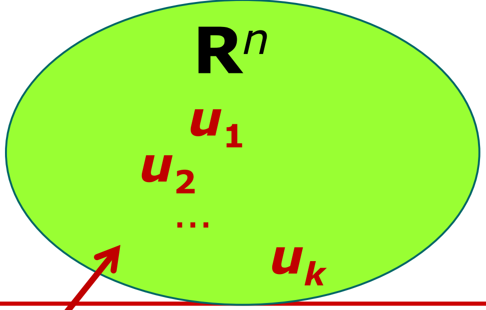
Section 3.2

Linear Combinations and Linear Spans

Objective

- What is a linear combination?
- How to express a vector as a linear combination?
- What is a linear span?

What is a linear combination?



Definition 3.2.1

$\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}$: a fixed set of vectors in $\mathbf{R}^n$

$c_1, c_2, \ldots, c_k$: real numbers

$$\boxed{c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k}}$$

is called a linear combination of $\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}$.

Example $\boldsymbol{u_1} = (2, 1, 0)$ $\boldsymbol{u_2} = (-3, 0, 1)$

$c_1 = 1, \; c_2 = 1$

$1(2, 1, 0) + 1(-3, 0, 1) = (-1, 1, 1)$

a specific linear combination

$c_1 = s, \; c_2 = t$

$s(2, 1, 0) + t(-3, 0, 1)$ general linear combination with parameters s and t

Can every vector be expressed as a linear combination of a given set of vectors?

Example 3.2.2.1

$\boldsymbol{u}_1 = (2, 1, 3)$, $\boldsymbol{u}_2 = (1, -1, 2)$ and $\boldsymbol{u}_3 = (3, 0, 5)$.

(a) $\boldsymbol{v} = (3, 3, 4)$

is a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$.

(3, 3, 4) can be expressed as

$$a(2, 1, 3) + b(1, -1, 2) + c(3, 0, 5)$$

(b) $\boldsymbol{w} = (1, 2, 4)$

is not a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$.

(1, 2, 4) cannot be expressed as

$$a(2, 1, 3) + b(1, -1, 2) + c(3, 0, 5)$$

How to express a vector as a specific linear combination of a given set of vectors?

Example 3.2.2.1(a)

$\boldsymbol{u}_1 = (2, 1, 3)$ $\boldsymbol{u}_2 = (1, -1, 2)$ $\boldsymbol{u}_3 = (3, 0, 5)$

Write $\boldsymbol{v} = a\boldsymbol{u}_1 + b\boldsymbol{u}_2 + c\boldsymbol{u}_3$

$$\boxed{(3, 3, 4) = a(2, 1, 3) + b(1, -1, 2) + c(3, 0, 5)}$$

solve for a, b, c

Equating components

$$\boxed{\begin{cases} 2a + b + 3c = 3 \\ a - b = 3 \\ 3a + 2b + 5c = 4 \end{cases}}$$

1st component
2nd component
3rd component

So we obtain a linear system in variables a, b, c

$$\boxed{\begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix} = a\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + b\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + c\begin{pmatrix} 3 \\ 0 \\ 5 \end{pmatrix}}$$

vector equation form of the linear system (P.43)

How to express a vector as a specific linear combination of a given set of vectors?

Example 3.2.2.1(a)

$\boldsymbol{u_1} = (2, 1, 3)$ $\boldsymbol{u_2} = (1, -1, 2)$ $\boldsymbol{u_3} = (3, 0, 5)$

$$\boxed{(3, 3, 4) = a(2, 1, 3) + b(1, -1, 2) + c(3, 0, 5)}$$

can find specific values for a, b, c

$$\left(\begin{array}{ccc|c} 2 & 1 & 3 & 3 \\ 1 & -1 & 0 & 3 \\ 3 & 2 & 5 & 4 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{ccc|c} 2 & 1 & 3 & 3 \\ 0 & -\frac{3}{2} & -\frac{3}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{array}\right)$$

system is consistent

So (3, 3, 4) is a linear combination of $\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}$.

To write (3, 3, 4) as a specific linear combination:

general solution of LS : a $=2 - t$, b $= -1 - t$, c $= t$

Take t = 0: a = 2, b = -1, c = 0

$$(3, 3, 4) = 2\boldsymbol{u_1} - \boldsymbol{u_2} + 0\boldsymbol{u_3}$$

Take t = 1: a = 1, b = -2, c = 1

$$(3, 3, 4) = \boldsymbol{u_1} - 2\boldsymbol{u_2} + \boldsymbol{u_3}$$

How to show that a vector cannot be expressed as a linear combination of a given set of vectors?

Example 3.2.2.1(b)

$\boldsymbol{u_1} = (2, 1, 3)$ $\boldsymbol{u_2} = (1, -1, 2)$ $\boldsymbol{u_3} = (3, 0, 5)$

Write $\boldsymbol{w} = a\boldsymbol{u_1} + b\boldsymbol{u_2} + c\boldsymbol{u_3}$

$$(1, 2, 4) = a(2, 1, 3) + b(1, -1, 2) + c(3, 0, 5)$$

$$\begin{aligned} 2a + b + 3c &= 1 \\ a - b &= 2 \\ 3a + 2b + 5c &= 4 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 1 & -1 & 0 & 2 \\ 3 & 2 & 5 & 4 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 0 & -\frac{3}{2} & -\frac{3}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 3 \end{array}\right)$$

system is inconsistent

(1, 2, 4) is not a linear combination of $\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}$.

How to express a general vector as a linear combination of a given set of vectors?

Example 3.2.2.2

standard basis vectors

Directional vectors of the x-axis, y-axis, z-axis

Every vector in $\mathbf{R}^3$

is a linear combination of the following vectors

$$\boxed{\mathbf{e}_1 = (1, 0, 0),\ \mathbf{e}_2 = (0, 1, 0),\ \mathbf{e}_3 = (0, 0, 1)}$$

Take a general 3-vector (x, y, z)

$$(x, y, z) = (x, 0, 0) + (0, y, 0) + (0, 0, z)$$

$$= x\,(1, 0, 0) + y\,(0, 1, 0) + z\,(0, 0, 1)$$

$$= x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$$

e.g. $(1, 2, 5) = 1\mathbf{e}_1 + 2\mathbf{e}_2 + 5\mathbf{e}_3$

Span preview

How many linear combinations of (2,1,0) and (-3,0,1) are there? Infinite

The set of all linear combinations of (2,1,0) and (-3,0,1)

$\{s(2, 1, 0) + t(-3, 0, 1) \mid s, t \in \mathbf{R}\}$

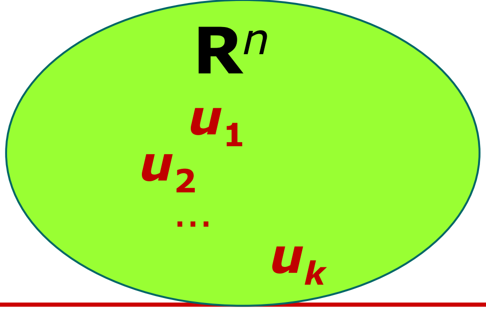
using set notation

We call it: the linear span of (2,1,0) and (-3,0,1)

using words (in terms of linear span)

We write it: span{(2,1,0) , (-3,0,1)}

using linear span notation



Definition 3.2.3

$\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$: k (finite) vectors in $\mathbf{R}^n$.

The set of all linear combinations of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$

$$\{c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_k\boldsymbol{u}_k \mid c_1, c_2, \ldots, c_k \text{ in } \mathbf{R}\}$$

$=$

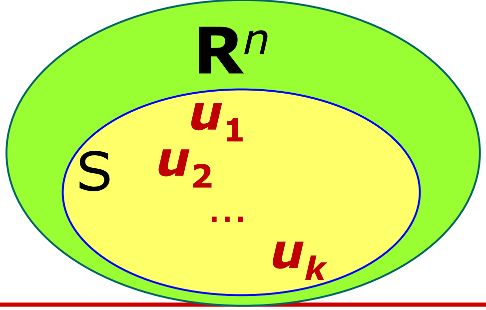
This set is called

the linear span of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$

"Linear span" is always used w.r.t. a set of vectors

This set is denoted by $\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$

What is a linear span?



Definition 3.2.3

$S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$: a (finite) subset of $\mathbf{R}^n$.

The set of all linear combinations of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$

$$\{c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_k\boldsymbol{u}_k \mid c_1, c_2, \ldots, c_k \text{ in } \mathbf{R}\}$$

$$= \operatorname{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\} = \operatorname{span}(S)$$

This set is called

the linear span of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$

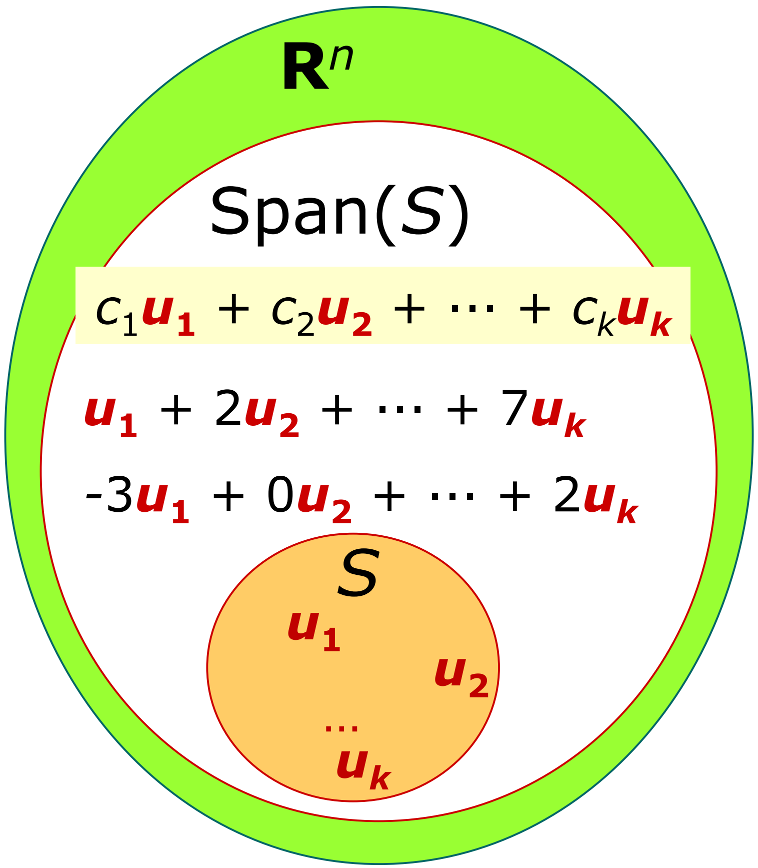
the linear span of S

"Linear span" is always used w.r.t. a set of vectors

This set is denoted by

Definition 3.2.3

$S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}\}$ a finite collection of vectors in $\mathbf{R}^n$



$\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k} \in \mathbf{R}^n$

$S \subseteq \mathbf{R}^n$

span(S) $\subseteq \mathbf{R}^n$

$S \subseteq$ span(S)

span(S) can be equal to $\mathbf{R}^n$ but not always.

Vectors belong to a linear span

Example 3.2.4.1

In Example 3.2.2.1,

$\boldsymbol{u_1} = (2, 1, 3)$, $\boldsymbol{u_2} = (1, -1, 2)$ and $\boldsymbol{u_3} = (3, 0, 5)$.

(a) $\boldsymbol{v} = (3, 3, 4)$ (b) $\boldsymbol{w} = (1, 2, 4)$

$\boldsymbol{v}$ is a linear combination of $\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}$.

$$\boldsymbol{v} \in \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$$

$\boldsymbol{w}$ is not a linear combination of $\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}$.

$$\boldsymbol{w} \notin \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$$

Express a linear span in explicit set notation form

Example 3.2.4.2

$S = \{(1, 0, 0, -1), (0, 1, 1, 0)\} \subseteq \mathbf{R}^4$ $\boxed{\text{span}(S) \subseteq \mathbf{R}^4}$

$$\text{span}(S) = \text{span}\{(1, 0, 0, -1), (0, 1, 1, 0)\}$$ linear span form

$$= \{a(1, 0, 0, -1) + b(0, 1, 1, 0) \mid a, b \in \mathbf{R}\}$$

$$= \{(a, b, b, -a) \mid a, b \in \mathbf{R}\}$$ explicit form

A general vector in span(S):

$$a(1, 0, 0, -1) + b(0, 1, 1, 0) = (a, b, b, -a).$$

Section 3.2

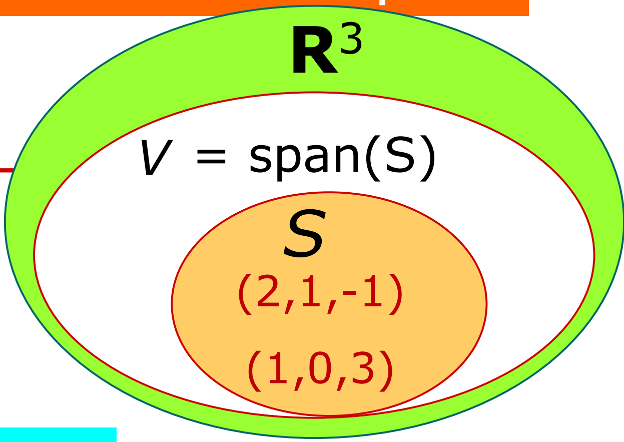
Linear Combinations and Linear Spans

Objective

- How to express a linear span in explicit set notation?
- How to express a set notation as a linear span?
- How to show a linear span is (is not) equal to $\mathbf{R}^n$?
- How to show a linear span is contained in another?

Express an explicit set notation form as linear span

Example 3.2.4.3



Let $V = \{\, (2a + b,\ a,\ 3b-a) \mid a, b \in \mathbf{R}\} \subseteq \mathbf{R}^3$.

explicit form

Rewrite the general form:

$(2a + b,\ a,\ 3b-a) = a\,(2, 1,-1) + b\,(1, 0, 3)$.

So $V = \text{span}\{(2, 1,-1), (1, 0, 3)\}$.

linear span form

The subset V is spanned by (2, 1,-1), (1, 0, 3)

(2, 1,-1), (1, 0, 3) spans the subset V.

How to show a linear span equal to $\mathbf{R}^n$?

$\mathbf{R}^n$

Span(S)

Example 3.2.4.4

To show: span{(1, 0, 1), (1, 1, 0), (0, 1, 1)} = $\mathbf{R}^3$

Same as showing:

every vector (x, y, z) in $\mathbf{R}^3$ can be written as linear combination of the three vectors

Write $(x, y, z) = a\,(1, 0, 1) + b\,(1, 1, 0) + c\,(0, 1, 1)$

Convert into linear system

$$\begin{cases} a + b = x \\ b + c = y \\ a + c = z \end{cases}$$

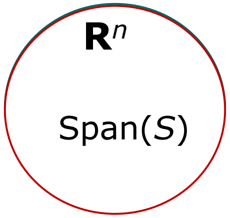
a, b, c are variables

x, y, z are treated as constants.

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & x \\ 0 & 1 & 1 & y \\ 1 & 0 & 1 & z \end{array}\right)$$

To show:
The system is consistent

How to show a linear span equal to $\mathbf{R}^n$?



Example 3.2.4.4

$(x, y, z) = a\,(1, 0, 1) + b\,(1, 1, 0) + c\,(0, 1, 1)$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & x \\ 0 & 1 & 1 & y \\ 1 & 0 & 1 & z \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{ccc|c} 1 & 1 & 0 & x \\ 0 & 1 & 1 & y \\ 0 & 0 & 2 & z - x + y \end{array}\right)$$

The system is consistent regardless of the values of x, y, z.

So we can always solve for a, b, c for any vector (x, y, z).

Every (x, y, z) in $\mathbf{R}^3$ is a linear combination of the three given vectors

So span{(1, 0, 1), (1, 1, 0), (0, 1, 1)} = $\mathbf{R}^3$

How to show a linear span equal to $\mathbf{R}^n$?

Example 3.2.4.4

Solve a, b, c in terms of x, y, z

$$(x, y, z) = a\,(1, 0, 1) + b\,(1, 1, 0) + c\,(0, 1, 1)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & x \\ 0 & 1 & 1 & y \\ 0 & 0 & 2 & z - x + y \end{array}\right) \Rightarrow \begin{cases} a + b = x \\ b + c = y \\ 2c = z - x + y \end{cases}$$

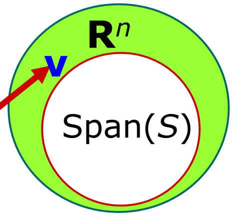
Solution: $c = \boxed{\frac{-x+y+z}{2}}$ $b = \boxed{\frac{x+y-z}{2}}$ $a = \boxed{\frac{x-y+z}{2}}$

$$(x, y, z) = \left(\frac{x-y+z}{2}\right)(1,0,1) + \left(\frac{x+y-z}{2}\right)(1,1,0) + \left(\frac{-x+y+z}{2}\right)(0,1,1)$$

e.g. $(1,2,5) = 2(1,0,1) + (-1)(1,1,0) + 3(0,1,1)$

Every (x, y, z) can be expressed as a linear combination of (1, 0, 1), (1, 1, 0) and (0, 1, 1) in exactly one way.

How to show a linear span not equal to $\mathbf{R}^n$?



Example 3.2.4.5

To show: $\text{span}\{(1,1,1), (1,2,0), (2,1,3), (2,3,1)\} \neq \mathbf{R}^3$

$$(x, y, z) = a\,(1, 1, 1) + b\,(1, 2, 0) + c\,(2, 1, 3) + d\,(2, 3, 1)$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 2 & 2 & x \\ 1 & 2 & 1 & 3 & y \\ 1 & 0 & 3 & 1 & z \end{array}\right) \xrightarrow{\text{G.E.}} \left(\begin{array}{cccc|c} 1 & 1 & 2 & 2 & x \\ 0 & 1 & -1 & 1 & y-x \\ 0 & 0 & 0 & 0 & y+z-2x \end{array}\right)$$

The system is inconsistent when $y + z - 2x \neq 0$.

e.g. $x = 1, y = 0, z = 0$

So $(1, 0, 0) \notin \text{span}\{(1,1,1), (1,2,0), (2,1,3), (2,3,1)\}$

How to determine whether a linear span is equal to $\mathbf{R}^n$ or not?

Discussion 3.2.5

span$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\} = \mathbf{R}^n$?

$\mathbf{u}_1 = (a_{11}, a_{12}, \ldots, a_{1n})$,
$\mathbf{u}_2 = (a_{21}, a_{22}, \ldots, a_{2n})$,
$\vdots$
$\mathbf{u}_k = (a_{k1}, a_{k2}, \ldots, a_{kn})$.

Consider the linear system

$$\underbrace{\left(\begin{array}{cccc|c} a_{11} & a_{21} & \cdots & a_{k1} & \\ a_{12} & a_{22} & \cdots & a_{k2} & \\ \vdots & \vdots & & \vdots & \\ a_{1n} & a_{2n} & \cdots & a_{kn} & \end{array}\right)}_{\mathbf{A}} \xrightarrow{\text{G.E.}} \underbrace{\left(\begin{array}{cccc|c} * & * & \cdots & * & \\ 0 & * & \cdots & * & \\ \vdots & \ddots & * & \vdots & \\ 0 & \cdots & 0 & * & \end{array}\right)}_{\mathbf{R}} \quad \text{REF}$$

R has no zero row
system is always consistent
span$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\} = \mathbf{R}^n$

R has a zero row
system may be inconsistent
span$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\} \neq \mathbf{R}^n$

Tutorial 3 Q43

A condition for a linear span to be not equal to $\mathbf{R}^n$

Theorem 3.2.7

Let S = {$\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$} be a set of vectors in $\mathbf{R}^n$.

If $k < n$, then S cannot span $\mathbf{R}^n$. span(S) $\neq \mathbf{R}^n$

More rows than columns

$$\underbrace{\left(\begin{array}{cccc|c} a_{11} & a_{21} & \cdots & a_{k1} & x \\ a_{12} & a_{22} & \cdots & a_{k2} & y \\ \vdots & \vdots & & \vdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{kn} & z \end{array}\right)}_{\mathbf{A}} \xrightarrow{\text{.E.}} \underbrace{\left(\begin{array}{cccc|c} * & * & \cdots & * & * \\ 0 & * & \cdots & * & * \\ \vdots & \ddots & * & \vdots & \vdots \\ 0 & \ldots & & 0 & * \end{array}\right)}_{\mathbf{R}} \quad \text{REF}$$

The REF **R** of **A** must have a zero row,
so the system may be inconsistent,
and span(S) $\neq \mathbf{R}^n$.

A condition for a linear span to be not equal to $\mathbf{R}^n$

Theorem 3.2.7

Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ be a set of vectors in $\mathbf{R}^n$.

If $k < n$, then S cannot span $\mathbf{R}^n$. $\text{span}(S) \neq \mathbf{R}^n$

Example 3.2.8

$\text{span}\{\ \boldsymbol{u}\ \} \neq \mathbf{R}^2$ since $k = 1 < n = 2$

$\text{span}\{\ \boldsymbol{u}\ \} \neq \mathbf{R}^3$ since $k = 1 < n = 3$

$\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2\ \} \neq \mathbf{R}^3$ since $k = 2 < n = 3$

Every linear span contains the zero vector

Theorem 3.2.9.1

Let $S = \{ \boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k \}$ ← any set

The zero vector $\mathbf{0} \in \text{span}(S)$.

Proof

$$c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \ldots + c_k\boldsymbol{u}_k \in \text{span}(S)$$

for any $c_1, c_2, \ldots, c_k$ in $\mathbf{R}$

In particular

$$\underbrace{0\boldsymbol{u}_1 + 0\boldsymbol{u}_2 + \ldots + 0\boldsymbol{u}_k}_{\mathbf{0}} \in \text{span}(S)$$

$$\mathbf{0} \in \text{span}(S)$$

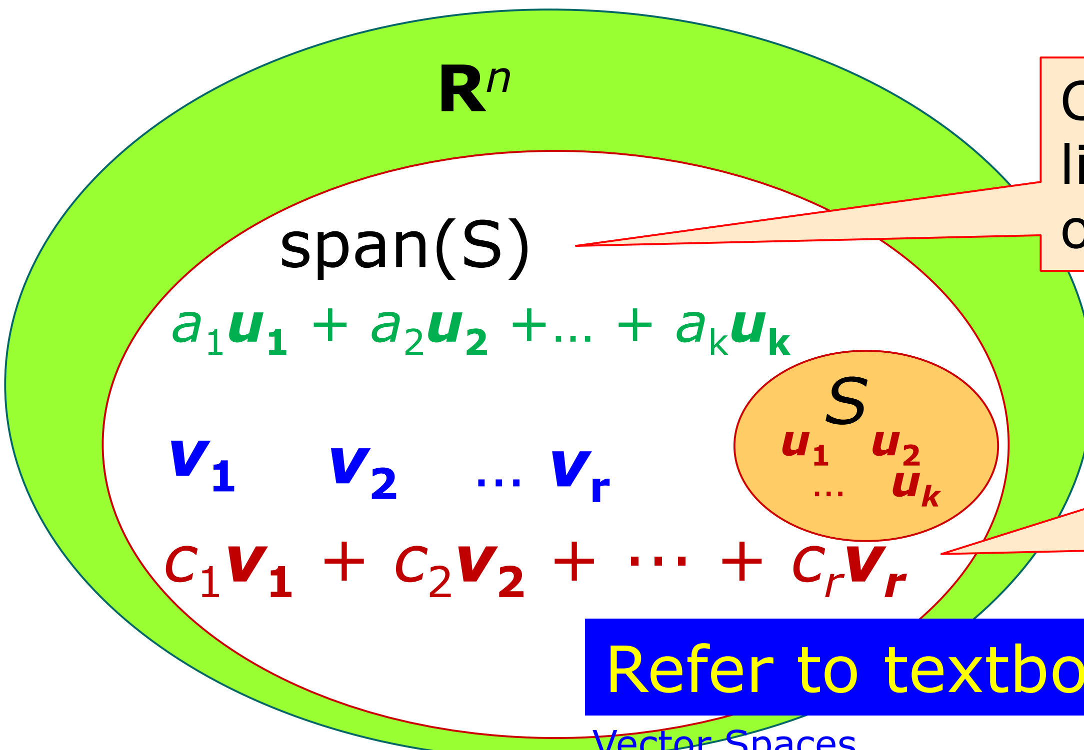
Any linear combination of vectors in a linear span is again a vector in the linear span.

Theorem 3.2.9.2

Let S = { $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ } $\subseteq \mathbf{R}^n$

If $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r \in \text{span}(S)$ and $c_1, c_2, \ldots, c_r \in \mathbf{R}$

then $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_r\boldsymbol{v}_r \in \text{span}(S)$



Contains linear combinations of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$

linear combinations of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r$

Refer to textbook for proof

Any linear combination of vectors in a linear span is again a vector in the linear span.

Theorem 3.2.9.2

Let S = { $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ } $\subseteq \mathbf{R}^n$

If $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r \in$ span(S) and $c_1, c_2, \ldots, c_r \in \mathbf{R}$

then $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_r\boldsymbol{v}_r \in$ span(S)

Consequent of theorem

if $\boldsymbol{u}$ and $\boldsymbol{v} \in$ span(S), then $\boldsymbol{u} + \boldsymbol{v} \in$ span(S).

Closure property under vector addition

if $\boldsymbol{u} \in$ span(S) and $c \in \boldsymbol{R}$, then $c\boldsymbol{u} \in$ span(S).

Closure property under scalar multiplication

Motivation

Example 3.2.11.1

$$\text{span}\boxed{\begin{array}{c}\boldsymbol{u_1} = (1, 0, 1)\\ \boldsymbol{u_2} = (1, 1, 2)\\ \boldsymbol{u_3} = (-1, 2, 1)\end{array}} \overset{?}{=} \text{span}\boxed{\begin{array}{l}\boldsymbol{v_1} = (1, 2, 3)\\ \boldsymbol{v_2} = (2, -1, 1)\end{array}}$$

How are the two linear spans related?

Given two sets A and B.

To show $A = B$: We check $A \subseteq B$ and $B \subseteq A$.

How to show span(S_1) ⊆ span(S_2)?

$\mathbf{u}_1 = (1, 0, 1)$ $\quad \mathbf{v}_1 = (1, 2, 3)$
$\mathbf{u}_2 = (1, 1, 2)$ $\quad \mathbf{v}_2 = (2, -1, 1)$
$\mathbf{u}_3 = (-1, 2, 1)$

Example 3.2.11.1

Show span$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} \subseteq$ span$\{\mathbf{v}_1, \mathbf{v}_2\}$:

Need to show: each $\mathbf{u}_i$ can be written as $a\mathbf{v}_1 + b\mathbf{v}_2$ for some real number a and b

$a\mathbf{v}_1 + b\mathbf{v}_2 = \mathbf{u}_1$ $\Downarrow$ $\begin{cases} a + 2b = 1 \\ 2a - b = 0 \\ 3a + b = 1 \end{cases}$

$a\mathbf{v}_1 + b\mathbf{v}_2 = \mathbf{u}_2$ $\Downarrow$ $\begin{cases} a + 2b = 1 \\ 2a - b = 1 \\ 3a + b = 2 \end{cases}$

$a\mathbf{v}_1 + b\mathbf{v}_2 = \mathbf{u}_3$ $\Downarrow$ $\begin{cases} a + 2b = -1 \\ 2a - b = 2 \\ 3a + b = 1 \end{cases}$

Need to show all three linear systems are consistent

$$\begin{array}{ccc} \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & -1 & 0 \\ 3 & 1 & 1 \end{array}\right) \\ \mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{u}_1 \end{array} \qquad \begin{array}{c} \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & -1 & 1 \\ 3 & 1 & 2 \end{array}\right) \\ \mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{u}_2 \end{array} \qquad \begin{array}{c} \left(\begin{array}{cc|c} 1 & 2 & -1 \\ 2 & -1 & 2 \\ 3 & 1 & 1 \end{array}\right) \\ \mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{u}_3 \end{array}$$

How to show span(S_1) $\subseteq$ span(S_2)?

$\boldsymbol{u_1} = (1, 0, 1)$ $\boldsymbol{v_1} = (1, 2, 3)$
$\boldsymbol{u_2} = (1, 1, 2)$ $\boldsymbol{v_2} = (2, -1, 1)$
$\boldsymbol{u_3} = (-1, 2, 1)$

Example 3.2.11.1

We can solve the three systems simultaneously:

$$\begin{array}{ccccc} \left(\begin{array}{cc|c|c|c} 1 & 2 & 1 & 1 & -1 \\ 2 & -1 & 0 & 1 & 2 \\ 3 & 1 & 1 & 2 & 1 \end{array}\right) \\ \boldsymbol{v_1} \quad \boldsymbol{v_2} \quad \boldsymbol{u_1} \quad \boldsymbol{u_2} \quad \boldsymbol{u_3} \end{array} \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left(\begin{array}{cc|c|c|c} 1 & 0 & \frac{1}{5} & \frac{3}{5} & \frac{3}{5} \\ 0 & 1 & \frac{2}{5} & \frac{1}{5} & -\frac{4}{5} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

All the three systems are consistent.

This shows each $\boldsymbol{u_i}$ can be written as $a\boldsymbol{v_1} + b\boldsymbol{v_2}$ for some real number a and b,

So span$\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\} \subseteq$ span$\{\boldsymbol{v_1}, \boldsymbol{v_2}\}$. Theorem 3.2.9.2

By solve the three systems, we get:

$$\boldsymbol{u}_1 = \tfrac{1}{5}\boldsymbol{v}_1 + \tfrac{2}{5}\boldsymbol{v}_2 \qquad \boldsymbol{u}_2 = \tfrac{3}{5}\boldsymbol{v}_1 + \tfrac{1}{5}\boldsymbol{v}_2 \qquad \boldsymbol{u}_3 = \tfrac{3}{5}\boldsymbol{v}_1 - \tfrac{4}{5}\boldsymbol{v}_2$$

How to show span(S_1) $\subseteq$ span(S_2)?

Theorem 3.2.10

Let $S_1 = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}\}$ and $S_2 = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_m}\}$ be subsets of $\mathbf{R}^n$.

Every linear combination of $\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}$ belongs to span(S_2)

Then

span(S_1) $\subseteq$ span(S_2)

if and only if

each $\boldsymbol{u_i}$ is a linear combination of $\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_m}$.

Every $\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}$ belongs to span(S_2)

How to show span(S_1) = span(S_2)?

Example 3.2.11.1

$$\text{span}\left\{\begin{array}{l} \boldsymbol{u_1} = (1, 0, 1) \\ \boldsymbol{u_2} = (1, 1, 2) \\ \boldsymbol{u_3} = (-1, 2, 1) \end{array}\right\} \overset{\text{to show}}{=} \text{span}\left\{\begin{array}{l} \boldsymbol{v_1} = (1, 2, 3) \\ \boldsymbol{v_2} = (2, -1, 1) \end{array}\right\}$$

Need to show

span{$\boldsymbol{u_1}$, $\boldsymbol{u_2}$, $\boldsymbol{u_3}$} ⊆ span{$\boldsymbol{v_1}$, $\boldsymbol{v_2}$}

$$\begin{array}{c} \left(\begin{array}{cc|c|c|c} 1 & 2 & 1 & 1 & -1 \\ 2 & -1 & 0 & 1 & 2 \\ 3 & 1 & 1 & 2 & 1 \end{array}\right) \\ \begin{array}{ccccc} \boldsymbol{v_1} & \boldsymbol{v_2} & \boldsymbol{u_1} & \boldsymbol{u_2} & \boldsymbol{u_3} \end{array} \end{array}$$

Check consistencies

span{$\boldsymbol{v_1}$, $\boldsymbol{v_2}$} ⊆ span{$\boldsymbol{u_1}$, $\boldsymbol{u_2}$, $\boldsymbol{u_3}$}

$$\begin{array}{c} \left(\begin{array}{ccc|c|c} 1 & 1 & -1 & 1 & 2 \\ 0 & 1 & 2 & 2 & -1 \\ 1 & 2 & 1 & 3 & 1 \end{array}\right) \\ \begin{array}{ccccc} \boldsymbol{u_1} & \boldsymbol{u_2} & \boldsymbol{u_3} & \boldsymbol{v_1} & \boldsymbol{v_2} \end{array} \end{array}$$

How to show span(S_1) $\neq$ span(S_2)?

Example 3.2.11.2

span $\begin{Bmatrix} \boldsymbol{u_1} = (1, 1, 0, 2) \\ \boldsymbol{u_2} = (1, 0, 0, 1) \\ \boldsymbol{u_3} = (0, 1, 0, 1) \end{Bmatrix}$ to show $\subseteq$ $\neq$ span $\begin{Bmatrix} \boldsymbol{v_1} = (1, 1, 1, 1) \\ \boldsymbol{v_2} = (-1, 1, -1, 1) \\ \boldsymbol{v_3} = (-1, 1, 1, -1) \end{Bmatrix}$

span$\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\} \subseteq$ span$\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$

Show that the augmented matrix ($\boldsymbol{v_1}\ \boldsymbol{v_2}\ \boldsymbol{v_3} \mid \boldsymbol{u_1} \mid \boldsymbol{u_2} \mid \boldsymbol{u_3}$) is consistent.

span$\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\} \neq$ span$\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$

Show that span$\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\} \not\subseteq$ span$\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$

Show that the augmented matrix ($\boldsymbol{u_1}\ \boldsymbol{u_2}\ \boldsymbol{u_3} \mid \boldsymbol{v_1} \mid \boldsymbol{v_2} \mid \boldsymbol{v_3}$) is inconsistent.

What is a redundant vector in span(S)?

$\mathbf{R}^n$

$\boldsymbol{u}_1$

$\boldsymbol{u}_2$

…

$\boldsymbol{u}_k$

Theorem 3.2.12

Suppose $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ are vectors taken from $\mathbf{R}^n$.

If $\boldsymbol{u}_k$ is a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}$,

then $\boldsymbol{u}_k = d_1\boldsymbol{u}_1 + d_2\boldsymbol{u}_2 + \cdots + d_{k-1}\boldsymbol{u}_{k-1}$

$$\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}\} = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}, \boldsymbol{u}_k\}$$

$$c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_{k-1}\boldsymbol{u}_{k-1} \iff c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_{k-1}\boldsymbol{u}_{k-1} + c_k\boldsymbol{u}_k$$

We say $\boldsymbol{u}_k$ is a "redundant" vector in span$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}, \boldsymbol{u}_k\}$. if u_k is a linear combination of ($u_1 \to u_{k-1}$)

If $\boldsymbol{u} \in$ span(S), then span(S) = span(S ∪ $\boldsymbol{u}$)

Geometrical meaning of linear span

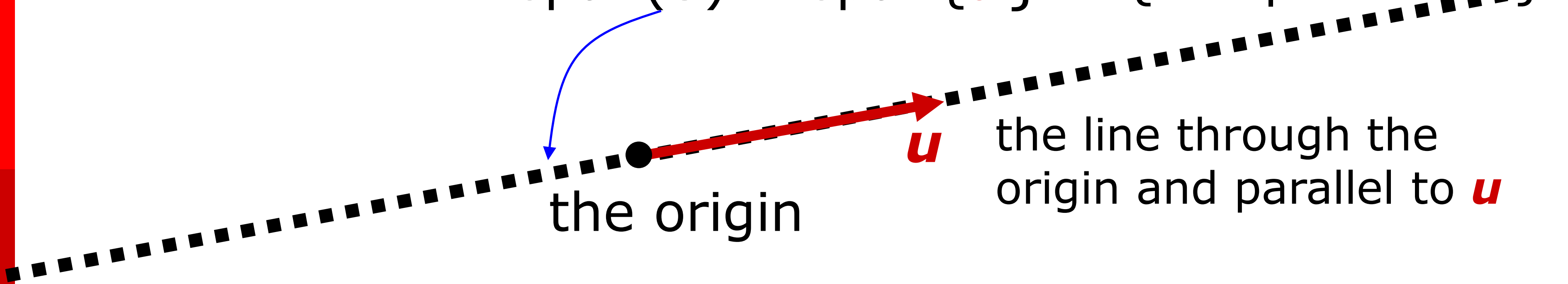
Discussion 3.2.14.1

span = extend across (Oxford Dictionary)

In $\mathbf{R}^2$ and $\mathbf{R}^3$

$S = \{\boldsymbol{u}\}$ ($\boldsymbol{u}$ is a non-zero vector)

$$\text{span}(S) = \text{span}\{\boldsymbol{u}\} = \{\, c\boldsymbol{u} \mid c \text{ in } \mathbf{R} \,\}$$



span(S) = span{$\boldsymbol{u}$} represents a line through the origin

Geometrical meaning of linear span

Discussion 3.2.14.2

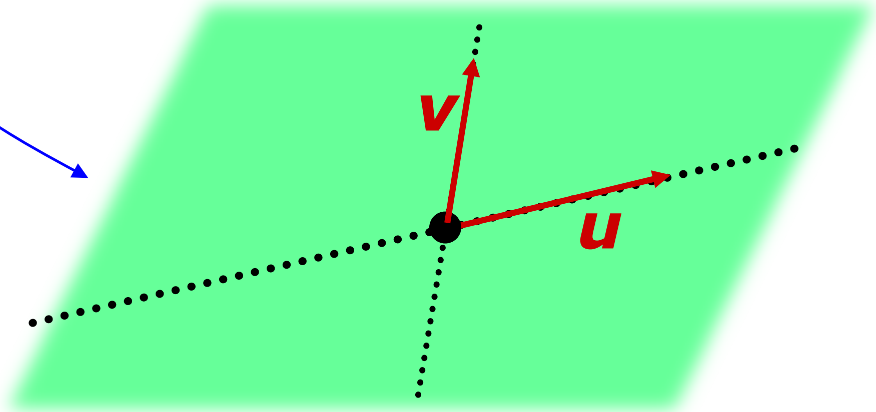
span = extend across (Oxford Dictionary)

In $\mathbf{R}^2$ and $\mathbf{R}^3$

$S = \{\boldsymbol{u}, \boldsymbol{v}\}$ ($\boldsymbol{u}$, $\boldsymbol{v}$ are two non-parallel vectors)

$$\text{span}(S) = \text{span}\{\boldsymbol{u}, \boldsymbol{v}\}$$
$$= \{ s\boldsymbol{u} + t\boldsymbol{v} \mid s, t \in \mathbf{R} \}$$

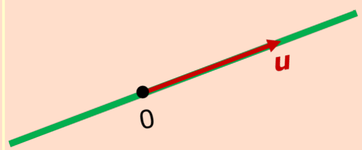
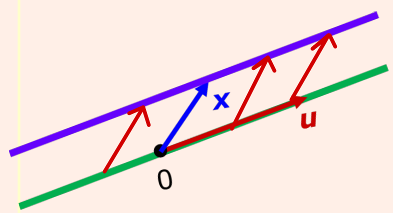
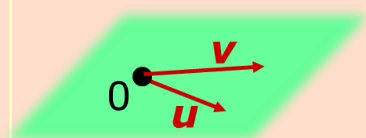
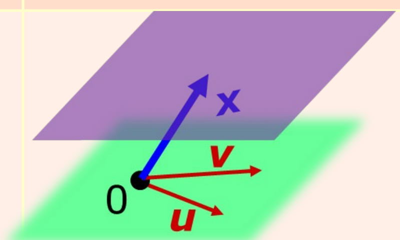
the plane containing the origin and parallel to $\boldsymbol{u}$ and $\boldsymbol{v}$



span(S) = span$\{\boldsymbol{u}, \boldsymbol{v}\}$ represents a plane through the origin

Lines and planes in terms of linear span

Discussion 3.2.15

Objects	Geometrical	Span	Set notation
Line through origin		$\text{span}\{\boldsymbol{u}\}$	$\{t\boldsymbol{u} \mid t \in \mathbf{R}\}$
Line not through origin		$\boldsymbol{x} + \text{span}\{\boldsymbol{u}\}$	$\{\boldsymbol{x} + t\boldsymbol{u} \mid t \in \mathbf{R}\}$ $\{\boldsymbol{x} + \boldsymbol{w} \mid \boldsymbol{w} \in \text{span}\{\boldsymbol{u}\}\}$
Plane through origin		$\text{span}\{\boldsymbol{u}, \boldsymbol{v}\}$	$\{t\boldsymbol{u} + s\boldsymbol{v} \mid t, s \in \mathbf{R}\}$
Plane not through origin		$\boldsymbol{x} + \text{span}\{\boldsymbol{u}, \boldsymbol{v}\}$	$\{\boldsymbol{x} + t\boldsymbol{u} + s\boldsymbol{v} \mid t, s \in \mathbf{R}\}$ $\{\boldsymbol{x} + \boldsymbol{w} \mid \boldsymbol{w} \in \text{span}\{\boldsymbol{u},\boldsymbol{v}\}\}$

Fill in the blanks

a vector in $\mathbb{R}^2$, a vector in $\mathbb{R}^3$, a line in $\mathbb{R}^3$, a plane in $\mathbb{R}^3$, the entire $\mathbb{R}^3$ space

1. A linear combination of two vectors in $\mathbb{R}^3$ is <u>a vector in $\mathbb{R}^3$</u>.
2. A linear combination of three vectors in $\mathbb{R}^3$ is <u>a vector in $\mathbb{R}^3$</u>.
3. A linear span of one non-zero vector in $\mathbb{R}^3$ is <u>a line in $\mathbb{R}^3$</u>.

Condition for the statement to be true

4. A linear span of two non-parallel vectors in $\mathbb{R}^3$ is <u>a plane in $\mathbb{R}^3$</u>.

3 vectors lie on the same plane

5. A linear span of three non-coplanar vectors in $\mathbb{R}^3$ is <u>the entire $\mathbb{R}^3$ space</u>.

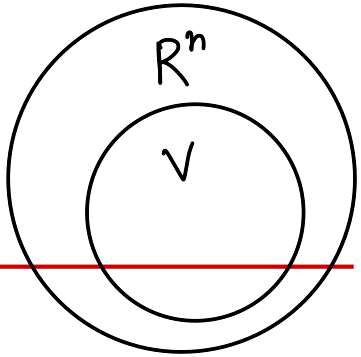
Section 3.3

Subspaces

Objective

- What is a subspace?
- How to show that a subset of $\mathbf{R}^n$ is a subspace?
- What are some subspaces of $\mathbf{R}^n$?
- What is a solution space of a LS?

What is a subspace of $\mathbf{R}^n$?



Definition 3.3.2

no condition

Let V be a subset of $\mathbf{R}^n$

condition applies

V is called a subspace of $\mathbf{R}^n$ provided …

there is a set $S = \{ \boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k} \}$ of $\mathbf{R}^n$

such that $V = \text{span}(S)$

condition of subspace

i.e. V can be expressed in linear span form.

$\text{span}\{v_1, v_2, v_3\}$

Every subspace of $\mathbf{R}^n$ is a subset of $\mathbf{R}^n$.

Not every subset of $\mathbf{R}^n$ is a subspace of $\mathbf{R}^n$.

$\{\mathbf{0}\}$ and $\mathbf{R}^n$ are subspaces of $\mathbf{R}^n$

Remark 3.3.3

condition of subspace $V = \text{span}(S)$

1. $\{\mathbf{0}\}$ is a subspace of $\mathbf{R}^n$. zero space

   Take $S = \{\mathbf{0}\}$

   $\{\mathbf{0}\} = \text{span}\{\mathbf{0}\}$

2. $\mathbf{R}^3$ is a subspace of $\mathbf{R}^3$.

   Take S to be standard basis vectors for $\mathbf{R}^3$

   $\boldsymbol{v}_1 = (1, 0, 0)$, $\boldsymbol{v}_2 = (0, 1, 0)$, $\boldsymbol{v}_3 = (0, 0, 1)$

   $\mathbf{R}^3 = \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ Refer to Example 3.2.2

$\mathbf{R}^n$ is a subspace of $\mathbf{R}^n$.

Take S to be standard basis vectors for $\mathbf{R}^n$

$\boldsymbol{e}_1 = (1, 0, \ldots, 0)$, $\boldsymbol{e}_2 = (0, 1, \ldots, 0)$, …, $\boldsymbol{e}_n = (0, \ldots, 0, 1)$

$\mathbf{R}^n = \text{span}\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$

How to show that a given subset is a subspace?

Example 3.3.4.1

$V_1 = \{ (a+4b, a) \mid a, b \in \mathbf{R} \}$ explicit form

$(a + 4b, a) = (a, a) + (4b, 0)$

$= a(1, 1) + b(4, 0)$ general linear combination

V_1 is the set of all linear combinations of $(1, 1)$ and $(4, 0)$

$\boxed{V_1 = \text{span}\{(1, 1), (4, 0)\}}$ linear span form

V_1 is a subspace of $\mathbf{R}^2$

In fact $V_1 = \mathbf{R}^2$ span of 2 vectors is a plane $= R^2$

How to show that a given subset is a subspace?

Example 3.3.4.2

$V_2 = \{ (x, y, z) \mid x + y - z = 0 \}$ implicit form

$V_2 = \{ (t - s, s, t) \mid s, t \in \mathbf{R} \}$ explicit form

$(t - s, s, t) = (t, 0, t) + (- s, s, 0)$

$= t(1, 0, 1) + s(-1, 1, 0)$

general linear combination

V_2 is the set of all linear combinations of $(1, 0, 1)$ and $(-1, 1, 0)$

$\boxed{V_2 = \text{span}\{(1, 0, 1), (-1, 1, 0)\}}$ linear span form

V_2 is a subspace of $\mathbf{R}^3$

In fact V_2 is a plane in $\mathbf{R}^3$.

How to show a given subset is not a subspace?

Example 3.3.4.3

$V_3 = \{ (1, a) \mid a \text{ in } \mathbf{R} \}$ subset of $\mathbf{R}^2$

need a parameter

$(1, a) = (1, 0) + (0, a) = (1, 0) + a(0, 1)$

linear Combi but not a general linear combination

V_3 is not a linear span of "any" set of vectors

"So" V_3 is not a subspace of $\mathbf{R}^2$

There is an easier way: Use theorem 3.2.9.1

$(0, 0) \notin V_3 = \{ (1, a) \mid a \text{ in } \mathbf{R} \}$

$\Rightarrow$ not a subspace of $\mathbf{R}^2$

If a subset of $\mathbf{R}^n$ does not contain the zero vector $\mathbf{0}$, then it is not a linear span.

How to show a given subset is not a subspace?

Example 3.3.4.4

$V_4 = \{ (x, y, z) \mid x^2 \le y^2 \le z^2 \}$ subset of $\mathbf{R}^3$

e.g. (1, 1, 2), (1, 1, -2), (0, 0, 0) $\in V_4$

Note: Having zero vector in a set V
does not guarantee V is a subspace

Take two vectors in V,
show that the sum is not in V.

Use theorem 3.2.9.2

$(1, 1, 2) + (1, 1, -2) = (2, 2, 0) \notin V_4$ Not a linear span

Violate the closure property of linear span
(theorem 3.2.9.2)

So V_4 is not a subspace of $\mathbf{R}^3$

Remark 3.3.5.1

The following are all the subspaces of $\mathbf{R}^2$:

a. $\{\mathbf{0}\}$ spanned by zero vector $\mathbf{0}$

b. any line that passes through the origin spanned by one non-zero vector $\boldsymbol{u}$

c. $\mathbf{R}^2$ spanned by two non-parallel vectors $\boldsymbol{u}, \boldsymbol{v}$

Why are there no other subspaces of $\mathbf{R}^2$?

$$V = \text{span}\{\, \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \ldots, \boldsymbol{u}_k \,\}$$

all parallel → a line

at least two not parallel → XY-plane $\mathbf{R}^2$

Geometrical interpretation of subspaces of $\mathbf{R}^3$

Remark 3.3.5.2

The following are all the subspaces of $\mathbf{R}^3$:

a. $\{\mathbf{0}\}$ — spanned by zero vector $\mathbf{0}$

b. any line through the origin — spanned by one non-zero vector $\boldsymbol{u}$

c. any plane containing the origin — spanned by two non-parallel vectors $\boldsymbol{u}, \boldsymbol{v}$

d. $\mathbf{R}^3$ — spanned by three vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ not lying on a plane

What is a solution space?

Theorem 3.3.6

Closure properties under vector addition and scalar multiplication

$\boldsymbol{Ax} = \mathbf{0}$

The solution set of a homogeneous linear system in n variables is a subspace of $\mathbf{R}^n$.

The solution set of every homogeneous LS can be written as a linear span

We call it the solution space of the system.

The solution set of non-homogeneous LS is not a subspace of $\mathbf{R}^n$.

Example 3.3.7

Homogeneous system

$$\begin{cases} x - 2y + 3z = 0 \\ 2x - 4y + 6z = 0 \\ 3x - 6y + 9z = 0 \end{cases}$$

general solution

$$\begin{cases} x = 2s - 3t \\ y = s \\ z = t \end{cases}$$

subspace of $\mathbf{R}^3$

linear span form

solution set

span{(2, 1, 0), (-3, 0, 1)} = {(2*s* – 3*t*, *s*, *t*) | *s*, *t* in **R**}

$s(2, 1, 0) + t(-3, 0, 1)$

general linear combination

Remark 3.3.8

Let V be a non-empty subset of $\mathbf{R}^n$.
Then

V is a subspace of $\mathbf{R}^n$

if and only if

for all $\boldsymbol{u}, \boldsymbol{v} \in V$ and $c, d \in \mathbf{R}$, $c\boldsymbol{u} + d\boldsymbol{v} \in V$.

closure properties under addition & scalar multiplication

This is the actual definition of subspaces in abstract linear algebra.

To show a subset V is a subspace,
(i) check that it contains the zero vector;
(ii) take two general vectors $\boldsymbol{u}$, $\boldsymbol{v}$ in V and $c, d \in \mathbf{R}$, show that $c\boldsymbol{u} + d\boldsymbol{v} \in V$.

To show subspace (or not)

To show a subset S of $\mathbf{R}^n$ is a subspace:

- Express S as a linear span
- Show that S is the solution set of a homogeneous system
- (For $\mathbf{R}^2$ and $\mathbf{R}^3$) show that S represents a line or plane through origin.

To show a subset S of $\mathbf{R}^n$ is not a subspace:

- Show that the zero vector is not in S
- Find $\mathbf{u}, \mathbf{v} \in S$ such that $\mathbf{u} + \mathbf{v} \notin S$
- Find $\mathbf{v} \in S$ and a scalar c such that $c\mathbf{v} \notin S$
- (For $\mathbf{R}^2$ and $\mathbf{R}^3$) show that S is not a line or plane through origin.