Section 2.3

Inverses of Square Matrices

Objectives

- What is an invertible matrix?
- What is the inverse of a matrix?
- What are some basic properties of invertible matrices?
- What are the powers of a matrix?

Two proving techniques:

- *Direct proof*
- *Proof by contradiction*

Motivation

Discussion 2.3.1

a, b real numbers such that $a \neq 0$

To solve the equation $ax = b$

inverse of a

$$x = b/a = (a^{-1})\cdot b$$

Let $\boldsymbol{A}$, $\boldsymbol{B}$ be two matrices.

To solve the matrix equation $\boldsymbol{AX} = \boldsymbol{B}$

Can we do this: $\boldsymbol{X} = \boldsymbol{B}/\boldsymbol{A}$?

We do not have "division" for matrices.

$$AX = B$$
$$A^{-1}AX = A^{-1}B$$
$$X = A^{-1}B$$

Can we find "inverses" for matrices "$\boldsymbol{A}^{-1}$" which have the similar property as a^{-1}?

What is an invertible matrix?

For ordinary numbers:
$a(a^{-1}) = 1 \qquad (a^{-1})a = 1$

Definition 2.3.2

$\boldsymbol{A}$: square matrix of order n.

Is $\mathbf{I}$ itself invertible?

$\boldsymbol{A}$ is invertible

if there exists a square matrix $\boldsymbol{B}$ of order n such that

$$\boldsymbol{AB} = \boldsymbol{I} \quad \text{and} \quad \boldsymbol{BA} = \boldsymbol{I}$$

The matrix $\boldsymbol{B}$ here is called an inverse of $\boldsymbol{A}$.

Does every matrix have an inverse? No

A square matrix is called singular if it has no inverse.

non-singular = invertible

What is an invertible matrix?

Example 2.3.3.1

$$\boldsymbol{A} = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$$

$$\boldsymbol{AB} = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \boldsymbol{I}$$

$$\boldsymbol{BA} = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \boldsymbol{I}$$

So $\boldsymbol{A}$ is invertible and $\boldsymbol{B}$ is an inverse of $\boldsymbol{A}$

Also $\boldsymbol{B}$ is invertible and $\boldsymbol{A}$ is an inverse of $\boldsymbol{B}$

A simple application

Example 2.3.3.2

2x1 variable column matrix $\begin{pmatrix} x \\ y \end{pmatrix}$

$$\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix} \boxed{\boldsymbol{X}} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

Linear system $\boldsymbol{AX} = \boldsymbol{b}$

$$\Rightarrow \boxed{\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}} \boldsymbol{X} = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

$$\Rightarrow \boxed{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \boldsymbol{X}} = \begin{pmatrix} 12 \\ 4 \end{pmatrix}$$

$$\Rightarrow \boldsymbol{X} = \begin{pmatrix} 12 \\ 4 \end{pmatrix}$$

Solution of the linear system

Given a matrix ***A***, how to find the inverse?

An example of a singular matrix

Example 2.3.3.3

No inverse

Show that $\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ is singular.

Proof by Contradiction

proving technique

Suppose $\boldsymbol{A}$ has an inverse: Let $\boldsymbol{B} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the inverse

assume the opposite of the claim

Represent the object

By definition of inverses, $\boldsymbol{BA} = \boldsymbol{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

using definition

On the other hand, $\boldsymbol{BA} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a+b & 0 \\ c+d & 0 \end{pmatrix}$

direct multiplication

The two results for $\boldsymbol{BA}$ contradict with each other.

arrive at a contradiction

Conclusion: $\boldsymbol{A}$ is singular.

Properties of invertible matrices

Remark 2.3.4.1 (Cancellation Law for Matrices)

Let $\boxed{A \text{ be an invertible}}$ matrix.

given condition to prove

$$\boxed{AB_1 = AB_2 \Rightarrow B_1 = B_2}$$

Is this true?

If A is not invertible, then the Cancellation Law may not hold.

$$C_1A = C_2A \Rightarrow C_1 = C_2$$

Prove it yourself

Direct Proof

Start from $AB_1 = AB_2$

$\Rightarrow A'AB_1 = A'AB_2$

$\Rightarrow IB_1 = IB_2$

$\Rightarrow B_1 = B_2$

Since A is invertible, let A' be an inverse of A.

introduce the inverse

How many inverses can a matrix have?

Theorem 2.3.5 Uniqueness of Inverses

If $\boldsymbol{B}$ and $\boldsymbol{C}$ are inverses of a square matrix $\boldsymbol{A}$, then $\boldsymbol{B} = \boldsymbol{C}$.

i.e. every invertible matrix has exactly one inverse

Direct Proof

$\boldsymbol{B}$ is an inverse of $\boldsymbol{A}$ $\Rightarrow \boldsymbol{BA} = \boldsymbol{I}$ and $\boxed{\boldsymbol{AB} = \boldsymbol{I}}$

given condition definition of inverse

$\boldsymbol{C}$ is an inverse of $\boldsymbol{A}$ $\Rightarrow \boxed{\boldsymbol{CA} = \boldsymbol{I}}$ and $\boldsymbol{AC} = \boldsymbol{I}$

$$\boldsymbol{AB} = \boldsymbol{I}$$

$$\Rightarrow \quad \boldsymbol{CAB} = \boldsymbol{CI}$$

$$\Rightarrow \quad \boldsymbol{IB} = \boldsymbol{C}$$

$$\Rightarrow \quad \boldsymbol{B} = \boldsymbol{C}$$

Notation of an inverse matrix

Notation 2.3.6

Let $\boldsymbol{A}$ be an invertible matrix.
By Theorem 2.3.5, we know that there is exactly one inverse of $\boldsymbol{A}$.

We use $\boldsymbol{A}^{-1}$ to denote this unique inverse of $\boldsymbol{A}$.

In example 2.3.3

$$\mathbf{A} = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix} \qquad \mathbf{A}^{-1} = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$$

How to show one matrix is the inverse of another?

Remark 2.3.7

If you are asked to show : $\boldsymbol{A}^{-1} = \boldsymbol{B}$
you just need to check

$$\boldsymbol{AB} = \boldsymbol{I} \quad \text{and} \quad \boldsymbol{BA} = \boldsymbol{I}$$

In fact, only need to check any one of these two conditions.
(See Theorem 2.4.12)

Example Given $\boldsymbol{A}^2 + \boldsymbol{A} = \boldsymbol{I}$ show : $\boldsymbol{A}^{-1} = \boldsymbol{A} + \boldsymbol{I}$

$$\boldsymbol{A}(\boldsymbol{A}+\boldsymbol{I}) = \boldsymbol{A}^2 + \boldsymbol{A} = \boldsymbol{I}$$

algebraic manipulation use given condition

$$(\boldsymbol{A}+\boldsymbol{I})\boldsymbol{A} = \boldsymbol{A}^2 + \boldsymbol{A} = \boldsymbol{I}$$

Conclusion : $\boldsymbol{A}^{-1} = \boldsymbol{A} + \boldsymbol{I}$

Example 2.3.8

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$ Show that if $ad - bc \neq 0$, then

determinant

formula

$$\mathbf{A}^{-1} = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix} = \begin{pmatrix} \frac{ad-bc}{ad-bc} & \frac{-ab+ba}{ad-bc} \\ \frac{cd-dc}{ad-bc} & \frac{-cb+da}{ad-bc} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{da-bc}{ad-bc} & \frac{db-bd}{ad-bc} \\ \frac{-ca+ac}{ad-bc} & \frac{-cb+ad}{ad-bc} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Conclusion : ***A*** is invertible and $\mathbf{A}^{-1} = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$

Invertibility and matrix operations

Theorem 2.3.9

The inverses can be expressed in terms of inverses of **A** and **B**

A, ***B*** : two invertible matrices (same size)
a : non-zero scalar

Matrix	Invertible?	Inverse
$a\boldsymbol{A}$	yes	$(a\boldsymbol{A})^{-1} = (1/a)\boldsymbol{A}^{-1}$
$\boldsymbol{A}^T$	yes	$(\boldsymbol{A}^T)^{-1} = (\boldsymbol{A}^{-1})^T$
$\boldsymbol{A}^{-1}$	yes	$(\boldsymbol{A}^{-1})^{-1} = \boldsymbol{A}$
$\boldsymbol{AB}$	yes	$(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$

Example

$$\mathbf{A} = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix} \qquad \mathbf{A}^{-1\,\mathbf{T}} = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}^{\mathbf{T}}$$

$$\mathbf{A}^T = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix} \qquad (\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$$

Remark 2.3.10

$$(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$$

Given $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_k$ are all invertible matrices of the same size.

1. The product $\boldsymbol{A}_1\boldsymbol{A}_2\ldots\boldsymbol{A}_k$ is an invertible matrix.

   This follows from Theorem 2.3.9.4

2. The inverse of $\boldsymbol{A}_1\boldsymbol{A}_2\ldots\boldsymbol{A}_n$ is

$$(\boldsymbol{A}_1\boldsymbol{A}_2\ldots\boldsymbol{A}_k)^{-1} = (\boldsymbol{A}_k)^{-1}\ldots(\boldsymbol{A}_2)^{-1}(\boldsymbol{A}_1)^{-1}$$

What are the powers of a matrix?

Definition 2.3.11

A : square matrix
n : nonnegative integer

Similar to ordinary number

We define $\boldsymbol{A}^n$ as follows:

$$\boldsymbol{A}^n = \underbrace{\boldsymbol{A}\boldsymbol{A} \ldots \boldsymbol{A}}_{\text{n times}} \qquad n \geq 1$$

$$\mathbf{A}^0 = \mathbf{I}$$

What about negative powers?

If **A** is invertible,

$$\mathbf{A}^{-n} = (\mathbf{A}^{-1})^n = \underbrace{\boldsymbol{A}^{-1}\boldsymbol{A}^{-1} \ldots \boldsymbol{A}^{-1}}_{\text{n times}}$$

Properties of matrix powers

Remark 2.3.13

1. $\boldsymbol{A}^r\boldsymbol{A}^s = \boldsymbol{A}^{r+s}$

   $\boldsymbol{A}^r\boldsymbol{A}^{-s} = \boldsymbol{A}^{r-s}$

Similar to ordinary number

2. $(\boldsymbol{A}^n)^{-1} = \boldsymbol{A}^{-n}$

inverse of n^{th} power

n^{th} power of inverse

Section 2.4

Elementary Matrices

Objectives

- What are elementary matrices?
- How are elementary matrices related to elementary row operations?
- How to find inverse of an elementary matrix?

Overview

- Perform e.r.o. R to a matrix $\boldsymbol{A}$ is the same as pre-multiply a certain square matrix $\boldsymbol{E}$ to $\boldsymbol{A}$

$$\boldsymbol{A} \xrightarrow{R} \boldsymbol{B} \qquad\qquad \boldsymbol{EA} = \boldsymbol{B}$$

- Every e.r.o. R has a "reverse" operation R'

"undo"

$$\boldsymbol{A} \xrightarrow{R} \boldsymbol{B} \xrightarrow{R'} \boldsymbol{A}$$

- R' is also an e.r.o.
- R' corresponds to a square matrix $\boldsymbol{E'}$
- $\boldsymbol{E'}$ is the inverse of $\boldsymbol{E}$

How to find the matrix E ?

$$\mathbf{I}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\xrightarrow{2R_2}\mathbf{E}=\begin{pmatrix}1&0&0\\0&2&0\\0&0&1\end{pmatrix}$$

$$\mathbf{I}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\xrightarrow{R_2\leftrightarrow R_3}\mathbf{E}=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$$

$$\mathbf{I}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\xrightarrow{R_3+2R_1}\mathbf{E}=\begin{pmatrix}1&0&0\\0&1&0\\2&0&1\end{pmatrix}$$

Which matrices have the same effect as e.r.o.?

Discussion 2.4.2.1

elementary row operations of the first type:

Multiply a row by a constant

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{pmatrix} \underset{2R_2}{\overset{\frac{1}{2}R_2}{\rightleftarrows}} \boldsymbol{B} = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 4 & -2 & 6 & 12 \\ 1 & 4 & 4 & 0 \end{pmatrix}$$

$$\boldsymbol{E} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{E}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{EA} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 4 & -2 & 6 & 12 \\ 1 & 4 & 4 & 0 \end{pmatrix} = \boldsymbol{B}$$

Elementary matrices of first type

Discussion 2.4.2.1

Let $\boldsymbol{A}$ be an m x n matrix.

Let $\boldsymbol{E}$ be a square matrix of order m:

$$\boldsymbol{E} = \begin{array}{c} \left(\begin{array}{ccc|c|ccc} 1 & & & & & & \\ & \ddots & & & & 0 & \\ & & 1 & & & & \\ \hline & & & c & & & \\ \hline & & & & 1 & & \\ & 0 & & & & \ddots & \\ & & & & & & 1 \end{array}\right) \end{array} \begin{array}{l} \\ \\ \\ \leftarrow i\text{th row} \\ \\ \\ \\ \end{array}$$

$\boldsymbol{EA}$: multiplying the ith row of $\boldsymbol{A}$ by c. cR_i

How to find inverse of an elementary matrix?

Discussion 2.4.2.1

Let $\boldsymbol{A}$ be an m x n matrix.

$$
\begin{array}{cc}
\boldsymbol{E} = \left(\begin{array}{ccc|c|ccc}
1 & & & & & & \\
 & \ddots & & & & 0 & \\
 & & 1 & & & & \\
\hline
 & & & c & & & \\
\hline
 & & & & 1 & & \\
 & 0 & & & & \ddots & \\
 & & & & & & 1
\end{array}\right) & \leftarrow i\text{th row} \\
\uparrow & \\
i\text{th column} &
\end{array}
$$

$$
\begin{array}{cc}
\boldsymbol{E}^{-1} = \left(\begin{array}{ccc|c|ccc}
1 & & & & & & \\
 & \ddots & & & & 0 & \\
 & & 1 & & & & \\
\hline
 & & & \frac{1}{c} & & & \\
\hline
 & & & & 1 & & \\
 & 0 & & & & \ddots & \\
 & & & & & & 1
\end{array}\right) & \leftarrow i\text{th row} \\
\uparrow & \\
i\text{th column} &
\end{array}
$$

$\boldsymbol{E}^{-1}\boldsymbol{A}$: multiplying the i^{th} row of $\boldsymbol{A}$ by $1/c$. $(1/c)R_i$

Which matrices have the same effect as e.r.o.?

Discussion 2.4.2.2

elementary row operations of the second type:

Interchange two rows

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{pmatrix} \underset{R_2 \leftrightarrow R_3}{\overset{R_2 \leftrightarrow R_3}{\rightleftarrows}} \boldsymbol{B} = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 1 & 4 & 4 & 0 \\ 2 & -1 & 3 & 6 \end{pmatrix}$$

$$\boldsymbol{E} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \boldsymbol{E}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\boldsymbol{EA} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 1 & 4 & 4 & 0 \\ 2 & -1 & 3 & 6 \end{pmatrix} = \boldsymbol{B}$$

Elementary matrices of second type

Discussion 2.4.2.2

Let $\boldsymbol{A}$ be an m x n matrix.

Let $\boldsymbol{E}$ be a square matrix of order m :

$$\boldsymbol{E} = \begin{array}{c} \left(\begin{array}{ccccccccccc}
1 & & & & & & & & & & \\
 & \ddots & & & & & & & & O & \\
 & & 1 & & & & & & & & \\
 & & & 0 & & & & 1 & & & \\
 & & & & 1 & & & & & & \\
 & & & & & \ddots & & & & & \\
 & & & & & & 1 & & & & \\
 & & & 1 & & & & 0 & & & \\
 & & & & & & & & 1 & & \\
 & O & & & & & & & & \ddots & \\
 & & & & & & & & & & 1
\end{array}\right) \end{array}
\begin{array}{l}
\\ \\ \\ \leftarrow i\text{th row} \\ \\ \\ \\ \leftarrow j\text{th row} \\ \\ \\ \\
\end{array}$$

$\boldsymbol{EA}$: interchanging the ith and jth rows of $\boldsymbol{A}$. $R_i \leftrightarrow R_j$

$$\boldsymbol{E}^{-1} = \boldsymbol{E}$$

How to find inverse of an elementary matrix?

Discussion 2.4.2.3

elementary row operations of the third type:

Add a multiple of a row to another row

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{pmatrix} \underset{\xleftarrow{R_3 - 2R_1}}{\xrightarrow{R_3 + 2R_1}} \boldsymbol{B} = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 3 & 4 & 8 & 6 \end{pmatrix}$$

$$\boldsymbol{E} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \qquad \boldsymbol{E}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

(3,1)-entry

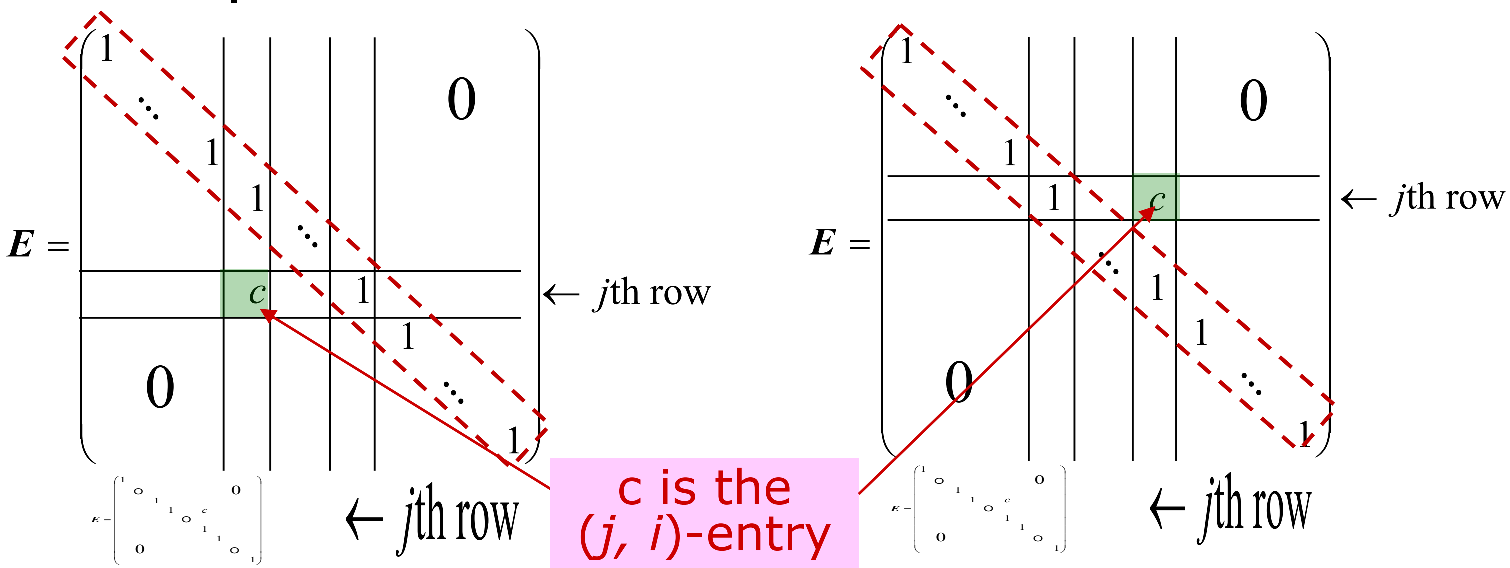
$$\boldsymbol{EA} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 3 & 4 & 8 & 6 \end{pmatrix} = \boldsymbol{B}$$

Elementary matrices of third type

Discussion 2.4.2.3

Let $\boldsymbol{A}$ be an m x n matrix.

$\boldsymbol{E}$ be a square matrix of order m as shown below



below diagonal if $i < j$ above diagonal if $i > j$

EA : adding c times of ith row to jth row of $\boldsymbol{A}$ $R_j + cR_i$

How to find inverse of an elementary matrix?

Discussion 2.4.2.3

$$E = \begin{array}{cl} \begin{pmatrix} 1 & & & & & & & & 0 \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & 1 & & & & & \\ & & & & \ddots & & & & \\ & & & c & & 1 & & & \\ & & & & & & 1 & & \\ & & & & & & & \ddots & \\ 0 & & & & & & & & 1 \end{pmatrix} & \begin{array}{l} \\ \\ \\ \\ \\ \leftarrow j\text{th row} \\ \\ \\ \\ \end{array} \\ \quad\quad \uparrow i\text{th column} \quad \uparrow j\text{th column} & \end{array}$$

Let A be an m x n matrix.

$$E^{-1} = \begin{array}{cl} \begin{pmatrix} 1 & & & & & & & & 0 \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & 1 & & & & & \\ & & & & \ddots & & & & \\ & & & -c & & 1 & & & \\ & & & & & & 1 & & \\ & & & & & & & \ddots & \\ 0 & & & & & & & & 1 \end{pmatrix} & \begin{array}{l} \\ \\ \\ \\ \\ \leftarrow j\text{th row} \\ \\ \\ \\ \end{array} \\ \quad\quad \uparrow i\text{th column} \quad \uparrow j\text{th column} & \end{array}$$

if $i < j$

$$E^{-1} = \begin{array}{cl} \begin{pmatrix} 1 & & & & & & & & 0 \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & 1 & & -c & & & \\ & & & & \ddots & & & & \\ & & & & & 1 & & & \\ & & & & & & 1 & & \\ & & & & & & & \ddots & \\ 0 & & & & & & & & 1 \end{pmatrix} & \begin{array}{l} \\ \\ \\ \leftarrow j\text{th row} \\ \\ \\ \\ \\ \\ \end{array} \\ \quad\quad \uparrow j\text{th column} \quad \leftarrow j\text{th row} & \end{array}$$

if $i > j$

R_j - cR_i

$E^{-1}A$: adding $-c$ times of ith row to jth row of A.

What are elementary matrices?

Definition 2.4.3 & Remark 2.4.4

A square matrix is called an elementary matrix if it can be obtained from an identity matrix by performing a single elementary row operation.

1. The matrices $\boldsymbol{E}$ in Discussion 2.4.2 are elementary matrices.

   Every elementary matrix is of one of the three types in Discussion 2.4.2.

2. All elementary matrices are invertible and their inverse are also elementary matrices.

Example 2.4.5

$$\mathbf{A}=\begin{pmatrix}0 & 4 & 2\\ -2 & 1 & -3\\ 1 & 0 & 2\end{pmatrix}\qquad \mathbf{B}=\begin{pmatrix}1 & 0 & 2\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}$$

A and ***B*** are row equivalent

i.e. ***B*** can be obtained from ***A*** by performing a series of e.r.o. and vice versa

$$\mathbf{A} \longrightarrow \longrightarrow \longrightarrow \cdots \longrightarrow \mathbf{B}$$

i.e. ***B*** can be obtained from ***A*** by pre-multiplying ***A*** with a series of elementary matrices and vice versa

$$E_n \ldots E_2 E_1 A = B$$

Elementary matrices and row equivalence

Example 2.4.5

$$\underbrace{\begin{pmatrix} 0 & 4 & 2 \\ -2 & 1 & -3 \\ 1 & 0 & 2 \end{pmatrix}}_{A} \xrightarrow[\mathbf{E}_1]{R_1 \leftrightarrow R_3} \underbrace{\begin{pmatrix} 1 & 0 & 2 \\ -2 & 1 & -3 \\ 0 & 4 & 2 \end{pmatrix}}_{E_1A} \xrightarrow[\mathbf{E}_2]{R_2 + 2R_1} \underbrace{\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 4 & 2 \end{pmatrix}}_{E_2E_1A}$$

$$\xrightarrow[\mathbf{E}_3]{R_3 - 4R_2} \underbrace{\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{pmatrix}}_{E_3E_2E_1A} \xrightarrow[\mathbf{E}_4]{-\frac{1}{2}R_3} \overbrace{\underbrace{\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}_{E_4E_3E_2E_1A}}^{B}$$

$$\mathbf{E}_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \mathbf{E}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{E}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{pmatrix} \quad \mathbf{E}_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}$$

$$E_4E_3E_2E_1A = B$$

Elementary matrices and row equivalence

Example 2.4.5

$$E_4E_3E_2E_1A = B$$

$$E_3E_2E_1A = E_4^{-1}B$$

$$\Rightarrow E_2E_1A = E_3^{-1}E_4^{-1}B$$

$$\Rightarrow E_1A = E_2^{-1}E_3^{-1}E_4^{-1}B$$

$$\Rightarrow \boxed{A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}B}$$

A and ***B*** are row equivalent

B in terms of ***A*** and elementary matrices

A in terms of ***B*** and elementary matrices

$$\mathbf{E}_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \mathbf{E}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{E}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{pmatrix} \quad \mathbf{E}_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}$$

$$\mathbf{E}_1^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \mathbf{E}_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{E}_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{pmatrix} \quad \mathbf{E}_4^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Section 2.4

Elementary Matrices

Objectives

- How to find the inverse of an invertible matrix?
- How to tell whether a matrix is invertible?
- What can we say about an invertible matrix?

Elementary matrices and row equivalence

Example 2.4.5

$\boldsymbol{A}$ and $\boldsymbol{B}$ are row equivalent

$$\boldsymbol{A} \rightarrow \;\rightarrow\; \rightarrow \;\ldots\; \rightarrow \boldsymbol{B}$$

$$\boldsymbol{E}_n \ldots \boldsymbol{E}_2\boldsymbol{E}_1\boldsymbol{A} = \boldsymbol{B}$$

$$\boldsymbol{A} \leftarrow \;\ldots\; \leftarrow \;\leftarrow\; \leftarrow \boldsymbol{B}$$

$$\boldsymbol{A} = \boldsymbol{E}_1^{-1}\boldsymbol{E}_2^{-1} \ldots \boldsymbol{E}_n^{-1}\boldsymbol{B}$$

Take note of the order

Remark 2.4.6

Proof of Theorem 1.2.7

If augmented matrices of two linear systems
are row equivalent,
then the two systems have the same set of solutions.

The idea is to use elementary matrices

Read up!

$\boldsymbol{Ax} = \boldsymbol{c}$ and $\boldsymbol{Bx} = \boldsymbol{d}$

$\boxed{\boldsymbol{E}_n \dots \boldsymbol{E}_2\boldsymbol{E}_1\boldsymbol{A}}$ $\boxed{\boldsymbol{E}_n \dots \boldsymbol{E}_2\boldsymbol{E}_1\boldsymbol{c}}$

How to find inverse matrix?

Example 2.4.9

Find the inverse of

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix} \qquad \boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

if it exists.

Form the 3x6 augmented matrix

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right)$$

Gauss-Jordan Elimination

How to find inverse matrix?

Example 2.4.9 Gauss-Jordan Elimination

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right)$$

$$\qquad\quad \boldsymbol{A} \qquad\qquad \boldsymbol{I}$$

$$\xrightarrow{R_3 + 2R_2} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right) \xrightarrow{-R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)$$

$$\xrightarrow[R_2 + 3R_3]{R_1 - 3R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right) \xrightarrow{R_1 - 2R_2} \begin{array}{c} \qquad \boldsymbol{I} \qquad\qquad\qquad \boldsymbol{A}^{-1} \\ \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right) \end{array}$$

RREF

$$\boldsymbol{E_k \cdots E_2 E_1 A = I} \;\Rightarrow\; \boldsymbol{E_k \cdots E_2 E_1 I = A^{-1}}$$

Why does it work?

Question:
What if the RREF is not $\boldsymbol{I}$?

Discussion 2.4.8

Suppose the RREF of $\boldsymbol{A}$ is the identity matrix

$\boldsymbol{A}$: invertible matrix of order n

$$\boldsymbol{A} \xrightarrow{R_1} \xrightarrow{R_2} \cdots \xrightarrow{R_k} \boldsymbol{I}$$

elementary matrices $\quad \boldsymbol{E_1} \quad \boldsymbol{E_2} \quad \boldsymbol{E_k}$

$$\boldsymbol{E_k}\cdots\boldsymbol{E_2}\boldsymbol{E_1}\boldsymbol{A} = \boldsymbol{I} \quad \Rightarrow \boldsymbol{E_k}\cdots\boldsymbol{E_2}\boldsymbol{E_1} = \boldsymbol{A}^{-1}$$

Form an n x $2n$ "augmented matrix" $(\ \boldsymbol{A} \mid \boldsymbol{I}\ )$

$$\boldsymbol{E_k}\cdots\boldsymbol{E_2}\boldsymbol{E_1}(\boldsymbol{A} \mid \boldsymbol{I})$$

$$(\boldsymbol{E_k}\cdots\boldsymbol{E_2}\boldsymbol{E_1}\boldsymbol{A} \mid \boldsymbol{E_k}\cdots\boldsymbol{E_2}\boldsymbol{E_1}\boldsymbol{I})$$

$$= (\boldsymbol{I} \mid \boldsymbol{A}^{-1})$$

applying e.r.o. to $(\ \boldsymbol{A} \mid \boldsymbol{I}\ )$
same as
applying e.r.o. to both $\boldsymbol{A}$ and $\boldsymbol{I}$

$$\boxed{(\ \boldsymbol{A} \mid \boldsymbol{I}\ ) \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} (\ \boldsymbol{I} \mid \boldsymbol{A}^{-1}\ )}$$

A very^3 important theorem

Theorem 2.4.7

Any 1 of the 4 statements implies the other 3.

Let $\boldsymbol{A}$ be a square matrix.

The following statements are equivalent

1. $\boldsymbol{A}$ is invertible.
2. The linear system $\boldsymbol{Ax} = \boldsymbol{0}$ has only the trivial solution.
3. The reduced row-echelon form of $\boldsymbol{A}$ is an identity matrix.
4. $\boldsymbol{A}$ can be expressed as a product of elementary matrices.

Equivalent Statements

Two equivalent statements

statement 1

Two implications

statement 2

Four equivalent statements

How many implications are there?

Do this for every pair of statements.

1 & 2	1 & 3	1 & 4
2 & 3	2 & 4	3 & 4

Twelve implications

1. $\boldsymbol{A}$ is invertible
2. $\boldsymbol{Ax} = \mathbf{0}$ has only trivial solution
3. RREF of $\boldsymbol{A}$ is $\boldsymbol{I}$
4. $\boldsymbol{A}$ a product of elementary matrices

What's it for?

1. $\boldsymbol{A}$ is invertible
2. $\boldsymbol{Ax} = \mathbf{0}$ has only trivial solution
3. RREF of $\boldsymbol{A}$ is $\boldsymbol{I}$
4. $\boldsymbol{A}$ a product of elementary matrices

Applications

(1) Given $\begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$ is invertible.

How many solutions does the linear system have?

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Ans: Only the trivial solution

Apply: Statement 1 $\Rightarrow$ Statement 2

(2) Given $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \xrightarrow{\text{G.J.E.}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Is $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ invertible?

Ans: Yes

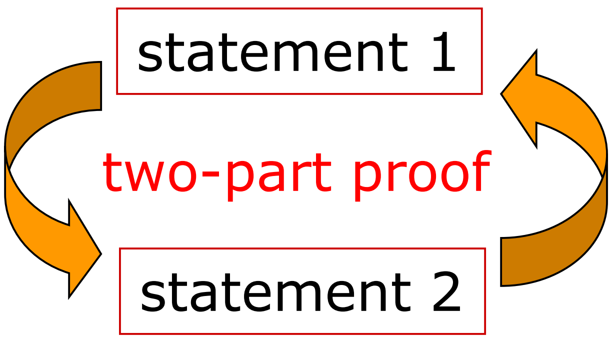
Apply: Statement 3 $\Rightarrow$ Statement 1

How to prove it?

1. $\boldsymbol{A}$ is invertible
2. $\boldsymbol{Ax} = \mathbf{0}$ has only trivial solution
3. RREF of $\boldsymbol{A}$ is $\boldsymbol{I}$
4. $\boldsymbol{A}$ a product of elementary matrices

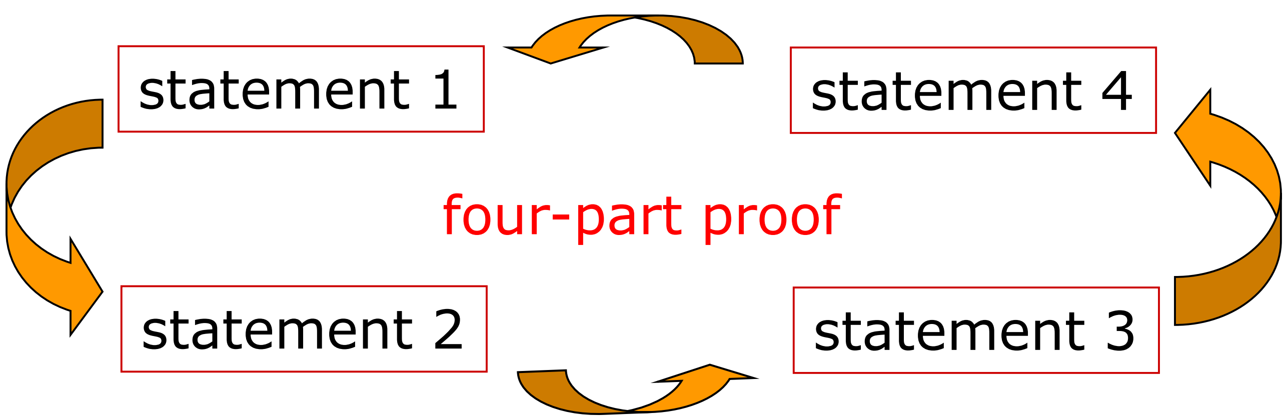
Proof

Two equivalent statements



Four equivalent statements

twelve-part proof ?



How to prove it?

Proof

1. $\boldsymbol{A}$ is invertible
2. $\boldsymbol{Ax} = \mathbf{0}$ has only trivial solution
3. RREF of $\boldsymbol{A}$ is $\boldsymbol{I}$
4. $\boldsymbol{A}$ a product of elementary matrices

(1 ⇒ 2) Start with $\boldsymbol{Au} = \mathbf{0}$ and show $\boldsymbol{u} = \mathbf{0}$

$u = A^{-1}0 = 0$

(2 ⇒ 3) Convert $\boldsymbol{Ax} = \mathbf{0}$ to augmented matrix $(\boldsymbol{A} \mid \mathbf{0})$ and consider the pivot columns of its RREF

$(A \mid 0) \xrightarrow{GJE} (R \mid 0)$

↳ all colums are pivot column

(3 ⇒ 4) Express the Gauss-Jordan Elimination from $\boldsymbol{A}$ to $\boldsymbol{I}$ in terms of elementary matrices

$A \to I \quad E_k \cdots E_2 E_1 A = I \Rightarrow A = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$

(4 ⇒ 1) Product of invertible matrices is invertible

$(A^{-1}B^{-1})^{-1} = BA$

How to tell whether a matrix is invertible?

Remark 2.4.10

To check whether a square matrix is invertible:

- Look at the RREF
  - RREF = $\boldsymbol{I}$ implies invertible
  - RREF $\neq \boldsymbol{I}$ implies not invertible
- Look at REF
  - REF has no zero row implies invertible
  - REF has zero rows implies not invertible

Example 2.4.11.1

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 4 \\ 3 & 6 & 3 \end{pmatrix} \xrightarrow{GE} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

$\boldsymbol{A}$ is not invertible.

How do all 2x2 invertible matrices look like?

Example 2.4.11.2

$$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Example 2.3.8

$\boldsymbol{A}$ is invertible if $ad - bc \neq 0$.

not quite the same!

$\boldsymbol{A}$ is invertible if and only if $ad - bc \neq 0$.

$\boxed{ad - bc \neq 0}$

Example 2.3.8

$\boxed{\boldsymbol{A} \text{ is invertible}}$

Theorem 2.4.7: 1 $\Rightarrow$ 3

1. $\boldsymbol{A}$ is invertible
3. RREF of $\boldsymbol{A}$ is $\boldsymbol{I}$

Read the solution in textbook

If we only know $\boldsymbol{AB} = \boldsymbol{I}$, can we say $\boldsymbol{A}$ and $\boldsymbol{B}$ are inverses of each other?

Theorem 2.4.12

Let $\boldsymbol{A}$, $\boldsymbol{B}$ be square matrices of the same size.

If $\boldsymbol{AB} = \boldsymbol{I}$,
then $\boldsymbol{BA} = \boldsymbol{I}$.
So $\boldsymbol{A}$ and $\boldsymbol{B}$ are invertible, $\boldsymbol{A}^{-1} = \boldsymbol{B}$, $\boldsymbol{B}^{-1} = \boldsymbol{A}$.

Outline of proof

$\boxed{\boldsymbol{AB} = \boldsymbol{I}} \longrightarrow \boxed{\text{1. } \boldsymbol{B} \text{ is invertible}} \longrightarrow \boxed{\text{2. } \boldsymbol{B}^{-1} = \boldsymbol{A}} \longrightarrow \boxed{\text{3. } \boldsymbol{BA} = \boldsymbol{I}}$

First prove $\boldsymbol{AB} = \boldsymbol{I} \Rightarrow \boldsymbol{B}$ is invertible

Theorem 2.4.12

Given $\boldsymbol{AB} = \boldsymbol{I}$ Show $\boldsymbol{B}$ is invertible

Consider the homogeneous system $\boldsymbol{Bx} = \mathbf{0}$.

$$
\begin{aligned}
\boldsymbol{Bu} &= \mathbf{0} && \text{Start with the system} \\
\Rightarrow \boldsymbol{ABu} &= \boldsymbol{A}\mathbf{0} && \text{algebraic manipulation} \\
\Rightarrow \boldsymbol{Iu} &= \mathbf{0} && \text{LHS: use given condition} \\
\boldsymbol{u} &= \mathbf{0}
\end{aligned}
$$

The system $\boldsymbol{Bx} = \mathbf{0}$ has only the trivial solution.
By Thm 2.4.7 ($2 \Rightarrow 1$), $\boldsymbol{B}$ is invertible.

Theorem 2.4.7
1. $\boldsymbol{A}$ is invertible.
2. $\boldsymbol{Ax} = \mathbf{0}$ has only the trivial solution.

Next prove $\boldsymbol{B}$ is invertible $\Rightarrow \boldsymbol{B}^{-1} = \boldsymbol{A}$ and $\boldsymbol{BA} = \boldsymbol{I}$

Theorem 2.4.12

Given $\boldsymbol{AB} = \boldsymbol{I}$

We have shown $\boldsymbol{B}$ is invertible

To show $\boldsymbol{B}^{-1} = \boldsymbol{A}$

and $\boldsymbol{BA} = \boldsymbol{I}$

$$
\begin{aligned}
& \boldsymbol{AB} = \boldsymbol{I} \\
\Rightarrow\ & \boldsymbol{AB}\boldsymbol{B}^{-1} = \boldsymbol{I}\boldsymbol{B}^{-1} \\
\Rightarrow\ & \boldsymbol{AI} = \boldsymbol{B}^{-1} \\
\Rightarrow\ & \boldsymbol{A} = \boldsymbol{B}^{-1} \\
\Rightarrow\ & \boldsymbol{BA} = \boldsymbol{BB}^{-1} \\
\Rightarrow\ & \boldsymbol{BA} = \boldsymbol{I}
\end{aligned}
$$

use given condition

use **B** is invertible

Elementary matrices and column operations

Summary 2.4.15-16

elementary column operations of the first type:

Multiply a column by a constant

elementary column operations of the second type:

Interchange two columns

elementary column operations of the third type:

Add a multiple of a column to another column

Perform e.c.o. C to a matrix **A** is the same as post-multiply a certain square matrix **E** to **A**

$$\mathbf{A} \xrightarrow{C} \mathbf{B} \qquad \boldsymbol{AE} = \boldsymbol{B}$$

$$\mathbf{I} \xrightarrow{C} \mathbf{E}$$

Section 2.5

Determinants

Objectives

- What is the determinant of a matrix?
- What is cofactor expansion?
- How to find determinant?

Motivation

Discussion 2.5.1

$$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$\boldsymbol{A}$ is invertible if and only if $\underbrace{ad - bc}_{\text{determinant}} \neq 0$.

det($\boldsymbol{A}$) $\quad \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$\boldsymbol{A}$: nxn square matrix

$\boldsymbol{A}$ is invertible if and only if "*determinant of* $\boldsymbol{A}$" $\neq 0$.

What is a 3x3 determinant?

Example 2.5.4.2

$$\boldsymbol{B} = \begin{pmatrix} -3 & -2 & 4 \\ 4 & 3 & 1 \\ 0 & 2 & 4 \end{pmatrix}$$

Define determinant "inductively"

3x3 determinant defined
in terms of 2x2 determinants

$$\det(\boldsymbol{B}) = (-3)\underbrace{\begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix}}_{\boldsymbol{M}_{11}} - (-2)\underbrace{\begin{vmatrix} 4 & 1 \\ 0 & 4 \end{vmatrix}}_{\boldsymbol{M}_{12}} + 4\underbrace{\begin{vmatrix} 4 & 3 \\ 0 & 2 \end{vmatrix}}_{\boldsymbol{M}_{13}}$$

Submatrices of $\boldsymbol{B}$ $\rightarrow$ $\boldsymbol{M}_{11}$, $\boldsymbol{M}_{12}$, $\boldsymbol{M}_{13}$

$$= -3(3 \times 4 - 1 \times 2) + 2(4 \times 4 - 1 \times 0) + 4(4 \times 2 - 3 \times 0)$$

$$= 34$$

What is a 4x4 determinant?

Example 2.5.4.3

4x4 determinant defined
in terms of 3x3 determinants

$$\mathbf{C} = \begin{pmatrix} 0 & -1 & 2 & 0 \\ 2 & -3 & 3 & -2 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 2 & -1 \end{pmatrix}$$

det($\boldsymbol{C}$)

$$0\begin{vmatrix} -3 & 3 & -2 \\ 2 & 4 & 0 \\ 0 & 2 & -1 \end{vmatrix} - (-1)\begin{vmatrix} 2 & 3 & -2 \\ 0 & 4 & 0 \\ 0 & 2 & -1 \end{vmatrix} + 2\begin{vmatrix} 2 & -3 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{vmatrix} - 0\begin{vmatrix} 2 & -3 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 2 \end{vmatrix}$$

$\boldsymbol{M}_{11}$ $\boldsymbol{M}_{12}$ $\boldsymbol{M}_{13}$ $\boldsymbol{M}_{14}$

in terms of 2x2 determinants | in terms of 2x2 determinants | in terms of 2x2 determinants | in terms of 2x2 determinants

What is an nxn determinant?

Definition 2.5.2

Let $\boldsymbol{A} = (a_{ij})$ be an $n \times n$ matrix.

If $\boldsymbol{A} = (a_{11})$ is a 1 x 1 matrix, then $\det(\boldsymbol{A}) = a_{11}$

For $n > 1$,
let $\boldsymbol{M}_{1j}$ be the $(n-1) \times (n-1)$ matrix obtained from $\boldsymbol{A}$ by deleting the 1*st* row and the *j*th column.

$A_{11} = \det(\boldsymbol{M}_{11})$ $A_{13} = \det(\boldsymbol{M}_{13})$

$A_{12} = -\det(\boldsymbol{M}_{12})$ $A_{14} = -\det(\boldsymbol{M}_{14})$

etc... cofactors of $\boldsymbol{A}$

The determinant of $\boldsymbol{A}$ is defined to be

$$\det(\mathbf{A}) = a_{11}A_{11} + a_{12}A_{12} + \ldots + a_{1n}A_{1n}$$

not practical for large matrices

cofactor expansion along row 1

What is an (i, j)-cofactor ?

Definition 2.5.2

Let $\boldsymbol{A} = (a_{ij})$ be an $n \times n$ matrix.

$\boldsymbol{M_{ij}}$:deleting ith row and jth column from $\boldsymbol{A}$

$$A_{ij} = (-1)^{i+j}\det(\boldsymbol{M_{ij}})$$ (i, j)-cofactor of $\boldsymbol{A}$

$A_{11} = \det(\boldsymbol{M_{11}})$ $A_{13} = \det(\boldsymbol{M_{13}})$

$A_{12} = -\det(\boldsymbol{M_{12}})$ $A_{14} = -\det(\boldsymbol{M_{14}})$

etc… cofactors of $\boldsymbol{A}$

How to compute determinant?

Theorem 2.5.6 (Cofactor Expansions)

$$\boxed{\det(\mathbf{A}) = a_{11}A_{11} + a_{12}A_{12} + \ldots + a_{1n}A_{1n}}$$

det($\boldsymbol{A}$) can be expressed as a cofactor expansion using any row or column of $\boldsymbol{A}$.

for any $i = 1, 2, \ldots, n$

$$\det(\mathbf{A}) = a_{i1}A_{i1} + a_{i2}A_{i2} + \ldots + a_{in}A_{in}$$

cofactor expansion along row i

for any $j = 1, 2, \ldots, n$

$$\det(\mathbf{A}) = a_{1j}A_{1j} + a_{2j}A_{2j} + \ldots + a_{nj}A_{nj}$$

cofactor expansion along column j

How to compute determinant

$$\mathbf{C} = \begin{pmatrix} 0 & -1 & 2 & 0 \\ 2 & -3 & 3 & -2 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 2 & -1 \end{pmatrix} = -2\begin{vmatrix} -1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 2 & -1 \end{vmatrix}$$

Example 2.5.7

$$\boldsymbol{B} = \begin{pmatrix} -3 & -2 & 4 \\ 4 & 3 & 1 \\ 0 & 2 & 4 \end{pmatrix}$$

(i, j)-cofactor of $\boldsymbol{B}$:

$$B_{ij} = (-1)^{i+j}\det(\boldsymbol{M_{ij}})$$

$$\det(\boldsymbol{B}) = \underbrace{-4\begin{vmatrix} -2 & 4 \\ 2 & 4 \end{vmatrix}}_{B_{21}} + \underbrace{3\begin{vmatrix} -3 & 4 \\ 0 & 4 \end{vmatrix}}_{B_{22}} \underbrace{-1\begin{vmatrix} -3 & -2 \\ 0 & 2 \end{vmatrix}}_{B_{23}} = 34$$

cofactor expansion along row 2

$$\det(\boldsymbol{B}) = \underbrace{4\begin{vmatrix} 4 & 3 \\ 0 & 2 \end{vmatrix}}_{B_{13}} \underbrace{-1\begin{vmatrix} -3 & -2 \\ 0 & 2 \end{vmatrix}}_{B_{23}} + \underbrace{4\begin{vmatrix} -3 & -2 \\ 4 & 3 \end{vmatrix}}_{B_{33}} = 34$$

cofactor expansion along column 3

Determinant of triangular matrix

Theorem 2.5.8 & Example 2.5.9

If $\boldsymbol{A}$ is a triangular matrix, diagonal matrix
then the determinant of $\boldsymbol{A}$ is equal to
the product of the diagonal entries of $\boldsymbol{A}$.

$$\begin{vmatrix} -1 & 3.5 & 14 \\ 0 & 5 & -26 \\ 0 & 0 & 2 \end{vmatrix} = (-1) \times 5 \times 2 = -10$$

$$\begin{vmatrix} -2 & 0 & 0 \\ 99 & 0 & 0 \\ 10 & -4.5 & 10 \end{vmatrix} = (-2) \times 0 \times 10 = 0$$

Suppose we want to prove certain property holds for all (specific type of) square matrices

Mathematical Induction

Show: Property P holds for all square matrices.

We can try to prove the following:

1. P works for all 1×1 matrices   Base case
2. Show that, if P works for all $k \times k$ matrices, then P works for all $(k+1)\times(k+1)$ matrices

Inductive step

Mathematical Induction

works for 1×1 $\Rightarrow$ works for 2×2 $\Rightarrow$ works for 3×3

$\Rightarrow$ … $\Rightarrow$ works for $n\times n$ $\Rightarrow$ …

Repeatedly, we have shown that P works for all square matrices

Determinant and transpose

Theorem 2.5.10

If $\boldsymbol{A}$ is a square matrix, then $\det(\boldsymbol{A}) = \det(\boldsymbol{A}^T)$.

Example

$$\det(\mathbf{C}) = \begin{vmatrix} 0 & -1 & 2 & 0 \\ 2 & -3 & 3 & -2 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 2 & -1 \end{vmatrix} = \det(\mathbf{C}^T) = \begin{vmatrix} 0 & 2 & 0 & 0 \\ -1 & -3 & 2 & 0 \\ 2 & 3 & 4 & 2 \\ 0 & -2 & 0 & -1 \end{vmatrix}$$

Prove by mathematical induction

Let P be the property: $\det(\boldsymbol{A}) = \det(\boldsymbol{A}^T)$

Base case P works for 1 x 1 matrices

Inductive step

We assume $\det(\boldsymbol{B}) = \det(\boldsymbol{B}^T)$ for any k x k matrix $\boldsymbol{B}$

Show $\det(\boldsymbol{A}) = \det(\boldsymbol{A}^T)$ for any (k + 1) x (k + 1) matrix $\boldsymbol{A}$

$\det(\boldsymbol{B}) = \det(\boldsymbol{B}^T)$ for 3 x 3 matrix $\boldsymbol{B}$
□ $\det(\boldsymbol{A}) = \det(\boldsymbol{A}^T)$ for 4 x 4 matrix $\boldsymbol{A}$

Example 2.5.11

$$\det(\boldsymbol{C}) = \begin{vmatrix} 0 & -1 & 2 & 0 \\ 2 & -3 & 3 & -2 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 2 & -1 \end{vmatrix} \qquad \det(\boldsymbol{C}^T) = \begin{vmatrix} 0 & 2 & 0 & 0 \\ -1 & -3 & 2 & 0 \\ 2 & 3 & 4 & 2 \\ 0 & -2 & 0 & -1 \end{vmatrix}$$

$\boldsymbol{M}_{12}$ $\qquad$ $\boldsymbol{M}_{12}{}^{T}$

cofactor expansion along row 1 $\qquad$ cofactor expansion along column 1

Inductive step

4x4 $\qquad$ 3x3 $\qquad$ 3x3 $\qquad$ 3x3 $\qquad$ 3x3

$$\det(\boldsymbol{C}) = 0\det(\boldsymbol{M}_{11}) - (-1)\det(\boldsymbol{M}_{12}) + 2\det(\boldsymbol{M}_{13}) - (0)\det(\boldsymbol{M}_{14})$$

$$\Vert \qquad \Vert \qquad \Vert \qquad \Vert \qquad \Vert$$

$$\det(\boldsymbol{C}^T) = 0\det(\boldsymbol{M}_{11}{}^T) - (-1)\det(\boldsymbol{M}_{12}{}^T) + 2\det(\boldsymbol{M}_{13}{}^T) - (0)\det(\boldsymbol{M}_{14}{}^T)$$