for $(\{\mathbf{true}, \mathbf{false}\}, \oplus)$

Table 1: Cayley tables.

Another example of a group is the familiar $(\mathbb{Z}, +)$, that is, the set of integers with the usual addition operator. Closure and Associativity are true for integers under addition, and thus axioms (A1) and (A2) are satisfied. The *identity* is 0, since $a + 0 = 0 + a = a$ for all $a \in \mathbb{Z}$. Finally, for each $a \in \mathbb{Z}$, its inverse is $-a$, since $a + (-a) = (-a) + a = 0$. Thus, $(\mathbb{Z}, +)$ is a group. But since $\mathbb{Z}$ is infinite, it is not possible to draw its Cayley table.

Q9. Which of the following is a group?

A. $(R, \circ)$, where $R = \{R_{120}, R_{240}, R_{360}\}$, and R_θ means "rotate an equilateral triangle[1] θ degrees clockwise around its center O" (see Figure 5), and $\circ$ means function composition, ie. $R_{\theta_2} \circ R_{\theta_1}$ means "rotate clockwise by θ_1, followed by θ_2". Its Cayley table is shown below:

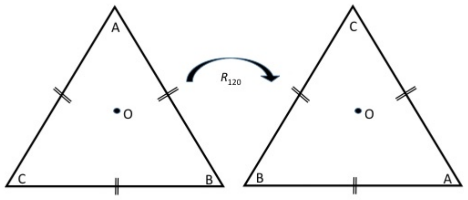


Figure 5: Rotation of an equilateral triangle by R_{120}.

$\circ$	R_{120}	R_{240}	R_{360}
R_{120}	R_{240}	R_{360}	R_{120}
R_{240}	R_{360}	R_{120}	R_{240}
R_{360}	R_{120}	R_{240}	R_{360}

~~B.~~ $(\mathbb{Z}, \times)$, that is, the set of integers with the multiplication operator. — no inverse. $\rightarrow 3 \times \frac{1}{3} = 1$ — not integer.

~~C.~~ $(\{\mathbf{true}, \mathbf{false}\}, \vee)$, whose Cayley table is: — no inverse for true.

identity = false.

$\vee$	**true**	**false**
true	**true**	**true**
false	**true**	**false**

~~D.~~ All of the above. $p \vee F = p$.

~~E.~~ None of the above.

[1] An equilateral triangle is a triangle in which all 3 sides have equal length, and all 3 angles are 60°

14. Which of the following sets is/are countable?

A. The set of all partitions of $\mathbb{Z}$.

B. The set of all partial orders on $\mathbb{Z}$.

C. The set of all functions $\mathbb{Z} \to \mathbb{Z}$.

D. The set $\mathbb{Z}^*$ of all strings over $\mathbb{Z}$.

E. The set of all simple undirected graphs whose vertex set is a finite subset of $\mathbb{Z}$.

A: 1. $\mathbb{Z}$ is countable.

2. Let A be a countable infinite set. Then $P(A)$ is uncountable. ⇐ similar concepts.

3. Each $S \subseteq \mathbb{Z}$ corresponds to a partition $\{S\} \cup \{\{x\} : x \in \mathbb{Z} \setminus S\}$ of $\mathbb{Z}$

4. This correspondence goes bad only

   3.1 when $S = \emptyset$, in which case the anticipated "partition" fails to be a partition;

   3.2 when $|S| = 1$, which make counterexamples to injectivity

5. There are only countably many bad cases $\emptyset, \{1\}, \{-1\}, \{2\}, \{-2\}, \ldots$

6. So they are negligible in the uncountability proof using prop. 10.3.6.

B: Each $S \subseteq \mathbb{Z}$ corresponds to a partial order on $\mathbb{Z}$ in which all the elements of S are ordered in the usual way, and no element of $\mathbb{Z} \setminus S$ is comparable to anything else.

This correspondence goes bad only when $|S| = 1$. Counterexample to injectivity.

There are countably many bad cases

$\{0\}, \{1\}, \{-1\} \ldots$

So they are negligible in the uncountability proof using 10.3.6.

C. The set of all infinite sequences over $\{0, 1\}$, or equivalently, the set of all functions $\mathbb{Z}_{\geq 0} \to \{0, 1\}$ is uncountable

As $\mathbb{Z}$ and $\mathbb{Z}_{\geq 0}$ are both infinite & countable, they have same cardinality

There are more elements in $\mathbb{Z}$ than in $\{0, 1\}$.

So there are more choices of image for a function $\mathbb{Z} \to \mathbb{Z}$ than for a function $\mathbb{Z}_{\geq 0} \to \{0, 1\}$

$\therefore$ uncountable.

3 pairs of socks for each colour of rainbow (42 total) Socks cannot be differentiated from each other

possible outcomes where Potts find a matching pair given randomly pick 5 socks?

$42 = 3 \times 2 \times 7$

num of ways without matching pair $\binom{7}{5} = \frac{7 \cdot 6}{2 \cdot 1} = 21$

Total number of ways $= {}^{5+7-1}C_6 = \binom{11}{6}$

7 colours.
5 socks.

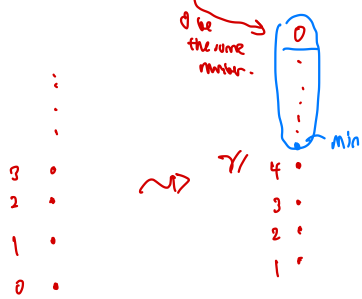
RED GREEN Yellow
xx | | xx | ...

5 x's
6 dividers.

num of ways with matching pair $= \binom{11}{6} - 21$

$= 441$.

D3. A *well-order* on a set A is a total order on A with respect to which every nonempty subset of A has a smallest element. Our Well-Ordering Principle states that the usual order on $\mathbb{Z}_{\geq 0}$ is a well-order. Under this order, below any number there are only finitely many numbers. Find a well-order without this property, i.e., find a well-order on $\mathbb{Z}_{\geq 0}$ with respect to which some number has infinitely many numbers below it.



$\emptyset \neq S \subseteq \mathbb{Z}_{\geq 0}$

Case 1: $0 \notin S$

Then $\min_{\preceq} S = \min_{\leq} S$ exists.

Case 2: $0 \in S$

Case 2A: $S = \{0\}$

Then $\min_{\preceq} S = 0$.

Case 2B: $\exists x \in \mathbb{Z}^+ \; x \in S$.

Then $\emptyset \neq S \setminus \{0\} \subseteq \mathbb{Z}^+$.

So $\min_{\preceq}(S \setminus \{0\})$ exists. is equal to $\min_{\preceq} S$.

9. [AY2015/16 Semester 1 Exam Question]

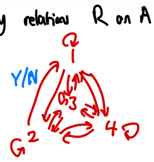
Let $A = \{1, 2, 3, 4\}$. Since each element of $P(A \times A)$ is a subset of $A \times A$, it is a binary relation on A. ($P(S)$ denotes the powerset of S.)

Assuming each relation in $P(A \times A)$ is equally likely to be chosen, what is the probability that a randomly chosen relation is (a) reflexive? (b) symmetric?

Can you generalize your answer to any set A with n elements?

How many relations R on A are there?

In general if $|A| = n$.



2^{16} ↖ 2 choices

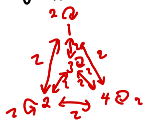
$\underbrace{2 \times 2 \times 2 \times \cdots}_{n \times n \text{ times}} \to R \subseteq A \times A$

(x, y) $x, y \in A$

$\therefore 2^{n^2}$

$\underbrace{n \text{ choices} \quad n \text{ choices}}_{n \times n \text{ choices}}$

How many symmetric relation R on A are there?



2^{10}

n loops. $\binom{n}{2}$ double-headed arrow

$\underbrace{2 \times 2 \times \cdots \times 2}_{n \text{ times}} \times \underbrace{2 \times \cdots \times 2}_{\binom{n}{2} \text{ times}} = 2^{n + \binom{n}{2}}$.

What is the probability that a randomly chosen relation on A is symmetric?

$\dfrac{2^{10}}{2^{16}} = 2^{-6}$ $\qquad$ $\dfrac{2^{n + \binom{n}{2}}}{2^{n^2}}$.

CS1231/CS12

3. For a set A and a function $f: A \to A$, define

$$C_f = \{f^{-1}(\{y\}) : y \in A \text{ and } \boxed{f(y) = y}\}.$$

Which of the following is/are true for all sets A and all functions $f: A \to A$? (idempotent)

(i) If $f \circ f = f$, then C_f is a partition of A. TRUE

(ii) If C_f is a partition of A, then $f \circ f = f$. TRUE

A. (i) and (ii).

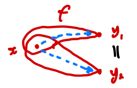
B. (i) only.

C. (ii) only.

D. None.

(i) (0) $f^{-1}(\{y\}) \neq \emptyset \Leftarrow y \in f^{-1}(\{y\}) \Leftarrow f(y) = y$

(1) $f^{-1}(\{y_1\}) \cap f^{-1}(\{y_2\}) \neq \emptyset \underset{\text{want}}{\Rightarrow} f^{-1}(\{y_1\}) = f^{-1}(\{y_2\})$



$x \in f^{-1}(\{y_1\}) \cap f^{-1}(\{y_2\}) \Rightarrow y_2 = f(x) = y_1$

(2) $\forall x \in A\ \exists y \in A\ (x \in f^{-1}(\{y\}) \wedge f(y) = y)$

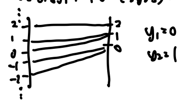
↳ $y = f(x)$ $\qquad$ $f \circ f = f$

$f(y) = f(f(x)) = (f \circ f)(x) = f(x) = y$

If it satisfy property of partition.

Possible that

$|f^{-1}(\{y_1\})| \neq |f^{-1}(\{y_2\})|$

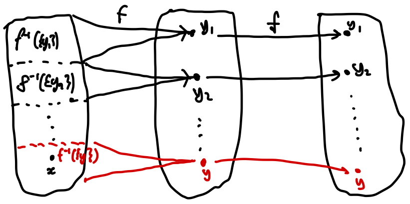


(ii)

$f(y_1) = y_1$

$f(y_2) = y_2$

$f(y) = y$



$(f \circ f)(x) = f(f(x)) = f(y) = y = f(x)$.

18. [8 marks]

$\{id_A^{-1}(\{y\}) : y \in A \text{ and } id(y) = y\} = \{\{y\} : y \in A\}$

Let A be a set and C be a partition of A. Prove that there exists a function $f: A \to A$ such that

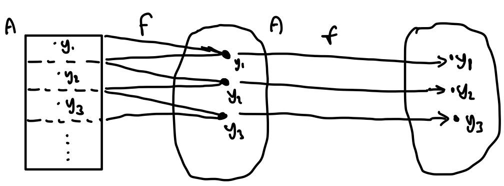
$$f \circ f = f \quad \text{and} \quad C = \{f^{-1}(\{y\}) : y \in A \text{ and } f(y) = y\}.$$

$\forall A\ \forall C\ \exists f\ (\ldots)$

verify $y_S \in S$.

$\forall x \in A\ \ (f \circ f)(x) = f(f(x)) = f(y_S) = y_S = f(x)$.

↳ $S \in C$ containing x ↳ $x \in S \to f(x) = y_S$



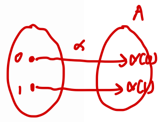
For each $S \in C$, pick $\boxed{y_S \in S}$

Define $f: A \to A$ by setting, for each $x \in A$ and each $S \in C$,

$f(x) = y_S \Leftrightarrow x \in S$.

19. [6 marks]

Let A be a set. Let S be the set of all functions $\{0,1\} \to A$, i.e.,



$(x,y) : x,y \in A$

$$S = \{\alpha \mid \alpha : \{0,1\} \to A\}.$$

Prove that $|S| = |A^2|$ according to Cantor's definition of same-cardinality.

essential info:
2 elems of A : $\alpha(0)$ $\alpha(1)$.

$$f(\alpha) = (\alpha(0), \alpha(1)).$$

1. We define a bijection $f: S \to A \times A$ that satisfies $f(\alpha) = (a_0, a_1)$ where $\alpha \in S$, $\alpha(0) = a_0 \in A$, $\alpha(1) = a_1 \in A$.

2. (f is injective)
   - 2.1. Suppose not. Then $\exists \alpha, \alpha' \in S : f(\alpha) = f(\alpha') \wedge \boxed{\alpha \neq \alpha'}$
   - 2.2. Then $f(\alpha) = f(\alpha') \Rightarrow (\alpha(0), \alpha(1)) = (\alpha'(0), \alpha'(1))$
   - 2.3. α and α' share same domain and codomain (by membership of S)
   - 2.4. α and α' bring the same inputs to the outputs (line 2.2)
   - 2.5. So $\alpha = \alpha'$ which contradicts 2.1
   - 2.6. $\therefore$ f is injective.

every (a_0, a_1) is f of smth.

$f(w) = (w(0), w(1)) = (a_0, a_1)$

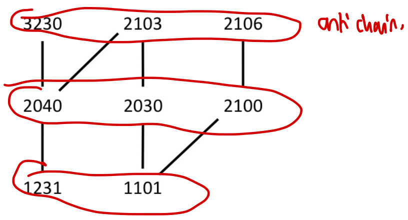
3. (f is surjective)
   - 3.1. Take any $(a_0, a_1) \in A \times A$.
   - 3.2. We define the function $w : \{0,1\} \to A$ that satisfies $w(0) = a_0$, $w(1) = a_1$.
   - 3.3. Then this $w \in S$ because it satisfies the definition of S.
   - 3.4. So $\forall (a_0, a_1) \in A \times A\ (\exists a : f(\alpha) = (a_0, a_1))$ (want to show)

4. By 2 and 3, f is bijective.

5. Since bijection from S to $A \times A = A^2$, $|S| = |A^2|$

19. [Total: 4 marks]

Consider the partial order R on $M = \{1101, 1231, 2030, 2040, 2100, 2103, 2106, 3230\}$ with the following Hasse diagram.



(a) Write down all partitions $\mathcal{C}_0$ of M with at most 3 components of which all the components are *chains* with respect to R, i.e.,

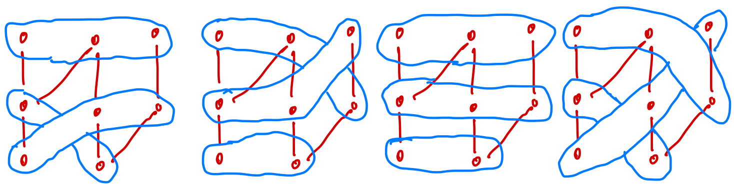
$$\forall S \in \mathcal{C}_0\ \forall x, y \in S\ (x\,R\,y \text{ or } y\,R\,x).$$

(b) Write down all partitions $\mathcal{C}_1$ of M with at most 3 components of which all the components are *antichains* with respect to R, i.e.,

not comparable.

$$\forall S \in \mathcal{C}_1\ \forall x, y \in S\ (x \neq y \Rightarrow \sim(x\,R\,y \text{ or } y\,R\,x)).$$

No justification is needed for this question.



8. Let R be a relation on a non-empty set A. Consider the following statements:

(I) If R is reflexive, then R^{-1} is reflexive.

(II) If R is symmetric, then R^{-1} is symmetric.

(III) If R is transitive, then R^{-1} is transitive.

$R^{-1} := \{(y,x) : (x,y) \in R\}$

"Reverse direction of arrows".

A. Only (II) is true.

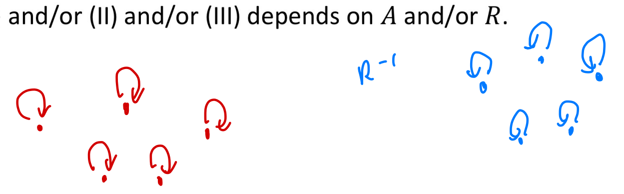
B. Only (I) and (II) are true.

C. Only (II) and (III) are true.

D. (I), (II) and (III) are all true.

E. The truth of (I) and/or (II) and/or (III) depends on A and/or R.

(I) R R^{-1}



TRUE.

(II) R R^{-1}

TRUE

(III) R R^{-1}

TRUE

$\therefore$ D.

17. **[Total: 12 marks]**

Let P be a partial order on a non-empty set A. Let R be another relation on A, and suppose that $R \subseteq P$. Let $\tilde{R}$ be the reflexive closure of R and let T be the transitive closure of $\tilde{R}$. Prove that:

(a) T is a partial order on A. [6 marks]

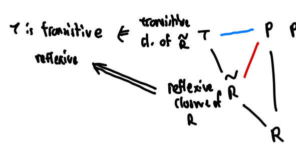
(b) If T' is another partial order on A such that $R \subseteq T'$, then $T \subseteq T'$. [6 marks]

Recall from Tutorial 8 that the reflexive closure of a relation is the smallest reflexive relation on the same set that contains this relation as a subset. Similarly, the transitive closure of a relation is the smallest transitive relation on the same set that contains this relation as a subset.

(a) Definition: Let R be a relation on a set A.
Then the reflexive/transitive closure of R
is a relation $\tilde{R}$ on A s.t.

(1) $\tilde{R} \supseteq R$ and $\tilde{R}$ is reflexive/transitive.

(2) If S is a relation on A s.t. $S \supseteq R$ and S is reflexive/transitive, then $S \supseteq \tilde{R}$.

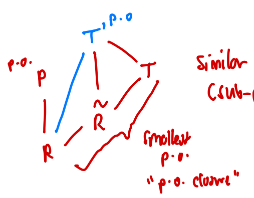


partial order = reflexive, antisym, transitive

Reflexive.
$\therefore$ bigger than $\tilde{R}$

Transitive
$\therefore$ bigger than T.

shows antisym.

(b) 

Similar to (a).
(sub-proof).

Antisym Let $x, y \in A$ s.t. xTy and yTx

$\therefore$ xPy and yPx.
as $T \subseteq P$.

$\therefore$ $x = y$ as P is antisym.

2. Let $A = \{1,2,3\}$. Define $g, h \colon A \to A$ by setting, for all $x \in A$,

$$g(x) = \begin{cases} 1, & \text{if } x = 2; \\ 2, & \text{if } x = 1; \\ x, & \text{otherwise,} \end{cases} \qquad h(x) = \begin{cases} 2, & \text{if } x = 3; \\ 3, & \text{if } x = 2; \\ x, & \text{otherwise.} \end{cases}$$

What is the order of the function $g \circ (g \circ h)^{-1} \circ h \circ h \circ g \circ h \circ h$?

A. 1.

B. 2.

C. 3.

D. 4.

similar

7. Let $A = \{1, 2, 3\}$. The *order* of a bijection $f \colon A \to A$ is defined to be the smallest $n \in \mathbb{Z}^+$ such that

$$\underbrace{f \circ f \circ \ldots \circ f}_{n\text{-many } f\text{'s}} = \mathrm{id}_A.$$

Define functions $g, h \colon A \to A$ by setting, for all $x \in A$,

$$g(x) = \begin{cases} 1, & \text{if } x = 2; \\ 2, & \text{if } x = 1; \\ x, & \text{otherwise,} \end{cases} \qquad h(x) = \begin{cases} 2, & \text{if } x = 3; \\ 3, & \text{if } x = 2; \\ x, & \text{otherwise.} \end{cases}$$

$h(1) = 1.$

$g(h(1)) = 2.$

Find the orders of g, h, $g \circ h$, and $h \circ g$.