2. The Logic of Compound Statements (aka Propositional Logic) Summary

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Summary

2.1 Logical Form and Logical Equivalence

- Statements; Compound Statements; Statement Form (Propositional Form)
- Logical Equivalence; Tautologies and Contradictions

2.2 Conditional Statements

- Conditional Statements; If-Then as Or
- Negation, Contrapositive, Converse and Inverse
- Only If and the Biconditional; Necessary and Sufficient Conditions

2.3 Valid and Invalid Arguments

- Argument; Valid and Invalid Arguments
- Modus Ponens and Modus Tollens
- Rules of Inference
- Fallacies

Reference: Epp's Chapter 2 The Logic of Compound Statements

Definition 2.1.1 (Statement)

A **statement** (or **proposition**) is a sentence that is true or false, but not both.

Definition 2.1.2 (Negation)

If p is a statement variable, the **negation** of p is "not p" or "it is not the case that p" and is denoted $\sim p$.

Definition 2.1.3 (Conjunction)

If p and q are statement variables, the **conjunction** of p and q is "p and q", denoted $p \wedge q$.

Definition 2.1.4 (Disjunction)

If p and q are statement variables, the **disjunction** of p and q is "p or q", denoted $p \vee q$.

Summary

Definition 2.1.5 (Statement Form/Propositional Form)

A **statement form** (or **propositional form**) is an expression made up of statement variables and logical connectives that becomes a statement when actual statements are substituted for the component statement variables.

Definition 2.1.6 (Logical Equivalence)

Two statement forms are called **logically equivalent** if, and only if, they have identical truth values for each possible substitution of statements for their statement variables.

The logical equivalence of statement forms P and Q is denoted by $\boldsymbol{P \equiv Q}$.

Definition 2.1.7 (Tautology)

A **tautology** is a statement form that is always true regardless of the truth values of the individual statements substituted for its statement variables. A statement whose form is a tautology is a **tautological statement**.

Definition 2.1.8 (Contradiction)

A **contradiction** is a statement form that is always false regardless of the truth values of the individual statements substituted for its statement variables. A statement whose form is a contradiction is a **contradictory statement**.

Summary

Theorem 2.1.1 Logical Equivalences

Given any statement variables *p*, *q* and *r*, a tautology **true** and a contradiction **false**:

1	Commutative laws	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2	Associative laws	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3	Distributive laws	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4	Identity laws	$p \wedge \textbf{true} \equiv p$	$p \vee \textbf{false} \equiv p$
5	Negation laws	$p \vee {\sim}p \equiv \textbf{true}$	$p \wedge {\sim}p \equiv \textbf{false}$
6	Double negative law	${\sim}({\sim}p) \equiv p$	
7	Idempotent laws	$p \wedge p \equiv p$	$p \vee p \equiv p$
8	Universal bound laws	$p \vee \textbf{true} \equiv \textbf{true}$	$p \wedge \textbf{false} \equiv \textbf{false}$
9	De Morgan's laws	${\sim}(p \wedge q) \equiv {\sim}p \vee {\sim}q$	${\sim}(p \vee q) \equiv {\sim}p \wedge {\sim}q$
10	Absorption laws	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11	Negation of **true** and **false**	${\sim}\textbf{true} \equiv \textbf{false}$	${\sim}\textbf{false} \equiv \textbf{true}$

*: Note that there is no ambiguity in $p \wedge q \wedge r$ as it is equivalent to $(p \wedge q) \wedge r$ and $p \wedge (q \wedge r)$. Likewise for $\vee$.

Summary

Definition 2.2.1 (Conditional)

If p and q are statement variables, the **conditional** of q by p is "if p then q" or "p implies q", denoted $p \to q$.
It is false when p is true and q is false; otherwise it is true.
We called p the hypothesis (or antecedent) and q the conclusion (or consequent).

Definition 2.2.2 (Contrapositive)

The **contrapositive** of a conditional statement "if p then q" is "if ~q then ~p".
Symbolically, the contrapositive of $p \to q$ is $\sim q \to \sim p$.

Definition 2.2.3 (Converse)

The **converse** of a conditional statement "if p then q" is "if q then p".
Symbolically, the converse of $p \to q$ is $q \to p$.

Definition 2.2.4 (Inverse)

The **inverse** of a conditional statement "if p then q" is "if ~p then ~q".
Symbolically, the inverse of $p \to q$ is $\sim p \to \sim q$.

Summary

$p \rightarrow q$ $\equiv$ $\sim p \vee q$ Implication law

$p \rightarrow q$ $\equiv$ $\sim q \rightarrow \sim p$

conditional statement *contrapositive*

$q \rightarrow p$ $\equiv$ $\sim p \rightarrow \sim q$

converse *inverse*

Note that: $p \rightarrow q$ $\not\equiv$ $q \rightarrow p$

Summary

Definition 2.2.5 (Only If)

If p and q are statements,

"p only if q" means "if not q then not p"

Or, equivalently,

"if p then q"

Definition 2.2.6 (Biconditional)

Given statement variables p and q, the **biconditional** of p and q is "p if, and only if, q" and is denoted $p \leftrightarrow q$.

It is true if both p and q have the same truth values and is false if p and q have opposite truth values.

The words *if and only if* are sometimes abbreviated *iff*.

Definition 2.2.7 (Necessary and Sufficient Conditions)

If r and s are statements,

"r is a sufficient condition for s" means "if r then s"

"r is a necessary condition for s" means "if not r then not s" (or "if s then r")

Order of operations:

$\sim$	$\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$
not	*and*	*or*	*if-then/implies*	*if and only if*

Coequal in order

Coequal in order

Performed first

Performed last

Summary

2.3 Valid and Invalid Arguments

Definition 2.3.1 (Argument)

An **argument** (**argument form**) is a sequence of statements (statement forms). All statements in an argument (argument form), except for the final one, are called **premises** (or **assumptions** or **hypothesis**). The final statement (statement form) is called the **conclusion**. The symbol •, which is read "therefore", is normally placed just before the conclusion.

To say that an argument form is **valid** means that no matter what particular statements are substituted for the statement variables in its premises, if the resulting premises are all true, then the conclusion is also true.

Definition 2.3.2 (Sound and Unsound Arguments)

An argument is called **sound** if, and only if, it is valid and all its premises are true.

An argument that is not sound is called **unsound**.

Summary

2.3 Valid and Invalid Arguments

Table 2.3.1 Rules of Inference

Rule of inference		
Modus Ponens	$p \rightarrow q$ p • q	
Modus Tollens	$p \rightarrow q$ $\sim q$ • $\sim p$	
Generalization	p • $p \vee q$	q • $p \vee q$
Specialization	$p \wedge q$ • p	$p \wedge q$ • q
Conjunction	p q • $p \wedge q$	

Rule of inference		
Elimination	$p \vee q$ $\sim q$ • p	$p \vee q$ $\sim p$ • q
Transitivity	$p \rightarrow q$ $q \rightarrow r$ • $p \rightarrow r$	
Proof by Division Into Cases	$p \vee q$ $p \rightarrow r$ $q \rightarrow r$ • r	
Contradiction Rule	$\sim p \rightarrow$ **false** • p	

proof by direct proof.

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contradiction