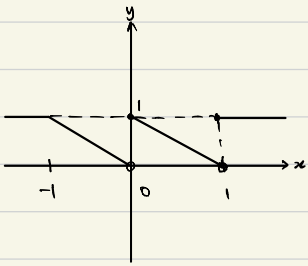
1.



When $x=1$,

$$\lim_{x\to 1^+} f(x) = 1$$

$$\lim_{x\to 1^-} f(x) = 0$$

$\therefore$ limit at $x=1$ does not exist.

graph is continuous from the right but discontinuous from the left.

When $x=-1$,

$$\lim_{x\to -1^+} f(x) = 1$$

$$\lim_{x\to -1^-} f(x) = 1$$

$\therefore \lim_{x\to 1} f(x) = 1$

graph is continuous at $x=-1$.

When $x=0$,

$$\lim_{x\to 0^+} f(x) = 1$$

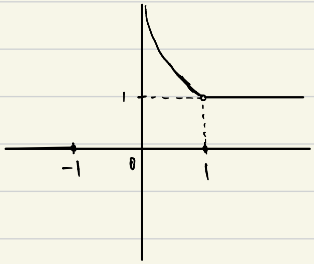
$$\lim_{x\to 0^-} f(x) = 0$$

$\therefore$ limit at $x=0$ does not exist

graph is continuous from the right but discontinuous from the left.

Discontinuities are non-removable as the values does not converge to a point from the left and right side.

2.



$x = 1$:

$\lim_{x \to 1^+} f(x) = 1$

$\lim_{x \to 1^-} f(x) = 1$

$\therefore \lim_{x \to 1} f(x) = 1$

graph is discontinuous at $x = 1$.

$x = 0$:

$\lim_{x \to 0^+} f(x) = \infty$

$\lim_{x \to 0^-} f(x) =$ undefined

$\therefore \lim_{x \to 0} f(x)$ does not exist

graph is discontinuous at $x = 0$

$x = -1$:

$\lim_{x \to -1^-} f(x) = 0$

$\lim_{x \to 1^+} f(x) =$ undefined

$\lim_{x \to 1} f(x)$ does not exist

graph is discontinuous at $x = -1$

discontinuity at $x = 1$ is removable as the limit exist.

3. $\lim_{t \to t_0} f(t) = -7$

$\lim_{t \to t_0} g(t) = 0$

a. $\lim_{t \to t_0} 3f(t)$

$= 3 \lim_{t \to t_0} f(t)$

$= 3(-7)$

$= -21$

b. $\lim_{t \to t_0} (f(t))^2$

$= \lim_{t \to t_0} f(t) \cdot f(t)$

$= \lim_{t \to t_0} f(t) \cdot \lim_{t \to t_0} f(t)$

$= (-7)(-7)$

$= 49.$

c. $\lim_{t \to t_0} f(t) \cdot \lim_{t \to t_0} g(t)$

$= (-7)(0)$

$= 0$

d. $\lim_{t \to t_0} \frac{f(t)}{g(t)-7}$, since all limit we known,

$= \frac{\lim_{t \to t_0} f(t)}{\lim_{t \to t_0} g(t) - 7} = \frac{-7}{0-7}$

$= 1$

e. $\lim_{t \to t_0} \cos(g(t))$

$= \cos\left(\lim_{t \to t_0} g(t)\right)$

$= \cos(0)$

$= 1$

f. $\lim_{t \to t_0} |f(t)|$

$= \left|\lim_{t \to t_0} f(t)\right|$

$= |-7|$

$= 7$

g. $\lim_{t \to t_0} (f(t) + g(t))$

$= \lim_{t \to t_0} f(t) + \lim_{t \to t_0} g(t)$

$= -7 + 0$

$= -7$

h. $\lim_{t \to t_0} \frac{1}{f(t)}$

$= \frac{1}{\lim_{t \to t_0} f(t)}$

$= \frac{1}{-7}$

$= -\frac{1}{7}$

4. $\lim_{x \to 0} f(x) = \frac{1}{2}$, $\lim_{x \to 0} g(x) = \sqrt{2}$

a. $\lim_{x \to 0} -g(x)$

$= - \lim_{x \to 0} g(x)$

$= -\sqrt{2}.$

b. $\lim_{x \to 0} g(x) \cdot \lim_{x \to 0} f(x)$

$= \frac{1}{2}\sqrt{2}.$

c. $\lim_{x \to 0} f(x) + \lim_{x \to 0} g(x)$

$= \frac{1}{2} + \sqrt{2}$

$= \frac{\sqrt{2}}{2}.$

d. $\frac{1}{\lim_{x \to 0} f(x)}$

$= \frac{1}{\frac{1}{2}}$

$= 2.$

e. $x + f(x).$

$\lim_{x \to 0} x + \lim_{x \to 0} f(x)$

$= 0 + \frac{1}{2}$

$= \frac{1}{2}$

f. $\lim_{x \to 0} \frac{f(x) \cdot \cos x}{x - 1}$

$= \frac{\lim_{x \to 0} f(x) \cdot \lim_{x \to 0} \cos x}{\lim_{x \to 0} x - 1}$

$= \frac{\frac{1}{2} \cdot (1)}{0 - 1}$

$= -\frac{1}{2}$

5. $\lim_{x \to 0} \left(\frac{4 - g(x)}{x} \right) = 1$

assume $\lim_{x \to 0} g(x)$ exist,

$$\frac{4 - \lim_{x \to 0} g(x)}{\lim_{x \to 0} x} = 1$$

$$4 - \lim_{x \to 0} g(x) = \lim_{x \to 0} x$$

$$\lim_{x \to 0} g(x) = 4 - \lim_{x \to 0} x$$

$$= 4 - 0$$

$$= 4 \,//$$

6. $\lim_{x \to -4} \left(x \underbrace{\lim_{x \to 0} g(x)}_{\text{constant}} \right) = 2.$

$$\lim_{x \to -4} x \cdot \lim_{x \to 0} g(x) = 2$$

$$\lim_{x \to 0} g(x) = \frac{2}{-4}$$

$$= -\frac{1}{2} \,//$$

7.

a. $f(x) = x^{\frac{1}{3}} = \sqrt[3]{x}$

continuous at $(-\infty, \infty)$

b. $g(x) = x^{7/4} = \sqrt[4]{x^7}$

$\therefore x^7 \geq 0$

$x \geq 0$

continuous at $[0, \infty)$

c. $h(x) = x^{-2/3}$

$= \sqrt[3]{\frac{1}{x^2}}$

$\therefore x^2 \neq 0$

$x \neq 0$

continuous at $(-\infty, 0) \cup (0, \infty)$

d. $k(x) = x^{-1/6}$

$= \sqrt[6]{\frac{1}{x}}$

$\frac{1}{x} \geq 0$ and $x \neq 0$ $\therefore x > 0$

$\therefore$ continuous at $(0, \infty)$.

8. a. $f(x) = \tan x$.

continuous when $\cos x \neq 0$

or $x \neq k\pi$.

b. $g(x) = \csc x$

$= \frac{1}{\sin x}$

continuous when $\sin x \neq 0$

$\therefore x \neq 2k\pi$

c. $h(x) = \frac{\cos x}{x - \pi}$

continuous when $x - \pi \neq 0$

$x \neq \pi$.

d. $k(x) = \frac{\sin x}{x}$

continuous when $x \neq 0$

9. a. $\lim_{x \to 0} \frac{x^2-4x+4}{x^3+5x^2-14}$

$= \frac{0^2-4(0)+4}{0^3+5(0)^2-14}$

$= \frac{4}{14}$ //

b. $\lim_{x \to 2} \frac{x^2-4x+4}{x^3+5x-14}$.

$= \frac{2^2-4(2)+4}{2^3+5(2)-14}$

$= \frac{4-8+4}{8+10-14}$

$= \frac{0}{4}$

$= 0$

10. a. $\lim_{x \to 0} \frac{x^2+x}{x^5+2x^4+x^3}$

$= \frac{\cancel{x}(x+1)}{x^{\cancel{3}2}(x^2+2x+1)}$

$= \frac{\cancel{x+1}}{x^2(x+1)^{\cancel{2}}}$

$= \frac{1}{x^2(x+1)}$

$= 0$ as $x \to 0$.

b. $\lim_{x \to -1} \frac{x^2+x}{x^5-2x^4+x^3}$

$= \frac{(-1)^2+(-1)}{(-1)^5-2(-1)^4+(-1)^3}$

$= \frac{1-1}{-1-2-1}$

$= \frac{0}{-4}$

$= 0$

11. $\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x} \cdot \frac{1+\sqrt{x}}{1+\sqrt{x}}$

$= \lim_{x\to 1} \frac{(\cancel{1-x})}{(\cancel{1-x})(1+\sqrt{x})}$

$= \lim_{x\to 1} \frac{1}{1+\sqrt{x}}$

$= 2.$

12. $\lim_{x\to a} \frac{x^2-a^2}{x^4-a^4}$

$= \frac{(x-a)(x+a)}{(x^2-a^2)(x^2+a^2)}$

$= \frac{\cancel{(x-a)}\cancel{(x+a)}}{\cancel{(x-a)}\cancel{(x+a)}(x^2+a^2)}$

$= \frac{1}{x^2+a^2}$

$= \frac{1}{a^2+a^2}$ as $x\to a$

$= \frac{1}{2a^2}$ //

13. $\lim_{h\to 0} \frac{(x+h)^2-x^2}{h}$

$= \frac{(x+h-x)(x+h+x)}{h}$

$= \frac{\cancel{h}(2x+h)}{\cancel{h}}$

$= 2x+h$

$= 2x$ as $h\to 0$ //

14. $\lim_{x\to 0} \frac{(x+h)^2-x^2}{h}$

$= \frac{(x+h-x)(x+h+x)}{h}$

$= \frac{\cancel{h}(2x+h)}{\cancel{h}}$

$= 2x+h$

$= h$ as $x\to 0$.

15. $\lim_{x\to 0} \dfrac{\frac{1}{2+x} - \frac{1}{2}}{x}$

$= \left(\frac{1}{2+x} - \frac{1}{2}\right)\frac{1}{x}$

$= \left(\frac{2-(2+x)}{2(2+x)}\right)\frac{1}{x}$

$= \left(\frac{2-2-x}{2(2+x)}\right)\frac{1}{x}$

$= \left(\frac{4-x}{4+2x}\right)\frac{1}{x}$

$= \frac{4-x}{x(4+2x)}$

$= \frac{4-x}{4x+2x^2}$

$= 0$?

16. $\lim_{x\to 0} \dfrac{(2+x)^3 - 8}{x}$

$(a^3-b^3) = (a-b)(a^2+ab+b^2)$

$= \frac{(2+x)^3 - 2^3}{x}$

$= \frac{(\cancel{2}+\cancel{x}-\cancel{2})((2+x)^2+(2+x)(2)+2^2)}{\cancel{x}}$

$= ((2+x)^2 + 2(2+x) + 4)$

$= ((2)^2 + 2(2) + 4)$ as $x \to 0$

$= 4+4+4$

$= 12$

17. $\lim_{x\to 1} \dfrac{x^{1/3}-1}{\sqrt{x}-1}$

$= \lim_{x\to 1} \frac{(x^{1/3}-1)}{(\sqrt{x}-1)} \cdot \frac{(x^{2/3}+x^{1/3}+1)(\sqrt{x}+1)}{(\sqrt{x}+1)(x^{2/3}+x^{1/3}+1)}$

$= \lim_{x\to 1} \frac{(x-1)(\sqrt{x}+1)}{(x-1)(x^{2/3}+x^{1/3}+1)}$

$= \lim_{x\to 1} \frac{\sqrt{x}+1}{x^{2/3}+x^{1/3}+1}$

$= \frac{1+1}{1+1+1} = \frac{2}{3}$

18. $\lim_{x\to 64} \dfrac{x^{2/3}-16}{\sqrt{x}-8}$

$= \frac{(x^{1/3}-4)(x^{1/3}+4)}{\sqrt{x}-8} \cdot \frac{(x^{2/3}+4x^{1/3}+16)(\sqrt{x}+8)}{(\sqrt{x}+8)(x^{2/3}+4x^{1/3}+16)}$

$= \frac{(x-64)(x^{1/3}+4)(\sqrt{x}+8)}{(x-64)(x^{2/3}+4x^{1/3}+16)}$

$= \lim_{x\to 64} \frac{(x^{1/3}+4)(\sqrt{x}+8)}{x^{2/3}+4x^{1/3}+16}$

$= \frac{8}{3}$

19. $\lim_{x\to 0} \frac{\tan(2x)}{\tan(\pi x)} = \frac{2}{\pi}$.

$= \lim_{x\to 0} \frac{\frac{\sin 2x}{\cos 2x}}{\frac{\sin \pi x}{\cos \pi x}}$

$= \lim_{x\to 0} \frac{\sin 2x \cdot \cos \pi x}{\cos 2x \cdot \sin \pi x}$

$= \lim_{x\to 0} \left(\frac{\sin 2x}{2x}\right)\left(\frac{\cos \pi x}{\cos 2x}\right)\left(\frac{\pi x}{\sin \pi x}\right)\left(\frac{2\cancel{x}}{\pi \cancel{x}}\right)$ $\to 1 \quad \to 1 \quad \to 1$

$= 1 \cdot 1 \cdot 1 \cdot \frac{2}{\pi} = \frac{2}{\pi}$.

20. $\lim_{x\to \pi^-} \csc x$

$= \lim_{x\to\pi^-} \frac{1}{\sin x}$

$= \infty$

21. $\lim_{x\to\pi} \sin\left(\frac{x}{2} + \sin x\right)$

$= \sin\left(\lim_{x\to\pi} \frac{x}{2} + \lim_{x\to\pi} \sin x\right)$.

$= \sin\left(\frac{\pi}{2} + 0\right)$

$= \sin\frac{\pi}{2} = 1$

22. $\lim_{x\to\pi} \cos^2(x - \tan x)$.

$= \lim_{x\to\pi} \cos^2\left(x - \frac{\sin x}{\cos x}\right)$

$= \cos^2\left(\lim_{x\to\pi} x - \frac{\lim_{x\to\pi} \sin x}{\lim_{x\to\pi} \cos x}\right)$

$= \cos^2\left(\pi - \frac{0}{-1}\right)$

$= \cos^2(\pi)$

$= 1$

23. $\lim_{x\to 0} \frac{8x}{3\sin x - x}$.

$= \frac{1}{\frac{3\sin x - x}{8x}}$

$= \frac{1}{\frac{3\sin x}{8x} - \frac{x}{8x}}$

$= \frac{1}{\frac{3}{8}\cdot\frac{\sin x}{x} - \frac{1}{8}}$. as $x \to 0$

$= \frac{1}{\frac{3}{8}(1) - \frac{1}{8}}$.

$= 4$.

24. $\lim_{x\to 0} \frac{\cos 2x - 1}{\sin x}$

$= \frac{\cancel{1} - 2\sin^{\cancel{2}} x \cancel{-1}}{\cancel{\sin x}}$

$= -2\sin x$ as $x \to 0$

$= 0$.

25. $\lim_{x\to 0^+} (4g(x))^{1/3} = 2$

$4^{1/3} \left(\lim_{x\to 0^+} g(x)\right)^{1/3} = 2.$

$\lim_{x\to 0^+} g(x) = \left(\frac{2}{\sqrt[3]{4}}\right)^3$

$= \frac{8}{4} = 2.$

26. $\lim_{x\to\sqrt{5}} \frac{1}{x+g(x)} = 2.$

$\frac{1}{\lim_{x\to\sqrt{5}} x + \lim_{x\to\sqrt{5}} g(x)} = 2.$

$\frac{1}{2} = \sqrt{5} + \lim_{x\to\sqrt{5}} g(x)$

$\lim_{x\to\sqrt{5}} g(x) = \frac{1}{2} - \sqrt{5}.$

27. $\lim_{x\to 1} \frac{3x^2+1}{g(x)} = \infty$

$\frac{3\lim_{x\to 1} x^2 + 1}{\lim_{x\to 1} g(x)} = \infty.$

$\frac{4}{\lim_{x\to 1} g(x)} = \infty.$

$\therefore \lim_{x\to 1} g(x) = 0.$

28. $\lim_{x\to -2} \frac{5-x^2}{\sqrt{g(x)}} = 0$

$\frac{5-\lim_{x\to -2} x^2}{\sqrt{\lim_{x\to -2} g(x)}} = 0$

$\frac{5-4}{\sqrt{\lim_{x\to -2} g(x)}} = 0$

$\frac{1}{\sqrt{\lim_{x\to -2} g(x)}} = 0.$

$\left(\frac{1}{0}\right)^2 = \lim_{x\to -2} g(x)$

$\lim_{x\to 2} g(x) = \infty.$

29. $f(x) = x^3 - x - 1$

a when $x = -1$,

$f(x) = -1 - 1 - 1$

$= -3 < 0$

when $x = 2$,

$f(x) = 2^3 - 2 - 1$

$= 5 > 0$

$\therefore$ By Intermediate value theorem, there exist a point $f(x) = 0$ between $x = -1$ and $x = 2$ //

b $x = 1.324718$.

c $x = 1.324717957$.

31. a. $f(\theta) = \theta^3 - 2\theta + 2$

when $\theta = -2$,

$f(\theta) = (-2)^3 - 2(-2) + 2$

$= -2$

when $\theta = 0$,

$f(\theta) = 2$.

$\therefore$ By Intermediate value theorem, $\exists c \in [-2, 2]$ s.t. $f(c) = 0$.

b. $x = -1.769292$.

c. $x = -1.769292354$.

31. $f(x) = \dfrac{x(x^2-1)}{|x^2-1|}$

No.

$\lim_{x\to -1^+} f(x) = 1$

$\lim_{x\to -1^-} f(x) = -1$

$\therefore \lim_{x\to -1} f(x)$ does not exist.

Similarly,

$\lim_{x\to 1^+} f(x) = 1$

$\lim_{x\to 1^-} f(x) = -1$

$\therefore \lim_{x\to 1} f(x)$ does not exist.

32. $f(x) = \sin\left(\frac{1}{x}\right)$

At $x=0$, graph of $f(x)$ is oscillating between -1 and 1, thus $\lim_{x\to 0} f(x)$ does not exist,

$\therefore$ $f(x)$ has no continuous extension at $x=0$.

33. $f(x) = \dfrac{x-1}{x-\sqrt[4]{x}}$, $x=1$.

when $x=1$,

$f(1) = \frac{1}{3}$. , extended.

34. $g(\theta) = \dfrac{5\cos\theta}{4\theta-2\pi}$, $x = \frac{\pi}{2}$.

when $\theta = \frac{\pi}{2}$,

$g(\theta) = -\frac{5}{4}$, extended

35. $h(t) = (1+|t|)^{1/t}$, $t=0$.

Function cannot be extended

from the right or left.

36. $k(x) = \dfrac{x}{1-2^{|x|}}$, $x=0$

Function cannot be extended ?

37. $\lim_{x\to\infty} \frac{2x+3}{5x+7} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$

$= \frac{2+\cancel{\frac{3}{x}}^{\to 0}}{5+\cancel{\frac{7}{x}}^{\to 0}}$ as $x\to\infty$

$= \frac{2}{5}$ //

38. $\lim_{x\to-\infty} \frac{2x^2+3}{5x^2+7} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$

$= \frac{2+\cancel{\frac{3}{x^2}}^{\to 0}}{5+\cancel{\frac{7}{x^2}}^{\to 0}}$ as $x\to-\infty$

$= \frac{2}{5}$.

39. $\lim_{x\to-\infty} \frac{x^2-4x+8}{3x^3} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}}$

$= \frac{\cancel{\frac{1}{x}}^{\to 0} - \cancel{\frac{4}{x^2}}^{\to 0} + \cancel{\frac{8}{x^3}}^{\to 0}}{3}$ as $x\to-\infty$.

$= 0$

40. $\lim_{x\to\infty} \frac{1}{x^2-7x+1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$

$= \frac{\cancel{\frac{1}{x^2}}^{\to 0}}{1-\cancel{\frac{7}{x}}^{\to 0}+\cancel{\frac{1}{x^2}}^{\to 0}}$

$= 0$ as $x\to\infty$.

41. $\lim_{x\to-\infty} \frac{x^2-7x}{x+1}$

$= \lim_{x\to-\infty} \frac{x(x-7)}{x+1} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$

$= \frac{x-7}{1+\cancel{\frac{1}{x}}^{\to 0}}$

$= -\infty$. ✓

42. $\lim_{x\to\infty} \frac{x^4+x^3}{12x^3+128} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}}$

$= \frac{\overset{\infty}{\cancel{x}}+1}{12+\cancel{\frac{128}{x^3}}^{\to 0}}$

$= \infty$. ✓

43. $\lim_{x\to\infty} \frac{\sin x}{\lfloor x \rfloor} \le \lim_{x\to\infty} \frac{1}{\lfloor x \rfloor} = 0$.

since $\lfloor x \rfloor \to \infty$
as $x\to\infty$

$-1 \le \cos x \le 1$.

44. $\lim_{x\to\infty} \frac{\cos x - 1}{x} \le \lim_{x\to\infty} \frac{2}{x} = 0$

45. $\lim_{x\to\infty} \frac{x+\sin x+2\sqrt{x}}{x+\sin x}$

$= 1 + \lim_{x\to\infty} \frac{2\sqrt{x}}{x+\sin x} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$

$= \frac{2\frac{\sqrt{x}}{x}}{1+\frac{\sin x}{x}}$

Since $\sqrt{x} \le x$,

then $x\to\infty$, $\frac{\sqrt{x}}{x}\to 0$.

Since $-1 \le \sin x \le 1$,

then $\lim_{x\to\infty} \frac{\sin x}{x} \le \lim_{x\to\infty} \frac{1}{x} = 0$.

$\therefore \lim_{x\to\infty} \frac{x+\sin x+2\sqrt{x}}{x+\sin x}$

$= 1$ //

46. $\lim_{x\to\infty} \frac{x^{2/3}+x^{-1}}{x^{2/3}+\cos^2 x} \cdot \frac{x^{-2/3}}{x^{-2/3}}$ $\to \frac{1}{\sqrt[3]{x^2}}$

$= \frac{1+(\frac{1}{x})(x^{-2/3})}{1+x^{-2/3}\cos^2 x}$

as $x\to\infty$, $\frac{1}{\sqrt[3]{x^2}} \to 0$

$\therefore x^{-2/3}\to 0$.

as $x\to\infty$, since $-1 \le \cos x \le 1$,

$\therefore \lim_{x\to\infty} \cos^2 x \le \lim_{x\to\infty} 1$

$= 1$

$\therefore \lim_{x\to\infty} \frac{x^{2/3}+x^{-1}}{x^{2/3}+\cos^2 x} = 1$ //

47. a. $\lim_{x \to 3} \frac{x^2+4}{x-3} = \infty$

$\therefore x = 3$

b. $\lim_{x \to 1} \frac{x^2-x-2}{(x-1)^2}$

$= \infty$

$\therefore x = 1$

c. $\frac{x^2+x-6}{x^2+2x-8}$

$\therefore x^2+2x-8=0$

$(x-2)(x+4)=0$

$x=2$ or $x=-4$

$\therefore \lim_{x \to 2} \frac{x^2+x-6}{x^2+2x-8} = \infty$

$\lim_{x \to -4} \frac{x^2+x-6}{x^2+2x-8} = \infty$

$\therefore$ eqn: $x=2$, $x=-4$ //

48 a. $\lim_{x\to\infty} \frac{1-x^2}{x^2+1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$

$= \frac{\frac{1}{x^2}-1}{1+\frac{1}{x^2}}$ when $x \to \infty$

$= -\frac{1}{1}$

$= -1$

$\therefore y = -1$.

b. $\lim_{x\to\infty} \frac{\sqrt{x}+4}{\sqrt{x+4}} \cdot \frac{\frac{1}{\sqrt{x}}}{\frac{1}{\sqrt{x}}}$

$\frac{1+\frac{4}{\sqrt{x}}}{\sqrt{1+\frac{4}{x}}} = 1$

$\therefore y = 1$.

c. $g(x) = \frac{\sqrt{x^2+4}}{x} \cdot \frac{\sqrt{\frac{1}{x^2}}}{\sqrt{\frac{1}{x^2}}}$

$\lim_{x\to\infty} \frac{\sqrt{1+\frac{4}{x^2}}}{1}$

$= 1$.

$y = 1$

$\lim_{x\to-\infty} g(x) = -1$

$y = -1$

d. $\lim_{x\to\infty} \sqrt{\frac{x^2+9}{9x^2+1}} \cdot \frac{\sqrt{\frac{1}{x^2}}}{\sqrt{\frac{1}{x^2}}}$

$\lim_{x\to\infty} \sqrt{\frac{1+\frac{9}{x^2}}{9+\frac{1}{x^2}}}$

$= \frac{1}{3}$.

49. $y = \dfrac{\sqrt{16-x^2}}{x-2}$

D: $x \neq 2$ R:

$16-x^2 \geq 0$

$-x^2 \geq -16$

$x^2 \leq 16$

$x \leq 4$

$\therefore$ D: $(-\infty, 2) \cup (2, 4]$

50.

Advance exercises.

1. $L = L_0\sqrt{1-\frac{v^2}{c^2}}$ $c = 3\times10^8\ m/s$

L increases when v increases.

$\lim_{v\to c^-}\left(L_0\sqrt{1-\frac{v^2}{c^2}}\right)$

$= L_0\sqrt{1-1}$

$= \cancel{L_0}\ 0$

Since $L \in \mathbb{R}$,

then $1-\frac{v^2}{c^2} \geq 0$

$\therefore$ when $v^2 > c^2$,

or $v > c$, when $\lim_{v\to c^+} L$,

then $1-\frac{v^2}{c^2} < 0$,

$\therefore L \notin \mathbb{R}$.

L is undefined.

if $v > c$

2. $y = kx$

Suppose $y = \frac{\sqrt{x}}{2}$,

a) $\left|\frac{\sqrt{x}}{2} - 1\right| < 0.2$

$(y - y_0) < 0.2$

$-0.2 < \frac{\sqrt{x}}{2} - 1 < 0.2$

b) Similar.

3. $y = 10 + (t-70) \times 10^{-4}$

$\left| 10 + (t-70) \times 10^{-4} - 10 \right| < 0.0005$

4. $V = \pi 6^2 h = 36\pi h$

$|36\pi h - 1000| \leq 10.$

11. $\lim_{x\to 0^+} f(x) = A$

$\lim_{x\to 0^-} f(x) = B$

a. $\lim_{x\to 0^+} f(x^3-x)$

when $x \to 0^+$,

$\underline{0 < x^3} < x < 1$

$(x^3 - x) \to 0^-$ when $x \to 0^+$

smaller bigger <0

$\lim_{x\to 0^+} f(x^3-x) = \lim_{x\to 0^-} f(y) = B.$

b. $\lim_{x\to 0^-} f(x^2-x)$

when $x \to 0^-$

$-1 < x < x^3 < 0$

$\therefore (x^3 - x) \to 0^+$ when $x \to 0^-$

$\therefore \lim_{x\to 0^-} f(x^3-x) = \lim_{x\to 0^+} f(y) = A.$

c. $\lim_{x\to 0^+} f(x^2-x^4)$

$0 < x^4 < x^2 < 1$

$\therefore (x^2-x^4) \to 0^+$ as $x \to 0^+$

$\therefore \lim_{x\to 0^+} f(x^2-x^4) = \lim_{x\to 0^+} f(y) = A.$

d. $\lim_{x\to 0^-} f(x^2-x^4)$

$0 < x^4 < x^2 < 1$

$\therefore (x^2-x^4) \to 0^+$ as $x \to 0^+$

$\therefore \lim_{x\to 0^+} f(x^2-x^4) = \lim_{x\to 0^+} f(y) = A.$

$0^- \quad 0^+$

12. a. $\lim_{x\to c} f(x)$ exist

$\lim_{x\to c} g(x)$ not exist

$\lim_{x\to c} (f(x)+g(x))$ does not exist?

True.

By contradiction,

Suppose $\lim_{x\to a} (f(x)+g(x))$ exist,

then $\lim_{x\to a} (f(x)+g(x)) - \lim_{x\to a} f(x)$

$= \lim_{x\to a} (f(x)+g(x) - f(x))$

$= \lim_{x\to a} g(x)$ exist

$\therefore$ Contradiction

b. $\lim_{x\to c} f(x)$ not exist

$\lim_{x\to c} g(x)$ not exist

$\therefore \lim_{x\to c} (f(x)+g(x))$ not exist

False.

By counterexample,

Suppose $f(x) = \frac{1}{x}$, $g(x) = -\frac{1}{x}$,

then neither $\lim_{x\to 0} f(x)$ or $\lim_{x\to 0} g(x)$ exist

but $\lim_{x\to 0} (f(x)+g(x))$

$= \lim_{x\to 0} (\frac{1}{x} - \frac{1}{x})$

$= \lim_{x\to 0} 0 = 0$ exist //

c. If f is continuous at x, then so is $|f|$.

True.

let $g(x) = |x|$ is continuous

as $g(f(x)) = |f(x)|$ is continuous.

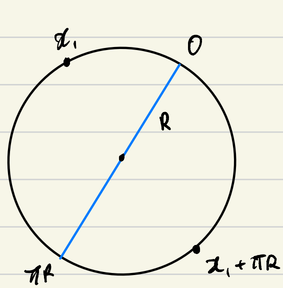
d. $|f|$ is continuous at c, then so is f.

Suppose $f(x) = \begin{cases} -1, & x \le 0 \\ 1, & x > 0 \end{cases}$

$\therefore$ discontinuous at $x=0$.

However, $|f(x)| = 1$ is continuous at $x=0$.

17.



noon midnight

$T(x_1) - T(x_1+\pi R) > 0$ temp diff

midnight noon

$T(x_2) - T(x_2+\pi R) < 0$ temp diff.

Since ΔT is continuous,

$\therefore \exists x \in \mathbb{R} \, (T(x_3) - T(x_3+\pi R) = 0)$

By intermediate value theorem.

18. $\lim_{x \to c} (f(x)+g(x)) = 3$.

and $\lim_{x \to c} (f(x)-g(x)) = -1$.

Find $\lim_{x \to c} f(x)g(x)$

$\lim_{x \to c} (f(x)+g(x)) + \lim_{x \to c} (f(x)-g(x))$

$= \lim_{x \to c} (f(x) \cancel{+g(x)} + f(x) \cancel{-g(x)})$

$= \lim_{x \to c} (2f(x))$

$= 2\lim_{x \to c} f(x) = 3+(-1)$

$= 2$

$\therefore \lim_{x \to c} f(x) = 1$.

$\lim_{x \to c} (f(x)+g(x)) \quad \lim_{x \to c} (f(x)-g(x))$

$= \lim_{x \to c} (\cancel{f(x)} + g(x) - \cancel{f(x)} + g(x))$

$= \lim_{x \to c} (2g(x))$

$= 2\lim_{x \to c} g(x) = 3-(-1)$

$= 4$

$\therefore \lim_{x \to c} g(x) = 2$.

$\therefore \lim_{x \to c} f(x)g(x) = 2$ //

19. $ax^2+2x-1=0$

$a>-1$, $a\neq 0$.

roots: $r_+(a)=\dfrac{-1+\sqrt{1+a}}{a}$

$r_-(a)=\dfrac{-1-\sqrt{1+a}}{a}$

a. $a\to 0$

$$\lim_{a\to 0} r_+(a)=\lim_{a\to 0}\frac{-1+\sqrt{1+a}}{a}\cdot\frac{-1-\sqrt{1+a}}{-1-\sqrt{1+a}}$$

$$=\lim_{a\to 0}\frac{(-1)^2-(1+a)}{a(-1-\sqrt{1+a})}$$

$$=\frac{\cancel{-a}}{-\cancel{a}(1+\sqrt{1+a})}$$

$$=\frac{1}{1+\sqrt{1+a}} \text{ as } a\to 0.$$

$$=\frac{1}{2}$$

$a\to -1^+$

$$\lim_{a\to -1^+}\frac{-1+\sqrt{1+a}}{a}$$

$$=\frac{-1+0}{-1}$$

$$=1$$

b. $\lim_{a\to 0}\dfrac{-1-\sqrt{1+a}}{a}\cdot\dfrac{-1+\sqrt{1+a}}{-1+\sqrt{1+a}}$

$$=\frac{(-1)^2-(1+a)}{a(-1+\sqrt{1+a})}$$

$$=\frac{-\cancel{a}}{-\cancel{a}(1-\sqrt{1+a})}$$

$$=\frac{1}{1-\sqrt{1+a}}$$

$\therefore$ as $a\to 0^-$, $\lim_{a\to 0^-} r_-(a)\to\infty$

as $a\to 0^+$, $\lim_{a\to 0^+} r_-(a)\to -\infty$

$\therefore$ limit does not exist..

$$\lim_{a\to -1^+}\frac{-1-\sqrt{1+a}}{a}$$

$$=\frac{1}{1-\sqrt{1+a}}$$

$$=1$$

2e. $x + 2\cos x = y$

let $x = 0$,

then $y = 2$.

let $x = \pi$,

then $y = -\pi - 2$

$\therefore$ Since between $x = 0$ and $x = \pi$,

y went from < 0 to > 0

$\therefore$ by IVT,

$f(x)$ must take on every

value between $[-\pi - 2, 2]$

$\therefore \exists c \in [-\pi, 0]$ s.t.

$f(c) = 0$.

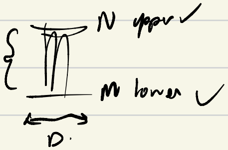
$\therefore$ c is a solution to y.

21. $f \rightarrow$ bounded from above, set D.

$\forall x \in D, \exists N \in D$ s.t. $f(x) \leq N$

N is an upper bound.

f is bounded on D.



a. f is bounded iff $\exists B$ s.t. $|f(x)| \leq B$ for all x

$\therefore$ $M \leq f(x) \leq N$

let B be max $\{|M|, |N|\}$

Then $|f(x)| \leq B$.

If $|f(x)| \leq B$,

then $-B \leq f(x) \leq B$

$f(x) \geq -B$

$f(x) \leq B \Rightarrow f(x)$ is bounded on D with $N = B$.

and $M = -B$ a lower bound.

22. Max $\{a, b\}$ and min $\{a, b\}$

a.

$$\max\{a, b\} = \frac{a+b}{2} + \frac{|a-b|}{2} = a \text{ if } a \geq b$$

$$= b \text{ if } b \geq a$$

$\Downarrow$

gives larger of 2 no.

$\therefore$ $a \geq b$,

$a - b \geq 0$,

$\therefore |a-b| = a-b$

$\therefore \max\{a, b\} = \frac{a+b}{2} + \frac{|a-b|}{2}$

$= \frac{a+b}{2} + \frac{a-b}{2}$

$= \frac{2a}{2}$

$2a$.

$a \geq b$

$a - b$.

$b \geq a$.

$b - a \geq 0$ $\quad a - b < 0$

$|b-a| \geq 0$.

$|b-a| = +(b-a)$.

$|a-b| = -(a-b)$

$\Downarrow$

$= (b-a)$

$\therefore \max\{a, b\}$

$= \frac{a+b}{2} + \frac{|a-b|}{2}$

$= \frac{a+b}{2} + \frac{b-a}{2}$

$= \frac{2b}{2}$

$= b$.

b. $\min\{a, b\} = \frac{a+b}{2} - \frac{|a+b|}{2}$.

$$\lim_{x \to c} \frac{\sin f(x)}{f(x)} = 1 \qquad \lim_{x \to c} f(x) = 0$$

$$\lim_{x \to \infty} \frac{\sin x}{x} = 0$$

$\sin x \quad -1 \leq \sin x \leq 1,$

$\lim \quad -\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$

$-\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}.$

$\therefore \lim_{x \to \infty} \left(-\frac{1}{x}\right) = 0$

use trigo for $x \to 0$.

$\lim_{x \to \infty} \left(\frac{1}{x}\right) = 0$

$\therefore$ By squeeze theorem,

$\lim_{x \to \infty} \frac{\sin x}{x} = 0$

23. $\lim_{x \to 0} \frac{\sin(1-\cos x)}{x} \cdot \frac{1-\cos x}{1-\cos x} \cdot \boxed{\frac{1+\cos x}{1+\cos x}}$

$\lim_{x \to 0} \frac{\sin(1-\cos x)}{(1-\cos x)} \cdot \left(\frac{1-\cos^2 x}{x(1+\cos x)}\right) \rightarrow 1 \cdot \lim_{x \to 0} \frac{\sin^2 x}{x(1+\cos x)}$

$= \frac{\sin(1-\cos x)}{1-\cos x} \times \left(\frac{1-\cos x}{x}\right)$

$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{1+\cos x}$

$= 1 \times \lim_{x \to 0} \left(\frac{1-\sqrt{1-\sin^2 x}}{x}\right)$

$= 1 \cdot \left(\frac{0}{2}\right) = 0.$

?

24. $\lim_{x\to0^+} \frac{\sin x}{\sin\sqrt{x}} \cdot \frac{x}{x}$

$= \frac{\sin x}{x} \cdot \frac{x}{\sin\sqrt{x}}$

$= 1 \cdot \lim_{x\to0^+} \frac{x}{\sin\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}}$

$= 1 \cdot \lim_{x\to0^+} \left[\frac{\sqrt{x}}{\sin\sqrt{x}} \cdot \frac{x}{\sqrt{x}} \right]$

$\frac{\sqrt{x}}{\sin\sqrt{x}}$

$\lim_{x\to0^+} \frac{\sin x}{\sin\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}} \cdot \frac{x}{x}$ %

$= \lim_{x\to0^+} \underbrace{\frac{\sin x}{x}}_{\to 1} \cdot \frac{\sqrt{x}}{\sin\sqrt{x}} \cdot \frac{x}{\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}}$

$= \lim_{x\to0^+} \frac{1}{\frac{\sin\sqrt{x}}{\sqrt{x}}} \cdot \frac{x\sqrt{x}}{x} = \sqrt{x}$

$\lim_{x\to0^+} \sqrt{x} \to 0$

$= 0$

$\lim_{x\to0} \frac{\sin(x^2+x)}{x} \cdot \frac{x^2+x}{x^2+x}$

$= \frac{(x^2+x)}{x} \cdot \frac{\sin(x^2+x)}{x^2+x}$

$= \lim_{x\to0} \frac{x^2+x}{x} \cdot 1$

$= \lim_{x\to0} (x+1)$

$= 1.$

27. $\lim_{x\to0} \frac{\sin(x^2-4)}{x-2} \cdot \frac{x^2-4}{x^2-4}$

$= \lim_{x\to0} \frac{x^2-4}{x-2} \cdot 1$

$= \frac{(\cancel{x-2})(x+2)}{(\cancel{x-2})}$

$= \lim_{x\to0} (x+2)$

$= 2$ /

25. $\lim_{x\to0} \frac{\sin(\sin x)}{x} \cdot \frac{\sin x}{\sin x}$

$= \frac{\sin(\sin x)}{\sin x} \cdot \frac{\sin x}{x}$

$= 1 \cdot 1$ as $x\to0$.

$= 1.$

$\lim_{x\to9} (\sqrt{x}-3) = 0.$

28. $\lim_{x\to9} \frac{\sin(\sqrt{x}-3)}{x-9} \cdot \frac{\sqrt{x}-3}{\sqrt{x}-3}$

$= \lim_{x\to9} \frac{\sqrt{x}-3}{x-9} \cdot 1 \cdot \frac{\sqrt{x}+3}{\sqrt{x}+3}$

$= \frac{(\cancel{x-9})}{(\cancel{x-9})(\sqrt{x}+3)}$

$= \frac{1}{6}$ //

30. $y = x + x\sin\frac{1}{x}$.

As $x \to \infty$,

$\frac{1}{x} \to 0$

$\sin\frac{1}{x} \to 0$

$\therefore 1 + \sin\frac{1}{x} \to 1$

As $x \to \pm\infty$,

$y = x + x\sin\left(\frac{1}{x}\right)$

$= x\left(1 + \sin\left(\frac{1}{x}\right)\right) \to x$.

$\therefore y = x$

33. $1 < a < b$

$\frac{a}{x} + x = \frac{1}{x-b}$. Show that it is solvable.

LHS: $\frac{a}{x} + x$.

$= \frac{a}{x} + \frac{x^2}{x}$

$= \frac{a + x^2}{x}$

$=$ c

34. a. $\lim_{x\to 0} \frac{\sqrt{a+bx}-1}{x} = 2$

$\therefore \lim_{x\to 0} \frac{\sqrt{a+bx}-1}{x}$

$$\frac{\lim_{x\to 0^+}(\sqrt{a+bx}-1)}{\lim_{x\to 0^+}(x)} = 2 \quad \lim\left(\frac{\sqrt{a+bx}-1}{x}\right)$$

$$\lim_{x\to 0^+}(\sqrt{a+bx}-1) = 2 \cdot \lim_{x\to 0^+}(x)$$

$$\sqrt{a}-1 = 0.$$

$$\sqrt{a} = 1$$

$$a = 1.$$

$$\therefore \lim_{x\to 0} \frac{\sqrt{1+bx}-1}{x} \cdot \frac{\sqrt{1+bx}+1}{\sqrt{1+bx}+1}$$

$$= \frac{(1+bx)-1}{x(\sqrt{1+bx}+1)}$$

$$= \frac{bx}{x(\sqrt{1+bx}+1)}$$

$$= \frac{b}{\sqrt{1+bx}+1} \quad \text{as } x\to 0.$$

$$= \frac{b}{\sqrt{1}+1}$$

$$= \frac{b}{2} = 2.$$

$$b = 4$$

34. b. $\lim_{x \to 1} \frac{\tan(ax-a)+b-2}{x-1} = 3$

$= \frac{\lim_{x\to 1}(\tan(ax-a)+b-2)}{\lim_{x\to 1}(x-1)} = 3.$

$\lim_{x\to 1}(\tan(ax-a)+b-2) = \lim\left(\frac{\tan(ax-a)+b-2}{x-1}\right) \cdot \lim(x-1)$

$\lim_{x\to 1}(\tan(ax-a)+b-2) = 3 \cdot \lim_{x\to 1}(x-1)$

$0+b-2 = 0$

$b = 2$

$\lim_{x\to 1} \frac{\tan(ax-a)}{x-1} = 3.$

$= \frac{\frac{\sin(ax-a)}{\cos(ax-a)}}{x-1}$

$= \frac{\sin(ax-a)}{(x-1)\cos(ax-a)} \cdot \frac{ax-a}{ax-a}$

$= \frac{\sin(\cancel{ax-a})}{\cancel{ax-a}} \cdot \frac{ax-a}{x-1}$

$= \lim_{x\to 1} \frac{ax-a}{x-1} = \frac{a(\cancel{x-1})}{\cancel{x-1}}$

$= \lim_{x\to 1} a.$

$= a = 3.$

35. $\lim_{x \to 1} \frac{x^{2/3} - 1}{1 - \sqrt{x}}$

$= \frac{x^{2/3} - 1}{1 - x^{1/2}}$

$= \frac{(x^{1/3} - 1)(x^{1/3} + 1)}{1 - \sqrt{x}}$

$= \frac{(x^{1/3} - 1)(x^{1/3} + 1)}{1 - \sqrt{x}} \cdot \frac{1 + \sqrt{x}}{1 + \sqrt{x}}$

$= \frac{(x^{1/3} - 1)(x^{1/3} + 1)(1 + \sqrt{x})}{1 - x}$

36. $\lim_{x \to 0} \frac{|3x+4| - |x| - 4}{x}$

$= \frac{|3x-4|}{x} - \frac{|x|}{x} - \frac{4}{x}$

$3x - 4 \le |3x-4|$ and $x \le |x|$

$-x < |x| = x$

$\frac{(x^{1/3} - 4)(x^{1/3} + 4)}{(\sqrt{x} - 8)} \cdot \frac{(x^{2/3} + 4x^{1/3} + 16)(\sqrt{x} + 8)}{(\sqrt{x} + 8)(x^{2/3} + 4x^{1/3} + 16)}$

$= \frac{}{(x - 64)(x^{2/3} + 4x^{1/3} + 16)}$

$\underbrace{(x^{1/3} - 4)}_{a} \underbrace{(x^{2/3} + 4x^{1/3} + 16)}_{b}$

$\underbrace{(x^{1/3} x^{2/3} + \cancel{4x^{2/3}} + \cancel{16x^{1/3}} - \cancel{4x^{2/3}} - \cancel{16x^{1/3}} - 64)}_{}$

$(x - 64)$

c

$a \cdot b = c$

$b = \frac{c}{a}$

$$\lim_{x\to 1} \frac{x^{2/3}-1}{1-\sqrt{x}}$$

$$= \frac{x^{2/3}-1}{1-x^{1/2}}$$

$$= \frac{(x^{1/3}-1)(x^{1/3}+1)}{1-\sqrt{x}} \cdot \frac{(1+\sqrt{x})}{(1+\sqrt{x})}$$

$$= \frac{(x^{1/3}-1)(x^{1/3}+1)(1+\sqrt{x})}{(1-x)}$$

$$-\frac{(x-1)}{x^{1/3}-1}$$

$$= -\frac{\cancel{(x-1)}\,(x^{1/3}+1)(1+\sqrt{x})}{\cancel{(x-1)}\,(x^{2/3}+x^{1/3}+1)}$$

$$\begin{array}{r|l} & x^{2/3}+x^{1/3}+1 \\ x^{1/3}-1 & x-1 \\ & -(x-x^{2/3}) \\ \hline & x^{2/3}-1 \\ & -(x^{2/3}-x^{1/3}) \\ \hline & x^{1/3}-1 \\ \hline & 0 \end{array}$$

$$= -\frac{(x^{1/3}+1)(1+\sqrt{x})}{(x^{2/3}+x^{1/3}+1)}$$

$$= -\frac{2\cdot 2}{3}$$

$$= -\frac{4}{3}.$$

3b $\lim_{x \to 0} \frac{|3x+4|-|x|-4}{x}$

$\lim_{x \to 0^+} \frac{3x \cancel{+4} - x - \cancel{4}}{x}$

$= \frac{2\cancel{x}}{\cancel{x}}$

$= 2$ //

$\lim_{x \to 0^-} \frac{|3x+4|-|x|-4}{x}$

$\frac{|-3x+4|-|-x|-4}{-x}$

$\frac{|-3x+4| -x-4}{-x}$

$\therefore |-3x+4| \leq |-3x|+|4|$

When $x \to 0^-$

$3x+4 > 0$

$|3x+4|-4 < 0$

$\therefore$ $3x < 0$

$\therefore$

$\frac{-4-|x|+|3x+4|}{x} = \frac{|3x+4|-4}{x} - \frac{|x|}{x}$

$\lim_{x \to 0^-} \left(\frac{|3x+4|-4}{x} - \frac{|x|}{x} \right)$

$\lim_{x \to 0^-} \left(\frac{|3x+4|-4}{x} \right) + 1$

$= \frac{3\cancel{x}}{\cancel{x}} + 1$

$= 4$

when $x = -1$

$\frac{|3(-1)+4|-4}{-1} - \frac{|-1|}{-1}$

$-(1-4) + 1$

$+3+1$

$= 4.$