Answers/Solutions of Exercise 5 (Version: August 18, 2014)

1. (a) $||\boldsymbol{u}|| = \sqrt{13}$, $||\boldsymbol{v}|| = \sqrt{2}$, $d(\boldsymbol{u}, \boldsymbol{v}) = \sqrt{5}$, $\boldsymbol{u} \cdot \boldsymbol{v} = 5$, $\theta = \cos^{-1}(\frac{5}{\sqrt{26}}) \approx 11.3°$.

   (b) $||\boldsymbol{u}|| = \sqrt{2}$, $||\boldsymbol{v}|| = \sqrt{10}$, $d(\boldsymbol{u}, \boldsymbol{v}) = \sqrt{20}$, $\boldsymbol{u} \cdot \boldsymbol{v} = -4$, $\theta = \cos^{-1}(\frac{-4}{\sqrt{20}}) \approx 153.4°$.

   (c) $||\boldsymbol{u}|| = \sqrt{14}$, $||\boldsymbol{v}|| = \sqrt{13}$, $d(\boldsymbol{u}, \boldsymbol{v}) = \sqrt{27}$, $\boldsymbol{u} \cdot \boldsymbol{v} = 0$, $\theta = 90°$.

   (d) $||\boldsymbol{u}|| = 2$, $||\boldsymbol{v}|| = \sqrt{10}$, $d(\boldsymbol{u}, \boldsymbol{v}) = \sqrt{14}$, $\boldsymbol{u} \cdot \boldsymbol{v} = 0$, $\theta = 90°$.

2. (a) $(1, 1, 0, 0) - (1, -1, 0, 0) = (0, 2, 0, 0)$ so $|AB| = \sqrt{(0^2 + 2^2 + 0^2 + 0^2)} = 2$. Likewise $|BC| = \sqrt{3}$ and $|AC| = \sqrt{3}$.

   (b) $\boldsymbol{u} = AB = (1, -1, 0, 0) - (1, 1, 0, 0) = (0, -2, 0, 0)$, $\boldsymbol{v} = AC = (2, 0, 0, 1) - (1, 1, 0, 0) = (1, -1, 0, 1)$. So the angle between AB and AC is $\cos^{-1}(\frac{\boldsymbol{u} \cdot \boldsymbol{v}}{|AB||AC|}) = \cos^{-1}(\frac{1}{\sqrt{3}}) \approx 54.7°$.

   (c) Easily verified that $2(2)(\sqrt{3})(\frac{1}{\sqrt{3}}) = 4 + 3 - 3$.

3. Let $\boldsymbol{u} = (u_1, u_2, \ldots, u_n\}$, $\boldsymbol{v} = (v_1, v_2, \ldots, v_n\}$ and $\boldsymbol{w} = (w_1, w_2, \ldots, w_n\}$.

   (a) $\boldsymbol{u} \cdot \boldsymbol{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n = v_1u_1 + v_2u_2 + \cdots + v_nu_n = \boldsymbol{v} \cdot \boldsymbol{u}$.

   (b)
$$\begin{aligned}(\boldsymbol{u} + \boldsymbol{v}) \cdot \boldsymbol{w} &= (u_1 + v_1,\ u_2 + v_2,\ \ldots,\ u_n + v_n) \cdot \boldsymbol{w} \\ &= (u_1 + v_1)w_1 + (u_2 + v_2)w_2 + \cdots + (u_n + v_n)w_n \\ &= (u_1w_1 + v_1w_1) + (u_2w_2 + v_2w_2) + \cdots + (u_nw_n + v_nw_n) \\ &= (u_1w_1 + u_2w_2 + \cdots + u_nw_n) + (v_1w_1 + v_2w_2 + \cdots + v_nw_n) \\ &= \boldsymbol{u} \cdot \boldsymbol{w} + \boldsymbol{v} \cdot \boldsymbol{w}.\end{aligned}$$

   (c)
$$\begin{aligned}(c\boldsymbol{u}) \cdot \boldsymbol{v} &= (cu_1,\ cu_2,\ \ldots,\ cu_n) \cdot \boldsymbol{v} \\ &= cu_1v_1 + cu_2v_2 + \cdots + cu_nv_n = c(u_1v_1 + u_2v_2 + \cdots + u_nv_n) = c(\boldsymbol{u} \cdot \boldsymbol{v}).\end{aligned}$$

   The proof for $\boldsymbol{u} \cdot (c\boldsymbol{v}) = c(\boldsymbol{u} \cdot \boldsymbol{v})$ is similar.

   (d)
$$\begin{aligned}||c\boldsymbol{u}|| = \sqrt{(c\boldsymbol{u}) \cdot (c\boldsymbol{u})} &= \sqrt{c^2(\boldsymbol{u} \cdot \boldsymbol{u})} \quad \text{(by (c))} \\ &= \sqrt{c^2}\sqrt{\boldsymbol{u} \cdot \boldsymbol{u}} = |c|\, ||\boldsymbol{u}||\end{aligned}$$

4. (a) If $\boldsymbol{u} = \boldsymbol{0}$, then it is obvious. Assume $\boldsymbol{u} \neq \boldsymbol{0}$. Let $a = \boldsymbol{u} \cdot \boldsymbol{u}$, $b = 2(\boldsymbol{u} \cdot \boldsymbol{v})$, $c = \boldsymbol{v} \cdot \boldsymbol{v}$ and let t be any real number.

$$\begin{aligned}0 &\leq (t\boldsymbol{u} + \boldsymbol{v}) \cdot (t\boldsymbol{u} + \boldsymbol{v}) \\ &= t^2(\boldsymbol{u} \cdot \boldsymbol{u}) + 2t(\boldsymbol{u} \cdot \boldsymbol{v}) + (\boldsymbol{v} \cdot \boldsymbol{v}) \\ &= at^2 + bt + c\end{aligned}$$

Thus the polynomial $at^2 + bt + c$ has either no real roots or repeated roots. This means that $b^2 - 4ac \leq 0$. In other words,

$$\begin{aligned} & 4(\boldsymbol{u} \cdot \boldsymbol{v})^2 \leq 4(\boldsymbol{u} \cdot \boldsymbol{u})(\boldsymbol{v} \cdot \boldsymbol{v}) \\ \Rightarrow\ & (\boldsymbol{u} \cdot \boldsymbol{v})^2 \leq (\boldsymbol{u} \cdot \boldsymbol{u})(\boldsymbol{v} \cdot \boldsymbol{v}) \\ \Rightarrow\ & |(\boldsymbol{u} \cdot \boldsymbol{v})| \leq \sqrt{\boldsymbol{u} \cdot \boldsymbol{u}}\sqrt{\boldsymbol{v} \cdot \boldsymbol{v}} = ||\boldsymbol{u}||\,||\boldsymbol{v}|| \end{aligned}$$

(b) $$\begin{aligned} ||\boldsymbol{u} + \boldsymbol{v}||^2 = (\boldsymbol{u} + \boldsymbol{v}) \cdot (\boldsymbol{u} + \boldsymbol{v}) &= \boldsymbol{u} \cdot \boldsymbol{u} + \boldsymbol{v} \cdot \boldsymbol{v} + 2\boldsymbol{u} \cdot \boldsymbol{v} \\ &\leq \boldsymbol{u} \cdot \boldsymbol{u} + \boldsymbol{v} \cdot \boldsymbol{v} + 2|\boldsymbol{u} \cdot \boldsymbol{v}| \\ &\leq ||\boldsymbol{u}||^2 + ||\boldsymbol{v}||^2 + 2||\boldsymbol{u}||||\boldsymbol{v}|| && \text{by (a)} \\ &= (||\boldsymbol{u}|| + ||\boldsymbol{v}||)^2 \end{aligned}$$

So $||\boldsymbol{u} + \boldsymbol{v}|| \leq ||\boldsymbol{u}|| + ||\boldsymbol{v}||$. Geometrically, this means that in any triangle, the sum of the length of any two sides is always greater than or equal to the length of the third side.

(c) In (b), substitute $\boldsymbol{u}$ and $\boldsymbol{v}$ by $\boldsymbol{u} - \boldsymbol{v}$ and $\boldsymbol{v} - \boldsymbol{w}$ respectively. We have $||\boldsymbol{u} - \boldsymbol{w}|| \leq ||\boldsymbol{u} - \boldsymbol{v}|| + ||\boldsymbol{v} - \boldsymbol{w}||$. So $d(\boldsymbol{u}, \boldsymbol{w}) \leq d(\boldsymbol{u}, \boldsymbol{v}) + d(\boldsymbol{v}, \boldsymbol{w})$.

5. (a) $$\begin{aligned} ||\boldsymbol{u} + \boldsymbol{v}||^2 + ||\boldsymbol{u} - \boldsymbol{v}||^2 &= (\boldsymbol{u} + \boldsymbol{v}) \cdot (\boldsymbol{u} + \boldsymbol{v}) + (\boldsymbol{u} - \boldsymbol{v}) \cdot (\boldsymbol{u} - \boldsymbol{v}) \\ &= 2(\boldsymbol{u} \cdot \boldsymbol{u}) + 2(\boldsymbol{v} \cdot \boldsymbol{v}) + 2\boldsymbol{u} \cdot \boldsymbol{v} - 2\boldsymbol{u} \cdot \boldsymbol{v} \\ &= 2||\boldsymbol{u}||^2 + 2||\boldsymbol{v}||^2 \end{aligned}$$

Geometric interpretation: For a parallelogram with $\boldsymbol{u}$ and $\boldsymbol{v}$ as sides, the sum of the squares of the four sides is equal to the sum of squares of the two diagonals.

(b) $\frac{1}{4}(\boldsymbol{u} + \boldsymbol{v}) \cdot (\boldsymbol{u} + \boldsymbol{v}) - \frac{1}{4}(\boldsymbol{u} - \boldsymbol{v}) \cdot (\boldsymbol{u} - \boldsymbol{v}) = \frac{1}{4}(2\boldsymbol{u} \cdot \boldsymbol{v} + 2\boldsymbol{u} \cdot \boldsymbol{v}) = \boldsymbol{u} \cdot \boldsymbol{v}$

6. (a) $\{\,(x, y) \mid x + y = 0\,\}$, the line $y = -x$ in $\mathbb{R}^2$.

(b) $\{\,(x, y, z) \mid x + 3z = 0\,\}$, the plane $x + 3z = 0$ in $\mathbb{R}^3$.

(c) $\{\,(x, y, z, w) \mid x - y + z - w = 0\,\}$

7. (a) Let (w, x, y, z) be any vector in $W^\perp$.

$$\left\{\begin{array}{r} (1, 0, 1, 1) \cdot (w, x, y, z) = 0 \\ (1, -1, 0, 2) \cdot (w, x, y, z) = 0 \\ (1, 2, 3, -1) \cdot (w, x, y, z) = 0 \end{array}\right. \quad \Leftrightarrow \quad \left\{\begin{array}{rrrrl} w & & +\ y & +\ \ z & = 0 \\ w & -\ \ x & & +2z & = 0 \\ w & +2x & +3y & -\ \ z & = 0 \end{array}\right.$$

A general solution of the linear system is $w = -s - t$, $x = -s + t$, $y = s$, $z = t$ where $s, t \in \mathbb{R}$. So $W^\perp = \{\, s(-1, -1, 1, 0) + t(-1, 1, 0, 1) \mid s, t \in \mathbb{R}\,\}$.

(b) Let $\{\boldsymbol{w_1}, \ldots, \boldsymbol{w_k}\}$ be a basis for W.

$$\boldsymbol{u} \in W^{\perp} \quad \Leftrightarrow \quad \begin{cases} \boldsymbol{w_1} \cdot \boldsymbol{u} = 0 \\ \quad \vdots \\ \boldsymbol{w_k} \cdot \boldsymbol{u} = 0 \end{cases} \quad \Leftrightarrow \quad \begin{pmatrix} \boldsymbol{w_1} \\ \vdots \\ \boldsymbol{w_k} \end{pmatrix} \boldsymbol{u}^{\mathrm{T}} = \boldsymbol{0}$$

So $W^{\perp}$ is a solution set of a homogeneous system of linear equations. By Theorem 3.3.6, $W^{\perp}$ is a subspace of $\mathbb{R}^n$.

Alternative proof: Since $\boldsymbol{w} \cdot \boldsymbol{0} = 0$ for all $\boldsymbol{w} \in W$, $\boldsymbol{0} \in W^{\perp}$. So $W^{\perp}$ is nonempty. Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be any vectors in $W^{\perp}$, i.e. $\boldsymbol{w} \cdot \boldsymbol{u} = 0$ and $\boldsymbol{w} \cdot \boldsymbol{v} = 0$ for all $\boldsymbol{w} \in W$, and let $a, b \in \mathbb{R}$. Then for all $\boldsymbol{w} \in W$, $\boldsymbol{w} \cdot (a\boldsymbol{u} + b\boldsymbol{v}) = a(\boldsymbol{w} \cdot \boldsymbol{u}) + b(\boldsymbol{w} \cdot \boldsymbol{v}) = 0 + 0 = 0$. Hence $\boldsymbol{u} + \boldsymbol{v} \in W^{\perp}$. By Remark 3.3.8, $W^{\perp}$ is a subspace of $\mathbb{R}^n$.

8. (a) Clearly $\mathrm{span}(T) \subseteq \mathrm{span}(S)$. We just need to show $\mathrm{span}(S) \subseteq \mathrm{span}(T)$. Since $\boldsymbol{u_1} = \boldsymbol{v_3}$, $\boldsymbol{u_2} = \frac{3}{5}\boldsymbol{v_1} + \frac{4}{5}\boldsymbol{v_2}$, $\boldsymbol{u_3} = \frac{4}{5}\boldsymbol{v_1} - \frac{3}{5}\boldsymbol{v_2}$, $\mathrm{span}(S) \subseteq \mathrm{span}(T)$ follows.

   (b) Since S is orthonormal, $\boldsymbol{v_1} \cdot \boldsymbol{v_1} = \frac{9}{25}(\boldsymbol{u_2} \cdot \boldsymbol{u_2}) + \frac{16}{25}(\boldsymbol{u_3} \cdot \boldsymbol{u_3}) + \frac{24}{25}(\boldsymbol{u_2} \cdot \boldsymbol{u_3}) = 1$. Likewise, it can be shown that $\boldsymbol{v_2} \cdot \boldsymbol{v_2} = \boldsymbol{v_3} \cdot \boldsymbol{v_3} = 1$, $\boldsymbol{v_1} \cdot \boldsymbol{v_2} = \boldsymbol{v_1} \cdot \boldsymbol{v_3} = \boldsymbol{v_2} \cdot \boldsymbol{v_3} = 0$. Hence T is also orthonormal.

9. $$\begin{aligned} ||\boldsymbol{u_1} + \cdots + \boldsymbol{u_n}||^2 &= (\boldsymbol{u_1} + \cdots + \boldsymbol{u_n}) \cdot (\boldsymbol{u_1} + \cdots + \boldsymbol{u_n}) \\ &= (\boldsymbol{u_1} \cdot \boldsymbol{u_1}) + \cdots + (\boldsymbol{u_n} \cdot \boldsymbol{u_n}) \quad \text{since } \boldsymbol{u_i} \cdot \boldsymbol{u_j} = 0 \text{ for } i \neq j \\ &= ||\boldsymbol{u_1}||^2 + \cdots + ||\boldsymbol{u_n}||^2 \end{aligned}$$

   For $n = 2$, it is Pythagoras' Theorem.

10. (a) It is easy to check that $\boldsymbol{u_i} \cdot \boldsymbol{u_j} = 0$ for $i \neq j$.

    (b) $S' = \{\, \frac{1}{\sqrt{10}}(1, 2, 2, -1),\ \frac{1}{2}(1, 1, -1, 1),\ \frac{1}{2}(-1, 1, -1, -1),\ \frac{1}{\sqrt{10}}(-2, 1, 1, 2) \,\}$

    (c) Yes.

    (d) $(\boldsymbol{w})_S = (\frac{3}{10}, \frac{1}{2}, -1, \frac{9}{10})$ and $(\boldsymbol{w})_{S'} = (\frac{3}{\sqrt{10}}, 1, -2, \frac{9}{\sqrt{10}})$.

    (e) A vector $\boldsymbol{v}$ is orthogonal to V if and only if $\boldsymbol{v} = t(-1, \frac{1}{2}, \frac{1}{2}, 1)$ for some $t \in \mathbb{R}$, i.e. $\boldsymbol{v} \in \mathrm{span}\{(-1, \frac{1}{2}, \frac{1}{2}, 1)\}$.

    (f) $(\frac{9}{5}, \frac{1}{10}, \frac{11}{10}, \frac{6}{5})$

11. (a) It is easy to check that $\boldsymbol{u_i} \cdot \boldsymbol{u_j} = 0$ for $i \neq j$.

    (b) For any $\boldsymbol{x} \in \mathbb{R}^3$, by Theorem 5.2.8,

    $$\boldsymbol{x} = \frac{\boldsymbol{x} \cdot \boldsymbol{u_1}}{\boldsymbol{u_1} \cdot \boldsymbol{u_1}}\boldsymbol{u_1} + \frac{\boldsymbol{x} \cdot \boldsymbol{u_2}}{\boldsymbol{u_2} \cdot \boldsymbol{u_2}}\boldsymbol{u_2} + \frac{\boldsymbol{x} \cdot \boldsymbol{u_3}}{\boldsymbol{u_3} \cdot \boldsymbol{u_3}}\boldsymbol{u_3} = \boldsymbol{v} + \boldsymbol{w}$$

    where $\boldsymbol{v} = \dfrac{\boldsymbol{x} \cdot \boldsymbol{u_1}}{\boldsymbol{u_1} \cdot \boldsymbol{u_1}}\boldsymbol{u_1} + \dfrac{\boldsymbol{x} \cdot \boldsymbol{u_2}}{\boldsymbol{u_2} \cdot \boldsymbol{u_2}}\boldsymbol{u_2} \in V$ and $\boldsymbol{w} = \dfrac{\boldsymbol{x} \cdot \boldsymbol{u_3}}{\boldsymbol{u_3} \cdot \boldsymbol{u_3}}\boldsymbol{u_3} \in W$.

(i) $\boldsymbol{v} = (\frac{1}{3}, -\frac{1}{3}, \frac{1}{3})$ and $\boldsymbol{w} = (-\frac{1}{3}, \frac{1}{3}, \frac{2}{3})$.

(ii) $\boldsymbol{v} = (1, 1, 0)$ and $\boldsymbol{w} = (0, 0, 0)$.

12. (a) $\{\frac{1}{\sqrt{2}}(1, 0, 1),\ \frac{1}{\sqrt{3}}(-1, 1, 1),\ \frac{1}{\sqrt{6}}(1, 2, -1)\}$

    (b) $\{\frac{1}{\sqrt{3}}(1, 1, 1),\ \frac{1}{\sqrt{6}}(1, -2, 1),\ \frac{1}{\sqrt{2}}(1, 0, -1)\}$

13. $\{\frac{1}{\sqrt{5}}(2, 1, 0, 0),\ \frac{1}{\sqrt{30}}(-1, 2, 0, 5),\ \frac{1}{\sqrt{10}}(1, -2, -2, 1),\ \frac{1}{\sqrt{15}}(1, -2, 3, 1)\}$

14. (a) A general solution to $x + y - z = 0$ is $x = t - s$, $y = s$, $z = t$ where $s, t \in \mathbb{R}$. So $\{(-1, 1, 0),\ (1, 0, 1)\}$ is a basis for the solution space. Using Gram-Schmidt process, we transform this basis into an orthonormal basis $\{\frac{1}{\sqrt{2}}(-1, 1, 0),\ \frac{1}{\sqrt{6}}(1, 1, 2)\}$.

    (b) $(\frac{1}{3}, -\frac{2}{3}, -\frac{1}{3})$

    (c) Since $(1, 1, -1)$ is orthogonal to the plane $x+y-z = 0$, it is orthogonal to the vectors in the basis obtained in (a). So $\{\frac{1}{\sqrt{2}}(-1, 1, 0),\ \frac{1}{\sqrt{6}}(1, 1, 2), \frac{1}{\sqrt{3}}(1, 1, -1)\}$ is an orthonormal basis for $\mathbb{R}^3$.

15. (a) We first show that $\boldsymbol{u_2}$ and $\boldsymbol{u_5}$ are linear combinations of $\boldsymbol{u_1}, \boldsymbol{u_3}, \boldsymbol{u_4}$.

$$\left(\begin{array}{ccc|c|c} 1 & 1 & 3 & 1 & -1 \\ 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 \end{array}\right) \xrightarrow[\text{Eliminaiton}]{\text{Gauss-Jordan}} \left(\begin{array}{ccc|c|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) = \boldsymbol{R}$$

Thus $\boldsymbol{u_2} = -\boldsymbol{u_1} - \boldsymbol{u_3} + \boldsymbol{u_4}$ and $\boldsymbol{u_5} = 2\boldsymbol{u_3} - \boldsymbol{u_4}$. We have $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_3}, \boldsymbol{u_4}\} = W$. Since the first three columns of the matrix $\boldsymbol{R}$ above are linearly independent, by Theorem 4.1.11, $\boldsymbol{u_1}, \boldsymbol{u_3}, \boldsymbol{u_4}$ are linearly independent. Hence they form a basis for W.

    (b) $\boldsymbol{v_1} = \boldsymbol{u_1}(1, 1, 0, 0)$

$$\boldsymbol{v_2} = \boldsymbol{u_3} - \frac{\boldsymbol{u_3} \cdot \boldsymbol{v_1}}{\boldsymbol{v_1} \cdot \boldsymbol{v_1}}\boldsymbol{v_1} = (\tfrac{1}{2}, -\tfrac{1}{2}, 1, 0)$$

$$\boldsymbol{v_3} = \boldsymbol{u_4} - \frac{\boldsymbol{u_4} \cdot \boldsymbol{v_1}}{\boldsymbol{v_1} \cdot \boldsymbol{v_1}}\boldsymbol{v_1} - \frac{\boldsymbol{u_4} \cdot \boldsymbol{v_2}}{\boldsymbol{v_2} \cdot \boldsymbol{v_2}}\boldsymbol{v_2} = (\tfrac{1}{3}, -\tfrac{1}{3}, -\tfrac{1}{3}, 1)$$

So $\{\frac{1}{\sqrt{2}}(1, 1, 0, 0),\ \frac{1}{\sqrt{6}}(1, -1, 2, 0),\ \frac{1}{\sqrt{12}}(1, -1, -1, 3)\}$ is an orthonormal basis for W.

    (c) $$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 3 & 1 & 1 & 1 \end{pmatrix} \xrightarrow[\text{Eliminaiton}]{\text{Gaussian}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

Let $\boldsymbol{w} = (0,0,0,1)$. Then $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_3}, \boldsymbol{u_4}, \boldsymbol{w}\} = \mathbb{R}^4$. We continue the Gram-Schmidt Process in (b):

$$\boldsymbol{v_4} = \boldsymbol{w} - \frac{\boldsymbol{w}\cdot\boldsymbol{v_1}}{\boldsymbol{v_1}\cdot\boldsymbol{v_1}}\boldsymbol{v_1} - \frac{\boldsymbol{w}\cdot\boldsymbol{v_2}}{\boldsymbol{v_2}\cdot\boldsymbol{v_2}}\boldsymbol{v_2} - \frac{\boldsymbol{w}\cdot\boldsymbol{v_3}}{\boldsymbol{v_3}\cdot\boldsymbol{v_3}}\boldsymbol{v_3} = (-\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}).$$

Thus $\{\frac{1}{\sqrt{2}}(1,1,0,0),\ \frac{1}{\sqrt{6}}(1,-1,2,0),\ \frac{1}{\sqrt{12}}(1,-1,-1,3),\ \frac{1}{2}(-1,1,1,1)\}$ is an orthonormal basis for $\mathbb{R}^4$.

16. When $a=1$, $\boldsymbol{V} = \text{span}\{(1,1,1)\}$ and hence $\left\{\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)\right\}$ is an orthonormal basis for $\boldsymbol{V}$. The projection of $(5,3,1)$ onto V is $(3,3,3)$.

   Suppose $a \neq 1$. Let $\boldsymbol{v_1} = (1,1,1)$ and $\boldsymbol{v_2} = (1,a,a) - \frac{(1,a,a)\cdot(1,1,1)}{(1,1,1)\cdot(1,1,1)}(1,1,1) = \frac{1-a}{3}(2,-1,-1)$. Then $\left\{\frac{1}{||\boldsymbol{v_1}||}\boldsymbol{v_1}, \frac{1}{||\boldsymbol{v_2}||}\boldsymbol{v_2}\right\} = \left\{\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)\right\}$ is an orthonormal basis for $\boldsymbol{V}$. The projection of $(5,3,1)$ onto V is $(5,2,2)$.

   An alternatively basis for $a \neq 1$: We can write $\boldsymbol{V} = \text{span}\{(1,0,0),\ (0,1,1)\}$ and hence $\left\{(1,0,0),\ \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\}$ is an orthonormal basis for $\boldsymbol{V}$.

17. (a) $\boldsymbol{w_1} = \frac{1}{\sqrt{3}}(1,1,1,0)^{\mathrm{T}}$, $\boldsymbol{w_2} = (0,0,0,1)^{\mathrm{T}}$, $\boldsymbol{w_3} = \frac{1}{\sqrt{6}}(-1,-1,2,0)^{\mathrm{T}}$.

   (b) $\boldsymbol{u_1} = \sqrt{3}\boldsymbol{w_1}$, $\boldsymbol{u_2} = \sqrt{3}\boldsymbol{w_1} + \boldsymbol{w_2}$, $\boldsymbol{u_3} = \frac{1}{\sqrt{3}}\boldsymbol{w_1} + \boldsymbol{w_2} + \frac{\sqrt{2}}{\sqrt{3}}\boldsymbol{w_3}$.

   (c) By (b), $\boldsymbol{A} = \begin{pmatrix}\boldsymbol{u_1} & \boldsymbol{u_2} & \boldsymbol{u_3}\end{pmatrix} = \begin{pmatrix}\boldsymbol{w_1} & \boldsymbol{w_2} & \boldsymbol{w_3}\end{pmatrix}\begin{pmatrix}\sqrt{3} & \sqrt{3} & \frac{1}{\sqrt{3}} \\ 0 & 1 & 1 \\ 0 & 0 & \frac{\sqrt{2}}{\sqrt{3}}\end{pmatrix}$.

   Let $\boldsymbol{Q} = \begin{pmatrix}\boldsymbol{w_1} & \boldsymbol{w_2} & \boldsymbol{w_3}\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ 0 & 1 & 0\end{pmatrix}$ and $\boldsymbol{R} = \begin{pmatrix}\sqrt{3} & \sqrt{3} & \frac{1}{\sqrt{3}} \\ 0 & 1 & 1 \\ 0 & 0 & \frac{\sqrt{2}}{\sqrt{3}}\end{pmatrix}$.

   Then the columns of $\boldsymbol{Q}$ are orthonormal, $\boldsymbol{R}$ is an upper triangular matrix and $\boldsymbol{A} = \boldsymbol{QR}$.

18. Suppose $\boldsymbol{u} = \boldsymbol{n_1} + \boldsymbol{p_1} = \boldsymbol{n_2} + \boldsymbol{p_2}$ where $\boldsymbol{n_1}, \boldsymbol{n_2}$ are orthogonal to V and $\boldsymbol{p_1}, \boldsymbol{p_2} \in V$. We need to show that $\boldsymbol{n_1} = \boldsymbol{n_2}$ and $\boldsymbol{p_1} = \boldsymbol{p_2}$.

   Observe that $\boldsymbol{n_i}\cdot\boldsymbol{p_j} = 0$ for $i,j = 1,2$.

   By $\boldsymbol{n_1} + \boldsymbol{p_1} = \boldsymbol{n_2} + \boldsymbol{p_2}$, we have $\boldsymbol{n_1} - \boldsymbol{n_2} = \boldsymbol{p_2} - \boldsymbol{p_1}$. Thus

$$\begin{aligned} ||\boldsymbol{n_1} - \boldsymbol{n_2}||^2 &= (\boldsymbol{n_1} - \boldsymbol{n_2})\cdot(\boldsymbol{n_1} - \boldsymbol{n_2}) \\ &= (\boldsymbol{n_1} - \boldsymbol{n_2})\cdot(\boldsymbol{p_2} - \boldsymbol{p_1}) \\ &= \boldsymbol{n_1}\cdot\boldsymbol{p_2} - \boldsymbol{n_1}\cdot\boldsymbol{p_1} - \boldsymbol{n_2}\cdot\boldsymbol{p_2} + \boldsymbol{n_2}\cdot\boldsymbol{p_1} = 0. \end{aligned}$$

By Theorem 5.1.5.5, $\boldsymbol{n_1}-\boldsymbol{n_2}=\mathbf{0}$ and hence $\boldsymbol{n_1}=\boldsymbol{n_2}$. Also, $\boldsymbol{p_2}-\boldsymbol{p_1}=\boldsymbol{n_1}-\boldsymbol{n_2}=\mathbf{0}$ and hence $\boldsymbol{p_1}=\boldsymbol{p_2}$.

19. (a) $(\boldsymbol{Au})\cdot\boldsymbol{v}=(\boldsymbol{Au})^{\mathrm{T}}\boldsymbol{v}=\boldsymbol{u}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{v}=\boldsymbol{u}^{\mathrm{T}}\boldsymbol{Av}=\boldsymbol{u}\cdot(\boldsymbol{Av})$.

    (b) Since $\boldsymbol{A}(\boldsymbol{Aw})=\boldsymbol{A}^2\boldsymbol{w}=\boldsymbol{Aw}$, $\boldsymbol{Aw}\in V$.

    Let $\boldsymbol{v}=\boldsymbol{w}-\boldsymbol{Aw}$. For any $\boldsymbol{u}\in V$,

$$\boldsymbol{u}\cdot\boldsymbol{v}=\boldsymbol{u}\cdot\boldsymbol{w}-\boldsymbol{u}\cdot(\boldsymbol{Aw})=\boldsymbol{u}\cdot\boldsymbol{w}-(\boldsymbol{Au})\cdot\boldsymbol{w}=\boldsymbol{u}\cdot\boldsymbol{w}-\boldsymbol{u}\cdot\boldsymbol{w}=\mathbf{0}.$$

    So $\boldsymbol{v}$ is orthogonal to V.

    Since we can write $\boldsymbol{w}=\boldsymbol{Aw}+\boldsymbol{v}$ where $\boldsymbol{Aw}\in V$ and $\boldsymbol{v}$ is orthogonal to V, $\boldsymbol{Aw}$ is the projection of $\boldsymbol{w}$ onto V.

20. (a) False. For example, let $\boldsymbol{u}=(1,0)$, $\boldsymbol{v}=(0,1)$, $\boldsymbol{w}=(2,0)$.

    (b) True. Since $\boldsymbol{w}$ is orthogonal to both $\boldsymbol{u}$ and $\boldsymbol{v}$, $||\boldsymbol{u}+\boldsymbol{w}||=\sqrt{||\boldsymbol{u}||^2+||\boldsymbol{w}||^2}$ and $||\boldsymbol{v}+\boldsymbol{w}||=\sqrt{||\boldsymbol{v}||^2+||\boldsymbol{w}||^2}$.

    (c) True. Since $\boldsymbol{u}$ is orthogonal to both $\boldsymbol{v}$ and $\boldsymbol{w}$, $\boldsymbol{u}\cdot(\boldsymbol{v}+\boldsymbol{w})=\boldsymbol{u}\cdot\boldsymbol{v}+\boldsymbol{u}\cdot\boldsymbol{w}=0$.

    (d) False. For example, let $\boldsymbol{u}=(1,0)$, $\boldsymbol{v}=(0,1)$, $\boldsymbol{w}=(2,0)$.

21. (a) The line is spanned by $(1,1)$. The projection of $(1,5)$ onto the line is $\frac{(1,5)\cdot(1,1)}{(1,1)\cdot(1,1)}(1,1)=(3,3)$. So the distance from $(1,5)$ to the line is $d((1,5),\,(3,3))=||(1,5)-(3,3)||=||(-2,2)||=\sqrt{8}$.

    (b) The standard method is first to find the projection $\boldsymbol{p}$ of $\boldsymbol{w}$ onto the plane $2x+y-2z=0$. Then the distance from $\boldsymbol{w}$ to the plane is $d(\boldsymbol{w},\boldsymbol{p})$. However, the computation is quite tedious. In the following, we present an alternative method:

    The distant from the point $\boldsymbol{w}=(1,0,-2)$ to the plane $2x+y-2z=0$ is equal to the length of the projection of $\boldsymbol{w}$ onto the line perpendicular to the plane, i.e. the line spanned by $\boldsymbol{u}=(2,1,-2)$. So the distant is

$$\left|\left|\frac{\boldsymbol{w}\cdot\boldsymbol{u}}{\boldsymbol{u}\cdot\boldsymbol{u}}\boldsymbol{u}\right|\right|=\frac{|\boldsymbol{w}\cdot\boldsymbol{u}|}{\boldsymbol{u}\cdot\boldsymbol{u}}||\boldsymbol{u}||=2.$$

    (c) The line is spanned by $(1,2,2)$. The projection of $(1,0,-2)$ onto the line is $\frac{(1,0,-2)\cdot(1,2,2)}{(1,2,2)\cdot(1,2,2)}(1,2,2)=(\frac{1}{3},\,\frac{2}{3},\,\frac{2}{3})$. So the distance from $(1,0,-2)$ to the line is $d((1,0,-2),\,(\frac{1}{3},\,\frac{2}{3},\,\frac{2}{3}))=||(1,0,-2)-(\frac{1}{3},\,\frac{2}{3},\,\frac{2}{3})||=||(\frac{2}{3},\,-\frac{2}{3},\,-\frac{8}{3})||=\sqrt{8}$.

22. (a) $\begin{cases} C+\ \ D=3\\ C+2D=5\\ C+3D=6\end{cases}$

(b) $C = \frac{5}{3}$ and $D = \frac{3}{2}$.

23. Let x, y and z be the amount of money that Jack, Jim and John received respectively.

    (a) The conditions are

    $$\begin{cases} x + 2y \phantom{{}+z} = 300 \\ \phantom{x+{}} y + z = 300 \\ x \phantom{{}+2y} - 2z = 300. \end{cases}$$

    The system is inconsistent. So there are no solution to the distribution problem.

    (b) The least squares solution to the system in (a) is $x = 200 + 2t$, $y = 100 - t$ and $z = t$ where t is arbitrary. However, to make sure that x, y and z are all non-negative, we need to have $0 \le t \le 100$.

24. (a) It is easy to check.

    (b) $x = 1$, $y = -\frac{2}{3}$, $z = 1$.

25. (a) $x = \frac{1}{3}$, $y = \frac{1}{3}$, $z = \frac{1}{3}$.

    (b) $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

26. (a) $\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & -2 \\ 1 & 2 & 3 \end{pmatrix} \xrightarrow[\text{Eliminaiton}]{\text{Gaussian}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

    So $\{(1, 0, 1), (0, 1, -2)\}$ is a basis for V.

    (b) (i) Applying the Gram-Schmidt Process to the basis obtained in (a), we obtain an orthonormal basis $\{\boldsymbol{u_1}, \boldsymbol{u_2}\}$ for the column space of $\boldsymbol{A}$ where $\boldsymbol{u_1} = \frac{1}{\sqrt{2}}(1, 0, 1)$ and $\boldsymbol{u_2} = \frac{1}{\sqrt{3}}(1, 1, -1)$. Then the projection of $\boldsymbol{w}$ onto V is $(\boldsymbol{w} \cdot \boldsymbol{u_1})\boldsymbol{u_1} + (\boldsymbol{w} \cdot \boldsymbol{u_2})\boldsymbol{u_2} = (\frac{4}{3}, \frac{1}{3}, \frac{2}{3})$.

    (ii) The least squares solution to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{4}{3} \\ -\frac{1}{3} \end{pmatrix}$.

    Since $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} \frac{4}{3} \\ -\frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{4}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$, the projection of $\boldsymbol{w}$ onto V is $(\frac{4}{3}, \frac{1}{3}, \frac{2}{3})$.

27. (a) For both (i) and (ii), $\boldsymbol{x} = (2, 1)^{\mathrm{T}}$.

(b) Let $\boldsymbol{v}$ be a solution of $\boldsymbol{Ax} = \boldsymbol{b}$, i.e. $\boldsymbol{Av} = \boldsymbol{b}$. Since $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{Av} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{b}$, $\boldsymbol{v}$ is also a solution of $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{Ax} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{b}$. Then

$$\begin{aligned}\text{the solution set of } (\boldsymbol{Ax} = \boldsymbol{b}) &= \{\boldsymbol{u} + \boldsymbol{v} \mid \boldsymbol{u} \in \text{the nullspace of } \boldsymbol{A}\} \\ &= \{\boldsymbol{u} + \boldsymbol{v} \mid \boldsymbol{u} \in \text{the nullspace of } \boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\} \\ &= \text{the solution set of } (\boldsymbol{A}^{\mathrm{T}}\boldsymbol{Ax} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{b}).\end{aligned}$$

28. (a) It is easy to check that U and V are orthogonal. Since $\dim(\mathbb{R}^3) = 3$, by Remark 5.2.6, U and V are bases for $\mathbb{R}^3$.

(b) $U' = \{\frac{1}{\sqrt{5}}(2, 1, 0),\ (0, 0, 1),\ \frac{1}{\sqrt{5}}(-1, 2, 0)\}$

$V' = \{\frac{1}{\sqrt{5}}(0, -1, 2),\ \frac{1}{\sqrt{6}}(-1, 2, 1),\ \frac{1}{\sqrt{30}}(5, 2, 1)\}$

(c) $\boldsymbol{P} = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \end{pmatrix}$ and $\boldsymbol{Q} = \begin{pmatrix} -\frac{1}{5} & \frac{2}{\sqrt{5}} & -\frac{2}{5} \\ 0 & \frac{1}{\sqrt{6}} & \frac{5}{\sqrt{30}} \\ \frac{12}{5\sqrt{6}} & \frac{1}{\sqrt{30}} & -\frac{1}{5\sqrt{6}} \end{pmatrix}$.

(d) Yes.

29. Let $\boldsymbol{R} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$.

(a) $\boldsymbol{R}^{\mathrm{T}}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 + \frac{\sqrt{3}}{2} \\ \frac{1}{2} - \sqrt{3} \end{pmatrix}$

(b) $\boldsymbol{R}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{\sqrt{3}}{2} \\ \frac{1}{2} + \sqrt{3} \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1 \ \Leftrightarrow\ \begin{pmatrix} 1 & 1 \end{pmatrix}\boldsymbol{R}\begin{pmatrix} x' \\ y' \end{pmatrix} = 1 \ \Leftrightarrow\ (1+\sqrt{3})x' + (1-\sqrt{3})y' = 2$

30. $\boldsymbol{A} = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$

31. (a) $(\boldsymbol{u})_{S_1} = (1, 4)$, $(\boldsymbol{v})_{S_1} = (-1, 1)$, $(\boldsymbol{u})_{S_1} \cdot (\boldsymbol{v})_{S_1} = 3$.

$(\boldsymbol{u})_{S_2} = (-\frac{7}{3}, \frac{5}{3})$, $(\boldsymbol{v})_{S_2} = (-1, 0)$, $(\boldsymbol{u})_{S_2} \cdot (\boldsymbol{v})_{S_2} = \frac{7}{3}$.

$(\boldsymbol{u})_{S_3} = (\frac{5}{\sqrt{2}}, \frac{3}{\sqrt{2}})$, $(\boldsymbol{v})_{S_3} = (0, \sqrt{2})$, $(\boldsymbol{u})_{S_3} \cdot (\boldsymbol{v})_{S_3} = 3$.

Note that $(\boldsymbol{u})_{S_1} \cdot (\boldsymbol{v})_{S_1} = (\boldsymbol{u})_{S_3} \cdot (\boldsymbol{v})_{S_3} \neq (\boldsymbol{u})_{S_2} \cdot (\boldsymbol{v})_{S_2}$. See (b) for an explanation.

(b) Let $\boldsymbol{P}$ be the transition matrix from S to T. Since S and T are orthonormal bases, P is orthogonal, i.e. $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{P} = \boldsymbol{I}$. (To use the transition matrix, it is more convenient to write the coordinate vectors as column vectors, i.e. we use $[\boldsymbol{u}]_S$, $[\boldsymbol{v}]_S$, $[\boldsymbol{u}]_T$ and $[\boldsymbol{v}]_T$ in the following computation.)

$$\begin{aligned}[\boldsymbol{u}]_T \cdot [\boldsymbol{v}]_T = ([\boldsymbol{u}]_T)^{\mathrm{T}}[\boldsymbol{v}]_T &= (\boldsymbol{P}[\boldsymbol{u}]_S)^{\mathrm{T}}(\boldsymbol{P}[\boldsymbol{v}]_S) \\ &= ([\boldsymbol{u}]_S)^{\mathrm{T}}\boldsymbol{P}^{\mathrm{T}}\boldsymbol{P}[\boldsymbol{v}]_S = ([\boldsymbol{u}]_S)^{\mathrm{T}}[\boldsymbol{v}]_S = [\boldsymbol{u}]_S \cdot [\boldsymbol{v}]_S.\end{aligned}$$

32. (a) $||\boldsymbol{Au}||^2 = (\boldsymbol{Au})^{\mathrm{T}}(\boldsymbol{Au}) = \boldsymbol{u}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{Au} = \boldsymbol{u}^{\mathrm{T}}\boldsymbol{u} = ||\boldsymbol{u}||^2$. Since both $||\boldsymbol{u}||$ and $||\boldsymbol{Au}||$ are nonnegative, we have $||\boldsymbol{Au}|| = ||\boldsymbol{u}||$.

(b) $d(\boldsymbol{Au}, \boldsymbol{Av}) = ||\boldsymbol{Au} - \boldsymbol{Av}|| = ||\boldsymbol{A}(\boldsymbol{u} - \boldsymbol{v})|| = ||\boldsymbol{u} - \boldsymbol{v}|| = d(\boldsymbol{u}, \boldsymbol{v})$

(c) $(\boldsymbol{Au}) \cdot (\boldsymbol{Av}) = (\boldsymbol{Au})^{\mathrm{T}}\boldsymbol{Av} = \boldsymbol{u}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{Av} = \boldsymbol{u}^{\mathrm{T}}\boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{v}$. So

$$\begin{aligned}\text{the angle between } \boldsymbol{u} \text{ and } \boldsymbol{v} &= \cos^{-1}\left(\frac{\boldsymbol{u} \cdot \boldsymbol{v}}{||\boldsymbol{u}||\,||\boldsymbol{v}||}\right) \\ &= \cos^{-1}\left(\frac{(\boldsymbol{Au}) \cdot (\boldsymbol{Av})}{||\boldsymbol{Au}||\,||\boldsymbol{Av}||}\right) \\ &= \text{the angle between } \boldsymbol{Au} \text{ and } \boldsymbol{Av}.\end{aligned}$$

33. (a) Since $\boldsymbol{A}$ is invertible, by Question 3.30(b)(i), T is linearly independent. So T is a basis for $\mathbb{R}^n$ by Theorem 3.6.7.

(b) See Question 5.32.

(c) Yes.

34. (a) True. Note that $\boldsymbol{c_i} \cdot \boldsymbol{c_j} = 0$ if $i \neq j$ and $\boldsymbol{c_i} \cdot \boldsymbol{c_i} = 1$.

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} = \begin{pmatrix} \boldsymbol{c_1}^{\mathrm{T}} \\ \vdots \\ \boldsymbol{c_k}^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} \boldsymbol{c_1} & \cdots & \boldsymbol{c_k} \end{pmatrix} = \begin{pmatrix} \boldsymbol{c_1} \cdot \boldsymbol{c_1} & \cdots & \boldsymbol{c_1} \cdot \boldsymbol{c_k} \\ \vdots & & \vdots \\ \boldsymbol{c_k} \cdot \boldsymbol{c_1} & \cdots & \boldsymbol{c_k} \cdot \boldsymbol{c_k} \end{pmatrix} = \boldsymbol{I}_k.$$

(b) False. For example, let $\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$.

(c) False. For example, let $\boldsymbol{A} = \boldsymbol{I}_2$, $\boldsymbol{B} = -\boldsymbol{I}_2$.

(d) True. $(\boldsymbol{AB})^{\mathrm{T}}(\boldsymbol{AB}) = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{AB} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{B} = \boldsymbol{I}$.