Section 3.6

Dimensions

Objective

- What is the dimension of a vector space?
- How to compute dimension for a vector space?
- What are some equivalent conditions for a set to be a basis for a vector space?

Number of vectors in a basis

Theorem 3.6.1

Let V be a vector space which has a basis $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}\}$ with k vectors.

1. Any subset of V with more than k vectors is always linearly dependent.
2. Any subset of V with less than k vectors cannot span V.

Recall Thm 3.4.7:
Any subset of $\mathbf{R}^n$ with more than n vectors is linearly dep.

Recall Thm 3.2.7:
Any subset of $\mathbf{R}^n$ with less than n vectors cannot span $\mathbf{R}^n$.

Number of vectors in a basis

Theorem 3.6.1 & Remark 3.6.2

Let V be a vector space which has a basis
$S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ with k vectors.

1. Any subset of V with more than k vectors is always linearly dependent.
2. Any subset of V with less than k vectors cannot span V.

> k : too many vectors to be a basis

< k : too few vectors to be a basis

All bases for a vector space have the same number of vectors

What is dimension of a vector space?

Definition 3.6.3

The dimension of a vector space V
denoted by $\dim(V)$
is the number of vectors in a basis for V.

Recall:
The basis for zero space is defined to be the empty set.
The number of vector in this “basis” is 0.

$\dim(\{\mathbf{0}\}) = 0$

Geometrical meaning of dimension

Example 3.6.4.1-3

1. The dimension of $\mathbf{R}^n$ is n,

   i.e. $\dim(\mathbf{R}^n) = n$.

2. Except $\{\mathbf{0}\}$ and $\mathbf{R}^2$, all subspaces of $\mathbf{R}^2$ are lines through the origin span$\{\mathbf{u}\}$

   they are of dimension 1.

3. Except $\{\mathbf{0}\}$ and $\mathbf{R}^3$, all subspaces of $\mathbf{R}^3$ are either lines through the origin span$\{\mathbf{u}\}$

   they are of dimension 1,

   or planes containing the origin, span$\{\mathbf{u}, \mathbf{v}\}$

   they are of dimension 2.

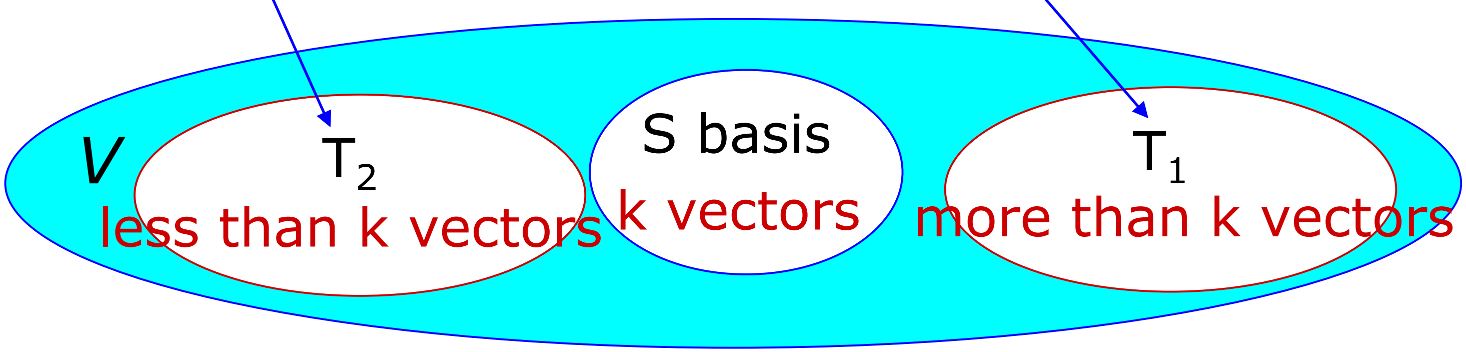
Number of vectors in a basis → dimension of the vector space

Theorem 3.6.1

Let V be a vector space which has a basis $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}\}$ with k vectors. dim V = k

1. Any subset of V with more than k vectors is always linearly dependent.

2. Any subset of V with less than k vectors cannot span V.



Example 3.6.4.4

Not the same as the "dimension" of the vectors in the subspace

Find a basis for and determine the dimension of the subspace $W = \{\boxed{(x, y, z)} \mid y = z\}$ of $\mathbf{R}^3$.

Note: $\dim(W) \neq 3$

Explicit: $(x, y, y) = x(1, 0, 0) + y(0, 1, 1)$

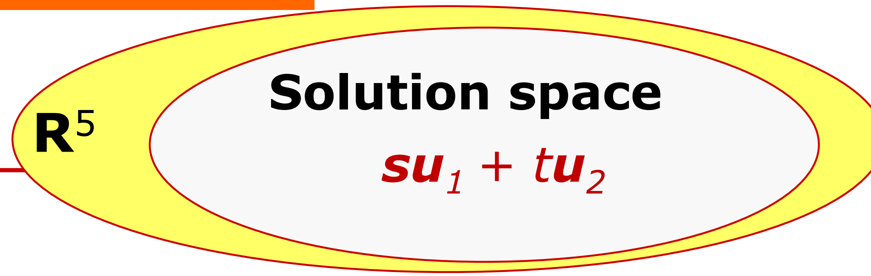
So $W = \text{span}\underbrace{\{(1, 0, 0), (0, 1, 1)\}}_{\text{linearly independent}}$

basis for W : $\{(1, 0, 0), (0, 1, 1)\}$

$\dim(W) = 2$

Dimension of solution space

Example 3.6.6



Find a basis for and determine the dimension of the solution space of the homogeneous system

$$\left\{\begin{array}{rcrcrcrcrcl} 2v & + & 2w & - & x & & & + & z & = & 0 \\ -v & - & w & + & 2x & - & 3y & + & z & = & 0 \\ & & & & x & + & y & + & z & = & 0 \\ v & + & w & - & 2x & & & - & z & = & 0 \end{array}\right. \xrightarrow{\text{general solution}} \begin{pmatrix} v \\ w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -s-t \\ s \\ -t \\ 0 \\ t \end{pmatrix} = s\underbrace{\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{\boldsymbol{u}_1} + t\underbrace{\begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}}_{\boldsymbol{u}_2}$$

linearly indep.

solution space = span$\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$

basis for the solution space = $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$

dim(solution space) = 2

no. of parameters in the general solution

Dimension of solution space

Discussion 3.6.5 (Example)

homogeneous system with 6 variables: u, v, w, x, y, z

Gaussian Elimination

general solution with 4 parameters: s, t, r, q

$$\begin{pmatrix} u \\ v \\ w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s+5t-3r \\ s \\ t \\ 2r-3q \\ r \\ q \end{pmatrix} = s\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 5 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + r\begin{pmatrix} -3 \\ 0 \\ 0 \\ 2 \\ 1 \\ 0 \end{pmatrix} + q\begin{pmatrix} 0 \\ 0 \\ 0 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

$$s\boldsymbol{u}_1 + t\boldsymbol{u}_2 + r\boldsymbol{u}_3 + q\boldsymbol{u}_4$$

linearly independent

the solution space= span$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4\}$

$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4\}$ is a basis for the solution space

dim(solution space) = 4

Discussion 3.6.5

homogeneous system $\longrightarrow$ row echelon form ***R***

number of non-pivot columns in ***R***

||

number of parameters in general solution

||

number of vectors in basis for solution space

||

the dimension of the solution space

Showing a set form a basis (alternative ways)

Theorem 3.6.7

Let V be a vector space of dimension k and S a subset of V.

The following are equivalent:

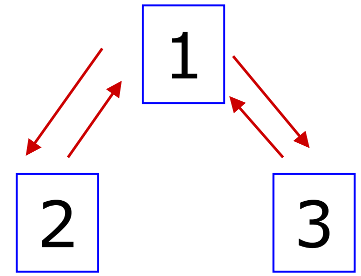
1. S is a basis for V
2. S is linearly independent and $|S| = k$ $= \dim(V)$
3. S spans V and $|S| = k$ $= \dim(V)$

To show S is a basis for V :

S lin. indep S spans V	or	S lin. indep $|S| = \dim V$	or	S spans V $|S| = \dim V$

The proof

1. S is a basis for V.
2. S is lin indep and $|S| = k$.
3. S spans V and $|S| = k$.



Theorem 3.6.7

"$1 \Rightarrow 2$" and "$1 \Rightarrow 3$" is immediate.

$2 \Rightarrow 1$: (prove by contradiction)

Assume that S is not a basis for V

Given S is linearly independent and $|S| = k$.

So $\text{span}(S) \neq V$.

There is a vector $\boldsymbol{u}$ in V and $\boldsymbol{u} \notin \text{span}(S)$.

$\boldsymbol{u}$ is not redundant in span(S)

Let $S' = S \cup \{\boldsymbol{u}\}$ $\Rightarrow$ S' is linearly indep.

$k + 1$ vectors

see Theorem 3.4.10

$\Rightarrow$ S' is linearly dep.

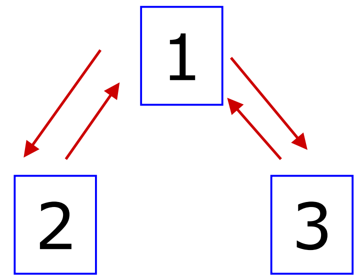
see Theorem 3.6.1.1

Contradiction

So S is a basis for V

The proof

1. S is a basis for V.
2. S is lin indep and $|S| = k$.
3. S spans V and $|S| = k$.



Theorem 3.6.7

$3 \Rightarrow 1$: (prove by contradiction)

Assume S not a basis for V

Given S spans V and $|S| = k$.

S is linearly dependent.

There is a redundant vector $\boldsymbol{v}$ in S.

see Theorem 3.2.12

Let $S'' = S - \{\boldsymbol{v}\}$ $\Rightarrow$ span(S'') = span(S) $= V$

$k - 1$ vectors

Contradiction

$\Rightarrow$ span(S'') $\neq V$

see Theorem 3.6.1.2

So S is a basis for V

Showing a set form a basis (alternative ways)

Example 3.6.8

Show that

$\boldsymbol{u}_1 = (2, 0, -1)$, $\boldsymbol{u}_2 = (4, 0, 7)$ and $\boldsymbol{u}_3 = (-1, 1, 4)$

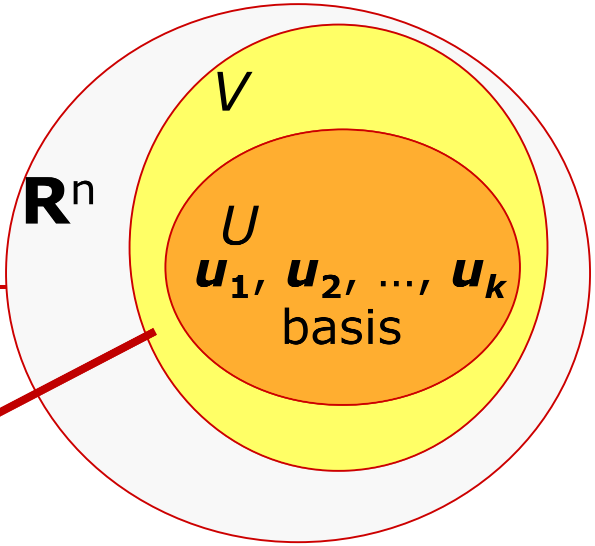
form a basis for $\mathbf{R}^3$.

Since dim $\mathbf{R}^3 = 3$,
we only need to show the set of 3 vectors is
either linear independent or spans $\mathbf{R}^3$.

If we don't know the dimension of a vector space V,
to show a set is a basis for V,
we still need to check
the set is both linear independent and spans V.

Dimensions give the "size" of subspaces of $\mathbf{R}^n$

Theorem 3.6.9



Let U and V be subspaces of $\mathbf{R}^n$

We say: U is a subspace of V.

(i) If $U \subseteq V$, then $\dim(U) \leq \dim(V)$

(ii) If $U \subseteq V$ and $U \neq V$, then $\dim(U) < \dim(V)$

For (i), $\dim(U) = k$

$\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ are k lin. indep. vectors in V

So $k \leq \dim(V)$

For (ii), suppose $\dim(U) = \dim(V)$

Then $\dim(V) = k$

contradiction

So $V = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\} = U$.

Dimensions give the "size" of subspaces of $\mathbf{R}^n$

Example 3.6.10

Given V a plane in $\mathbf{R}^3$ containing the origin.

Suppose U is a subspace of V such that $U \neq V$.
What can we say about U?

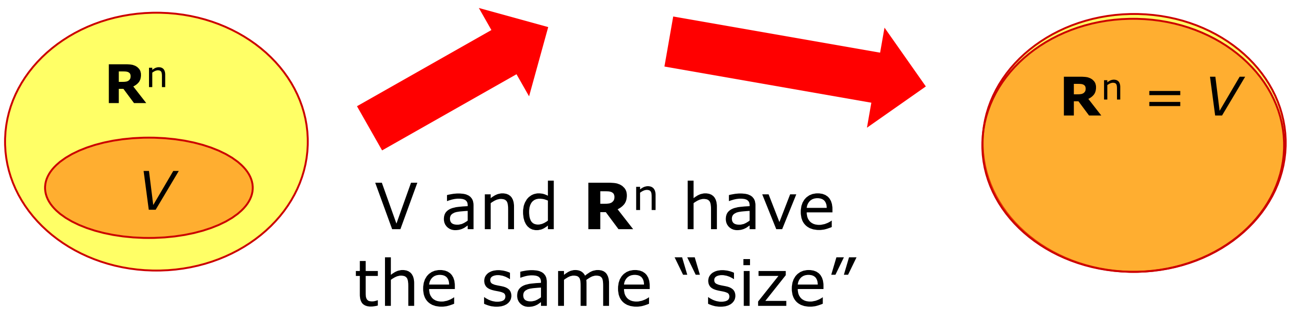
V is of dimension 2.
By Theorem 3.6.9, $\dim(U) < 2$.
So
either $\dim(U) = 0 \quad \Leftrightarrow U = \{\mathbf{0}\}$
or $\dim(U) = 1 \quad \Leftrightarrow U =$ a line through the origin

True or False

Let U and V be subspaces of $\mathbf{R}^n$

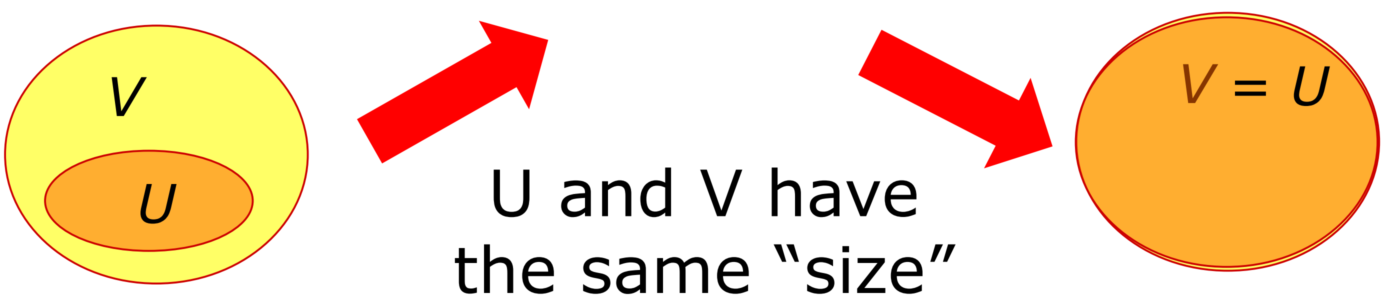
No subspace of $\mathbf{R}^n$ has dimension n, except $\mathbf{R}^n$ itself.

A. If $\dim(V) = n$, then $V = \mathbf{R}^n$ True



Theorem 3.6.9.2

B. If $U \subseteq V$ and $\dim(U) = \dim(V)$, then $U = V$ True



A very^3 important theorem (revisited)

Theorem 3.6.11

A is an n x n matrix.

The following statements are equivalent:

1. ***A*** is invertible.
2. The linear system $\boldsymbol{Ax} = \mathbf{0}$ has only the trivial solution.
3. The reduced row-echelon form of ***A*** is an identity matrix.
4. ***A*** can be expressed as a product of elementary matrices.
5. $\det(\boldsymbol{A}) \neq 0$.
6. The rows of ***A*** form a basis for $\mathbf{R}^n$.
7. The columns of ***A*** form a basis for $\mathbf{R}^n$.

1. $\boldsymbol{A}$ is invertible

2. $\boldsymbol{Ax} = \mathbf{0}$ has only trivial solution

7. The columns of $\boldsymbol{A}$ form a basis for $\mathbf{R}^n$

Example

Suppose we know $\begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$ is invertible.

Then we know that the linear system

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$ has only trivial solution

Write the linear system in vector equation form:

$$x\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + y\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$ has only zero coefficients

$x = y = z = 0$

We conclude that $\left\{\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\}$ is linearly independent

hence form a basis for $\mathbf{R}^3$

1. $\boldsymbol{A}$ is invertible
5. det $\boldsymbol{A} \neq 0$
7. The columns of $\boldsymbol{A}$ form a basis for $\mathbf{R}^n$
6. The rows of $\boldsymbol{A}$ form a basis for $\mathbf{R}^n$

Example

Suppose we know $\begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$ is invertible.

Then we know that the determinant $\begin{vmatrix} 1 & 2 & 1 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{vmatrix} \neq 0$

Then the transpose determinant $\begin{vmatrix} 1 & 3 & 2 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} \neq 0$

So $\begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ is invertible.

So the columns $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \right\}$ form a basis for $\mathbf{R}^3$

So the rows $\{(1\ 2\ 1), (3\ 1\ 0), (2\ 0\ 1)\}$ form a basis for $\mathbf{R}^3$

Alternative method to check basis for $\mathbf{R}^n$

Example 3.6.12 (Determinant method)

$\boldsymbol{u_1} = (1, 1, 1)$, $\boldsymbol{u_2} = (-1, 1, 2)$, $\boldsymbol{u_3} = (1, 0, 1)$

Is $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ a basis for $\mathbf{R}^3$? YES

$$\begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \\ 1 & 0 & 1 \end{vmatrix} = 3 \neq 0$$

$\boldsymbol{u_1} = (1, 1, 1, 1)$, $\boldsymbol{u_2} = (1, -1, 1, -1)$,
$\boldsymbol{u_3} = (0, 1, -1, 0)$, $\boldsymbol{u_4} = (2, 1, 1, 0)$

Is $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\}$ a basis for $\mathbf{R}^4$? NO

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 2 & 1 & 1 & 0 \end{vmatrix} = 0$$

Cannot use this method to check basis for subspaces of $\mathbf{R}^n$

Section 3.7

Transition Matrices

Objective

- What is a transition matrix?
- How to compute transition matrices?
- What is the relation between coordinate vectors w.r.t. different bases?

From one basis to another

Example 3.7.4.1

$S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ basis for $\mathbf{R}^3$

$\boldsymbol{u}_1 = (1, 0, -1)$, $\boldsymbol{u}_2 = (0, -1, 0)$, $\boldsymbol{u}_3 = (1, 0, 2)$.

$T = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ basis for $\mathbf{R}^3$

$\boldsymbol{v}_1 = (1, 1, 1)$, $\boldsymbol{v}_2 = (1, 1, 0)$, $\boldsymbol{v}_3 = (-1, 0, 0)$.

Given $(\boldsymbol{w})_S = (2, -1, 2)$. Find $(\boldsymbol{w})_T$.

2.3 meter $\longrightarrow$ 7.5 feet

$$\begin{tikzcd} (\boldsymbol{w})_S \arrow[r] \arrow[rr, bend right] & \boldsymbol{w} \arrow[r] & (\boldsymbol{w})_T \end{tikzcd}$$

Is there a direct method?

Coordinate vector notation: $(\boldsymbol{v})_S$ & $[\boldsymbol{v}]_S$

Notation 3.7.1

$S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}\}$: a basis for a vector space V

$\boldsymbol{v}$: a vector in V

Write $\boldsymbol{v} = c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k}$

Then $(\boldsymbol{v})_S = (c_1, c_2, \ldots, c_k)$ row form of coordinate vector

$$[\boldsymbol{v}]_S = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}$$

column form of coordinate vector

We need to pre-multiply the coordinate-vector by a $k \times k$ matrix

From one basis to another

Discussion 3.7.2

$$[\boldsymbol{w}]_S = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} \qquad [\boldsymbol{w}]_T = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_k \end{pmatrix}$$

$S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ and $T = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ two bases for a vector space V.

Take a vector $\boldsymbol{w}$ in V

Relation between $[\boldsymbol{w}]_S$ and $[\boldsymbol{w}]_T$?

$\boldsymbol{w}$ in terms of $\boldsymbol{u}_i$

$\boldsymbol{w}$ in terms of $\boldsymbol{v}_i$

We will show that

does not depend on $\boldsymbol{w}$

$[\boldsymbol{w}]_T = \boldsymbol{P}\,[\boldsymbol{w}]_S$ for some fixed k x k matrix $\boldsymbol{P}$

transition matrix

Finding transition matrix from S to T

Definition 3.7.3

Read Discussion 3.7.2 to see why it works

$S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ and $T = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$

two bases for a vector space V.

1. Express each $\boldsymbol{u}_i$ as linear combination of $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$
2. Form the (column) coordinate vectors w.r.t. T

$$[\boldsymbol{u}_1]_T = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{k1} \end{pmatrix} \quad [\boldsymbol{u}_2]_T = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{k2} \end{pmatrix} \quad \cdots \quad [\boldsymbol{u}_k]_T = \begin{pmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{kk} \end{pmatrix}$$

3. Form the matrix $\mathbf{P} = ([\boldsymbol{u}_1]_T \ [\boldsymbol{u}_2]_T \ \ldots \ [\boldsymbol{u}_k]_T)$

$$P = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{pmatrix}$$

transition matrix from S to T

4. $\mathbf{P}\,[\boldsymbol{w}]_S = [\boldsymbol{w}]_T$ for any vector $\boldsymbol{w}$ in V.

From one basis to another

Example 3.7.4.1

$S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ basis for $\mathbf{R}^3$

$\boldsymbol{u_1} = (1, 0, -1)$, $\boldsymbol{u_2} = (0, -1, 0)$, $\boldsymbol{u_3} = (1, 0, 2)$.

$T = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ basis for $\mathbf{R}^3$

$\boldsymbol{v_1} = (1, 1, 1)$, $\boldsymbol{v_2} = (1, 1, 0)$, $\boldsymbol{v_3} = (-1, 0, 0)$.

(a) Find the transition matrix from S to T.

$P = \left([\boldsymbol{u}_1]_T \quad [\boldsymbol{u}_2]_T \quad [\boldsymbol{u}_3]_T\right)$

(b) $\boldsymbol{w}$ a vector in $\mathbf{R}^3$ with $(\boldsymbol{w})_S = (2, -1, 2)$.

Find $(\boldsymbol{w})_T$.

$[\boldsymbol{w}]_T = P\,[\boldsymbol{w}]_S$

Finding transition matrix

$S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$
$T = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$

Example 3.7.4.1(a)

$$\boldsymbol{u_1} = a_{11}\boldsymbol{v_1} + a_{21}\boldsymbol{v_2} + a_{31}\boldsymbol{v_3}$$
$$\boldsymbol{u_2} = a_{12}\boldsymbol{v_1} + a_{22}\boldsymbol{v_2} + a_{32}\boldsymbol{v_3}$$
$$\boldsymbol{u_3} = a_{13}\boldsymbol{v_1} + a_{23}\boldsymbol{v_2} + a_{33}\boldsymbol{v_3}$$

find $a_{11}, a_{21}, \ldots, a_{33}$

Convert to three linear systems:

$$\begin{cases} a_{11} + a_{21} - a_{31} = 1 \\ a_{11} + a_{21} = 0 \\ a_{11} = -1 \end{cases} \qquad \begin{cases} a_{12} + a_{22} - a_{32} = 0 \\ a_{12} + a_{22} = -1 \\ a_{12} = 0 \end{cases} \qquad \begin{cases} a_{13} + a_{23} - a_{33} = 1 \\ a_{13} + a_{23} = 0 \\ a_{13} = 2 \end{cases}$$

$$\begin{array}{c} \left(\begin{array}{ccc|c|c|c} 1 & 1 & -1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 & 0 & 2 \end{array}\right) \\ \boldsymbol{v}_1 \quad \boldsymbol{v}_2 \quad \boldsymbol{v}_3 \quad \boldsymbol{u}_1 \quad \boldsymbol{u}_2 \quad \boldsymbol{u}_3 \end{array} \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \begin{array}{c} \left(\begin{array}{ccc|c|c|c} 1 & 0 & 0 & -1 & 0 & 2 \\ 0 & 1 & 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & -1 & -1 & -1 \end{array}\right) \\ [\boldsymbol{u}_1]_T [\boldsymbol{u}_2]_T [\boldsymbol{u}_3]_T \end{array}$$

T

S transition matrix from S to T

Finding $(\boldsymbol{w})_T$ form $(\boldsymbol{w})_S$

Example 3.7.4.1(b)

$(\boldsymbol{w})_S = (2, -1, 2)$

$$[\boldsymbol{w}]_T = (\text{Transition matrix from S to T})[\boldsymbol{w}]_S$$

$$\boldsymbol{P} = \begin{pmatrix} -1 & 0 & 2 \\ 1 & -1 & -2 \\ -1 & -1 & -1 \end{pmatrix}$$

$$[\boldsymbol{w}]_T = \boldsymbol{P}[\boldsymbol{w}]_S = \begin{pmatrix} -1 & 0 & 2 \\ 1 & -1 & -2 \\ -1 & -1 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix}$$

So $(\boldsymbol{w})_T = (2, -1, -3)$.

From S to T and from T to S

Example 3.7.4.2

$\boldsymbol{P}\,[\boldsymbol{w}]_S = [\boldsymbol{w}]_T$ for any vector $\boldsymbol{w}$

$\boldsymbol{Q}\,[\boldsymbol{w}]_T = [\boldsymbol{w}]_S$ for any vector $\boldsymbol{w}$

$S = \{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ $\quad \boldsymbol{u}_1 = (1, 1),\ \boldsymbol{u}_2 = (1, -1).$

$T = \{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ $\quad \boldsymbol{v}_1 = (1, 0),\ \boldsymbol{v}_2 = (1, 1).$

two bases for $\mathbf{R}^2$

transition matrix from S to T

$$\begin{cases} \mathbf{u}_1 = 0\mathbf{v}_1 + \mathbf{v}_2 \\ \mathbf{u}_2 = 2\mathbf{v}_1 - \mathbf{v}_2 \end{cases}$$

$$[\mathbf{u}_1]_T = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad [\mathbf{u}_2]_T = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\mathbf{P} = \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}$$

transition matrix from T to S

$$\begin{cases} \mathbf{v}_1 = \frac{1}{2}\mathbf{u}_1 + \frac{1}{2}\mathbf{u}_2 \\ \mathbf{v}_2 = \mathbf{u}_1 + 0\mathbf{u}_2 \end{cases}$$

$$[\mathbf{v}_1]_S = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \quad [\mathbf{v}_2]_S = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\boldsymbol{Q} = \begin{pmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{pmatrix}$$

$\mathbf{P} \longleftrightarrow \boldsymbol{Q}$ inverse of each other

The inverse of transition matrix

Theorem 3.7.5

S and T : two bases of a vector space

$\boldsymbol{P}$: the transition matrix from S to T.

1. $\boldsymbol{P}$ is invertible.
2. $\boldsymbol{P}^{-1}$ is the transition matrix from T to S.

$S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$, $T = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ bases

$\mathbf{P} = ([\boldsymbol{u}_1]_T \; [\boldsymbol{u}_2]_T \; \ldots \; [\boldsymbol{u}_k]_T)$ $\Rightarrow$ $\boldsymbol{P}$ is invertible

$[\boldsymbol{u}_1]_T \; [\boldsymbol{u}_2]_T \; \ldots \; [\boldsymbol{u}_k]_T$ are linearly independent

Let $\boldsymbol{Q}$ be the transition matrix from T to S.

$\mathbf{Q} = ([\boldsymbol{v}_1]_S \; [\boldsymbol{v}_2]_S \; \ldots \; [\boldsymbol{v}_k]_S)$

To show $\boldsymbol{QP} = \boldsymbol{I}$

The proof: two observations

Theorem 3.7.5

Let $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}\}$ basis

Obs. 1

$$[\boldsymbol{u}_1]_S = \begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix} \quad [\boldsymbol{u}_2]_S = \begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix} \quad \cdots \quad [\boldsymbol{u}_k]_S = \begin{pmatrix}0\\0\\0\\\vdots\\1\end{pmatrix}$$

standard basis vectors

$$\boldsymbol{u_1} = 1\boldsymbol{u_1} + 0\boldsymbol{u_2} + \ldots + 0\boldsymbol{u_k}$$

Obs. 2

any mxn matrix ***A***

$[\boldsymbol{u}_\mathbf{i}]_S$

$$\begin{pmatrix} a_{11} & \cdots & a_{1,i-1} & a_{1i} & a_{1,i+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2,i-1} & a_{2,i} & a_{2,i+1} & \cdots & a_{2n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{m,i-1} & a_{m,i} & a_{m,i+1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix}0\\\vdots\\0\\1\\0\\\vdots\\0\end{pmatrix} = \begin{pmatrix}a_{1j}\\a_{2j}\\0\\\vdots\\a_{mj}\end{pmatrix}$$

i^{th} column of ***A***

i^{th} coordinate

The proof

$$[\boldsymbol{u}_1]_S = \begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix} \quad [\boldsymbol{u}_2]_S = \begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix} \quad \cdots \quad [\boldsymbol{u}_k]_S = \begin{pmatrix}0\\0\\0\\\vdots\\1\end{pmatrix}$$

Theorem 3.7.5

To show $\boldsymbol{QP} = \boldsymbol{I}$

Examine the i^{th} column of $\boldsymbol{QP}$ for $i = 1, 2, \ldots, k$

i^{th} column of $\boldsymbol{A} = \boldsymbol{A}\,[\boldsymbol{u}_i]_S$

$$i^{th} \text{ column of } \boldsymbol{QP} = \boldsymbol{QP}\,[\boldsymbol{u}_i]_S = \boldsymbol{Q}[\boldsymbol{u}_i]_T = [\boldsymbol{u}_i]_S = \begin{pmatrix}0\\0\\1\\\vdots\\0\end{pmatrix}$$

$\boldsymbol{P}$: transition matrix from S to T

$\boldsymbol{Q}$: transition matrix from T to S

$$\boldsymbol{QP} = \begin{pmatrix}1 & 0 & & 0\\0 & 1 & & 0\\0 & 0 & \ddots & 0\\\vdots & \vdots & & \vdots\\0 & 0 & & 1\end{pmatrix} = \boldsymbol{I}$$

So $\boldsymbol{P}$ is invertible and $\boldsymbol{P}^{-1} = \boldsymbol{Q}$