NATIONAL UNIVERSITY OF SINGAPORE

SEMESTER 1, 2021/2022

MA1521 Calculus for Computing **Tutorial 9**

1. Find the limits:

(a) $\displaystyle\lim_{(x,y)\to(2,-3)} \left(\frac{1}{x}+\frac{1}{y}\right)^2$

(Thomas' Calculus (14$^{\text{th}}$ edition), p. 773, Problem 4)

(b) $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ x\neq y}} \frac{x-y+2\sqrt{x}-2\sqrt{y}}{\sqrt{x}-\sqrt{y}}$

(Thomas' Calculus (14$^{\text{th}}$ edition), p. 774, Problem 17)

(c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{1-\cos(xy)}{xy}$

(Thomas' Calculus (14$^{\text{th}}$ edition), p. 774, Problem 22)

2. Show that the following limits do not exist.

(a) $\displaystyle\lim_{(x,y)\to(1,-1)} \frac{xy+1}{x^2-y^2}$

(Thomas' Calculus (14$^{\text{th}}$ edition), p. 774, Problem 50)

(b) $\displaystyle\lim_{(x,y)\to(1,1)} \frac{\tan y - y\tan x}{y-x}$

(Thomas' Calculus (14$^{\text{th}}$ edition), p. 774, Problem 54)

3. The inequality
$$1-\frac{x^2y^2}{3}<\frac{\tan^{-1}xy}{xy}<1$$
holds for (x, y) "close to $(0, 0)$". What can you say about $\lim_{(x,y)\to(0,0)} \frac{\tan^{-1}xy}{xy}$?

(Thomas' Calculus (14$^{\text{th}}$ edition), p. 775, Problem 59)

4. Find $\partial f/\partial x$ and $\partial f/\partial y$ for the following functions f.
   (a) $f(x, y) = (xy - 1)^2$

   (Thomas' Calculus (14$^{\text{th}}$ edition), p. 785, Problem 5)

   (b) $f(x, y) = e^{xy} \ln y$

   (Thomas' Calculus (14$^{\text{th}}$ edition), p. 785, Problem 16)

   (c) $f(x, y) = x^y$

   (Thomas' Calculus (14$^{\text{th}}$ edition), p. 785, Problem 19)

5. For each of the following functions, determine f_{xy}.
   (a) $f(x, y) = y + x^2y + 4y^3 - \ln(y^2 + 1)$

   (Thomas' Calculus (14$^{\text{th}}$ edition), p. 786, Problem 61(d))

   (b) $f(x, y) = x \ln(xy)$

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6. Show that the function $f(x, y) = \ln(x^2 + y^2)$ satisfies the two-dimensional Laplace equation
$$f_{xx} + f_{yy} = 0.$$

(Thomas' Calculus (14$^{\text{th}}$ edition), p. 787, Problem 86)

7. Find the absolute maxima and minima of the functions on the given domains.
   (a) $D(x, y) = x^2 - xy + y^2 + 1$ on the closed triangular plate in the first quadrant bounded by the lines $x = 0$, $y = 4$ and $y = x$.

   (Thomas' Calculus (14$^{\text{th}}$ edition), p. 823, Problem 32)

(b) Find two numbers a and b with $a \leq b$ such that

$$\int_a^b (6 - x - x^2)dx$$

has its largest values.

(Thomas' Calculus (14$^{\text{th}}$ edition), p. 823, Problem 39)

8. Let

$$f(x, y) = \begin{cases} \dfrac{xy^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$$

Show that $f_x(0, 0)$ and $f_y(0, 0)$ exist, but f is not differentiable at $(0, 0)$.

(Thomas' Calculus (14$^{\text{th}}$ edition), p. 788, Problem 101)

9. Find the extreme values of $f(x, y) = xy$ subject to the constraint $g(x, y) = x^2 + y^2 - 10 = 0$.

(Thomas' Calculus (14$^{\text{th}}$ edition), p. 832, Problem 2)

10. (a) Find the extreme values of $f(x, y) = (xy)^2$ subject to the constraint that $g(x, y) = x^2 + y^2 - r^2 = 0$ and deduce that for positive real numbers a, b,

$$\sqrt{ab} \leq \frac{a + b}{2}.$$

(b) (Optional) The Lagrange multiplier method works for functions of n variables. Find the extreme values of

$$f(x_1, x_2, \cdots, x_n) = (x_1 x_2 \cdots x_n)^n$$

subject to the constraint that

$$g(x_1, x_2, \cdots, x_n) = x_1^n + x_2^n + \cdots + x_n^n - r^n = 0$$

for some $r > 0$ and deduce that for positive real numbers $a_1, a_2, \cdots, a_n$,

$$(a_1 a_2 \cdots a_n)^{1/n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

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