NATIONAL UNIVERSITY OF SINGAPORE

SEMESTER 1, 2021/2022

MA1521 Calculus for Computing **Tutorial 9**

1. Find the limits:

   (a) $\displaystyle\lim_{(x,y)\to(2,-3)} \left(\frac{1}{x}+\frac{1}{y}\right)^2$

   (Thomas' Calculus (14^{th} edition), p. 773, Problem 4)

   (b) $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ x\neq y}} \frac{x-y+2\sqrt{x}-2\sqrt{y}}{\sqrt{x}-\sqrt{y}}$

   (Thomas' Calculus (14^{th} edition), p. 774, Problem 17)

   (c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{1-\cos(xy)}{xy}$

   (Thomas' Calculus (14^{th} edition), p. 774, Problem 22)

2. Show that the following limits do not exist.

   (a) $\displaystyle\lim_{(x,y)\to(1,-1)} \frac{xy+1}{x^2-y^2}$

   (Thomas' Calculus (14^{th} edition), p. 774, Problem 50)

   (b) $\displaystyle\lim_{(x,y)\to(1,1)} \frac{\tan y - y\tan x}{y-x}$

   (Thomas' Calculus (14^{th} edition), p. 774, Problem 54)

1. Find the limits:

(a) $\displaystyle\lim_{(x,y)\to(2,-3)} \left(\frac{1}{x}+\frac{1}{y}\right)^2$

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(b) $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ x\neq y}} \frac{x-y+2\sqrt{x}-2\sqrt{y}}{\sqrt{x}-\sqrt{y}}$

(Thomas' Calculus (14$^{\text{th}}$ edition), p. 774, Problem 17)

(c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{1-\cos(xy)}{xy}$

(Thomas' Calculus (14$^{\text{th}}$ edition), p. 774, Problem 22)

a) $\displaystyle\lim_{(x,y)\to(2,-3)} \left(\frac{1}{x}+\frac{1}{y}\right)^2$

$= \left(\frac{1}{2}+\frac{1}{-3}\right)^2$

$= \left(\frac{1}{6}\right)^2 = \frac{1}{36}$ //

b) $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ x\neq y}} \underbrace{\frac{x-y+2\sqrt{x}-2\sqrt{y}}{\sqrt{x}-\sqrt{y}}}$

$= \dfrac{(\cancel{x-y})(\sqrt{x}+\sqrt{y})}{\cancel{x-y}} + \dfrac{2\cancel{\sqrt{x}-\sqrt{y}}}{\cancel{\sqrt{x}-\sqrt{y}}}$

$= \sqrt{x}+\sqrt{y}+2.$

$\therefore$ as $(x,y)\to(0,0)$, lim $\to 2.$

c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{1-\cos xy}{xy}.$

let $u=xy$,

$= \displaystyle\lim_{u\to 0} \frac{1-\cos u}{u}$

By L'hopital rule,

$= \displaystyle\lim_{u\to 0} \frac{\sin u}{1}$

$= 0.$

2. Show that the following limits do not exist.

(a) $\displaystyle\lim_{(x,y)\to(1,-1)} \frac{xy+1}{x^2-y^2}$

(Thomas' Calculus (14th edition), p. 774, Problem 50)

(b) $\displaystyle\lim_{(x,y)\to(1,1)} \frac{\tan y - y\tan x}{y-x}$

(Thomas' Calculus (14th edition), p. 774, Problem 54)

a) let $y = 2x$

$$\lim_{x\to 1} \frac{x\cdot 2x+1}{x^2-(2x)^2} = \frac{2+1}{1-4} = -\frac{3}{3} = -1.$$

let $y = -2x$

$$\lim_{x\to 1} \frac{x\cdot -2x+1}{x^2-(-2x)^2} = \frac{-2+1}{1-4} = \frac{-1}{-3} = \frac{1}{3}$$

$\therefore$ $f(x,y)$ has different limits along 2 different path as $(x,y)\to(1,-1)$

$\therefore$ $\lim_{(x,y)\to(1,-1)} f(x,y)$ does not exist.

b) $\displaystyle\lim_{(x,y)\to(1,1)} \frac{\tan y - y\tan x}{y-x}$

let $y = 2x$.

$$\lim_{x\to 1} \frac{\tan 2x - 2x\tan x}{x} = \tan 2 - 2\tan 1 \quad — ①$$

let $y = -2x$.

$$\lim_{x\to 1} \frac{\tan(-2x)+2x\tan x}{-2x-x} = \frac{\tan -2 + 2\tan 1}{-3} \quad — ②$$

Since ① ≠ ②,

$f(x,y)$ has diff. limits along 2 diff. path as $(x,y)\to(1,-1)$

$\therefore$ $\lim_{(x,y)\to(1,-1)} f(x,y)$ does not exist.

3. The inequality

$$1 - \frac{x^2y^2}{3} < \frac{\tan^{-1} xy}{xy} < 1$$

holds for (x, y) "close to $(0, 0)$". What can you say about $\lim\limits_{(x,y)\to(0,0)} \dfrac{\tan^{-1} xy}{xy}$?

(Thomas' Calculus (14th edition), p. 775, Problem 59)

4. Find $\partial f/\partial x$ and $\partial f/\partial y$ for the following functions f.

(a) $f(x, y) = (xy - 1)^2$

(Thomas' Calculus (14th edition), p. 785, Problem 5)

(b) $f(x, y) = e^{xy} \ln y$

(Thomas' Calculus (14th edition), p. 785, Problem 16)

(c) $f(x, y) = x^y$

(Thomas' Calculus (14th edition), p. 785, Problem 19)

5. For each of the following functions, determine f_{xy}.

(a) $f(x, y) = y + x^2y + 4y^3 - \ln(y^2 + 1)$

(Thomas' Calculus (14th edition), p. 786, Problem 61(d))

(b) $f(x, y) = x \ln(xy)$

(Thomas' Calculus (14th edition), p. 786, Problem 61(d))

6. Show that the function $f(x, y) = \ln(x^2 + y^2)$ satisfies the two-dimensional Laplace equation

$$f_{xx} + f_{yy} = 0.$$

(Thomas' Calculus (14th edition), p. 787, Problem 86)

7. Find all the local maxima, local minima and saddle points of the following functions:

(a) $f(x, y) = x^2 + 2xy$

(b) $f(x, y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$

3. The inequality

$$1 - \frac{x^2y^2}{3} < \frac{\tan^{-1} xy}{xy} < 1$$

holds for (x, y) "close to $(0, 0)$". What can you say about $\lim_{(x,y)\to(0,0)} \frac{\tan^{-1} xy}{xy}$?

(Thomas' Calculus (14^{th} edition), p. 775, Problem 59)

By Squeeze theorem,

since $\lim_{(x,y)\to(0,0)} \left(1 - \frac{x^2y^2}{3}\right) = 1$

and $\lim_{(x,y)\to(0,0)} 1 = 1$

$\therefore \lim_{(x,y)\to(0,0)} \frac{\tan^{-1}(xy)}{xy} = 1.$

4. Find $\partial f/\partial x$ and $\partial f/\partial y$ for the following functions f.

(a) $f(x, y) = (xy - 1)^2$

(Thomas' Calculus (14^{th} edition), p. 785, Problem 5)

(b) $f(x, y) = e^{xy} \ln y$

(Thomas' Calculus (14^{th} edition), p. 785, Problem 16)

(c) $f(x, y) = x^y$

(Thomas' Calculus (14^{th} edition), p. 785, Problem 19)

a) $\frac{\partial f}{\partial x} = 2(xy-1) \cdot y$

$\frac{\partial f}{\partial y} = 2(xy-1) \cdot x.$

b) $\frac{\partial f}{\partial x} = e^{xy} \cdot y \cdot \ln y.$

$\frac{\partial f}{\partial y} \cdot (e^{xy} \cdot y \ln y) + (e^{xy} \frac{1}{y})$

c) $\frac{\partial f}{\partial x} = y x^{y-1}$

$\frac{\partial f}{\partial y} = x^y \ln x.$

5. For each of the following functions, determine f_{xy}.

(a) $f(x,y) = y + x^2y + 4y^3 - \ln(y^2+1)$

(Thomas' Calculus (14th edition), p. 786, Problem 61(d))

(b) $f(x,y) = x\ln(xy)$

(Thomas' Calculus (14th edition), p. 786, Problem 61(d))

a) $f(x,y) = y + x^2y + 4y^3 - \ln(y^2+1)$.

$f_x = 2xy$.

$f_{xy} = 2x$.

b) $f(x,y) = x\ln(xy)$

$f_x = (\ln xy) + (x \frac{y}{xy})$

$= \ln xy + 1$

$f_{xy} = \frac{x}{xy}$

$= \frac{1}{y}$

6. Show that the function $f(x,y) = \ln(x^2+y^2)$ satisfies the two-dimensional Laplace equation

$$f_{xx} + f_{yy} = 0.$$

$f_x = \frac{2x}{x^2+y^2}$ $\qquad$ $f_y = \frac{2y}{x^2+y^2}$

$f_{xx} = \frac{2(x^2+y^2) - (2x)(2x)}{(x^2+y^2)^2}$ $\qquad$ $f_{yy} = \frac{2(x^2+y^2) - (2y)(2y)}{(x^2+y^2)^2}$

$\therefore f_{xx} + f_{xy}$

$= \frac{2(x^2+y^2) - (2x)(2x)}{(x^2+y^2)^2} + \frac{2(x^2+y^2) - (2y)(2y)}{(x^2+y^2)^2}$

$= \frac{2x^2+2y^2 - 4x^2}{(x^2+y^2)^2} + \frac{2x^2+2y^2-4y^2}{(x^2+y^2)^2}$

$= \frac{\cancel{2y^2} - \cancel{2x^2} + \cancel{2x^2} - \cancel{2y^2}}{(x^2+y^2)^2}$

$= 0$ //

7. Find all the local maxima, local minima and saddle points of the following functions:

(a) $f(x,y) = x^2 + 2xy$

(b) $f(x,y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$

1. f has a **local maximum** at (a,b) if $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a,b).
2. f has a **local minimum** at (a,b) if $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a,b).
3. f has a **saddle point** at (a,b) if $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a,b).

a) $f_x = 2x + 2y$.

$f_{xx} = 2$

$f_y = 2x$

$f_{yy} = 0$

$f_{xy} = 2$.

Since $f_{xx} > 0$, there is no local maxima.

For local min:

$f_{xx} = 2 > 0$,

$f_{xx} \cdot f_{yy} - f_{xy}^2 > 0$

$2 \cdot 0 - 2^2 > 0$

$-4 < 0$

$\therefore$ there is no local min.

Since $f_{xx}f_{yy} - f^2_{xy} < 0$

$\therefore$ $f(x,y)$ has a saddle point at $(0,0,0)$

$f_x(a,b) = f_y(a,b) = 0$.

$2a + 2b = 2a = 0$

$\therefore a = 0,\ b = 0$.

b) $f(x,y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$.

$f_x = 3x^2 + 6x$

$f_{xx} = 6x + 6$

$f_y = 3y^2 - 6y$

$f_{yy} = 6y - 6$.

$f_{xy} = 0$

$f_x(a,b) = f_y(a,b) = 0$

$3a^2 + 6b = 3a^2 - 6b = 0$

$3a^2 + 6b = 0$.

$3a^2 = -6b$

$\therefore -6b - 6b = 0$.

$-12b = 0$.

$b = 0$.

$a = 0$.

$f_{xx} = 6 > 0$

$\therefore$ no local maxima

$f_{xx}f_{yy} - f^2_{xy}$

$= (6 \cdot 0 + 6) \cdot (6 \cdot 0 - 6) - 0^2$

$= 6 \cdot -6$

$= -36 < 0$

$\therefore$ no local min.

$\therefore$ Saddle point at $(0,0,0)$.

8. Find two numbers a and b with $a \leq b$ such that

$$\int_a^b (6 - x - x^2)dx$$

has its largest values.

(Thomas' Calculus (14^{th} edition), p. 823, Problem 39)

9. Let

$$f(x, y) = \begin{cases} \dfrac{xy^2}{x^2 + y^4}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$$

Show that $f_x(0, 0)$ and $f_y(0, 0)$ exist, but f is not differentiable at $(0, 0)$.

(Thomas' Calculus (14^{th} edition), p. 788, Problem 101)

8. Find two numbers a and b with $a \leq b$ such that

$$\int_a^b (6 - x - x^2)dx$$

has its largest values. → maximum.

(Thomas' Calculus (14th edition), p. 823, Problem 39)

$$\int_a^b 6 - x - x^2 \, dx$$

$$= 6x - \frac{x^2}{2} - \frac{x^3}{3} \Big|_a^b$$

$$= \left(6b - \frac{b^2}{2} - \frac{b^3}{3}\right) - \left(6a - \frac{a^2}{2} - \frac{a^3}{3}\right)$$

$\therefore$ Let $f(x,y) = 6y - \frac{y^2}{2} - \frac{y^3}{3} - 6x + \frac{x^2}{2} + \frac{x^3}{3}$.

$f_x = -6 + x + x^2$

$f_{xx} = 1 + 2x$

$f_y = 6 - y - y^2$

$f_{yy} = -1 - 2y$

$f_{xy} = 0$

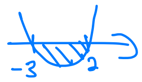
$f_x(c,d) = f_y(c,d) = 0$

$-6 + c + c^2 = 6 - d - d^2 = 0$

$\therefore c = d$.

For max:

$f_{xx} = -6 + x + x^2 < 0 \rightarrow (x+3)(x-2) < 0$



$-3 < x < 2$

and $f_{xx} f_{yy} - f_{xy}^2 > 0$

$(1+2x)(-1-2y) > 0$

$\therefore$ When $y = x$,

$(1+2x)(1+2x) < 0$

$(1+2x)^2 < 0$

$1 + 2x < 0$

$2x < -1$

$x < -\frac{1}{2}$

$\therefore -3 < x < -\frac{1}{2}$

$a = -2, \ b = -2$

9. Let

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

Show that $f_x(0,0)$ and $f_y(0,0)$ exist, but f is not differentiable at $(0,0)$.

(Thomas' Calculus (14th edition), p. 788, Problem 101)

$$f_x = \begin{cases} \dfrac{(x^2+y^4)x - (2x)(xy^2)}{(x^2+y^4)^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$f_y = \begin{cases} \dfrac{(x^2+y^4)(2y) - (4y)(xy^2)}{(x^2+y^4)^2}, & (x,y) \neq (0,0). \\ 0, & (x,y) = (0,0). \end{cases}$$

$\therefore$ $f_x(0,0) = 0$ and $f_y(0,0) = 0$ exists.