Matrix

$$\boxed{A = (a_{ij})_{m\times n}}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

row matrix: $(1, 2, 0)$

col. matrix: $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

Sqr. matrix: $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}_{n\times n}$
of order 3

diagonal of matrix: $\boxed{a_{11}, a_{22}, \ldots, a_{nn}}$

diagonal entries: a_{ii}, $i = 1, \ldots, n$

non-diagonal entries: a_{ij}, $i \neq j$

anti-diagonal: $a_{1n}, a_{2,n-1}, \ldots, a_{n1}$.

Definition. Two matrices are **equal** if

- they have the same size (same number of rows, same number of columns), and
- all the corresponding entries are the same.

Let $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$ be matrices.

- **Addition**: $A + B = (a_{ij} + b_{ij})_{m\times n}$.
- **Subtraction**: $A - B = (a_{ij} - b_{ij})_{m\times n}$.
- **Scalar multiplication**: $cA = (ca_{ij})_{m\times n}$.

Theorem. Let A, B, C be matrices of the same size.

- $A - B = A + (-B)$.
- Commutative Law for Matrix Addition:
  - $A + B = B + A$.
- Associative Law for Matrix Addition:
  - $(A + B) + C = A + (B + C)$.
- Let $\mathbf{0}$ be the zero matrix of the same size as A.
  - $\mathbf{0} + A = A$; $A - A = \mathbf{0}$; $0A = \mathbf{0}$; $c\mathbf{0} = \mathbf{0}$.
- Distributive Law for Scalar Multiplication over Addition:
  - $c(A + B) = cA + cB$.
  - $(c + d)A = cA + dA$.
- $c(dA) = (cd)A$, $1A = A$.

Square matrix: $A_{n\times n}$

diagonal matrix: (also sqr.)

$$A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}.$$

can be anything.

Scalar matrix: (also sqr.) (also diag.)

$$A = \begin{pmatrix} c & 0 & \cdots & 0 \\ 0 & c & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c \end{pmatrix}, \text{ where } c \text{ is a constant.}$$

Identity matrix: (also sqr.) (also diag.) (also scalar)

$$I_n = A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Zero matrix.

$$O_{m\times n} = A = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{m\times n}.$$

Symmetric: (sqr. matrix)

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & a_{34} & \cdots & a_{3n} \\ a_{41} & a_{42} & a_{43} & a_{44} & \cdots & a_{4n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & \cdots & a_{nn} \end{pmatrix}$$

eg. $\begin{pmatrix} 2 & 1 & 6 & -2 \\ 1 & 3 & 0 & -1 \\ 6 & 0 & 0 & 0 \\ -2 & -1 & 0 & 1 \end{pmatrix}$

$A = A^T$.

triangular matrix:

Upper triangular: (sqr. matrix)

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & a_{24} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & a_{34} & \cdots & a_{3n} \\ 0 & 0 & 0 & a_{44} & \cdots & a_{4n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix}.$$

Lower triangular: (sqr. matrix)

$$A = \begin{pmatrix} a_{11} & 0 & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & 0 & \cdots & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & \cdots & a_{nn} \end{pmatrix}.$$

Definition. Let $A = (a_{ij})_{m\times p}$ and $B = (b_{ij})_{p\times n}$.

- AB is the $m \times n$ matrix such that its (i, j)-entry is
  - $\boxed{a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj} = \sum_{k=1}^{p} a_{ik}b_{kj}}$

Note: No. of columns of A = the no. of rows of B.

1. ith row of A: $\begin{pmatrix} & & \vdots & \\ a_{i1} & a_{i2} & \cdots & a_{ip} \\ & & \vdots & \end{pmatrix}$
2. jth column of B: $\begin{pmatrix} & b_{1j} & \\ & b_{2j} & \\ \cdots & \vdots & \cdots \\ & b_{pj} & \end{pmatrix}$
3. Multiply componentwise and add the products.

Theorem.

- Let A, B, C be $m \times p$, $p \times q$, $q \times n$ matrices, resp.
  - Associative Law: $A(BC) = (AB)C$.
- Let A be $m \times p$ matrix, B_1, B_2 be $p \times n$ matrices.
  - Distributive Law: $A(B_1 + B_2) = AB_1 + AB_2$.
- Let A_1, A_2 be $m \times p$ matrices, B be $p \times n$ matrix.
  - Distributive Law: $(A_1 + A_2)B = A_1B + A_2B$.
- Let A be $m \times p$ and B be $p \times n$. For constant c,
  - $c(AB) = (cA)B = A(cB)$.
- Let A be an $m \times n$ matrix.
  - $A\mathbf{0}_{n\times p} = \mathbf{0}_{m\times p}$; $\mathbf{0}_{p\times m}A = \mathbf{0}_{p\times n}$.
  - $AI_n = A$; $I_mA = A$.

Not Commutative.

- $A^m A^n = A^{m+n}$, $(A^m)^n = A^{mn}$, $m, n \in \mathbb{Z}^+$.

power of $A_{n\times n}$:

$$\boxed{A^k = \begin{cases} I_n & \text{if } k = 0, \\ \underbrace{AA\cdots A}_{k \text{ times}} & \text{if } k \geq 1. \end{cases}}$$

$(AB)^m \neq A^m B^m$

if $AB = BA$ then

$(AB)^m = A^m B^m$

Representation of Linear System:

Let $\boldsymbol{A} = (a_{ij})_{m\times n}$, $\boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$.

- Then $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is the linear system of
  - m linear equations in n variables $x_1, \ldots, x_n$,
  - a_{ij} are the coefficients, and b_i are the constants.
- Let $\boldsymbol{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$.
  - $x_1 = u_1, \ldots, x_n = u_n$ is a solution to the system

    $\Leftrightarrow \boldsymbol{A}\boldsymbol{u} = \boldsymbol{b}$

    $\Leftrightarrow \boldsymbol{u}$ is a solution to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$.

Transpose

- Let $\boldsymbol{A} = (a_{ij})_{m\times n}$ be a matrix.
  - The **transpose** of $\boldsymbol{A}$ is the $n \times m$ matrix $\boldsymbol{A}^{\mathrm{T}}$ (or $\boldsymbol{A}^{\mathrm{t}}$)
    - whose (i,j)-entry is a_{ji}.

Theorem. Let $\boldsymbol{A}$ be an $m \times n$ matrix.

- $(\boldsymbol{A}^{\mathrm{T}})^{\mathrm{T}} = \boldsymbol{A}$.
- $\boldsymbol{A}$ is symmetric $\Leftrightarrow \boldsymbol{A} = \boldsymbol{A}^{\mathrm{T}}$.
- Let c be a scalar. Then $(c\boldsymbol{A})^{\mathrm{T}} = c\boldsymbol{A}^{\mathrm{T}}$.
- Let $\boldsymbol{B}$ be $m \times n$. Then $(\boldsymbol{A}+\boldsymbol{B})^{\mathrm{T}} = \boldsymbol{A}^{\mathrm{T}} + \boldsymbol{B}^{\mathrm{T}}$. ∴
- Let $\boldsymbol{B}$ be $n \times p$. Then $(\boldsymbol{A}\boldsymbol{B})^{\mathrm{T}} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}$. ∵

Inverse of (Sqr.) matrix

Definition. Let $\boldsymbol{A}$ be a **square matrix** of order n.

- If there exists a square matrix $\boldsymbol{B}$ of order n so that
  - $\boxed{\boldsymbol{A}\boldsymbol{B} = \boldsymbol{I}_n}$ and $\boxed{\boldsymbol{B}\boldsymbol{A} = \boldsymbol{I}_n}$ → suffice to only proof one side.

  then $\boldsymbol{A}$ is called **invertible**, and $\boldsymbol{B}$ is an **inverse** of $\boldsymbol{A}$.
- If $\boldsymbol{A}$ is not **invertible**, $\boldsymbol{A}$ is called **singular**.

Note: Non-square matrix is neither invertible nor singular.

Theorem. Let $\boldsymbol{A}$ be a square matrix.

- If $\boldsymbol{A}$ is invertible, then its inverse is unique.

  $AA^{-1} = A^{-1}A = I$

Definition. Let $\boldsymbol{A}$ be an invertible matrix.

- For any positive integer k, $\boxed{\boldsymbol{A}^{-k} = (\boldsymbol{A}^{-1})^k}$

Theorem. Let $\boldsymbol{A}, \boldsymbol{B}$ be invertible matrices of same size.

- Let $c \neq 0$. $c\boldsymbol{A}$ is invertible, and $(c\boldsymbol{A})^{-1} = \frac{1}{c}\boldsymbol{A}^{-1}$.
- $\boldsymbol{A}^{\mathrm{T}}$ is invertible, and $(\boldsymbol{A}^{\mathrm{T}})^{-1} = (\boldsymbol{A}^{-1})^{\mathrm{T}}$.
- $\boldsymbol{A}^{-1}$ is invertible, and $(\boldsymbol{A}^{-1})^{-1} = \boldsymbol{A}$.
- $\boldsymbol{A}\boldsymbol{B}$ is invertible, and $(\boldsymbol{A}\boldsymbol{B})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$.

Cancellation Law. Let $\boldsymbol{A}$ be an invertible matrix.

- $\cancel{\boldsymbol{A}}\boldsymbol{B}_1 = \cancel{\boldsymbol{A}}\boldsymbol{B}_2 \Rightarrow \boldsymbol{B}_1 = \boldsymbol{B}_2$.
- $\boldsymbol{C}_1\cancel{\boldsymbol{A}} = \boldsymbol{C}_2\cancel{\boldsymbol{A}} \Rightarrow \boldsymbol{C}_1 = \boldsymbol{C}_2$.

Cancel on same side.

Elementary Matrices.

- **Definition.** A square matrix is called an **elementary matrix** if it can be obtained from the identity matrix by performing a single elementary row operation.
  - cR_i, where $c \neq 0$: $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
  - $R_i \leftrightarrow R_j$, where $i \neq j$, $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$
  - $R_i + cR_j$, where $i \neq j$, $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & c \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Theorem.

- Let $\boldsymbol{E}$ be the elementary matrix obtained
  - by performing an elementary row operation to $\boldsymbol{I}_m$.

  Then for any $m \times n$ matrix $\boldsymbol{A}$, $\boldsymbol{E}\boldsymbol{A}$ can be obtained
  - by performing same elementary row operation to $\boldsymbol{A}$.

Let $\boldsymbol{A}$ be an $m \times n$ matrix.

- $\boldsymbol{I}_m \xrightarrow{cR_i} \boldsymbol{E} \Rightarrow \boldsymbol{A} \xrightarrow{cR_i} \boldsymbol{E}\boldsymbol{A}$.
- $\boldsymbol{I}_m \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{E} \Rightarrow \boldsymbol{A} \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{E}\boldsymbol{A}$.
- $\boldsymbol{I}_m \xrightarrow{R_i + cR_j} \boldsymbol{E} \Rightarrow \boldsymbol{A} \xrightarrow{R_i + cR_j} \boldsymbol{E}\boldsymbol{A}$.

Theorem. Every elementary matrix is invertible.

- The inverse of an elementary matrix is elementary. (of same type)

  literally inverse the operation

  $\boldsymbol{I} \xrightarrow{cR_i} \boldsymbol{E} \Rightarrow \boldsymbol{I} \xrightarrow{\frac{1}{c}R_i} \boldsymbol{E}^{-1}$.

  $\boldsymbol{I} \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{E} \Rightarrow \boldsymbol{I} \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{E}^{-1}$. (So $\boldsymbol{E} = \boldsymbol{E}^{-1}$)

  $\boldsymbol{I} \xrightarrow{R_i + cR_j} \boldsymbol{E} \Rightarrow \boldsymbol{I} \xrightarrow{R_i + (-c)R_j} \boldsymbol{E}^{-1}$.

- **Theorem.** Two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are row equivalent

  $\Leftrightarrow$ there exist elementary matrices $\boldsymbol{E}_1, \boldsymbol{E}_2, \ldots, \boldsymbol{E}_k$

  such that $\boldsymbol{B} = \boldsymbol{E}_k\boldsymbol{E}_{k-1}\cdots\boldsymbol{E}_2\boldsymbol{E}_1\boldsymbol{A}$.

- **Remarks.** Suppose for elementary matrices $\boldsymbol{E}_i$,
  - $\boldsymbol{B} = \boldsymbol{E}_k\boldsymbol{E}_{k-1}\cdots\boldsymbol{E}_2\boldsymbol{E}_1\boldsymbol{A}$.
    - $\boldsymbol{A} \xrightarrow{\boldsymbol{E}_1} \bullet \xrightarrow{\boldsymbol{E}_2} \bullet \to \cdots \to \bullet \xrightarrow{\boldsymbol{E}_{k-1}} \bullet \xrightarrow{\boldsymbol{E}_k} \boldsymbol{B}$.
    - $\boldsymbol{A} \xleftarrow{\boldsymbol{E}_1^{-1}} \bullet \xleftarrow{\boldsymbol{E}_2^{-1}} \bullet \leftarrow \cdots \leftarrow \bullet \xleftarrow{\boldsymbol{E}_{k-1}^{-1}} \bullet \xleftarrow{\boldsymbol{E}_k^{-1}} \boldsymbol{B}$.

  $\therefore\ \boldsymbol{A} = \boldsymbol{E}_1^{-1}\boldsymbol{E}_2^{-1}\cdots\boldsymbol{E}_{k-1}^{-1}\boldsymbol{E}_k^{-1}\boldsymbol{B}$.

Main Theorem for Invertible Matrices

- **Theorem.** Let $\boldsymbol{A}$ be a square matrix. Then the followings are equivalent:
  1. $\boldsymbol{A}$ is an invertible matrix. A^{-1}
  2. Linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ has a unique solution. $Ax = b$ → unique.
  3. Linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ has only the trivial solution. $Ax = 0$ → trivial only.
  4. The reduced row-echelon form of $\boldsymbol{A}$ is $\boldsymbol{I}$. → REF of $A = I$.
  5. $\boldsymbol{A}$ is the product of elementary matrices.

Theorem. Let $\boldsymbol{A}$ be an invertible matrix.

- The RREF of $(\boldsymbol{A} \mid \boldsymbol{I})$ is $\boxed{(\boldsymbol{I} \mid \boldsymbol{A}^{-1})}$

Theorem. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be square matrices of the same size. If $\boldsymbol{A}\boldsymbol{B} = \boldsymbol{I}$, then

- $\boldsymbol{A}$ and $\boldsymbol{B}$ are invertible, and $\boldsymbol{A}^{-1} = \boldsymbol{B}$, $\boldsymbol{B}^{-1} = \boldsymbol{A}$.

Corollary and Exercise. Let $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_k$ be square matrices of the same size.

- $\boldsymbol{A}_1\boldsymbol{A}_2\cdots\boldsymbol{A}_k$ is invertible $\Leftrightarrow$ all $\boldsymbol{A}_i$ are invertible.
- $\boldsymbol{A}_1\boldsymbol{A}_2\cdots\boldsymbol{A}_k$ is singular $\Leftrightarrow$ some $\boldsymbol{A}_i$ are singular.

Determinant

Definition. Let $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

- The **determinant** of $\boldsymbol{A}$ is $\boxed{\det(\boldsymbol{A}) = |\boldsymbol{A}| = ad - bc}$

Therefore, $\boldsymbol{A}$ is invertible $\Leftrightarrow \det(\boldsymbol{A}) \neq 0$.

Definition. If $\boldsymbol{A} = (a)$, it is natural to set $\det(\boldsymbol{A}) = a$.

Properties & Exercises. Let $\boldsymbol{A}, \boldsymbol{B}$ be 2×2 matrices.

- $\det(\boldsymbol{I}_2) = 1$.
- $\boldsymbol{A} \xrightarrow{cR_i} \boldsymbol{B} \Rightarrow \det(\boldsymbol{B}) = \underline{c}\det(\boldsymbol{A})$. * Scalar multiple
- $\boldsymbol{A} \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{B} \Rightarrow \det(\boldsymbol{B}) = \underline{-}\det(\boldsymbol{A})$. Interchange
- $\boldsymbol{A} \xrightarrow{R_i + cR_j} \boldsymbol{B} \Rightarrow \det(\boldsymbol{B}) = \det(\boldsymbol{A})$. Constant multiple.

Notation. Let $\boldsymbol{A} = (a_{ij})_{n \times n}$.

- Let $\boldsymbol{M}_{ij}$ be the submatrix obtained from $\boldsymbol{A}$ by deleting the ith row and jth column.
- If $\boldsymbol{A} = (a_{ij})_{3\times3}$, then $\det(\boldsymbol{A})$ is given by
  - $a_{11}\det(\boldsymbol{M}_{11}) - a_{12}\det(\boldsymbol{M}_{12}) + a_{13}\det(\boldsymbol{M}_{13})$
- * Let $A_{ij} = (-1)^{i+j}\det(\boldsymbol{M}_{ij})$, the (i,j)**-cofactor** of $\boldsymbol{A}$.
- If $\boldsymbol{A} = (a_{ij})_{3\times3}$, then
  - $\boxed{\det(\boldsymbol{A}) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}}$

Finding Determinant

- Find $\det(\boldsymbol{A})$ if $\boldsymbol{A}$ is a square matrix of order n.
  - If $\boldsymbol{A}$ has a zero row/column, then $\det(\boldsymbol{A}) = 0$.
  - If $\boldsymbol{A}$ is triangular, $\det(\boldsymbol{A}) = a_{11}\cdots a_{nn}$. product of diagonal entries.
  - Suppose that $\boldsymbol{A}$ is not triangular.
    - If $n = 2$, use formula $\det(\boldsymbol{A}) = a_{11}a_{22} - a_{12}a_{21}$. others.
    - If a row/coln has many 0, use cofactor expansion. → along that row/col.
    - Otherwise, use ele. row operations to get REF:
      - $\det(\boldsymbol{EA}) = \det(\boldsymbol{E})\det(\boldsymbol{A})$. Via gaussian elimination
- Note the following formulas (exercises for the last two): → either same or negative (depends on how many time ⇔ interchanging row is used).
  - $\det(\boldsymbol{A}) = \det(\boldsymbol{A}^{\mathrm{T}})$.
  - $\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B})$.
  - $\det(c\boldsymbol{A}) = \boxed{c^n}\det(\boldsymbol{A})$, where $\boldsymbol{A}$ is $n \times n$. * $n=4$ $\det(-A) = (-1)^4\det(A)$ need multiply each row by c.
  - $\det(\boldsymbol{A}^{-1}) = \det(\boldsymbol{A})^{-1}$ if $\boldsymbol{A}$ is invertible.

Definition. Let $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. Cofactor.

- $$\det(\boldsymbol{A}) = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Notation. Let $\boldsymbol{A} = (a_{ij})_{n \times n}$.

- Let $\boldsymbol{M}_{ij}$ be the submatrix obtained from $\boldsymbol{A}$ by deleting the ith row and jth column.
- If $\boldsymbol{A} = (a_{ij})_{3\times3}$, then $\det(\boldsymbol{A})$ is given by
  - $a_{11}\det(\boldsymbol{M}_{11}) - a_{12}\det(\boldsymbol{M}_{12}) + a_{13}\det(\boldsymbol{M}_{13})$
- * Let $A_{ij} = (-1)^{i+j}\det(\boldsymbol{M}_{ij})$, the (i,j)**-cofactor** of $\boldsymbol{A}$.
- If $\boldsymbol{A} = (a_{ij})_{3\times3}$, then
  - $\boxed{\det(\boldsymbol{A}) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}}$

An Alternative Formula

- Let $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.

$$\begin{aligned} \det(\boldsymbol{A}) &= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) \\ &\quad + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\ &= (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}) \\ &\quad -(a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} + a_{13}a_{22}a_{31}). \end{aligned}$$

- The positive terms come from the
  - 3 (broken) diagonals from the top left to bottom right.

  The negative terms come from the (Same but opposite)
  - 3 (broken) diagonals from the top right to bottom left.

Definition. Let $\boldsymbol{A} = (a_{ij})_{n\times n}$. Its **determinant** is:

- If $n = 1$, define $\det(\boldsymbol{A}) = a_{11}$.
- If $n > 1$, let A_{ij} be its (i,j)-cofactor, define (recursive)
  - $\det(\boldsymbol{A}) = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}$.

- The diagonal expansion no longer works for $n \geq 4$

Theorem. Suppose $\boldsymbol{A} = (a_{ij})_{n\times n}$ is **upper triangular**. / lower triangular.

- Then $\det(\boldsymbol{A}) = a_{11}a_{22}\cdots a_{nn}$. (the diagonal)

REF via gaussian elimination

Theorem. $\det(\boldsymbol{I}) = 1$. For any square matrices,

- If $\boldsymbol{A} \xrightarrow{cR_i} \boldsymbol{B}$, then $\det(\boldsymbol{B}) = c\det(\boldsymbol{A})$.
  - In particular, If $\boldsymbol{I} \xrightarrow{cR_i} \boldsymbol{E}$, then $\det(\boldsymbol{E}) = c$.
- If $\boldsymbol{A} \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{B}$, then $\det(\boldsymbol{B}) = -\det(\boldsymbol{A})$.
  - In particular, if $\boldsymbol{I} \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{E}$, then $\det(\boldsymbol{E}) = -1$.
- If $\boldsymbol{A} \xrightarrow{R_i + cR_j} \boldsymbol{B}$, then $\det(\boldsymbol{B}) = \det(\boldsymbol{A})$.
  - In particular, if $\boldsymbol{I} \xrightarrow{R_i + cR_j} \boldsymbol{E}$, then $\det(\boldsymbol{E}) = 1$.

Theorem. Let $\boldsymbol{A}$ be a square matrix.

- For any elementary matrix $\boldsymbol{E}$ of the same order,
  - $\det(\boldsymbol{EA}) = \det(\boldsymbol{E})\det(\boldsymbol{A})$. (to determine invertibility of square matrix).

Theorem. Let $\boldsymbol{A}$ be a square matrix of order n.

- Let A_{ij} denote the (i,j)-cofactor of $\boldsymbol{A}$.

Then for any i and j,

- $\det(\boldsymbol{A}) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$.
- $\det(\boldsymbol{A}) = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}$.

In evaluating the determinant using cofactor expansion,

- expand along the row or column with the most zeros.

Example. $\boldsymbol{A} = \begin{pmatrix} 0 & -1 & 2 & 0 \\ 2 & -3 & 3 & -2 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 2 & -1 \end{pmatrix}$.

$$\begin{aligned} \det(\boldsymbol{A}) &= a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} + a_{41}A_{41} \\ &= 2\cdot(-1)^{2+1}\begin{vmatrix} -1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 2 & -1 \end{vmatrix} \\ &= -2\cdot(-1)\cdot(-1)^{3+3}\begin{vmatrix} -1 & 2 \\ 2 & 4 \end{vmatrix} \\ &= -16. \end{aligned}$$

0 0 0

Theorem. Suppose a square matrix $\boldsymbol{A}$ has a zero row.

- Then $\det(\boldsymbol{A}) = 0$.

Theorem. $\det(\boldsymbol{A}) = 0 \Leftrightarrow \boldsymbol{A}$ is **singular**. **.

- Equivalently, $\det(\boldsymbol{A}) \neq 0 \Leftrightarrow \boldsymbol{A}$ is **invertible**. **

Theorem. Let $\boldsymbol{A}, \boldsymbol{B}$ be square matrices of the same size.

- Then $\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B})$.

Theorem. For any square $\boldsymbol{A}$, $\det(\boldsymbol{A}) = \det(\boldsymbol{A}^{\mathrm{T}})$.

Adjoint Matrix

Definition. Let $\boldsymbol{A}$ be a square matrix of order n. The **(classical) adjoint** (or **adjugate**, or **adjunct**) of $\boldsymbol{A}$ is

- $\boxed{\mathbf{adj}(\boldsymbol{A}) = (A_{ji})_{n\times n}}$

where $\underline{A_{ij}}$ is the (i,j)-cofactor of $\boldsymbol{A}$.

Theorem. Let $\boldsymbol{A}$ be a square matrix. Then

- $\boldsymbol{A}[\mathbf{adj}(\boldsymbol{A})] = \det(\boldsymbol{A})\boldsymbol{I}$.

Theorem. Let $\boldsymbol{A}$ be a square matrix. Then

- $\boldsymbol{A}[\mathbf{adj}(\boldsymbol{A})] = \det(\boldsymbol{A})\boldsymbol{I}$.
- $[\mathbf{adj}(\boldsymbol{A})]\boldsymbol{A} = \det(\boldsymbol{A})\boldsymbol{I}$. (Exercise!)

If $\boldsymbol{A}$ is invertible, then $\boldsymbol{A}^{-1} = \dfrac{1}{\det(\boldsymbol{A})}\mathbf{adj}(\boldsymbol{A})$.

Cramer's Rule. Let $\boldsymbol{A}$ be an invertible matrix of order n.

- For every column matrix $\boldsymbol{b}$ of size $n \times 1$, the linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ has a unique solution
  - $\boldsymbol{x} = \dfrac{1}{\det(\boldsymbol{A})} \begin{pmatrix} \det(\boldsymbol{A}_1) \\ \vdots \\ \det(\boldsymbol{A}_n) \end{pmatrix}$, column matrix.

  $\boldsymbol{A}_j$ is obtained from $\boldsymbol{A}$ by replacing its jth coln by $\boldsymbol{b}$.

Example. Let $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

- Suppose that $\boldsymbol{A}$ is invertible. $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ implies
  - $\boldsymbol{x} = \dfrac{1}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \begin{pmatrix} \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} \\ \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix} \end{pmatrix}$

Vectors

Euclidean Spaces

- **Definition.** An n**-vector** or **ordered** n**-tuple** of real numbers is $\boldsymbol{v} = (v_1, v_2, \ldots, v_i, \ldots, v_n)$.
  - $v_i \in \mathbb{R}$ is the i**th component** or i**th coordinate** of $\boldsymbol{v}$.

Let $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$ and $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$.

1. $\boldsymbol{u}$ and $\boldsymbol{v}$ are **equal** if $u_i = v_i$ for all $i = 1, \ldots, n$.
2. The n-vector $\boldsymbol{0} = (0, 0, \ldots, 0)$ is the **zero vector**.
3. Let $c \in \mathbb{R}$. The **scalar multiple** $c\boldsymbol{v}$ is
   - $c\boldsymbol{v} = (cv_1, cv_2, \ldots, cv_n)$.
4. The **negative** of $\boldsymbol{v}$ is $(-1)\boldsymbol{v}$, denoted by $-\boldsymbol{v}$.
5. The **addition** $\boldsymbol{u} + \boldsymbol{v}$ is
   - $\boldsymbol{u} + \boldsymbol{v} = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)$.
6. The **subtraction** $\boldsymbol{u} - \boldsymbol{v}$ is
   - $\boldsymbol{u} - \boldsymbol{v} = (u_1 - v_1, u_2 - v_2, \ldots, u_n - v_n)$.

Notation. An n-vector $(v_1, v_2, \ldots, v_n)$ can be viewed as

- a **row matrix** (**row vector**) $\begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}$,
- a **column matrix** (**column vector**) $\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$.

Properties. Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ be n-vectors and $c, d \in \mathbb{R}$.

- $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$.
- $(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})$.
- $\boldsymbol{v} + \boldsymbol{0} = \boldsymbol{v}$ and $\boldsymbol{v} + (-\boldsymbol{v}) = \boldsymbol{0}$.
- $c(\boldsymbol{u} + \boldsymbol{v}) = c\boldsymbol{u} + c\boldsymbol{v}$.
- $(c + d)\boldsymbol{v} = c\boldsymbol{v} + d\boldsymbol{v}$.
- $c(d\boldsymbol{v}) = (cd)\boldsymbol{v}$.
- $1\boldsymbol{v} = \boldsymbol{v}$. (Verification is left as exercise.)

Implicit vs Explicit forms

Linear Systems

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \quad \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Its general soln

e.g. $\{(x,y,z) \mid x+y+z=0 \text{ and } x-y+2z=1\}$

Straight line in $\mathbb{R}^2$

$\{(x,y) \mid ax+by=c\}$ e.g. $\{(\frac{c-bt}{a}, t), t \in \mathbb{R}\}$

A **straight line** in $\mathbb{R}^2$ is determined by a point (x_0, y_0) on the line, and its direction vector $(a, b) \neq \boldsymbol{0}$.

- A point on the line is of the form $(x_0, y_0) + t(a, b)$.

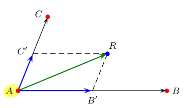
Explicit form of the line: (general explicit form)

- $\{(x_0 + ta, y_0 + tb) \mid t \in \mathbb{R}\}$.

Plane in $\mathbb{R}^3$

$\{(x,y,z) \mid ax+by+cz=d\}$ e.g. $\{(\frac{d-bs-ct}{a}, s, t) \mid s,t \in \mathbb{R}\}$

Three **non-collinear** points A, B, C determines a plane.



$$\begin{aligned} \boldsymbol{r} - \boldsymbol{a} = \overrightarrow{AR} &= \overrightarrow{AB'} + \overrightarrow{AC'} \\ &= s\overrightarrow{AB} + t\overrightarrow{AC} \\ &= s\boldsymbol{u} + t\boldsymbol{v}. \end{aligned}$$

- $\boxed{\boldsymbol{r} = \boldsymbol{a} + s\boldsymbol{u} + t\boldsymbol{v}, \quad s, t \in \mathbb{R}}$ (general explicit form)

A **plane** in $\mathbb{R}^3$ can be explicitly represented as

- $\{(x_0, y_0, z_0) + s(a_1, b_1, c_1) + t(a_2, b_2, c_2) \mid s, t \in \mathbb{R}\}$,

(x_0, y_0, z_0) is a point on the plane, and (a_1, b_1, c_1) & (a_2, b_2, c_2) are non-parallel vectors parallel to the plane.

A **straight line** in $\mathbb{R}^3$ is the intersection of two non-parallel planes. An implicit form is

- $\{(x, y, z) \mid a_1x + b_1y + c_1z = d_1 \ \& \ a_2x + b_2y + c_2z = d_2\}$,

a_i, b_i, c_i not all zero, and the planes are not parallel.

A **straight line** in $\mathbb{R}^3$ is determined by a point (x_0, y_0, z_0) on the line, and its direction vector $(a, b, c) \neq \boldsymbol{0}$.

- A point on the line: $(x_0, y_0, z_0) + t(a, b, c)$.

Explicit form: $\{(x_0 + ta, y_0 + tb, z_0 + tc) \mid t \in \mathbb{R}\}$.

In order to have an implicit form, we need to find two non-parallel planes $ax + by + cz = d$ containing the line.

Definition. Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$ be vectors in $\mathbb{R}^n$.

- A **linear combination** of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$ has the form
  - $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k$,

  where $c_1, c_2, \ldots, c_k \in \mathbb{R}$.

In particular, $\boldsymbol{0}$ is a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$:

- $\boldsymbol{0} = 0\boldsymbol{v}_1 + 0\boldsymbol{v}_2 + \cdots + 0\boldsymbol{v}_k$.

Definition. Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ be a subset of $\mathbb{R}^n$.

- The **set of all linear combinations** of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$
  - $\{c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k \mid c_1, c_2, \ldots, c_k \in \mathbb{R}\}$

  is called the **linear span** of S (or $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$).
- It is denoted by $\text{span}(S)$ or $\text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$.

$\boldsymbol{v}$ is a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$

$\Leftrightarrow \boldsymbol{v} \in \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$.

Criterion for $\text{span}(S) = \mathbb{R}^n$

- Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\} \subseteq \mathbb{R}^n$.
  1. View each $\boldsymbol{v}_j$ as a column vector.
  2. Let $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_k \end{pmatrix}$.
  3. Find a row-echelon form $\boldsymbol{R}$ of $\boldsymbol{A}$.
     - If $\boldsymbol{R}$ has a zero row, then $\text{span}(S) \neq \mathbb{R}^n$.
     - If $\boldsymbol{R}$ has no zero row, then $\text{span}(S) = \mathbb{R}^n$.

REF → no zero row = does not span $\mathbb{R}^n$ → inf. sols

Theorem. Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ be a subset of $\mathbb{R}^n$.

- If $k < n$, then $\text{span}(S) \neq \mathbb{R}^n$.

Theorem. Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ be a subset of $\mathbb{R}^n$.

- $\boldsymbol{0} \in \text{span}(S)$, where $\boldsymbol{0}$ is the zero vector in $\mathbb{R}^n$.
- Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r \in \text{span}(S)$, $c_1, c_2, \ldots, c_r \in \mathbb{R}$.
  - $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_r\boldsymbol{v}_r \in \text{span}(S)$.

Remarks. In particular,

- Since $\boldsymbol{0} \in \text{span}(S)$, $\text{span}(S) \neq \varnothing$. → smth always exist in span S.
- $\boldsymbol{v} \in \text{span}(S)$ and $c \in \mathbb{R} \Rightarrow c\boldsymbol{v} \in \text{span}(S)$.
  - $\text{span}(S)$ is **closed** under scalar multiplication.
- $\boldsymbol{u} \in \text{span}(S)$ and $\boldsymbol{v} \in \text{span}(S) \Rightarrow \boldsymbol{u} + \boldsymbol{v} \in \text{span}(S)$.
  - $\text{span}(S)$ is **closed** under addition.

Theorem. Given two subsets of $\mathbb{R}^n$:

- $S_1 = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$, $S_2 = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m\}$.

Then $\text{span}(S_1) \subseteq \text{span}(S_2)$

$\Leftrightarrow$ Every $\boldsymbol{u}_i$ is a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$.

Theorem. Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k-1}, \boldsymbol{v}_k \in \mathbb{R}^n$.

- If $\boldsymbol{v}_k$ is a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k-1}$, then
  - $\text{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k-1}\} = \text{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k-1}, \boldsymbol{v}_k\}$.

extra.

Properties of Linear Spans

- Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ be a subset of $\mathbb{R}^n$.

$$\boldsymbol{v} \in \text{span}(S) \Leftrightarrow \boldsymbol{v} = c_1\boldsymbol{v}_1 + \cdots + c_k\boldsymbol{v}_k \text{ for some } c_i \in \mathbb{R}$$

$$\Leftrightarrow \begin{pmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_k \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = \boldsymbol{v}.$$

1. View each $\boldsymbol{v}_j$ as a column vector.
2. Let $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_k \end{pmatrix}$.
3. Check if the linear system $\boldsymbol{Ax} = \boldsymbol{v}$ is consistent.
   - If $\boldsymbol{Ax} = \boldsymbol{v}$ is consistent, then $\boldsymbol{v} \in \text{span}(S)$.
   - If $\boldsymbol{Ax} = \boldsymbol{v}$ is inconsistent, then $\boldsymbol{v} \notin \text{span}(S)$.

Subspaces

- **Definition.** Let V be a subset of $\mathbb{R}^n$. Then V is called a **subspace** of $\mathbb{R}^n$ if there exist $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k \in \mathbb{R}^n$ s.t.
  - $V = \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$.

  More precisely,
  - V is the **subspace spanned** by $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$;
  - $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ **spans** the subspace V.
- **Remark.**
  - Let $\boldsymbol{0} \in \mathbb{R}^n$ be the zero vector. Then
    - $\{\boldsymbol{0}\} = \text{span}\{\boldsymbol{0}\}$ is the **zero space**.
  - Let $\boldsymbol{e}_i$ denote the n-vector whose ith coordinate is 1 and elsewhere 0, e.g., $\boldsymbol{e}_2 = (0, 1, 0, \ldots, 0)$.
    - Then for every $\boldsymbol{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$
      - $\boldsymbol{v} = v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + \cdots + v_n\boldsymbol{e}_n$.

  $\mathbb{R}^n = \text{span}\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ is a subspace of $\mathbb{R}^n$.

n vectors.

Subspaces of $\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^3$

- Subspaces of $\mathbb{R}^1$:
  - $\{0\}$,
  - $\mathbb{R}$.
- Subspaces of $\mathbb{R}^2$:
  - $\{\boldsymbol{0}\} = \{(0, 0)\}$,
  - A straight line passing through the origin $(0, 0)$,
  - $\mathbb{R}^2$.
- Subspaces of $\mathbb{R}^3$:
  - $\{\boldsymbol{0}\} = \{(0, 0, 0)\}$,
  - A straight line passing through the origin $(0, 0, 0)$,
  - A plane containing the origin $(0, 0, 0)$,
  - $\mathbb{R}^3$.

  A subspace of $\mathbb{R}^i$, $i = 1, 2, 3$, is always the solution set of a homogeneous linear system.

Theorem. The **solution set** of a **homogeneous** linear system of n variables is a **subspace** of $\mathbb{R}^n$.

Linear Independence

- **Definition.** Let $S = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ be a subset of $\mathbb{R}^n$.
  - The equation $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k = \boldsymbol{0}$

  has a **trivial solution** $c_1 = c_2 = \cdots = c_k = 0$.

  1. If the equation has a **non-trivial solution**, then
     - S is a **linearly dependent set**,
     - $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k$ are **linearly dependent**.

     There exist $c_1, c_2, \dots, c_k \in \mathbb{R}$ not all zero such that
     - $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k = \boldsymbol{0}$.
  2. If the equation has **only the trivial solution**, then
     - S is a **linearly independent set**,
     - $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k$ are **linearly independent**.

     $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k = \boldsymbol{0} \Rightarrow c_1 = \cdots = c_k = 0.$

Properties

- Let S_1, S_2 be finite subsets of $\mathbb{R}^n$ such that $S_1 \subseteq S_2$.
  - S_1 linearly dependent $\Rightarrow$ S_2 linearly dependent.
  - S_2 linearly independent $\Rightarrow$ S_1 linearly independent.
- $c\boldsymbol{0} = \boldsymbol{0}$ has infinitely many solutions $c \in \mathbb{R}$.
  - $\{\boldsymbol{0}\}$ is linearly dependent.
  - If $\boldsymbol{0} \in S$ $(\subseteq \mathbb{R}^n)$ then S is linearly dependent.
- Let $\boldsymbol{v} \in \mathbb{R}^n$. Then $c\boldsymbol{v} = \boldsymbol{0} \Leftrightarrow c = 0$ or $\boldsymbol{v} = \boldsymbol{0}$.
  - $\{\boldsymbol{v}\}$ is linearly independent $\Leftrightarrow \boldsymbol{v} \neq \boldsymbol{0}$.
- Let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$. Then

  $\{\boldsymbol{u}, \boldsymbol{v}\}$ is linearly dependent $\Leftrightarrow$ $\boldsymbol{u} = a\boldsymbol{v}$ for some $a \in \mathbb{R}$
  or $\boldsymbol{v} = a\boldsymbol{u}$ for some $a \in \mathbb{R}$

Properties

- **Theorem.** Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k\} \subseteq \mathbb{R}^n$, $k \geq 2$.
  - S is linearly dependent

    $\Leftrightarrow$ there exists $\boldsymbol{v}_i$ such that it is a linear combination of other vectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{i-1}, \boldsymbol{v}_{i+1}, \dots, \boldsymbol{v}_k$.
  - S is linearly independent

    $\Leftrightarrow$ no vector in S can be written as a linear combination of other vectors.
- **Remarks.** Suppose $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k\}$ is linearly dependent. Let $V = \text{span}(S)$.
  - Some $\boldsymbol{v}_i \in S$ is a linear combination of other vectors.
  - Remove $\boldsymbol{v}_i$ from S and repeat the procedure until we obtain a linearly independent set S'.
  - Then $\text{span}(S') = V$ and S' has no "redundant vector" to span V.

Theorem. Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k\} \subseteq \mathbb{R}^n$.

- If $k > n$, then S is linearly dependent.

Theorem. Suppose $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k\} \subseteq \mathbb{R}^n$ is linearly independent.

- If $\boxed{\boldsymbol{v}_{k+1} \in \mathbb{R}^n}$ is not in $\text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k\}$.
  - then $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k, \boldsymbol{v}_{k+1}\}$ is linearly independent.

Definition. A set V is called a **vector space** if

- V is a $\boxed{\text{subspace of } \mathbb{R}^n}$ for some positive integer n.

If W and V are vector spaces such that $W \subseteq V$,

- then W is a **subspace** of V.

Definition. Let $S = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ be a subset of a vector space V. Then S is called a **basis** (plural **bases**) for V if

- S is **linearly independent**, and $\text{span}(S) = V$.

Remarks.

- A basis for a vector space V contains
  - smallest possible number of vectors that spans V,
  - largest possible number of vectors that is l.indept.
- For convenience, $\varnothing$ is said to be the **basis** for $\{\boldsymbol{0}\}$. nothing span 0.
- Other than $\{\boldsymbol{0}\}$, any vector space has infinitely many different bases.

Theorem. Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k\}$ be a subset of a vector space V. Then the following are equivalent:

- S is a basis for V.
- Every vector $\boldsymbol{v} \in V$ can be **uniquely** expressed as
  - $\boldsymbol{v} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k$, $c_i \in \mathbb{R}$.

Coordinate Vector

- **Definition.** Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k\}$ be a basis for a vector space V.
  - For every $\boldsymbol{v} \in V$, there exist unique $c_1, \dots, c_k \in \mathbb{R}$ s.t.
    - $\boldsymbol{v} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k$.

    $c_1, c_2, \dots, c_k$ are the **coordinates of $\boldsymbol{v}$ relative** to S.
    - $(c_1, c_2, \dots, c_k)$ is the **coordinate vector** of $\boldsymbol{v}$ **relative** to the basis S, denoted by $(\boldsymbol{v})_S$.
- **Remark.** The order of $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k$ is fixed.
  - Let $S_1 = \{(1,1), (-1,1)\}$ be a basis for $\mathbb{R}^2$. (Check!)
    - Let $\boldsymbol{v} = 2(1,1) + 3(-1,1) = (-1,5)$.
      - Then $(\boldsymbol{v})_{S_1} = (2,3)$.

    Let $S_2 = \{(-1,1), (1,1)\}$. Then $(\boldsymbol{v})_{S_2} = (3,2)$.

Standard Basis

- **Definition.** Let $E = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_n\}$ be a subset of $\mathbb{R}^n$,
  - $\boldsymbol{e}_1 = (1, 0, \dots, 0),\ \boldsymbol{e}_2 = (0, 1, \dots, 0), \dots, \boldsymbol{e}_n = (0, 0, \dots, 1)$.

1. Let $\boldsymbol{v} = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$. Then
   - $\boldsymbol{v} = v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + \cdots + v_n\boldsymbol{e}_n$.

   ✓ So $\text{span}(E) = \mathbb{R}^n$.
2. Suppose that $c_1\boldsymbol{e}_1 + c_2\boldsymbol{e}_2 + \cdots + c_n\boldsymbol{e}_n = \boldsymbol{0}$. Then
   - $(c_1, c_2, \dots, c_n) = (0, 0, \dots, 0)$.

   ✓ So E is linearly independent.

$\boldsymbol{E}$ is called the **standard basis** for $\mathbb{R}^n$. simplest basis.

- For any $\boldsymbol{v} = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$,
  - $(\boldsymbol{v})_E = (v_1, v_2, \dots, v_n) = \boldsymbol{v}$.

Properties

- **Theorem.** Let S be a basis for a vector space V.
  - $(\boldsymbol{v})_S = \boldsymbol{0} \Leftrightarrow \boldsymbol{v} = \boldsymbol{0}$.
  - For any $c \in \mathbb{R}$ and $\boldsymbol{v} \in$ ~~$\mathbb{R}$~~ V, $(c\boldsymbol{v})_S = c(\boldsymbol{v})_S$.
  - For any $\boldsymbol{u}, \boldsymbol{v} \in V$, $(\boldsymbol{u} + \boldsymbol{v})_S = (\boldsymbol{u})_S + (\boldsymbol{v})_S$.

Theorem. Let S be a basis for a vector space V.
- For any $\boldsymbol{u}, \boldsymbol{v} \in V$, $\boldsymbol{u} = \boldsymbol{v} \Leftrightarrow (\boldsymbol{u})_S = (\boldsymbol{v})_S$.
- For any $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_r \in V$ and $c_1, c_2, \dots, c_r \in \mathbb{R}$,
  - $(c_1\boldsymbol{v}_1 + \cdots + c_r\boldsymbol{v}_r)_S = c_1(\boldsymbol{v}_1)_S + \cdots + c_r(\boldsymbol{v}_r)_S$.

Criterion for Bases

- Let $S = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ be a subset of $\mathbb{R}^n$.
  - If $k > n$, then S is linearly dependent.
  - If $k < n$, then $\text{span}(S) \neq \mathbb{R}^n$.

  If S is a basis, then $k = n$.
- **Theorem.** Let V be a vector space having a basis with k vectors.
  - Any subset of V of $> k$ vectors is linearly dependent.
  - Any subset of V of $< k$ vectors cannot span V.
- **Corollary.** All bases of a vector space have same size.
  - To be more precise, if S_1 and S_2 are two bases of a vector space V,
    - then $|S_1| = |S_2|$.

Theorem. Let S be a basis for a vector space V.
- Suppose $|S| = k$. Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_r \in V$.
  1. $\boldsymbol{v}_1, \dots, \boldsymbol{v}_r$ are linearly independent

     $\Leftrightarrow (\boldsymbol{v}_1)_S, \dots, (\boldsymbol{v}_r)_S$ are linearly independent.
  2. $\text{span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_r\} = V$

     $\Leftrightarrow \text{span}\{(\boldsymbol{v}_1)_S, \dots, (\boldsymbol{v}_r)_S\} = \mathbb{R}^k$.

Dimension

- **Definition.** Let V be a vector space and S a basis for V.
  - The **dimension** of V is $\dim(V) = |S|$.
- **Examples.**
  - $\varnothing$ is a (the) basis for $\{\boldsymbol{0}\}$.
    - Then $\dim(\{\boldsymbol{0}\}) = |\varnothing| = 0$.
  - $\mathbb{R}^n$ has the standard basis $E = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_n\}$.
    - Then $\dim(\mathbb{R}^n) = n$.
  - In $\mathbb{R}^2$ and $\mathbb{R}^3$, every straight line through the origin is of the form $\text{span}\{\boldsymbol{v}\}$ with $\boldsymbol{v} \neq \boldsymbol{0}$.
    - The dimension of such a straight line is 1.
  - In $\mathbb{R}^3$, every plane containing the origin is of the form $\text{span}\{\boldsymbol{u}, \boldsymbol{v}\}$, where $\boldsymbol{u}, \boldsymbol{v}$ are linearly independent. not parallel.
    - The dimension of such a plane is 2.

Properties

- **Theorem.** Let S be a subset of a vector space V. The following are equivalent:
  1. S is a basis for V.
  2. S is linearly independent, and $|S| = \dim(V)$.
  3. S spans V, and $|S| = \dim(V)$.
- To check whether a subset S is a basis for a vector space V, simply check any **two** of the following three conditions:
  - S is linearly independent,
  - S spans V,
  - $|S| = \dim(V)$.

Theorem. Let $\boldsymbol{A}$ be a square matrix of order n. Then the following are equivalent:
1. $\boldsymbol{A}$ is invertible.
2. $\boldsymbol{Ax} = \boldsymbol{b}$ has a unique solution.
3. $\boldsymbol{Ax} = \boldsymbol{0}$ has only the trivial solution.
4. The reduced row-echelon form of $\boldsymbol{A}$ is $\boldsymbol{I}_n$.
5. $\boldsymbol{A}$ is a product of elementary matrices.
6. $\det(\boldsymbol{A}) \neq 0$.
7. The rows of $\boldsymbol{A}$ form a basis for $\mathbb{R}^n$.
8. The columns of $\boldsymbol{A}$ form a basis for $\mathbb{R}^n$.

Dimension of Solution Space

- Let $\boldsymbol{Ax} = \boldsymbol{0}$ be a homogeneous linear system.
  - Recall that the solution set is a vector space V.

  Let $\boldsymbol{R}$ be a row-echelon form of $\boldsymbol{A}$.

no. of non-pivot colns of $\boldsymbol{R}$
$=$ no. of arbitrary parameters in soln
$=$ the dimension of V.

Theorem. Let U be a subspace of a vector space V.
- $U = V \Leftrightarrow \dim(U) = \dim(V)$.

Transition Matrix

- **Definition.** Let V be a vector space, and
  - $S = \{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k\}$ and T be bases for V.

  $\begin{pmatrix} [\boldsymbol{u}_1]_T & \cdots & [\boldsymbol{u}_k]_T \end{pmatrix}$ is the **transition matrix** from S to T.
  - Denote it by $\boldsymbol{P}$. Then $\boldsymbol{P}[\boldsymbol{w}]_S = [\boldsymbol{w}]_T$ for all $\boldsymbol{w} \in V$.

Properties

- **Theorem.** Let S and T be bases for a vector space V.
  - Let $\boldsymbol{P}$ be the transition matrix from S to T. Then
    - $\boldsymbol{P}$ is an invertible matrix.
    - $\boldsymbol{P}^{-1}$ is the transition matrix from T to S.

Random Notes Chpt 4 onwards.