MA2001 CHAP 2 Proofs. And Stuff

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1) Matrices Size

- Square Matrix
  - Diagonal (Non-diagonal Entries are zero)
    - Scalar (Diagonal Entries are equal)
  - Symmetric (symmetric WRT diagonal)
  - Upper Triangular (All entries below diagonal are zero)
  - Lower Triangular
  - Triangular (Upper + Lower)

2) Matrix Operations

- Equal Matrics $\Leftrightarrow$ Same Size $(m \times n)$ and entries $[(i,j)$ - entry of$]$
- Addition, Subtraction, Scalar Multiplication.
  - Properties (A, B, C are same size Matrices)

1. $A-B = A+(-B)$	4. $0+A=A$, $A-A=0$, $0A=0$, $c0=0$ $(0 = 0$ Matrix$)$
2. $A+B = B+A$	5. $c(A+B) = cA + cB$ $(c+d)(A) = cA + dA$
3. $(A+B)+C = A+(B+C)$	6. $c(dA) = (cd)A$

- Matrix Multiplication.
  - Properties

1. $A(BC) = (AB)C$	4. $c(AB) = (cA)B = A(cB)$
2. $A(B_1+B_2) = AB_1 + AB_2$	5. $A0 = 0$ $0A = 0$
3. $(A_1+A_2)B = A_1B + A_2B$	$AI_n = I_n$ $I_mA = I_m$

- Powers of square Matrices.
  - Properties.

1. $A^0 = I_n$	4. $(AB)^n \neq A^nB^n$ for $n \geq 2$
2. $A^mA^n = A^{m+n}$	
3. $(A^m)^n = A^{mn}$	

- Matrix Representation.

$$\begin{array}{ccc|l} \multicolumn{3}{c}{\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix}} & \begin{array}{l} \rightarrow a_1 \\ \rightarrow a_2 \end{array} \\ \downarrow & \downarrow & \downarrow & \\ b_1 & b_2 & b_3 & \end{array}$$

- Representation of Linear System
  - $Ax = b \leftarrow$ Constant $\quad (= ()x + ()y + ()z)$
    $\uparrow\uparrow$
    Coefficient
    - Variable
  - Linear system has more than one solution $\Rightarrow$ It has infinitely many solution

- Transpose
  - $A = (a_{ij})_{m \times n} \quad A^T = (a_{ji})_{n \times m}$
  - Properties

1. $(A^T)^T = A$
2. A is symmetric $\Leftrightarrow A^T = A$
3. $c(A^T) = (cA)^T$
4. $(A+B)^T = A^T + B^T$
5. $(AB)^T = B^T A^T$

- Inverses of square Matrices.
  - If there exists B such that $AB = I_n \quad BA = I_n$, A is invertibe and B is the inverse (opposite : Singular)
  - Find the Inverse. Let inverse $= \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, Prove Singular. Use contradiction and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$
  - Properties

1. If A invertible, then its inverse is unique
2. $AB_1 = AB_2 \Rightarrow B_1 = B_2$ (Only if A is Invertible)
   $C_1 A = C_2 A \Rightarrow C_1 = C_2$
3. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, A is invertible $\Leftrightarrow ad - bc \neq 0$
   If A is invertible, $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
4. $c \neq 0$, cA is invertible, $(cA)^{-1} = \frac{1}{c} A^{-1}$
5. A^T is invertible, $(A^T)^{-1} = (A^{-1})^T$
6. A^{-1} is invertible, $(A^{-1})^{-1} = A$
7. AB is invertible, $(AB)^{-1} = B^{-1} A^{-1}$
8. $(A^k)^{-1} = (A^{-1})^k$, where k is positive integer and A is Invertible

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3) Elementary Matrices.

- Elementary Operation $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \cdots$

- Elementary Matrices.
  - └ Square Matrix that can be obtained from I_n with one elementary row operation

$$\underset{A}{\underset{\shortparallel}{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}} \quad \underset{B}{\underset{\shortparallel}{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}}} \quad \underset{C}{\underset{\shortparallel}{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & c \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}}$$

- Connection to Matrix Multiplication.
  - AD ⇒ Multiply 3rd row of D by c
  - BD ⇒ Swapping 2nd and 4th row of D
  - CD ⇒ Adding $c \times 4$th row to second row
  - Theorem : From I_n to A, B, C, ABC does the same action on D

$$I_n \xrightarrow{R_i + cR_j} C \Rightarrow D \xrightarrow{R_i + cR_j} CD$$

- Invertibility
  - Every Elementary matrix is invertible
  - If A and B are row equivalent,

$$B = E_k E_{k-1} \cdots E_2 E_1 A$$

$$A = E_1^{-1} E_2^{-1} \cdots E_{k-1}^{-1} E_k^{-1} B$$

  - Augmented Matrices of Two linear systems are row equivalent ⇒ Two systems have same solution

- Main theorem

Let A be a square matrix. Then the followings are equivalent

1. A is an invertible matrix
2. $Ax = b$ has one solution
3. $Ax = 0$ has trivial solution
4. RREF of A $= I_n$
5. A is product of elementary matrices

- Find Inverse
  - $(A \mid I_n) \to (I_n \mid A^{-1})$
    - $\uparrow$ RREF of $(A \mid I)$
  - A square matrix is invertible $\Leftrightarrow$ RREF is I | Singular: Not I
    - All columns in REF are pivot | Not all
    - All rows in REF are nonzero | Not all
  - If $AB = I$ and A, B are square matrices $\Rightarrow$ A and B are invertible
    - $A^{-1} = B$, $B^{-1} = A$
  - Let $A_1, A_2, \ldots A_k$ be square matrices of same size
    - $A_1 A_2 \ldots A_k$ is invertible $\Leftrightarrow$ All A_i are invertible
    - $A_1 A_2 \ldots A_k$ is singular $\Leftrightarrow$ Some A_i are singular.
  - Verify $AB = I$. Use theorem $A^{-1} = B$, $BA = I$ is not needed.

- Column Operations,
  - Same as row but instead of AD, DA and applies to column.

4) Determinant

- Determinant of 2x2 Matrix
  - Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det(A) = |A| = ad - bc$
  - ① A is invertible $\Leftrightarrow \det(A) \neq 0$, Singlar $\Leftrightarrow \det(A) = 0$
  - If $A = (a)$, $\det(A) = a$
  - ② Properties (2x2)
    1. $\det(I_2) = 1$
    2. $A \xrightarrow{cR_i} B \Rightarrow \det(B) = c\det(A)$
    3. $A \xrightarrow{R_i \leftrightarrow R_j} B \Rightarrow \det(B) = -\det(A)$
    4. $A \xrightarrow{R_i + cR_j} B \Rightarrow \det(B) = \det(A)$
  - Application

    Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and it be invertible

    $A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$

- Determinant of 3x3 Matrix.
  - ① + ②
  - If A is invertible, there exist

    $A \xrightarrow{ero1} A_1 \xrightarrow{ero2} A_2 \rightarrow \dots \rightarrow A_{k-1} \xrightarrow{erok} A_k = I$ $\qquad A \rightarrow \rightarrow \rightarrow I \Rightarrow \det(A) = -\frac{1}{3}$

    Then $\det(A)$ can be evaluated backwards.
  - $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ $\quad \det(A) = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$
  - Let M_{ij} be submatrix by deleting i^{th} row and j^{th} column

    If $A = (a_{ij})_{3\times3}$, $\det(A) = a_{11}\det(M_{11}) - a_{12}\det(M_{12}) + a_{13}\det(M_{13})$

    ↳ $A_{ij} = (-1)^{i+j} \det(M_{ij})$, (i,j)-cofactor of A.

    ↳ $\det(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$
  - Alternative Formula

    $\det(A) = (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}) - (a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} + a_{13}a_{22}a_{31})$

- Elementary Row Operation and det (3×3) *

- $I \xrightarrow{cR_i} E \Rightarrow \det(E) = c\cdot\det(I) = c$ — cR_i
  $A \xrightarrow{cR_i} EA \Rightarrow \det(EA) = c.\det(A) = \det(E)\det(A)$
- $I \xrightarrow{R_i \leftrightarrow R_j} E \Rightarrow \det(E) = -\det(I) = -1$ — $R_i \Leftrightarrow R_j$
  $A \longrightarrow EA \Rightarrow \det(EA) = -\det(A) = \det(E)\det(A)$
- $I \xrightarrow{R_i + cR_j} E \Rightarrow \det(E) = \det(I) = 1$ — $R_i + cR_j$
  $A \longrightarrow EA \Rightarrow \det(EA) = \det(A) = \det(E)\det(A)$

(A)

- $\det(EA) = \det(E)\det(A)$
- Let R be RREF of A,

$$R = E_k E_{k-1} \dots E_2 E_1 A$$
$$\det(R) = \det(E_k)\det(E_{k-1})\dots\det(E_2)\det(E_1)\det(A)$$

If A is invertible, $R = I$, $\det(R) = 1$
- $\det(A) = [\det(E_k)\det(E_{k-1})\dots\det(E_2)\det(E_1)]^{-1}$

If A is singular, then last row of R is zero
- $R \xrightarrow{2R_3} R \Rightarrow 2\det(R) = \det(R) \Rightarrow \det(R) = 0 \quad \therefore \det(A) = 0$

- Determinant for all square Matrices
  - Let $A = (a_{ij})_{n\times n}$

Determinant : If $n = 1$, $\det(A) = a_{11}$

Else, Let A_{ij} be its (i,j)-cofactor

$$\det(A) = a_{11}A_{11} + a_{12}A_{12} + \dots + a_{1n}A_{1n}$$

- Properties :

(A)

- If square Matrix A has a zero row, $\Rightarrow \det(A) = 0$
  $A \xrightarrow{2R_i} A \Rightarrow \det(A) = 2\det(A) \Rightarrow \det(A) = 0$
- If A and B are row equivalent, $(B = E_k \dots E_1 A$, $\det(B) = \det(E_k)\dots\det(E_1)\det(A))$
  $\det(A) = 0 \Leftrightarrow \det(B) = 0$ $\quad \det(A) \neq 0 \Leftrightarrow \det(B) \neq 0$
- $\det(A) = 0 \Leftrightarrow A$ is singular $\Rightarrow$ RREF is not I and has a zero row.
  $\det(A) \neq 0 \Leftrightarrow A$ is invertible $\Rightarrow A$ is row equivalent to I
- AB. If A is singular, AB is singular.
- For any square A, $\det(A) = \det(A^T)$
- Suppose $(a_{ij})_{n\times n}$ is upper/lower Triangular, $\det(A) = a_{11}a_{22}\dots a_{nn}$

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- Cofactor Expansion.
  - Let A be square matrix of order n and A_{ij} be (i,j)-cofactor of A

Then for any i and j, $\det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \dots + a_{in}A_{in}$

$\det(A) = a_{1j}A_{1j} + a_{2j}A_{2j} + \dots + a_{nj}A_{nj}$

↑ Choose row/column with most number of 0

- Finding determinant (A is a square Matrix)
  - If A has a zero row/column, $\det(A) = 0$
  - If A is triangular, $\det(A) = a_{11} \cdots a_{nn}$
  - If A is not triangular,
    - If $n = 2$, use $\det(A) = a_{11}a_{22} - a_{12}a_{21}$
    - If 0 is present, use cofactor
    - Else use REF $\det(EA) = \det(E)\det(A)$
  - $\det(A) = \det(A^T)$ $\qquad$ $\det(cA) = c^n \det(A)$, $n \times n$
  - $\det(AB) = \det(A)\det(B)$ $\qquad$ $\det(A^{-1}) = \det(A)^{-1}$, if A is invertible

- Adjoint Matrix
  - $\mathrm{adj}(A) = (A_{ji})_{n\times n}$ where A_{ij} is the (i,j)-cofactor of A
  - $A^{-1} = [\det(A)]^{-1}\,\mathrm{adj}(A)$
  - $A\,[\mathrm{adj}(A)] = \det(A)\,I$

    $[\mathrm{adj}(A)]\,A = \det(A)\,I$

- Cramer's rule
  - Let A be an invertible of order n.
    - For every column matrix b of size $n \times 1$, the linear system $Ax = b$ has a unique solution.

$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\qquad x = \dfrac{1}{\det(A)} \begin{pmatrix} \det(A_1) \\ \vdots \\ \det(A_n) \end{pmatrix}$ $\qquad$ A_j is obtained from A by replacing its jth coln by b.

(Like $x = \det(A_1)$, $y = \det(A_2)$)