NATIONAL UNIVERSITY OF SINGAPORE

SEMESTER 1, 2021/2022

MA1521 Calculus for Computing **Tutorial 5**

1. Show that $|\cos x - 1| \leq |x|$ for all $x \in \mathbf{R}$.

   Hint: Apply the Mean Value Theorem with $f(x) = \cos x$.

   (Thomas' Calculus (14^{th} edition), p. 214, Problem 63)

2. Let $f(x) = 1/x$ and suppose $b > a > 0$. The Mean Value Theorem states that there is a $c \in (a, b)$ such that
$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$
   Determine c explicitly in this case.

   (Thomas' Calculus (14^{th} edition), p. 214, Problem 55 (Modified))

3. Find values of a and b such that the function
$$f(x) = \frac{ax + b}{x^2 - 1}$$
   has a local extreme value of 1 at $x = 3$.

   (Thomas' Calculus (14^{th} edition), p. 260, Problem 14 (Modified))

4. Find the following limits using L'Hôpital's Rule:

   (a) $\displaystyle\lim_{x\to\infty} \left(\frac{x+2}{x-1}\right)^x$,

   (Thomas' Calculus (14^{th} edition), p. 432, Problem 61)

   (b) $\displaystyle\lim_{x\to\infty} \frac{2^x - 3^x}{3^x + 4^x}$.

   (Thomas' Calculus (14^{th} edition), p. 432, Problem 71)

5. Complete the following problems using only the definition of Riemann integrals:

(a) Show that

$$\int_0^1 \sin x dx \leq 1.$$

(b) Use $\sin x \leq x$ for $x \geq 0$ to show that

$$\int_0^1 \sin x \, dx \leq \frac{1}{2}.$$

(Thomas' Calculus (14^{th} edition), p. 294, Problem 79(Modified))

Note: The actual value of the left hand side is $\cos 0 - \cos 1 = 0.4596\cdots$.

6. Evaluate the following integrals:

(a) $\displaystyle\int_{-\pi/3}^{\pi/3} \sin^2 t \, dt$

(Thomas' Calculus (14^{th} edition), p. 305, Problem 14)

(b) $\displaystyle\int_{-3}^{-1} \frac{y^5 - 2y}{y^3} \, dy.$

(Thomas' Calculus (14^{th} edition), p. 305, Problem 22)

7. Find the derivative of the following functions:

(a) $\displaystyle f(x) = x \int_a^{x^2} \sin(t^3) \, dt,$

(b) $\displaystyle g(x) = \int_0^{\sin x} \frac{1}{\sqrt{1-t^2}} \, dt, \quad |x| < \frac{\pi}{2}.$

(Thomas' Calculus (14^{th} edition), p. 305, Problem 42,45)

8. (a) Let $a > 0$ and $f(x) = ax^2 + 2bx + c$. Show that $f(x)$ has an absolute minimum at $x = -b/a$ and deduce that

$$f(x) \geq 0 \quad \text{if and only if} \quad b^2 - ac \leq 0.$$

Note: The statement A if and only if B means that you need to establish the following: If A holds then B holds AND if B holds then A holds.

(Thomas' Calculus (14^{th} edition), p. 265, Problem 31)

(b) Using the polynomial

$$f(x) = (a_1 x + b_1)^2 + (a_2 x + b_2)^2 + \cdots + (a_n x + b_n)^2,$$

show that

$$(a_1 b_1 + a_2 b_2 + \cdots + a_n b_b)^2 \leq (a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2).$$

(Thomas' Calculus (14^{th} edition), p. 265, Problem 32 (Modified))

1. Show that $|\cos x - 1| \leq |x|$ for all $x \in \mathbf{R}$.

   Hint: Apply the Mean Value Theorem with $f(x) = \cos x$.

   (Thomas' Calculus (14^{th} edition), p. 214, Problem 63)

Let $f(x) = \cos x$.

By Mean Value Theorem,

$\exists c \in (a,b)$, $f'(c) = \frac{f(b)-f(a)}{b-a}$

$-\sin c = \frac{f(b)-f(a)}{b-a}$

Since $-1 \leq \sin x \leq 1$

$1 \geq -\sin x \geq -1$

$\therefore |-\sin x| \leq 1$

$\therefore |-\sin c| = \left|\frac{f(b)-f(a)}{b-a}\right|$

$\frac{|\cos b - \cos a|}{|b-a|} = |-\sin c| \leq 1$

$\therefore \frac{|\cos b - \cos a|}{|b-a|} \leq 1$

$|\cos b - \cos a| \leq |b-a|$

Let $b = x$, $a = 0$,

$|\cos x - 1| \leq |x - 0|$

$|\cos x - 1| \leq |x|$ //, $x \in \mathbb{R}$.

2. Let $f(x) = 1/x$ and suppose $b > a > 0$. The Mean Value Theorem states that there is a $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

   Determine c explicitly in this case.

   (Thomas' Calculus (14^{th} edition), p. 214, Problem 55 (Modified))

$f(x) = \frac{1}{x}$, Suppose $b > a > 0$,

By mean value Theorem,

$\exists c \in (a,b)$ s.t.

$f'(c) = \frac{f(b)-f(a)}{b-a}$.

Let $b = 2$, $a = 1$,

$f'(c) = \frac{f(2)-f(1)}{2-1}$

$= \frac{1}{2} - 1$

$= -\frac{1}{2}$.

$\therefore f'(x) = -\frac{1}{x^2}$

$\therefore$ By comparison, when $x = c$,

$c^2 = 2$

$c = \sqrt{2}$. // (Since $c \in (a,b)$).

$c = \sqrt{ab}$

c is the geometric mean of a and b.

3. Find values of a and b such that the function

$$f(x) = \frac{ax+b}{x^2-1}$$

has a local extreme value of 1 at $x = 3$. $f'(3) = 0$

(Thomas' Calculus (14th edition), p. 260, Problem 14 (Modified))

$f(3) = 1$

$$f'(x) = \frac{(x^2-1)(ax+b)' - (ax+b)(x^2-1)'}{(x^2-1)^2}$$

$$= \frac{(x^2-1)(a) - (ax+b)(2x)}{(x^2-1)^2}$$

$$= \frac{ax^2 - a - (2ax^2 + 2bx)}{(x^2-1)^2}$$

$$= \frac{ax^2 - a - 2ax^2 - 2bx}{(x^2-1)^2}$$

$$= -\frac{ax^2 + a + 2bx}{(x^2-1)^2}$$

When $x=3$, $f'(x)=0$,

$$-\frac{9a + a + 2(3)b}{(9-1)^2} = 0$$

$$10a + 6b = 0$$

$$a = -\frac{6}{10}b$$

$$a = -\frac{3}{5}b$$

When $x=3$, $f(x)=1$,

$$\frac{a(3)+b}{3^2-1} = 1$$

$$3a + b = 8$$

$$3\left(-\frac{3}{5}b\right) + b = 8$$

$$-\frac{4}{5}b = 8$$

$$b = -10$$

$$a = 6$$

4. Find the following limits using ~~L'Hôpital's Rule~~:

(a) $\displaystyle\lim_{x\to\infty}\left(\frac{x+2}{x-1}\right)^{x}$,

(Thomas' Calculus (14th edition), p. 432, Problem 61)

(b) $\displaystyle\lim_{x\to\infty}\frac{2^x-3^x}{3^x+4^x}$.

(Thomas' Calculus (14th edition), p. 432, Problem 71)

a) $\displaystyle\lim_{x\to\infty}\left(\frac{x+2}{x-1}\right)^{x}$

$= \displaystyle\lim_{x\to\infty} e^{x\ln\left(\frac{x+2}{x-1}\right)}$

$= e^{\lim_{x\to\infty} \frac{\ln\left(\frac{x+2}{x-1}\right)}{x^{-1}}}$

$= e^{\lim_{x\to\infty} \frac{\ln(x+2)-\ln(x-1)}{x^{-1}}}$

$= e^{\lim_{x\to\infty} -\left(\frac{1}{x+2}-\frac{1}{x-1}\right)\frac{x^2}{1}}$

$= e^{\lim_{x\to\infty}\left(\frac{3x^2}{(x+1)(x+2)}\right)}$

$= e^{3\lim_{x\to\infty}\left(\frac{x^2}{(x+1)(x+2)}\right)}$

$= e^{3\left(\lim_{x\to\infty}\frac{2x}{2x+1}\right)}$

$= e^{3\left(\lim_{x\to\infty}\left(\frac{2}{2+\frac{1}{x}}\right)\right)}$

$= e^{3(1)}$

$= e^3$

Let $y = \left(\frac{x+2}{x-1}\right)^x$

$\ln y = x\ln\left(\frac{(x+2)}{(x-1)}\right)$

$= \dfrac{\ln\left(\frac{x+2}{x-1}\right)}{\frac{1}{x}}$

$\displaystyle\lim_{x\to\infty}\frac{1}{x}=0.$

$\displaystyle\lim_{x\to\infty}\ln\left(\frac{x+2}{x-1}\right)$

$= \displaystyle\lim_{x\to\infty}\ln\left(\frac{\cancel{x}}{\cancel{x}}\left(\frac{1+\frac{2}{x}}{1-\frac{1}{x}}\right)\right)$

$= \ln(1)$

$= 0$

By l'hopital,

$\displaystyle\lim_{x\to\infty}\frac{\ln(x+2)-\ln(x-1)}{\left(\frac{1}{x}\right)}$ ← split up, easier differentiation

$= \displaystyle\lim_{x\to\infty}\frac{\frac{1}{(x+2)}-\frac{1}{(x-1)}}{-\frac{1}{x^2}}$

$= \displaystyle\lim_{x\to\infty}\frac{x^2}{-1}\cdot\frac{x-1-(x+2)}{(x+2)(x-1)}$

$= -\displaystyle\lim_{x\to\infty}\frac{\cancel{x^2}}{\cancel{x^2}}\cdot\frac{-3}{\left(1+\frac{2}{x}\right)\left(1-\frac{1}{x}\right)} = 3$

$\therefore \displaystyle\lim_{x\to\infty}\ln y = 3 \Rightarrow \lim_{x\to\infty} y = e^3$ //

b) $\displaystyle\lim_{x\to\infty}\frac{2^x-3^x}{3^x-4^x}\cdot\frac{\frac{1}{3^x}}{\frac{1}{3^x}}$

$= \displaystyle\lim_{x\to\infty}\frac{\frac{2^x}{3^x}-1}{1-\frac{4^x}{3^x}}$

$= \displaystyle\lim_{x\to\infty}\frac{\left(\frac{2}{3}\right)^x-1}{1-\left(\frac{4}{3}\right)^x}$ //

By l'Hôpital's rule,

$= \displaystyle\lim_{x\to\infty}\frac{\ln\frac{2}{3}\cdot\left(\frac{2}{3}\right)^x}{\ln\frac{4}{3}\cdot\left(\frac{4}{3}\right)^x}$

$= \dfrac{\ln\frac{2}{3}}{\ln\frac{4}{3}}\displaystyle\lim_{x\to\infty}\left(\frac{\frac{2}{3}}{\frac{4}{3}}\right)^x$

$= \dfrac{\ln\frac{2}{3}}{\ln\frac{4}{3}}\displaystyle\lim_{x\to\infty}\left(\frac{1}{2}\right)^x$

$= \dfrac{\ln\frac{2}{3}}{\ln\frac{4}{3}}\displaystyle\lim_{x\to\infty}\frac{1}{2^x}$ $\to 0$

$= 0.$

a^x

a^x here

use this

$y = \left(\frac{2}{3}\right)^x \quad \ln y = x\ln\frac{2}{3}$

$\displaystyle\lim_{x\to\infty}\ln y = \ln\left(\frac{2}{3}\right)\lim_{x\to\infty}x$

$= -\infty$

$\therefore\ y = e^{\lim_{x\to\infty}x\ln\left(\frac{2}{3}\right)}$

$= 0.$

$y = \left(\frac{4}{3}\right)^x : \ \ln y = x\ln\frac{4}{3}$

$\displaystyle\lim_{x\to\infty}\ln y$

$= \ln\frac{4}{3}\displaystyle\lim_{x\to\infty}x = \infty$

or $\Rightarrow \displaystyle\lim_{x\to\infty}y$

$= e^{\lim_{x\to\infty}x\left(\ln\left(\frac{4}{3}\right)\right)}$

$= \infty.$

$\displaystyle\lim_{x\to\infty}\frac{2^x-3^x}{3^x+4^x}$

approach not so good.

$\displaystyle\lim_{x\to\infty}\frac{2^x}{3^x} = \lim_{x\to\infty}\frac{(\ln 2)\,2^x}{(\ln 3)\,3^x}$

$= \left(1-\frac{\ln 2}{\ln 3}\right)\displaystyle\lim_{x\to\infty}\frac{2^x}{3^x}=0$

$\therefore \displaystyle\lim_{x\to\infty}\frac{2^x}{3^x}=0$

Since $1-\frac{\ln 2}{\ln 3}\neq 0$

Assume limit $\frac{2^x}{3^x}$ exist, prove this first

$= \displaystyle\lim_{x\to\infty}\frac{\left(\left(\frac{2}{3}\right)^x-1\right)}{\left(\frac{3}{4}\right)^x+1}$ $\left(\dfrac{3^x}{4^x}\right)$

$= 0$ as $x\to\infty.$

Very Important!

$y = r^x = \{$

5. Complete the following problems using only the definition of Riemann integrals:

Riemann sum not tested

(a) Show that
$$\int_0^1 \sin x dx \le 1.$$

(b) Use $\sin x \le x$ for $x \ge 0$ to show that
$$\int_0^1 \sin x\, dx \le \frac{1}{2}.$$

before fundamental theorem of calculus → can use if an came out.

(Thomas' Calculus (14th edition), p. 294, Problem 79(Modified))

Note: The actual value of the left hand side is $\cos 0 - \cos 1 = 0.4596\cdots$.

a) $\int_0^1 \sin x\, dx \le 1.$

$P = \{a, a+\frac{b-a}{n}, \dots, a+(n-1)\frac{b-a}{n}, b\}.$

$S_p = \sum_{j=1}^{n} f(a+(\frac{b-a}{n})j)(\Delta x_j)$

$= \sum_{j=1}^{n} \sin(a+(\frac{b-a}{n})j)(\frac{b-a}{n})$

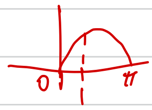
$= \frac{b-a}{n}\sum_{j=1}^{n} \sin(a+(\frac{b-a}{n})j).$

when $a=0$, $b=1$,

$= \frac{1}{n}\sum_{j=1}^{n} \sin(j(\frac{1}{n}))$

$\int_0^1 \sin x\, dx = \lim_{\|P\|\to 0} S_p = \lim_{n\to\infty}\left(\frac{1}{n}\sum_{j=1}^{n}\sin(j(\frac{1}{n}))\right) = 0 \le 1.$

$\sin x \ge 0$ for $0 \le x \le 1$



$|\sin x| \le 1 \Rightarrow \sin x \le 1 \quad \therefore \sin x \ge 0$

$\int_0^1 \sin x\, dx \le \int_0^1 1\, dx = 1\cdot(1-0)$
$= 1$

(b) Use $\sin x \le x$ for $x \ge 0$ to show that

$$\int_0^1 \sin x\,dx \le \frac{1}{2}.$$

(Thomas' Calculus (14th edition), p. 294, Problem 79(Modified))

Note: The actual value of the left hand side is $\cos 0 - \cos 1 = 0.4596\cdots$.

$\sin x \le x$ for $0 \le x \le 1$

$\int_0^1 \sin x\,dx = \int_0^1 x\,dx$

$= \frac{1^2}{2} - \frac{0^2}{2} = \frac{1}{2}$

$\sin x \le x$ for $x \ge 0$

$\therefore \int_0^1 x\,dx$

$P = \{a, a+\frac{b-a}{n}, \dots, a+(n-1)\frac{b-a}{n}, b\}.$

$\therefore S_P = \sum_{j=1}^{n} f(a+(\frac{b-a}{n})j)(\frac{b-a}{n})$

$= \sum_{j=1}^{n}(a+\frac{b-a}{n}j)(\frac{b-a}{n}).$

$= (\frac{b-a}{n})(\sum_{j=1}^{n} a + \sum_{j=1}^{n}\frac{b-a}{n}j)$

$= \frac{b-a}{n}(an + \frac{b-a}{n}(\frac{n(n+1)}{2}))$

$= a(b-a) + \frac{(b-a)^2(n+1)}{2n}$

when $b=1$, $a=0$,

$= \frac{n+1}{2n}.$

$\therefore \int_0^1 x\,dx = \lim_{\|P\|\to 0} S_P = \lim_{n\to\infty}(\frac{n+1}{2n}\cdot\frac{\frac{1}{n}}{\frac{1}{n}})$

$= \lim_{n\to\infty}(\frac{1+\frac{1}{n}}{2+\frac{1}{n}})$

$= \frac{1}{2}$ //

$\therefore$ Since $\sin x \le x$,

then $\int_0^1 \sin x\,dx \le \int_0^1 x\,dx$

$\therefore \int_0^1 \sin x\,dx \le \frac{1}{2}.$ //

6. Evaluate the following integrals:

(a) $\int_{-\pi/3}^{\pi/3} \sin^2 t \, dt$ double angle

$\cos 2t = 1 - 2\sin^2 t$

$\sin^2 t = \frac{1-\cos 2t}{2}$

(Thomas' Calculus (14th edition), p. 305, Problem 14)

(b) $\int_{-3}^{-1} \frac{y^5 - 2y}{y^3} \, dy.$

(Thomas' Calculus (14th edition), p. 305, Problem 22)

a) $\int_{-\pi/3}^{\pi/3} \sin^2 t \, dt$

$\cos 2t = 1 - 2\sin^2 t$

$\frac{1-\cos 2t}{2} = \sin^2 t$

$\int \sin^2 t \, dt$

$= \int \frac{1-\cos 2t}{2} \, dt$

$= 2\int 1 - \cos 2t \, dt$

$= 2\int 1 \, dt - 2\int \cos 2t \, dt$

$= 2t - \frac{1}{2}\sin 2t + w$

$\therefore \int_{-\pi/3}^{\pi/3} \sin^2 t \, dt = \left[2t - \frac{1}{2}\sin 2t\right]_{-\pi/3}^{\pi/3}$

$= \left(2\frac{\pi}{3} - \frac{1}{2}\sin 2\frac{\pi}{3}\right) - \left(2\left(-\frac{\pi}{3}\right) - \frac{1}{2}\sin 2\left(-\frac{\pi}{3}\right)\right)$

$= \frac{2}{3}\pi - \frac{1}{2}\sin\left(\frac{2}{3}\pi\right) + \frac{2}{3}\pi + \frac{1}{2}\sin\left(-\frac{2}{3}\pi\right)$

$= \frac{4}{3}\pi - \sin\frac{2}{3}\pi$

$= \frac{4}{3}\pi - \frac{\sqrt{3}}{2}$ //

Compute indefinite integral first.

$\int \sin^2 t \, dt = \int \frac{1-\cos 2t}{2} \, dt$

$= \frac{t}{2} - \frac{\sin 2t}{4}$

$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \sin^2 t \, dt = \frac{t}{2} - \frac{\sin 2t}{4} \Big|_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$

$= \frac{\pi}{6} - \frac{\sqrt{3}}{8}$

$- \left(-\frac{\pi}{6} + \frac{\sqrt{3}}{8}\right)$

$= \frac{\pi}{3} - \frac{\sqrt{3}}{4}$

b) $\int_{-3}^{-1} \frac{y^5 - 2y}{y^3} \, dy.$

$\frac{y^{5-2}}{y^{\cancel{3}}} - \frac{2\cancel{y}}{y^{3-2}}$

$\int y^2 \, dy - 2\int \frac{1}{y^2} \, dy$

$= \frac{y^3}{3} - 2\left(-\frac{1}{y}\right)$

$= \frac{y^3}{3} + \frac{2}{y}$ ✓

$\therefore \int_{-3}^{-1} \frac{y^5 - 2y}{y^3} \, dy = \left[\frac{y^3}{3} + \frac{2}{y}\right]_{-3}^{-1}$

$= \left(\frac{(-1)^3}{3} + \frac{2}{-1}\right) - \left(\frac{(-3)^3}{3} + \frac{2}{(-3)}\right)$

$= \left(-\frac{1}{3} - \frac{1}{2}\right) - \left(-9 - \frac{2}{3}\right)$

$= \frac{15}{2}$ // $\frac{22}{3}$ somehow?

7. Find the derivative of the following functions:

(a) $f(x) = x\int_a^{x^2} \sin(t^3)\,dt$,

(b) $g(x) = \int_0^{\sin x} \frac{1}{\sqrt{1-t^2}}\,dt, \quad |x| < \frac{\pi}{2}.$ — a way to define $\sin(x)$.

(Thomas' Calculus (14th edition), p. 305, Problem 42,45)

7. a) $f(x) = x\int_a^{x^2} \sin(t^3)\,dt$. ← wrong integration.

$= x\left[3t^2\cos t^3\right]_a^{x^2}$

$= x\left[(3(x^2)^2\cos(x^2)^3 - 3a^2\cos a^3\right]$.

$= x(3x^4\cos x^6 - 3a^2\cos a^3)$.

$= 3x^5\cos x^6 - 3a^2\cos a^3 x$ ✗

Let $H(x) = \int_a^x \sin(t^3)\,dt$.

$H'(x) = \sin(x^3)$

$H(x^2) = \int_a^{x^2}\sin(t^3)\,dt$

$f'(x) = (xH(x^2))'$

$= x(H(x^2))' + H(x^2)$

$= x(H'(x^2)(2x)) + H(x^2)$

$= 2x^2\sin(x^6)$

$+ \underline{\int_a^{x^2}\sin(t^3)\,dt}$

transcendental function with a power and cannot use substitution since there is no function beside it.

do not integrate this.

b) $g(x) = \int_0^{\sin x}\frac{1}{\sqrt{1-t^2}}\,dt, \quad |x| < \frac{\pi}{2}$.

$= \left[\sin^{-1} t\right]_0^{\sin x}$

$= \sin^{-1}(\sin x) - \sin^{-1}(0)$

$= x - 0$

$= x$.

$G(x) = \int_0^x \frac{1}{\sqrt{1-t^2}}\,dt \Rightarrow G'(x) = \frac{1}{\sqrt{1-x^2}}$ pt 1

Let $g(x) = G(\sin x)$

$g'(x) = (G(\sin x))' = G'(\sin x)(\cos x)$

$= \frac{1}{\sqrt{1-\sin^2 x}}\cdot\cos x = \frac{1}{\cos x}\cdot\cos x$

$= 1$

$g(x) = x + C$

$\int_0^{\sin x}\frac{1}{\sqrt{1-t^2}}\,dt = x + C - (0 + C)$ by pt 2

$= x$

$u = \sin x$, $x = \sin^{-1} u$, "$\sin^{-1} u = \int_0^u \frac{1}{\sqrt{1-t^2}}\,dt$"

$\sin u$ is the inverse of $\sin^{-1} u$.

Similar to $\ln x = \int_1^x \frac{1}{t}\,dt$.

e^x is inverse of $\ln x$

8. (a) Let $a > 0$ and $f(x) = ax^2 + 2bx + c$. Show that $f(x)$ has an absolute minimum at $x = -b/a$ and deduce that

$$f(x) \geq 0 \quad \text{if and only if} \quad b^2 - ac \leq 0.$$

Note: The statement A if and only if B means that you need to establish the following: If A holds then B holds AND if B holds then A holds.

(Thomas' Calculus (14th edition), p. 265, Problem 31)

$a > 0$, $f(x) = ax^2 + 2bx + c$

$f(x)$ has an absolute minimum at $x = -\frac{b}{a}$.

$f'(x) = 2ax + 2b = 0$.

$2ax = -2b$

$x = -\frac{b}{a}$

Second deri'v cause easy function

$f''(x) = 2a$

Since $a > 0$, $f''(x) > 0$ for all x

$\therefore$ Since the $f(x)$ only has one critical point $x = -\frac{b}{a}$, it is thus the absolute minimum of $f(x)$.

Let $f(x) \geq 0$,

$ax^2 + 2bx + c \geq 0$

when $x = -\frac{b}{a}$, $f(x) \geq 0$ for $f(x) \geq 0$ for any $x \in \mathbb{R}$,

$a\left(-\frac{b}{a}\right)^2 + 2b\left(-\frac{b}{a}\right) + c \geq 0$

$\frac{b^2}{a^2}a - 2\frac{b^2}{a} + c \geq 0$.

$\frac{b^2}{a} - \frac{2b^2}{a} + c \geq 0$.

$-a \cdot \left(-\frac{b^2}{a} + c\right) \geq 0 \cdot -a$.

$b^2 - ac \leq 0$ (shown)

Let $b^2 - ac \leq 0$.

when $x = -\frac{b}{a}$,

$a\left(-\frac{b}{a}\right)^2 + 2b\left(-\frac{b}{a}\right) + c$

$= \frac{b^2}{a} - \frac{2b^2}{a} + c$

$= -\frac{b^2}{a} + c$

Since $b^2 - ac \leq 0$

$\Rightarrow -\frac{b^2}{a} + c \geq 0$

$\therefore f(x) \geq 0$ //

Complete sqr:

$f(x) = ax^2 + 2bx + c$

$= a\left(x^2 + \frac{2b}{a}x\right) + c$

complete sqr.

$= \underbrace{a\left(x^2 + \frac{2b}{a}x + \left(\frac{b}{a}\right)^2\right)}_{\text{perfect sqr.}} + c - \left(\frac{b}{a}\right)^2$

$= a\left(x + \frac{b}{a}\right)^2 + c - a\frac{b^2}{a^2}$

$= a\left(x + \frac{b}{a}\right)^2 + \frac{ac - b^2}{a}$, $a > 0$

when $x = -\frac{b}{a}$, $f\left(-\frac{b}{a}\right) = \frac{ac - b^2}{a}$

and this is a absolute minimum.

If $x \neq -\frac{b}{a}$, $f(x) > \frac{ac - b^2}{a}$,

$f(x) \geq 0 \Leftrightarrow \frac{ac - b^2}{a} \geq 0$

$\Leftrightarrow b^2 - ac \leq 0$

(shown)

$f'(x) = 2ax + 2b$.

$f'(u) = 0 \Rightarrow 2au + 2b = 0$

$\Rightarrow u = -\frac{b}{a}$,

$f(x) = ax^2 + 2bx + c$

as $x \to \infty$: $f(x) \to \infty$

as $x \to -\infty$: $f(x) \to \infty$

Show this.

$f(x) = x^2\left(a + \frac{2b}{x} + \frac{c}{x^2}\right)$

$\to \infty$.

$\therefore f$ has an abs. min.

$f\left(-\frac{b}{a}\right) = \frac{ac - b^2}{a} \geq 0$.

$\Leftrightarrow b^2 - ac \leq 0$.

(b) Using the polynomial

$$f(x) = (a_1x + b_1)^2 + (a_2x + b_2)^2 + \cdots + (a_nx + b_n)^2,$$

show that

$$(a_1b_1 + a_2b_2 + \cdots + a_nb_b)^2 \le \underbrace{(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2)}_{\text{maximum}}.$$

(Thomas' Calculus (14^{th} edition), p. 265, Problem 32 (Modified))

← Cauchy-Schwarz inequality.

b) $f(x) = (a_1x+b_1)^2 + (a_2x+b_2)^2 + \dots + (a_nx+b_n)^2$

$f'(x) = 2(a_1x+b_1)(a_1) + 2(a_2x+b_2)(a_2) + \dots + 2(a_nx+b_n)(a_n) = 0$

$a_1(a_1x+b_1) + \dots + a_n(a_nx+b_n) = 0$

$a_1^2x + a_1b_1 + \dots + a_n^2x + a_nb_n = 0$

$x(a_1^2 + \dots a_n^2) = -(a_1b_1 + \dots + a_nb_n)$

$x = \dfrac{-(a_1b_1 + \dots + a_nb_n)}{(a_1^2 + \dots + a_n^2)}$

$f''(x) = a_1^2 + a_2^2 + \dots + a_n^2$

$\therefore$ For any x, when $f'(x)=0$, x is a critical point

$\therefore f(x) = \left(a_1\left(-\dfrac{(a_1b_1 + \dots + a_nb_n)}{(a_1^2 + \dots + a_n^2)}\right) + b_1\right)^2 + \dots$

$(a_1b_1 + \dots + a_nb_n)^2 \le (a_1^2 + \dots + a_n^2)(b_1^2 + \dots + b_n^2)$

Let $f(x) = (a_1x + b_1)^2 + \dots + (a_nx + b_n)^2$

$= (a_1^2 + a_2^2 + \dots + a_n^2)x^2$

$+ 2(a_1b_1 + \dots + a_nb_n)x$

$+ (b_1^2 + \dots + b_n^2)$

$= Ax^2 + 2Bx + C$

$(a_jx + b_j)^2$

$= a_j^2x^2 + 2a_jb_jx + b_j^2$

By (a): Since $f(x) \ge 0$, $B^2 - AC \le 0$ since $b^2 - ac \le 0$

$(a_1b_1 + \dots + a_nb_n)^2 \le (a_1^2 + \dots + a_n^2)(b_1^2 + \dots + b_n^2)$ //