CS1231S Tutorial 7: Functions

National University of Singapore

2021/22 Semester 1

Background

Definition. The *range* or the *image* of a function $f\colon A \to B$ is a defined to be

$$\{f(x) : x \in A\}.$$

Definition. For each $x \in \mathbb{Q}$, define $\lfloor x \rfloor$ to be the largest integer n such that $n \leqslant x$. Dually, for each $x \in \mathbb{Q}$, define $\lceil x \rceil$ to be the smallest integer n such that $n \geqslant x$.

Questions for discussion on the LumiNUS Forum

Answers to these questions will not be provided.

> [...] "[S]et" turns out to have many meanings, so that the purported foundation of all of Mathematics upon set theory totters. But there are other possibilities. For example, the membership relation for sets can often be replaced by the composition operation for functions. This leads to an alternative foundation for Mathematics upon categories [...] Mac Lane 1986

D1. Find a function whose codomain is infinite but whose range is finite.

D2. (a) Define a function $\mathbb{Z} \to \mathbb{Z}^+$ that is neither injective nor surjective.
  (b) Define a function $\mathbb{Z} \to \mathbb{Z}^+$ that is injective but not surjective.
  (c) Define a function $\mathbb{Z} \to \mathbb{Z}^+$ that is surjective but not injective.
  (d) Define a function $\mathbb{Z} \to \mathbb{Z}^+$ that is both injective and surjective.

You may use anything to define these functions, but you must give a precise definition.

D3. Let $A = \{1, 2, 3\}$ and $B = \{4, 5\}$. List out all functions $A \to B$. How many such functions are there? How many of them are injections? How many of them are surjections? How many of them are bijections?

Tutorial questions

1. Let $f, g\colon \mathbb{Q} \to \mathbb{Q}$ defined by setting, for each $x \in \mathbb{Q}$,

$$f(x) = \lfloor x \rfloor + 1 \quad \text{and} \quad g(x) = \lceil x \rceil .$$

$\mathbb{Z}$ $\mathbb{Z}$ No. $f(x) \neq g(x)$.

What is the range of f? What is the range of g? Is $f = g$? Why?

2. Let $A = \{\text{s}, \text{u}\}$. Define a function $\operatorname{len}\colon A^* \to \mathbb{Z}_{\geqslant 0}$ by setting $\operatorname{len}(\sigma)$ to be the length of σ for each $\sigma \in A^*$.

   (a) What is $\operatorname{len}(\text{suu})$? 3
   (b) What is $\operatorname{len}(\{\varepsilon, \text{ss}, \text{uu}, \text{ssss}\})$? $\{0, 2, 2, 4\}$.
   (c) What is $\operatorname{len}^{-1}(\{3\})$? all str. length 3.

possible: setwise preimage for the whole set.

(d) Does len^{-1} exist? Explain your answer. No.

3. Which of the functions defined in the following are injective? Which are surjective? Prove that your answers are correct. If a function defined below is both injective and surjective, then find a formula for the inverse of the function. Here we denote by Bool the set $\{\mathbf{true}, \mathbf{false}\}$.

$$f\colon \mathbb{Q} \to \mathbb{Q}; \quad x \mapsto 12x + 31, \qquad g\colon \mathrm{Bool}^2 \to \mathrm{Bool}; \quad (p, q) \mapsto p \wedge {\sim}q, \qquad h\colon \mathrm{Bool}^2 \to \mathrm{Bool}^2; \quad (p, q) \mapsto (p \wedge q, p \vee q),$$

$$k\colon \mathbb{Z} \to \mathbb{Z}; \quad x \mapsto \begin{cases} x, & \text{if } x \text{ is even;} \\ 2x - 1, & \text{if } x \text{ is odd.} \end{cases}$$

piecewise
2 cases.

Surjective? NO. → contradiction.

For this question, you may use without proof the fact that every integer is either odd or even, but not both.

4. **Motivation.** As alluded to by Mac Lane at the beginning, here we devise a way to tell whether a function is an identity function by composing it with other functions, instead of looking at elements of some set.

Let A be a set and $f\colon A \to A$. Suppose $g \circ f = g$ for all functions g with domain A. Show that $\boxed{f = \mathrm{id}}$. must be injective.

smallest set of f

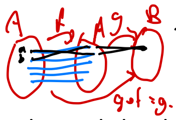
$A \to A$
$x \mapsto x$

① Domain
② codomain
③ Everything in Domain maps to image of codomain.

$g = Id_A$

$Id_A \circ f = f$.

Let g be any function



Transposition $\Rightarrow g \circ f(a) = g(f(a)) = g(b) = c$. $\to$ g must satisfy. $f = id_A$.

5. Let $f\colon B \to C$.

f, g injective.
$g \circ f \to$ injective?
$x_1, x_2 \in B$, $g \circ f(x_1) = g \circ f(x_2)$
$g(f(x_1)) = g(f(x_2))$
Since g is injective, $f(x_1) = f(x_2)$
Since f is injective, $x_1 = x_2$ (claim).

(a) Suppose f is injective. Show that $g \circ f$ is injective whenever g is an injective function with domain C.

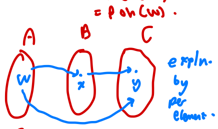
(b) Suppose we have a function g with domain C such that $g \circ f$ is injective. Show that f is injective.

$g \circ f$ injective.
$\to f$ is injective?
Assume f not injective.
$\therefore f(x_1) = f(x_2) \wedge x_1 \neq x_2$
But $g(f(x_1)) = g(f(x_2))$
$g \circ f(x_1) = g \circ f(x_2)$
Making $g \circ f$ not injective
$\therefore$ By contradiction, f is injective.

In particular, this will tell us that f is injective if and only if one can left-compose it with some function to give an injection.

6. Let $f\colon B \to C$.

f, h are surjective.
$\to f \circ h$ is surjective.
Any $y \in C$.
By surjectivity of f, $x \in B$ s.t. $y = f(x)$.
By surjectivity of h, $w \in A$ s.t. $x = h(w)$
Then, $y = f(x) = f(h(w)) = f \circ h(w)$.



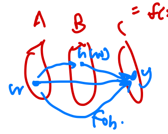
explain by per element.

For all elements in B, some x maps to it

(a) Suppose f is surjective. Show that $f \circ h$ is surjective whenever h is a surjective function with codomain B.

(b) Suppose we have a function h with codomain B such that $f \circ h$ is surjective. Show that f is surjective.

$f \circ h$ surjective
$\to f$ surjective?
Any $y \in C$
By surjectivity of $f \circ h$, $w \in A$ s.t. $y = f \circ h(w)$
Let $x = h(w)$,
Then $x \in B$, $y = f \circ h(w) = f(h(w)) = f(x)$



In particular, this will tell us that f is surjective if and only if one can right-compose it with some function to give a surjection.

7. Let $A = \{1, 2, 3\}$. The *order* of a bijection $f\colon A \to A$ is defined to be the smallest $n \in \mathbb{Z}^+$ such that

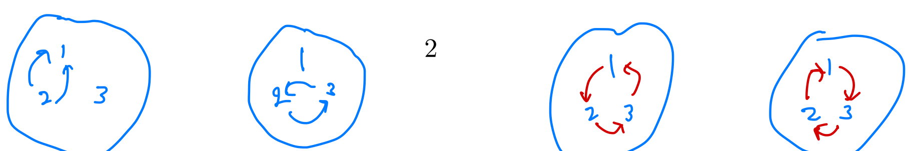
$$\underbrace{f \circ f \circ \ldots \circ f}_{n\text{-many } f\text{'s}} = \mathrm{id}_A.$$

Define functions $g, h\colon A \to A$ by setting, for all $x \in A$,

$$g(x) = \begin{cases} 1, & \text{if } x = 2; \\ 2, & \text{if } x = 1; \\ x, & \text{otherwise,} \end{cases} \qquad h(x) = \begin{cases} 2, & \text{if } x = 3; \\ 3, & \text{if } x = 2; \\ x, & \text{otherwise.} \end{cases}$$

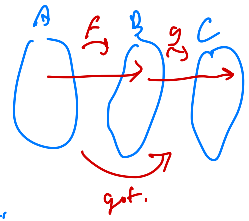
$h(1) = 1$.
$g(h(1)) = 2$.

Find the orders of g, h, $g \circ h$, and $h \circ g$.



g order 2 $\quad$ h order 2 $\quad$ $g \circ h$. order 3. $\quad$ $h \circ g$. order 3.

$z = g\circ f(x)$
$z = g(f(x))$
$g^{-1}(z) = f(x).$
$(f^{-1}\circ g^{-1})(z) = x.$ def. of $f^{-1}\circ g^{-1}$.



Check domain & codomain are equal

Then $(g\circ f)^{-1} = f^{-1}\circ g^{-1}$. by def of $(g\circ f)^{-1}$!

8. Let A, B, C be sets. Show that $\boxed{(g \circ f)^{-1}} = \boxed{f^{-1} \circ g^{-1}}$ for all bijections $f\colon A \to B$ and all bijections $g\colon B \to C$.

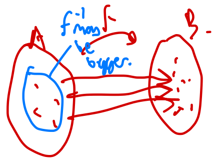
   In particular, this shows that invertible functions are closed under composition.

9. Let $f\colon A \to B$ be a function. Let $X \subseteq A$ and $Y \subseteq B$.

   (a) Is it always the case that $X \subseteq f^{-1}(f(X))$? Is it always the case that $f^{-1}(f(X)) \subseteq X$? Justify your answer. Yes! $x\in X$, then $f(x)\in f(X)$ by def. of $f(X)$; $x \in f^{-1}(f(X))$

   (b) Is it always the case that $Y \subseteq f(f^{-1}(Y))$? Is it always the case that $f(f^{-1}(Y)) \subseteq Y$? Justify your answer.

No! (pt 2)



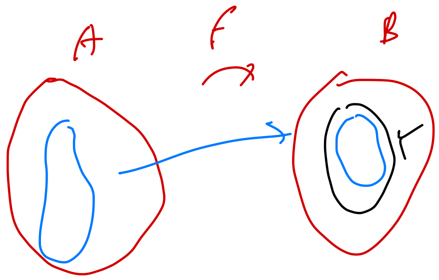
$f: \{-1, 1\} \to \{0\}$

where $f(-1) = 0 = f(1)$, $X = \{1\}$.

$f(X) = \{f(1)\} = \{0\}$.

Since $f(-1) = 0$,

$-1 \in f^{-1}(\{0\}) = f^{-1}(f(X))$



$f f^{-1}(Y) \subseteq Y.$ f may not be surjective

Yes.

$Y \subseteq f(f^{-1}(Y))$ False.

Y is inside this image

Please do this and the next tutorial by this weekend

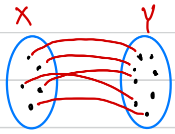
Notes on functions.

$f: X \to Y$.

↗ Domain set ↖ Co-Domain set.

① Every element in X is related to Y.

$$\forall x \in X, \exists y \in Y \, (xRy).$$

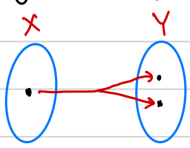


② No element in X is related to more than one element in Y.

$$\forall x \in X, \exists y \in Y \, (xRy \to \forall z \in Y \, (z \neq y, x \not R z))$$

There is a unique elem in Y that is related to x.

$\underbrace{\text{unique}}$ set of all unique elements $\Rightarrow$ range of f



This cannot happen!

Tutorial questions

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$$f(x) = \lfloor x \rfloor + 1 \quad \text{and} \quad g(x) = \lceil x \rceil\,.$$

$\mathbb{Z}$ $\mathbb{Z}$ No. $f(x) \neq g(x)$.

What is the range of f? What is the range of g? Is $f = g$? Why?

set of all values of f taken together.

$\{ y \in Y \mid y = f(x)$, for some x in $X \}$.

all positive integers and 0.

2. Let $A = \{\mathrm{s}, \mathrm{u}\}$. Define a function $\mathrm{len}\colon A^* \to \mathbb{Z}_{\geqslant 0}$ by setting $\mathrm{len}(\sigma)$ to be the length of σ for each $\sigma \in A^*$.

A set that is related to A.

   (a) What is $\mathrm{len}(\mathrm{suu})$? 3
   (b) What is $\mathrm{len}(\{\varepsilon, \mathrm{ss}, \mathrm{uu}, \mathrm{ssss}\})$? $\{0, 2, 2, 4\}$.
   (c) What is $\mathrm{len}^{-1}(\{3\})$? all str. length 3.
   (d) Does len^{-1} exist? Explain your answer.

a) 3.

b) $\{0, 2, 2, 4\}$.

c) $\{$ sss, ssu, sus, uss, suu, usu, uus, uuu $\}$

d) if $\mathrm{len}(\mathrm{ss}) = 2 = \mathrm{len}(\mathrm{uu})$,

then len is not injective as $\mathrm{ss} \neq \mathrm{uu}$.

$\therefore$ len^{-1} does not exist.

$\forall x \in X (F(x_1) = F(x_2) \rightarrow x_1 = x_2)$

$\forall y \in Y, \exists x \in X (F(x) = y)$

3. Which of the functions defined in the following are injective? Which are surjective? Prove that your answers are correct. If a function defined below is both injective and surjective, then find a formula for the inverse of the function. Here we denote by Bool the set $\{\mathbf{true}, \mathbf{false}\}$.

$$f\colon \mathbb{Q} \to \mathbb{Q}; \quad x \mapsto 12x + 31, \qquad g\colon \text{Bool}^2 \to \text{Bool}; \quad (p,q) \mapsto p \wedge {\sim} q, \qquad h\colon \text{Bool}^2 \to \text{Bool}^2; \quad (p,q) \mapsto (p \wedge q, p \vee q),$$

$$k\colon \mathbb{Z} \to \mathbb{Z}; \quad x \mapsto \begin{cases} x, & \text{if } x \text{ is even;} \\ 2x - 1, & \text{if } x \text{ is odd.} \end{cases}$$

Surjective? N

For this question, you may use without proof the fact that every integer is either odd or even, but not both.

f: injective

$\forall x_1, x_2 \in \mathbb{Q} (f(x_1) = f(x_2) \Rightarrow x_1 = x_2)$

surjective.

$\forall y \in \mathbb{Q}, \exists x \in \mathbb{Q} (f(x) = y)$

$y = 12x + 31$

$x = \frac{y - 31}{12}$

Define $f^{-1} : \mathbb{Q} \to \mathbb{Q}$ by setting,

$f^{-1}(y) = \frac{y-31}{12}$. $\Leftrightarrow f(x)$ whenever $x, y \in \mathbb{Q}$.

g: not injective.

(true, true) → false.

(false, true) → false

surjective.

$\forall y \in \text{Bool}, \exists x \in (\text{Bool})^2 (g(x) = y)$

Any values of x will give either true or false.

h: not injective

(false, true) → (true, false).

(true, false) → (true, false) but (false, true) ≠ (true, false).

not surjective as there is never a time when we get (true, false).

$k : \mathbb{Z} \to \mathbb{Z}$

$x \mapsto \begin{cases} x, & \text{even} \\ 2x-1, & \text{odd} \end{cases}$

1. (Show that if x is even, then $k(x)$ is even).
   1.1 Let x be an even integer
   1.2 Then $k(x) = x$ by definition of k.
   1.3 So $k(x)$ is even.
2. (Show that if x is odd, then $k(x)$ is odd).
   2.1 Let x be odd integer
   2.2 Then $k(x) = 2x - 1 = 2(x-1)+1$, where $x-1$ is an integer.
   2.3. $k(x)$ is odd.
3. (Since every integer is odd and even but not both, then for every $x \in \mathbb{Z}$)
   3.1 x is even iff $k(x)$ is even
   3.2 x is odd iff $k(x)$ is odd.
4. (injectivity)
   4.1 Let $x_1, x_2 \in \mathbb{Z}$ such that $k(x_1) = k(x_2)$
   4.2 Case 1: $k(x_1)$ is even,
      4.2.1. Then both x_1 and x_2 are even.
      4.2.2. So $x_1 = k(x_1) = k(x_2) = x_2$ by definition of k.

4.3 Case 2: $k(x_1)$ is odd.

4.3.1 Then both x_1 and x_2 are odd.

4.3.2. So $2x_1-1 = k(x_1) = k(x_2) = 2x_2-1$

4.3.3. $\therefore x_1 = x_2$.

4.4. Since $k(x_1)$ is either even or odd, then $x_1 = x_2$ in any case.

5. (Surjectivity)

5.1 (By contradiction)

5.1.1 Suppose k is surjective,

5.1.2 $3 \in \mathbb{Z}$

5.1.3 $\therefore \exists x \in \mathbb{Z}$ s.t. $k(x) = 3$.

5.1.4. $k(x) = 3 = 2 \times 1 + 1$ is odd.

5.1.5. $\therefore x$ is odd

5.1.6. $3 = k(x) = 2x-1$ by definition of k

5.1.7. $x = (3+1)/2 = 2 = 2 \times 1$, which is even.

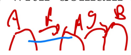
5.1.8. Contradicts that no integer is both even and odd.

5.2. $\therefore k$ is not surjective

4. **Motivation.** As alluded to by Mac Lane at the beginning, here we devise a way to tell whether a function is an identity function by composing it with other functions, instead of looking at elements of some set.

Let A be a set and $f\colon A \to A$. Suppose $\underbrace{g \circ f = g}_{\text{smallest set of } f}$ for all functions g with domain A. Show that $\boxed{f = \mathrm{id}_A}$. must be injection. ① Domain ② codomain

$A \to A$



Trans

5. Let $f\colon B \to C$.

   (a) Suppose f is injective. Show that $g \circ f$ is injective whenever g is an injective function with domain C.

   (b) Suppose we have a function g with domain C such that $g \circ f$ is injective. Show that f is injective.

   In particular, this will tell us that f is injective if and only if one can left-compose it with some function to give an injection.

6. Let $f\colon B \to C$.

   (a) Suppose f is surjective. Show that $f \circ h$ is surjective whenever h is a surjective function with codomain B.

   (b) Suppose we have a function h with codomain B such that $f \circ h$ is surjective. Show that f is surjective.

   In particular, this will tell us that f is surjective if and only if one can right-compose it with some function to give a surjection.

7. Let $A = \{1, 2, 3\}$. The *order* of a bijection $f\colon A \to A$ is defined to be the smallest $n \in \mathbb{Z}^+$ such that

$$\underbrace{f \circ f \circ \ldots \circ f}_{n\text{-many } f\text{'s}} = \mathrm{id}_A.$$

Define functions $g, h\colon A \to A$ by setting, for all $x \in A$,

$$g(x) = \begin{cases} 1, & \text{if } x = 2; \\ 2, & \text{if } x = 1; \\ x, & \text{otherwise,} \end{cases} \qquad h(x) = \begin{cases} 2, & \text{if } x = 3; \\ 3, & \text{if } x = 2; \\ x, & \text{otherwise.} \end{cases}$$

Find the orders of g, h, $g \circ h$, and $h \circ g$.

8. Let A, B, C be sets. Show that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ for all bijections $f\colon A \to B$ and all bijections $g\colon B \to C$.

In particular, this shows that invertible functions are closed under composition.

9. Let $f\colon A \to B$ be a function. Let $X \subseteq A$ and $Y \subseteq B$.

(a) Is it always the case that $X \subseteq f^{-1}(f(X))$? Is it always the case that $f^{-1}(f(X)) \subseteq X$? Justify your answer. Yes! $x \in X$, then $f(x) \in f(X)$

(b) Is it always the case that $Y \subseteq f(f^{-1}(Y))$? Is it always the case that $f(f^{-1}(Y)) \subseteq Y$? Justify your answer.

f^{-1} f B

CS1231S Tutorial 7: Functions
Solutions

National University of Singapore

2021/22 Semester 1

1. Let $f, g\colon \mathbb{Q} \to \mathbb{Q}$ defined by setting, for each $x \in \mathbb{Q}$,

$$f(x) = \lfloor x \rfloor + 1 \quad \text{and} \quad g(x) = \lceil x \rceil .$$

What is the range of f? What is the range of g? Is $f = g$? Why?

Solution. The range of f and the range of g are both $\mathbb{Z}$, because

$$f(n-1) = \lfloor n-1 \rfloor + 1 = (n-1) + 1 = n = \lceil n \rceil = g(n)$$

for all $n \in \mathbb{Z}$. We know $f \neq g$ because $f(0) = \lfloor 0 \rfloor + 1 = 0 + 1 = 1 \neq 0 = \lceil 0 \rceil = g(0)$.

2. Let $A = \{\mathrm{s}, \mathrm{u}\}$. Define a function $\mathrm{len}\colon A^* \to \mathbb{Z}_{\geqslant 0}$ by setting $\mathrm{len}(\sigma)$ to be the length of σ for each $\sigma \in A^*$.

   (a) What is $\mathrm{len}(\mathrm{suu})$?
   (b) What is $\mathrm{len}(\{\varepsilon, \mathrm{ss}, \mathrm{uu}, \mathrm{ssss}\})$?
   (c) What is $\mathrm{len}^{-1}(\{3\})$?
   (d) Does len^{-1} exist? Explain your answer.

   Solution.

   (a) $\mathrm{len}(\mathrm{suu}) = 3$.
   (b) $\mathrm{len}(\{\varepsilon, \mathrm{ss}, \mathrm{uu}, \mathrm{ssss}\}) = \{0, 2, 4\}$.
   (c) $\mathrm{len}^{-1}(\{3\}) = \{\mathrm{sss}, \mathrm{ssu}, \mathrm{sus}, \mathrm{suu}, \mathrm{uss}, \mathrm{usu}, \mathrm{uus}, \mathrm{uuu}\}$.
   (d) $\mathrm{len}(\mathrm{ss}) = 2 = \mathrm{len}(\mathrm{uu})$ but $\mathrm{ss} \neq \mathrm{uu}$. So len is not injective. Thus len^{-1} does not exist by Theorem 9.3.19.

3. Which of the functions defined in the following are injective? Which are surjective? Prove that your answers are correct. If a function defined below is both injective and surjective, then find a formula for the inverse of the function. Here we denote by Bool the set $\{\mathbf{true}, \mathbf{false}\}$.

$$\begin{aligned} f\colon \mathbb{Q} &\to \mathbb{Q}; \\ x &\mapsto 12x + 31, \end{aligned} \qquad \begin{aligned} g\colon \mathrm{Bool}^2 &\to \mathrm{Bool}; \\ (p, q) &\mapsto p \wedge {\sim} q, \end{aligned} \qquad \begin{aligned} h\colon \mathrm{Bool}^2 &\to \mathrm{Bool}^2; \\ (p, q) &\mapsto (p \wedge q, p \vee q), \end{aligned}$$

$$\begin{aligned} k\colon \mathbb{Z} &\to \mathbb{Z}; \\ x &\mapsto \begin{cases} x, & \text{if } x \text{ is even;} \\ 2x - 1, & \text{if } x \text{ is odd.} \end{cases} \end{aligned}$$

Solution.

- 1. Note that for all $x, y \in \mathbb{Q}$,

  $$y = 12x + 31 \quad \Leftrightarrow \quad x = (y - 31)/12.$$

  2. Define $f^* \colon \mathbb{Q} \to \mathbb{Q}$ by setting, for all $y \in \mathbb{Q}$,

  $$f^*(y) = (y - 31)/12.$$

  3. Then whenever $x, y \in \mathbb{Q}$,

  $$y = f(x) \quad \Leftrightarrow \quad x = f^*(y).$$

  4. Thus f^* is the inverse of f.
  5. Hence f is both injective and surjective by Theorem 9.3.19. □

- 1. $g(\textbf{false}, \textbf{true}) = \textbf{false} = g(\textbf{false}, \textbf{false})$, where $(\textbf{false}, \textbf{true}) \neq (\textbf{false}, \textbf{false})$.
  2. So g is not injective.
  3. $g(\textbf{true}, \textbf{false}) = \textbf{true}$.
  4. Lines 1 and 3 show that every element of the codomain Bool is in the range of g.
  5. This says g is surjective. □

- 1. $h(\textbf{true}, \textbf{false}) = (\textbf{false}, \textbf{true}) = h(\textbf{false}, \textbf{true})$, where $(\textbf{true}, \textbf{false}) \neq (\textbf{false}, \textbf{true})$.
  2. So h is not injective.
  3. If $p, q, r \in$ Bool such that $h(p, q) = (\textbf{true}, r)$, then

     3.1. $p \wedge q = \textbf{true}$ by the definition of h;
     3.2. $\therefore$ $p = \textbf{true}$
     3.3. $\therefore$ $r = p \vee q = \textbf{true}$ by the definition of h.

  4. So $(\textbf{true}, \textbf{false})$ in the codomain is not in the range of h.
  5. Thus h is not surjective. □

- 1. We first show that if x is an even integer, then $k(x)$ is even.
     1.1. Let x be an even integer.
     1.2. Then $k(x) = x$ by the definition of k.
     1.3. So $k(x)$ is even.
  2. Next we show that if x is an odd integer, then $k(x)$ is odd.
     2.1. Let x be an odd integer.
     2.2. Then $k(x) = 2x - 1 = 2(x - 1) + 1$, where $x - 1$ is an integer.
     2.3. So $k(x)$ is odd.
  3. Since every integer is either even or odd but not both, lines 1 and 2 tell us that, for every $x \in \mathbb{Z}$,
     3.1. x is even if and only if $k(x)$ is even; and
     3.2. x is odd if and only if $k(x)$ is odd.
  4. Now we show that k is injective.
     4.1. Let $x_1, x_2 \in \mathbb{Z}$ such that $k(x_1) = k(x_2)$.
     4.2. Case 1: $k(x_1)$ is even.
          4.2.1. Then both x_1 and x_2 are even by line 3.1.
          4.2.2. So $x_1 = k(x_1) = k(x_2) = x_2$ by the definition of k.
     4.3. Case 2: $k(x_1)$ is odd.
          4.3.1. Then both x_1 and x_2 are odd by line 3.2.
          4.3.2. So $2x_1 - 1 = k(x_1) = k(x_2) = 2x_2 - 1$ by the definition of k.
          4.3.3. Thus $x_1 = x_2$.
     4.4. Since $k(x_1)$ is either even or odd, we conclude that $x_1 = x_2$ in any case.
  5. Finally, we show that k is not surjective.
     5.1. We prove this by contradiction.
          5.1.1. Suppose k is surjective.

5.1.2. Note 3 is in the codomain $\mathbb{Z}$.
5.1.3. Use the surjectivity of k to find $x \in \mathbb{Z}$ such that $k(x) = 3$.
5.1.4. Note $k(x) = 3 = 2 \times 1 + 1$ is odd.
5.1.5. So x is odd by line 3.2.
5.1.6. Thus $3 = k(x) = 2x - 1$ by the choice of x and the definition of k.
5.1.7. Solving gives $x = (3 + 1)/2 = 2 = 2 \times 1$, which is even.
5.1.8. This contradicts line 5.1.5 as no integer is both even and odd.

5.2. Hence k is not surjective. □

4. Let A be a set and $f\colon A \to A$. Suppose $g \circ f = g$ for all functions g with domain A. Show that $f = \mathrm{id}_A$.

Solution.

1. The domain and the codomain of f and id_A are all equal to A.
2. We show $f(a) = \mathrm{id}_A(a)$ for all $a \in A$.
   2.1. Pick $a \in A$.
   2.2. Let $B = \{0, 1\}$ and $g\colon A \to B$ such that for all $x \in A$,

$$g(x) = \begin{cases} 1, & \text{if } x = a; \\ 0, & \text{otherwise.} \end{cases}$$

   2.3. As the domain of g is A, our supposition on f implies $g \circ f = g$.

2.4.	Thus	$g(f(a)) = (g \circ f)(a)$	by the definition of $g \circ f$;
2.5.		$= g(a)$	as $g \circ f = g$;
2.6.		$= 1$	by the definition of g.
2.7.	Hence	$f(a) = a$	by the definition of f;
2.8.		$= \mathrm{id}_A(a)$	by the definition of id_A. □

Alternative solution.

1.	$f = \mathrm{id}_A \circ f$	by Example 9.2.3;
2.	$= \mathrm{id}_A$	by our supposition, as id_A is a function with domain A. □

5. Let $f\colon B \to C$.

(a) Suppose f is injective. Show that $g \circ f$ is injective whenever g is an injective function with domain C.

(b) Suppose we have a function g with domain C such that $g \circ f$ is injective. Show that f is injective.

Solution.

(a)
1. Suppose f is injective.
2. Let g be an injective function with domain C.
3. Take $x_1, x_2 \in B$ such that $(g \circ f)(x_1) = (g \circ f)(x_2)$.

4.	Then	$g(f(x_1)) = g(f(x_2))$	by the definition of $g \circ f$;
5.	$\therefore$	$f(x_1) = f(x_2)$	as g is injective;
6.	$\therefore$	$x_1 = x_2$	as f is injective. □

(b)
1. Suppose g is a function with domain C such that $g \circ f$ is injective.
2. Let $x_1, x_2 \in B$ such that $f(x_1) = f(x_2)$.
3. Then $(g \circ f)(x_1) = g(f(x_1)) = g(f(x_2)) = (g \circ f)(x_2)$ by the definition of $g \circ f$.
4. So $x_1 = x_2$ as $g \circ f$ is injective by the choice of g. □

6. Let $f\colon B \to C$.

(a) Suppose f is surjective. Show that $f \circ h$ is surjective whenever h is a surjective function with codomain B.

(b) Suppose we have a function h with codomain B such that $f \circ h$ is surjective. Show that f is surjective.

Solution.

(a) 1. Suppose f is surjective.
2. Let h be a surjective function with codomain B.
3. Take any $y \in C$.
4. Apply the surjectivity of f to find $x \in B$ such that $y = f(x)$.
5. Apply the surjectivity of h to find w in the domain of h such that $x = h(w)$.
6. Then $y = f(x) = f(h(w)) = (f \circ h)(w)$ by the definition of $f \circ h$. $\square$

(b) 1. Suppose h is a function with codomain B such that $f \circ h$ is surjective.
2. Take any $y \in C$.
3. Apply the surjectivity of $f \circ h$ to find w in the domain of h such that $y = (f \circ h)(w)$.
4. Let $x = h(w)$.
5. Then $x \in B$ and $y = (f \circ h)(w) = f(h(w)) = f(x)$ by the definition of $f \circ h$. $\square$

7. Let $A = \{1, 2, 3\}$. The *order* of a bijection $f\colon A \to A$ is defined to be the smallest $n \in \mathbb{Z}^+$ such that

$$\underbrace{f \circ f \circ \ldots \circ f}_{n\text{-many } f\text{'s}} = \mathrm{id}_A.$$

Define functions $g, h\colon A \to A$ by setting, for all $x \in A$,

$$g(x) = \begin{cases} 1, & \text{if } x = 2; \\ 2, & \text{if } x = 1; \\ x, & \text{otherwise,} \end{cases} \qquad h(x) = \begin{cases} 2, & \text{if } x = 3; \\ 3, & \text{if } x = 2; \\ x, & \text{otherwise.} \end{cases}$$

Find the orders of g, h, $g \circ h$, and $h \circ g$.

Solution. The orders are respectively 2, 2, 3 and 3.

8. Let A, B, C be sets. Show that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ for all bijections $f\colon A \to B$ and all bijections $g\colon B \to C$.

Solution.

1. 1.1. domain of $f^{-1} \circ g^{-1}$ = domain of g^{-1} by the definition of $\circ$;
   1.2. = codomain of g by the definition of inverses;
   1.3. = codomain of $g \circ f$ by the definition of $\circ$.
2. 2.1. codomain of $f^{-1} \circ g^{-1}$ = codomain of f^{-1} by the definition of $\circ$;
   2.2. = domain of f by the definition of inverses;
   2.3. = domain of $g \circ f$ by the definition of $\circ$.
3. For all $x \in A$ and all $z \in C$,

   3.1. $z = (g \circ f)(x)$
   3.2. $\Leftrightarrow$ $z = g(f(x))$ by the definition of $g \circ f$;
   3.3. $\Leftrightarrow$ $g^{-1}(z) = f(x)$ by the definition of g^{-1};
   3.4. $\Leftrightarrow$ $f^{-1}(g^{-1}(z)) = x$ by the definition of f^{-1};
   3.5. $\Leftrightarrow$ $(f^{-1} \circ g^{-1})(z) = x$ by the definition of $f^{-1} \circ g^{-1}$.

4. So $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ by the definition of $(g \circ f)^{-1}$. $\square$

9. Let $f\colon A \to B$ be a function. Let $X \subseteq A$ and $Y \subseteq B$.

(a) Is it always the case that $X \subseteq f^{-1}(f(X))$? Is it always the case that $f^{-1}(f(X)) \subseteq X$? Justify your answer.

(b) Is it always the case that $Y \subseteq f(f^{-1}(Y))$? Is it always the case that $f(f^{-1}(Y)) \subseteq Y$? Justify your answer.

Solution.

(a) First we show it is always the case that $X \subseteq f^{-1}(f(X))$.

1. Let $x \in X$.
2. Then $f(x) \in f(X)$ by the definition of $f(X)$.
3. So $x \in f^{-1}(f(X))$ by the definition of $f^{-1}(f(X))$. $\square$

Next we show it is possible that $f^{-1}(f(X)) \nsubseteq X$.

1. Consider $f\colon \{-1, 1\} \to \{0\}$ where $f(-1) = 0 = f(1)$, and $X = \{1\}$.
2. Note $f(X) = \{f(1)\} = \{0\}$.
3. Since $f(-1) = 0$, we know $-1 \in f^{-1}(\{0\}) = f^{-1}(f(X))$.
4. As $-1 \notin \{1\} = X$, we deduce that $f^{-1}(f(X)) \nsubseteq X$. $\square$

There are many other counterexamples.

(b) First we show it is always the case that $f(f^{-1}(Y)) \subseteq Y$.

1. Take any $y \in f(f^{-1}(Y))$.
2. Then the definition of $f(f^{-1}(Y))$ gives some $x \in f^{-1}(Y)$ such that $y = f(x)$.
3. Now as $x \in f^{-1}(Y)$, we get $y' \in Y$ which makes $y' = f(x)$.
4. Since f is a function, this implies $y = f(x) = y' \in Y$, as required. $\square$

Next we show it is possible that $Y \nsubseteq f(f^{-1}(Y))$.

1. Consider $f\colon \{0\} \to \{-1, 1\}$ where $f(0) = 1$, and $Y = \{-1\}$.
2. Note that no $x \in \{0\}$ makes $f(x) = -1$.
3. So $f^{-1}(Y) = \varnothing$ by the definition of $f^{-1}(Y)$.
4. This entails $f(f^{-1}(Y)) = \varnothing \not\supseteq \{-1\} = Y$. $\square$

There are many other counterexamples.