<u>Answers/Solutions of Exercise 6</u> (Q9-30) (Version: November 7, 2014)

9. (a) Diagonalizable. Let $\boldsymbol{P} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$.

   (b) Not diagonalizable.

   (c) Diagonalizable. Let $\boldsymbol{P} = \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$.

   (d) Diagonalizable. Let $\boldsymbol{P} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

   (e) Not diagonalizable.

   (f) Diagonalizable. Let $\boldsymbol{P} = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 3 & 3 \\ 1 & 0 & 0 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

   (g) Not diagonalizable.

   (h) Diagonalizable. Let $\boldsymbol{P} = \begin{pmatrix} -1 & 1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

   (i) Diagonalizable. Let $\boldsymbol{P} = \begin{pmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 9 \\ 0 & 1 & 4 & 12 \\ 1 & 1 & 3 & 8 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$.

   (j) Diagonalizable. Let $\boldsymbol{P} = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

10. (a) Eigenvalues are $-i$ and i.

   Let $\boldsymbol{P} = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$.

   (b) Eigenvalues are $2-i$ and $2+i$.

   Let $\boldsymbol{P} = \begin{pmatrix} 1+i & 1-i \\ 2 & 2 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 2-i & 0 \\ 0 & 2+i \end{pmatrix}$.

   (c) Eigenvalues are 0, $2-i$ and $2+i$.

   Let $\boldsymbol{P} = \begin{pmatrix} 1 & 1+3i & 1-3i \\ 0 & 5i & -5i \\ 0 & 5 & 5 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2-i & 0 \\ 0 & 0 & 2+i \end{pmatrix}$.

11. (a) Let $\boldsymbol{P} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

(b) $\boldsymbol{A}^{10} = \begin{pmatrix} 1 & 0 & 4^{10}-1 \\ 0 & 4^{10} & 0 \\ 0 & 0 & 4^{10} \end{pmatrix}$

(c) For example, let $\boldsymbol{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and $\boldsymbol{B} = \boldsymbol{P}\boldsymbol{C}\boldsymbol{P}^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. Then $\boldsymbol{B}^2 = \boldsymbol{A}$.

12. Let $\boldsymbol{P} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$ and $\boldsymbol{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Then the matrix $\boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1} = \begin{pmatrix} -1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ has the required eigenvalues and eigenvectors.

13. The matrix is diagonalizable if and only if $a \neq b$.

14. (a) The eigenvalues are 2, 0, 1 and -1.

(b) $\boldsymbol{u_1}$ is an eigenvector associated with 2.
$\boldsymbol{u_2}$ is an eigenvector associated with 0.
$\boldsymbol{u_3} + \boldsymbol{u_4}$ is an eigenvector associated with 1.
$\boldsymbol{u_3} - \boldsymbol{u_4}$ is an eigenvector associated with -1.

(c) Note that $\boldsymbol{u_1}$, $\boldsymbol{u_2}$, $\boldsymbol{u_3}$, $\boldsymbol{u_3}+\boldsymbol{u_4}$, $\boldsymbol{u_3}+\boldsymbol{u_4}$ are linearly independent eigenvectors. By Theorem 6.2.3, $\boldsymbol{B}$ is diagonalizable.

Alternatively Solution: Since $\boldsymbol{B}$ has 4 distinct eigenvalues, by Theorem 6.2.7, $\boldsymbol{B}$ is diagonalizable.

15. (a) (i) $\boldsymbol{B} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} \Rightarrow \boldsymbol{B}^n = \underbrace{(\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P})(\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P})\cdots(\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P})}_{n \text{ times}} = \boldsymbol{P}^{-1}\boldsymbol{A}^n\boldsymbol{P}$

So $\boldsymbol{A}^n$ is similar to $\boldsymbol{B}^n$.

(ii) $\boldsymbol{B} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} \Rightarrow \boldsymbol{B}^{-1} = (\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P})^{-1} = \boldsymbol{P}^{-1}\boldsymbol{A}^{-1}\boldsymbol{P}$
So $\boldsymbol{A}^{-1}$ is similar to $\boldsymbol{B}^{-1}$.

(iii) Suppose there exists an invertible matrix $\boldsymbol{Q}$ such that $\boldsymbol{Q}^{-1}\boldsymbol{A}\boldsymbol{Q}$ is a diagonal matrix. Let $\boldsymbol{R} = \boldsymbol{P}^{-1}\boldsymbol{Q}$. Then $\boldsymbol{R}$ is invertible and $\boldsymbol{R}^{-1}\boldsymbol{B}\boldsymbol{R} = \boldsymbol{Q}^{-1}\boldsymbol{P}\boldsymbol{B}\boldsymbol{P}^{-1}\boldsymbol{Q} = \boldsymbol{Q}^{-1}\boldsymbol{A}\boldsymbol{Q}$ is a diagonal matrix.

(b) Since $\boldsymbol{A}$ is a triangular matrix, its eigenvalues are 0, 1 and -1. Also it is easy to find from the characteristic equation of $\boldsymbol{B}$ that the eigenvalues of $\boldsymbol{B}$ are 0, 1 and -1. By Theorem 6.2.7, both $\boldsymbol{A}$ and $\boldsymbol{B}$ are diagonalizable. So there exist invertible matrices $\boldsymbol{R}$ and $\boldsymbol{Q}$ such that

$$\boldsymbol{R}^{-1}\boldsymbol{A}\boldsymbol{R} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \boldsymbol{Q}^{-1}\boldsymbol{B}\boldsymbol{Q}.$$

Let $\boldsymbol{P} = \boldsymbol{R}\boldsymbol{Q}^{-1}$. Then $\boldsymbol{P}$ is invertible matrix and $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{Q}\boldsymbol{R}^{-1}\boldsymbol{A}\boldsymbol{R}\boldsymbol{Q}^{-1} = \boldsymbol{B}$.

16. (a) Let $\boldsymbol{A} = (a_{ij})_{n\times n}$. Then $a_{1i} + a_{2i} + \cdots + a_{ni} = 1$ for $i = 1, 2, \ldots, n$.

(i) $$\boldsymbol{A}^{\mathrm{T}} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} a_{11} + a_{21} + \cdots + a_{n1} \\ a_{12} + a_{22} + \cdots + a_{n2} \\ \vdots \\ a_{1n} + a_{2n} + \cdots + a_{nn} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Thus 1 is an eigenvalue of $\boldsymbol{A}^{\mathrm{T}}$. By 3c, 1 is an eigenvalue of $\boldsymbol{A}$.

(ii) By 3c, λ is an eigenvalue of $\boldsymbol{A}^{\mathrm{T}}$.

Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ be a eigenvector of $\boldsymbol{A}^{\mathrm{T}}$ associated with the eigenvalue λ, i.e. $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{x} = \lambda\boldsymbol{x}$. Choose $k \in \{1, 2, \ldots, n\}$ such that $|x_k| = \max\{\, |x_i| \mid i = 1, 2, \ldots, n \,\}$, i.e. $|x_k| \geq |x_i|$ for $i = 1, 2, \ldots, n$. Since $\boldsymbol{x}$ is a nonzero vector, $|x_k| > 0$.

By comparing the kth coordinate of both sides of $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{x} = \lambda\boldsymbol{x}$, we have

$$\begin{aligned} & a_{1k}x_1 + a_{2k}x_2 + \cdots + a_{nk}x_n = \lambda x_k \\ \Rightarrow\ & |\lambda|\,|x_k| = |a_{1k}x_1 + a_{2k}x_2 + \cdots + a_{nk}x_n| \\ & \qquad \leq |a_{1k}x_1| + |a_{2k}x_2| + \cdots + |a_{nk}x_n| \\ & \qquad \leq a_{1k}|x_1| + a_{2k}|x_2| + \cdots + a_{nk}|x_n| \quad (\because a_{ij} \geq 0 \text{ for all } i, j) \\ & \qquad \leq (a_{1k} + a_{2k} + \cdots + a_{nk})|x_k| \\ & \qquad = |x_k| \\ \Rightarrow\ & |\lambda| \leq 1. \end{aligned}$$

(b) (i) Yes.

(ii) Let $\boldsymbol{P} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.95 & 0 \\ 0 & 0 & 0.9 \end{pmatrix}$.

17. Let a_n (respectively, b_n) be the number of customers who pay late (respectively, early) in month n. Then for $n = 1, 2, \ldots,$

$$\begin{cases} a_n = \frac{1}{2}a_{n-1} + \frac{2}{10}b_{n-1} \\ b_n = \frac{1}{2}a_{n-1} + \frac{8}{10}b_{n-1}. \end{cases}$$

Let $\boldsymbol{x}_n = \begin{pmatrix} a_n \\ b_n \end{pmatrix}$ and $\boldsymbol{A} = \begin{pmatrix} \frac{1}{2} & \frac{1}{5} \\ \frac{1}{2} & \frac{4}{5} \end{pmatrix}$. Then $\boldsymbol{x_n} = \boldsymbol{A}\boldsymbol{x}_{n-1} = \cdots = \boldsymbol{A}^{n-1}\boldsymbol{x_1}$ where $\boldsymbol{x_1} = \begin{pmatrix} 0 \\ 10000 \end{pmatrix}$.

By Algorithm 6.2.4, we find a matrix $\boldsymbol{P} = \begin{pmatrix} 2 & 1 \\ 5 & -1 \end{pmatrix}$ such that $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0.3 \end{pmatrix}$. Then

$$\boldsymbol{x_n} = \boldsymbol{P} \begin{pmatrix} 1 & 0 \\ 0 & 0.3^{n-1} \end{pmatrix} \boldsymbol{P}^{-1}\boldsymbol{x_1} = \frac{10000}{7} \begin{pmatrix} 2 - 2(0.3)^{n-1} \\ 5 + 2(0.3)^{n-1} \end{pmatrix}.$$

So the number of customers that will pay on time in April is $b_4 = \frac{10000}{7}[5 + 2(0.3)^3] = 7220$.

The number of customers that will pay on time will stabilize in the long run and $\lim\limits_{n\to\infty} b_n = \frac{50000}{7} \approx 7143$.

18. Let a_n, b_n and c_n be the percentage of customers choosing brand A, B and C, respectively, after n months. Then for $n = 1, 2, ...,$

$$\begin{cases} a_n = 0.97a_{n-1} + 0.01b_{n-1} + 0.02c_{n-1} \\ b_n = 0.01a_{n-1} + 0.97b_{n-1} + 0.02c_{n-1} \\ c_n = 0.02a_{n-1} + 0.02b_{n-1} + 0.96c_{n-1}. \end{cases}$$

Let $\boldsymbol{x_n} = \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}$ and $\boldsymbol{A} = \begin{pmatrix} 0.97 & 0.01 & 0.02 \\ 0.01 & 0.97 & 0.02 \\ 0.02 & 0.02 & 0.96 \end{pmatrix}$.

Then $\boldsymbol{x_n} = \boldsymbol{A}\boldsymbol{x}_{n-1} = \cdots = \boldsymbol{A}^n\boldsymbol{x_0}$ where $\boldsymbol{x_0} = \begin{pmatrix} 100 \\ 0 \\ 0 \end{pmatrix}$.

By Algorithm 6.2.4, we find $\boldsymbol{P} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & 0 & 2 \end{pmatrix}$ such that $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.96 & 0 \\ 0 & 0 & 0.94 \end{pmatrix}$.

Then

$$\boldsymbol{x_n} = \boldsymbol{P}\begin{pmatrix}1 & 0 & 0\\ 0 & 0.96^n & 0\\ 0 & 0 & 0.94^n\end{pmatrix}\boldsymbol{P}^{-1}\boldsymbol{x_0} = \frac{50}{3}\begin{pmatrix}2+3\cdot 0.96^n + 0.94^n\\ 2-3\cdot 0.96^n + 0.94^n\\ 2-2\cdot 0.94^n\end{pmatrix}.$$

The present market shares are $\frac{50}{3}[2+3\cdot 0.96^4+0.94^4]\% \approx 88.8\%$, $\frac{50}{3}[2-3\cdot 0.96^4+0.94^4]\% \approx 3.9\%$ and $\frac{50}{3}[2-2\cdot 0.94^4]\% \approx 7.3\%$ for brand A, B and C, respectively.

The market shares will stabilize after a long run and $\lim\limits_{n\to\infty}\boldsymbol{x_n} = \begin{pmatrix}\frac{100}{3}\\ \frac{100}{3}\\ \frac{100}{3}\end{pmatrix}$.

19. Note that $e^x = 1 + \dfrac{1}{1!}x + \dfrac{1}{2!}x^2 + \cdots = \displaystyle\sum_{n=0}^{\infty}\frac{1}{n!}x^n$ for $x \in \mathbb{R}$.

(a) Since $\boldsymbol{A}^n = \begin{pmatrix}1 & 0 & 0\\ 0 & 2^n & 0\\ 0 & 0 & 3^n\end{pmatrix}$ for $n = 1, 2, \ldots,$

$$e^{\boldsymbol{A}} = \begin{pmatrix}1+\frac{1}{1!}+\frac{1}{2!}+\cdots & 0 & 0\\ 0 & 1+\frac{1}{1!}2+\frac{1}{2!}2^2+\cdots & 0\\ 0 & 0 & 1+\frac{1}{1!}3+\frac{1}{2!}3^2+\cdots\end{pmatrix} = \begin{pmatrix}e & 0 & 0\\ 0 & e^2 & 0\\ 0 & 0 & e^3\end{pmatrix}.$$

(b) Let $\boldsymbol{P} = \begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix}2 & 0\\ 0 & 4\end{pmatrix}$. Since $\boldsymbol{A}^n = \boldsymbol{P}\begin{pmatrix}2^n & 0\\ 0 & 4^n\end{pmatrix}\boldsymbol{P}^{-1}$ for $n = 1, 2, \ldots,$

$$e^{\boldsymbol{A}} = \boldsymbol{P}\begin{pmatrix}1+\frac{1}{1!}2+\frac{1}{2!}2^2+\cdots & 0\\ 0 & 1+\frac{1}{1!}4+\frac{1}{2!}4^2+\cdots\end{pmatrix}\boldsymbol{P}^{-1} = \frac{1}{2}\begin{pmatrix}e^4+e^2 & e^4-e^2\\ e^4-e^2 & e^4+e^2\end{pmatrix}.$$

(c) Let $\boldsymbol{P} = \begin{pmatrix}-1 & 1 & 1\\ -1 & 2 & 0\\ 1 & 0 & 2\end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$. Since $\boldsymbol{A}^n =$

$$\boldsymbol{P}\begin{pmatrix}(-1)^n & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\boldsymbol{P}^{-1} \text{ for } n = 1, 2, \ldots,$$

$$\begin{aligned} e^{\boldsymbol{A}} &= \boldsymbol{P}\begin{pmatrix}1 - \frac{1}{1!} + \frac{1}{2!} - \cdots & 0 & 0\\ 0 & 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots & 0\\ 0 & 0 & 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots\end{pmatrix}\boldsymbol{P}^{-1}\\ &= \begin{pmatrix} e^{-1} & \frac{1}{2}(e - e^{-1}) & \frac{1}{2}(e - e^{-1})\\ -e + e^{-1} & \frac{1}{2}(3e - e^{-1} & \frac{1}{2}(e - e^{-1})\\ e - e^{-1} & \frac{1}{2}(-e + e^{-1}) & \frac{1}{2}(e + e^{-1})\end{pmatrix}.\end{aligned}$$

20. In the following, we use the procedure discussed in Example 6.2.11.2.

(a) Let $\boldsymbol{x_n} = \begin{pmatrix} a_n \\ a_{n+1}\end{pmatrix}$ and $\boldsymbol{A} = \begin{pmatrix} 0 & 1\\ -2 & 3\end{pmatrix}$. Then $\boldsymbol{x_n} = \boldsymbol{A}\boldsymbol{x_{n-1}} = \cdots = \boldsymbol{A}^n\boldsymbol{x_0}$.
Let $\boldsymbol{P} = \begin{pmatrix} 1 & 1\\ 2 & 1\end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 2 & 0\\ 0 & 1\end{pmatrix}$. Thus

$$\begin{aligned}\begin{pmatrix} a_n\\ a_{n+1}\end{pmatrix} = \boldsymbol{x_n} &= \boldsymbol{P}\begin{pmatrix} 2^n & 0\\ 0 & 1\end{pmatrix}\boldsymbol{P}^{-1}\boldsymbol{x_0}\\ &= \begin{pmatrix} 1 & 1\\ 2 & 1\end{pmatrix}\begin{pmatrix} 2^n & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 1\\ 2 & 1\end{pmatrix}^{-1}\begin{pmatrix} 0\\ 1\end{pmatrix} = \begin{pmatrix} 2^n - 1\\ 2^{n+1} - 1\end{pmatrix}.\end{aligned}$$

Thus $a_n = 2^n - 1$.

(b) Let $\boldsymbol{x_n} = \begin{pmatrix} a_n \\ a_{n+1}\end{pmatrix}$ and $\boldsymbol{A} = \begin{pmatrix} 0 & 1\\ 2 & 1\end{pmatrix}$. Then $\boldsymbol{x_n} = \boldsymbol{A}\boldsymbol{x_{n-1}} = \cdots = \boldsymbol{A}^n\boldsymbol{x_0}$.
Let $\boldsymbol{P} = \begin{pmatrix} 1 & -1\\ 2 & 1\end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 2 & 0\\ 0 & -1\end{pmatrix}$. Thus

$$\begin{aligned}\begin{pmatrix} a_n\\ a_{n+1}\end{pmatrix} = \boldsymbol{x_n} &= \boldsymbol{P}\begin{pmatrix} 2^n & 0\\ 0 & (-1)^n\end{pmatrix}\boldsymbol{P}^{-1}\boldsymbol{x_0}\\ &= \begin{pmatrix} 1 & -1\\ 2 & 1\end{pmatrix}\begin{pmatrix} 2^n & 0\\ 0 & (-1)^n\end{pmatrix}\begin{pmatrix} 1 & -1\\ 2 & 1\end{pmatrix}^{-1}\begin{pmatrix} 1\\ 0\end{pmatrix} = \begin{pmatrix} \frac{1}{3}[2^n + 2(-1)^n]\\ \frac{1}{3}[2^{n+1} - 2(-1)^n]\end{pmatrix}.\end{aligned}$$

Thus $a_n = \frac{1}{3}[2^n + 2(-1)^n]$.

21. Use cofactor expansion along the first row:

$$d_n = \begin{vmatrix} 3 & 1 & & & & 0 \\ 1 & 3 & 1 & & & \\ & 1 & 3 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 3 & 1 \\ 0 & & & & 1 & 3 \end{vmatrix}_{n\times n}$$

$$= 3\begin{vmatrix} 3 & 1 & & 0 \\ 1 & 3 & \ddots & \\ & 1 & \ddots & \ddots & \\ & & \ddots & 3 & 1 \\ 0 & & & 1 & 3 \end{vmatrix}_{(n-1)\times(n-1)} - \begin{vmatrix} 1 & 1 & & 0 \\ 0 & 3 & \ddots & \\ & 1 & \ddots & \ddots & \\ & & \ddots & 3 & 1 \\ 0 & & & 1 & 3 \end{vmatrix}_{(n-1)\times(n-1)}.$$

The first determinant above is d_{n-1}. By using cofactor expansion along the first column, we find that the second determinant is d_{n-2}. So

$$d_n = 3d_{n-1} - d_{n-2}.$$

Note that $d_1 = 3$ and $d_2 = \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} = 8$.

By the procedure discussed in Example 6.2.11.2, we obtain

$$d_n = \left(\frac{5+3\sqrt{5}}{10}\right)\left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{5-3\sqrt{5}}{10}\right)\left(\frac{3-\sqrt{5}}{2}\right)^n.$$

22. Consider the vector equation

$$a_1\boldsymbol{u_1} + a_2\boldsymbol{u_2} + \cdots + a_m\boldsymbol{u_m} + b_1\boldsymbol{v_1} + b_2\boldsymbol{v_2} + \cdots + b_p\boldsymbol{v_p} = \mathbf{0}. \tag{1}$$

Pre-multiplying $\boldsymbol{A}$ to both side of (1), we have

$$a_1\lambda_1\boldsymbol{u_1} + a_2\lambda_2\boldsymbol{u_2} + \cdots + a_m\lambda_m\boldsymbol{u_m} + b_1\mu\boldsymbol{v_1} + b_2\mu\boldsymbol{v_2} + \cdots + b_p\mu\boldsymbol{v_p} = \mathbf{0}. \tag{2}$$

Subtracting (2) by μ times of (1), we obtain

$$a_1(\lambda_1-\mu)\boldsymbol{u_1} + a_2(\lambda_2-\mu)\boldsymbol{u_2} + \cdots + a_m(\lambda_m-\mu)\boldsymbol{u_m} = \mathbf{0}.$$

Since $\boldsymbol{u_1}$, $\boldsymbol{u_2}$, $\dots$, $\boldsymbol{u_m}$ are linearly independent, $a_1(\lambda_1-\mu)=0$, $a_2(\lambda_2-\mu)=0$, $\dots$, $a_m(\lambda_m-\mu)=0$. As $\lambda_i \neq \mu$ for $i = 1, 2, \dots, m$, we have $a_1 = 0$, $a_2 = 0$, $\dots$, $a_m = 0$.

Substituting $a_1 = 0$, $a_2 = 0$, ..., $a_m = 0$ into (2), we have

$$b_1\boldsymbol{v_1} + b_2\boldsymbol{v_2} + \cdots + b_p\boldsymbol{v_p} = \boldsymbol{0}.$$

Since $\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_p}$ are linearly independent, $b_1 = 0$, $b_2 = 0$, ..., $b_p = 0$.

We have shown that the vector equation (1) has only the trivial solution. Thus $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_m}, \boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_p}\}$ is linearly independent.

23. (a) True. Let $\boldsymbol{P}$ be an invertible matrix that diagonalizes $\boldsymbol{A}$, i.e. $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{D}$ where $\boldsymbol{D}$ is a diagonalizable matrix. Then

$$\boldsymbol{D} = \boldsymbol{D}^{\mathrm{T}} = (\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P})^{\mathrm{T}} = \boldsymbol{P}^{T}\boldsymbol{A}^{\mathrm{T}}(\boldsymbol{P}^{-1})^{\mathrm{T}} = \boldsymbol{P}^{T}\boldsymbol{A}^{\mathrm{T}}(\boldsymbol{P}^{\mathrm{T}})^{-1}.$$

Thus the matrix $(\boldsymbol{P}^{\mathrm{T}})^{-1}$ diagonalizes $\boldsymbol{A}^{\mathrm{T}}$.

(b) False. For example, $\boldsymbol{A} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$ are both diagonalizable but $\boldsymbol{A} + \boldsymbol{B} = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$ is not diagonalizable.

(c) False. For example, $\boldsymbol{A} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ are both diagonalizable but $\boldsymbol{AB} = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$ is not diagonalizable.

24. (a) Let $\boldsymbol{P} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$. Then $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$.

(b) Let $\boldsymbol{P} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$. Then $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} -2 & 0 \\ 0 & 4 \end{pmatrix}$.

(c) Let $\boldsymbol{P} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$. Then $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2+\sqrt{2} & 0 \\ 0 & 0 & 2-\sqrt{2} \end{pmatrix}$.

(d) Let $\boldsymbol{P} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$. Then $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

(e) Let $\boldsymbol{P} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix}$. Then $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$.

(f) Let $\boldsymbol{P} = \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$. Then $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$.

(g) Let $\boldsymbol{P} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$. Then $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

(h) Let $\boldsymbol{P} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{12}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{12}} & \frac{1}{2} \\ 0 & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{12}} & \frac{1}{2} \\ 0 & 0 & \frac{3}{\sqrt{12}} & \frac{1}{2} \end{pmatrix}$. Then $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$.

25. (a) Since $(\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})^{\mathrm{T}} = \boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$, $\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$ is symmetric. Hence $\boldsymbol{I} - \boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$ is also symmetric and thus is orthogonally diagonalizable.

(b) When $\boldsymbol{u} = (1, -1, 1)^{\mathrm{T}}$, $\boldsymbol{I} - \boldsymbol{u}\boldsymbol{u}^{\mathrm{T}} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$.

Let $\boldsymbol{P} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}$. Then $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$.

26. By the given conditions, we have $\boldsymbol{A}^{\mathrm{T}} = \boldsymbol{A}$, $\boldsymbol{A}\boldsymbol{u} = \lambda\boldsymbol{u}$ and $\boldsymbol{A}\boldsymbol{v} = \mu\boldsymbol{v}$. We compute $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{u}$ in two ways:

$$\begin{aligned} \boldsymbol{v}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{u} &= \boldsymbol{v}^{\mathrm{T}}(\lambda\boldsymbol{u}) = \lambda\,\boldsymbol{v}^{\mathrm{T}}\boldsymbol{u} = \lambda(\boldsymbol{v}\cdot\boldsymbol{u}), \\ \boldsymbol{v}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{u} &= \boldsymbol{v}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{u} = (\boldsymbol{A}\boldsymbol{v})^{\mathrm{T}}\boldsymbol{u} = (\mu\boldsymbol{v})\boldsymbol{u} = \mu\,\boldsymbol{v}^{\mathrm{T}}\boldsymbol{u} = \mu(\boldsymbol{v}\cdot\boldsymbol{u}). \end{aligned}$$

Thus $\lambda(\boldsymbol{v}\cdot\boldsymbol{u}) = \mu(\boldsymbol{v}\cdot\boldsymbol{u})$ which implies $(\lambda - \mu)(\boldsymbol{v}\cdot\boldsymbol{u}) = 0$. Since $\lambda \neq \mu$, we have $\boldsymbol{v}\cdot\boldsymbol{u} = 0$.

27. Since

$$E_1 = \{(x, y, z)^{\mathrm{T}} \mid x + y - z = 0\} = \mathrm{span}\{(-1, 1, 0)^{\mathrm{T}},\ (1, 0, 1)^{\mathrm{T}}\},$$

$\{(-1, 1, 0)^{\mathrm{T}},\ (1, 0, 1)^{\mathrm{T}}\}$ is a basis for E_1.

Let $\boldsymbol{u}$ be an eigenvector associated with -1. Since $\boldsymbol{A}$ is symmetric, by Question 6.26, $\boldsymbol{u}$ is orthogonal to E_1, i.e. $\boldsymbol{u}$ is perpendicular to $x + y - z = 0$. Hence $\boldsymbol{u}$ is a scalar multiple of $(1, 1, -1)^{\mathrm{T}}$. This means

$$E_{-1} = \mathrm{span}\{(1, 1, -1)^{\mathrm{T}}\}$$

and $\{(1, 1, -1)^{\mathrm{T}}\}$ is a basis for E_{-1}.

Let $\boldsymbol{P} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{AP} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Hence

$$\boldsymbol{A} = \boldsymbol{P} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \boldsymbol{P}^{-1} = \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix}.$$

28. Suppose the eigenvalues associated with the eigenspaces span$\{(1,0,1,0)^{\mathrm{T}}, (1,1,1,1)^{\mathrm{T}}\}$ and span$\{(1,1,-1,-1)^{\mathrm{T}}, (1,-1,-1,1)^{\mathrm{T}}\}$ are λ and μ respectively.

Let $\boldsymbol{P} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 0 & 1 & -1 & 1 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{PA} = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix}$. So

$$\boldsymbol{A} = \boldsymbol{P} \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix} \boldsymbol{P}^{-1} = \begin{pmatrix} \frac{1}{2}(\lambda+\mu) & 0 & \frac{1}{2}(\lambda-\mu) & 0 \\ 0 & \frac{1}{2}(\lambda+\mu) & 0 & \frac{1}{2}(\lambda-\mu) \\ \frac{1}{2}(\lambda-\mu) & 0 & \frac{1}{2}(\lambda+\mu) & 0 \\ 0 & \frac{1}{2}(\lambda-\mu) & 0 & \frac{1}{2}(\lambda+\mu) \end{pmatrix}$$

which is a symmetric matrix.

Alternative Solution: Since

$$\begin{aligned} (1,0,1,0) \cdot (1,1,-1,-1) &= 0, \\ (1,0,1,0) \cdot (1,-1,-1,1) &= 0, \\ (1,1,1,1) \cdot (1,1,-1,-1) &= 0, \\ (1,1,1,1) \cdot (1,-1,-1,1) &= 0, \end{aligned}$$

any vector from span$\{(1,0,1,0)^{\mathrm{T}}, (1,1,1,1)^{\mathrm{T}}\}$ is orthogonal to any vector from span$\{(1,1,-1,-1)^{\mathrm{T}}, (1,-1,-1,1)^{\mathrm{T}}\}$.

Take any orthonormal bases $\{\boldsymbol{u_1}, \boldsymbol{u_2}\}$ and $\{\boldsymbol{v_1}, \boldsymbol{v_2}\}$ for span$\{(1,0,1,0)^{\mathrm{T}}, (1,1,1,1)^{\mathrm{T}}\}$ and span$\{(1,1,-1,-1)^{\mathrm{T}}, (1,-1,-1,1)^{\mathrm{T}}\}$ respectively. By the observation above, $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{v_1}, \boldsymbol{v_2}\}$ is orthonormal. Let $\boldsymbol{P} = \begin{pmatrix} \boldsymbol{u_1} & \boldsymbol{u_2} & \boldsymbol{v_1} & \boldsymbol{v_2} \end{pmatrix}$. Then $\boldsymbol{P}$ is an orthogonal matrix that diagonalizes $\boldsymbol{A}$. By Theorem 6.3.4, $\boldsymbol{A}$ is symmetric.

29. (a) Since $\boldsymbol{Au} = 4\boldsymbol{u}$, $\boldsymbol{u}$ is an eigenvector of $\boldsymbol{A}$ associated with the eigenvalue 4.

(b) $\boldsymbol{v} \cdot \boldsymbol{u} = 0 \quad \Rightarrow \quad a + b + c + d = 0$.

Thus $\boldsymbol{Av} = \boldsymbol{0} = 0\boldsymbol{v}$, $\boldsymbol{v}$ is an eigenvector of $\boldsymbol{A}$ associated with the eigenvalue 0.

(c) Since $\boldsymbol{P}$ is an orthogonal matrix, $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \cdot (a_i, b_i, c_i, d_i) = 0$ for $i = 1, 2, 3$. By (a), the first column of $\boldsymbol{P}$ is the eigenvector of $\boldsymbol{A}$ associated with the eigenvalue 4. By (b), the other four columns of $\boldsymbol{P}$ are eigenvectors of $\boldsymbol{A}$ associated with the eigenvalue 0. So

$$\boldsymbol{P}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

30. (a) True. Since $\boldsymbol{A}$ and $\boldsymbol{B}$ are orthogonally diagonalizable, they are both symmetric. Then $\boldsymbol{A} + \boldsymbol{B}$ is also symmetric and hence orthogonally diagonalizable.

(b) False. For example, $\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are both orthogonally diagonalizable but $\boldsymbol{AB} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is not orthogonally diagonalizable.