NATIONAL UNIVERSITY OF SINGAPORE

CS1231/CS1231S – DISCRETE STRUCTURES

(Semester 2: AY2020/21)

Time Allowed: 2 Hours

INSTRUCTIONS

1. This assessment paper contains **TWENTY ONE (21)** questions (excluding question 0) in **THREE (3)** parts and comprises **TEN (10)** printed pages.
2. Answer ALL questions.
3. This is an **OPEN BOOK** assessment.
4. The maximum mark of this assessment is 100.

Question	Max. mark
Q0	3
Part A: Q1 – 10	20
Part B: Q11 – 16	18
Part C: Q17	5
Part C: Q18	8
Part C: Q19	6
Part C: Q20	20
Part C: Q21	20
Total	**100**

Section A: 41 marks.

Section B: 59 marks.

5. You are to submit a single pdf file (size $\leq$ 20MB) to your submission folder on LumiNUS.
6. The number of pages in your file should not exceed 6.
7. Your submitted file should be named after your **Student Number** (eg: A1234567X.pdf) and your Student Number should be written on the first page of your file.
8. <u>Do NOT write your name</u> anywhere in your submitted file.

——— END OF INSTRUCTIONS ———

0. **Check that you have done the following:**
   (a) Submission folder consists of a **single pdf file** and no other files. [1 mark]
   (b) File named correctly with **Student Number** (eg: A1234567X.pdf). [1 mark]
   (c) Student Number written **on top of the first page** of submitted file. [1 mark]

Part A: Multiple Choice Questions [Total: 10×2 = 20 marks]

Each multiple choice question (MCQ) is worth **TWO marks** and has exactly **one** correct answer. You are advised to write your answers on a **single line or two lines** to conserve space. For example:

1. A   2. B   3. C   4. D   ...

Please write in **CAPITAL LETTERS**.

1. Which of the following is/are true?
   (i) To prove that $\forall n \in \mathbb{Z}_{\geqslant 0}\ P(n)$ is true, where each $P(n)$ is a proposition, it suffices to
   - show that $P(0)$, $P(1)$ are true; and
   - show that $\forall k \in \mathbb{Z}_{\geqslant 0}\ \big(P(k) \Rightarrow P(k+1)\big)$ is true.

   (ii) To prove that $\forall n \in \mathbb{Z}_{\geqslant 0}\ P(n)$ is true, where each $P(n)$ is a proposition, it suffices to
   - show that $P(0)$ is true; and
   - show that $\forall k \in \mathbb{Z}_{\geqslant 0}\ \big(P(k) \Rightarrow P(k+2)\big)$ is true. X

   A. (i) and (ii).
   B. (i) only.
   C. (ii) only.
   D. None.

B.

bijection for g and h.

2. Let $A = \{1,2,3\}$. Define $g, h\colon A \to A$ by setting, for all $x \in A$,

$$g(x) = \begin{cases} 1, & \text{if } x = 2; \\ 2, & \text{if } x = 1; \\ x, & \text{otherwise,} \end{cases} \qquad h(x) = \begin{cases} 2, & \text{if } x = 3; \\ 3, & \text{if } x = 2; \\ x, & \text{otherwise.} \end{cases}$$

$h = h^{-1}$.

→ input num

times applied to :-

What is the order of the function $g \circ (g \circ h)^{-1} \circ \cancel{h \circ h} \circ g \circ \cancel{h \circ h}$? ← start from.

$f f^{-1} = x$.

identity.

A. 1.
B. 2.
C. 3.
D. 4.

⇓ order = repeated?

x^{sm}

min no. of $g \circ h \circ \ldots$

$h^{-1}og$ goh

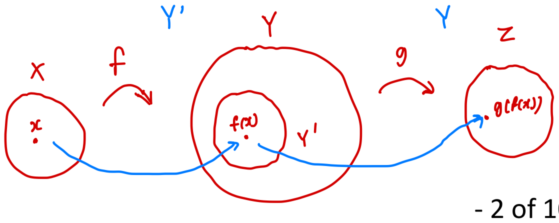
2	1
3	2
1	3

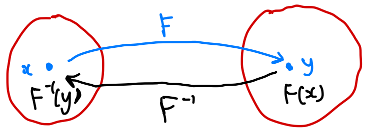
A × C

$\underbrace{g \circ g \circ g}_{= g} \Rightarrow 3$

$\underline{goh}$

$g \circ f$: range of f is a subset of domain of g.





D

3. For a set A and a function $f: A \to A$, define (set of all f^{-1})

$$C_f = \{f^{-1}(\{y\}) : y \in A \text{ and } f(y) = y\}.$$

(such that)

Which of the following is/are true for all sets A and all functions $f: A \to A$?

(i) If $f \circ f = f$, then C_f is a partition of A. (no empty set; 2 comp intersect → 2 comp equal; all comp union tgt)

(ii) If C_f is a partition of A, then $f \circ f = f$.

A. (i) and (ii).

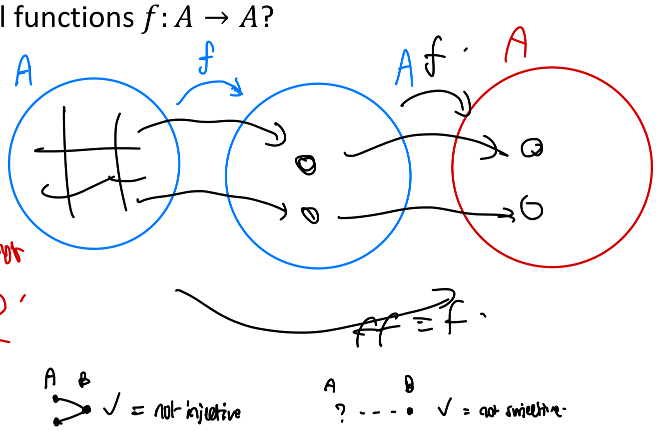
B. (i) only.

C. (ii) only.

D. None.

f is idempotent. $f(f(x)) = f(x)$.

A



4. Which of the following is/are true for all functions $f: A \to B$ and all finite subsets $X \subseteq A$, assuming $f(X)$ is finite?

(i) $|f(X)| \leqslant |X|$. range and not co-domain. range can be smaller than codomain.

(ii) $|f(X)| \geqslant |X|$.

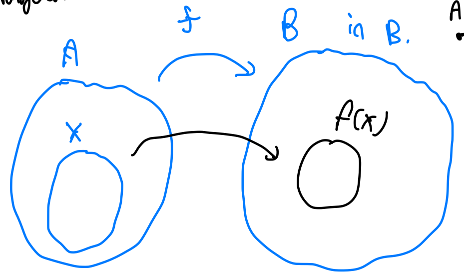
every element in A is related to some element in B; no element in A is related to more than one element in B.

A. (i) and (ii).

B. (i) only.

C. (ii) only.

D. None.



B.

5. Which of the following is/are true for all functions $f: A \to B$ and all finite subsets $Y \subseteq B$, assuming $f^{-1}(Y)$ is finite? (may not exist)

~~(i) $|f^{-1}(Y)| \leqslant |Y|$.~~

(ii) $|f^{-1}(Y)| \geqslant |Y|$.

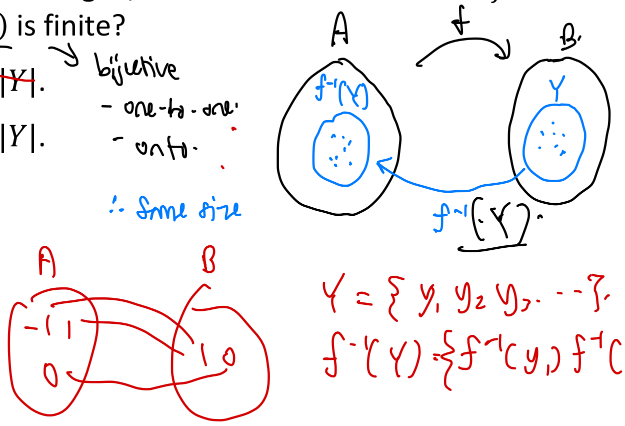
bijective - one-to-one - onto. ∴ same size

A. (i) and (ii).

B. (i) only.

C. (ii) only.

D. None.



$Y = \{y_1, y_2, y_3, \ldots\}$. A.

$f^{-1}(Y) = \{f^{-1}(y_1), f^{-1}(y_2) \ldots\}$.

6. Let Bool $= \{\mathbf{true}, \mathbf{false}\}$. Define $f: \text{Bool}^2 \to \text{Bool}^2$ by setting, for all $p, q \in \text{Bool}$,

$$f(p, q) = ({\sim}p \wedge q, p \wedge {\sim}q).$$

Which of the following is true?

A. f is injective and surjective.

B. f is injective but not surjective.

C. f is surjective but not injective.

D. f is neither injective nor surjective.

p	q	¬p∧q	p∧¬q
T	T	F	F
T	F	F	T
F	T	T	F
F	F	F	F

D.

∴ 2 diff input → same output ∴ not injective.

T, T is not a possible output → not surjective.

A $\{\}$ → infinite. not infinite.   infinite. A. is infinite.

7. Which of the following is/are true?

(i) If A is a set of sets that has an element that is infinite, then A is infinite. ✗

(ii) If A is a set of sets that has a subset that is infinite, then A is infinite. ✓

A. (i) and (ii).

B. (i) only.

C. (ii) only.

D. None.

C

8. Define $f : \mathcal{P}(\mathbb{Z}_{\geqslant 0}) \setminus \{\emptyset\} \to \mathbb{Z}_{\geqslant 0}$ by setting $f(S)$ to be the smallest element of S whenever $S \in \mathcal{P}(\mathbb{Z}_{\geqslant 0}) \setminus \{\emptyset\}$. Which of the following is/are true?

(i) The function f has an inverse. ✗ no bijection as is not one-to-one.

(ii) $f^{-1}(\{n\})$ is uncountable for some $n \in \mathbb{Z}_{\geqslant 0}$. ✗ ~~not uncountable.~~ uncountable'

may not exist.

A. (i) and (ii).

B. (i) only.

C. (ii) only.

D. None.

$P(\mathbb{Z}_{\geq 0}) \setminus \{\emptyset\}$ $\mathbb{Z}_{\geq 0}$

uncountable f countable

S . S' . . $f(S)$. $\{n\}$

$f^{-1}(\{n\})$ f^{-1} : one to one correspondence.

D.

9. How many permutations of **AprilFools!** are there?

A. 9!

B. 11!

C. $\frac{11!}{2}$

D. $\frac{11!}{4}$

E. $\frac{11!}{24}$

2 0
2 l.

D.

10. How many **walks of length 3** are there from vertex a to vertex d in the directed graph given below?

A. 6

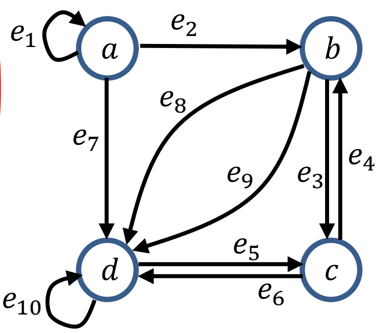
B. 7

C. 8

D. 9

E. 10

$$\begin{array}{c|c} & \begin{matrix} a & b & c & d \end{matrix} \\ \hline \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$



. $e_1 e_1 e_7$
. $e_1 e_2 e_8$
. $e_1 e_2 e_9$
. $e_2 e_3 e_6$
. $e_7 e_{10} e_{10}$
. $e_2 e_8 e_{10}$
. $e_2 e_9 e_{10}$
. $e_7 e_5 e_6$
$e_1 e_7 e_{10}$

D

Part B: Multiple Response Questions [Total: 6×3 = 18 marks]

Each multiple response question (MRQ) is worth **THREE marks** and may have one answer or multiple answers. Write out **all** correct answers. For example, if you think that A, B, C are the correct answers, write A, B, C.

Only if you get all the answers correct will you be awarded three marks. **No partial credit will be given for partially correct answers.**

You are advised to write your answers on **a single line** to conserve space. For example:

11. A,B   12. B,D   13. C   14. A,B,C,D   …

Please write in **CAPITAL LETTERS**.

11. Recall that $\{s,u\}^*$ denotes the set of all strings over $\{s,u\}$. Define a subset $T \subseteq \{s,u\}^*$ recursively as follows.

ssuu. ssussuuu.

- $su \in T$. (base clause)
- If $\sigma, \tau \in T$, then $s\sigma u \in T$ and $s\sigma\tau u \in T$. (recursion clause) |S| = |u|
- Membership for T can always be demonstrated by (finitely many) successive applications of clauses above. (minimality clause)

su
ssuu
ssussuuu
sssuuu
s ssuu ssussuuuu

susuu.

Which of the following is/are in T?

A. ssssusuuusuu

B. ~~ssussususuuu~~

C. ~~ssussuusssuu~~ 5 v 7s.

D. ~~ussususussuu~~

12. Define a subset $S \subseteq \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$ recursively as follows.

- $(1,0) \in S$. (base clause)
- If $(m_1, n_1), (m_2, n_2) \in S$, then (recursion clause) will be repeated

$$(m_1 + 1, n_1 + 1) \in S \quad \text{and} \quad (m_1 + m_2 + 1, n_1 + n_2 + 2) \in S.$$

- Membership for S can always be demonstrated by (finitely many) successive applications of clauses above. (minimality clause)

$a_0 = (1,0)$
$a_1 = (2,1)$
$a_2 = (3,2)$
$a_3 = (4,3)$
$a_4 = (5,4)$
$a_5 = (6,5)$
$a_6 = (7,6)$ pattern.

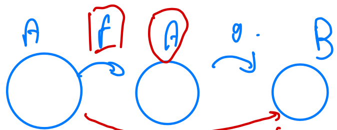
Which of the following is/are in S?

A. $(12,31)$.

B. $(31,12)$.

C. $(1231,1230)$. ✓

D. $(2021,1231)$.



codomain is not A forg

identity function : bijective

13. Which of the following is/are true for all sets A and all functions $f\colon A \to A$? $g \circ f$

A. ~~$f = \mathrm{id}_A$~~ if and only if $g \circ f = g$ for some function g with domain A.

B. $f = \mathrm{id}_A$ if and only if $g \circ f = g$ for all injective functions g with domain A.

C. $f = \mathrm{id}_A$ if and only if $g \circ f = g$ for all surjective functions g with domain A.

D. $f = \mathrm{id}_A$ if and only if $g \circ f = g$ for all bijective functions g with domain A.

E. $f = \mathrm{id}_A$ if and only if $\mathrm{id}_A \circ f = \mathrm{id}_A$.

$g(1,2)$ 1

$f(1,2)$ 1

$\mathrm{id}_A(f(x)) = \mathrm{id}_A(x)$

$f(x)$: $y = x$

$g(x)$: $y = 2x$.

$f(g(x)) = 2x$

14. Which of the following sets is/are countable?

A. The set of all partitions of $\mathbb{Z}$. → infinite uncountable partitions?

infinitely many ways to divide the partition. like the power set.

B. The set of all partial orders on $\mathbb{Z}$. → infinite uncountable partitions.

C. The set of all functions $\mathbb{Z} \to \mathbb{Z}$. → infinite uncountable.

D. The set $\mathbb{Z}^*$ of all strings over $\mathbb{Z}$. countable.

E. The set of all simple undirected graphs whose vertex set is a finite subset of $\mathbb{Z}$.

15. Given the following statement:

{ } 18. {...} {...}.

$$\exists x \in \mathbb{Z}\,\big(P(x) \wedge Q(x)\big) \equiv \big(\exists x \in \mathbb{Z}\, P(x)\big) \wedge \big(\exists x \in \mathbb{Z}\, Q(x)\big)$$

what are the predicates $P(x)$ and $Q(x)$ that can make the above statement false?

A. $P(x) = Q(x)$. essentially $\exists x \in \mathbb{Z}(P(x)) \equiv \exists x \in \mathbb{Z}(P(x))$. $T \equiv T \wedge T$.

Truth val.

B. $P(x) =$ "x is prime"; $Q(x) =$ "x is composite". $F \equiv T \wedge T$.

C. $P(x) =$ "x is divisible by 3"; $Q(x) =$ "x is divisible by 5". $T \equiv T \wedge T$.

D. $P(x) =$ "x is even"; $Q(x) =$ "x is odd". zero is even

E. $P(x) =$ "x is even"; $Q(x) =$ "x is an even prime". 2 $T \equiv T \wedge T$.

16. Given a complete graph K_{99} and a complete bipartite graph $K_{80,60}$, which of the following statements is/are true? vertices - even vertices. even.

A. K_{99} is an Eulerian graph.

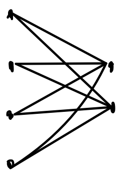
B. K_{99} is a Hamiltonian graph.

C. $K_{80,60}$ is an Eulerian graph. no circuit for eulerian.

D. $K_{80,60}$ is a Hamiltonian graph.

Hamiltonian graph: contains hamiltonian circuits. — simple circuit that includes every vertex of g. → each vertex once

Euler graph : contains euler circuits. — circuit that contains every vertex and every edge of g. traverse once



Part C: There are 5 questions in this part [Total: 59 marks]

17. [5 marks]

Define a sequence $a_1, a_2, a_3, \ldots$ by setting, for each $n \in \mathbb{Z}^+$,

$$a_1 = \frac{1}{10} \quad \text{and} \quad a_{n+1} = a_n + \frac{1}{(3n+2)(3n+5)}.$$

Prove using (usual or strong) induction that, for all $n \in \mathbb{Z}^+$,

$$a_n = \frac{n}{2(3n+2)}.$$

Marks will not be given for proofs that do not involve induction.

18. [8 marks]

Let A be a set and $\mathcal{C}$ be a partition of A. Prove that there exists a function $f: A \to A$ such that

$$f \circ f = f \quad \text{and} \quad \mathcal{C} = \{f^{-1}(\{y\}) : y \in A \text{ and } f(y) = y\}.$$

19. [6 marks]

Let A be a set. Let S be the set of all functions $\{0,1\} \to A$, i.e.,

$$S = \{\alpha \mid \alpha: \{0,1\} \to A\}.$$

Prove that $|S| = |A^2|$ according to Cantor's definition of same-cardinality.

17. [5 marks]

Define a sequence $a_1, a_2, a_3, \ldots$ by setting, for each $n \in \mathbb{Z}^+$,

$$a_1 = \frac{1}{10} \quad \text{and} \quad a_{n+1} = a_n + \frac{1}{(3n+2)(3n+5)}.$$

Prove using (usual or strong) induction that, for all $n \in \mathbb{Z}^+$,

$$a_n = \frac{n}{2(3n+2)}.$$

Marks will not be given for proofs that do not involve induction.

1. Let $P(n)$ be the proposition " $a_n = \frac{n}{2(3n+2)}$ ".

2. (base step)
$P(1)$ is $\frac{1}{2(3(1)+2)} = \frac{1}{2(5)} = \frac{1}{10} = a_1$ (from question).
$\therefore$ $P(1)$ is true.

3. (induction step). Suppose $P(k)$ is true where $k \in \mathbb{Z}^+$, then

$$P(k) = \frac{k}{2(3k+2)} = a_k$$

3.1 (show that $P(k+1)$ is true)

$$P(k+1) = a_{k+1} = a_k + \frac{1}{(3k+2)(3k+5)} \quad \text{(from question)}$$

$$= \frac{k}{2(3k+2)} + \frac{1}{(3k+2)(3k+5)} \quad \text{(from induction step).}$$

$$= \frac{(3k+5)k + 2}{2(3k+2)(3k+5)}$$

$$= \frac{3k^2+5k+2}{2(3k+2)(3k+5)}$$

$$= \frac{\cancel{(3k+2)}(k+1)}{2\cancel{(3k+2)}(3k+5)}$$

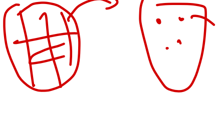
$$= \frac{(k+1)}{2(3k+5)}$$

$$= \frac{k+1}{2(3k+3+2)}$$

$$= \frac{k+1}{2(3(k+1)+2)}$$

4. $\therefore$ By MI, since $P(k) \Rightarrow P(k+1)$, then $\forall n \in \mathbb{Z}^+ \left(a_n = \frac{n}{2(3n+2)}\right)$.

A each C A



some flaw

18. [8 marks]

Let A be a set and $\mathcal{C}$ be a partition of A. Prove that there exists a function $f : A \to A$ such that

existential proof.

$$f \circ f = f \quad \text{and} \quad \mathcal{C} = \{f^{-1}(\{y\}) : y \in A \text{ and } f(y) = y\}.$$

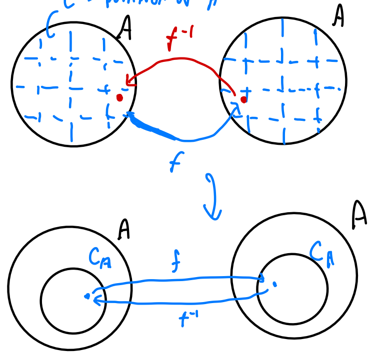
set of $\quad f^{-1}$

$f \circ f = f$

$f(f(x)) = f(x)$.

$f(y) = y$.

C = partition of A.



bijection.

disjoint -

$x \in f^{-1}(A)$

$f(x) \in A$

$x \in f^{-1}(\{y\})$

$f(x) \in \{y\}$

$\therefore f(x) = y$

whenever $x \in f^{-1}(\{y\})$.

1. Suppose $S \in \mathcal{C}$ where $\mathcal{C}$ is the partition of A.
   1.1. Then let $f : A \to A$ such that $f(x) = y$.
      1.1.1 Then $f \circ f = f(f(x)) = f(y)$
   1.2 By definition of $\mathcal{C}$, $f(y) = y = f(x)$.
   1.3. Hence, $f \circ f = f$.

Date No.

C = partition of A.

Prove $\exists\ f : A \to A$ s.t. $f \circ f = f$, $C = \{f^{-1}(\{y\}) : y \in A$ and $f(y) = y\}$.

$A = \bigsqcup_{\alpha \in I} C_\alpha$ $\quad$ I = index set (a set used to 'label' another set)

$\bigsqcup$ = disjoint union

$f(C_\alpha) = a$ representative of C_α, say a.

19. [6 marks]

all function domain of 2 numbers.

Let A be a set. Let S be the set of all functions $\{0,1\} \to A$, i.e.,

$\alpha \mid \alpha : \{0,1\} \to A.$

$$S = \{\alpha \mid \alpha : \{0,1\} \to A\}.$$

Prove that $|S| = |A^2|$ according to Cantor's definition of same-cardinality.

prove $f : S \to A \times A$ is bijective.

form.

$\llcorner$ A has the same cardinality as B iff there is a function f from A to B that is one-to-one and onto

$\vdots$	$\vdots$
$1 \times d$	$2 \times d$
$1 \times c$	$2 \times c$
$1 \times b$	$2 \times b$
$1 \times a$	$2 \times a$

1. Let f be a function such that $f \in S = \{\alpha \mid \alpha : \{0,1\} \to A\}$.

2. Then $f : \{0,1\} \to A$.

   2.1 Since every $f \in S$ is unique

$f(g) = (a, b)$

20. Counting and Probability [Total: 20 marks]

<u>Note that working is not required for parts (a) to (d)</u>.

(a) How many integer solutions for x_1, x_2, x_3 and x_4 does the following equation have, given that $x_i \geq 2^i + i$, for $1 \leq i \leq 4$?

$$\boldsymbol{x_1 + x_2 + x_3 + x_4 = 56.}$$

Write your answer as a single number. [3 marks]

(b) On a die there are 6 numbers. We call 4, 5, 6 the big numbers and 1, 2, 3 the small numbers. Given a loaded die in which the probability of rolling any fixed big number is twice the probability of rolling any fixed small number, answer the following questions.

(i) What is the probability of rolling a 6? Write your answer as a single fraction. [1 mark]

(ii) If two such loaded dice are rolled, what is the expected value of the maximum of the two dice? Write your answer as a single fraction. [3 marks]

(c) There are three urns U_1, U_2 and U_3. Urn U_k $(1 \leq k \leq 3)$ contains k red balls and $k+1$ blue balls.

(i) If you draw 2 balls at random from U_2 without replacement, what is the probability of drawing **at least one red ball**? Write your answer as a single fraction. [2 marks]

(ii) Four words "I", "CAN", "DO" and "IT" are separately written on 4 slips of paper and concealed, each having an equal chance of being selected.

You select a slip of paper at random and reveal the word written on it. The length of the word, k, directs you to urn U_k to pick a ball. If the ball picked is **blue**, what is the probability that it comes from U_2? Write your answer as a single fraction or a percentage rounded to 4 significant figures. [3 marks]

(d) Aaron loves Hawaii and is always looking forward to spending his holiday there. He is intrigued to hear that new bridges will be constructed on the islands of Maui, Moloka'i and Lana'i, making a total of 13 bridges between these islands. Each bridge connects two islands.

If there are 11 bridges with one end on Maui and 6 with one end on Lana'i, (i) how many bridges are there connecting Maui and Moloka'i, and (ii) how many bridges are there connecting Moloka'i and Lana'i? [4 marks]

(e) You are to pick 14 numbers from 1 through 20. Is it true that no matter how you pick the 14 numbers, there is always a pair of numbers such that one is three times the other? Explain your answer. [4 marks]

(a) How many integer solutions for x_1, x_2, x_3 and x_4 does the following equation have, given that $x_i \geq 2^i + i$, for $1 \leq i \leq 4$?

$$x_1 + x_2 + x_3 + x_4 = 56.$$

Write your answer as a single number. [3 marks]

$x_1 \geq 2^1+1 = 3.$
$x_2 \geq 2^2+2 = 6.$
$x_3 \geq 2^3+3 = 11$
$x_4 \geq 2^4+4 = 20.$

bijective principle.
multiset.

$\therefore 56-20-11-6-3 = 16.$

$C^{16+4-1}_{16} = C^{19}_{16} = 3876$ // 969.

r-combi w/ repeat.

(b) On a die there are 6 numbers. We call 4, 5, 6 the big numbers and 1, 2, 3 the small numbers. Given a loaded die in which the probability of rolling any fixed big number is twice the probability of rolling any fixed small number, answer the following questions.

(i) What is the probability of rolling a 6? Write your answer as a single fraction. [1 mark]

(ii) If two such loaded dice are rolled, what is the expected value of the maximum of the two dice? Write your answer as a single fraction. [3 marks]

i) $\frac{2}{9}$

ii)

cases	probability.
(1,1)	$\frac{1}{9}\cdot\frac{1}{9}$
(1,2)(2,1)(2,2)	$\frac{1}{9}\cdot\frac{1}{9}+\frac{1}{9}\cdot\frac{1}{9}+\frac{1}{9}\cdot\frac{1}{9}$
(1,3)×2, (2,3)×2, (3,3)	$\frac{5}{81}$
(1,4)×2, (2,4)×2, (3,4)×2, (4,4)	$\frac{16}{81}$
(1,5)×2, (2,5)×2, (3,5)×2, (4,5)×2, (5,5)	$\frac{24}{81}$
(1,6)×2, (2,6)×2, (3,6)×2, (4,6)×2, (5,6)×2, (6,6)×2.	$\frac{32}{81}$

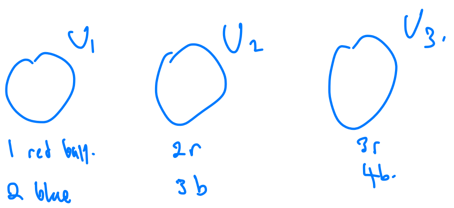
E.V. = $\frac{398}{81}$.

(c) There are three urns U_1, U_2 and U_3. Urn U_k $(1 \leq k \leq 3)$ contains k red balls and $k+1$ blue balls.

(i) If you draw 2 balls at random from U_2 without replacement, what is the probability of drawing **at least one red ball**? Write your answer as a single fraction. [2 marks]

(ii) Four words "I", "CAN", "DO" and "IT" are separately written on 4 slips of paper and concealed, each having an equal chance of being selected.

You select a slip of paper at random and reveal the word written on it. The length of the word, k, directs you to urn U_k to pick a ball. If the ball picked is **blue**, what is the probability that it comes from U_2? Write your answer as a single fraction or a percentage rounded to 4 significant figures. [3 marks]



i) P(at least one red ball) = P(choose 2) − P(no red ball)
$= \left(\frac{5}{5}\right)\left(\frac{4}{4}\right) - \left(\frac{3}{5}\right)\left(\frac{2}{4}\right)$
$= \frac{7}{10}.$

ii)

Word	length		probability
I	1	$\rightarrow U_1$	$\frac{1}{4}$
do	2	$\rightarrow U_2$	$\frac{1}{4}$
it	2	$\rightarrow U_2$	$\frac{1}{4}$
Can	3	$\rightarrow U_3$.	$\frac{1}{4}$

$\left.\frac{1}{4}, \frac{1}{4}\right\} \frac{2}{4}.$

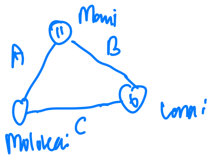
$\therefore P(U_2 \mid \text{blue}) = \dfrac{P(U_2 \cap \text{blue})}{P(\text{blue})}$

$= \dfrac{\frac{2}{4} \times \frac{3}{5}}{\frac{1}{4}\times\frac{2}{3} + \frac{2}{4}\times\frac{3}{5} + \frac{1}{4}\times\frac{4}{7}}$

$= \frac{63}{128}.$

(d) Aaron loves Hawaii and is always looking forward to spending his holiday there. He is intrigued to hear that new bridges will be constructed on the islands of Maui, Moloka'i and Lana'i, making a total of 13 bridges between these islands. Each bridge connects two islands.

If there are 11 bridges with one end on Maui and 6 with one end on Lana'i, (i) how many bridges are there connecting Maui and Moloka'i, and (ii) how many bridges are there connecting Moloka'i and Lana'i? [4 marks]



i) 7.

ii) 2.

$\therefore A+B+C=13.$

$A+B=11 \quad A=11-B. \qquad A=7$

$B+C=6 \quad C=6-B. \qquad C=2.$

$\therefore 11-\cancel{B+B}+6-B=13.$

$17-B=13 \Rightarrow B=4.$

(e) You are to pick 14 numbers from 1 through 20. Is it true that no matter how you pick the 14 numbers, there is always a pair of numbers such that one is three times the other? Explain your answer. [4 marks]

1 2 3 4 5 6 7 8 9 10

11 12 13 14 15 16 17 18 19 20

No.

Counter-example: {2, 3, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 20}.

There exist a set that contains no pair of numbers such that one is 3 times the other.

21. Graphs and Trees [Total: 20 marks]

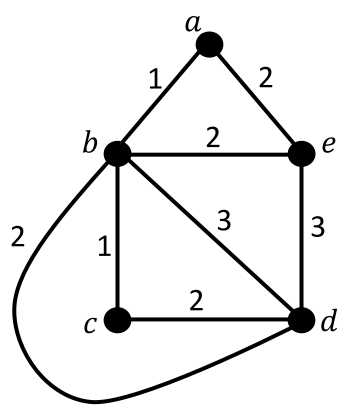
(a) The **pre-order traversal** and **post-order traversal** of a full binary tree with 9 vertices are given below:

Pre-order: **U C A N D O I T !**

Post-order: **A N C I T O ! D U**

Draw the full binary tree (clearly!). [3 marks]

(b) In the following weighted graph, how many minimum spanning trees are there? You do not need to provide working or diagrams. [3 marks]



(c) A *height-balanced binary tree* (or simply *balanced binary tree*) is a binary tree in which the heights of the left and right subtrees under any vertex differ by not more than one.

Draw all balanced <u>full</u> binary trees with 9 vertices. [4 marks]

(d) A *regular graph* is a simple undirected graph where every vertex has the same degree.
A *2-regular graph* is a regular graph where every vertex has degree 2.

Prove or disprove the following statement:

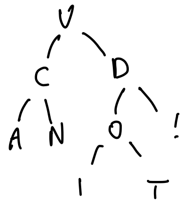
All 2-regular graphs are connected graphs. [3 marks]

(a) The **pre-order traversal** and **post-order traversal** of a full binary tree with 9 vertices are given below:

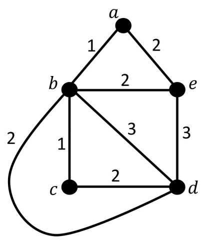
Pre-order: **U C A N D O I T !** 9 elems.

Post-order: **A N C I T O ! D U**

Draw the full binary tree (clearly!). [3 marks]



(b) In the following weighted graph, how many minimum spanning trees are there? You do not need to provide working or diagrams. [3 marks]



abced 2x2

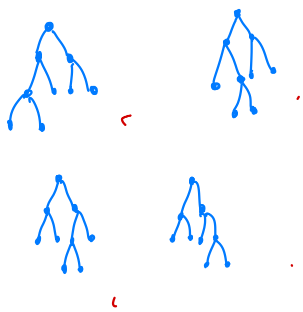
abced =4

doesn't matter where start.

(c) A *height-balanced binary tree* (or simply *balanced binary tree*) is a binary tree in which the heights of the left and right subtrees under any vertex differ by not more than one.

Draw all balanced full binary trees with 9 vertices. [4 marks]

4

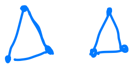


(d) A *regular graph* is a simple undirected graph where every vertex has the same degree.
A *2-regular graph* is a regular graph where every vertex has degree 2.

Prove or disprove the following statement:

All 2-regular graphs are connected graphs. [3 marks]

No.



∴ not connected but have 2 degree

21. Graphs and Trees (continue…)

(e) Assume that all graphs in this question are simple undirected graphs.

Given two simple undirected graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ where $V_1 \cap V_2 = \emptyset$ and $E_1 \cap E_2 = \emptyset$, the *graph union*, *graph join* and *graph product* are defined as follows:

Graph union: The union $G_\cup = G_1 \cup G_2$ has vertex set $V_\cup = V_1 \cup V_2$ and edge set $E_\cup = E_1 \cup E_2$.

Graph join: The join $G_+ = G_1 + G_2$ has vertex set $V_+ = V_1 \cup V_2$ and edge set $E_+ = E_1 \cup E_2 \cup$ {all edges connecting every vertex in V_1 with every vertex in V_2}.

Graph product: The product $G_\times = G_1 \times G_2$ has vertex set $V_\times = V_1 \times V_2$ (Cartesian product of V_1 and V_2) and two vertices $(\alpha, \beta), (\gamma, \delta) \in V_\times$ are connected by an edge if and only if the vertices $\alpha, \beta, \gamma, \delta$ satisfy the following (with $\sim$ denoting "is adjacent to"): $(\alpha = \gamma \ \wedge \ \beta \sim \delta) \vee (\beta = \delta \ \wedge \ \alpha \sim \gamma)$

Given the following graphs G_1 and G_2, draw their union graph (1 mark), join graph (2 marks) and product graph (4 marks). You should label the vertices clearly on your graphs. [7 marks]

=== END OF PAPER ===

(e) Assume that all graphs in this question are simple undirected graphs.

Given two simple undirected graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ where $V_1 \cap V_2 = \emptyset$ and $E_1 \cap E_2 = \emptyset$, the *graph union*, *graph join* and *graph product* are defined as follows:

Graph union:
The union $G_\cup = G_1 \cup G_2$ has vertex set $V_\cup = V_1 \cup V_2$ and edge set $E_\cup = E_1 \cup E_2$.

Graph join:
The join $G_+ = G_1 + G_2$ has vertex set $V_+ = V_1 \cup V_2$ and edge set $E_+ = E_1 \cup E_2 \cup$ {all edges connecting every vertex in V_1 with every vertex in V_2}.

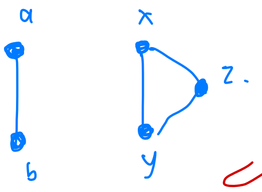
Graph product:
The product $G_\times = G_1 \times G_2$ has vertex set $V_\times = V_1 \times V_2$ (Cartesian product of V_1 and V_2) and two vertices $(\alpha, \beta), (\gamma, \delta) \in V_\times$ are connected by an edge if and only if the vertices $\alpha, \beta, \gamma, \delta$ satisfy the following (with $\sim$ denoting "is adjacent to"):

$$(\alpha = \gamma \wedge \beta \sim \delta) \vee (\beta = \delta \wedge \alpha \sim \gamma)$$

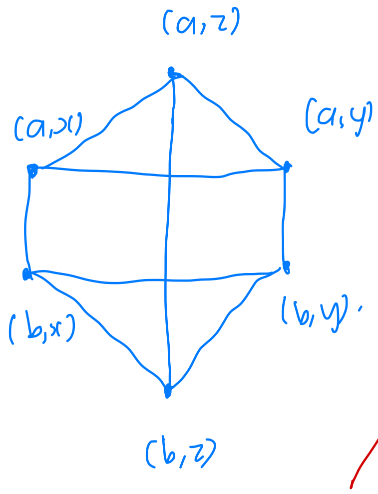
$\{ (a,x)\ (a,z),\ (a,y)$
$(b,x)\ (b,z),\ (b,y) \}$

Given the following graphs G_1 and G_2, draw their union graph (1 mark), join graph (2 marks) and product graph (4 marks). You should label the vertices clearly on your graphs. [7 marks]

G_u :



$G_\times$:



G_+ :