EXERCISES FOR CHAPTER 6: DIAGONALIZATION

Question 6.1 to Question 6.23 are exercises for Sections 6.1 and 6.2.

1. For each of the following,

(i) find the characteristic equation of $\boldsymbol{A}$; $\det(\lambda I - A) = 0$

(ii) find all the eigenvalues of $\boldsymbol{A}$; and λ

(iii) find a basis for the eigenspace associated with each eigenvalues of $\boldsymbol{A}$.

$E_\lambda : (\lambda I - A)x = 0$
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(a) $\boldsymbol{A} = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$,

(b) $\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$,

(c) $\boldsymbol{A} = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}$,

(d) $\boldsymbol{A} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$,

(e) $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$,

(f) $\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 0 \\ 9 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$,

(g) $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$,

(h) $\boldsymbol{A} = \begin{pmatrix} -1 & 1 & 1 \\ -2 & 2 & 1 \\ 2 & -1 & 0 \end{pmatrix}$,

(i) $\boldsymbol{A} = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 2 & 2 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$,

(j) $\boldsymbol{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$.

2. Let $\boldsymbol{A}$ be a 2×2 matrix. Suppose $\lambda^2 + m\lambda + n$ is the characteristic polynomial of $\boldsymbol{A}$.

(a) Show that $m = -\operatorname{tr}(\boldsymbol{A})$ (see Question 2.11) and $n = \det(\boldsymbol{A})$.

(b) Show that $\boldsymbol{A}^2 + m\boldsymbol{A} + n\boldsymbol{I} = \boldsymbol{0}$. sub in?

3. Let λ be an eigenvalue of a square matrix $\boldsymbol{A}$.

(a) Show that λ^n is an eigenvalue of $\boldsymbol{A}^n$ where n is a positive integer. ?

(b) If $\boldsymbol{A}$ is invertible, show that $\frac{1}{\lambda}$ is an eigenvalue of $\boldsymbol{A}^{-1}$ (by Theorem 6.1.8, $\lambda \neq 0$).

(c) Show that λ is an eigenvalue of $\boldsymbol{A}^{\mathrm{T}}$.

4. Let $\boldsymbol{A}$ be a square matrix such that $\boldsymbol{A}^2 = \boldsymbol{A}$.

(a) Show that if λ is an eigenvalue of $\boldsymbol{A}$, then $\lambda = 0$ or 1.

(b) Find all 2×2 matrices $\boldsymbol{A}$ such that $\boldsymbol{A}^2 = \boldsymbol{A}$ and $\boldsymbol{A}$ has eigenvalues 0 and 1.

5. Let $\boldsymbol{A}$ be a nonzero $n \times n$ matrix such that $\boldsymbol{A}^2 = \boldsymbol{0}$.

(a) Show that if λ is an eigenvalue of $\boldsymbol{A}$, then $\lambda = 0$.

(b) Can $\boldsymbol{A}$ be diagonalizable? Justify your answer.

(c) Let $\boldsymbol{u}$ be a vector in $\mathbb{R}^n$ such that $\boldsymbol{A}\boldsymbol{u} \neq \boldsymbol{0}$. Prove that $\boldsymbol{u}$ and $\boldsymbol{A}\boldsymbol{u}$ are linearly independent.

(d) For $n = 2$, show that there exists an invertible 2×2 matrix $\boldsymbol{P}$ such that

$$\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

6. Let $\boldsymbol{A} = \begin{pmatrix} 0 & -1 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

(a) Show that -1 is an eigenvalue of $\boldsymbol{A}$. $\det(\lambda I - A)$.

(b) Show that $\dim(E_{-1}) = 2$. $E_\lambda \Rightarrow (\lambda I - A)x = 0 \Rightarrow$ span

(c) Find a 3×3 matrix $\boldsymbol{B}$ such that -3 is an eigenvalue of $\boldsymbol{B}\boldsymbol{A}$. ?

7. Let $\boldsymbol{A} = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}$.

(a) Show that 2 is an eigenvalue of $\boldsymbol{A}$. $\det(\lambda I - A) = 0 \rightarrow \lambda = 2$.

(b) Find a basis for the eigenspace associated with 2. $E_2 \rightarrow (\lambda I - A)x = 0$.

(c) If $\boldsymbol{B}$ is another 3×3 matrix with an eigenvalue λ such that the dimension of the eigenspace associated with λ is 2, prove that $2 + \lambda$ is an eigenvalue of the matrix $\boldsymbol{A} + \boldsymbol{V}$.

8. Let $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ be a basis for $\mathbb{R}^n$ and let $\boldsymbol{A}$ be an $n \times n$ matrix such that $\boldsymbol{A}\boldsymbol{u}_i = \boldsymbol{u}_{i+1}$ for $i = 1, 2, \ldots, n-1$ and $\boldsymbol{A}\boldsymbol{u}_n = \boldsymbol{0}$. Show that the only eigenvalue of $\boldsymbol{A}$ is 0 and find all the eigenvectors of $\boldsymbol{A}$.

9. For each matrix $\boldsymbol{A}$ in Question 6.1,

(i) determine whether $\boldsymbol{A}$ is diagonalizable; and

(ii) if $\boldsymbol{A}$ is diagonalizable, find a matrix $\boldsymbol{P}$ that diagonalizes $\boldsymbol{A}$ and determine $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$.

10. (This question is only for students who are familiar with computations using complex numbers.) Each matrix $\boldsymbol{A}$ below has complex eigenvalues. Following the discussion in Remark 6.2.5.1, find a matrix $\boldsymbol{P}$ that diagonalizes $\boldsymbol{A}$ and determine $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$.

(a) $\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, (b) $\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}$, (c) $\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{pmatrix}$.

11. Let $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

(a) Find a matrix $\boldsymbol{P}$ that diagonalizes $\boldsymbol{A}$.

(b) Compute $\boldsymbol{A}^{10}$.

(c) Find a matrix $\boldsymbol{B}$ such that $\boldsymbol{B}^2 = \boldsymbol{A}$.

12. Find a 3×3 matrix which has eigenvalues $1, 0$ and -1 with corresponding eigenvectors $(0,1,1)^{\mathrm{T}}$, $(1,-1,1)^{\mathrm{T}}$ and $(1,0,0)^{\mathrm{T}}$ respectively.

13. Determine the values of a and b so that the matrix $\begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix}$ is diagonalizable.

14. Let $\boldsymbol{B}$ be a 4×4 matrix and $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4\}$ is a basis for $\mathbb{R}^4$. Suppose

$$\boldsymbol{B}\boldsymbol{u}_1 = 2\boldsymbol{u}_2, \quad \boldsymbol{B}\boldsymbol{u}_2 = \boldsymbol{0}, \quad \boldsymbol{B}\boldsymbol{u}_3 = \boldsymbol{u}_4, \quad \boldsymbol{B}\boldsymbol{u}_4 = \boldsymbol{u}_3.$$

(a) Write down all the eigenvalues of $\boldsymbol{B}$.

(b) For each eigenvalue of $\boldsymbol{B}$, write down one eigenvector associated with it.

(c) Is $\boldsymbol{B}$ a diagonalizable matrix? Justify your answer.

15. Two square matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are said to be *similar* if there exists an invertible matrix $\boldsymbol{P}$ such that $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{B}$.

(a) Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are similar matrices.

(i) Show that $\boldsymbol{A}^n$ is similar to $\boldsymbol{B}^n$ for all positive integer n.

(ii) If $\boldsymbol{A}$ is invertible, show that $\boldsymbol{B}$ is invertible and $\boldsymbol{A}^{-1}$ is similar to $\boldsymbol{B}^{-1}$.

(iii) If $\boldsymbol{A}$ is diagonalizable, show that $\boldsymbol{B}$ is diagonalizable.

(b) Show that the following two matrices are similar:

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

16. A square matrix $(a_{ij})_{n\times n}$ is called a *stochastic matrix* if all the entries are non-negative and the sum of entries of each column is 1, i.e., $a_{1i} + a_{2i} + \cdots + a_{ni} = 1$ for $i = 1, \ldots, n$.

(a) Let $\boldsymbol{A}$ be a stochastic matrix.

(i) Show that 1 is an eigenvalue of $\boldsymbol{A}$.

(ii) If λ is an eigenvalue of $\boldsymbol{A}$, then $|\lambda| \le 1$.

(b) Let $\boldsymbol{B} = \begin{pmatrix} 0.95 & 0 & 0 \\ 0.05 & 0.95 & 0.05 \\ 0 & 0.05 & 0.95 \end{pmatrix}$.

(i) Is $\boldsymbol{B}$ a stochastic matrix?

(ii) Find a 3×3 invertible matrix $\boldsymbol{P}$ that diagonalizes $\boldsymbol{B}$.

17. A utility company finds that, in general, if a customer pays a bill late one month, there is about $\frac{1}{2}$ of the time that person will pay before the due date next month; and if a customer pays early one month, there are about $\frac{8}{10}$ of the time that person pays early again the following month. In January, the company finds that all 10 000 customers pay their bills on time. Estimate the number of customers that will pay on time in April. Will the number of customers that pay on time stabilizes in the long run? If so, estimate the number of customers that pay on time each month in the long run.

18. In a large city, the soft-drink market was 100% dominated by brand A. Four months ago, two new brands B and C were introduced to the market. According to the market research, for each month, about 1% and 2% of the customers of brand A switch to brands B and C respectively; about 1% and 2% of the customers of brand B switch to brands A and C respectively; and about 2% and 2% of the customers of brand C switch to brands A and B respectively. Compute the present market shares of the three brands of soft drink. Will the market shares stabilize in the long run if the trend continues? If so, estimate the market shares in the long run.

19. Let $\boldsymbol{A}$ be a square matrix. The *exponential* of $\boldsymbol{A}$ is defined o be the matrix

$$e^{\boldsymbol{A}} = \boldsymbol{I} + \boldsymbol{A} + \frac{1}{2!}\boldsymbol{A}^2 + \frac{1}{3!}\boldsymbol{A}^3 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}\boldsymbol{A}^n.$$

For each of the following, compute $e^{\boldsymbol{A}}$.

(a) $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, (b) $\boldsymbol{A} = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$, (c) $\boldsymbol{A} = \begin{pmatrix} -1 & 1 & 1 \\ -2 & 2 & 1 \\ 2 & -1 & 0 \end{pmatrix}$.

20. Following the procedure discussed in Example 6.2.11.2, solve the following recurrence relations.

(a) $a_n = 3a_{n-1} - 2a_{n-2}$ with $a_0 = 0$ and $a_1 = 1$.

(b) $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 1$ and $a_1 = 0$.

21. Let d_n be the determinant of the following $n \times n$ matrix:

$$\begin{pmatrix} 3 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 3 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 3 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 3 \end{pmatrix}.$$

Show that $d_n = 3d_{n-1} - d_{n-2}$. Hence, or otherwise, find d_n.

22. (This is an induction step for proving Remark 6.2.5.3.) Let $\boldsymbol{A}$ be a square matrix of order n. By Theorem 6.2.3, to diagonalize $\boldsymbol{A}$, we need to find n linearly independent eigenvectors. Suppose we already have m $(< n)$ linearly independent eigenvectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$, say $\boldsymbol{A}\boldsymbol{u}_i = \lambda_i \boldsymbol{u}_i$ for $i = 1, 2, \ldots, m$ where $\lambda_1, \lambda_2, \ldots, \lambda_m$ are not necessarily distinct.

For a new eigenvalue μ ($\mu \neq \lambda_i$ for $i = 1, 2, \ldots, m$) of $\boldsymbol{A}$, let $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_p\}$ be a basis for the eigenspace E_μ. Prove that $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m, \boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_p\}$ is linearly independent.

(*Hint*: Consider the vector equation

$$a_1\boldsymbol{u}_1 + a_2\boldsymbol{u}_2 + \cdots + a_m\boldsymbol{u}_m + b_1\boldsymbol{v}_1 + b_2\boldsymbol{v}_2 + \cdots + b_p\boldsymbol{v}_p = \boldsymbol{0}.$$

By using the property of eigenvectors, show that

$$a_1(\lambda_1 - \mu)\boldsymbol{u}_1 + a_2(\lambda_2 - \mu)\boldsymbol{u}_2 + \cdots + a_m(\lambda_m - \mu)\boldsymbol{u}_m = \boldsymbol{0}.$$

Then make use of the linearly independent assumption on $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$, as well as $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_p$, to finish the proof.)

23. Determine which of the following statements are true. Justify your answer.

(a) If $\boldsymbol{A}$ is a diagonalizable matrix, then $\boldsymbol{A}^{\mathrm{T}}$ is diagonalizable.

(b) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are diagonalizable matrices of the same size, then $\boldsymbol{A} + \boldsymbol{B}$ is diagonalizable.

(c) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are diagonalizable matrices of the same size, then $\boldsymbol{AB}$ is diagonalizable.

Question 6.24 to Question 6.34 are exercises for Sections 6.3 and 6.4.

24. For each of the following, find a matrix $\boldsymbol{P}$ that orthogonally diagonalizes $\boldsymbol{A}$ and determine $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{AP}$.

(a) $\boldsymbol{A} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$, (b) $\boldsymbol{A} = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$,

(c) $\boldsymbol{A} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$, (d) $\boldsymbol{A} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$,

(e) $\boldsymbol{A} = \begin{pmatrix} 0 & -2 & 1 \\ -2 & 3 & -2 \\ 1 & -2 & 0 \end{pmatrix}$, (f) $\boldsymbol{A} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$,

(g) $\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, (h) $\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$.

25. Let $\boldsymbol{u}$ be a column matrix.

(a) Show that $\boldsymbol{I} - \boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$ is orthogonally diagonalizable.

(b) Find a matrix $\boldsymbol{P}$ that orthogonally diagonalizes $\boldsymbol{I} - \boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$ if $\boldsymbol{u} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

26. Let $\boldsymbol{A}$ be a symmetric matrix. If $\boldsymbol{u}$ and $\boldsymbol{v}$ are two eigenvectors of $\boldsymbol{A}$ associated with eigenvalues λ and μ, respectively, where $\lambda \neq \mu$, show that $\boldsymbol{u} \cdot \boldsymbol{v} = 0$.
(*Hint*: Compute $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{u}$ in two different ways.)

27. Let $\boldsymbol{A}$ be a 3×3 symmetric matrix with two eigenvalues 1 and -1. Suppose the eigenspace associated with the eigenvalue 1 represents the plane $x + y - z = 0$. Determine the matrix $\boldsymbol{A}$.

28. Let $\boldsymbol{A}$ be a 4×4 matrix with eigenspaces given by $\mathrm{span}\{(1,0,1,0)^{\mathrm{T}}, (1,1,1,1)^{\mathrm{T}}\}$ and $\mathrm{span}\{(1,1,-1,-1)^{\mathrm{T}}, (1,-1,-1,1)^{\mathrm{T}}\}$. Show that $\boldsymbol{A}$ is symmetric.

29. Let $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ and $\boldsymbol{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

(a) Show that $\boldsymbol{u}$ is an eigenvector of $\boldsymbol{A}$.

(b) Let $\boldsymbol{v} = (a,b,c,d)^{\mathrm{T}}$ be a nonzero vector. Show that if $\boldsymbol{v} \cdot \boldsymbol{u} = 0$, then $\boldsymbol{v}$ is an eigenvector of $\boldsymbol{A}$.

(c) Suppose $\boldsymbol{P} = \begin{pmatrix} \frac{1}{2} & a_1 & a_2 & a_3 \\ \frac{1}{2} & b_1 & b_2 & b_3 \\ \frac{1}{2} & c_1 & c_2 & c_3 \\ \frac{1}{2} & d_1 & d_2 & d_3 \end{pmatrix}$ is an orthogonal matrix. Find $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{AP}$.

30. Determine which of the following statements are true. Justify your answer.

(a) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are orthogonally diagonalizable matrices of the same size, then $\boldsymbol{A}+\boldsymbol{B}$ is orthogonally diagonalizable.

(b) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are orthogonally diagonalizable matrices of the same size, then $\boldsymbol{AB}$ is orthogonally diagonalizable.

31. For each of the quadratic forms below,

(i) rewrite the form using matrix notation; and

(ii) simplify the form by following the procedure in Discussion 6.4.4 and Example 6.4.5.

(a) $Q_1(x, y) = 5x^2 + 5y^2 - 4xy$.

(b) $Q_2(x, y) = 7x^2 + 6y^2 + 5z^2 - 4xy + 4yz$. (*Hint*: For Part (ii), 3 is one of the eigenvalues of the corresponding matrix.)

32. (a) Let λ_1, λ_2 and λ_3 be real numbers such that $\lambda_1 \le \lambda_2 \le \lambda_3$.

(i) Show that λ_1 is the minimum value of $\lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2$ for all real numbers x_1, x_2, x_3 satisfying $x_1^2 + x_2^2 + x_3^2 = 1$.

(ii) Show that λ_3 is the maximum value of $\lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2$ for all real numbers x_1, x_2, x_3 satisfying $x_1^2 + x_2^2 + x_3^2 = 1$.

(b) For each of the following 3×3 matrices $\boldsymbol{Q}$, find the minimum and maximum values of $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{Qu}$ for all unit vectors $\boldsymbol{u}$ (i.e., $\|\boldsymbol{u}\| = 1$) in $\mathbb{R}^3$.

(i) $\boldsymbol{Q} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, (ii) $\boldsymbol{Q} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$.

33. For each of the following quadratic equations, find the quadratic form associated with it, name the conic section and give its equation in the standard form.

(a) $x^2 + 2y^2 - 2x + 8y + 8 = 0$,

(b) $x^2 - 4x + 4y + 4 = 0$,

(c) $2x^2 - 4xy - y^2 + 8 = 0$,

(d) $x^2 + xy + y^2 = 6$,

(e) $11x^2 + 24xy + 4y^2 - 15 = 0$,

(f) $9x^2 - 4xy + 6y^2 - 10x - 20y - 5 = 0$,

(g) $9x^2 + 6xy + y^2 - 10\sqrt{10}\,x + 10\sqrt{10}\,y + 90 = 0$.

34. Let $\boldsymbol{A}$ be a 2×2 symmetric matrix such that the eigenvalues of $\boldsymbol{A}$ are 1 and 4. The equation

$$\begin{pmatrix} x & y \end{pmatrix} \boldsymbol{A} \begin{pmatrix} x \\ y \end{pmatrix} = 8$$

represents a non-degenerate conic section in $\mathbb{R}^2$. Name the conic section and write its equation in the standard form. Justify your answer.