CS1231S: Discrete Structures

Tutorial #10: Counting and Probability II

(Week 12: 1 – 5 November 2021)

I. Discussion Questions

You are strongly encouraged to discuss D1 – D3 on LumiNUS forum. No answers will be provided.

D1. Suppose a random sample of 2 lightbulbs is selected from a group of 8 bulbs in which 3 are defective, what is the expected value of the number of defective bulbs in the sample? Let X represent of the number of defective bulbs that occur on a given trial, where $X = 0,1,2$. Find $E[X]$.

D2. How many **injective functions** are there from a set A with m elements to a set B with n elements, where $m \leq n$?

D3. How many **surjective functions** are there from a 5-element set A to a 3-element set B?

II. Tutorial Questions

1. [CS1231 Past Year's Exam Question]
   You wish to select five persons from seven men and six women to form a committee that includes at least three men.
   (a) In how many ways can you form the committee?
   (b) If you randomly choose five persons to form the committee, what is the probability that you will get a committee with at least three men? Give your answer correct to 4 significant figures.

2. Think of a set with $m + n$ elements as composed of two parts, one with m elements and the other with n elements. Give a **combinatorial argument** to show that

$$\binom{m+n}{r} = \binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \cdots + \binom{m}{r}\binom{n}{0}$$

where $m, n \in \mathbb{Z}^+$, $r \leq m$ and $r \leq n$.

Call the above equation (A). Using equation (A), prove that for all integers $n \geq 0$,

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2.$$

II. Tutorial Questions

7M 6W

1. [CS1231 Past Year's Exam Question]
   You wish to select five persons from seven men and six women to form a committee that includes at least three men.
   (a) In how many ways can you form the committee?
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a) ${}^{7}C_{3} \cdot {}^{10}C_{2}$

interesting results.

$3M2W + 4M1W + 5M.$

Choose item from specific gp → general gp. ⇒ end up having extra terms.

b) $\dfrac{(a)}{{}^{13}C_{5}}$.

58.74%

2. Think of a set with $m + n$ elements as composed of two parts, one with m elements and the other with n elements. Give a **combinatorial argument** to show that

$$\binom{m+n}{r} = \binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \cdots + \binom{m}{r}\binom{n}{0}$$ formula.

where $m, n \in \mathbb{Z}^+$, $r \le m$ and $r \le n$.

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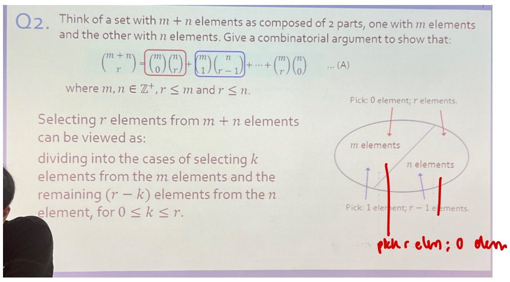
$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2. \quad \text{— (B)}$$



$$\binom{m+n}{r} = \binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \cdots + \binom{m}{r}\binom{n}{0} \quad \text{— (A).}$$

let $m = n$, $r = n$.

Then, from (A),

$$\binom{2n}{n} = \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \cdots + \binom{n}{n-1}\binom{n}{1} + \binom{n}{n}\binom{n}{0}$$

$\binom{n}{r} = \binom{n}{n-r}$ (Lecture 12 eg. 8)

$=$ (B).

3. [AY2020/21 Semester 2 Exam Question]
How many integer solutions for x_1, x_2, x_3 and x_4 does the following equation have, given that $x_i \geq 2^i + i$, for $1 \leq i \leq 4$?

$$\boldsymbol{x_1 + x_2 + x_3 + x_4 = 56.}$$

4. Find the term independent of x in the expansion of

$$\left(2x^2 + \frac{1}{x}\right)^9$$

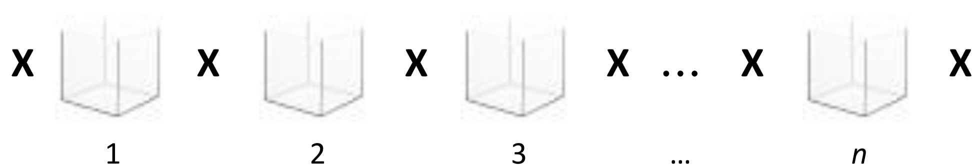
5. Let's revisit Tutorial #9 Question 5:

   Given n boxes numbered 1 to n, each box is to be filled with either a white ball or a blue ball such that at least one box contains a white ball and boxes containing white balls must be consecutively numbered. What is the total number of ways this can be done?

   Last week, the answer given was

   For k $(1 \leq k \leq n)$ consecutively numbered boxes that contain white balls, there are $n - k + 1$ ways. Therefore, total number of ways is $\sum_{k=1}^{n}(n - k + 1) = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

   Now, let's use another approach to solve this problem. Draw crosses on the side of the boxes as shown below. How do you use these crosses?



6. [AY2020/21 Semester 2 Exam Question]
On a die there are 6 numbers. We call 4, 5, 6 the big numbers and 1, 2, 3 the small numbers. Given a loaded die in which the probability of rolling any fixed big number is twice the probability of rolling any fixed small number, answer the following questions.

   (i) What is the probability of rolling a 6? Write your answer as a single fraction.

   (ii) If two such loaded dice are rolled, what is the expected value of the maximum of the two dice? Write your answer as a single fraction.

7. An urn contains five balls numbered 1, 2, 2, 8 and 8. If a person selects a set of two balls at random, what is the expected value of the sum of the numbers on the balls?

3. [AY2020/21 Semester 2 Exam Question] 963

Multiset Qn

How many integer solutions for x_1, x_2, x_3 and x_4 does the following equation have, given that $x_i \geq 2^i + i$, for $1 \leq i \leq 4$?

$$\boldsymbol{x_1 + x_2 + x_3 + x_4 = 56.}$$

$x_1 \geq 2^1 + 1 = 3$

$x_2 \geq 2^2 + 2 = 6$

$x_3 \geq 2^3 + 3 = 11$

$x_4 \geq 2^4 + 4 = 20$

$\therefore$ $x_1 \geq 3$

$x_2 \geq 6$

$x_3 \geq 11$

$x_4 \geq 20$

$$\binom{4+16-1}{16}$$

$r \leftarrow$ combi

repition allowed

$3 + 6 + 11 + 20 = 40$

$\therefore 56 - 40 = 16$.

$\underline{\underline{x_i \geq 0}}$ every bucket can have 0 balls.

Multiset : $n = 4$, $r = 16$.

$$\underbrace{x_1' + x_2' + x_3' + x_4'}_{} = 16$$

$\binom{r+n-1}{r}$ $\quad {}^{4+16-1}C_{16}$.

4. Find the term independent of x in the expansion of

$$\left(2x^2 + \frac{1}{x}\right)^9$$

$(a+b)^n$

$\binom{n}{k} a^{n-k} b^k$

$= \binom{9}{k} (2x^2)^{9-k} \left(\frac{1}{x}\right)^k$

Let $\underline{k = 6}$

$\binom{9}{6} (2x^2)^{9-6} \left(\frac{1}{x}\right)^6$

$= \binom{9}{6} 2^3 \cancel{x^6} \cdot \frac{1}{\cancel{x^6}}$

$= 2^3 \binom{9}{6}$

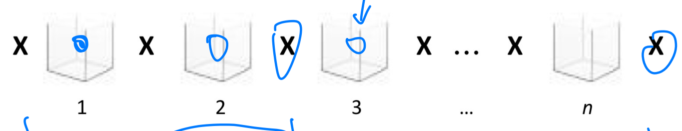
5. Let's revisit Tutorial #9 Question 5:

   Given n boxes numbered 1 to n, each box is to be filled with either a white ball or a blue ball such that at least one box contains a white ball and boxes containing white balls must be consecutively numbered. What is the total number of ways this can be done?

   Last week, the answer given was

   For k $(1 \leq k \leq n)$ consecutively numbered boxes that contain white balls, there are $n-k+1$ ways. Therefore, total number of ways is $\sum_{k=1}^{n}(n-k+1) = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

   Now, let's use another approach to solve this problem. Draw crosses on the side of the boxes as shown below. How do you use these crosses?



n+1 cross, choose any 2 cross. for white ball to be in between

$^{n+1}C_2$

6. [AY2020/21 Semester 2 Exam Question] 2 2 1 1 1
   On a die there are 6 numbers. We call 4, 5, 6 the big numbers and 1, 2, 3 the small numbers. Given a loaded die in which the probability of rolling any fixed big number is twice the probability of rolling any fixed small number, answer the following questions.

   (i) What is the probability of rolling a 6? Write your answer as a single fraction.

   (ii) If two such loaded dice are rolled, what is the expected value of the maximum of the two dice? Write your answer as a single fraction.

list down all probability! 1 2 1 2

i) $\frac{2}{9}$

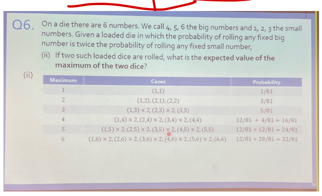
ii) Expected value :

$$\sum_{k=1}^{n} a_k P_k = a_1P_1 + a_2P_2 + a_3P_3 + \dots + a_nP_n$$

$$= 1\cdot\frac{1}{9}\cdot\frac{1}{9} + 2\cdot 2\cdot\frac{1}{9}\cdot\frac{1}{9} + 3\cdot 3\cdot\frac{1}{9}\cdot\frac{1}{9} + 3\cdot 4\cdot\frac{1}{9}\cdot\frac{2}{9} + 1\cdot 4\cdot\frac{2}{9}\cdot\frac{2}{9} + 3\cdot 5\cdot\frac{1}{9}\cdot\frac{2}{9} + 2\cdot 5\cdot\frac{2}{9}\cdot\frac{2}{9} + 3\cdot 6\cdot\frac{1}{9}\cdot\frac{2}{9} + 3\cdot 6\cdot\frac{2}{9}\cdot\frac{2}{9}$$

$=$

Value | P(value)



Q6. On a die there are 6 numbers. We call 4, 5, 6 the big numbers and 1, 2, 3 the small numbers. Given a loaded die in which the probability of rolling any fixed big number is twice the probability of rolling any fixed small number,

(ii) If two such loaded dice are rolled, what is the **expected value of the maximum of the two dice?**

(ii)

Maximum	Cases	Probability
1	(1,1)	1/81
2	(1,2), (2,1), (2,2)	3/81
3	(1,3) × 2, (2,3) × 2, (3,3)	5/81
4	(1,4) × 2, (2,4) × 2, (3,4) × 2, (4,4)	12/81 + 4/81 = 16/81
5	(1,5) × 2, (2,5) × 2, (3,5) × 2, (4,5) × 2, (5,5)	12/81 + 12/81 = 24/81
6	(1,6) × 2, (2,6) × 2, (3,6) × 2, (4,6) × 2, (5,6) × 2, (6,6)	12/81 + 20/81 = 32/81

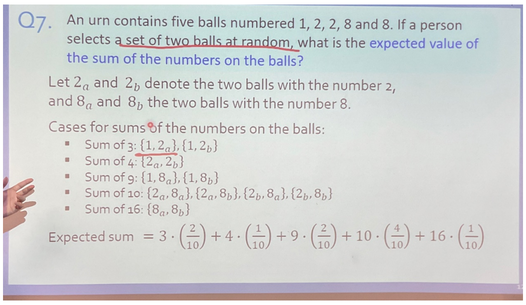
$$EV = \frac{1}{81}\cdot 1 + \frac{3}{81}\cdot 2 + \frac{16}{81}\cdot 3 + \frac{24}{81}\cdot 4 + \frac{32}{81}\cdot 5$$

$$= \frac{398}{81}.$$

different balls

7. An urn contains five balls numbered 1, 2, 2, 8 and 8. If a person selects a set of two balls at random, what is the expected value of the sum of the numbers on the balls? set w/o permutation

Expected value: $\sum_{k=1}^{n} a_k p_k = a_1p_1 + a_2p_2 + a_3p_3 + \dots + a_np_n$



$1+2 = 3$

$2+2 = 4$

$2+8 = 10$

$8+8 = 16$

$8+1 = 9$

$$= 3 \cdot \left(\frac{1}{5} \cdot \frac{2}{4} + \frac{2}{5} \cdot \frac{1}{4}\right)$$
$$+ 4 \cdot \left(\frac{2}{5} \cdot \frac{1}{4}\right) \cdot \cancel{2} \rightarrow$$ no need to permutation, it's the same?
$$+ 10 \cdot \left(\frac{2}{5} \cdot \frac{2}{4} + \frac{2}{5} \cdot \frac{2}{4}\right)$$
$$+ 16 \left(\frac{2}{5} \cdot \frac{1}{4}\right) \cdot \cancel{2}$$
$$+ 9 \left(\frac{1}{5} \cdot \frac{2}{4} + \frac{2}{5} \cdot \frac{1}{4}\right)$$
$$= 8.4$$

8. One urn contains 10 red balls and 25 green balls, and a second urn contains 22 red balls and 15 green balls. A ball is chosen as follows: First an urn is selected by tossing a loaded coin with probability 0.4 of landing heads up and probability of 0.6 of landing tails up. If the coin lands heads up, the first urn is chosen; otherwise, the second urn is chosen. Then a ball is picked at random from the chosen urn.

Write your answers correct to three significant figures.

(a) What is the probability that the chosen ball is green? ✓

(b) If the chosen ball is green, what is the probability that it was picked from the first urn? what we want to find.

given that

10R 25G   22R 15G.

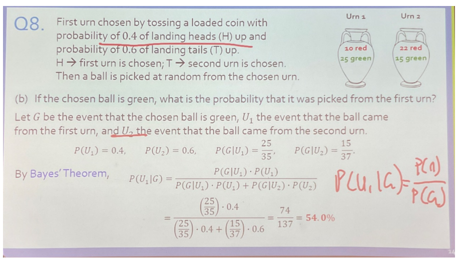
0.4 H   0.6 T

a) $0.4 \times \frac{25}{35} + 0.6 \times \frac{15}{22+15}$

$= 0.529$. or $\frac{137}{259}$.

b) $0.4 \times (a)$. $\times \frac{25}{35}$. $= 0.540$.

$$P(1 \mid G) = \frac{P(1 \cap G)}{P(G)} = \frac{0.4 \times \frac{25}{35}}{(a)}$$



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   Write your answers correct to three significant figures.

   (a) What is the probability that the chosen ball is green?

   (b) If the chosen ball is green, what is the probability that it was picked from the first urn?

9. [AY2015/16 Semester 1 Exam Question]

   Let $A = \{1, 2, 3, 4\}$. Since each element of $P(A \times A)$ is a subset of $A \times A$, it is a binary relation on A. ($P(S)$ denotes the powerset of S.)

   Assuming each relation in $P(A \times A)$ is equally likely to be chosen, what is the probability that a randomly chosen relation is (a) reflexive? (b) symmetric?

   Can you generalize your answer to any set A with n elements?

10. Let's revisit Tutorial #9 Question 1:

    In a certain tournament, the first team to win four games wins the tournament. Suppose there are two teams A and B, and team A wins the first two games. How many ways can the tournament be completed?

    The solution given last week uses a possibility tree to depict the 15 ways. Now, let's approach this problem using combination.

    Let us define a function $W(a, b)$ to be the number of ways the tournament can be completed if team A has to win a more games to win, while team B has to win b more games to win. Hence,

$$W(a,b) = \begin{cases} 1, & \text{if } a = 0 \text{ or } b = 0; \\ W(a, b-1) + W(a-1, b), & \text{if a} > 0 \text{ and } b > 0. \end{cases}$$

    We may express $W(a, b)$ as a simple combination formula as follows:

$$W(a,b) = \binom{a+b}{a}.$$

    Verify the above.

    Now, we denote the function $T(n, k)$ to be the number of ways the tournament can be completed, given that the first team to win n games wins the tournament, and team A wins the first k ($k \leq n$) games.

    Derive a simple combination formula for $T(n, k)$ (hint: relate function T to function W), and hence solve $T(4, 2)$ which is the problem in Tutorial #9 Question 1.

9. [AY2015/16 Semester 1 Exam Question]

subset of $A \times A \in P(A \times A)$.

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$$A \times A = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), (4,4)\} \quad 16.$$

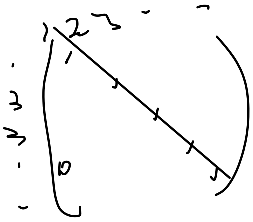
$$\frac{2^{16}-2}{2^{16}} \qquad \frac{2^{n^2-n}}{2^{n^2}}$$

$$|P(A \times A)| = 2^{16}$$

$$= \{\emptyset, \{(1,1)\}, \ldots \quad (2^{n^2-n})$$
$$\{(1,1), (1,2)\}, \ldots\}.$$

a) Reflexive: (a,a)

$$p = \frac{2^{12}}{2^{16}}.$$



b) Symmetric: $xRy \to yRx$

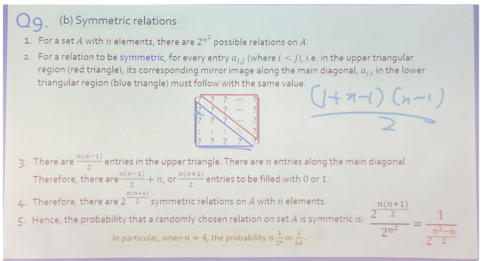
$$c = \frac{n^2-n}{2} + n$$

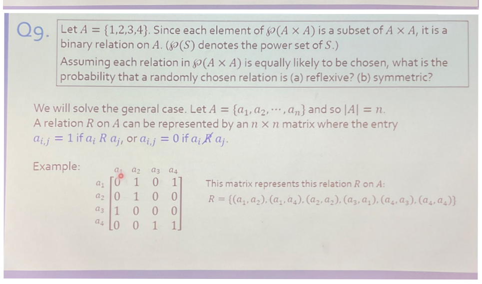
n is diagonal $= 4$.

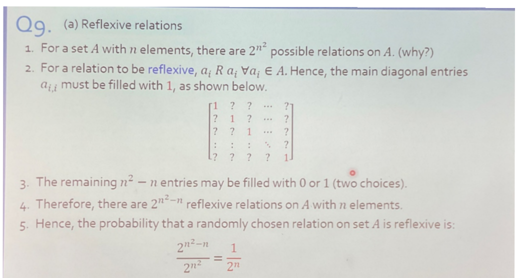
General form:

$$\frac{1}{2^{\frac{n^2-n}{2}}}.$$

$$= \frac{2^{\frac{4^2-4}{2}+4}}{2^{16}} = \frac{2^{10}}{2^{14}}.$$







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$$= \binom{a+b}{a}$$

$W(a, b-1-1)$

$b-1-1-1 \ldots$

if $a=0$, $= \binom{b}{0} = 1$

if $b=0$ $= \binom{a}{a} = 1$

commutative

else $= \binom{a+(b-1)}{a} + \binom{(a-1)+b}{a} = \binom{a+b}{a}$

$\binom{a+b-1}{a-1} + \binom{a+b-1}{a}$

pascal's formula

$$T(n,k) = W(n-k, n)$$

$$= \binom{n-k+n}{n-k}$$

$$= \binom{2n-k}{n-k} = \cancel{\binom{2n-k}{n}}$$

$$\therefore T(4,2) = \binom{2\cdot 4-2}{4-2}$$

$$= \binom{6}{2}$$

$$= {}^6C_2.$$