EXERCISES FOR CHAPTER 5: ORTHOGONALITY

Question 5.1 to Question 5.20 are exercises for Sections 5.1 and 5.2.

1. For each of the following, find $\|\boldsymbol{u}\|$, $\|\boldsymbol{v}\|$, $d(\boldsymbol{u},\boldsymbol{v})$, $\boldsymbol{u}\cdot\boldsymbol{v}$ and the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$.

(a) $\boldsymbol{u}=(2,3)$ and $\boldsymbol{v}=(1,1)$. $\sqrt{u_1^2+u_2^2}$ $\sqrt{(u_1-v_1)^2+(u_2-v_2)^2}$ $u_1v_1+u_2v_2+\dots$ $\cos^{-1}\left(\frac{u\cdot v}{\|u\|\|v\|}\right)$ //

(b) $\boldsymbol{u}=(1,-1)$ and $\boldsymbol{v}=(-1,3)$.

(c) $\boldsymbol{u}=(1,2,3)$ and $\boldsymbol{v}=(0,-3,2)$.

(d) $\boldsymbol{u}=(1,-1,1,-1)$ and $\boldsymbol{v}=(2,1,1,2)$.

2. Consider a triangle in $\mathbb{R}^4$ with vertices $A=(1,1,0,0)$, $B=(1,-1,0,0)$ and $C=(2,0,0,1)$.

(a) Find the lengths of the sides of the triangle. $AB=\|A-B\|$ $AC=\|A-C\|$ $BC=\|B-C\|$.

(b) Find the angle between AB and AC. let $u=A-B$ and $v=A-C$, $\cos^{-1}\frac{u\cdot v}{\|u\|\|v\|}$.

(c) Verify the cosine rule: $2|AB||AC|\cos\theta=|AB|^2+|AC|^2-|BC|^2$, where θ is the angle between AB and AC. use a and b.

3. Complete the proof of Theorem 5.1.5:

Let $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$ be vectors in $\mathbb{R}^n$ and c a scalar. Show that

(a) $\boldsymbol{u}\cdot\boldsymbol{v}=\boldsymbol{v}\cdot\boldsymbol{u}$; $u\cdot v=u_1v_1+u_2v_2$ where $v\cdot u=v_1u_1+v_2u_2$ ∴ same.

(b) $(\boldsymbol{u}+\boldsymbol{v})\cdot\boldsymbol{w}=\boldsymbol{u}\cdot\boldsymbol{w}+\boldsymbol{v}\cdot\boldsymbol{w}$ and $\boldsymbol{w}\cdot(\boldsymbol{u}+\boldsymbol{v})=\boldsymbol{w}\cdot\boldsymbol{u}+\boldsymbol{w}\cdot\boldsymbol{v}$;

(c) $(c\boldsymbol{u})\cdot\boldsymbol{v}=\boldsymbol{u}\cdot(c\boldsymbol{v})=c(\boldsymbol{u}\cdot\boldsymbol{v})$; trivial.

(d) $\|c\boldsymbol{u}\|=|c|\,\|\boldsymbol{u}\|$.

4. Let $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ be any three vectors in $\mathbb{R}^n$. Prove the following inequalities.

(a) $|\boldsymbol{u}\cdot\boldsymbol{v}|\le\|\boldsymbol{u}\|\,\|\boldsymbol{v}\|$ (the *Cauchy-Schwarz Inequality*). $u\cdot v=u_1v_1+u_2v_2$ $\|u\|=\sqrt{u_1^2+u_2^2}$

(b) $\|\boldsymbol{u}+\boldsymbol{v}\|\le\|\boldsymbol{u}\|+\|\boldsymbol{v}\|$ (the *Triangle Inequality*). $\|v\|=\sqrt{v_1^2+v_2^2}$

(c) $d(\boldsymbol{u},\boldsymbol{w})\le d(\boldsymbol{u},\boldsymbol{v})+d(\boldsymbol{v},\boldsymbol{w})$.

Interpret the result in Part (b) geometrically in $\mathbb{R}^2$. $(u\cdot v)^2=(u_1^2+u_2^2)(u^2+v_2^2)$

5. Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be any two vectors in $\mathbb{R}^n$. Prove the following equalities.

(a) $\|\boldsymbol{u}+\boldsymbol{v}\|^2+\|\boldsymbol{u}-\boldsymbol{v}\|^2=2\|\boldsymbol{u}\|^2+2\|\boldsymbol{v}\|^2$.

(b) $\boldsymbol{u}\cdot\boldsymbol{v}=\frac{1}{4}\|\boldsymbol{u}+\boldsymbol{v}\|^2-\frac{1}{4}\|\boldsymbol{u}-\boldsymbol{v}\|^2$.

Interpret the result in Part (a) geometrically in $\mathbb{R}^2$.

6. For each of the following vectors, find all vectors that are orthogonal to it.

(a) $(1,1)$, $\cos 90^\circ=\frac{(1,1)\cdot v}{\|(1,1)\|\|v\|}\to v_1+v_2=0$. $(0,0)$ or $(1,-1)$ or $(-1,1)$ on a graph.
(b) $(1,0,3)$, $v_1+3v_3=0$. $3v_3=-v_1$ $(0,0)$, $(3,1)$, $(-3,-1)$
(c) $(1,-1,1,-1)$. $v_1-v_2+v_3-v_4=0$ $v_1-v_2=v_4-v_3$ //

Interpret the results in Part (a) and (b) geometrically.

7. Let W be a subspace of $\mathbb{R}^n$. Define $W^\perp=\{\boldsymbol{u}\in\mathbb{R}^n \mid \boldsymbol{u}\text{ is orthogonal to }W\}$.

(a) Let $W=\operatorname{span}\{(1,0,1,1),(1,-1,0,2),(1,2,3,-1)\}$. Find $W^\perp$. RREF $\left(\begin{array}{c|c} u\cdot v_1 & 0\\ u\cdot v_2 & 0\\ u\cdot v_3 & 0\end{array}\right)\to u?$

(b) Show that $W^\perp$ is a subspace of $\mathbb{R}^n$. (*Hint*: Show that $W^\perp$ is a solution set of a homogeneous system of linear equations.) shown by (a).

8. Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$, where $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$ are vectors in $\mathbb{R}^3$, and let $T = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$, where

$$\boldsymbol{v}_1 = \tfrac{3}{5}\boldsymbol{u}_2 + \tfrac{4}{5}\boldsymbol{u}_3, \quad \boldsymbol{v}_2 = \tfrac{4}{5}\boldsymbol{u}_2 - \tfrac{3}{5}\boldsymbol{u}_3 \quad \text{and} \quad \boldsymbol{v}_3 = \boldsymbol{u}_1.$$

(a) Show that $\text{span}(S) = \text{span}(T)$. general solu ?.

(b) If S is orthogonal, show that T is also orthogonal.

9. Let $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_n\}$ be an orthogonal set of vectors in a vector space. Show that

$\|u_1+u_2\|^2 = \|u_1\|^2 + \|u_2\|^2$ pythagoras theorem?

$$\|\boldsymbol{u}_1 + \boldsymbol{u}_2 + \cdots + \boldsymbol{u}_n\|^2 = \|\boldsymbol{u}_1\|^2 + \|\boldsymbol{u}_2\|^2 + \cdots + \|\boldsymbol{u}_n\|^2.$$

$\forall i,j \in \mathbb{Z}^+, i \neq j, i,j \leq n, u_i \cdot u_j = 0$ ∴ eqn valid.

For $n = 2$, interpret the result geometrically in $\mathbb{R}^2$.

10. Let $\boldsymbol{u}_1 = (1,2,2,-1)$, $\boldsymbol{u}_2 = (1,1,-1,1)$, $\boldsymbol{u}_3 = (-1,1,-1,-1)$, $\boldsymbol{u}_4 = (-2,1,1,2)$.

(a) Show that $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4\}$ is an orthogonal set. $u_1\cdot u_2 = 0$. $u_1 \cdot u_3 = 0$ $u_1\, u_4 = 0$. $u_2 \cdot u_4 = 0$ $u_2\cdot u_3 = 0$ $u_3 \cdot u_4 = 0$.

(b) Obtain an orthonormal set S' by normalizing $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4$. $\left\{\frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2\|}, \frac{u_3}{\|u_3\|}, \frac{u_4}{\|u_4\|}\right\}$

(c) Is S' an orthonormal basis for $\mathbb{R}^4$? yes. orthogonal → linear indep.

$(w)_S = \left(\frac{w\cdot u_1}{\|u_1\|^2}, \frac{w\cdot u_2}{\|u_2\|^2}, \frac{w\cdot u_3}{\|u_3\|^2}, \frac{w\cdot u_4}{\|u_4\|^2}\right)$

$(w)_{S'} = (w\cdot v_1, w\cdot v_2, w\cdot v_3, w \cdot v_4)$

(d) If $\boldsymbol{w} = (0,1,2,3)$, find $(\boldsymbol{w})_S$ and $(\boldsymbol{w})_{S'}$.

(e) Let $V = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$. Find all vectors that are orthogonal to V. $\begin{pmatrix} v\cdot u_1 = 0 \\ v \cdot u_2 = 0 \\ v\cdot u_3 = 0 \end{pmatrix} \to$ RREF.

(f) Find the projection of $\boldsymbol{w}$ onto V. $p = \frac{w\cdot u_1}{\|u_1\|^2} + \frac{w\cdot u_2}{\|u_2\|^2} + \frac{w \cdot u_3}{\|u_3\|^2}$.

11. Let $\boldsymbol{u}_1 = (-2,-4,1)$, $\boldsymbol{u}_2 = (3,-1,2)$ and $\boldsymbol{u}_3 = (1,-1,-2)$. $u_1\cdot u_2 = 0$. $u_1 \cdot u_3 = 0$.

(a) Show that $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ is an orthogonal basis for $\mathbb{R}^3$. → is orthogonal? ⇒ $u_2\cdot u_3 = 0$

(b) Let $V = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ and $W = \text{span}\{\boldsymbol{u}_3\}$. Write each of the following vectors as a sum of two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ such that $\boldsymbol{v} \in V$ and $\boldsymbol{w} \in W$:

(i) $(0,0,1)$. $(au_1 + bu_2) + cu_3$. RREF (ii) $(1,1,0)$.

12. Use Gram-Schmidt Process to transform each of the following bases for $\mathbb{R}^3$ to an orthonormal basis.

(a) $\{(1,0,1), (0,1,2), (2,1,0)\}$.

(b) $\{(1,1,1), (1,-1,1), (1,1,-1)\}$.

13. Use Gram-Schmidt Process to transform the following basis for $\mathbb{R}^4$ to an orthonormal basis:

$$\{(2,1,0,0), (-1,0,0,1), (2,0,-1,1), (0,0,1,1)\}.$$

$v_1 = u_1$
$v_2 = u_2 - \frac{u_2 \cdot v_1}{\|v_1\|^2} v_1$
$v_3 = u_3 - \frac{u_3\cdot v_1}{\|v_1\|^2}\cdot v_1 - \frac{u_3 \cdot v_2}{\|v_2\|^2}\cdot v_2$

14. (a) Find an orthonormal basis for the solution space of the equation $x + y - z = 0$.

gen soln → gram schmidt → normalisation

$p = (w\cdot v_1)v_1 + (w\cdot v_2)v_2 + \dots + (w\cdot v_n)v_n$

(b) Find the projection of $(1,0,-1)$ onto the plane $x + y - z = 0$.

(c) Extend the set obtained in Part (a) to an orthonormal basis for $\mathbb{R}^3$.

15. Let $W = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4, \boldsymbol{u}_5\}$ be a subspace of $\mathbb{R}^4$ where

$$\boldsymbol{u}_1 = (1,1,0,0), \ \boldsymbol{u}_2 = (1,0,0,1), \ \boldsymbol{u}_3 = (1,0,1,0), \ \boldsymbol{u}_4 = (3,1,1,1), \ \boldsymbol{u}_5 = (-1,-1,1,-1).$$

(a) Show that $\{\boldsymbol{u}_1, \boldsymbol{u}_3, \boldsymbol{u}_4\}$ is a basis for W. $(u_1\ u_2\ u_3\ u_4\ u_5) \xrightarrow{REF}$

(b) Apply the Gram-Schmidt Process to transform $\{\boldsymbol{u}_1, \boldsymbol{u}_3, \boldsymbol{u}_4\}$ into an orthonormal basis for W.

(c) Extend the set obtained in Part (b) to an orthonormal basis for $\mathbb{R}^4$. extend first → normalise the basis by $\frac{u_i}{\|u_i\|}$.

16. Let $V = \text{span}\{(1,1,1), (1,a,a)\}$, where a is a real number.

(a) Find an orthonormal basis for V. gram smith process → normalise.

(b) Compute the projection of $(5,3,1)$ onto V. $p = (w.v_1)v_1 + (w.v_2)v_2.$

17. Let $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, $\boldsymbol{u}_1 = (1,1,1,0)^{\mathrm{T}}$, $\boldsymbol{u}_2 = (1,1,1,1)^{\mathrm{T}}$ and $\boldsymbol{u}_3 = (0,0,1,1)^{\mathrm{T}}$.

(a) Use the Gram-Schmidt Process to transform]$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ into an orthonormal basis $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3\}$ for the column space of $\boldsymbol{A}$. (Do not change the order of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$ when applying the Gram-Schmidt Process.)

(b) Write each of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$ as a linear combination of $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$. REF .

(c) Hence, or otherwise, write $\boldsymbol{A} = \boldsymbol{QR}$ where $\boldsymbol{Q}$ is a 4×3 matrix with orthonormal columns and $\boldsymbol{R}$ is a 3×3 upper triangular matrix with positive entries along its diagonal.

(The process of writing a matrix in the form described in Part (c) is called the *QR factorization*. It is widely used in computer algorithms for various computations concerning matrices.)

18. Prove the uniqueness of (orthogonal) projection:

Let V be a subspace of $\mathbb{R}^n$ and $\boldsymbol{u}$ a vector in $\mathbb{R}^n$. Show that $\boldsymbol{u}$ can be written uniquely as $\boldsymbol{u} = \boldsymbol{n} + \boldsymbol{p}$ such that $\boldsymbol{n}$ is a vector orthogonal to V and $\boldsymbol{p}$ is a vector in V.

(*Hint*: We need to prove that if $\boldsymbol{u} = \boldsymbol{n}_1 + \boldsymbol{p}_1 = \boldsymbol{n}_2 + \boldsymbol{p}_2$, where $\boldsymbol{n}_1, \boldsymbol{n}_2$ are orthogonal to V and $\boldsymbol{p}_1, \boldsymbol{p}_2 \in V$, then $\boldsymbol{n}_1 = \boldsymbol{n}_2$ and $\boldsymbol{p}_1 = \boldsymbol{p}_2$.

19. (All vectors in this question are written as column vectors.) Let $\boldsymbol{A}$ be a square matrix of order n such that $\boldsymbol{A}^2 = \boldsymbol{A}$ and $\boldsymbol{A}^{\mathrm{T}} = \boldsymbol{A}$.

(a) For any two vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$, show that $(\boldsymbol{Au}) \cdot \boldsymbol{v} = \boldsymbol{u} \cdot (\boldsymbol{Av})$.

(b) For any vector $\boldsymbol{w} \in \mathbb{R}^n$, show that $\boldsymbol{Aw}$ is the projection of $\boldsymbol{w}$ onto the subspace $V = \{\boldsymbol{u} \in \mathbb{R}^n \mid \boldsymbol{Au} = \boldsymbol{u}\}$ of $\mathbb{R}^n$.

20. Determine which of the following statements are true. Justify your answer.

(a) If $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are vectors in $\mathbb{R}^n$ such that $\|\boldsymbol{u}\| = \|\boldsymbol{v}\|$, then $\|\boldsymbol{u} + \boldsymbol{w}\| = \|\boldsymbol{v} + \boldsymbol{w}\|$.

(b) If $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are vectors in $\mathbb{R}^n$ such that $\|\boldsymbol{u}\| = \|\boldsymbol{v}\|$ and $\boldsymbol{w}$ is orthogonal to both $\boldsymbol{u}$ and $\boldsymbol{v}$, then $\|\boldsymbol{u} + \boldsymbol{w}\| = \|\boldsymbol{v} + \boldsymbol{w}\|$.

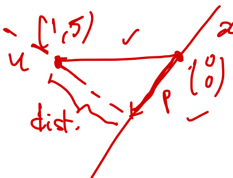
(c) If $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are vectors in $\mathbb{R}^n$ such that $\boldsymbol{u}$ is orthogonal to both $\boldsymbol{v}$ and $\boldsymbol{w}$, then $\boldsymbol{u}$ and $\boldsymbol{v} + \boldsymbol{w}$ are orthogonal.

(d) If $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are vectors in $\mathbb{R}^n$ such that $\boldsymbol{u}, \boldsymbol{v}$ are orthogonal and $\boldsymbol{v}, \boldsymbol{w}$ are orthogonal, then $\boldsymbol{u}$ and $\boldsymbol{w}$ are orthogonal.

Question 5.21 to Question 5.34 are exercises for Sections 5.3 and 5.4.

21. (a) In $\mathbb{R}^2$, find the distance from the point $(1,5)$ to the line $x - y = 0$.

(b) In $\mathbb{R}^3$, find the distance from the point $(1,0,-2)$ to the plane $2x + y - 2z = 0$.



gn sol. of eqn: span{$\binom{1}{1}$}

$\Rightarrow p = \left(\dfrac{\binom{1}{5} \cdot \binom{1}{1}}{\|\binom{1}{1}\|^2}\right)\binom{1}{1}$

$\Rightarrow$ pythagoras.

1. gn sol
2. projection using vector/vector space.
3. pythagoras.

(c) In $\mathbb{R}^3$, find the distance from the point $(1,0,-2)$ to the line $x=t$, $y=2t$ and $z=2t$ for $t \in \mathbb{R}$.

22. There are two costs involved if we want to publish a book. C is a fixed cost due to typesetting and editing and D is the printing and binding cost for each additional book we want to produce.

Suppose we expect b, the total cost of producing t books to be a linear function of t. We shall apply the least squares method (see Example 5.3.5, Theorem 5.3.10 and Example 5.3.11.2) to find a straight line $b = C + Dt$ that "best fits" the following set of data:

$\underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix}}_{b.}$ $3 = C+D$, $5 = C+2D$, $6 = C+3D$

$$b_1 = 3 \text{ when } t_1 = 1, \quad b_2 = 5 \text{ when } t_2 = 2 \quad \text{and} \quad b_3 = 6 \text{ when } t_3 = 3.$$

(a) Write down a linear system with three equations and two variables using the data set.

$\therefore$ u is a sol to $A^TAx = A^Tb$

(b) Obtain the least squares solution for C and D.

23. A father wishes to distribute an amount of money among his three sons Jack, Jim and John.

(a) Show that it is not possible to have a distribution such that the following conditions are all satisfied.

300 = Jack + 2Jim   300 = Jim + John   Jack = 2John + 300   300 = Jack − 2John.

(i) The amount Jack receives plus twice the amount Jim receives is \$300.

(ii) The amount Jim receives plus the amount John receives is \$300.

(iii) Jack receives \$300 more than twice of what John receives.

$$\begin{array}{ccc} \text{Jack} & \text{Jim} & \text{John} \end{array}$$
$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 300 \\ 1 & 0 & 1 & 300 \\ 1 & 0 & -2 & 300 \end{array}\right)$$

→ REF inconsistent.

(b) Since there is no solution to the distribution problem above, find a least squares solution. (Make sure that your least squares solution is feasible. For example, one cannot give a negative amount of money to anybody.)

$A^TAx = A^Tb$ s.t. u is sol to the eqn

24. Consider the following linear system:

$$\begin{cases} x + y + z = 1 \\ \quad\; y + z = 1 \\ x - y - z = 1 \\ \qquad\;\; z = 1. \end{cases}$$

(a) Show that the linear system is inconsistent. REF

(b) Find the least squares solution to the linear system. $A^TAx = A^Tb.$

25. Let $\boldsymbol{A} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, $\boldsymbol{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

(a) Find the least squares solution to the linear system $\boldsymbol{Ax} = \boldsymbol{b}$. $A^TAx = A^Tb.$

(b) By the result in Part (a), compute the projection of $\boldsymbol{b}$ onto the column space of $\boldsymbol{A}$.

$p = Au.$

26. Let $V = \text{span}\{\overset{v_1}{(1,0,1)}, \overset{v_2}{(2,1,0)}, \overset{v_3}{(0,1,-2)}, \overset{v_4}{(1,2,-3)}\}$.

(a) Find a basis for V. $\therefore$ $(v_1^T\ v_2^T\ v_3^T\ v_4^T) \to$ RREF $\to$ corresponding pivot col

(b) Find the projection of $\underset{w}{(1,1,1)}$ onto V using

(i) Theorem 5.2.15;   (ii) Theorem 5.3.8.

⇒ Orthogonalise the basis using gram schmidt

⇒ $p = \frac{w \cdot v_1}{\|v_1\|^2} v_1 + \dots$

$A^TAx = A^Tb$ where A: $b = w$.

then $p = Au.$

27. (a) Let $\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 0 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$.

(i) Solve the linear system $\boldsymbol{Ax} = \boldsymbol{b}$. RREF

(ii) Find the least squares solution to $\boldsymbol{Ax} = \boldsymbol{b}$. $A^TAx = A^Tb$

(b) Suppose a linear system $\boldsymbol{Ax} = \boldsymbol{b}$ is consistent. Show that the solution set of $\boldsymbol{Ax} = \boldsymbol{b}$ is equal to the solution set of $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{Ax} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{b}$.

(*Hint*: You need Theorem 4.3.6 and the result of Question 4.25(a).)

28. Let E be the standard basis for $\mathbb{R}^3$,

$$U = \{(2,1,0),(0,0,1),(-1,2,0)\} \quad \text{and} \quad V = \{(0,-1,2),(-1,2,1),(5,2,1)\}.$$

(a) Check that U and V are both orthogonal basis for $\mathbb{R}^3$. check if both are orthogonal.

(b) Find two orthonormal basis U' and V' by normalizing the vectors in U and V. normalize $\frac{u}{\|u\|}$.

(c) Find $\boldsymbol{P}$ and $\boldsymbol{Q}$, the transition matrices from E to U' and U' to V' respectively. $P = U'^T I$ $Q = V'^T U'$

(d) Let $\boldsymbol{R} = \boldsymbol{QP}$. Is $\boldsymbol{R}$ the transition matrix from E to V'? $UP[w]_E = Q[w]_u$

$Q[w]_u = [w]_v$

29. Suppose an $x'y'$-coordinate system is obtained from the xy-coordinate system by an anti-clockwise rotation through an angle $\theta = \pi/3$. $[v]_T = P^T[v]_S$ (2,1)

(a) Let P be the point such that its xy-coordinates are $(2,1)$. Find the $x'y'$-coordinates of P.

(b) Let Q be the point such that its $x'y'$-coordinates are $(2,1)$. Find the xy-coordinates of Q.

(c) Let L be the line $x + y = 1$. Write down the equation of L using the $x'y'$-coordinates.

30. Suppose an $x'y'z'$-coordinate system is obtained from the xyz-coordinates system by an anti-clockwise rotation about the z-axis through an angle θ. Let $\boldsymbol{u} = (x,y,z)^{\mathrm{T}}$ and $\boldsymbol{u}' = (x',y',z')^{\mathrm{T}}$ be the xyz-coordinates and $x'y'z'$-coordinates, respectively, of he same point. Find a 3×3 matrix $\boldsymbol{A}$ such that $\boldsymbol{Au} = \boldsymbol{u}'$.

(*Hint*: The z-axis is fixed under the rotation.)

31. (a) Let $S_1 = \{(1,0),(0,1)\}$, $S_2 = \{(1,-1),(2,1)\}$ and $S_3 = \left\{(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}),(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})\right\}$. Clearly, S_1, S_2 and S_3 are three bases for $\mathbb{R}^2$.

Let $\boldsymbol{u} = (1,4)$ and $\boldsymbol{v} = (-1,1)$. Compute $(\boldsymbol{u})_{S_i}$, $(\boldsymbol{v})_{S_i}$ and $(\boldsymbol{u})_{S_i}\cdot(\boldsymbol{v})_{S_i}$ for $i = 1,2,3$. What do you observe?

(b) Prove that if S and T are two orthonormal bases for a vector space V, then for any vectors $\boldsymbol{u},\boldsymbol{v} \in V$, $(\boldsymbol{u})_S\cdot(\boldsymbol{v})_S = (\boldsymbol{u})_T\cdot(\boldsymbol{v})_T$.

32. (All vectors in this question are written as column vectors.) Let $\boldsymbol{A}$ be an orthogonal matrix of order n and let $\boldsymbol{u},\boldsymbol{v}$ be any two vectors in $\mathbb{R}^n$. Show that

(a) $\|\boldsymbol{u}\| = \|\boldsymbol{Au}\|$;

(b) $d(\boldsymbol{u},\boldsymbol{v}) = d(\boldsymbol{Au},\boldsymbol{Av})$; and

(c) the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ is equal to the angle between $\boldsymbol{Au}$ and $\boldsymbol{Av}$.

33. (All vectors in this question are written as column vectors.) Let $\boldsymbol{A}$ be an orthogonal matrix of order n and let $S = \{\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_n\}$ be a basis for $\mathbb{R}^n$.

(a) Show that $T = \{\boldsymbol{A}\boldsymbol{u}_1, \boldsymbol{A}\boldsymbol{u}_2, \dots, \boldsymbol{A}\boldsymbol{u}_n\}$ is a basis for $\mathbb{R}^n$.

(b) If S is orthogonal, show that T is orthogonal.

(c) If S is orthonormal, is T orthonormal?

34. Determine which of the following statements are true. Justify your answer.

(a) If $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{c}_1 & \boldsymbol{c}_2 & \cdots & \boldsymbol{c}_k \end{pmatrix}$ is an $n \times k$ matrix such that $\boldsymbol{c}_1, \boldsymbol{c}_2, \dots, \boldsymbol{c}_k$ are orthonormal, then $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} = \boldsymbol{I}_k$.

(b) If $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{c}_1 & \boldsymbol{c}_2 & \cdots & \boldsymbol{c}_k \end{pmatrix}$ is an $n \times k$ matrix such that $\boldsymbol{c}_1, \boldsymbol{c}_2, \dots, \boldsymbol{c}_k$ are orthonormal, then $\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}} = \boldsymbol{I}_n$.

(c) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are orthogonal matrices, then $\boldsymbol{A} + \boldsymbol{B}$ is an orthogonal matrix.

(d) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are orthogonal matrices, then $\boldsymbol{A}\boldsymbol{B}$ is an orthogonal matrix.