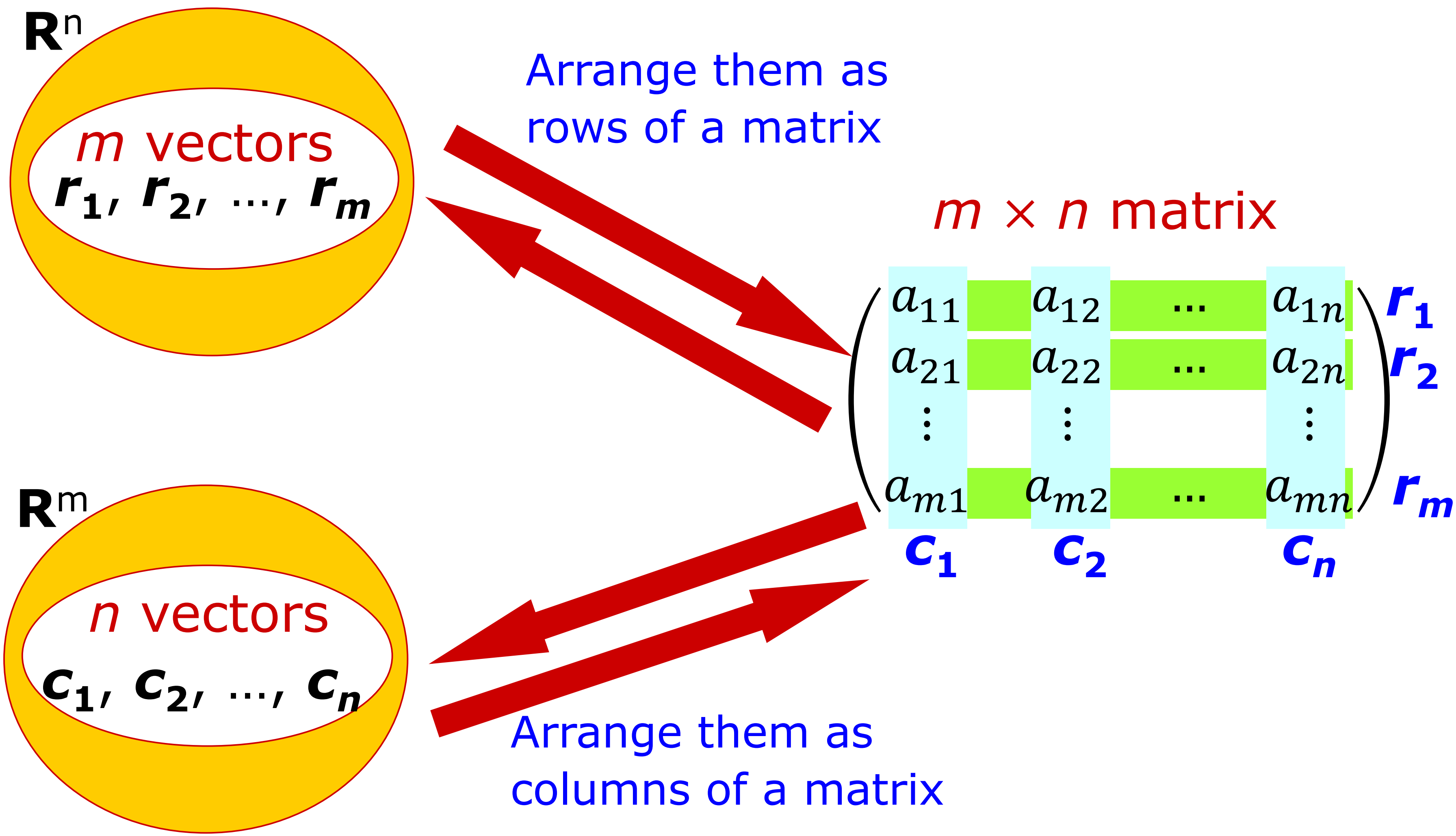
Section 4.1

Row Spaces and Column Spaces

Objectives

- What are row space and column space of a matrix?
- How to find bases for row /column spaces?
- How to use row /column spaces to find bases for vector spaces?
- How to extend a basis?
- What is the relation between column space and consistency of linear system?

Discussion 4.1.1



Example 4.1.4.1

$$\boldsymbol{A} = \begin{pmatrix} 2 & -1 & 0 \\ 1 & -1 & 3 \\ -5 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

rows of $\boldsymbol{A}$

$\boldsymbol{r_1} = (2, -1, 0)$
$\boldsymbol{r_2} = (1, -1, 3)$
$\boldsymbol{r_3} = (-5, 1, 0)$
$\boldsymbol{r_4} = (1, 0, 1)$

We call it the row space of $\boldsymbol{A}$

span{(2, −1, 0), (1, −1, 3), (−5, 1, 0), (1, 0, 1)} a subspace of $\mathbf{R}^3$

columns of $\boldsymbol{A}$

$$\mathbf{c}_1 = \begin{pmatrix} 2 \\ 1 \\ -5 \\ 1 \end{pmatrix} \quad \mathbf{c}_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{c}_3 = \begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix}$$

We call it the column space of $\boldsymbol{A}$

$$\text{span}\left\{ \begin{pmatrix} 2 \\ 1 \\ -5 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix} \right\}$$ a subspace of $\mathbf{R}^4$

Definition 4.1.2

$$A = \begin{array}{c} \\ \left(\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{array}\right) \\ \begin{array}{cccc} \boldsymbol{c_1} & \boldsymbol{c_2} & & \boldsymbol{c_n} \end{array} \end{array} \begin{array}{c} \boldsymbol{r_1} \\ \boldsymbol{r_2} \\ \\ \boldsymbol{r_m} \\ \\ \end{array}$$

row col

an m x n matrix

row

The row space of $\boldsymbol{A}$ $= \text{span}\{\boldsymbol{r_1}, \boldsymbol{r_2}, \ldots, \boldsymbol{r_m}\}$

$\neq$

$\{\boldsymbol{r_1}, \boldsymbol{r_2}, \ldots, \boldsymbol{r_m}\}$

a subspace of $\mathbf{R}^n$

col.

The column space of $\boldsymbol{A}$ $= \text{span}\{\boldsymbol{c_1}, \boldsymbol{c_2}, \ldots, \boldsymbol{c_n}\}$

$\neq$

$\{\boldsymbol{c_1}, \boldsymbol{c_2}, \ldots, \boldsymbol{c_n}\}$

a subspace of $\mathbf{R}^m$

Row space and column space

Remark 4.1.3

row space of $\boldsymbol{A}$ = column space of $\boldsymbol{A}^T$
column space of $\boldsymbol{A}$ = row space of $\boldsymbol{A}^T$

$$
\boldsymbol{A} = \begin{array}{cc}
\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} & \begin{array}{c} \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \\ \\ \boldsymbol{r}_m \end{array} \\
\begin{array}{cccc} \boldsymbol{c}_1 & \boldsymbol{c}_2 & \quad & \boldsymbol{c}_n \end{array} &
\end{array}
\qquad
\boldsymbol{A}^{\mathrm{T}} = \begin{array}{cc}
\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} & \begin{array}{c} \boldsymbol{c}_1 \\ \boldsymbol{c}_2 \\ \\ \boldsymbol{c}_n \end{array} \\
\begin{array}{cccc} \boldsymbol{r}_1 & \boldsymbol{r}_2 & \quad & \boldsymbol{r}_m \end{array} &
\end{array}
$$

Some special matrices

Row (column) space of zero matrix $\boldsymbol{0}$ = zero space

$$\mathbf{0} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

Row (column) space of n x n identity matrix $\boldsymbol{I}_n$ = $\mathbf{R}^n$

$$\boldsymbol{I}_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

Example 4.1.4.2

$$\boldsymbol{A} = \begin{pmatrix} 2 & -1 & 0 \\ 1 & -1 & 3 \\ -5 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Find a basis and the dimension for the row space

span$\{\boldsymbol{r_1}, \boldsymbol{r_2}, \boldsymbol{r_3}, \boldsymbol{r_4}\}$

basis = $\{\boldsymbol{r_1}, \boldsymbol{r_2}, \boldsymbol{r_3}, \boldsymbol{r_4}\}$?

Find a basis and the dimension for the column space

span$\{\boldsymbol{c_1}, \boldsymbol{c_2}, \boldsymbol{c_3}\}$

basis = $\{\boldsymbol{c_1}, \boldsymbol{c_2}, \boldsymbol{c_3}\}$?

These sets may be linearly dependent

There may be redundant vectors

not necessary

Discussion 4.1.6

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be row equivalent matrices.

$$\boldsymbol{A} \rightarrow \rightarrow \ldots \rightarrow \boldsymbol{B}$$

Row equivalence (r.e.) is an equivalence relation on matrices of the same size

- $\boldsymbol{A}$ is r.e. to itself
- If $\boldsymbol{A}$ is r.e. to $\boldsymbol{B}$, then $\boldsymbol{B}$ is r.e. to $\boldsymbol{A}$
- If $\boldsymbol{A}$ is r.e. to $\boldsymbol{B}$, and $\boldsymbol{B}$ is r.e. to $\boldsymbol{C}$, then $\boldsymbol{A}$ is r.e. to $\boldsymbol{C}$.

If two matrices **M** and **N** (of the same size) have the same reduced row echelon form, then **M** and **N** are row equivalent.

Row equivalent matrices have same row space

Theorem 4.1.7

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be row equivalent matrices.

Then

row space of $\boldsymbol{A}$ = row space of $\boldsymbol{B}$

elementary row operations

change the rows of a matrix

but do not change the row space of a matrix.

Idea of proof

Theorem 4.1.7

Let $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$ be rows of a matrix.

We need to show that

1. $\text{span}\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_i, \ldots, \boldsymbol{a}_n\}$
$= \text{span}\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, c\boldsymbol{a}_i, \ldots, \boldsymbol{a}_n\}$

2. $\text{span}\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_i, \ldots, \boldsymbol{a}_j, \ldots, \boldsymbol{a}_n\}$
$= \text{span}\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_j, \ldots, \boldsymbol{a}_i, \ldots, \boldsymbol{a}_n\}$

3. $\text{span}\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_i, \ldots, \boldsymbol{a}_n\}$
$= \text{span}\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_i + c\boldsymbol{a}_j, \ldots, \boldsymbol{a}_n\}$

Row equivalent matrices have same row space

Example 4.1.8.1

$$\boldsymbol{A} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 4 \\ \frac{1}{2} & 1 & 2 \end{pmatrix} \quad \boldsymbol{B} = \begin{pmatrix} \frac{1}{2} & 1 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{pmatrix} \quad \boldsymbol{C} = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{pmatrix} \quad \boldsymbol{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{A} \xrightarrow{R_1 \leftrightarrow R_3} \boldsymbol{B} \xrightarrow{2R_1} \boldsymbol{C} \xrightarrow{R_1 - R_2} \boldsymbol{D}$$

$\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$ are row equivalent to one another

So their row spaces are all the same

In particular

span{(0, 0, 1), (0, 2, 4), (1/2, 1, 2)} row space of $\boldsymbol{A}$

= span{(1, 0, 0), (0, 2, 4), (0, 0, 1)}. row space of $\boldsymbol{D}$

Finding basis for row space

Example 4.1.8.2

$$A = \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} R = \begin{array}{c} \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ \boxed{0 \quad 0} & \frac{3}{2} & -3 & \frac{3}{2} \\ \boxed{0 \quad 0 \quad 0} & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{array}{c} \boldsymbol{r_1} \\ \boldsymbol{r_2} \\ \boldsymbol{r_3} \\ \end{array} \end{array}$$

row echelon form

The row space of ***A*** **=** The row space of ***R***

span$\{\boldsymbol{r_1}, \boldsymbol{r_2}, \boldsymbol{r_3}, \boldsymbol{0}\}$

$$\text{span}\{(2,2,-1,0,1),\ (0,0,\tfrac{3}{2},-3,\tfrac{3}{2}),\ (0,0,0,3,0)\}$$

The three non-zero rows $\boldsymbol{r_1}, \boldsymbol{r_2}, \boldsymbol{r_3}$ of ***R*** are linearly indep.

So $\{\boldsymbol{r_1}, \boldsymbol{r_2}, \boldsymbol{r_3}\}$ is a basis for the row space of ***A***

Remark 4.1.9

$\boldsymbol{A} \longrightarrow \boldsymbol{R}$ (row-echelon form)

The set of nonzero rows of **R** $\{\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_k\}$

is a basis for the row space of **A**.

spans the row space of **R**

spans the row space of **A**

linearly independent

Note that this basis may not contain the original rows of **A**

Discussion 4.1.10

Can we take the non-zero columns of a row-echelon form to form a basis for the column space?

Is this a basis for the column space of $\boldsymbol{A}$?

$$\boldsymbol{A} = \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \boldsymbol{R} = \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ 0 & 0 & \frac{3}{2} & -3 & \frac{3}{2} \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

not **linearly indep**

BAD NEWS: Row equivalent matrices may have different column spaces

Discussion 4.1.10

Elementary row operations may not preserve the column space of a matrix.

$\boldsymbol{A} \ \rightarrow \ \rightarrow \ \dots \ \rightarrow \ \boldsymbol{B}$

row sp A = row sp B
col. sp A ≠ col. sp B

For example, $\boldsymbol{A} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \boldsymbol{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

A and ***B*** are row equivalent but their column spaces are different.

The column space of **A** = span$\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$

The column space of **B** = span$\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$

GOOD NEWS: Row equivalent matrices preserve linear dependency of the columns

Example 4.1.12.1

$$\boldsymbol{A} = \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \boldsymbol{R} = \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ 0 & 0 & \frac{3}{2} & -3 & \frac{3}{2} \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

1. The 1st, 3rd and 5th columns of ***R*** are linearly dependent.

   Correspondingly,
   the 1st, 3rd and 5th columns of ***A*** are linearly dependent.

GOOD NEWS: Row equivalent matrices preserve linear dependency of the columns

Example 4.1.12.2

$$\boldsymbol{A} = \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \boldsymbol{R} = \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ 0 & 0 & \frac{3}{2} & -3 & \frac{3}{2} \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

2. The 1st, 3rd and 4th columns of $\boldsymbol{R}$ are linearly independent.

   Correspondingly, the 1st, 3rd and 4th columns of $\boldsymbol{A}$ are linearly independent.

Row equivalent matrices preserve linear dependency of the columns

Theorem 4.1.11

$$A \xleftrightarrow{\text{row equivalent}} B$$

column space of $\boldsymbol{A}$ may not be equal column space of $\boldsymbol{B}$

A set of columns of $\boldsymbol{A}$ is linearly independent $\Leftrightarrow$ corresponding columns of $\boldsymbol{B}$ are linearly independent

linearly dependent — linearly dependent

a column of $\boldsymbol{A}$ is redundant — corresponding column of $\boldsymbol{B}$ is redundant

A set of columns of $\boldsymbol{A}$ form a basis for the column space of $\boldsymbol{A}$ $\Leftrightarrow$ corresponding columns of $\boldsymbol{B}$ form a basis for the column space of $\boldsymbol{B}$

Finding basis for column space

Example 4.1.12.2

linearly indep.
pivot columns

$$\boldsymbol{A} = \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \boldsymbol{R} = \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ 0 & 0 & \frac{3}{2} & -3 & \frac{3}{2} \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

non-pivot columns
redundant

The 1st, 3rd and 4th columns of $\boldsymbol{R}$
form a basis for the column space of $\boldsymbol{R}$.

Correspondingly,
the 1st, 3rd and 4th columns of $\boldsymbol{A}$
form a basis for the column space of $\boldsymbol{A}$.

Finding basis for column space

Remark 4.1.13

$$\boldsymbol{A} \longrightarrow \boldsymbol{R} \text{ (row-echelon form)}$$

pivot columns

$\{\boldsymbol{c}_1^*, \boldsymbol{c}_2^*, \ldots, \boldsymbol{c}_k^*\} \longleftarrow \{\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_k\}$

columns of ***A*** that correspond to the pivot columns in ***R***.

basis for the column space of ***A***

basis for the column space of ***R***

may not be basis for the column space of ***A***

Idea of proof of Theorem 4.1.11

Remark

row operations preserve linear relations among columns

$$\boxed{C_1 + C_3 = C_4}$$

$$\begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \\ \end{array}$$
$$\begin{pmatrix} a & \# & b & a+b \\ c & \# & d & c+d \\ e & \# & f & e+f \end{pmatrix}$$

kR_1 Type 1

$$\begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \\ \end{array}$$
$$\begin{pmatrix} ka & \# & kb & k(a+b) \\ c & \# & d & c+d \\ e & \# & f & e+f \end{pmatrix}$$

$R_1 \leftrightarrow R_3$ Type 2

$$\begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \\ \end{array}$$
$$\begin{pmatrix} e & \# & f & e+f \\ c & \# & d & c+d \\ a & \# & b & a+b \end{pmatrix}$$

$R_2 + kR_1$ Type 3

$$\begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \\ \end{array}$$
$$\begin{pmatrix} a & \# & b & a+b \\ c+ka & \# & d+kb & (c+d)+k(a+b) \\ e & \# & f & e+f \end{pmatrix}$$

Example 4.1.14.1

Find a basis for span{$\boldsymbol{u_1}$, $\boldsymbol{u_2}$, $\boldsymbol{u_3}$, $\boldsymbol{u_4}$, $\boldsymbol{u_5}$, $\boldsymbol{u_6}$}

$$\boldsymbol{u_1} = (1, 2, 2, 1)$$
$$\boldsymbol{u_2} = (3, 6, 6, 3)$$
$$\boldsymbol{u_3} = (4, 9, 9, 5)$$
$$\boldsymbol{u_4} = (-2, -1, -1, 1)$$
$$\boldsymbol{u_5} = (5, 8, 9, 4)$$
$$\boldsymbol{u_6} = (4, 2, 7, 3)$$

Arrange the vectors as rows of a matrix

Row space method

Column space method

Arrange the vectors as columns of a matrix

Application: finding basis for linear span

Example 4.1.14.1 (Row space method)

Place the vectors in the form of rows in a 6 x 4 matrix.

$$\boldsymbol{A} = \begin{array}{c} \begin{pmatrix} 1 & 2 & 2 & 1 \\ 3 & 6 & 6 & 3 \\ 4 & 9 & 9 & 5 \\ -2 & -1 & -1 & 1 \\ 5 & 8 & 9 & 4 \\ 4 & 2 & 7 & 3 \end{pmatrix} \end{array} \begin{array}{l} \boldsymbol{u_1} \\ \boldsymbol{u_2} \\ \boldsymbol{u_3} \\ \boldsymbol{u_4} \\ \boldsymbol{u_5} \\ \boldsymbol{u_6} \end{array} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

row space of $\boldsymbol{A}$ $= \text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}, \boldsymbol{u_5}, \boldsymbol{u_6}\}$

{(1, 2, 2, 1), (0, 1, 1, 1), (0, 0, 1, 1)} is a basis

not from the original rows

Application: finding basis for linear span

Example 4.1.14.1 (Column space method)

Place the vectors in the form of columns in a 4 x 6 matrix.

$$\boldsymbol{B} = \begin{array}{c} \begin{pmatrix} 1 & 3 & 4 & -2 & 5 & 4 \\ 2 & 6 & 9 & -1 & 8 & 2 \\ 2 & 6 & 9 & -1 & 9 & 7 \\ 1 & 3 & 5 & 1 & 4 & 3 \end{pmatrix} \\ \begin{matrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{u}_3 & \boldsymbol{u}_4 & \boldsymbol{u}_5 & \boldsymbol{u}_6 \end{matrix} \end{array} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 1 & 3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

column space of $\boldsymbol{B}$ $= \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4, \boldsymbol{u}_5, \boldsymbol{u}_6\}$

Pivot columns: 1st, 3rd and 5th columns

{(1, 2, 2, 1), (4, 9, 9, 5), (5, 8, 9, 3)} is a basis

all from the original columns

Application: extend a set to a basis

Example 4.1.14.2

$S = \{(1, 4, -2, 5, 1), (2, 9, -1, 8, 2), (2, 9, -1, 9, 3)\}$

S is linearly independent.

Extend S to a basis for $\mathbf{R}^5$.

Different from finding a basis for $\mathbf{R}^5$

This means:
Add on non-redundant vectors to S
to form a basis for $\mathbf{R}^5$

Need two more vectors

Use row space method

Application: extend a set to a basis

Example 4.1.14.2

$S = \{(1, 4, -2, 5, 1), (2, 9, -1, 8, 2), (2, 9, -1, 9, 3)\}$

$$\boldsymbol{A} = \begin{pmatrix} 1 & 4 & -2 & 5 & 1 \\ 2 & 9 & -1 & 8 & 2 \\ 2 & 9 & -1 & 9 & 3 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \boldsymbol{R} = \begin{pmatrix} 1 & 4 & -2 & 5 & 1 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

1. Form a matrix $\boldsymbol{A}$ using the vectors in $\boldsymbol{S}$ as rows.
2. Reduce $\boldsymbol{A}$ to a row-echelon form $\boldsymbol{R}$.
3. Identify the non-pivot columns of $\boldsymbol{R}$.

   Look for columns without leading entries

   the 3rd and the 5th columns

Application: extend a set to a basis

Example 4.1.14.2

form a basis for $\mathbf{R}^5$

complete $\boldsymbol{R}$ to a 5x5 matrix

$$\boldsymbol{A} = \begin{pmatrix} 1 & 4 & -2 & 5 & 1 \\ 2 & 9 & -1 & 8 & 2 \\ 2 & 9 & -1 & 9 & 3 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \boldsymbol{R} = \begin{pmatrix} 1 & 4 & -2 & 5 & 1 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$(0 \;\; 0 \;\; x \;\; * \;\; *)$$
$$(0 \;\; 0 \;\; 0 \;\; 0 \;\; y)$$

are not redundant in row space of $\boldsymbol{A}$ $\left\{ \begin{array}{l} \text{E.g. } (0\ 0\ 1\ 0\ 0) \\ \text{E.g. } (0\ 0\ 0\ 0\ 1) \end{array} \right.$

4. Form (row) vectors with leading entries at the non-pivot columns.

5. {Row vectors in $\boldsymbol{A}$} ∪ {vectors from Step 4 } form a basis for $\mathbf{R}^n$

$$\boxed{\begin{array}{c} \{ (1, 4, -2, 5, 1), (2, 9, -1, 8, 2), (2, 9, -1, 9, 3), \\ (0, 0, 1, 0, 0), (0, 0, 0, 0, 1) \} \end{array}}$$

Revision on Bases

S = {(2, −1, 0), (1, −1, 3), (−5, 1, 0), (1, 0, 1)}

How to get a basis from S for **R**^{3}?

Throw out redundant vectors from S

Arrange the vectors as columns of a matrix
Look for pivot columns of the REF

$$T = \left\{ \begin{pmatrix} 2 \\ 1 \\ -5 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

How to extend T to a basis for **R**^{4}?

Add on non-redundant vectors to T

Arrange the vectors as rows of a matrix
Look for 'missing' leading entries of the REF

Solutions of linear system revisited

Ax = b

How do we tell whether this system has
(i) no solution, (ii) unique solution; (iii) infinite solutions ?

Approach 1: Form (**A** | **b**) and look at REF

Approach 2: If **A** is a square matrix

A is invertible ⇒ system has unique solution

A is singular ⇒ system has no or infinite solutions

Approach 3: **A** is any matrix

b belongs to column space of **A**
⇒ system has unique or infinite solutions

b does not belong to column space of **A**
⇒ system has no solution

Consistency of linear system and column space

Discussion 4.1.15

matrix multiply with vector

general linear combination of the column vectors

$$\begin{pmatrix} 2 & -1 & 0 \\ 1 & -1 & 3 \\ -5 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ -2 \\ 3 \end{pmatrix} \Leftrightarrow 1\begin{pmatrix} 2 \\ 1 \\ -5 \\ 1 \end{pmatrix} + 3\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ -2 \\ 3 \end{pmatrix}$$

matrix equation form

vector equation form

system has a solution

actual linear combination of the column vectors

this vector belongs to the column space

Discussion 4.1.15

$$\underbrace{\begin{pmatrix} 2 & -1 & 0 \\ 1 & -1 & 3 \\ -5 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}}_{\boldsymbol{A}} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\boldsymbol{x}} = \underbrace{\begin{pmatrix} -1 \\ 4 \\ -2 \\ 3 \end{pmatrix}}_{\boldsymbol{b}}$$

system $\boldsymbol{Ax} = \boldsymbol{b}$ has a solution

$\Updownarrow$

$\boldsymbol{b}$ can be written as a linear combination of the columns of $\boldsymbol{A}$

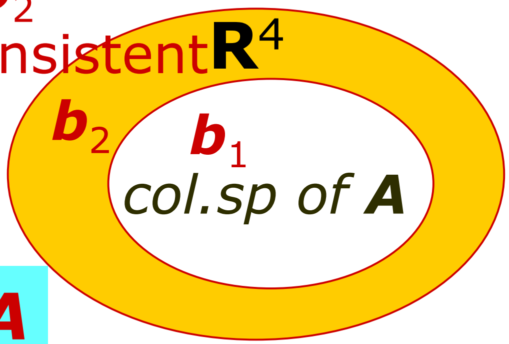
$$x\begin{pmatrix} 2 \\ 1 \\ -5 \\ 1 \end{pmatrix} + y\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ -2 \\ 3 \end{pmatrix}$$

$$x\boldsymbol{C}_1 + y\boldsymbol{C}_2 + z\boldsymbol{C}_3 = \boldsymbol{b}$$

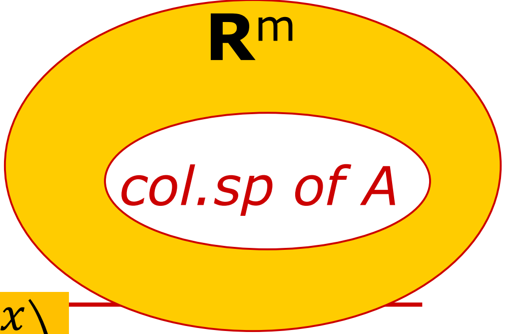
$\Updownarrow$

$\boldsymbol{b}$ belongs to the column space of $\boldsymbol{A}$

$\boldsymbol{Ax} = \boldsymbol{b}_2$
not consistent



$\boldsymbol{Ax} = \boldsymbol{b}_1$
consistent



Theorem 4.1.16

Let $\boldsymbol{A}$ be an m × n matrix.

The column space of $\boldsymbol{A}$ = $\{\ \boldsymbol{A}\boldsymbol{u}\ \mid \boldsymbol{u} \in \mathbf{R}^n \}$.

$\boldsymbol{u} = \begin{pmatrix} x \\ y \\ \vdots \\ z \end{pmatrix}$

$(\ \boldsymbol{c}_1 \mid \boldsymbol{c}_2 \mid \dots \mid \boldsymbol{c}_n\ )$ $\qquad$ $x\boldsymbol{c}_1 + y\boldsymbol{c}_2 + \dots + z\boldsymbol{c}_n$

Span$\{\boldsymbol{c}_1, \boldsymbol{c}_2, \dots, \boldsymbol{c}_n\}$ = $\left\{ \text{all linear combination of the column vectors of } \boldsymbol{A} \right\}$

A system of linear equation $\boldsymbol{Ax} = \boldsymbol{b}$ is consistent if and only if $\boldsymbol{b}$ lies in the column space of $\boldsymbol{A}$.

Section 4.2

Ranks

Objectives

- What is the rank of a matrix?
- What is the relation between rank and invertibility of a matrix?
- What is the relation between rank and consistency of linear system?

Dimension of row space and column space

Theorem 4.2.1

The row space and column space of a matrix have the same dimension.

Let ***A*** be a matrix with row-echelon form ***R***.

$$\boldsymbol{R} = \begin{pmatrix} \otimes & * & & & \\ & & \otimes & * & \\ & & & \ddots & \\ 0 & & & & \otimes \quad * \end{pmatrix}$$

dimension of row space of ***A***

= the number of nonzero rows

= the number of leading entries

dimension of column space of ***A***

= the number of pivot columns

= the number of leading entries

What is the rank of a matrix?

Definition 4.2.3

rank of a matrix :
dimension of its row space or column space.

Notation rank of matrix $\boldsymbol{A}$: rank($\boldsymbol{A}$)

If $\boldsymbol{R}$ is a row-echelon form of $\boldsymbol{A}$,

rank($\boldsymbol{A}$) = the number of nonzero rows of $\boldsymbol{R}$
= the number of leading entries in $\boldsymbol{R}$
= the number of pivot columns in $\boldsymbol{R}$

= largest number of linearly independent rows in $\boldsymbol{A}$

= largest number of linearly independent columns in $\boldsymbol{A}$

Example 4.2.4.1

Row (column) space of zero matrix $\mathbf{0}$ = zero space

rank($\mathbf{0}$) = 0

$$\mathbf{0} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

Row (column) space of n x n identity matrix $\boldsymbol{I_n}$ = $\mathbf{R}^n$

rank($\boldsymbol{I_n}$) = n

$$\boldsymbol{I}_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

Dimension is for vector space
Rank is for matrix

Example 4.2.4.3

$$A = \begin{array}{c} \begin{pmatrix} 1 & 2 & 2 & 1 \\ 3 & 6 & 6 & 3 \\ 4 & 9 & 9 & 5 \\ -2 & -1 & -1 & 1 \\ 5 & 8 & 9 & 4 \\ 4 & 2 & 7 & 3 \end{pmatrix} \\ \begin{array}{cccc} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 & \end{array} \end{array} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{array}{cc} \begin{pmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{array}{c} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \\ \\ \\ \\ \end{array} \end{array}$$

Basis for row space of $A = \{\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\}$

Basis for column space of $A = \{\mathbf{c}_1\ \mathbf{c}_2\ \mathbf{c}_3\}$

rank(A) = 3

DON'T Write: dim(A) = 3

Largest possible rank of a matrix

Example 4.2.4.4

What is the largest possible rank of a 5×3 matrix ?

The answer is 3

Find the largest possible number of pivot columns in a row-echelon form of a 5×3 matrix.

3 columns

What is the largest possible rank of a 3×5 matrix ?

The answer is 3

3 rows

Find the largest possible number of non-zero rows in a row-echelon form of a 3×5 matrix.

Largest possible rank of a matrix

Remark 4.2.5.1

For an $m \times n$ matrix $\boldsymbol{A}$, $\operatorname{rank}(\boldsymbol{A}) \leq \underbrace{\min\{m, n\}}_{\text{the smaller of the two numbers m and n}}$.

Example: $\boldsymbol{A}$ is 4×6

possible $\operatorname{rank}(\boldsymbol{A}) = 0, 1, 2, 3, 4$

$\boldsymbol{A}$ is full rank $\Leftrightarrow \operatorname{rank}(\boldsymbol{A}) = 4$

An $m \times n$ matrix $\boldsymbol{A}$ with $\operatorname{rank}(\boldsymbol{A}) = \min\{m, n\}$ is said to be of full rank.

Relation between rank and determinant of a matrix

Remark 4.2.5.2-3

A square matrix $\boldsymbol{A}$ is of full rank if and only if $\det(\boldsymbol{A}) \neq 0$.

$$\begin{tikzcd} \boldsymbol{A} \text{ is of full rank} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \det(\boldsymbol{A}) \neq 0 \arrow[d, leftrightarrow] \\ \text{RREF of } \boldsymbol{A} \text{ is } \boldsymbol{I} \arrow[r, leftrightarrow] & \boldsymbol{A} \text{ is invertible} \end{tikzcd}$$

$\operatorname{rank}(\boldsymbol{A}) = \operatorname{rank}(\boldsymbol{A}^T)$ for any matrix $\boldsymbol{A}$

row space of $\boldsymbol{A}$ = column space of $\boldsymbol{A}^T$

Relation between rank and consistency of system

Remark 4.2.6

A system $\boldsymbol{Ax} = \boldsymbol{b}$ is consistent $\Longleftrightarrow$

Last lecture:
$\boldsymbol{b} \in$ column space of $\boldsymbol{A}$

if and only if
the coefficient matrix $\boldsymbol{A}$
and the augmented matrix $(\ \boldsymbol{A} \mid \boldsymbol{b}\ )$
have the same rank.

not a pivot column

$$\left(\begin{array}{cccccc|c} \otimes & * & & & & & * \\ & & \otimes & * & & & * \\ \mathbf{0} & & & \ddots & & & * \\ & & & & \otimes & * & * \\ 0 & \cdots & & \cdots & & 0 & 0 \end{array}\right)$$

system is consistent
$\mathrm{rank}(\boldsymbol{A}) = \mathrm{rank}(\ \boldsymbol{A} \mid \boldsymbol{b})$

a pivot column

$$\left(\begin{array}{cccccc|c} \otimes & * & & & & & * \\ & & \otimes & * & & & * \\ \mathbf{0} & & & \ddots & & & * \\ & & & & \otimes & * & * \\ 0 & \cdots & & \cdots & & 0 & \otimes \end{array}\right)$$

system is inconsistent
$\mathrm{rank}(\ \boldsymbol{A}\ ) < \mathrm{rank}(\ \boldsymbol{A} \mid \boldsymbol{b})$

Example 4.2.7

$$\begin{cases} 2x - y = 1 \\ x - y + 3z = 0 \\ -5x + y = 0 \\ x + z = 0 \end{cases}$$

coefficient matrix

$$\boldsymbol{A} = \begin{pmatrix} 2 & -1 & 0 \\ 1 & -1 & 3 \\ -5 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

augmented matrix

$$(\boldsymbol{A} \mid \boldsymbol{b}) = \left(\begin{array}{ccc|c} 2 & -1 & 0 & 1 \\ 1 & -1 & 3 & 0 \\ -5 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 2 & -1 & 0 & 1 \\ 1 & -1 & 3 & 0 \\ -5 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{ccc|c} \multicolumn{3}{c|}{\overbrace{\begin{matrix} 1 & 0 & 1 \end{matrix}}^{\text{REF of } \boldsymbol{A}}} & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 1 & 0 \\ \boxed{0 \quad 0 \quad 0} & \boxed{1} \end{array}\right)$$

REF of $\boldsymbol{A}$ $\quad \operatorname{rank}(\boldsymbol{A}) = 3$

REF of $(\boldsymbol{A}|\boldsymbol{b})$ $\quad \operatorname{rank}(\boldsymbol{A}|\boldsymbol{b}) = 4$

The system is inconsistent.

Rank of a product of two matrices

$\text{rank}(\boldsymbol{AB}) \le \text{rank}(\boldsymbol{A})$
$\text{rank}(\boldsymbol{AB}) \le \text{rank}(\boldsymbol{B})$

Theorem 4.2.8

$\text{rank}(\boldsymbol{AB}) \le \min\{\ \text{rank}(\boldsymbol{A}), \text{rank}(\boldsymbol{B})\ \}$

$\boldsymbol{A}$: m×n
$\boldsymbol{B}$: n×p

Proof

Let $\boldsymbol{B} = (\boldsymbol{b}_1\ \boldsymbol{b}_2\ \ldots\ \boldsymbol{b}_p)$

$\boldsymbol{AB} = (\boldsymbol{Ab}_1\ \boldsymbol{Ab}_2\ \ldots\ \boldsymbol{Ab}_p)$ see Notation 2.2.15

where $\boldsymbol{Ab}_i$ is the i^{th} column of $\boldsymbol{AB}$.

$\boldsymbol{Ab}_i \in$ column space of $\boldsymbol{A}$ By Theorem 4.1.16

$$\underbrace{\text{span}\{\boldsymbol{Ab}_1, \boldsymbol{Ab}_2, \ldots, \boldsymbol{Ab}_p\}}_{=\ \text{column space of } \boldsymbol{AB}} \subseteq \text{column space of } \boldsymbol{A}$$

$\Downarrow$ By Theorem 3.2.10

$$\underbrace{\dim(\text{column space of } \boldsymbol{AB})}_{\text{rank}(\boldsymbol{AB})} \le \underbrace{\dim(\text{column space of } \boldsymbol{A})}_{\text{rank}(\boldsymbol{A})}$$

Rank of a product of two matrices

$\text{rank}(\boldsymbol{AB}) \le \text{rank}(\boldsymbol{A})$

$\text{rank}(\boldsymbol{AB}) \le \text{rank}(\boldsymbol{B})$

Theorem 4.2.8

$$\text{rank}(\boldsymbol{AB}) \le \min\{\ \text{rank}(\boldsymbol{A}),\ \text{rank}(\boldsymbol{B})\ \}$$

Proof

$\text{rank}(\boldsymbol{AB}) \le \text{rank}(\boldsymbol{A})$

Also need to show: $\text{rank}(\boldsymbol{AB}) \le \text{rank}(\boldsymbol{B})$

we have $\text{rank}(\boldsymbol{B}^T\boldsymbol{A}^T) \le \text{rank}(\boldsymbol{B}^T)$

$$\begin{array}{ccc} \text{rank}(\boldsymbol{B}^T\boldsymbol{A}^T) & \le & \text{rank}(\boldsymbol{B}^T) \\ \| & & \| \\ \text{rank}((\boldsymbol{AB})^T) & & \\ \| & & \\ \text{rank}(\boldsymbol{AB}) & \le & \text{rank}(\boldsymbol{B}) \end{array}$$

Therefore

$\text{rank}(\boldsymbol{AB}) \le \min\{\text{rank}(\boldsymbol{A}), \text{rank}(\boldsymbol{B})\}$.

column space of $\boldsymbol{AB} \subseteq$ column space of $\boldsymbol{A}$

From proof of thm 4.2.8

Quiz Time

row space of $\boldsymbol{AB} \subseteq$ row space of $\boldsymbol{B}$

column space of $(\boldsymbol{AB})^T \subseteq$ column space of $\boldsymbol{B}^T$

column space of $\boldsymbol{B}^T\boldsymbol{A}^T \subseteq$ column space of $\boldsymbol{B}^T$

A. True
B. False

Section 4.3

Nullspaces and Nullities

Objectives

- What is the nullspace and nullity of a matrix?
- What is the Dimension Theorem?
- What is the relation between nullspace and solution set of a linear system?

What is the nullspace and nullity of a matrix?

Definition 4.3.1

A : $m \times n$ matrix

nullspace of ***A*** subspace of $\mathbf{R}^n$

is the solution space of the homogeneous system of linear equations $\boldsymbol{Ax} = \mathbf{0}$

nullity of ***A*** a number $\leq n$

is the dimension of the nullspace of ***A***

denoted by nullity(***A***)

Number of parameters in the general solution

Nullspace of a matrix ***A***	same	Solution space of a linear system $\boldsymbol{Ax} = \mathbf{0}$
all the vectors in $\mathbf{R}^n$ that are "killed" by ***A***		all the vectors in $\mathbf{R}^n$ that satisfy $\boldsymbol{Ax} = \mathbf{0}$

Basis for the nullspace

Example 4.3.3.1

$$\text{span}\left\{\begin{pmatrix}-1\\1\\0\\0\\0\end{pmatrix},\begin{pmatrix}-1\\0\\-1\\0\\1\end{pmatrix}\right\}$$

Find a basis for the nullspace of the matrix

$$\boldsymbol{A}=\left(\begin{array}{ccccc|c}2&2&-1&0&1&0\\-1&-1&2&-3&1&0\\0&0&1&1&1&0\\1&1&-2&0&-1&0\end{array}\right)\xrightarrow{\text{G.E.}}\left(\begin{array}{ccccc|c}1&1&0&0&1&0\\0&0&1&0&1&0\\0&0&0&1&0&0\\0&0&0&0&0&0\end{array}\right)$$

write all vectors as columns

The general solution of $\boldsymbol{Ax}=\mathbf{0}$

$$\mathbf{x}=\begin{pmatrix}-s-t\\s\\-t\\0\\t\end{pmatrix}=s\begin{pmatrix}-1\\1\\0\\0\\0\end{pmatrix}+t\begin{pmatrix}-1\\0\\-1\\0\\1\end{pmatrix}$$

$\text{nullity}(\boldsymbol{A})=2$

basis for the nullspace of $\boldsymbol{A}$

Example 4.3.3.2

$$\boldsymbol{B} = \begin{pmatrix} 2 & 1 & -5 & 1 \\ -1 & -1 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \begin{pmatrix} 1 & 0 & 0 & -\frac{7}{9} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{4}{9} \end{pmatrix}$$

$\text{rank}(\boldsymbol{B}) = 3$

general solution of $\boldsymbol{Bx} = \mathbf{0}$

$$\mathbf{x} = \begin{pmatrix} \frac{7}{9}t \\ -\frac{1}{3}t \\ \frac{4}{9}t \\ t \end{pmatrix} = \frac{1}{9}t \begin{pmatrix} 7 \\ -3 \\ 4 \\ 9 \end{pmatrix}$$

$\text{nullity}(\boldsymbol{B}) = 1$ basis for the nullspace of $\boldsymbol{B}$

$\text{rank}(\boldsymbol{B}) + \text{nullity}(\boldsymbol{B}) = 3 + 1 = 4$

$=$ the number of columns of $\boldsymbol{B}$

Dimension Theorem for Matrices

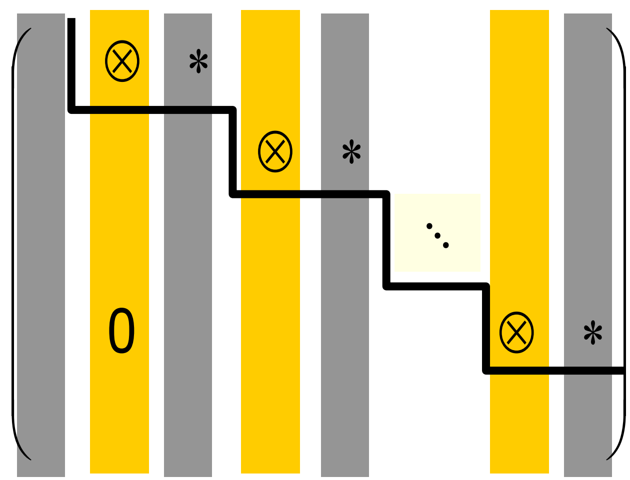
Theorem 4.3.4

If $\boldsymbol{A}$ is a matrix with n columns, then

$$\text{rank}(\boldsymbol{A}) + \text{nullity}(\boldsymbol{A}) = n.$$

row-echelon form

$\boldsymbol{R} =$



pivot columns

(correspond to basis for column space of $\boldsymbol{A}$) rank($\boldsymbol{A}$)

non-pivot columns

(correspond to parameters in general solutions)

nullity($\boldsymbol{A}$)

Applying Dimension Theorem

Example 4.3.5.2

In each of the following cases,
find rank($\boldsymbol{A}$), nullity($\boldsymbol{A}$) and nullity($\boldsymbol{A}^T$).

Size of $\boldsymbol{A}$	# column of $\boldsymbol{A}$	# column of $\boldsymbol{A}^T$	rank($\boldsymbol{A}$) rank($\boldsymbol{A}^T$)	nullity($\boldsymbol{A}$)	nullity($\boldsymbol{A}^T$)
3×4	4	3	3	1	0
7×5	5	7	2	3	5
3×2	2	3	0	2	3

rank($\boldsymbol{A}$) + nullity($\boldsymbol{A}$) = # column of $\boldsymbol{A}$

rank($\boldsymbol{A}^T$) + nullity($\boldsymbol{A}^T$) = # column of $\boldsymbol{A}^T$

homogeneous linear system $\left\{\begin{array}{rcrcrcrcrcr} & & & & 2x_3 & + & 4x_4 & + & 2x_5 & = & 0 \\ x_1 & + & 2x_2 & + & 4x_3 & + & 5x_4 & + & 3x_5 & = & 0 \\ -2x_1 & - & 4x_2 & - & 5x_3 & - & 4x_4 & + & 3x_5 & = & 0 \end{array}\right\} (L_0)$

Example 1.4.7 (revisited)

Non-homogeneous linear system:

$$\left\{\begin{array}{rcrcrcrcrcr} & & & & 2x_3 & + & 4x_4 & + & 2x_5 & = & 8 \\ x_1 & + & 2x_2 & + & 4x_3 & + & 5x_4 & + & 3x_5 & = & -9 \\ -2x_1 & - & 4x_2 & - & 5x_3 & - & 4x_4 & + & 3x_5 & = & 6 \end{array}\right\} (L)$$

solutions of (L_0)

general solution of (L) not solutions of (L) a solution of (L)

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -29 - 2s + 3t \\ s \\ 8 - 2t \\ t \\ -4 \end{pmatrix} = \underbrace{s\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 3 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}}_{\text{general solution of } (L_0)} + \boxed{\begin{pmatrix} -29 \\ 0 \\ 8 \\ 0 \\ -4 \end{pmatrix}}$$

can be replaced by any other solution of (L)

Exercise 2 Q9

Suppose the homogeneous system $\boldsymbol{Ax} = \mathbf{0}$ has non-trivial solutions. ← $\boldsymbol{u}$ is a non-trivial solution

Show that the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ has either no solution or infinitely many solutions.

Idea of proof

We already know $\boldsymbol{Ax} = \boldsymbol{b}$ has either:

- No solution
- Exactly one solution ← $\boldsymbol{v}$ is a solution
- Infinitely many solutions

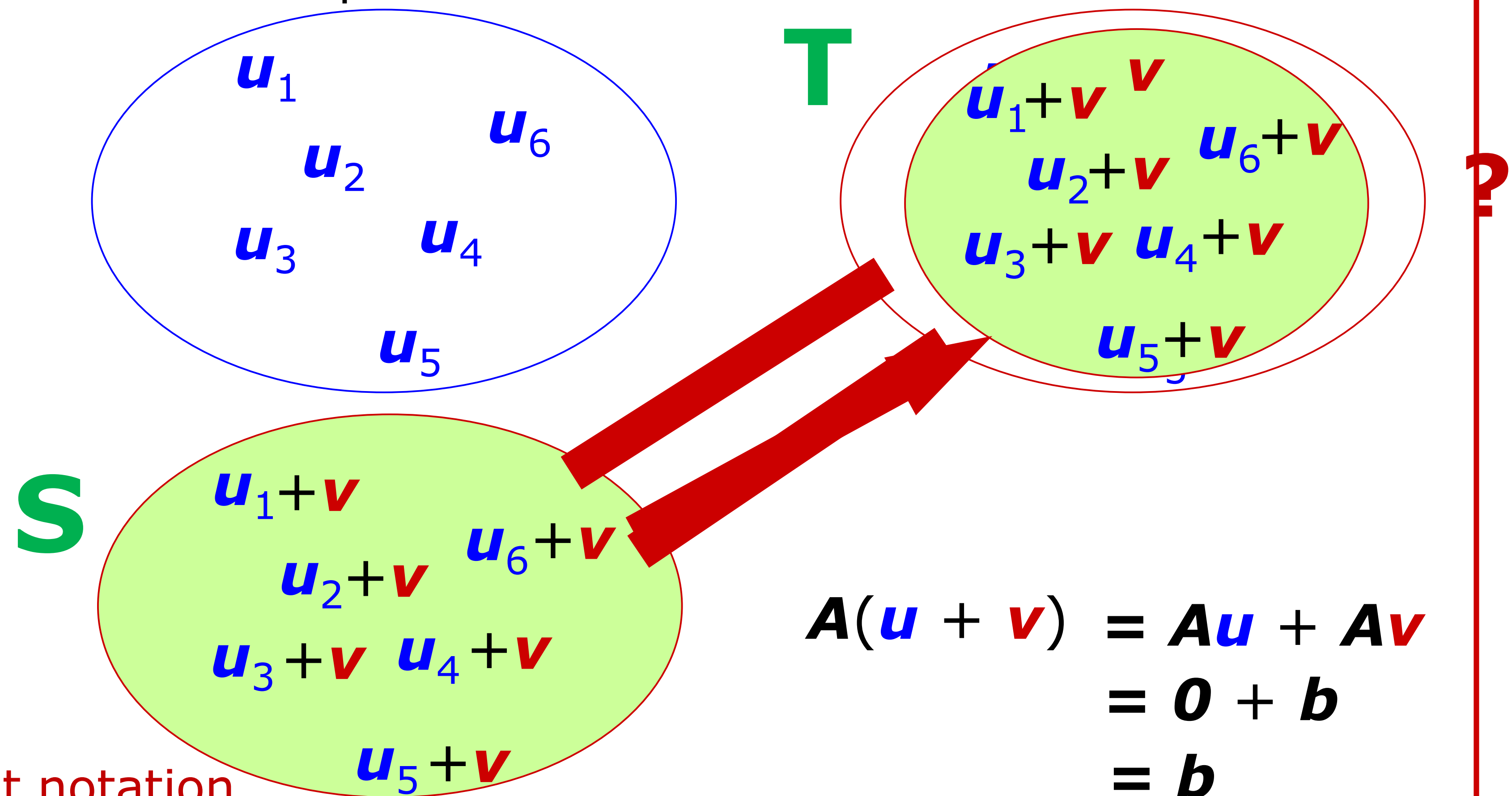
Not possible

$\boldsymbol{u} + \boldsymbol{v}$ is also a solution of $\boldsymbol{Ax} = \boldsymbol{b}$

Solution set of non-homogeneous system

Theorem 4.3.6 (Diagram version)

solution space of $\boldsymbol{Ax} = \mathbf{0}$ solution set of $\boldsymbol{Ax} = \boldsymbol{b}$



$$\boldsymbol{A}(\boldsymbol{u} + \boldsymbol{v}) = \boldsymbol{Au} + \boldsymbol{Av} = \mathbf{0} + \boldsymbol{b} = \boldsymbol{b}$$

set notation

$\{\ \boldsymbol{u} + \boldsymbol{v} \mid \boldsymbol{u}$ is a solution of the system $\boldsymbol{Ax} = \mathbf{0}\ \}$

Solution set of non-homogeneous system

Theorem 4.3.6

Suppose the system of linear equations $\boldsymbol{Ax} = \boldsymbol{b}$ has a (particular) solution $\boldsymbol{v}$.

The solution set of $\boldsymbol{Ax} = \boldsymbol{b}$

$= \{ \underbrace{\boldsymbol{u}}_{\text{vary}} + \underbrace{\boldsymbol{v}}_{\text{fix}} \mid \boldsymbol{u}$ is an element of the nullspace of $\boldsymbol{A}\}$

The general solution of $\boldsymbol{Ax} = \boldsymbol{b}$ can be given by

(the general solution of $\boldsymbol{Ax} = \boldsymbol{0}$) + $\boldsymbol{v}$

If we know the general solution of $\boldsymbol{Ax} = \boldsymbol{0}$
and one particular solution of $\boldsymbol{Ax} = \boldsymbol{b}$,
then we have the general solution for $\boldsymbol{Ax} = \boldsymbol{b}$.

Solution set of non-homogeneous system

Example 4.3.8

linear system $\boldsymbol{Ax} = \boldsymbol{b}$

one particular solution

$$\boldsymbol{A} = \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{pmatrix} \qquad \boldsymbol{b} = \begin{pmatrix} 0 \\ 0 \\ 3 \\ -3 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

By Example 4.3.3.1,

the nullspace of $\boldsymbol{A}$ = $\left\{ s\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \,\middle|\, s, t \text{ in } \mathbf{R} \right\}$

solution space of $\boldsymbol{Ax} = \boldsymbol{0}$

solution set of $\boldsymbol{Ax} = \boldsymbol{b}$ $\left\{ s\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \,\middle|\, s, t \text{ in } \mathbf{R} \right\}$

The proof

Theorem 4.3.6

$\boxed{\text{T = the solution set of } \boldsymbol{Ax} = \boldsymbol{b}}$

$\boxed{S = \{ \boldsymbol{u} + \boldsymbol{v} \mid \boldsymbol{u} \text{ is an element of the nullspace of } \boldsymbol{A}\}}$

We want to show: T = S

Need to show: T ⊆ S and S ⊆ T

T ⊆ *S*

Show every solution of $\boldsymbol{Ax} = \boldsymbol{b}$ has the form $\boldsymbol{u} + \boldsymbol{v}$

Next slide

S ⊆ T

Show every $\boldsymbol{u} + \boldsymbol{v}$ is a solution of $\boldsymbol{Ax} = \boldsymbol{b}$

Substitute $\boldsymbol{u} + \boldsymbol{v}$ for $\boldsymbol{x}$ in $\boldsymbol{Ax} = \boldsymbol{b}$

T = the solution set of $\boldsymbol{Ax} = \boldsymbol{b}$

The proof

$S = \{\, \boldsymbol{u} + \boldsymbol{v} \mid \boldsymbol{u}$ is an element of the nullspace of $\boldsymbol{A}\}$

a solution of $\boldsymbol{Ax} = \boldsymbol{b}$

Theorem 4.3.6

To show $\mathrm{T} \subseteq S$: element-chasing method

Let $\boldsymbol{w} \in \mathrm{T}$ Want to show: $\boldsymbol{w} \in S$

i.e. Given $\boldsymbol{Aw} = \boldsymbol{b}$

i.e. To show $\boldsymbol{w}$ can be written as $\boldsymbol{u} + \boldsymbol{v}$

We have $\boldsymbol{Av} = \boldsymbol{b}$

i.e. To show $\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{v}$

i.e. To show $\boldsymbol{w} - \boldsymbol{v} = \boldsymbol{u}$

$$\begin{aligned} &\boldsymbol{A}(\boldsymbol{w} - \boldsymbol{v}) \\ &= \boldsymbol{Aw} - \boldsymbol{Av} \\ &= \boldsymbol{b} - \boldsymbol{b} = \boldsymbol{0} \end{aligned}$$

i.e. To show $\boldsymbol{w} - \boldsymbol{v}$ is an element of the nullspace of $\boldsymbol{A}$

i.e. To show $\boldsymbol{A}(\boldsymbol{w} - \boldsymbol{v}) = \boldsymbol{0}$

Hence $\mathrm{T} \subseteq S$.

Solution set of non-homogeneous system

Remark 4.3.7

Suppose the system of linear equations $\boldsymbol{Ax} = \boldsymbol{b}$ has a (particular) solution $\boldsymbol{v}$.

The solution set of $\boldsymbol{Ax} = \boldsymbol{b}$

$= \{ \boldsymbol{u} + \boldsymbol{v} \mid \boldsymbol{u}$ is an element of the nullspace of $\boldsymbol{A} \}$

Let $\boldsymbol{Ax} = \boldsymbol{b}$ be a consistent linear system. Then

$\boldsymbol{Ax} = \boldsymbol{b}$ has exactly one solution
if and only if
the nullspace of $\boldsymbol{A}$ is equal to $\{\mathbf{0}\}$