EXERCISES FOR CHAPTER 1: LINEAR SYSTEMS AND GAUSSIAN ELIMINATION

Questions 1.1 to Question 1.15 are exercises for Sections 1.1 to 1.3.

1. Which of the following are linear equations in x_1, x_2 and x_3? In Parts (i)–(l), m is a constant.

(a) $\sqrt{3}x_1 - x_2 - x_3 = 0,$ linear

(b) $x_1x_3 + 2x_2 + x_1 = 4,$ non-linear

(c) $x_1 = -7x_2 + 3x_3,$ linear.

(d) $x_1 + 2x_2 + x_3^2 = 1,$ non. linear.

(e) $\sqrt{x_1} - 2x_2 + x_3 = 1,$ linear

(f) $x_1 + \frac{1}{2}x_2 + \frac{1}{3}x_3 = 1.333\pi$ linear.

(g) $2^{x_1+x_2+x_3} = 5,$ non-linear-

(h) $\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = \frac{1}{10},$ non-linear.

(i) $x_1 + x_2 + x_3 = \cos(m),$ linear.

(j) $\cos(x_1) + \cos(x_2) + \cos(x_3) = \cos(m),$ non-linear

(k) $3x_1 + x_2 - x_3^m = 0,$ non-linear. (unless m=1).

(l) $mx_1 - m^2x_2 = 9.$ linear.

2. Write down a general solution for each of the following linear equations.

(a) $2x + 5y = 0,$

(b) $8w - 2x - 5y + 6z = -1.5,$

(c) $3x_1 - 8x_2 + 2x_3 + x_4 - 4x_5 = 1.$

3. (a) Find a linear equation in variables x and y that has a general solution $x = 1 + 2t$ and $y = t$ where t is an arbitrary parameter.

(b) Show that $x = t$ and $y = \frac{1}{2}t - \frac{1}{2}$, where t is an arbitrary parameter, is also a general solution for the equation in Part (a).

4. (a) Find a linear equation in the variables x, y and z that has a general solution

$$\begin{cases} x = 3 - 4s + t \\ y = s \\ z = t \end{cases} \qquad \text{where } s, t \text{ are arbitrary parameters.}$$

(b) Express a general solution for the equation in Part (a) in two other different ways.

(c) Write down a linear system of two different nonzero linear equations such that the system has the same general solution as in Part (a).

5. (a) Give a geometrical interpretation for the linear equation $x + y + z = 1$.

(b) Give geometrical interpretations for the linear equation $x - y = 0$ in (i) the xy-plane; and (ii) the xyz-space.

(c) Give a geometrical interpretation for the solutions to the system of linear equations

$$\begin{cases} x + y + z = 1 \\ x - y = 0. \end{cases}$$

6. Consider the system of linear equations

$$\begin{cases} 3x + 4y - 5z = -8 \\ x - 2y + z = 2. \end{cases}$$

2. a) $2x+5y=0$

$x=-\frac{5}{2}y.$

$\left\{\begin{pmatrix}-\frac{5}{2}\\1\end{pmatrix}y \;\middle|\; y\in\mathbb{R}\right\}$

b) $8w-2x+5y+6z=-1.5$

$w=\frac{-1.5}{8}+\frac{1}{4}x-\frac{5}{8}y-\frac{6}{8}z.$

$\left\{\begin{pmatrix}-3/16\\0\\0\\0\end{pmatrix}+\begin{pmatrix}1/4\\1\\0\\0\end{pmatrix}x+\begin{pmatrix}-5/8\\0\\1\\0\end{pmatrix}y+\begin{pmatrix}-3/4\\0\\0\\1\end{pmatrix}z \;\middle|\; x,y,z\in\mathbb{R}\right\}.$

c) $3x_1-8x_2+2x_3+x_4-4x_5=1.$

$x_1=1+\frac{8}{3}x_2-\frac{2}{3}x_3-\frac{1}{3}x_4+\frac{4}{3}x_5$

$$\left\{\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}+\begin{pmatrix}8/3\\1\\0\\0\\0\end{pmatrix}x_2+\begin{pmatrix}-2/3\\0\\1\\0\\0\end{pmatrix}x_3+\begin{pmatrix}-1/3\\0\\0\\1\\0\end{pmatrix}x_4+\begin{pmatrix}4/3\\0\\0\\0\\1\end{pmatrix}x_5 \;\middle|\; x_2,x_3,x_4,x_5\in\mathbb{R}\right\},$$

3. a) $x=1+2t.$

$y=t.$

$\left(\begin{array}{cc|c}1&0&1+2t\\0&1&t\end{array}\right),$

$x=1+2y$

$\left\{\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}2\\1\end{pmatrix}y \;\middle|\; y\in\mathbb{R}\right\}.$

(y)

$x = t$

$y = \frac{1}{2}t - \frac{1}{2}$

$y = \frac{1}{2}x - \frac{1}{2}$.

$y + \frac{1}{2} = \frac{1}{2}x$

$2y + 1 = x$.

$x = 2y + 1$.

$\therefore \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} y \mid y \in \mathbb{R} \right\} = 3(a)$.

4. a)
$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3-4s+t \\ 0 & 1 & 0 & s \\ 0 & 0 & 1 & t \end{array}\right).$$

$x = 3 - 4y + z$.

$z = x - 3 + 4y$.

b)

$\left\{ \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix} y + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} z \mid y, z \in \mathbb{R} \right\}$.

$\left\{ \begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} x + \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} y \mid x, y \in \mathbb{R} \right\}$.

c)
$$\begin{cases} x = 6 - 4s + t \\ y + z = s + t - 3. \end{cases}$$

5. a) plane, cutting z-axis at 1.

b) i) line at 45° from origin

ii) plane at 45° between x and y-axis.

c)
$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{-R_1+R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & -1 & -1 \end{array}\right)$$

$$-2y - z = -1.$$

$$y = \frac{-1+z}{-2}$$

$$y = \frac{1}{2} - \frac{1}{2}z.$$

$$x = 1 - y - z$$

$$x = 1 - \frac{1}{2} + \frac{1}{2}z - z.$$

$$x = \frac{1}{2} - \frac{1}{2}z.$$

$$\left\{ \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} + \begin{pmatrix} -1/2 \\ -1/2 \end{pmatrix} z \mid z \in \mathbb{R} \right\}.$$

∴ line

6. a) $\begin{cases} 3x+4y-5y=-8 \\ x-2y+z=2. \end{cases}$

$$\left(\begin{array}{ccc|c} 3 & 4 & -5 & -8 \\ 1 & -2 & 1 & 2 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{3}R_1+R_2} \left(\begin{array}{ccc|c} 3 & 4 & -5 & -8 \\ 0 & -\frac{10}{3} & \frac{8}{3} & \frac{14}{3} \end{array}\right).$$

Can also substitute in.

$$-\frac{10}{3}y+\frac{8}{3}z=\frac{14}{3}.$$

$$-\frac{10}{3}y=\frac{14}{3}-\frac{8}{3}z.$$

$$y=-\frac{7}{5}+\frac{4}{5}z.$$

$$y=\frac{1}{5}(-7+4z)$$

$$3x=-8-4y+5z$$

$$x=-\frac{8}{3}-\frac{4}{3}y+\frac{5}{3}z.$$

$$x=-\frac{8}{3}-\frac{4}{3}\left(-\frac{7}{5}+\frac{4}{5}z\right)+\frac{5}{3}z.$$

$$x=-\frac{4}{5}+\frac{3}{5}z.$$

$$x=\frac{1}{5}(-4+3z).$$

$\therefore$ if $z=t$,

$$y=\frac{1}{5}(-7+4t)$$

$$x=\frac{1}{5}(-4+3t).$$

b) when $t=0$,

$x=-\frac{4}{5}$

$y=-\frac{7}{5}.$

$z=0.$

when $t=1$,

$x=-\frac{1}{5}.$

$y=-\frac{3}{5},$

$z=1.$

7. $\left(\begin{array}{cc|c} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{array}\right)$

a) parallel.

b) intersect at one point.

c) Same line

8. $\left(\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right)$

a) parallel.

b) intersect at one point.

c) same plane.

9. a) yes. A linear system will have no solution if there is a incorrect line. Etc. $1=0$ Inconsistent, Contradictory

b) yes. Equations could be the same that reduce to $0=0$.

c) no. If the system has one soln, then the number of equations must be equal to or more than the unknowns.

d) yes. Equations could be the same that reduce to $0=0$.

11.

$A = \left(\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 0 & 0 \end{array}\right)$

$A = \left\{ \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} -3 \\ 1 \end{pmatrix} \lambda \mid \lambda \in \mathbb{R} \right\}$.

$B = \left(\begin{array}{cc|c} 1 & 3 & 2 \\ 1 & 3 & 2 \end{array}\right) \xrightarrow{-P_1+P_2} \left(\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 0 & 0 \end{array}\right)$

$\therefore B = A = \left\{ \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} -3 \\ 1 \end{pmatrix} \lambda \mid \lambda \in \mathbb{R} \right\}$

$C = \left(\begin{array}{cc|c} 5 & 15 & 10 \\ 1 & 3 & 2 \end{array}\right) \xrightarrow{-\frac{1}{5}P_1+P_2} \left(\begin{array}{cc|c} 5 & 15 & 10 \\ 0 & 0 & 0 \end{array}\right) \xrightarrow{\frac{1}{5}P_1} \left(\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 0 & 0 \end{array}\right)$

$C = A$.

$D = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 2 & 6 & 4 \end{array}\right)$. $\xrightarrow{P_1 \leftrightarrow P_2} \left(\begin{array}{cc|c} 2 & 6 & 4 \\ 0 & 0 & 0 \end{array}\right) \xrightarrow{\frac{1}{2}P_1} \left(\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 0 & 0 \end{array}\right)$

$= A'$

11.

$A = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \to B = \left(\begin{array}{ccc|c} 3 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 9 & 9 \end{array}\right)$

$\xrightarrow[-P_2+P_1]{-P_3+P_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)$

$\xrightarrow[6P_2,\ 9P_3]{3P_1} \left(\begin{array}{ccc|c} 3 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 9 & 9 \end{array}\right)$.

12. a) i) Neither.

ii) $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 5. \\ 0 & 0 & 1 & 3. \\ 1 & 1 & 0 & 4 \end{array}\right)$

$\xrightarrow{-P_1+P_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 5. \\ 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & -1 \end{array}\right)$

$\xrightarrow{P_2 \leftrightarrow P_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 5. \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right)$

$\therefore$ $x_1 = 5.$
$x_2 = -1$
$x_3 = 3.$

b) i) echelon.

ii) $\left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$

$x_1 = 0.$
$x_2 = 0.$
$x_3 = 0.$

c) i) Reduced echelon, / Both

ii) $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$

$\therefore$ no soln.

d) i) echelon

ii) $\left(\begin{array}{cccc|c} -2 & 0 & -1 & -7 & 8 \\ 0 & 3 & 0 & 3 & 2 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right)$

$x_3 = -1.$

$x_2 = \frac{2+3}{3} = \frac{5}{3}.$

$x_1 = \frac{8-7+\frac{5}{3}}{-2}.$

$x_1 = -\frac{4}{3}.$

e) $\left(\begin{array}{ccccc|c} 1 & 0 & 2 & -2 & 3 & -2 \\ 0 & 0 & 1 & 1 & 3 & 2. \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 5 & 5. \end{array}\right)$

i) neither.

ii) $x_4 = 5 - 5x_5.$

$x_3 = 2 - 3x_5 - (5 - 5x_5).$

$x_3 = -3 + 2x_5.$

$x_2 = x_2$

$x_1 = -2 - 3x_5 + 2(5 - 5x_5)$
$-2(-3 + 2x_5)$

$x_1 = 14 - 17x_5.$

$\left\{ \begin{pmatrix} 14 \\ 0 \\ -3 \\ 5 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} -17 \\ 0 \\ 2 \\ -5 \\ 1 \end{pmatrix} x_5 \;\middle|\; x_2, x_5 \in \mathbb{R} \right\}.$

f) i) Both.

ii) $\left(\begin{array}{cccccc|c} 1 & 0 & -2 & 0 & 2 & 0 & -2 \\ 0 & 1 & 0 & 0 & 2 & 0 & 4. \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right)$

$x_6 = 1$

$x_4 = 1 + x_5$

$x_2 = 4 - 2x_5.$

$x_1 = -2 - 2x_5 + 2x_3.$

$\therefore \left\{ \begin{pmatrix} -2 \\ 4 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} -2 \\ -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} x_5 \;\middle|\; x_3, x_5 \in \mathbb{R} \right\}.$

13

A: $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 4 & 6 & 0 \end{array}\right) \rightarrow$ plane.

B. $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 3 & 6 & 9 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \rightarrow$ line

C: $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right) \rightarrow$ point. $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$A \neq B$

$A \neq C.$

$C \neq B$

row equivalence $\leftrightarrow$ system have same solution set

14.

$\left\{ \begin{pmatrix} d \\ e \\ f \end{pmatrix} \right\}$.

$a, b, c \neq 0$. // System is a plane.

$d + e + f > 0$. //

15.

a) $\left(\begin{array}{ccc|c} a & b & c & d \\ e & f & g & h \end{array}\right)$

1. $\left(\begin{array}{ccc|c} 1 & 0 & * & * \\ 0 & 1 & * & * \end{array}\right)$.

7. $\left(\begin{array}{ccc|c} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$,

2. $\left(\begin{array}{ccc|c} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{array}\right)$

8. $\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$,

3. $\left(\begin{array}{ccc|c} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$

9. $\left(\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 0 & 0 & 0 \end{array}\right)$

4. $\left(\begin{array}{ccc|c} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right)$

5. $\left(\begin{array}{ccc|c} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$

10. $\left(\begin{array}{ccc|c} 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{array}\right)$

6. $\left(\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$

11. $\left(\begin{array}{ccc|c} 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{array}\right)$

b) 3., 5., 6., 7.

c) none.

16. a) $\left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ -1 & 2 & -1 & 1 \\ 2 & 0 & 3 & -2 \end{array}\right) \xrightarrow[-2P_1+P_3]{P_1+P_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 3 & 1 & 5 \\ 0 & -2 & -1 & -10 \end{array}\right) \xrightarrow{\frac{2}{3}P_2+P_3} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 3 & 1 & 5 \\ 0 & 0 & -1/3 & -20/3 \end{array}\right)$

$\therefore x_3 = 20.$

$x_2 = -5.$

$x_1 = -31.$

b) $\left(\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 1 \\ 2 & -1 & -1 & 0 & 2 \\ -2 & 0 & 3 & -1 & 3 \end{array}\right) \xrightarrow[2P_1+P_3]{-2P_1+P_2} \left(\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -2 & 0 \\ 0 & -2 & 3 & 1 & 5 \end{array}\right) \xrightarrow{2P_2+P_3} \left(\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -2 & 0 \\ 0 & 0 & 1 & -3 & 5 \end{array}\right)$

$y = 5+3z.$

$x = 2z + (5+3z)$

$= 5+5z.$

$w = 1 - z + (5+5z)$

$= 6+4z.$

$\therefore \left\{ \begin{pmatrix} 6 \\ 5 \\ 5 \\ 0 \end{pmatrix} + \begin{pmatrix} 4 \\ 5 \\ 3 \\ 1 \end{pmatrix} z \;\middle|\; z \in \mathbb{R} \right\}.$

c) $\left(\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 1 \\ 2 & 6 & 9 & 5 & 5 \\ -1 & -3 & 3 & 0 & 5 \end{array}\right) \xrightarrow[P_1+P_3]{-2P_1+P_2} \left(\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 1 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & 6 & 2 & 6 \end{array}\right) \xrightarrow{-2P_2+P_3} \left(\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 1 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

$\therefore x_3 = 1 - \frac{1}{3}x_4.$

$x_1 = 1 - 2x_4 - 3\left(1 - \frac{1}{3}x_4\right) - 3x_2.$

$= -2 - x_4 - 3x_2.$

$\left\{ \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} -1 \\ 0 \\ -1/3 \\ 1 \end{pmatrix} x_4 \;\middle|\; x_2, x_4 \in \mathbb{R} \right\}.$

d) $\left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 1 & -1 & -1 & -1 \\ 2 & -4 & -5 & 1 \end{array}\right) \xrightarrow[-2P_1+P_3]{-P_1+P_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & -2 & -3 & -5 \\ 0 & -6 & -9 & -7 \end{array}\right) \xrightarrow{-3P_2+P_3} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & -2 & -3 & -5 \\ 0 & 0 & 0 & 8 \end{array}\right)$

$\therefore$ no soln.

e) $\left(\begin{array}{ccc|c} 1 & -1 & 2 & 6 \\ 2 & 2 & -5 & 3 \\ 2 & 5 & 1 & 9 \end{array}\right) \xrightarrow[-2P_1+P_3]{-2P_1+P_2} \left(\begin{array}{ccc|c} 1 & -1 & 2 & 6 \\ 0 & 4 & -9 & -9 \\ 0 & 7 & -3 & -3 \end{array}\right) \xrightarrow{-\frac{7}{4}P_2+P_3} \left(\begin{array}{ccc|c} 1 & -1 & 2 & 6 \\ 0 & 4 & -9 & -9 \\ 0 & 0 & \frac{63}{4} & \frac{63}{4} \end{array}\right)$

$\therefore w = 1$

$v = \frac{581}{16}$

$u = \frac{123}{16}.$

f) $\left(\begin{array}{cccc|c} 1 & 5 & -2 & 6 & 0 \\ 2 & -2 & 1 & 3 & 1 \\ 1 & -7 & 3 & -3 & 1 \\ 5 & 1 & 0 & 12 & 2 \end{array}\right) \xrightarrow[\substack{-P_1+P_3 \\ -5P_1+P_4}]{-2P_1+P_2} \left(\begin{array}{cccc|c} 1 & 5 & -2 & 6 & 0 \\ 0 & -12 & 5 & -9 & 1 \\ 0 & -12 & 5 & -9 & 1 \\ 0 & -24 & 10 & -18 & 2 \end{array}\right) \xrightarrow[-2P_2+P_4]{-P_2+P_3} \left(\begin{array}{cccc|c} 1 & 5 & -2 & 6 & 0 \\ 0 & -12 & 5 & -9 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

$x = \frac{1 + 9z - 5y}{-12}.$

$x = -\frac{1}{12} - \frac{3}{4}z + \frac{5}{12}y.$

$w = -6z + 2y - 5\left(-\frac{1}{12} - \frac{3}{4}z + \frac{5}{12}y\right)$

$= -\frac{9}{4}z - \frac{1}{12}y + \frac{5}{12}.$

$\therefore \left\{ \begin{pmatrix} 5/12 \\ -1/12 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1/12 \\ 5/12 \\ 1 \\ 0 \end{pmatrix} y + \begin{pmatrix} -9/4 \\ -3/4 \\ 0 \\ 1 \end{pmatrix} z \;\middle|\; y, z \in \mathbb{R} \right\}.$

g) $\left(\begin{array}{ccccc|c} 0 & 2 & 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 1 & -1 & 2 \\ 4 & 6 & 1 & 4 & -3 & 8 \\ 2 & 2 & 0 & 1 & -1 & 2 \end{array}\right) \xrightarrow[P_2 \leftrightarrow P_4]{P_1 \leftrightarrow P_3} \left(\begin{array}{ccccc|c} 4 & 6 & 1 & 4 & -3 & 8 \\ 2 & 2 & 0 & 1 & -1 & 2 \\ 0 & 2 & 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 1 & -1 & 3 \end{array}\right)$

$\xrightarrow{-\frac{1}{2}P_1+P_2} \left(\begin{array}{ccccc|c} 4 & 6 & 1 & 4 & -3 & 8 \\ 0 & -1 & -0.5 & -1 & 0.5 & -2 \\ 0 & 2 & 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 1 & -1 & 3 \end{array}\right) \xrightarrow[P_2+P_4]{2P_2+P_3} \left(\begin{array}{ccccc|c} 4 & 6 & 1 & 4 & -3 & 8 \\ 0 & -1 & -0.5 & -1 & 0.5 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.5 & 0 & -0.5 & 1 \end{array}\right)$

$\xrightarrow{P_3 \leftrightarrow P_4} \left(\begin{array}{ccccc|c} 4 & 6 & 1 & 4 & -3 & 8 \\ 0 & -1 & -0.5 & -1 & 0.5 & -2 \\ 0 & 0 & -0.5 & 0 & -0.5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$

$\therefore x_3 = \frac{1 + \frac{1}{2}x_5}{-0.5}$

$x_3 = -2 - x_5$

$x_2 = -\left(-2 + \frac{1}{2}x_5 + x_4 + \frac{1}{2}x_3\right)$

$= 2 - \cancel{\frac{1}{2}x_5} - x_4 - \frac{1}{2}\left(-2 - \cancel{x_5}\right).$

$= 3 - x_4$

$x_1 = \frac{8 + 3x_5 - 4x_4 - x_3 - 6x_2}{4}$

$x_1 = 2 + \frac{3}{4}x_5 - x_4 + \frac{(-2-x_5)}{4} - \frac{6}{4}(3 - x_4)$

$= -3 - \frac{1}{2}x_5 - \frac{5}{2}x_4.$

$\left\{ \begin{pmatrix} -3 \\ 3 \\ -2 \\ 0 \end{pmatrix} + \begin{pmatrix} -5/2 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} x_4 + \begin{pmatrix} -1/2 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} x_5 \;\middle|\; x_4, x_5 \in \mathbb{R} \right\}.$

17.

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 6 \\ 2 & 2 & -5 & 3 \\ 2 & 5 & 1 & 9 \end{array}\right)$$

$$\xrightarrow[-2\rho_1+\rho_3]{-2\rho_1+\rho_2} \left(\begin{array}{ccc|c} 1 & -1 & 2 & 6 \\ 0 & 4 & -9 & -9 \\ 0 & 7 & -3 & -3 \end{array}\right)$$

$$\xrightarrow{-\frac{7}{4}\rho_2+\rho_3} \left(\begin{array}{ccc|c} 1 & -1 & 2 & 6 \\ 0 & 4 & -9 & -9 \\ 0 & 0 & 51/4 & 51/4 \end{array}\right).$$

$z^2 = 1 \rightarrow z = 1$

$y^2 = 0 \rightarrow y = 0$

$x^2 = 4. \rightarrow x = \pm 2,$

18.

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 3 & -1 & -2 & 0 \end{array}\right)$$

$$\xrightarrow{-3\rho_1+\rho_2} \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right)$$

$\therefore \quad \sin\phi = -\tan\phi$

$\cos\theta = \tan\phi - \tan\phi$

$= 0.$

$\therefore \quad \phi = \pi.$

$\theta = \frac{\pi}{2}.$

19.

$$\left(\begin{array}{ccc|c} -1 & -1 & -1 & -5 \\ -1 & 1 & -1 & -1 \\ 1 & -1 & -2 & -5 \end{array}\right)$$

$$\xrightarrow[\rho_1+\rho_3]{-\rho_1+\rho_2} \left(\begin{array}{ccc|c} -1 & -1 & -1 & -5 \\ 0 & 2 & 0 & 4 \\ 0 & -2 & -3 & -10 \end{array}\right)$$

$$\xrightarrow{\rho_2+\rho_3} \left(\begin{array}{ccc|c} -1 & -1 & -1 & -5 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & -3 & -6 \end{array}\right)$$

$\therefore$

$P_3 = 2$

$P_2 = 2$

$P_1 = -(-5 + 2 + 2)$

$P_1 = 1$

20 a) We can only form at most 4 equations:

total volume in = total volume out.

$x_4 + x_3 = 520 + 480.$

$x_4 + x_1 = 610 + 450.$

$x_1 + 310 = x_2 + 640.$

$x_2 + 600 = x_3 + 330.$

Such that

$$\left(\begin{array}{cccc|c} 0 & 0 & 1 & 1 & 1000 \\ 1 & 0 & 0 & 1 & 1060 \\ 1 & -1 & 0 & 0 & 330 \\ 0 & 1 & -1 & 0 & -270 \end{array}\right)$$

$$= \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1060 \\ 1 & -1 & 0 & 0 & 330 \\ 0 & 1 & -1 & 0 & -270 \\ 0 & 0 & 1 & 1 & 1000 \end{array}\right).$$

$$\xrightarrow{-R_1 + R_2} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1060 \\ 0 & -1 & 0 & -1 & -730 \\ 0 & 1 & -1 & 0 & -270 \\ 0 & 0 & 1 & 1 & 1000. \end{array}\right)$$

$$\xrightarrow{R_2 + R_3} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1060. \\ 0 & -1 & 0 & -1 & -730 \\ 0 & 0 & -1 & -1 & -1000 \\ 0 & 0 & 1 & 1 & 1000 \end{array}\right)$$

$$\xrightarrow{R_3 + R_4} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1060 \\ 0 & -1 & 0 & 1 & -730 \\ 0 & 0 & -1 & -1 & -1000 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\therefore$ Not enough info, 3 eqn, 4 unknowns.

b) $x^4 + x^3 = 1000.$

$x^3 = 1000 - 500$

$= 500$ //

$-x_2 = -730 - 500$

$x_2 = 1230.$ //

$x_1 = 1060 - 500$

$x_1 = 560.$

22. a) i) $\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1+\delta & 1 \end{array}\right)$

ii) $\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & -1-\delta & 1 \end{array}\right)$

$\xrightarrow{-P_1+P_2} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & \delta & 1 \end{array}\right)$

$\xrightarrow{-P_1+P_2} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -2-\delta & 1 \end{array}\right)$

$\therefore\ y = \frac{1}{\delta}$

$x = -\frac{1}{\delta}$

$y = \frac{1}{-2-\delta}$

$x = \frac{1}{2+\delta}.$

$\delta = 0.0001,\ y = 10000,\ x = -10000$

$\delta = 0.0002,\ y = 5000,\ x = -5000$

$y = -0.4999750012,\ x = 0.4999750012.$

$y = -0.4999\ldots,\ x = 0.4999\ldots$

b) large effect   small effect

c) For (i), graph is exponential towards 0. Thus, small value change in δ will result in a great change in x or y.

For (ii), graph is translated 2 units to the right, thus, the exponential change occurs towards the value -2 instead, hence small value change around 0 for δ would not affect much for x or y.

22. a) $\left(\begin{array}{ccc|c} 1 & 1 & a & 0 \\ 1 & a & 1 & 0 \\ a & 1 & 1 & 0 \end{array}\right)$

$\xrightarrow[-aP_1+P_3]{-P_1+P_2} \left(\begin{array}{ccc|c} 1 & 1 & a & 0 \\ 0 & a-1 & 1-a & 0 \\ 0 & 1-a & 1-a^2 & 0 \end{array}\right)$

$\xrightarrow{P_2+P_3} \left(\begin{array}{ccc|c} 1 & 1 & a & 0 \\ 0 & a-1 & 1-a & 0 \\ 0 & 0 & 2-a^2-a & 0 \end{array}\right)$: $x=0,\ y=0,\ z=0.$

$\therefore$ i) no value of a.

ii) $a \in \mathbb{R}$.

iii) no value of a.

b) $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & a & 2 & 2 \\ 4 & 4 & a^2 & 2a \end{array}\right)$

$\xrightarrow[-4P_1+P_3]{-2P_1+P_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & a-2 & 0 & 0 \\ 0 & 0 & a^2-4 & 2a-4 \end{array}\right)$

$\therefore\ z = \frac{2a-4}{a^2-4}$

$y = 0.$

$x = 1 - \frac{2a-4}{a^2-4}.$

i) $a^2-4=0.$
$a^2 = 4$
$a = \pm 2.$

ii) $\{a : a \in \mathbb{R} \setminus \pm 2, 0\}.$

iii) $a^2-4 = 2a-4$
$a^2-2a = 0$
$a(a-2) = 0.$
$\therefore\ a = 0$ ~~or a = 2~~.

c) $\left(\begin{array}{cc|c} a & a & 1 \\ a & -|a| & 1 \end{array}\right),$

$\xrightarrow{-P_1+P_2} \left(\begin{array}{cc|c} a & a & 1 \\ 0 & -a-|a| & 0 \end{array}\right),$

$y = 0.$
$x = \frac{1}{a}.$

$\therefore$ i) $a = 0$

ii) $a < 0.$

iii) $-a-|a| = 0.$

$a \geq 0.$

23.
$$\left(\begin{array}{ccc|c} a & 0 & b & 2 \\ a & a & 4 & 4 \\ 0 & a & 2 & b \end{array}\right)$$

$$\xrightarrow{-P_1+P_2} \left(\begin{array}{ccc|c} a & 0 & b & 2 \\ 0 & a & 4-b & 2 \\ 0 & a & 2 & b \end{array}\right)$$

$$\xrightarrow{-P_2+P_3} \left(\begin{array}{ccc|c} a & 0 & b & 2 \\ 0 & a & 4-b & 2 \\ 0 & 0 & b-2 & b-2 \end{array}\right)$$

$\therefore \quad z = 1.$

$y = \dfrac{2-(4-b)}{a}$

$\therefore \quad y = \dfrac{b-2}{a}.$

$\therefore \quad x = \dfrac{2-b}{a}.$

a) $a = 0.$ $\quad b \neq 2$

b) $b \in \mathbb{R} \setminus 2, \; a \in \mathbb{R} \setminus 0.$ — other vals of $b \Rightarrow$ non constant

c) $b = 2 \leftarrow$ set $\quad$ requires $a \neq 0.$

d) $b = 2, \; a = 0.$
requires $\quad$ set

1 soln $\rightarrow$ all pivots

no soln $\rightarrow$ last column pivot.

∞ solns 1 param $\rightarrow$ 1 non-pivot.

∞ solns 2 param. $\rightarrow$ 2 non-pivot

24.
$$\left(\begin{array}{ccc|c} a & a & a & c \\ 0 & b & b & a \\ 0 & 0 & c & b \end{array}\right)$$

$z = \dfrac{b}{c}$

$y = \dfrac{a - \frac{b^2}{c}}{b}.$

$x = \dfrac{1}{a}\left(c - a\left(\dfrac{b}{c}\right) - a\left(\dfrac{a - \frac{b^2}{c}}{b}\right)\right).$

a) $a = 0, \; b = 0, \; c = 0.$

b) $a \neq 0, \; b \neq 0, \; c \neq 0.$

c) $c = b = a = 0.$

2.5. a) trivial

b) trivial.

c) non-trivial,

d) non-trivial.

26. a) $3W = y \rightarrow 3w - y = 0.$

$8w = 2z \rightarrow 8w - 2z = 0$

$2x = 2y + z \rightarrow 2x - 2y - z = 0$

b) $\therefore \left(\begin{array}{cccc|c} 2 & -2 & -1 & 0 & 0 \\ 0 & 0 & -2 & 8 & 0 \\ 0 & -1 & 0 & 3 & 0 \end{array}\right).$

$= \left(\begin{array}{cccc|c} 2 & -2 & -1 & 0 & 0 \\ 0 & -1 & 0 & 3 & 0 \\ 0 & 0 & -2 & 8 & 0 \end{array}\right)$

$-2z = -8w$

$z = 4w.$

$y = 3z$

$= 12w$

$x = 4w + 24w$

$= 28w.$

$\therefore \left\{ \begin{pmatrix} 28 \\ 12 \\ 4 \\ 1 \end{pmatrix} w \mid w \in \mathbb{R} \right\}.$

c) when $w=1$, $x = 28$, $y = 12$, $z = 4$.

27.

$$\left(\begin{array}{ccc|c} a & b & c & 0 \\ d & e & f & 0 \end{array}\right)$$

$$\xrightarrow{-\frac{d}{a}r_1 + r_2} \left(\begin{array}{ccc|c} a & b & c & 0 \\ 0 & e-\frac{d}{a}b & f-\frac{d}{a}c & 0 \end{array}\right)$$

$$\therefore y = \underbrace{\frac{f-\frac{d}{a}c}{e-\frac{d}{a}b}}_{k} z$$

$$y = kz.$$

$$x = \frac{-cz - by}{a}.$$

$$x = -\frac{c}{a}z - \frac{b}{a}\left(\frac{f-\frac{d}{a}c}{e-\frac{d}{a}b}z\right).$$

$$x = \left(-\frac{c}{a} - \frac{b}{c}\left(\frac{f-\frac{d}{a}c}{e-\frac{d}{a}b}\right)\right)z$$

$$x = k'z.$$

$$\therefore \left\{ \begin{pmatrix} k' \\ k \\ 1 \end{pmatrix} z \;\middle|\; z \in \mathbb{R} \right\}$$

$\therefore$ If $x = x_0$, $y = y_0$, $z = z_0$ is a solution,
since $k \in \mathbb{R}$, then $x = kx_0$, $y = ky_0$, $z = kz_0$
is also a solution as the general solution is based on z.

28.

$$\left(\begin{array}{ccc|c} a_1 & b_1 & c_1 & 0 \\ a_2 & b_2 & c_2 & 0 \\ a_3 & b_3 & c_3 & 0 \end{array}\right)$$

RREF:

1 $\left(\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right)$ point.

2 $\left(\begin{array}{ccc|c} 1 & 0 & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$ none

3 $\left(\begin{array}{ccc|c} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{array}\right)$ line.

4 $\left(\begin{array}{ccc|c} 1 & * & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{array}\right)$ line.

5 $\left(\begin{array}{ccc|c} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$ none.

6 $\left(\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ plane.

7 $\left(\begin{array}{ccc|c} 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ plane.

8 $\left(\begin{array}{ccc|c} 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ line.

9 $\left(\begin{array}{ccc|c} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ none

10 $\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ plane.

11 $\left(\begin{array}{ccc|c} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{array}\right)$ line.

12 $\left(\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$ none.

13 $\left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$ none

29 $\begin{pmatrix}0\\0\\0\end{pmatrix}$ $\begin{pmatrix}1\\1\\1\end{pmatrix}$.

Solution: $\left\{ \begin{pmatrix}0\\0\\0\end{pmatrix} + \begin{pmatrix}1\\1\\1\end{pmatrix}x \;\middle|\; x \in \mathbb{R} \right\}$

$\therefore \quad hz = k$

$z = \frac{k}{h}$.

$y = \frac{g - fz}{e}$

$y = \frac{g}{e} - \frac{f}{e}\left(\frac{k}{h}\right)$

$x = \frac{d - cz - by}{a}$

$x = \frac{d}{a} - \frac{c}{a}\left(\frac{k}{h}\right) - \frac{b}{a}\left(\frac{g}{e} - \frac{f}{e}\left(\frac{k}{h}\right)\right)$

30. a) True. eg. $2x - y = 0$, $x = 1$, $y = 2$.

b) False. eg. $2x - y = 4$, x and y cannot be 0 at the same time

c) False. eg. line at origin

d) False, eg. line at origin

e) True defn

f) False eg. line at origin.

g) True. by defn of homogeneous

(a) For any real number t, verify that $x = \frac{1}{5}(-4+3t)$, $y = \frac{1}{5}(-7+4t)$, $z = t$ is a solution to the linear system.

(b) Write down two particular solutions to the system.

7. Each equation in the following linear system represents a line in the xy-plane:

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \\ a_3x + b_3y = c_3, \end{cases}$$

where $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ are constants and for each $i = 1, 2, 3$, a_i, b_i are not both zero. Discuss the relative positions of the three lines when the system

(a) has no solution;

(b) has only one solution;

(c) has infinitely many solutions.

8. Each equation in the following linear system represents a plane in the xyz-space.

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3, \end{cases}$$

where $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2, d_3$ are constants and for each $i = 1, 2, 3$, a_i, b_i, c_i are not all zero. Discuss the relative positions of the three planes when the linear system

(a) has no solution;

(b) has only one solution;

(c) has infinitely many solutions.

9. (a) Does an inconsistent linear system with more unknowns than equations exist?

(b) Does a linear system which has only one solution, but more equations than unknowns, exist?

(c) Does a linear system which has only one solution, but more unknowns than equations, exist?

(d) Does a linear system which has infinitely many solutions, but more equations than unknowns, exist?

Justify your answers.

10. Show that the following augmented matrices are row equivalent to each other.

$$\boldsymbol{A} = \left(\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 0 & 0 \end{array}\right), \quad \boldsymbol{B} = \left(\begin{array}{cc|c} 1 & 3 & 2 \\ 1 & 3 & 2 \end{array}\right), \quad \boldsymbol{C} = \left(\begin{array}{cc|c} 5 & 15 & 10 \\ 1 & 3 & 2 \end{array}\right), \quad \boldsymbol{D} = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 2 & 6 & 4 \end{array}\right).$$

11. Find a series of elementary row operations that bring the augmented matrix $\boldsymbol{A}$ to the augmented matrix $\boldsymbol{B}$ where

$$\boldsymbol{A} = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \quad \text{and} \quad \boldsymbol{B} = \left(\begin{array}{ccc|c} 3 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 9 & 9 \end{array}\right).$$

12. For each of the following augmented matrices, (i) determine whether the matrix is in row-echelon form, reduced row-echelon form, both, or neither; and (ii) find a system of linear systems corresponding to the augmented matrix and then solve the system (if possible). You may assume that the variables are x_1, x_2, x_3, etc.

(a) $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 1 & 1 & 0 & 4 \end{array}\right)$, (b) $\left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$,

(c) $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$, (d) $\left(\begin{array}{cccc|c} -2 & 0 & -1 & -7 & 8 \\ 0 & 3 & 0 & 3 & 2 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right)$,

(e) $\left(\begin{array}{ccccc|c} 1 & 0 & 2 & -2 & 3 & -2 \\ 0 & 0 & 1 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 5 & 5 \end{array}\right)$, (f) $\left(\begin{array}{cccccc|c} 1 & 0 & -2 & 0 & 2 & 0 & -2 \\ 0 & 1 & 0 & 0 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right)$.

13. Determine whether the following augmented matrices are row equivalent to each other.

$$\boldsymbol{A} = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 4 & 6 & 0 \end{array}\right), \quad \boldsymbol{B} = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 3 & 6 & 9 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right), \quad \boldsymbol{C} = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right).$$

14. The following is the reduced row-echelon form of the augmented matrix of a linear system:

$$\left(\begin{array}{ccc|c} a & 0 & 0 & d \\ 0 & b & 0 & e \\ 0 & 0 & c & f \end{array}\right),$$

where a, b, c, d, e, f are constants. Write down the necessary conditions on a, b, c, d, e, f so that the solution set of the linear system is a plane in the three dimensional space that does not contain the origin.

15. Consider the augmented matrix

$$\left(\begin{array}{ccc|c} a & b & c & d \\ e & f & g & h \end{array}\right),$$

where a, b, c, d, e, f, g, h are constants.

(a) Write down all possible reduced row-echelon forms for the augmented matrix.

(b) Which of the reduced row-echelon forms in Part (a) represent inconsistent systems?

(c) Which of the reduced row-echelon forms in Part (a) are row equivalent to each other?

Question 1.16 to Question 1.30 are exercises for Sections 1.4 and 1.5.

16. Solve each of the following systems by Gaussian Elimination or Gauss-Jordan Elimination.

(a) $\begin{cases} x_1 + x_2 + 2x_3 = 4 \\ -x_1 + 2x_2 - x_3 = 1 \\ 2x_1 + 3x_3 = -2 \end{cases}$

(b) $\begin{cases} w - x + z = -1 \\ 2w - x - y = 2 \\ -2w + 3y - z = 3 \end{cases}$

(c) $\begin{cases} x_1 + 3x_2 + 3x_3 + 2x_4 = 1 \\ 2x_1 + 6x_2 + 9x_3 + 5x_4 = 5 \\ -x_1 - 3x_2 + 3x_3 = 5 \end{cases}$

(d) $\begin{cases} x + y + 2z = 4 \\ x - y - z = -1 \\ 2x - 4y - 5z = 1 \end{cases}$

(e) $\begin{cases} u - v + 2w = 6 \\ 2u + 2v - 5w = 3 \\ 2u + 5v + w = 9 \end{cases}$

(f) $\begin{cases} w + 5x - 2y + 6z = 0 \\ 2w - 2x + y + 3z = 1 \\ w - 7x + 3y - 3z = 1 \\ 5w + x + 12z = 2 \end{cases}$

(g) $\begin{cases} 2x_2 + x_3 + 2x_4 - x_5 = 4 \\ x_2 + x_4 - x_5 = 3 \\ 4x_1 + 6x_2 + x_3 + 4x_4 - 3x_5 = 8 \\ 2x_1 + 2x_2 + x_4 - x_5 = 2 \end{cases}$

17. Solve the following system of non-linear equations:

$$\begin{cases} x^2 - y^2 + 2z^2 = 6 \\ 2x^2 + 2y^2 - 5z^2 = 3 \\ 2x^2 + 5y^2 + z^2 = 9. \end{cases}$$

18. Solve the following system of non-linear equations if $0 \le \theta < 2\pi$ and $0 \le \phi < 2\pi$:

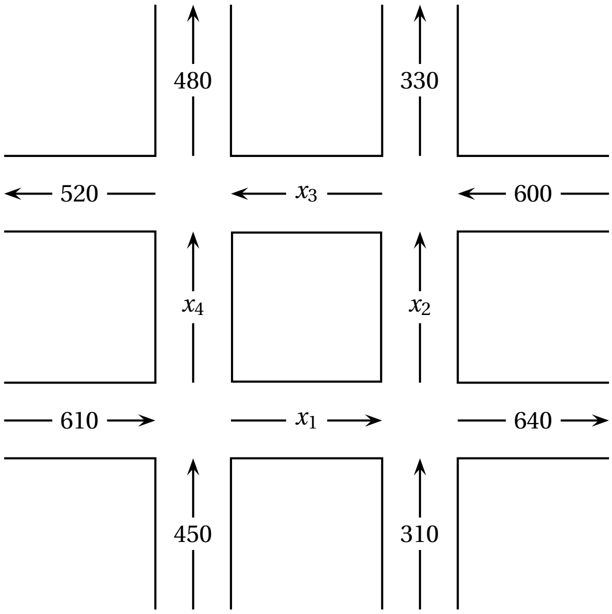
$$\begin{cases} \cos(\theta) - \sin(\phi) - \tan(\phi) = 0 \\ 3\cos(\theta) - \sin(\phi) - 2\tan(\phi) = 0. \end{cases}$$

19. In a three-commodity market, the supply and demand for each commodity depend on the prices of the commodities. Let D_1, D_2, D_3 be the respective demands for products 1,2 and 3, S_1, S_2, S_3 the respective supplies and P_1, P_2, P_3 the respective prices for the commodities. Suppose the market can be described by the linear equations

$$\begin{aligned} D_1 &= -2P_1 + P_2 + P_3 + 4, & S_1 &= -P_1 + 2P_2 + 2P_3 - 1, \\ D_2 &= P_1 + 2P_2 + P_3 - 1, & S_2 &= 2P_1 + P_2 + 2P_3 - 2, \\ D_3 &= 2P_1 + P_2 + P_3 + 4, & S_3 &= P_1 + 2P_2 + 3P_3 - 1. \end{aligned}$$

Use Gaussian Elimination or Gauss-Jordan Elimination to find the *equilibrium solution* (where supplies equal to demands).

20. In the downtown section of a certain city, two sets of one-way streets intersect as shown in the following:



The average hourly volume of traffic entering and leaving this section during rush hour is given in the diagram.

(a) Do we have enough information to find the traffic volume x_1, x_2, x_3, x_4? Explain your answer.

(b) Given that $x_4 = 500$, find x_1, x_2, x_3.

(The average hourly volume of traffic entering an intersection must be equal to the volume that leaving.)

21. Consider the two linear systems

$$\text{(i)} \quad \begin{cases} x + y = 0 \\ x + (1+\delta)y = 1 \end{cases} \qquad \text{and} \qquad \text{(ii)} \quad \begin{cases} x + y = 0 \\ x - (1+\delta)y = 1 \end{cases}$$

where δ is a real number close to zero.

(a) Solve the two linear systems for $\delta = 0.0001$ and 0.0002.

(b) Compare the effect on the solution to the two systems when we change δ from 0.0001 to 0.0002.

(c) Explain your answer in Part (b) geometrically. (*Hint*: Plot the lines to see the difference between the two systems.)

(This question shows that solutions of some linear systems can be largely affected by small changes in the coefficients, say, due to roundoff errors.)

22. For each of the following linear systems, determine the values of a such that the system has (i) no solution; (ii) only one solution; and (iii) infinitely many solutions.

$$\text{(a)} \quad \begin{cases} x + y + az = 0 \\ x + ay + z = 0 \\ ax + y + z = 0 \end{cases} \qquad \text{(b)} \quad \begin{cases} x + y + z = 1 \\ 2x + ay + 2z = 2 \\ 4x + 4y + a^2 z = 2a \end{cases}$$

$$\text{(c)} \quad \begin{cases} ax + ay = 1 \\ ax - |a|y = 1 \end{cases}$$

23. Determine the values of a and b so that the linear system

$$\begin{cases} ax \qquad\quad + bz = 2 \\ ax + ay + 4z = 4 \\ \qquad\ \ ay + 2z = b \end{cases}$$

(a) has no solution;

(b) has only one solution;

(c) has infinitely many solutions and a general solution has one arbitrary parameter;

(d) has infinitely many solutions and a general solution has two arbitrary parameters.

24. Determine the values of a, b and c so that the linear system

$$\begin{cases} ax + ay + az = c \\ \qquad\ \ by + bz = a \\ \qquad\qquad\quad cz = b \end{cases}$$

(a) has no solution;

(b) has only one solution;

(c) has infinitely many solutions.

25. Without using pencil and paper, determine which of the following homogeneous systems have non-trivial solutions.

(a) $\begin{cases} x - y = 0 \\ x + y = 0 \end{cases}$

(b) $\begin{cases} x - 4y + 6z = 0 \\ \quad\ 2y - 5z = 0 \\ \qquad\quad\ 3z = 0 \end{cases}$

(c) $\begin{cases} 3x_1 - x_2 + 11x_3 \qquad\quad = 0 \\ \ \ x_1 + x_2 - \ \ 7x_3 + \ \ x_4 = 0 \\ 5x_1 + x_2 - \ \ 4x_3 + 2x_4 = 0 \end{cases}$

(d) $\begin{cases} \qquad ax + \qquad by + \qquad cz = 0 \\ \qquad dx + \qquad ey + \qquad fz = 0 \\ (a+d)x + (b+e)y + (c+f)z = 0 \end{cases}$

For Part (d), a, b, c, d, e, f are constants.

26. When propane gas burns, the propane combines with oxygen to form carbon dioxide and water:

$$w\,C_3H_8 + x\,O_2 \ \rightarrow\ y\,CO_2 + z\,H_2O$$

where w and x are the numbers of propane and oxygen molecules, respectively, required for the combustion; and y and z are the numbers of carbon dioxide and water molecules, respectively, produced.

(a) By equating the numbers of carbon, hydrogen and oxygen atoms, respectively, on both sides of the chemical equation, write down a homogeneous system of three equations in terms of w, x, y, z.

(b) Find a general solution for the homogeneous system obtained in Part (a).

(c) Find the (non-trivial) solution of w, x, y, z with smallest values. (Note that w, x, y, z are positive integers.)

27. Consider the homogeneous system of equations

$$\begin{cases} ax + by + cz = 0 \\ dx + ey + fz = 0 \end{cases}$$

where a, b, c, d, e, f are constants.

(a) Let $x = x_0$, $y = y_0$, $z = z_0$ be a solution to the system and k a constant. Show that $x = kx_0$, $y = ky_0$, $z = kz_0$ is also a solution to the system.

(b) Let $x = x_0$, $y = y_0$, $z = z_0$ and $x = x_1$, $y = y_1$, $z = z_1$ be two solutions to the system. Show that $x = x_0 + x_1$, $y = y_0 + y_1$, $z = z_0 + z_1$ is also a solution to the system.

28. Consider the homogeneous linear system

$$\begin{cases} a_1x + b_1y + c_1z = 0 \\ a_2x + b_2y + c_2z = 0 \\ a_3x + b_3y + c_3z = 0 \end{cases}$$

where $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ are constants. Determine all possible reduced row-echelon forms of the augmented matrix of the system and describe the geometrical meaning of the solutions obtained from various reduced row-echelon forms.

29. The following is the reduced row-echelon form of the augmented matrix of a linear system:

$$\left(\begin{array}{ccc|c} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & k \end{array}\right)$$

where $a, b, c, d, e, f, g, h, k$ are constants. Suppose the solution set of this system is represented by a line that passes through the origin and the point $(1, 1, 1)$. Find the values of $a, b, c, d, e, f, g, h, k$. Justify your answers.

30. Determine which of the following statements are true. Justify your answer.

(a) A homogeneous system can have a non-trivial solution.

(b) A non-homogeneous system can have a trivial solution.

(c) If a homogeneous system has the trivial solution, then it cannot have a non-trivial solution.

(d) If a homogeneous system has a non-trivial solution, then it cannot have a trivial solution.

(e) If a homogeneous system has a unique solution, then the solution has to be trivial.

(f) If a homogeneous system has the trivial solution, then the solution has to be unique.

(g) If a homogeneous system has a non-trivial solution, then there are infinitely many solutions to the system.