1. Let $A = (a_{ij})_{3\times 4}$, where $a_{ij} = 2i - 3j$, $\boldsymbol{B} = \boldsymbol{I}_4$, $\boldsymbol{C} = \boldsymbol{0}_{3\times 3}$,

$$\boldsymbol{D} = (d_{ij})_{4\times 3} \quad \text{where} \quad d_{ij} = \begin{cases} -1 & \text{if } i+j \text{ is even} \\ 1 & \text{if } i+j \text{ is odd.} \end{cases}$$

$$\boldsymbol{E} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}, \qquad \boldsymbol{F} = \begin{pmatrix} 5 & -1 \\ 9 & 1 \\ 2 & 0 \end{pmatrix}, \qquad \text{and} \qquad \boldsymbol{G} = \begin{pmatrix} 1 \\ -1 \\ 3 \\ 2 \end{pmatrix}.$$

Evaluate the following, wherever possible.

(a) $\boldsymbol{AD}$, (b) $\boldsymbol{DA} - 3\boldsymbol{B}$, (c) $\boldsymbol{D}^2$, (d) $\boldsymbol{E}^2 + \boldsymbol{C}^3$,

(e) $\boldsymbol{DE} + 2\boldsymbol{D}$, (f) $\boldsymbol{EA}$, (g) $\boldsymbol{DB}$, (h) $\boldsymbol{CF}$,

(i) $\boldsymbol{AG}$, (j) $\boldsymbol{FE}$, (k) $\boldsymbol{EF}$, (l) $\boldsymbol{CA}$,

(m) $\boldsymbol{E} - \boldsymbol{E}^{\mathrm{T}}$, (n) $\boldsymbol{F} - \boldsymbol{F}^{\mathrm{T}}$, (o) $\boldsymbol{GG}^{\mathrm{T}}$, (p) $\boldsymbol{G}^{\mathrm{T}}\boldsymbol{G}$.

1.

$$A = \begin{pmatrix} -1 & -4 & -7 & -10 \\ 1 & -2 & -5 & -8 \\ 3 & 0 & -3 & -6 \end{pmatrix}_{3\times 4} \quad D = \begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}_{4\times 3}$$

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{4\times 4} \quad E = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}, \quad F = \begin{pmatrix} 5 & -1 \\ 9 & 1 \\ 2 & 0 \end{pmatrix}, \quad \text{and} \quad G = \begin{pmatrix} 1 \\ -1 \\ 3 \\ 2 \end{pmatrix}.$$

$$C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

a) $$AD = \begin{pmatrix} 1-4+7-10 & -1+4-7+10 & 1-4+7-10 \\ -1-2+5-8 & 1+2-5+8 & -1-2+5-8 \\ -3+0+3-6 & 3-0-3+6 & -3+0+3-6 \end{pmatrix}$$

$$= \begin{pmatrix} -6 & 6 & -6 \\ -6 & 6 & -6 \\ -6 & 6 & -6 \end{pmatrix}$$

b) $DA - 3B$

$$DA = \begin{pmatrix} 1+1-3 & 4-2-0 & 7-5+3 & 10-8+6 \\ -1-1+3 & -4+2+0 & -7+5-3 & -10+8-6 \\ 1+1-3 & 4-2-0 & 7-5+3 & 10-8+6 \\ -1-1+3 & -4+2+0 & -7+5-3 & -10+8-6 \end{pmatrix}$$

$$DA = \begin{pmatrix} -1 & 2 & 5 & 8 \\ 1 & -2 & -5 & -8 \\ -1 & 2 & 5 & 8 \\ 1 & -2 & -5 & -8 \end{pmatrix} \qquad 3B = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$DA - 3B = \begin{pmatrix} -4 & 2 & 5 & 8 \\ 1 & -5 & -5 & -8 \\ -1 & 2 & 2 & 8 \\ 1 & -2 & -5 & -11 \end{pmatrix}$$

c) $D^2 = D \times D = \begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}_{4\times3}$

$\therefore$ not possible

d) $E^2 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$ $C^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$= \begin{pmatrix} 1 & 1+2 & 1+2+3 \\ 0 & 4 & 4+6 \\ 0 & 0 & 9 \end{pmatrix}$

$= \begin{pmatrix} 1 & 3 & 6 \\ 0 & 4 & 10 \\ 0 & 0 & 9 \end{pmatrix}$

$\therefore E^2 + C^2 = \begin{pmatrix} 1 & 3 & 6 \\ 0 & 4 & 10 \\ 0 & 0 & 9 \end{pmatrix}$

e)

$DE = \begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}_{4\times3} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}_{3\times3}, = \begin{pmatrix} -1 & -1+2 & -1+2-3 \\ 1 & 1-2 & 1-2+3 \\ -1 & -1+2 & -1+2-3 \\ 1 & 1-2 & 1-2+3 \end{pmatrix}$

$= \begin{pmatrix} -1 & 1 & -2 \\ 1 & -1 & 2 \\ -1 & 1 & -2 \\ 1 & -1 & 2 \end{pmatrix}$

$2D = \begin{pmatrix} -2 & 2 & -2 \\ 2 & -2 & 2 \\ -2 & 2 & -2 \\ 2 & -2 & 2 \end{pmatrix}$

$\therefore DE + 2D =$ Not possible.

f) $EA = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}_{3\times3} \cdot \begin{pmatrix} -1 & -4 & -7 & -10 \\ 1 & -2 & -5 & -8 \\ 3 & 0 & -3 & -6 \end{pmatrix}_{3\times4}$

$= \begin{pmatrix} -1+1+3 & -4-2+0 & -7-5-3 & -10-8-6 \\ 0+2+6 & 0-4+0 & 0-10-6 & 0-16-12 \\ 0+0+9 & 0+0+0 & 0+0-9 & 0+0-24 \end{pmatrix}$

$= \begin{pmatrix} 3 & -6 & -15 & -24 \\ 8 & -4 & -16 & -28 \\ 9 & 0 & -9 & -24 \end{pmatrix}$

g) $DB = \begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}_{4\times3} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{4\times4}$

$\therefore$ not possible.

h) $CF = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{3\times3} \cdot \begin{pmatrix} 5 & -1 \\ 9 & 1 \\ 2 & 0 \end{pmatrix}_{3\times2}$

$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

i) $AG = \begin{pmatrix} -1 & -4 & -7 & -10 \\ 1 & -2 & -5 & -8 \\ 3 & 0 & -3 & -6 \end{pmatrix}_{3\times4} \cdot \begin{pmatrix} 1 \\ -1 \\ 3 \\ 2 \end{pmatrix}_{4\times1}$

$= \begin{pmatrix} -1+4-21-20 \\ 1+2-15-16 \\ 3+0-9-12 \end{pmatrix}$

$= \begin{pmatrix} -38 \\ -28 \\ -18 \end{pmatrix}$

j) $FE = \begin{pmatrix} 5 & -1 \\ 9 & 1 \\ 2 & 0 \end{pmatrix}_{3\times2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}_{3\times3}$

$\therefore$ not possible.

k) $GF = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}_{3\times3} \begin{pmatrix} 5 & -1 \\ 9 & 1 \\ 2 & 0 \end{pmatrix}_{3\times2}$

$= \begin{pmatrix} 5+9+2 & -1+1+0 \\ 0+18+4 & 0+2+0 \\ 0+0+6 & 0+0+0 \end{pmatrix}$

$= \begin{pmatrix} 16 & 0 \\ 22 & 2 \\ 6 & 0 \end{pmatrix}$

l) $CA = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{3\times3} \begin{pmatrix} -1 & -4 & -7 & -10 \\ 1 & -2 & -5 & -8 \\ 3 & 0 & -3 & -6 \end{pmatrix}_{3\times4}$

$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

m) $E - E^T$

$E = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$ $E^T = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 3 \end{pmatrix}$

$\therefore E - E^T = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -2 \\ -1 & -2 & 0 \end{pmatrix}$

n) $F - F^T$

$F = \begin{pmatrix} 5 & -1 \\ 9 & 1 \\ 2 & 0 \end{pmatrix}$, $F^T = \begin{pmatrix} 5 & 9 & 2 \\ -1 & 1 & 0 \end{pmatrix}$

$\therefore$ not possible.

(o) $G \cdot G^T = \begin{pmatrix} 1 \\ -1 \\ 3 \\ 2 \end{pmatrix}_{4\times1} \cdot \begin{pmatrix} 1 & -1 & 3 & 2 \end{pmatrix}_{1\times4}$

$= (1+1+9+4)$

$= (15)$

(p) $G^TG = \begin{pmatrix} 1 & -1 & 3 & 2 \end{pmatrix}_{1\times4} \begin{pmatrix} 1 \\ -1 \\ 3 \\ 2 \end{pmatrix}_{4\times1}$

$= (15)$

2. Solve the following matrix equation for a, b, c and d:

$$\begin{pmatrix} a-b & a+c \\ -a+c & a+b-d \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}.$$

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 2 \\ -1 & 0 & 1 & 0 & 2 \\ 1 & 1 & 0 & -1 & 3 \end{array}\right)$$

$$\xrightarrow[\substack{P_1+P_3 \\ -P_1+P_4}]{-P_1+P_2} \left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 & 3 \\ 0 & 2 & 0 & -1 & 2. \end{array}\right)$$

$$\xrightarrow[-2P_2+P_4]{P_2+P_3} \left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 4 \\ 0 & 0 & -2 & -1 & 0 \end{array}\right)$$

$$\xrightarrow{P_3+P_4} \left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 4 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right)$$

$\therefore$ $d = 0.$

$c = 2.$

$b = -1$

$a = 0.$ N

3. The symbol $\sum$ is used to denote the sum of a sequence of numbers. For example,

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n,$$

$$\sum_{x=0}^{m} f(x) = f(0) + f(1) + \cdots + f(m),$$

$$\sum_{k=1}^{r} c_{ik} d_{kk} = c_{i1} d_{1j} + c_{i2} d_{2j} + \cdots + c_{ir} d_{rj}.$$

Let $\boldsymbol{A} = (a_{ij})$ be an $m \times n$ matrix and $\boldsymbol{B} = (b_{ij})$ an $n \times m$ matrix, with $m, n \geq 5$.

(a) Each of the following sums represents an entry of either $\boldsymbol{AB}$ or $\boldsymbol{BA}$. Determine which matrix product is involved and which entry of that product is represented in each case:

(i) $\sum_{k=1}^{n} a_{3k} b_{k4}$, (ii) $\sum_{p=1}^{n} a_{4p} b_{p1}$, (iii) $\sum_{q=1}^{m} a_{q2} b_{3q}$, (iv) $\sum_{x=1}^{m} b_{2x} a_{x5}$.

(b) Use the symbol $\sum$ to express the following entries symbolically.

(i) In $\boldsymbol{AB}$, the entry in the third row and second column.

(ii) In $\boldsymbol{BA}$, the entry in the fourth row and first column.

a) i) AB - (3, 4)

ii) AB - (4, 1)

iii) BA (3, 2)

iv) AB. (2, 5)

b) i) $\sum_{k=1}^{n} a_{3k} b_{k2}$

ii) $\sum_{k=1}^{n} b_{4k} a_{k1}$

4. Given $\boldsymbol{A} = (a_{ij})_{n \times p}$, $\boldsymbol{B} = (b_{ij})_{p \times q}$ and $\boldsymbol{C} = (c_{ij})_{p \times p}$, write down the (i, j) entries of

(a) $\boldsymbol{AB}$, (b) $\boldsymbol{C}^2$, (c) $\boldsymbol{AC}^{\mathrm{T}}$.

a) $a_{np} b_{pq}$

b) $c_{pp} c_{pp}$

c) $a_{np} c_{pp}$.

5. Find an example of a nonzero 3×3 matrix $\boldsymbol{A}$ such that $\boldsymbol{A}^{\mathrm{T}} = -\boldsymbol{A}$. What is the general form of the matrix $\boldsymbol{A}$?

$$A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = -A'$$

$\therefore$ general sol : $\begin{pmatrix} 0 & a & a \\ -a & 0 & 0 \\ -a & 0 & 0 \end{pmatrix}$ $a \in \mathbb{C}$.

6. Find examples of nonzero 3×3 matrices $\boldsymbol{A}, \boldsymbol{B}$ and $\boldsymbol{C}$ for each of the following cases:

(a) $\boldsymbol{AB} = \boldsymbol{0}$, (b) $\boldsymbol{AB} \neq \boldsymbol{BA}$, (c) $\boldsymbol{BA} = \boldsymbol{CA}$ but $\boldsymbol{B} \neq \boldsymbol{C}$.

a) $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. $B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

c) $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ B and C can be any.

7. Give an example of a 2×3 matrix $\boldsymbol{A}$ such that the solution set of the linear system $\boldsymbol{Ax} = \boldsymbol{0}$ is the plane $2x + 3y - z = 0$.

8. Let S be the set of points (x, y, z) of the form

$$\underset{2\times 3}{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}} \cdot \underset{3\times 1}{\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0.$$

8. Let S be the set of points (x, y, z) of the form

$$\begin{cases} x = t + 1 \\ y = t, \\ z = 3 \end{cases} \qquad \text{where } t \text{ is an arbitary parameter.}$$

(a) Describe S geometrically.

(b) Find an example of a 2×3 matrix $\boldsymbol{A}$ and a 2×1 matrix $\boldsymbol{b}$ such that the solution set of the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ is S. Give a geometric interpretation for the linear system.

a) plane

b) $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}_{2\times3}$ $\quad b = \begin{pmatrix} 4 \\ 4 \end{pmatrix}_{2\times1}$

$$x = t\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}.$$

9. Suppose the homogeneous system $\boldsymbol{Ax} = \boldsymbol{0}$ has non-trivial solution. Show that the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ has either no solution or infinitely many solutions.

Suppose $Ax = 0$ has non-trivial soln;

then either $A = 0$ or $x = 0$.

Suppose $A = 0$, then $Ax = b$ has no solution.

Suppose $x = 0$, then $A = b$. then A has infinite soln.

10. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $m \times n$ and $n \times p$ matrices respectively. Theorem 2.3.9, Theorem 2.4.7

(a) Suppose the homogeneous system $\boldsymbol{Bx} = \boldsymbol{0}$ has infinitely many solutions. How many solutions does the system $\boldsymbol{ABx} = \boldsymbol{0}$ have?

(b) Suppose $\boldsymbol{Bx} = \boldsymbol{0}$ has only the trivial solution. Can we tell how many solutions are there for $\boldsymbol{ABx} = \boldsymbol{0}$?

a) infinite.

b) ~~definite~~

No, we cannot tell.

a) $x_1, x_2 \dots$ $Bx = 0$.

$ABx_1 = A(Bx_1) = A0 = 0$.

$\therefore$ Sols of $Bx = 0$ = Sols of $ABx = 0$.

b) $Bx = 0$ e.g. B is unit matrix.

11. Let $\boldsymbol{A} = (a_{ij})_{n\times n}$ be a square matrix. The *trace* of A, denoted by $\mathrm{tr}(\boldsymbol{A})$, is defined to be the sum of the entries on the diagonal of $\boldsymbol{A}$, i.e.

$$\mathrm{tr}(\boldsymbol{A}) = a_{11} + a_{22} + \cdots + a_{nn} = \sum_{i=1}^{n} a_{ii}.$$

(a) Find the trace of each of the following square matrices.

(i) $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, (ii) $\begin{pmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{pmatrix}$, (iii) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 3 & 5 & 0 \\ 1 & 3 & 5 & 7 \end{pmatrix}$.

(b) Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be any square matrices of the same size. Show that

$$\mathrm{tr}(\boldsymbol{A}+\boldsymbol{B}) = \mathrm{tr}(\boldsymbol{A}) + \mathrm{tr}(\boldsymbol{B}).$$

(c) Let $\boldsymbol{A}$ be any square matrix and c a scalar. Show that

$$\mathrm{tr}(c\boldsymbol{A}) = c\,\mathrm{tr}(\boldsymbol{A}).$$

(d) Let $\boldsymbol{C}$ and $\boldsymbol{D}$ be $m \times n$ and $n \times m$ matrices respectively. Show that

$$\mathrm{tr}(\boldsymbol{CD}) = \mathrm{tr}(\boldsymbol{DC}).$$

(e) Show that there are no square matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ such that $\boldsymbol{AB} - \boldsymbol{BA} = \boldsymbol{I}$.

a) i) 2

ii) -6

iii) 16

b)

Suppose $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ $B = \begin{bmatrix} d & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & f \end{bmatrix}$.

Then $A+B = \begin{bmatrix} a+d & 0 & 0 \\ 0 & b+e & 0 \\ 0 & 0 & c+f \end{bmatrix}$.

$\therefore\ tr(A+B) = a+d+b+e+c+f$

$= a+b+c+d+e+f$

$= tr(A) + tr(B)$

c) Suppose $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

then $cA = \begin{pmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix}$

such that $tr(cA) = c+c+c$

$= 3c$

$= c3$

$= c\, tr(A)$.

d) Suppose $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ 3x2

$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ 2x3

then $CD = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ 3x3

$DC = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ 2x2

$\therefore\ tr(CD) = 2 = tr(DC)$.

e) Suppose A and B are sqr matrix.

then let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

12. A square matrix $\boldsymbol{A}$ is called *orthogonal* if

$$\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}} = \boldsymbol{I} \quad \text{and} \quad \boldsymbol{A}^{\mathrm{T}}\boldsymbol{A} = \boldsymbol{I}.$$

(a) Determine which of the following matrices are orthogonal:

(i) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, (ii) $\begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix}$.

(b) Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are orthogonal matrices of the same size. Show that $\boldsymbol{AB}$ is orthogonal.

a) i) $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$A^T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

$\therefore A \cdot A^T = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \neq I.$

Not orthogonal.

ii) $A = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix}$

$A^T = \begin{pmatrix} \frac{3}{4} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix}$

$\frac{9}{20} + \frac{16}{25} = \frac{9 \cdot 25 + 16 \cdot 20}{20 \cdot 25}$

$\therefore A \cdot A^T = \begin{pmatrix} \frac{3}{4} \cdot \frac{3}{5} + \frac{4}{5} \cdot \frac{4}{5} & \frac{3}{4} \cdot \frac{4}{5} + \frac{4}{5} \cdot -\frac{3}{5} \\ \frac{4}{5} \cdot \frac{3}{4} + -\frac{3}{5} \cdot \frac{4}{5} & \frac{4}{5} \cdot \frac{4}{5} + -\frac{3}{5} \cdot \frac{3}{5} \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = A^T \cdot A.$ Assumption

$\therefore$ orthogonal.

13. A square matrix $\boldsymbol{A}$ is called *nilpotent* if $\boldsymbol{A}^k = \boldsymbol{0}$ for some positive integer k.

(a) Show that $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ are nilpotent.

(b) Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be square matrices of the same size such that $\boldsymbol{AB} = \boldsymbol{BA}$ and $\boldsymbol{A}$ is nilpotent. Show that $\boldsymbol{AB}$ is nilpotent.

(c) If $\boldsymbol{AB} \neq \boldsymbol{BA}$ in Part (b), must $\boldsymbol{AB}$ be nilpotent?

a) let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

then $A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$

$\therefore$ nilpotent.

let $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

then $B^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$

$\therefore$ nilpotent.

b) $AB = BA = I$.

Since A is nilpotent

(c) If $AB \neq BA$ in Part (b), must AB be nilpotent?

14. Let $\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Consider the matrix equation

$$\boldsymbol{AX} = \boldsymbol{XA} \tag{2.5}$$

where $\boldsymbol{X}$ is a 2×2 unknown matrix.

(a) Determine which of the following matrices satisfy the Equation (2.5):

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \to A., \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \to O., \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \to I.$$

(b) Prove that if $\boldsymbol{P}$ and $\boldsymbol{Q}$ satisfy Equation (2.5), then $\boldsymbol{P}+\boldsymbol{Q}$ and $\boldsymbol{PQ}$ also satisfy Equation (2.5).

(c) Find conditions n p, q, r, s which determine precisely when $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ satisfy the Equation (2.5).

a) $X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

$X = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

b)

c) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} p+r & q+s \\ r & s \end{pmatrix} = \begin{pmatrix} p & p+q \\ r & r+s \end{pmatrix}$

$p+r = p$

$q+s = p+q \Rightarrow s = p$

$r = r$

$s = r+s.$

$\therefore r = 0.$

$\therefore$ Condition: $s = p \in \mathbb{R}$, $r = 0.$

$q \in \mathbb{R}$

(2.3).

15. (a) Let $D = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$ where a, b, c are real numbers. Show that, for all positive integer k,

$$\boldsymbol{D}^k = \begin{pmatrix} a^k & 0 & 0 \\ 0 & b^k & 0 \\ 0 & 0 & c^k \end{pmatrix}.$$

odd.

(b) Find a diagonal matrix $\boldsymbol{A}$ such that $\boldsymbol{A}^{5} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

(c) Find all diagonal matrices $\boldsymbol{B}$ such that $\boldsymbol{B}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{pmatrix}$.

a)

$$D = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

then $D^2 = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \cdot \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$

$= \begin{pmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix}$

Suppose $D^k = \begin{pmatrix} a^k & 0 & 0 \\ 0 & b^k & 0 \\ 0 & 0 & c^k \end{pmatrix}$

then $D^{k+1} = D^k \cdot D.$

$= \begin{pmatrix} a^k & 0 & 0 \\ 0 & b^k & 0 \\ 0 & 0 & c^k \end{pmatrix} \cdot \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$

$= \begin{pmatrix} a^{k+1} & 0 & 0 \\ 0 & b^k & 0 \\ 0 & 0 & c^k \end{pmatrix}$

$\therefore$ By MI, $D^k = \begin{pmatrix} a^k & 0 & 0 \\ 0 & b^k & 0 \\ 0 & 0 & c^k \end{pmatrix}$, $k \in \mathbb{Z}^+$

b)

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

c)

$$B = \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 2 & 0 \\ 0 & 0 & \pm 3 \end{pmatrix}$$

16. Let $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ be three nonzero $n \times n$ matrices. Suppose $\boldsymbol{AB} = \boldsymbol{BA}$ and $\boldsymbol{AC} = \boldsymbol{CA}$.

(a) Is $\boldsymbol{CB} = \boldsymbol{BC}$?

(b) Among the matrices $\cancel{\boldsymbol{ABC}}$, $\cancel{\boldsymbol{ACB}}$, $\cancel{\boldsymbol{BAC}}$, $\cancel{\boldsymbol{BCA}}$, $\cancel{\boldsymbol{CAB}}$, $\boldsymbol{CBA}$, which of them are equal to one another?

Justify your answers.

a) $CBA = BCA$

$CAB = BAC$

$ACB = BAC$

$\therefore$ not equal.

b) $ABC = BAC = BCA$

$ACB = CAB = CBA$

17. Consider the population of certain endangered species of wild animals: On the average, each adult will give birth to one baby each year; 50% of the new born babies will survive the first year; 60% of the one-year-old cubs will survive the second year and become adults; and 70% of the adults will survive each year.

Define $\boldsymbol{A} = \begin{pmatrix} 0 & 0 & 1 \\ 0.5 & 0 & 0 \\ 0 & 0.6 & 0.7 \end{pmatrix}$. Let x_0, y_0 and z_0 be the numbers of babies, one-year-old cubs and adults, respectively, at the end of a particular year.

(a) Let $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \boldsymbol{A} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$. What information do the numbers x_1, y_1 and z_1 give us?

↳ number of babies, one-year cubs and adults that survived that particular year.

(b) Let $\begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} = \boldsymbol{A}^n \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ where n is a positive integer. Interpret the numbers x_n, y_n and z_n.

↳ number of babies, cubs, and adults at the end of year n.

(c) Suppose initially, $x_0 = 0$, $y_0 = 0$ and $z_0 = 100$. What is the total population three years later.

$$\begin{pmatrix} 0 & 0 & 1 \\ 0.5 & 0 & 0 \\ 0 & 0.6 & 0.7 \end{pmatrix}^3 \begin{pmatrix} 0 \\ 0 \\ 100 \end{pmatrix}$$

$$= \begin{pmatrix} 49 \\ 35 \\ 643/10 \end{pmatrix}$$

$\therefore$ Total $= 49 + 35 + 643/10$.

$= 178.3$.

≈ 178.

18. Complete the proof of Theorem 2.2.6:

Let $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ be matrices of the same size and c, d scalars. Show that

(a) $\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{B}+\boldsymbol{A}$,

(b) $c(\boldsymbol{A}+\boldsymbol{B})=c\boldsymbol{A}+c\boldsymbol{B}$,

(c) $(c+d)\boldsymbol{A}=c\boldsymbol{A}+d\boldsymbol{A}$,

(d) $c(d\boldsymbol{A})=(cd)\boldsymbol{A}=d(c\boldsymbol{A})$,

(e) $\boldsymbol{A}+\boldsymbol{0}=\boldsymbol{0}+\boldsymbol{A}=\boldsymbol{A}$,

(f) $\boldsymbol{A}-\boldsymbol{A}=\boldsymbol{0}$,

(g) $0\boldsymbol{A}=\boldsymbol{0}$.

a) $A+B$

$= (a_{ij})_{m\times n} + (b_{ij})_{m\times n}$

$= (b_{ij})_{m\times n} + (a_{ij})_{m\times n}$

$= B+A$.

b) $c(A+B)$

$= c((a_{ij})_{m\times n} + (b_{ij})_{m\times n})$

$= c(a_{ij})_{m\times n} + c(b_{ij})_{m\times n}$

$= cA + cB$.

c) $(c+d)A$

$= (c+d)(a_{ij})_{m\times n}$.

$= c(a_{ij})_{m\times n} + d(a_{ij})_{m\times n}$

$= cA + dA$

d) $c(dA)$

$= c(d(a_{ij})_{m\times n})$

$= cd(a_{ij})_{m\times n}$

$= (cd)A$ $\quad = dc(a_{ij})_{m\times n}$

$= d(c(a_{ij})_{m\times n})$

$= d(cA)$

e) $A+0$

$= (a_{ij})_{m\times n} + (0_{ij})_{m\times n}$

$= (0_{ij})_{m\times n} + (a_{ij})_{m\times n}$

$= 0 + A$

$= A$ since $\forall i,j,\ 0_{ij} = 0$.

f) $A-A$

$= (a_{ij})_{m\times n} - (a_{ij})_{m\times n}$

$= 0$.

g) $0A$

$= (0_{ij})_{m\times n} \cdot (a_{ij})_{m\times n}$.

$= 0\,(a_{ij})_{m\times n}$

$= 0$.

19. Complete the proof of Theorem 2.2.11:

(a) If $\boldsymbol{A}, \boldsymbol{B}$ and $\boldsymbol{C}$ are $m\times p$, $p\times q$ and $q\times n$ matrices respectively, show that

$$\boldsymbol{A}(\boldsymbol{B}\boldsymbol{C})=(\boldsymbol{A}\boldsymbol{B})\boldsymbol{C}.$$

(b) If $\boldsymbol{A}, \boldsymbol{C}_1$ and $\boldsymbol{C}_2$ are $p\times n$, $m\times p$ and $m\times p$ matrices respectively, show that

$$(\boldsymbol{C}_1+\boldsymbol{C}_2)\boldsymbol{A}=\boldsymbol{C}_1\boldsymbol{A}+\boldsymbol{C}_2\boldsymbol{A}.$$

(c) If $\boldsymbol{A}, \boldsymbol{B}$ are $m\times p$, $p\times n$ matrices, respectively, and c is a scalar, show that

$$c(\boldsymbol{A}\boldsymbol{B})=(c\boldsymbol{A})\boldsymbol{B}=\boldsymbol{A}(c\boldsymbol{B}).$$

a) $A(BC)$

$= (a_{ij})_{m\times p}((b_{ij})_{p\times q} \cdot (c_{ij})_{q\times n})_{p\times n}$.

$= (a_{ij})_{m\times p}(b_{ij})_{p\times q}(c_{ij})_{q\times n}$

$= ((a_{ij})_{m\times p}(b_{ij})_{p\times q})_{p\times q}(c_{ij})_{q\times n}$

b)

20. Complete the proof of Theorem 2.2.22:

Let $\boldsymbol{A}$ be an $m \times n$ matrix.

(a) Show that $(\boldsymbol{A}^{\mathrm{T}})^{\mathrm{T}} = \boldsymbol{A}$.

(b) If $\boldsymbol{B}$ is an $m \times n$ matrix, show that $(\boldsymbol{A}+\boldsymbol{B})^{\mathrm{T}} = \boldsymbol{A}^{\mathrm{T}} + \boldsymbol{B}^{\mathrm{T}}$.

(c) If c is a scalar, show that $(c\boldsymbol{A})^{\mathrm{T}} = c\boldsymbol{A}^{\mathrm{T}}$.

21. Given that A is a 3×3 matrix such that

$$A\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} \quad \text{and} \quad A\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Find a matrix X such that

$$AX = \begin{pmatrix}1 & 0 & 4\\1 & 0 & 4\\1 & 0 & 7\end{pmatrix}.$$

Write $X = (x_1, x_2, x_3)$ where x_i is the ith column of X.

$AX = (Ax_1, Ax_2, Ax_3)$

$\Downarrow \quad \Downarrow \quad \Downarrow$

$\begin{pmatrix}1\\0\\0\end{pmatrix} \quad \begin{pmatrix}0\\0\\0\end{pmatrix} \quad 4\begin{pmatrix}1\\1\\1\end{pmatrix}+3\begin{pmatrix}0\\0\\1\end{pmatrix}.$

always the trivial.

$\Rightarrow 4\begin{pmatrix}1\\0\\0\end{pmatrix}+3\begin{pmatrix}0\\0\\1\end{pmatrix}$

let $A = \begin{pmatrix}a & b & c\\d & e & f\\g & h & i\end{pmatrix}$

$$\begin{pmatrix}a & b & c\\d & e & f\\g & h & i\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\therefore \begin{pmatrix}a\\d\\g\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\begin{pmatrix}1 & b & c\\1 & e & f\\1 & h & i\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\begin{pmatrix}c\\f\\i\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\therefore A = \underbrace{\begin{pmatrix}1 & b & 0\\1 & e & 0\\1 & h & 1\end{pmatrix}}_{3\times 3} \cdot \begin{pmatrix}S_1 & S_2 & S_3\\S_4 & S_5 & S_6\\S_7 & S_8 & S_9\end{pmatrix} = \begin{pmatrix}1 & 0 & 4\\1 & 0 & 4\\1 & 0 & 7\end{pmatrix}$$

$\therefore$

$1 \cdot S_1 + b \cdot S_4 = 1$ | $1 \cdot S_1 + e \cdot S_4 = 1$ | $1 \cdot S_1 + h \cdot S_4 + 1 \cdot S_7 = 1 \rightarrow 1 + S_7 = 1.$ $\therefore S_7 = 0.$

$1 \cdot S_2 + b \cdot S_5 = 0$ | $1 \cdot S_2 + e \cdot S_5 = 0$ | $1 \cdot S_2 + h \cdot S_5 + 1 \cdot S_8 = 0 \rightarrow 0 + S_8 = 0 \therefore S_8 = 0$

$1 \cdot S_3 + b \cdot S_6 = 4$ | $1 \cdot S_3 + e \cdot S_6 = 4$ | $1 \cdot S_3 + h \cdot S_6 + 1 \cdot S_9 = 7. \rightarrow 4 + S_9 = 7$ $\therefore S_9 = 3$

$e = b.$

$S_1 + bS_4 = 1 \rightarrow S_1 = 1, S_4 = 0$

$S_2 + bS_5 = 0. \rightarrow S_2 = 0, S_5 = 0.$

$S_3 + bS_6 = 4. \rightarrow S_3 = 4, S_6 = 0$

$$X = \begin{pmatrix}1 & 0 & 4\\0 & 0 & 0\\0 & 0 & 3\end{pmatrix} \quad //$$

22. Prove Remark 1.1.10:

Show that a linear system $\boldsymbol{Ax} = \boldsymbol{b}$ has either no solution, only one solution or infinitely many solutions.

(*Hint*: Suppose $\boldsymbol{Ax} = \boldsymbol{b}$ has two different solutions $\boldsymbol{u}$ and $\boldsymbol{v}$. Use $\boldsymbol{u}$ and $\boldsymbol{v}$ to construct infinitely many other solutions.)

Suppose $Ax=b$ has 2 diff. sols u, v,

$Au=b$, $Av=b$ and $u \neq v$.

$\forall t \in \mathbb{R}$, then $A(tu+(1-t)v) = tAu + (1-t)Av = tb + (1-t)b = b$.

$\therefore$ $tu+(1-t)v$ is a solution of $Ax=b$.

Since $t_1 u + (1-t_1)v \neq t_2 u + (1-t_2)v$

whenever $t_1 \neq t_2$,

there are infinite sols

23. Let $\boldsymbol{A}$ be an $m \times n$ matrix.

(a) Let $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ be $n \times p$ and $n \times q$ matrices respectively. Show that

$$\boldsymbol{A}\begin{pmatrix}\boldsymbol{B}_1 & \boldsymbol{B}_2\end{pmatrix} = \begin{pmatrix}\boldsymbol{AB}_1 & \boldsymbol{AB}_2\end{pmatrix}.$$

(In here, $\begin{pmatrix}\boldsymbol{B}_1 & \boldsymbol{B}_2\end{pmatrix}$ is an $n \times (p+q)$ matrix such that its jth column is equal to the jth column of $\boldsymbol{B}_1$ if $j \leq p$ and equal to the $(j-p)$th column of $\boldsymbol{B}_2$ if $j > p$.)

(b) Let $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ be $s \times m$ and $t \times m$ matrices respectively. Show that

$$\begin{pmatrix}\boldsymbol{D}_1 \\ \boldsymbol{D}_2\end{pmatrix}\boldsymbol{A} = \begin{pmatrix}\boldsymbol{D}_1\boldsymbol{A} \\ \boldsymbol{D}_2\boldsymbol{A}\end{pmatrix}.$$

(In here, $\begin{pmatrix}\boldsymbol{D}_1 \\ \boldsymbol{D}_2\end{pmatrix}$ is an $(s+t) \times m$ matrix such that its ith row is equal to the ith row of $\boldsymbol{D}_1$ if $i \leq s$ and equal to the $(i-s)$th row of $\boldsymbol{D}_2$ if $i > s$.)

24. Determine which of the following statements are true. Justify your answer.

(a) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are two diagonal matrices of the same size, then $\boldsymbol{AB} = \boldsymbol{BA}$.

(b) If $\boldsymbol{A}$ is a square matrix, then $\frac{1}{2}(\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}})$ is symmetric.

(c) For all matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, $(\boldsymbol{A}+\boldsymbol{B})^2 = \boldsymbol{A}^2 + \boldsymbol{B}^2 + 2\boldsymbol{AB}$.

(d) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are symmetric matrices of the same size, then $\boldsymbol{A} - \boldsymbol{B}$ is symmetric.

(e) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are symmetric matrices of the same size, then $\boldsymbol{AB}$ is symmetric.

(f) If $\boldsymbol{A}$ is a square matrix such that $\boldsymbol{A}^2 = \boldsymbol{0}$, then $\boldsymbol{A} = \boldsymbol{0}$.

(g) If $\boldsymbol{A}$ is a matrix such that $\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}} = \boldsymbol{0}$, then $\boldsymbol{A} = \boldsymbol{0}$.

b) ~~False~~ True.

Let $D = \frac{1}{2}(A + A^T)$

$D^T = [\frac{1}{2}(A + A^T)]^T$

$= \frac{1}{2}(A + A^T)^T$

$= \frac{1}{2}(A^T + (A^T)^T)$

$= \frac{1}{2}(A^T + A) = D.$

$\therefore D^T = D \Rightarrow$ Symmetric.

c) $(A+B)(A+B)$

$= A \cdot A + A \cdot B + B \cdot A + B \cdot B$

$= A^2 + AB + BA + B^2.$

$\therefore$ false.

f) ~~True~~ false ↑ same as above.

g) True

(i,i) entry of AA^T

$= a_{i1}a_{i1} + \cdots a_{in}a_{in}$

$= \sum_{k=1}^{n} a_{ik}^2$ (diagonal)

$\therefore AA^T = 0$ implies. $a_{ik} = 0$ for all i and k, i.e. $A = 0$.

25. Let $A = \begin{pmatrix} 2 & -1 & -1 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}$.

(a) Verify that $A^2 - 6A + 8I = 0$.

(b) Show that $A^{-1} = \frac{1}{8}(6I - A)$ without computing the inverse of A explicitly.

$$\begin{pmatrix} 2 & -1 & -1 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 2 & -1 & -1 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix} - 6 \begin{pmatrix} 2 & -1 & -1 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix} + 8 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 4 & -6 & -6 \\ 0 & 10 & 6 \\ 0 & 6 & 10 \end{pmatrix} - \begin{pmatrix} 12 & -6 & -6 \\ 0 & 18 & 6 \\ 0 & 6 & 18 \end{pmatrix} + \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} -8 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & -8 \end{pmatrix} + \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

$$= 0_{3\times 3}. \quad \square$$

b) $A^2 - 6A + 8I = 0$

$A - 6 + 8A^{-1} = 0$

$A^{-1} = \frac{1}{8}(6I - A)$.

26. Let A be a square matrix.

(a) Show that if $A^2 = 0$, then $I - A$ is invertible and $(I - A)^{-1} = I + A$.

(b) Show that if $A^3 = 0$, then $I - A$ is invertible and $(I - A)^{-1} = I + A + A^2$.

(c) If $A^n = 0$ for $n \geq 4$, is $I - A$ invertible?

a) $(I - A)(I + A)$

$= I + A - A - A^2$

$= I - A^2$

$= I$ since $A^2 = 0$.

$\therefore$ $I - A$ is invertible and $(I-A)^{-1} = I + A$

b) $(I - A)(I + A + A^2)$

$= I + \cancel{A} + \cancel{A^2} - \cancel{A} - \cancel{A^2} - A^3$.

$= I - A^3$

$= I$ since $A^3 = 0$.

$\therefore$ $I - A$ is invertible and $(I - A)^{-1} = I + A + A^2$.

c) $(I - A)(I + A + A^2 + \dots + A^{n-1})$

$= I + \cancel{A} + \cancel{A^2} + \dots + \cancel{A^{n-1}} - \cancel{A} - \cancel{A^2} - \cancel{A^3} - \dots - A^{n-1+1}$

$= I - A^n$

$= I$, $A^n = 0$ $\therefore$ Invertible.

27. (a) Give three examples of 2×2 matrices $\boldsymbol{A}$ such that $\boldsymbol{A}^2 = \boldsymbol{A}$.

(b) Let $\boldsymbol{A}$ be a square matrix such that $\boldsymbol{A}^2 = \boldsymbol{A}$. Show that $\boldsymbol{I} + \boldsymbol{A}$ is invertible and $(\boldsymbol{I}+\boldsymbol{A})^{-1} = \frac{1}{2}(2\boldsymbol{I} - \boldsymbol{A})$.

a) $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

b)

Let A be $\begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & \ddots & & & \vdots \\ x_{31} & & \ddots & & \vdots \\ \vdots & & & \ddots & \\ x_{n1} & \cdots & \cdots & \cdots & x_{nn} \end{pmatrix}_{n \times n}$

Let I be $\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & & & \\ 0 & & 1 & & \\ \vdots & & & \ddots & \\ & & & & 1 \end{pmatrix}_{n \times n}$.

Then $I + A = \begin{pmatrix} x_{11}+1 & x_{12} & \cdots & x_{1n} \\ x_{21} & \ddots & & \vdots \\ x_{31} & & \ddots & \vdots \\ \vdots & & & \\ x_{n1} & \cdots & \cdots & x_{nn}+1 \end{pmatrix}$

$(I+A)\left[\frac{1}{2}(2I-A)\right] = I$

$\parallel$

$\frac{1}{2}(I+A)(2I-A)$

$= \frac{1}{2}(2I + A - A^2)$

$= I.$

$\therefore \frac{1}{2}(2I - A)$ is its inverse.

28. Determine which of the following statements are true. Justify your answer.

(a) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are invertible matrices of the same size, then $\boldsymbol{A}+\boldsymbol{B}$ is also invertible.

(b) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are singular matrices of the same size, then $\boldsymbol{A}+\boldsymbol{B}$ is also singular.

a) True

b) False

29. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be invertible matrices of the same size. Suppose $\boldsymbol{A}+\boldsymbol{B}$ is invertible. Show that $\boldsymbol{A}^{-1}+\boldsymbol{B}^{-1}$ is invertible and $(\boldsymbol{A}+\boldsymbol{B})^{-1} = \boldsymbol{A}^{-1}(\boldsymbol{A}^{-1}+\boldsymbol{B}^{-1})^{-1}\boldsymbol{B}^{-1}$,

$$(A^{-1}+B^{-1})^{-1} = A(A+B)^{-1}B.$$

$$\therefore (A^{-1}+B^{-1})[A(A+B)^{-1}B]$$

$$= (A+B)^{-1}B + B^{-1}A(A+B)B^{-1}.$$

$$= (I+B^{-1}A)(A+B)^{-1}B$$

$$= (B+A)(A+B)^{-1}$$

$$= I.$$

30. Complete the proof of Theorem 2.3.9:

Let $\boldsymbol{A}, \boldsymbol{B}$ be two invertible matrices of the same size and c a nonzero scalar. Show that

(a) $c\boldsymbol{A}$ is invertible and $(c\boldsymbol{A})^{-1} = \frac{1}{c}\boldsymbol{A}^{-1}$.

(b) $\boldsymbol{A}^{-1}$ is invertible and $(\boldsymbol{A}^{-1})^{-1} = \boldsymbol{A}$.

(c) $\boldsymbol{AB}$ is invertible and $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$.

a) $(cA)^{-1} = c^{-1}A^{-1} = \frac{1}{c}A^{-1}$

31. (a) Let $\boldsymbol{A}, \boldsymbol{P}$ and $\boldsymbol{D}$ be square matrices of the same size such that $\boldsymbol{A} = \boldsymbol{PDP}^{-1}$. Show that $\boldsymbol{A}^k = \boldsymbol{PD}^k\boldsymbol{P}^{-1}$ for all positive integer k.

(b) Let $\boldsymbol{A} = \begin{pmatrix} -7 & 5 \\ -10 & 8 \end{pmatrix}$. Verify that

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1}.$$

Hence or otherwise, find $\boldsymbol{A}^{10}$.

a) $A = PDP^{-1}$

Then $A^k = (PDP^{-1})^k$

$= (PDP^{-1})(PDP^{-1}) \cdots (PDP^{-1})$

$= (PD\cancel{P^{-1}}\cancel{P}DP^{-1}\cancel{P}D\cancel{P^{-1}} \cdots \cancel{P}DP^{-1})$

$= (PD \cdots DP^{-1})$

$= PD^kP^{-1}$

b)

$$\underbrace{\begin{pmatrix} -7 & 5 \\ -10 & 8 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}}_{\begin{pmatrix} -2 & 3 \\ -2 & 6 \end{pmatrix}} = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}}_{\begin{pmatrix} -2 & 3 \\ -2 & 6 \end{pmatrix}}$$

$= \begin{pmatrix} -2 & 3 \\ -2 & 6 \end{pmatrix}$ //

$$A^{10} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}^{10}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1}$$ //

32. Express the matrix $\boldsymbol{A} = \begin{pmatrix} 5 & -2 & 6 & 0 \\ -2 & 1 & 3 & 1 \end{pmatrix}$ in the form $\boldsymbol{E}_1\boldsymbol{E}_2\cdots\boldsymbol{E}_n\boldsymbol{R}$ where $\boldsymbol{E}_1, \boldsymbol{E}_2, \ldots, \boldsymbol{E}_n$ are elementary matrices and $\boldsymbol{R}$ is the reduced row-echelon form of $\boldsymbol{A}$.

$$\begin{pmatrix} 5 & -2 & 6 & 0 \\ -2 & 1 & 3 & 1 \end{pmatrix}$$

$$\xrightarrow{R_2+\frac{2}{5}R_1} \begin{pmatrix} 5 & -2 & 6 & 0 \\ 0 & \frac{1}{5} & \frac{27}{5} & 1 \end{pmatrix} \qquad \therefore E_1 = \begin{pmatrix} 1 & 0 \\ \frac{2}{5} & 1 \end{pmatrix}$$

$$\xrightarrow{5R_2} \begin{pmatrix} 5 & -2 & 6 & 0 \\ 0 & 1 & 27 & 5 \end{pmatrix} \qquad E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$$

$$\xrightarrow{R_1+2R_2} \begin{pmatrix} 5 & 0 & 60 & 10 \\ 0 & 1 & 27 & 5 \end{pmatrix} \qquad E_3 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{5}R_1} \begin{pmatrix} 1 & 0 & 12 & 2 \\ 0 & 1 & 27 & 5 \end{pmatrix} \qquad E_4 \begin{pmatrix} \frac{1}{5} & 0 \\ 0 & 1 \end{pmatrix}$$

33. Let $\boldsymbol{A}$ be the 4×4 matrix obtained from $\boldsymbol{I}$ by the following sequence of elementary row operations:

$$\boldsymbol{I} \xrightarrow{\frac{1}{2}R_2} \bullet \xrightarrow{R_1 - R_2} \bullet \xrightarrow{R_2 \leftrightarrow R_4} \bullet \xrightarrow{R_3 + 3R_1} \boldsymbol{A}.$$

LN: A, B are row equivalent : $A \xrightarrow[E_1]{ero_1} \cdot \xrightarrow[E_2]{ero_2} \cdot \cdots \xrightarrow[E_k]{ero_k} B$. : $B \xrightarrow[E_k^{-1}]{ero_k^{-1}} \cdot \longrightarrow \cdot \longrightarrow \cdots \xrightarrow[E_1^{-1}]{ero_1^{-1}} A$

(a) Write $\boldsymbol{A}$ as a product of four elementary matrices.

(b) Write $\boldsymbol{A}^{-1}$ as a product of four elementary matrices.

$B = E_k E_{k-1} \cdots E_1 A.$
$A = E_1^{-1} \cdots E_k^{-1} B$
$E_i = ero_i(I).$

a)

$$A = E_4 E_3 E_2 E_1$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A^{-1} = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1}$$

b)

ero_1^{-1} is $2R_2 \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

ero_2^{-1} is $R_1 + R_2 \longrightarrow \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

ero_3^{-1} is $R_2 \leftrightarrow R_4. \longrightarrow E_3^{-1} = E_3.$

ero_4^{-1} is $R_3 - 3R_1 \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

34. Let $\boldsymbol{A}, \boldsymbol{B}$ be 3×3 matrices such that $\boldsymbol{A} = \boldsymbol{E}_1\boldsymbol{E}_2\boldsymbol{E}_3\boldsymbol{E}_4\boldsymbol{B}$ where

$$\boldsymbol{E}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \boldsymbol{E}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}, \quad \boldsymbol{E}_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{E}_4 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) Describe how $\boldsymbol{A}$ is obtained from $\boldsymbol{B}$ by elementary row operations.

(b) If $\boldsymbol{A}$ is invertible, is $\boldsymbol{B}$ invertible? Justify your answer.

a)

$E_4 : R_1 - R_3.$

$E_3 : R_1 \leftrightarrow R_3$

$E_2 : R_3 + 2R_2.$

$E_1 : 2R_3$

b) Since E_i are invertible for all i, then B is invertible.

35. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be 4×4 matrices such that $\boldsymbol{E}_1\boldsymbol{E}_2\boldsymbol{A} = \boldsymbol{E}_3\boldsymbol{E}_4\boldsymbol{B}$ where

$$\boldsymbol{E}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{E}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\boldsymbol{E}_3 = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{E}_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Describe how $\boldsymbol{B}$ can be obtained from $\boldsymbol{A}$ by elementary row operations.

$A = E_2^{-1}E_1^{-1}E_3E_4B$

$E_4 : R_1 \leftrightarrow R_4$

$E_3 : R_1 + 2R_3$

$E_1^{-1} : \frac{1}{2}R_3$

$E_2^{-1} : R_1 + R_2$

36. (a) Let $\boldsymbol{A} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ where $ac \neq 0$. Express $\boldsymbol{A}$ as a product of three elementary row matrices.

(b) Let $\boldsymbol{B} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ -1 & -1 & 4 \end{pmatrix}$. Express $\boldsymbol{B}$ as a product of four elementary matrices.

a)

$$A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \xrightarrow{\frac{1}{a}R_1} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & c \end{pmatrix} \xrightarrow{\frac{1}{c}R_2} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix} \xrightarrow{R_1 - \frac{b}{a}R_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\therefore A = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}\begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix}$$

b)

$$B = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ -1 & -1 & 4 \end{pmatrix} \xrightarrow{R_3 + R_1} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 1 & 4 \end{pmatrix} \xrightarrow{R_3 - R_2} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 - 3R_3} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 - 2R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\therefore B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & +1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & +3 \\ 0 & 0 & 1 \end{pmatrix}\cancel{\begin{pmatrix} 1 & +2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}.$$

37. For each of the following matrices, determine if the matrix is invertible. Also, for each of the invertible matrices, find the inverse of the matrix.

(a) $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, (b) $\begin{pmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{pmatrix}$, (c) $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$,

(d) $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 4 \end{pmatrix}$, (e) $\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 2 & 6 & 3 \\ 1 & -2 & -6 & -4 \\ 1 & 1 & 1 & 0 \end{pmatrix}$, (f) $\begin{pmatrix} 1 & 3 & 1 & 1 \\ 2 & 5 & 2 & 2 \\ 1 & 3 & 8 & 9 \\ 1 & 3 & 2 & 2 \end{pmatrix}$.

a)

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 - R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 \end{array}\right) \xrightarrow{R_3 - R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{2}R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right) \xrightarrow[R_1 - R_3]{R_2 - R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right)$$

$\therefore$ invertible, inverse is $\begin{pmatrix} -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \end{pmatrix}$

b)

$$\left(\begin{array}{ccc|ccc} -1 & 3 & -4 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 1 & 0 \\ -4 & 2 & -9 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3 - 4R_1]{R_2 + 2R_1} \left(\begin{array}{ccc|ccc} -1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 10 & -7 & 2 & 1 & 0 \\ 0 & -10 & 7 & -4 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 + R_2} \left(\begin{array}{ccc|ccc} -1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 10 & -7 & 2 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 \end{array}\right)$$

$\therefore$ not invertible.

c)

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3 + \frac{1}{2}R_2} \left(\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1.5 & 0 & \frac{1}{2} & 1 \end{array}\right)$$

$$\xrightarrow[\frac{1}{2}R_2,\ \frac{1}{2}R_1]{\frac{2}{3}R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{3} & \frac{2}{3} \end{array}\right) \xrightarrow[R_1 - \frac{1}{2}R_3]{R_2 + \frac{1}{2}R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{6} & -\frac{1}{3} \\ 0 & 1 & 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 1 & 0 & \frac{1}{3} & \frac{2}{3} \end{array}\right)$$

$\therefore$ invertible, inverse $= \begin{pmatrix} \frac{1}{2} & -\frac{1}{6} & -\frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$

38. Solve the matrix equation $\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 3 & 2 \end{pmatrix} \boldsymbol{X} = \begin{pmatrix} 2 & 3 & 4 & 1 \\ 1 & 0 & 3 & 7 \\ 2 & 1 & 1 & 2 \end{pmatrix}$.

$$X = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 3 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 3 & 4 & 1 \\ 1 & 0 & 3 & 7 \\ 2 & 1 & 1 & 2 \end{pmatrix}$$

$$X = \begin{pmatrix} 4/7 & -1/7 & -1/7 \\ -2/7 & -3/7 & -4/7 \\ 1/7 & 5/7 & -2/7 \end{pmatrix} \begin{pmatrix} 2 & 3 & 4 & 1 \\ 1 & 0 & 3 & 7 \\ 2 & 1 & 1 & 2 \end{pmatrix}$$

$$X = \begin{pmatrix} 5/7 & 11/7 & 12/7 & -5/7 \\ 1/7 & -2/7 & -13/7 & -15/7 \\ 3/7 & 1/7 & 17/7 & 32/7 \end{pmatrix}.$$

39. A manufacturer makes three types of chairs A, B, C. The company has available 260 units of wood, 60 units of upholstery and 240 units of labor. The manufacture wants a production schedule that uses all of these resources. The various products require the following amounts of resources.

	A	B	C
Wood	4	4	3
Upholstery	0	1	2
Labor	2	4	5

(a) Find the inverse of the data matrix above and hence determine how many pieces of each product should be manufactured.

(b) If the amount of wood is increased by 10 units, how will this change the number of type C chairs produced?

$(1 \quad 0 \quad a)$

no. of wood = $4 \cdot A + 4 \cdot B + 3 \cdot C$

no. of uphols = $+ 1 \cdot B + 2 \cdot C$

no. of labor = $2 \cdot A + 4 \cdot B + 5 \cdot C$

$$\begin{pmatrix} 4 & 4 & 3 \\ 0 & 1 & 2 \\ 2 & 4 & 5 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} W \\ U \\ L \end{pmatrix} = \begin{pmatrix} 260 \\ 60 \\ 240 \end{pmatrix}$$

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 4 & 4 & 3 \\ 0 & 1 & 2 \\ 2 & 4 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 260 \\ 60 \\ 240 \end{pmatrix}.$$

$$(M \mid I) \xrightarrow{\sim} \text{RREF } (I \mid M^{-1})$$

$$\left(\begin{array}{ccc|ccc} 4 & 4 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 2 & 4 & 5 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 - \frac{1}{2}R_1} \left(\begin{array}{ccc|ccc} 4 & 4 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & 7/2 & -\frac{1}{2} & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 - 2R_2} \left(\begin{array}{ccc|ccc} 4 & 4 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & -1/2 & -1/2 & -2 & 1 \end{array}\right)$$

$$\xrightarrow{R_1 - 4R_2} \left(\begin{array}{ccc|ccc} 4 & 0 & -5 & 1 & -4 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1/2 & -1/2 & -2 & 1 \end{array}\right)$$

$$\xrightarrow{(-2)R_3} \left(\begin{array}{ccc|ccc} 4 & 0 & -5 & 1 & -4 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 4 & -2 \end{array}\right)$$

$$\xrightarrow{R_1 + 5R_3} \left(\begin{array}{ccc|ccc} 4 & 0 & 0 & 6 & 16 & -10 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 4 & -2 \end{array}\right)$$

$$\xrightarrow{R_2 - 2R_3} \left(\begin{array}{ccc|ccc} 4 & 0 & 0 & 6 & 16 & -10 \\ 0 & 1 & 0 & -2 & -7 & 4 \\ 0 & 0 & 1 & 1 & 4 & -2 \end{array}\right)$$

$$\xrightarrow{\frac{1}{4}R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/2 & 4 & -5/2 \\ 0 & 1 & 0 & -2 & -7 & 4 \\ 0 & 0 & 1 & 1 & 4 & -2 \end{array}\right)$$

M^{-1}.

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix} = M^{-1} \begin{pmatrix} 260 \\ 60 \\ 240 \end{pmatrix}$$

$$= \begin{pmatrix} 30 \\ 20 \\ 20 \end{pmatrix}$$

b)

$$\begin{pmatrix} A \\ B \\ \boxed{C} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \boxed{0} & 0 & 0 \end{pmatrix} \begin{pmatrix} W \\ n \\ L \end{pmatrix}$$

$M^{-1}.$

$W \to W + 10.$

$C \to C + 1 \cdot 10$

from M^{-1}

40. Determine the value(s) of a so that the matrix $\begin{pmatrix} 1 & 0 & a \\ 0 & a & 1 \\ a & 1 & 0 \end{pmatrix}$ is invertible. What is the inverse of the matrix if it exists?

$$A = \left(\begin{array}{ccc|ccc} 1 & 0 & a & 1 & 0 & 0 \\ 0 & a & 1 & 0 & 1 & 0 \\ a & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 - aR_1} \left(\begin{array}{ccc|ccc} 1 & 0 & a & 1 & 0 & 0 \\ 0 & a & 1 & 0 & 1 & 0 \\ 0 & 1 & -a^2 & -a & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 - \frac{1}{a}R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & a & 1 & 0 & 0 \\ 0 & a & 1 & 0 & 1 & 0 \\ 0 & 0 & -a^2-\frac{1}{a} & -a & -\frac{1}{a} & 1 \end{array}\right)$$

$\therefore$ A is invertible if $-a^2 - \frac{1}{a} \neq 0.$

$a^3 + 1 \neq 0.$

$a^3 \neq -1$

$a \neq \sqrt[3]{-1}.$

$a \neq -1$

$$\xrightarrow[\frac{1}{a}R_2]{-\frac{a}{(a^3+1)}R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & a & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{a} & 0 & \frac{1}{a} & 0 \\ 0 & 0 & 1 & \frac{a^2}{(a^3+1)} & \frac{1}{(a^3+1)} & -\frac{a}{(a^3+1)} \end{array}\right)$$

$$\xrightarrow[R_1 - aR_3]{R_2 - \frac{1}{a}R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1-\frac{a^3}{a^3+1} & \frac{-a}{a^3+1} & \frac{a^2}{(a^3+1)} \\ 0 & 1 & 0 & -\frac{a}{(a^3+1)} & \frac{1}{a}-\frac{1}{a(a^3+1)} & \frac{1}{(a^3+1)} \\ 0 & 0 & 1 & \frac{a^2}{(a^3+1)} & \frac{1}{(a^3+1)} & -\frac{a}{(a^3+1)} \end{array}\right)$$

inverse.

$$\frac{1}{1+a^3}\begin{pmatrix} 1 & -a & a^2 \\ -a & a^2 & 1 \\ a^2 & 1 & -a \end{pmatrix}$$

41. (a) Determine the values of a, b and c so that the homogeneous system

$$\begin{cases} x + y + z = 0 \\ ax + by + cz = 0 \\ a^2x + b^2y + cz^2 = 0 \end{cases}$$

has non-trivial solution.

(b) Write down the conditions so that the matrix $\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix}$ is invertible.

a) $$A = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ a & b & c & 0 \\ a^2 & b^2 & c^2 & 0 \end{array}\right)$$

$$\xrightarrow[R_3 - a^2R_1]{R_2 - aR_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & b-a & c-a & 0 \\ 0 & b^2-a^2 & c^2-a^2 & 0 \end{array}\right)$$

$$\xrightarrow{R_3 - (b+a)R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & b-a & c-a & 0 \\ 0 & 0 & \downarrow & 0 \end{array}\right)$$

$c^2 - a^2 - (c-a)(b+a).$

$\therefore$ A has non-trivial soln,

$c^2 - a^2 - (b+a)(c-a) = 0.$ or $(b-a) = 0.$

$(c-a)(c+a) - (b+a)(c-a) = 0$ or $a = b$

$(c-a)(c+\cancel{a} - b - \cancel{a}) = 0$

$(c-a)(c-b) = 0.$

$c = a$ or $c = b.$

b) $$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & (c-a)(c-b) \end{pmatrix}$$

is invertible iff A has only the trivial solution.

$(c-a)(c-b) \neq 0$

$\therefore c \neq a,\ c \neq b.$

$b - a \neq 0$

$\therefore b \neq a.$

a, b, c must be unique.

42. Prove Theorem 2.4.14:

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two square matrices of the same order. Prove that if $\boldsymbol{A}$ is singular then $\boldsymbol{AB}$ and $\boldsymbol{BA}$ are singular. (Since we use Theorem 2.4.14 to prove Theorem 2.5.22.2, we cannot use determinants to do this question. Work out the proof using the definition of inverses together with Theorem 2.4.12.)

$\Downarrow$

2.4.12. If $AB = I$ then A and B are invertible. and $A^{-1} = B$, $B^{-1} = A$.

A, B same size.

A is singular

proof by contradiction:

Suppose AB is invertible

$\therefore \exists\, C$ s.t. $(AB)C = I$

$A(BC) = I$.

matrix matrix

both must be invertible.

$\therefore$ Contradiction since A is singular.

$\therefore$ AB cannot be invertible. $\rightarrow$ singular.

Suppose BA is invertible.

$\exists\, D$ s.t. $D(BA) = I$.

$(DB)\cdot A = I$.

M A.

both invertible

$\therefore$ BA is singular.

43. Let $\boldsymbol{A}$ be an $m \times n$ matrix which is row equivalent to the following matrix:

$$\begin{pmatrix} \boldsymbol{R} \\ \boxed{0 \quad \cdots \quad 0} \end{pmatrix}$$

where the last row is a zero row and $\boldsymbol{R}$ is an $(m-1) \times n$ matrix. Show that there exists an $n \times 1$ matrix $\boldsymbol{b}$ such that the linear system $\boxed{\boldsymbol{Ax} = \boldsymbol{b}}$ is inconsistent.

(*Hint*: If $\boldsymbol{A}$ is row equivalent to a matrix $\boldsymbol{C}$, then $\boldsymbol{A} = \boldsymbol{E}_k \cdots \boldsymbol{E}_1 \boldsymbol{C}$ for some elementary matrices $\boldsymbol{E}_1, \ldots, \boldsymbol{E}_k$.

$$\begin{pmatrix} R \\ 0 \cdots 0 \end{pmatrix} x = d = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \quad \therefore \text{ Inconsistent.}$$

row equivalent

$\boxed{Ax = ?}$ inconsistent

$$A = E_n E_{n-1} \cdots E_1 \begin{pmatrix} R \\ 0 \cdots 0 \end{pmatrix}$$

$$\begin{pmatrix} R \\ 0 \cdots 0 \end{pmatrix} = E_1^{-1} E_2^{-1} \cdots E_n^{-1} A$$

$$E_1^{-1} E_2^{-1} \cdots E_k^{-1} A x = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \text{ inconsistent}$$

$$Ax = E_k \cdots E_1 \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

example b.

Let $\boldsymbol{A}$ be an $m \times n$ matrix which is row equivalent to the following matrix:

$$\begin{pmatrix} & \boldsymbol{R} & \\ 0 & \cdots & 0 \end{pmatrix}$$

where the last row is a zero row and $\boldsymbol{R}$ is an $(m-1) \times n$ matrix. Show that there exists an $n \times 1$ matrix $\boldsymbol{b}$ such that the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ is inconsistent.

44. Let $\boldsymbol{A}$ be an $m \times n$ matrix and $\boldsymbol{B}$ an $n \times m$ matrix.

(a) Suppose $\boldsymbol{A}$ is the matrix described in Question 2.43. Show that $\boldsymbol{AB}$ is singular.

(b) If $m > n$, can $\boldsymbol{AB}$ be invertible? Justify your answer. (*Hint*: How will a row-echelon form of $\boldsymbol{A}$ look like if $m > n$?)

(c) When $m = 2$ and $n = 3$, give an example of $\boldsymbol{A}$ and $\boldsymbol{B}$ such that $\boldsymbol{AB}$ is invertible.

a)

$$A = E_1 E_2 \cdots E_k \begin{pmatrix} & R & \\ 0 & \cdots & 0 \end{pmatrix}$$

$$AB = E_1 E_2 \cdots E_k \begin{pmatrix} & R & \\ 0 & \cdots & 0 \end{pmatrix} B.$$

AB is row equivalent to $\begin{pmatrix} & R & \\ 0 & \cdots & 0 \end{pmatrix} B = \begin{pmatrix} & RB & \\ 0 & \cdots & 0 \end{pmatrix}$

Since AB can never be equivalent to identity matrix for any B,

$\therefore$ AB is singular.

b) AB cannot be invertible because a REF of A can have at most n non-zero rows and $m>n$, a REF of a must have a zero-row.

$\therefore$ using (a), AB cannot be invertible.

c) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

$AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\therefore$ invertible.

45. Let $\boldsymbol{A}$ be a square matrix and let $\mathscr{R}_1, \mathscr{R}_2, \ldots, \mathscr{R}_n$ be elementary row operations such that

$$\boldsymbol{A} \xrightarrow{\mathscr{R}_1} \bullet \xrightarrow{\mathscr{R}_2} \bullet \overset{\cdots}{\cdots} \bullet \xrightarrow{\mathscr{R}_n} \boldsymbol{I}.$$

Suppose $\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_n$ are the elementary column operations (see Discussion 2.4.15) such that $\mathscr{R}_i$ and $\mathscr{C}_i$ correspond to the same elementary matrix for each i.
Show that

$$\boldsymbol{A} \xrightarrow{\mathscr{C}_n} \bullet \xrightarrow{\mathscr{C}_{n-1}} \bullet \overset{\cdots}{\cdots} \bullet \xrightarrow{\mathscr{C}_1} \boldsymbol{I}.$$

For $i = 1, 2, \ldots n$, let E_i be elementary matrix associated with the row operation R_i (and the column operation C_i)
Since A is reduced to I by the row operation $R_1, R_2, \ldots R_n$, we have

$$E_n \cdots E_2 E_1 A = I.$$

$\therefore$ A is invertible and $A^{-1} = E_n \cdots E_2 E_1$

$\because$ $A E_n \cdots E_2 E_1 = I.$

$\therefore$ $A \xrightarrow{C_n} \xrightarrow{C_{n-1}} \cdots \xrightarrow{C_1} I$

46. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two invertible matrices of the same size. In each of the following cases, describe how $\boldsymbol{B}^{-1}$ is related to $\boldsymbol{A}^{-1}$.

(a) $\boldsymbol{B}$ can be obtained from $\boldsymbol{A}$ by multiplying a nonzero constant c to a ith row.

(b) $\boldsymbol{B}$ can be obtained from $\boldsymbol{A}$ by interchanging two rows.

(c) $\boldsymbol{B}$ can be obtained from $\boldsymbol{A}$ by adding a multiple of a row to another row.

a) $B = EA$

$B^{-1} = A^{-1}E^{-1}$

post multiplying an elementary matrix on A is equivalent to doing a column operation on A.

$\therefore$ B^{-1} can be obtain from A^{-1} by multiplying $\frac{1}{c}$ on ith column of A^{-1}.

b) B^{-1} can be obtained from A^{-1} by interchanging 2 columns.

c) B^{-1} can be obtained from A^{-1} by adding $-c$ times jth column to ith column

if c times ith row was added to the jth row.

47. For each of the following matrices,

(a) $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, (b) $\begin{pmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{pmatrix}$, (c) $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$, (d) $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 4 \end{pmatrix}$.

A B C D

$\det(B) = 0$

$\therefore$ Not invertible.

(i) compute the determinant using cofactor expansion;

(ii) compute the determinant using the method discussed in Remark 2.5.16; and

(iii) if the matrix is invertible, compute the inverse using Theorem 2.5.25.

i)
$$\det(A) = (-1)^{1+1} A_{11} \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + (-1)^{1+2} A_{12} \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + (-1)^{1+3} A_{13} \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$$
$$= 0 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - (1)(-1) + 1$$
$$= 2.$$

$$\det(B) = -1 \begin{vmatrix} 4 & 1 \\ 2 & -9 \end{vmatrix} - (3) \begin{vmatrix} 2 & 1 \\ -4 & -9 \end{vmatrix} - 4 \begin{vmatrix} 2 & 4 \\ -4 & 2 \end{vmatrix}$$
$$= 38 + 42 - 80.$$
$$= 0.$$

$$\det(C) = 2 \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} - 0 \begin{vmatrix} 0 & -1 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 0 & 2 \\ 0 & -1 \end{vmatrix}$$
$$= 6$$

$$\det(D) = \begin{vmatrix} 2 & 0 & 0 \\ 2 & 3 & 0 \\ 2 & 3 & 4 \end{vmatrix} - 0 \cancel{\begin{vmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 4 \end{vmatrix}} + 0 \cancel{\begin{vmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 4 \end{vmatrix}} - \begin{vmatrix} 1 & 2 & 0 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{vmatrix}.$$
$$= 2 \begin{vmatrix} 3 & 0 \\ 3 & 4 \end{vmatrix} - \left(\begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix} - 2 \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} \right)$$
$$= 24 - 0$$
$$= 24.$$

ii) gaussian elimination.

$$\det(A) = -\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{vmatrix}$$
$$= -(1 \times 1 \times -2) = 2.$$

$$\det(B) = \begin{vmatrix} -1 & 3 & -4 \\ 0 & 10 & -7 \\ 0 & 0 & 0 \end{vmatrix}$$
$$= 0$$

$$\det(C) = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & \frac{3}{2} \end{vmatrix} = 2 \times 2 \times \frac{3}{2} = \frac{4 \times 3}{2} = \frac{12}{2} = 6.$$

$$\det(D) = \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{vmatrix}$$
$$= 1 \times 2 \times 3 \times 4$$
$$= 6 \times 4$$
$$= 24.$$

iii)
$$A^{-1} = \frac{1}{\det(A)} (\operatorname{adj}(A))$$
$$= \frac{1}{2} \begin{pmatrix} \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \end{pmatrix}^T = \frac{1}{2} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

48. Solve the following linear systems using Cramer's Rule.

(a) $\begin{cases} 9x + y = 8 \\ x - 9y = 10 \end{cases}$ (b) $\begin{cases} 9x - y = 0 \\ y - z = -1 \\ x + z = 2 \end{cases}$

(c) $\begin{cases} x + y + z = -1 \\ 2x - y - z = 4 \\ x + 2y - 3z = 7 \end{cases}$ (d) $\begin{cases} w - x = 0 \\ 2w + x - y = 0 \\ 3w + 2x + y - z = 1 \\ 4w + 3x + 2y + z = -1 \end{cases}$

a)
$$\begin{pmatrix} 9 & 1 \\ 1 & -9 \end{pmatrix} x = \begin{pmatrix} 8 \\ 10 \end{pmatrix}.$$
$$x = \frac{1}{\det(A)} \begin{pmatrix} \det(A_1) \\ \det(A_2) \end{pmatrix}$$
$$= \frac{1}{-82} \begin{pmatrix} \begin{vmatrix} 8 & 1 \\ 10 & -9 \end{vmatrix} \\ \begin{vmatrix} 9 & 8 \\ 1 & 10 \end{vmatrix} \end{pmatrix}$$
$$= \frac{1}{-82} \begin{pmatrix} -82 \\ 82 \end{pmatrix}$$
$$= \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

b)
$$\begin{pmatrix} 9 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$$
$$x = \frac{1}{\det(A)} \begin{pmatrix} \begin{vmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ 2 & 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 9 & 0 & 0 \\ 0 & -1 & -1 \\ 1 & 2 & 1 \end{vmatrix} \\ \begin{vmatrix} 9 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{vmatrix} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \end{pmatrix}$$

49. Let $\boldsymbol{A} = \begin{pmatrix} a & b & c \\ 0 & b & c \\ 0 & 0 & c \end{pmatrix}$.

(a) Find $\det(\boldsymbol{A})$.

(b) Determine the values of a, b, c for which $\boldsymbol{A}$ is invertible and find $\boldsymbol{A}^{-1}$.

a) $\det(A) = abc$.

b) $c \neq 0,\ b \neq 0,\ a \neq 0$.

$$A^{-1} = \frac{1}{abc}\begin{pmatrix} \begin{vmatrix} b & c \\ 0 & c \end{vmatrix} & -\begin{vmatrix} 0 & c \\ 0 & c \end{vmatrix} & \begin{vmatrix} 0 & b \\ 0 & 0 \end{vmatrix} \\ -\begin{vmatrix} b & c \\ 0 & c \end{vmatrix} & \begin{vmatrix} a & c \\ 0 & c \end{vmatrix} & -\begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} \\ \begin{vmatrix} b & c \\ b & c \end{vmatrix} & -\begin{vmatrix} a & c \\ 0 & c \end{vmatrix} & \begin{vmatrix} a & b \\ 0 & b \end{vmatrix} \end{pmatrix}^T$$

$$= \frac{1}{abc}\begin{pmatrix} bc & 0 & 0 \\ -bc & ac & 0 \\ 0 & -ac & ab \end{pmatrix}^T$$

$$= \frac{1}{abc}\begin{pmatrix} bc & -bc & 0 \\ 0 & ac & -ac \\ 0 & 0 & ab \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{a} & -\frac{1}{a} & 0 \\ 0 & \frac{1}{b} & -\frac{1}{b} \\ 0 & 0 & \frac{1}{c} \end{pmatrix}$$

50. Let

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & -1 \\ -2 & 1 & 0 & -2 \\ 0 & 0 & 2 & 1 \end{pmatrix}, \quad \boldsymbol{C} = \begin{pmatrix} -1 & 3 & 4 & -2 \\ 0 & 10 & 1 & 1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 2 \\ 4 \\ 6 \\ 8 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

(a) Find $\det(\boldsymbol{C})$.

(b) Without computing the matrix $\boldsymbol{AC}$, explain why the homogeneous linear system $\boldsymbol{ACx} = \boldsymbol{0}$ has infinitely many solutions.

a) $\det(C) = 0$

b) $\det(AC) = \det(A)\det(C) = 0$

$\therefore$ $ACx = 0$ has infinitely many sols

as there is a zero row.

51. Find all values of λ for which $\det(\boldsymbol{A}) = 0$.

(a) $\boldsymbol{A} = \begin{pmatrix} \lambda-2 & 1 \\ -5 & \lambda+4 \end{pmatrix}$,

(b) $\boldsymbol{A} = \begin{pmatrix} \lambda-4 & 0 & 0 \\ 0 & \lambda & 2 \\ 0 & 3 & \lambda-1 \end{pmatrix}$,

(c) $\begin{pmatrix} 1 & \lambda & \lambda & \lambda \\ 2 & \lambda & \lambda & \lambda \\ \lambda+1 & 1 & 2 & 0 \\ 4 & 0 & 1 & 2\lambda \end{pmatrix}$,

(d) $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 2-\lambda^2 & 2 & 3 \\ 2 & 3 & 1 & 5 \\ 2 & 3 & 1 & 9-\lambda^2 \end{pmatrix}$.

a)

$\det(A) = (\lambda-2)(\lambda+4)+5.$

$= \lambda^2+4\lambda-2\lambda-8+5.$

$= \lambda^2+2\lambda-3 = 0.$

$(\lambda-1)(\lambda+3)=0.$

$\therefore \lambda=1$ or $\lambda=-3.$

52. Show that $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & a^2 \end{vmatrix} = (b-a)(c-a)(c-b).$

$\begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & a^2 \end{pmatrix} \xrightarrow[R_3-R_1]{R_2-R_1} \begin{pmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{pmatrix}$

$\therefore \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$

$= (b-a)(c^2-a^2) - (c-a)(b^2-a^2)$

$= (b-a)(c-a)(c-b).$

53. Let $\boldsymbol{A}$ be a 4×4 matrix such that $\det(\boldsymbol{A}) = 9$. Find

(a) $\det(3\boldsymbol{A})$, (b) $\det(\boldsymbol{A}^{-1})$, (c) $\det(3\boldsymbol{A}^{-1})$, (d) $\det((3\boldsymbol{A})^{-1})$.

a) $\det(3A) = 3^4 \det(A) = 3^4 \cdot 9.$

$= 729.$

b) $\det(A^{-1}) = \dfrac{1}{\det(A)}$

$= \dfrac{1}{9}.$

c) $\det(3A^{-1}) = 3^4 \cdot \det(A^{-1})$

$= 3^4 \cdot \dfrac{1}{9} = 9.$

d) $\det((3A)^{-1}) = \left(\dfrac{1}{3}\right)^4 \det(A^{-1})$

$= \dfrac{1}{3^4} \cdot \dfrac{1}{9}.$

$= \dfrac{1}{729}.$

54. Let $\boldsymbol{A}, \boldsymbol{B}$ and $\boldsymbol{C}$ be matrices such that both $\boldsymbol{A}$ and $\boldsymbol{B}$ can be obtained from $\boldsymbol{C}$ by elementary row operations:

$$\boldsymbol{C} \xrightarrow{3R_2} \bullet \xrightarrow{R_3+2R_1} \boldsymbol{A}, \qquad \boldsymbol{C} \xrightarrow{R_1+R_2} \bullet \xrightarrow{R_2\leftrightarrow R_3} \bullet \xrightarrow{R_4-R_2} \boldsymbol{B}.$$

(a) Describe how $\boldsymbol{A}$ can be obtained from $\boldsymbol{B}$ by elementary row operations.

(b) Let $\boldsymbol{A} = \begin{pmatrix} 1 & -1 & 7 & \frac{1}{11} \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$. Find $\det(\boldsymbol{B})$.

a) $B \xrightarrow{R_4+R_2} \bullet \xrightarrow{R_2\leftrightarrow R_3} \bullet \xrightarrow{R_1-R_2} \bullet \xrightarrow{3R_2} \bullet \xrightarrow{R_3+2R_1} A.$

b) $A = \begin{pmatrix} 1 & -1 & 7 & \frac{1}{11} \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

$\det(A) = 1 \cdot 2 \cdot 3 \cdot -1$
$= -6.$

$\therefore \det(B) = (-1)(\frac{1}{3})(-6) = 2.$

55. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be 3×3 matrices such that

2nd, 1st, first

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\boldsymbol{A} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\boldsymbol{B}.$$

(a) Describe how $\boldsymbol{A}$ can be obtained from $\boldsymbol{B}$ by elementary row operations.

(b) If $\det(\boldsymbol{A}) = 4$, find $\det(\boldsymbol{B})$.

a) $B \xrightarrow{R_1-2R_3} \xrightarrow{R_3-3R_2} \xrightarrow{\frac{1}{2}R_2} \xrightarrow{R_1\leftrightarrow R_2} A.$

b) $\det(A) = 4.$
$\det(B) = (-1)(2)(4) = -8.$

56. Let $\boldsymbol{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ where $a, b, c, d, e, f, g, h, i$ are either 0 or 1. Find the largest possible value and the smallest possible value of $\det(\boldsymbol{A})$.

$\det(A) = aei + bfg + dhc$
$- ceg - fha - bdi$

$a = 0,\quad bfg + dhc - ceg - bdi$
$111 + 111 - 101 - 110$

$\therefore$ max $= 2$,

min $= -2$.

57. Let $\boldsymbol{A}$ be a 2×2 orthogonal matrix (see Question 2.12).

(a) Prove that $\det(\boldsymbol{A}) = \pm 1$.

(b) If $\det(\boldsymbol{A}) = 1$, show that $\boldsymbol{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ for some real number θ.

(c) If $\det(\boldsymbol{A}) = -1$, show that $\boldsymbol{A} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ for some real number θ.

(*Hint*: If a and b are real numbers such that $a^2 + b^2 = 1$, then $a = \cos\theta$ and $b = \sin\theta$ for some real number θ.)

a)

$$AA^T = I.$$

$$\therefore \det(A) = \det(A^T)$$

$$\therefore \det(A)^2 = \det(I)$$

$$= 1.$$

$$\det(A) = \pm 1 \quad //$$

b)

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\therefore \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\therefore a^2 + c^2 = ad - bc = 1. \quad //$$

58. Prove Theorem 2.5.12:

(a) Prove that the determinant of a square matrix with two identical rows is zero.

(*Hint*: First prove that the statement is true for 2×2 matrices. Then assume the statement is true for $k \times k$ matrices where $k \geq 2$. Let $\boldsymbol{A}$ be a $(k+1) \times (k+1)$ matrix such that the ith row and jth rows of $\boldsymbol{A}$ are the same. Take any $m = 1, 2, \ldots, k+1$ and $m \neq i, j$. Compute $\det(\boldsymbol{A})$ by expanding along the mth row of $\boldsymbol{A}$.)

(b) Prove that the determinant of a square matrix with two identical columns is zero.

59. Complete the proof of Theorem 2.5.15:

Let $\boldsymbol{A}$ be a square matrix.

(a) If $\boldsymbol{B}$ is a square matrix obtained from $\boldsymbol{A}$ by multiplying one row of $\boldsymbol{A}$ by a constant k, show that $\det(\boldsymbol{B}) = k\det(\boldsymbol{A})$.

(b) If $\boldsymbol{B}$ is a square matrix obtained from $\boldsymbol{A}$ by interchanging two rows of $\boldsymbol{A}$, show that $\det(\boldsymbol{B}) = -\det(\boldsymbol{A})$.

(c) Let $\boldsymbol{E}$ be an elementary matrix of the same size as $\boldsymbol{A}$. Show that $\det(\boldsymbol{E}\boldsymbol{A}) = \det(\boldsymbol{E})\det(\boldsymbol{A})$.

60. Let $\boldsymbol{A}$ be an $n \times n$ invertible matrix.

(a) Show that $\mathbf{adj}(\boldsymbol{A})$ is invertible.

(b) Find $\det(\mathbf{adj}(\boldsymbol{A}))$ and $(\mathbf{adj}(\boldsymbol{A}))^{-1}$.

(c) If $\det(\boldsymbol{A}) = 1$, show that $\mathbf{adj}(\mathbf{adj}(\boldsymbol{A})) = \boldsymbol{A}$.

a) $A\left[\frac{1}{\det(A)}\operatorname{adj}(A)\right] = I.$

$\Rightarrow \left[A\frac{1}{\det(A)}\right]\operatorname{adj}(A) = I.$

$\therefore$ adj (A) is Invertible

b) $\det(\operatorname{adj}(A)) = \det(A)^{n-1}$?

$\operatorname{adj}(A)^{-1} = \frac{1}{\det(A)} A.$

c) $\operatorname{adj}(A)^{-1} = \frac{1}{\det(\operatorname{adj}(A))}\operatorname{adj}(\operatorname{adj}(A))$

$\therefore \operatorname{adj}(\operatorname{adj}(A)) = \det(A)^{n-2} A.$?

If $\det(A) = 1$, $\operatorname{adj}(\operatorname{adj}(A)) = A$

61. Determine which of the following statements are true. Justify your answer.

(a) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are square matrices of the same size, then $\det(\boldsymbol{A}+\boldsymbol{B}) = \det(\boldsymbol{A}) + \det(\boldsymbol{B})$.

(b) If $\boldsymbol{A}$ is a square matrix, then $\det(\boldsymbol{A}+\boldsymbol{I}) = \det(\boldsymbol{A}^{\mathrm{T}}+\boldsymbol{I})$.

(c) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are square matrices of the same size such that $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{B}\boldsymbol{P}^{-1}$ for some invertible matrix $\boldsymbol{P}$, then $\det(\boldsymbol{A}) = \det(\boldsymbol{B})$.

(d) If $\boldsymbol{A}, \boldsymbol{B}$ and $\boldsymbol{C}$ are square matrices of the same size such that $\det(\boldsymbol{A}) = \det(\boldsymbol{B})$, then $\det(\boldsymbol{A}+\boldsymbol{C}) = \det(\boldsymbol{B}+\boldsymbol{C})$.

a) $\det(A+B) = 0$.

$\det(A) + \det(B) \neq 0$.

$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. $\therefore$ False.

b) $(A+I)^T$

$= A^T + I^T$

$= A^T + I$.

$\det(M) = \det(M^T)$

$\therefore$ True

c) $A = PBP^{-1}$

$\det(A) = \det(PBP^{-1})$

$= \cancel{\det(P)} \det(B) \cancel{\det(P^{-1})}$

$= \det(B)$.

$\therefore$ true.

d) $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

$B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

$\det(A) = \det(B)$.

$C \longrightarrow \det(A+C) = \det\left(2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 4$.

$\neq \det(B+C) = \det(0) = 0$.

$\therefore$ False.