CS1231S: Discrete Structures
Tutorial #1: Propositional Logic and Proofs
(Week 3: 23 – 27 August 2021)

Tutorials are meant to reinforce topics taught in lectures. Please try these questions on your own before coming to tutorial. In doing so, you may discover gaps in your understanding. Usually, a tutorial has a mix of easy, moderate and slightly challenging questions. It is perfectly fine if you cannot do some of the questions, but attempt them nonetheless, to at least get some partial solution.

You will be asked to present your answers I class. Your tutor's job is to guide you, not to provide the answers for you. Also, please keep in mind that the goal of tutorials is not to answer every question in the tutorial, but to clarify doubts and reinforce concepts. Solutions to all tutorial questions will be released at the end of the week after all tutorial classes are over, but please treat the solutions as a guide, for there are usually alternative ways of solving a problem.

You are also encouraged to raise your doubts or questions on **LumiNUS > Forum > Tutorials**.

Tutorials are important so attendance is taken and it contributes 5% of your final grade. If you miss a tutorial with valid reason (eg: due to illness), please submit your document (eg: medical certificate) to your respective tutor (softcopy or hardcopy) in advance or within three days after your absence and you will not be penalized for your absence. You are to stick with your officially assigned tutorial group (we give exception for the first week of tutorial for students who do not have any group yet), or your attendance will not be taken. If you need to join a different group for just once with valid reason, please email Aaron (tantc@comp.nus.edu.sg) in advance, citing your reason.

Every week, your attendance record will be posted on LumiNUS Gradebook and you are given a week to verify and report any discrepancy directly to your tutor, after which the record will not be changed.

Please take note of the above as it will not be repeated in subsequent tutorials.

I. Discussion Questions

To encourage discussion on LumiNUS forum, the questions in this section will not be discussed in tutorial. You may discuss them or post your answers on the LumiNUS forum.

D1. We discussed in class that English is an ambiguous language. Often, it also carries notions of perception, past experiences, cause and effect, etc. For example, in saying "today is rainy but hot", we use "but" because we often associate a rainy day as a cool day. In logic, we write "today is rainy" ∧ "today is hot".

Restate the following symbolically. You may introduce your own statement variables.

(a) $P(B) \leq P(A)$ whenever $B \subseteq A$. $P(B) \leq P(A) \leftrightarrow B \subseteq A$

(b) If $A - B$ is countable, then A is countable or B is uncountable. $A-B \subseteq C_0 \rightarrow A \subseteq C_0 \vee \neg(B \subseteq C_0)$

let C_0 be countable.

(c) An undirected graph (V, E) that is connected and acyclic must have $|E| = |V| - 1$.

$(V, E) \longrightarrow |E| = |V| - 1$.

one direction

D2. As mentioned above, we often rely on our past experiences and assumption in interpreting English sentences. For example, if you hear a mother utter this to her son:

$p \rightarrow q$

"If you behave, you get ice-cream."

Suppose the above statement is true. Which of the following statements is/are true?

(a) If the child behaves, he gets ice-cream. p is true, q is true. $p \rightarrow q$ T truth value ✓

(b) If the child does not behave, he does not get ice-cream. $\neg p \rightarrow \neg q$ T Inverse. truth value ✓

(c) If the child gets ice-cream, he behaves. p is false, q is false. $q \rightarrow p$ F converse.

(d) If the child does not get ice-cream, he does not behave. $\neg q \rightarrow \neg p$ T contrapositive. truth value ✓

Using p to represent "the child behaves" and q to represent "the child gets ice-cream", write out the above four statements in propositional logic, and relate them to what was discussed in lecture.

implication
$p \rightarrow q$, if p then q.

D3. Use the laws given in **Theorem 2.1.1 (Epp)** and the **implication law** to prove that the following are tautologies. ($p \vee \neg p$ ⇒ proposition which is always true).

(a) $(p \wedge q) \rightarrow p$ ✓

(b) $\big((p \vee q) \wedge {\sim}p\big) \rightarrow q$ ✓

(c) $\big((p \rightarrow q) \wedge p\big) \rightarrow q$ ✓ modus ponens.

(d) $\big({\sim}p \rightarrow (q \wedge {\sim}q)\big) \rightarrow p$ ✓ proof by contradiction.

Mathematical arguments are often constructed by using one implication (conditional statement) after another. Logically speaking, such an argument is constructed by using implications that are tautologies, like the ones above. For example, (c) is *modus ponens* and (d) is proof by contradiction. Part (b) is used in problems like Knights and Knaves in question 8 in this tutorial.

D4. Mala has hidden her treasure somewhere on her property. She left a note in which she listed five statements (a-e below) and challenged the reader to use them to figure out the location of the treasure.

(a) If this house is next to a lake, then the treasure is not in the kitchen. $a \rightarrow \neg b$

(b) If the tree in the front yard is an elm, then the treasure is in the kitchen. $c \rightarrow b$

(c) If the tree in the back yard is an oak, then the treasure is in the garage. $d \rightarrow e$

(d) The tree in the front yard us an elm or the treasure is buried under the flagpole. $c \vee f$

(e) The house is next to a lake. $a \equiv$ true.

Where has Mala hidden her treasure?

a ≡ T

a	b	¬b	a→¬b
T	T	F	F ✗
T	F	T	T ✓
F	T	F	T ✓
F	F	T	T ✓

c	b	c→b
T	T	T
T	F	F ✗
F	T	T
F	F	T

$a \rightarrow \neg b$
$c \rightarrow b$
$d \rightarrow e$
$c \vee f$
$a \equiv$ true.

b ≡ false
c ≡ false
f ≡ true

∴ treasure is under flagpole.

c	f	c∨f
T	T	T
T	F	T
F	T	T
F	F	F ✗

true

D5. Draw a truth table to show that the following statement is a tautology:

$$\big((a \to x) \wedge (b \to y)\big) \to \big((a \wedge b) \to (x \wedge y)\big).$$

This will be useful for Question 6b.

II. Additional Notes

Given an argument:

$$\begin{array}{l} p_1 \\ p_2 \\ \vdots \\ p_k \\ \therefore q \end{array}$$

where $p_1, p_2, \dots, p_k$ are the k premises and q the conclusion, we can say that "the argument is valid if and only if $(p_1 \wedge p_k \wedge \cdots \wedge p_k) \longrightarrow q$ is a tautology".

This serves as an alternative way to check whether an argument is valid, besides the critical row method shown in lecture. Go through the examples in the lecture yourself to verify the above.

III. Tutorial Questions

1. (Basics)
   a. What are the names of these logical connectives?

      (i) $\sim$ negation (ii) $\wedge$ conjunction (iii) $\vee$ disjunction (iv) $\to$ Conditional, Implication (v) $\leftrightarrow$ biconditional

   b. Given the statement $p \to q$, what is p called and what is q called? p is hypothesis, antecedent; q is conclusion, consequent.
      What is the negation of $p \to q$, i.e. $\sim(p \to q)$?
      What do you think are the common mistakes when negating $p \to q$? (These mistakes are still found in students' work towards the end of the semester.) $p \wedge \neg q$.
      mistakes: $p \to \neg q$, $q \to p$, $p \vee \neg q$.

   c. Use logical connectives to rewrite the following statements (the parts are independent of one another): then only sufficient $\to$
      (i) p is a sufficient condition for q. $p \to q$ (use TOS.)
      (ii) p is a necessary condition for q. $q \to p$ (when p does not happen, q will not happen.)
      (iii) p is a necessary and sufficient condition for q. $p \leftrightarrow q$
      (iv) p if q. $q \to p$
      (v) p only if q. $p \to q$
      (vi) p if and only if q. $p \leftrightarrow q$.

$\neg(p \to q)$
$\equiv \neg(\neg p \vee q)$ by Implication
$\equiv \neg(\neg p) \wedge \neg q$ by de morgan's
$\equiv p \wedge \neg q$ by double negation.

$p \to q \equiv \neg p \vee q$?

p	q	$p \to q$	$\neg p$	$\neg p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Same $\therefore \equiv$

If I eat an apple, then the apple is eaten
$p \to q$.
Similar.
I did not eat an apple or apple is eaten.
$\neg p \vee q$

2. Simplify the propositions below using the laws given in **Theorem 2.1.1 (Epp)** and the **implication law** (if necessary) with only negation (~), conjunction ($\wedge$) and disjunction ($\vee$) in your final answers. Supply a justification for every step.

   (For the first half of the module, we want students to cite justification for every step. This is to ensure that you do not arrive at the answer by coincidence. Only after you have gained sufficient experience then would we relax this and allow you to skip obvious steps, or combine multiple steps in a line.)

   a. $\sim a \rightarrow \sim(b \vee \sim a)$

      Aiken worked out the first step of his answer as follows:

      $\sim a \rightarrow \sim(b \vee \sim a)$

      $\equiv \sim\big(a \rightarrow \sim(b \vee \sim a)\big)$ $\quad \neg a \rightarrow \neg(b \vee \neg a) \equiv (\neg a) \rightarrow \neg(b \vee \neg a)$.

      negation priority is diff

      What mistake did he make?

   b. $\sim a \rightarrow \sim(b \vee \sim a)$

      Aiken has been notified of his mistake in part (a) and he re-worked his answer as shown below. You can verify that the answer is correct by comparing the truth tables of $\sim a \rightarrow \sim(b \vee \sim a)$ with a. However, he skipped a number of steps in his working and hence his answer will not be awarded full credit. Can you point out the omissions? (Note: To show that two logical statements are equivalent, we use $\equiv$, not $=$.)

      $\neg(p \vee q) \equiv \neg p \wedge \neg q$.

      $p \rightarrow q \equiv \neg p \vee q$.

      $\sim a \rightarrow \sim(b \vee \sim a)$ $\quad \equiv \neg b \wedge \neg(\neg a)$

      $\equiv \sim a \rightarrow (\sim b \wedge a)$ $\equiv \neg(\neg a) \vee (\neg b \wedge a)$ by De Morgan's law (step 1)

      $\equiv a \vee (\sim b \wedge a)$ $\equiv a \vee (a \wedge \neg b)$ by the implication law (step 2)

      $\equiv a$ by the absorption law (step 3)

      $\therefore$ Step 1 missing double negation

      Step 2 missing double negation

      Step 3 missing commutative law.

   and and or operation aren't associative.

   which bracket?

   c. $(x \wedge x \vee y) \rightarrow z$

   d. $(p \wedge q) \rightarrow q$

   e. $(p \rightarrow q) \rightarrow r$

   > You don't need to write "step x". This is for ease of reference in the answers that will be released after tutorials.

3. Prove, or disprove, that $(p \rightarrow q) \rightarrow r$ is logically equivalent to $p \rightarrow (q \rightarrow r)$.

4. Given the conditional statement "If $12x - 7 = 29$, then $x = 3$", write the **negation**, **contrapositive**, **converse** and **inverse** of the statement. Is the given conditional statement true? Is its converse true?

   In general, is it possible for the converse of a conditional statement to be true while the inverse of the same statement is false? Why?

doubting def^n of implication.

5. The conditional statement $p \to q$ is an important logical statement. Recall that it is defined by the following truth table:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Oftentimes, students are perplexed by this definition. The first two rows look reasonable, but the last two rows seem strange. However, this way of defining $p \to q$ actually gives us the nice intuitive property of the following statement:

$$\big((p \to q) \land (q \to r)\big) \to (p \to r)$$

which is the **transitive rule of inference** we studied in lecture (Lecture #2 slide 67):

$$\begin{array}{l} p \to q \\ q \to r \\ \therefore p \to r \end{array}$$

For example, given premises "if x is a square then x is a rectangle" and "if x is a rectangle than x is a quadrilateral", the conclusion is "if x is a square then x is a quadrilateral". We use such intuitive reasoning very often in our life.

Show that if we define the conditional statement alternatively as follows, then the transitive rule of inference would no longer hold.

Alternative 1: $\to_a$

p	q	$\cancel{p \to_a q}$
T	T	T
T	F	F
F	T	F
F	F	F

Alternative 2: $\to_b$

p	q	$\cancel{p \to_b q}$
T	T	T
T	F	F
F	T	T
F	F	F

Alternative 3: $\to_c$

p	q	$\cancel{p \to_c q}$
T	T	T
T	F	F
F	T	F
F	F	T

Iff p is true and q is true, then $p \to_a q$ is true

p	q	r	$p \to r$	$p \to_a q$	$q \to r$	$(p \to_a q) \land (q \to r) \not\equiv p \to r$	$p \to_b q$	$(p \to_b q) \land (q \to r) \not\equiv p \to r$	$p \to_c q$	$(p \to_c q) \land (q \to r) \not\equiv p \to r$
T	T	T	T	T	T	T	T	T	T	T
T	T	F	F	T	F	F	F	F	F	F
T	F	T	T	F	T	F	F	F	F	F
T	F	F	F	F	T	F	F	F	F	F
F	T	T	T	F	T	F	T	F	F	F
F	T	F	T	F	F	F	T	F	F	F
F	F	T	T	T	T	F	F	F	T	F
F	F	F	T	F	T	F	F	F	T	T

(p→q) ∧ (q→p) ∴ p→q is valid by specialization.

or X → Y by truth table.

6a. It is intuitively clear that $(p \leftrightarrow q) \rightarrow (p \rightarrow q)$ is true, i.e. the statement is a tautology. Can you prove this just by citing a rule of inference covered in class?

b. Prove that biconditional ("if and only if") is transitive, that is,

if a → b, b → c, ∴ a → c.

$$((p \leftrightarrow q) \wedge (q \leftrightarrow r)) \rightarrow (p \leftrightarrow r)$$

≡ (p→q) ∧ (q→p) ∧ (q→r) ∧ (r→q)
≡ (p→q) ∧ (q→r) ∧ (r→q) ∧ (q→p)
→ (p→r) ∧ (r→p) by DS.
≡ p ⟺ r.

∴ Since statement is a tautology, the biconditional is transitive.

7a. Given the following argument:

$p \vee (q \wedge r)$ ≡ (p ∨ q) ∧ (p ∨ r) by distributive.
$\sim p$ ≡ T ∴ p ≡ F   (F ∨ q) ∧ (F ∨ r) by substitution.
$\therefore q \wedge r$ ≡ T   (q ∨ F) ∧ (r ∨ F) by commutative.
∴ q ∧ r by identity

p ≡ F
q ≡ T
r ≡ T
q ∧ r ≡ T.

Without actually drawing the truth table, determine the values of p, q and r in the critical row(s) of the truth table. Is the argument valid?

F   T   T

↳ resultant premise are true

Yes. F ∨ T ≡ T, elimination.

b. Give a counterexample to show that the following argument is invalid.

$p \vee (q \wedge r)$ ≡ (p ∨ q) ∧ (p ∨ r)
$\sim(p \wedge q)$ ≡ ¬p ∨ ¬q by demorgan's
$\therefore r$

p ∨ (q ∧ r)
r ≡ T
∴ (p ∨ q) ∧ (p ∨ T) by distributive.
(p ∨ q) ∧ T by universal bound
p ∨ q by identity
¬(¬p ∧ ¬q) ≢ ¬(p ∧ q) ∴ invalid

Find a case to show that the statement is wrong.
p ≡ true
q ≡ false
r ≡ false
T ∨ (F ∧ F) ≡ T ∨ F ≡ T
~(T ∧ F) ≡ T.
F.
(premise is true & conclusion is false).

c. Determine whether the following argument is valid or invalid.

p → q ∨ r
¬(q ∨ r)
¬q ∧ ¬r

If I go to the beach, I will take my shades or my sunscreen.
I am taking my shades but not my sunscreen.   q ∧ ¬r
∴ I will go to the beach.   ∴ p.

q ≡ T   r ≡ F

p ≡ F   Conclusion: False ⇒ invalid

Note: sufficient → (B needs only A)
necessary ← (A cannot occur without B)

d. Determine whether the following argument is valid or invalid.

I will buy a new goat or a used Yugo.
If I buy both a new goat and a used Yugo, I will need a loan.
I bought a used Yugo but I don't need a loan.
∴ I didn't buy a new goat.

q ≡ T   r ≡ F

p ≡ F

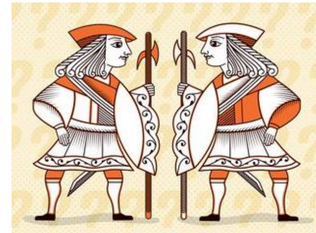
p ∨ q
(p ∧ q) → r
q ∧ ¬p
∴ ¬p.

critical row: p ≡ F, q ≡ T, r ≡ F.

argument is valid.

From truth table:
Find all arguments to be true.
∴ Since critical row is true, argument is true.

8. The island of Wantuutreewan is inhabited by exactly two types of people: **knights** who always tell the truth and **knaves** who always lie. Every native is a knight or a knave, <u>but not both</u>. You visit the island and have the following encounters with a few natives.



Let knights be True, knaves be False

a. You meet two natives *A* and *B*.

$A \wedge B \equiv T$

A says: Both of us are knights.
B says: A is a knave. $A \equiv F$

What are *A* and *B*?

b. You meet two natives *C* and *D*.

C says: Both of us are knaves.
D says nothing.

What are *C* and *D*?

Part (a) has been solved for you (see below). Study the solution, and use the same format in answering part (b).

Answer for part (a):

Proof (by contradiction).

tempting to say "contradiction" not valid cause contradiction req: $p \wedge \neg p$. but 'knave' is not a negation of 'knight'. ∴ 1.5 is req. before we arrive at conclusion in 1.6.

1. If *A* is a knight, then:
    - 1.1 What *A* says is true. (by definition of knight)
    - 1.2 ∴ *B* is a knight too. (that's what *A* says)
    - 1.3 ∴ What *B* says is true. (by definition of knight)
    - 1.4 ∴ *A* is a knave. (that's what *B* says)
    - 1.5 ∴ *A* is not a knight. (one is a knight or a knave, but not both)
    - 1.6 ∴ Contradiction to 1.
2. ∴ *A* is not a knight.
3. ∴ *A* is a knave. (one is a knight or a knave, but not both)
4. ∴ What *B* says is true.
5. ∴ *B* cannot be a knave. (as *B* has said something true)
6. ∴ *B* is a knight. (one is a knight or a knave)
7. Conclusion: *A* is a knave and *B* is a knight.

Notes:
- It is tempting to say "Contradiction" right after line 1.4. However, this is not valid because contradiction requires $p \wedge \sim p$, but 'knave' is <u>not</u> the negation of 'knight'. Hence line 1.5 is required before we arrive at the contradiction in 1.6.

9. Recall the definitions of even and odd integers in Lecture #1 slide 27:

> If n is an integer, then
> n is even if and only if $\exists k \in \mathbb{Z}$ s.t. $n = 2k$;
> n is odd if and only if $\exists k \in \mathbb{Z}$ s.t. $n = 2k + 1$.

Prove the following:

The product of any two odd integers is an odd integer.

10. Your classmate Smart came across this claim:

prove both directions

Let n be an integer. Then n^2 is odd if and only if n is odd.

a. Smart attempts to prove the above claim as follows:

Proof (by contradiction).

only proved 1 way.

1. Suppose n is an even integer.
2. Then $\exists k \in \mathbb{Z}$ s.t. $n = 2k$.
3. Squaring both sides, we get $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$.
4. Since k is an integer, so is $2k^2$.
5. Hence $n^2 = 2p$, with $p = 2k^2 \in \mathbb{Z}$.
6. Therefore, n^2 is even.
7. So, if n is even, then n^2 is even, which is the same as saying, if n^2 is odd, then n is odd.
8. Therefore, n^2 is odd if and only if n is odd.

Comment on Smart's proof. absent of reason that arrives at his statements

b. Write your own proof.

D3

a) $(p \wedge q) \rightarrow p$

p	q	$(p \wedge q)$	$(p \wedge q) \rightarrow p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

b) $((p \vee q) \wedge \neg p) \rightarrow q$

p	q	$p \vee q$	$\neg p$	$(p \vee q) \wedge \neg p$	
T	T	T	F	F	T
T	F	T	F	F	T
F	T	T	T	T	T
F	F	F	T	F	T

c) $((p \rightarrow q) \wedge p) \rightarrow q$

$((p \to q) \wedge p) \to q$

hypothesis p → conclusion q

p	q	$p \to q$	$(p \to q) \wedge p$	
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

d) $[\neg p \to (q \wedge \neg q)] \to p$

p	q	$\neg p$	$q \wedge \neg q$	$(\neg p \to (q \wedge \neg q))$	
T	T	F	F	T	T
T	F	F	F	T	T
F	T	T	F	F	T
F	F	T	F	F	T

D5

$[(a \to x) \wedge (b \to y)] \to ((a \wedge b) \to (x \wedge y))$

$((a \to x) \wedge (b \to y)) \to ((a \wedge b) \to (x \wedge y))$

$((a \to x) \wedge (b \to y)) \to ((a \wedge b) \to (x \wedge y))$. tautology

a	$\to$	x	$\wedge$	b	$\to$	y	$\to$	a	$\wedge$	b	$\to$	x	$\wedge$	y
T	T	T	T	T	T	T	T	T	T	T	T	T	T	T
T	T	T	F	T	F	F	T	T	T	T	F	T	F	F
T	F	F	F	T	T	T	T	T	T	T	F	F	F	T
T	F	F	F	T	F	F	T	T	T	T	F	F	F	F

T	T	T	T	F	T	T	T	T	F	F	T	T	T	T
T	T	T	T	F	T	F	T	T	F	F	T	T	F	F
T	F	F	F	F	T	T	T	T	F	F	T	F	F	T
T	F	F	F	F	T	F	T	T	F	F	T	F	F	F
F	T	T	T	T	T	T	T	F	F	T	T	T	T	T
F	T	T	F	T	F	F	T	F	F	T	T	T	F	F
F	T	F	T	T	T	T	T	F	F	T	T	F	F	T
F	T	F	F	T	F	F	T	F	F	T	T	F	F	F
F	T	T	T	F	T	T	T	F	F	F	T	T	T	T
F	T	T	T	F	T	F	T	F	F	F	T	T	F	F
F	T	F	T	F	T	T	T	F	F	F	T	F	F	T
F	T	F	T	F	T	F	T	F	F	F	T	F	F	F

2 a) $\underbrace{\neg a}_{p} \rightarrow \underbrace{\neg(b \vee \neg a)}_{q}$

We should not negate the entire statement as the truth value would be different.

By implication, $p \rightarrow q \equiv \neg p \vee q$,

$\neg(\neg a) \vee \neg(b \vee \neg a)$

By de morgan's,

$\neg(\neg a) \vee (\neg b \wedge \neg(\neg a))$

By double negation,

$a \vee (\neg b \wedge a)$.

By commutative,

$a \vee (a \wedge \neg b)$.

By absorbtion, a.

2c) $(x \wedge x \vee y) \rightarrow z$

d) $(p \wedge q) \rightarrow q$

e) $(p \rightarrow q) \rightarrow r$

c) $(x \wedge x \vee y) \rightarrow z$

$(x \vee y) \rightarrow z$ by idempotent.

$\neg(x \vee y) \vee z$ by implication

$(\neg x \wedge \neg y) \vee z$ by de morgan's

e) $(p \rightarrow q) \rightarrow r$

$\neg(\neg p \vee q) \vee r$ by implication ×

$(\neg(\neg p) \wedge \neg q) \vee r$ by demorgan

$(p \wedge \neg q) \vee r$ by double negation

d) $(p \wedge q) \rightarrow q$.

$\neg(p \wedge q) \vee q$ by implication.

$(\neg p \vee \neg q) \vee q$ by demorgan's.

$\neg p \vee (\neg q \vee q)$ by associative.

$\neg p \vee (q \vee \neg q)$ by commutative.

$\neg p \vee T$ by negation.

T by domination

3 $(p \to q) \to r \equiv p \to (q \to r)$?

LHS: $(p \to q) \to r$

$\equiv (p \wedge \neg q) \vee r$ by 2e)

RHS: $p \to (q \to r)$

$\neg p \vee (\neg q \vee r)$ by implication.

$(\neg p \vee \neg q) \vee r$ by association.

$p \wedge \neg q \equiv \neg p \vee \neg q$?

p	q	¬p	¬q	p∧¬q	¬p∨¬q
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	F	T
F	F	T	T	F	T

∴ Statement is not logically equivalent.

To disprove → just need a counterexample.

$(F \to T) \to F$

$T \to F \equiv F$

$F \to (T \to F)$

$\equiv F \equiv T$

or use truth table.

p = True　　q = True.

4. If $12x - 7 = 29$, then $x = 3$　True.

negation: $\neg(p \to q) \xrightarrow{\text{Implication}} \neg(\neg p \vee q) \xrightarrow{\text{de morgan's}} p \wedge \neg q$　False.

$12x - 7 = 29$ and $x \neq 3$.

Contrapositive　$\neg q \to \neg p \equiv p \to q$.

If $x \neq 3$, then $12x - 7 \neq 29$.　True (same as $p \to q$).

Contrapositive of each other

Converse　$q \to p$

If $x = 3$, then $12x - 7 = 29$.　True.

inverse.　$\neg p \to \neg q$.

If $12x - 7 \neq 29$, then $x \neq 3$.　True.

p	q	(p→q)	¬(p→q)
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	F

Converse:

p	q	q→p
T	T	T
T	F	T
F	T	F
F	F	T

Inverse:

¬p	¬q	¬p→¬q
F	F	T
F	T	T
T	F	F
T	T	T

From truth tables, not possible for when $q \to p$ (converse) is true and $\neg p \to \neg q$ (inverse) is false.

$((p \to q) \wedge (q \to r)) \to (p \to r)$

p q r

8. proof　knaves always lie, knights always true.

C: Both of us are knaves.

A

D :

Proof (by cases)

1. ~~Case 1~~: C is a knight.
   1.1. What C says is True (by definition of knight)
   1.2. ∴ C is a knave (What C says)
   1.3. ∴ C is not a knight (one is a knight or one is a knave but not both)
   1.4. ∴ Contradiction to 1.

Therefore, C is not a knight, thus

2. ~~Case 2~~: C is a knave
   2.1. What C says is False (by defn of knave)
   2.2. ∴ both C and D are not knaves (by negation of C's statement)
   2.3. ∴ D cannot be a knave (by statement 2 and 2.2)
   2.4. ∴ D is a knight (one is a knight or one is a knave but not both)
3. Conclusion: C is a knave and D is a knight

9. If n is an integer, then

n is even iff $\exists k \in \mathbb{Z}$ s.t. $n = 2k$; (there exist, so that)

n is odd iff $\exists k \in \mathbb{Z}$ s.t. $n = 2k+1$;

Prove: product of any 2 odd integers is an odd integer.

only used when its hard to go forward, need more info from conclusion side.

$x + y = z$, where x, y, z is odd.

By contradiction,

use $\Downarrow$ acts

also need $a > b$ and $a > 0$ reduce to 2 cases

1. Suppose z is odd, then
   $z = 2n+1$, $\exists n \in \mathbb{Z}$ s.t. $n = 2i$ — by definition of odd and even.
   1.1 $x = 2k+1$, $\exists k \in \mathbb{Z}$ s.t. $k = 2i$. — by definition of odd & even.
   1.2 $y = 2m+1$, $\exists m \in \mathbb{Z}$ s.t. $m = 2i$. — by definition of odd & even.
2. $2k+1 + 2m+1 = 2m + 2k + 2$ — by algebraic manipulation.
3. $2m + 2k + 2 = 2(m+k+1)$ — by factorization. by closure of int under + and ×.
4. ∴ $2(m+k+1)$ is even — by defn of even.
5. ∴ z is even — by defn of z.
6. ∴ contradiction to 1.
7. Conclusion: statement is false.

Direct proof

1. Take any 2 odd ints n, m.
2. $n = 2k+1$, $m = 2p+1$, for $k, p \in \mathbb{Z}$ by defn of odd
3. $nm = (2k+1)(2p+1)$
   $= (2k(2p+1)) + (2p+1)$
   $= (4kp + 2k) + 2p + 1$
   $= 2(2kp + k + p) + 1$
4. $q = 2kp + k + p \in \mathbb{Z}$
5. $nm = 2q+1$ ⇒ odd, by defn.
6. conclusion.

10. Let n be an integer. Then n^2 is odd iff n is odd

Proof (by modus ponens): $n \Rightarrow n^2$

1. If n is odd, then
   1.1. $\exists k \in \mathbb{Z}$ s.t. $n = 2k+1$ by defn of odd no.

1.2. $\therefore n^2 = (2k+1)^2$ by defn of n^2

1.3. $\therefore n^2 = 4k^2 + 4k + 1$ by algebraic manipulation

1.4. $\therefore n^2 = 2(2k^2+2k)+1$ by algebraic manipulation

1.5. $\therefore n^2 = 2m+1$, $m = 2k^2+2k$ by algebraic manipulation.

1.6. $\therefore n^2$ is odd by defn of odd numbers.

Since $k \in \mathbb{Z}$, $(2k^2+2k) \in \mathbb{Z}$ by closure of integers under $+$ and $\times$.

$n^2 \rightarrow n$

2. If n is even, then (modus tollens)

2.1 $\exists k \in \mathbb{Z}$ s.t. $n = 2a$ by defn of even no.

2.2 $\therefore n^2 = (2a)^2$ by defn of n^2.

2.3 $\therefore n^2 = 4a^2 = 2(2a^2)$ by algebra.

2.4 $\therefore n^2 = 2b$, $b = 2a^2$ by algebra.

2.5. $\therefore n^2$ is even by defn of even no.

3. By modus tollens, since even is negation of odd and $\neg(n \text{ is odd}) \rightarrow \neg(n^2 \text{ is odd})$,
$\therefore$ n^2 is odd $\rightarrow$ n is odd is true.

4. Since n^2 is odd $\rightarrow$ n is odd and n is odd $\rightarrow$ n^2 is odd are true,
$\therefore$ n is odd $\leftrightarrow$ n^2 is odd is true. $\blacksquare$

CS1231S: Discrete Structures

Tutorial #1: Propositional Logic and Proofs

Answers

III. Tutorial Questions

1. (Basics)
   a. What are the names of these logical connectives?

      (i) $\sim$ (ii) $\wedge$ (iii) $\vee$ (iv) $\rightarrow$ (v) $\leftrightarrow$

   b. Given the statement $p \rightarrow q$, what is p called and what is q called?
      What is the negation of $p \rightarrow q$, i.e. $\sim(p \rightarrow q)$?
      What do you think are the common mistakes when negating $p \rightarrow q$? (These mistakes are still found in students' work towards the end of the semester.)

   c. Use logical connectives to rewrite the following statements (the parts are independent of one another):
      (i) p is a sufficient condition for q.
      (ii) p is a necessary condition for q.
      (iii) p is a necessary and sufficient condition for q.
      (iv) p if q.
      (v) p only if q.
      (vi) p if and only if q.

Answers:

a. $\sim$ ("negation", informally "not");

   $\wedge$ ("conjunction", informally "and");

   $\vee$ ("disjunction", informally "or");

   $\rightarrow$ ("conditional", informally "implies" or "if then");

   $\leftrightarrow$ ("biconditional", informally "if and only if").

b. p is called the antecedent or hypotheses; q is called the consequent or conclusion.

   $\sim(p \rightarrow q) \equiv p \wedge \sim q$

   Common mistakes:
   $\sim(p \rightarrow q) \equiv p \rightarrow \sim q$; $\sim(p \rightarrow q) \equiv q \rightarrow p$; $\sim(p \rightarrow q) \equiv p \vee \sim q$.

c. (i) $p \rightarrow q$; (ii) $q \rightarrow p$; (iii) $p \leftrightarrow q$;

   (iv) $q \rightarrow p$; (v) $p \rightarrow q$; (vi) $p \leftrightarrow q$;

2. Simplify the propositions below using the laws given in **Theorem 2.1.1 (Epp)** and the **implication law** (if necessary) with only negation ($\sim$), conjunction ($\wedge$) and disjunction ($\vee$) in your final answers. Supply a justification for every step.

   (For the first half of the module, we want students to cite justification <u>for every step</u>. This is to ensure that you do not arrive at the answer by coincidence. Only after you have gained sufficient experience then would we relax this and allow you to skip obvious steps, or combine multiple steps in a line.)

   a. $\sim a \rightarrow \sim(b \vee \sim a)$

      Aiken worked out the first step of his answer as follows:
      $\sim a \rightarrow \sim(b \vee \sim a)$
      $\equiv \sim\big(a \rightarrow \sim(b \vee \sim a)\big)$

      What mistake did he make?

   b. $\sim a \rightarrow \sim(b \vee \sim a)$

      Aiken has been notified of his mistake in part (a) and he re-worked his answer as shown below. You can verify that the answer is correct by comparing the truth tables of $\sim a \rightarrow \sim(b \vee \sim a)$ with a. However, he skipped a number of steps in his working and hence his answer will not be awarded full credit. Can you point out the omissions? (Note: To show that two logical statements are equivalent, we use $\equiv$, not $=$.)

$\sim a \rightarrow \sim(b \vee \sim a)$	
$\equiv \sim a \rightarrow (\sim b \wedge a)$	by De Morgan's law (step 1)
$\equiv a \vee (\sim b \wedge a)$	by the implication law (step 2)
$\equiv a$	by the absorption law (step 3)

   c. $(x \wedge x \vee y) \rightarrow z$
   d. $(p \wedge q) \rightarrow q$
   e. $(p \rightarrow q) \rightarrow r$

You don't need to write "step x". This is for ease of reference in the answers that will be released after tutorials.

Answers

a. Aiken has mixed up the precedence of the connectives. The given statement is equivalent to $(\sim a) \rightarrow \sim(b \vee \sim a)$, not $\sim\big(a \rightarrow \sim(b \vee \sim a)\big)$.

b.

$\sim a \rightarrow \sim(b \vee \sim a)$	
$\equiv \sim a \rightarrow \big(\sim b \wedge \sim(\sim a)\big)$	by De Morgan's law (step 1)
$\equiv \sim a \rightarrow (\sim b \wedge a)$	by the double negative law
$\equiv \sim(\sim a) \vee (\sim b \wedge a)$	by the implication law (step 2)
$\equiv a \vee (\sim b \wedge a)$	by the double negative law
$\equiv a \vee (a \wedge \sim b)$	by the commutative law
$\equiv a$	by the absorption law (step 3)

c. The statement is ambiguous as $\wedge$ and $\vee$ are coequal in precedence. Add parentheses to disambiguate, i.e., $((x \wedge x) \vee y) \rightarrow z$ or $(x \wedge (x \vee y)) \rightarrow z$.
Question: Is $(a \wedge b \wedge c)$ ambiguous? Why or why not?

d. $(p \wedge q) \rightarrow q$

$\equiv {\sim}(p \wedge q) \vee q$	by the implication law
$\equiv ({\sim}p \vee {\sim}q) \vee q$	by De Morgan's law
$\equiv {\sim}p \vee ({\sim}q \vee q)$	by the associative law
$\equiv {\sim}p \vee (q \vee {\sim}q)$	by the commutative law
$\equiv {\sim}p \vee \textbf{true}$	by the negation law
$\equiv \textbf{true}$	by the universal bound law

Did you skip this commutative law step?

Reminder: We will use **true** and **false** instead of **t** and **c** for tautology and contradiction respectively.

e. $(p \rightarrow q) \rightarrow r$

$\equiv ({\sim}p \vee q) \rightarrow r$	by the implication law
$\equiv ({\sim}({\sim}p \vee q)) \vee r$	by the implication law
$\equiv ({\sim}({\sim}p) \wedge {\sim}q) \vee r$	by De Morgan's law
$\equiv (p \wedge {\sim}q) \vee r$	by the double negative law

3. Prove, or disprove, that $(p \rightarrow q) \rightarrow r$ is logically equivalent to $p \rightarrow (q \rightarrow r)$.

Answers

$(p \rightarrow q) \rightarrow r$ is not logically equivalent to $p \rightarrow (q \rightarrow r)$.

Counterexample: Let p, q and r be false. Then $(p \rightarrow q) \rightarrow r$ is false but $p \rightarrow (q \rightarrow r)$ is true.

For such questions, sometimes you do not know whether the given statement is true or false. In this case, you would probably have to do some trial-and-errors, or fill in the truth table:

p	q	r	$p \rightarrow q$	$(p \rightarrow q) \rightarrow r$	$q \rightarrow r$	$p \rightarrow (q \rightarrow r)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	T	T	T
T	F	F	F	T	T	T
F	T	T	T	T	T	T
F	T	F	T	F	F	T
F	F	T	T	T	T	T
F	F	F	T	F	T	T

You can see that there are two counterexamples: the sixth and eighth rows.
Certainly, filling out the whole truth table is tedious and should be avoided.

4. Given the conditional statement "If $12x - 7 = 29$, then $x = 3$", write the **negation**, **contrapositive**, **converse** and **inverse** of the statement. Is the given conditional statement true? Is its converse true?

   In general, is it possible for the converse of a conditional statement to be true while the inverse of the same statement is false? Why?

   Answers

   Negation: $12x - 7 = 29$ and $x \neq 3$.
   Contrapositive: If $x \neq 3$, then $12x - 7 \neq 29$.
   Converse: If $x = 3$, then $12x - 7 = 29$.
   Inverse: If $12x - 7 \neq 29$, then $x \neq 3$.
   Both the given conditional statement and its converse are true. (And hence the contrapositive and inverse are true too.)

   It is not possible for the converse of a conditional statement to be true while the inverse of the same statement is false, as they are logically equivalent to each other.

5. The conditional statement $p \to q$ is an important logical statement. Recall that it is defined by the following truth table:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

   Oftentimes, students are perplexed by this definition. The first two rows look reasonable, but the last two rows seem strange. However, this way of defining $p \to q$ actually gives us the nice intuitive property of the following statement:

$$\big((p \to q) \wedge (q \to r)\big) \to (p \to r)$$

   which is the **transitive rule of inference** we studied in lecture (Lecture #2 slide 67):

$$\begin{array}{l} \quad p \to q \\ \quad q \to r \\ \therefore p \to r \end{array}$$

   For example, given premises "if x is a square then x is a rectangle" and "if x is a rectangle than x is a quadrilateral", the conclusion is "if x is a square then x is a quadrilateral". We use such intuitive reasoning very often in our life.

   Show that if we define the conditional statement alternatively as follows, then the transitive rule of inference would no longer hold.

Alternative 1: $\rightarrow_a$

p	q	$p \rightarrow_a q$
T	T	T
T	F	F
F	T	F
F	F	F

Alternative 2: $\rightarrow_b$

p	q	$p \rightarrow_b q$
T	T	T
T	F	F
F	T	T
F	F	F

Alternative 3: $\rightarrow_c$

p	q	$p \rightarrow_c q$
T	T	T
T	F	F
F	T	F
F	F	T

Answers

Alternative 1: $\rightarrow_a$

p	q	r	$p \rightarrow_a q$	$q \rightarrow_a r$	$p \rightarrow_a r$	$(p \rightarrow_a q) \wedge (q \rightarrow_a r)$	$\big((p \rightarrow_a q) \wedge (q \rightarrow_a r)\big) \rightarrow_a (p \rightarrow_a r)$
T	T	F	T	F	F	F	F

Therefore, $\big((p \rightarrow_a q) \wedge (q \rightarrow_a r)\big) \rightarrow_a (p \rightarrow_a r)$ is not a tautology.

Alternative 2: $\rightarrow_b$

p	q	r	$p \rightarrow_b q$	$q \rightarrow_b r$	$p \rightarrow_b r$	$(p \rightarrow_b q) \wedge (q \rightarrow_b r)$	$\big((p \rightarrow_b q) \wedge (q \rightarrow_b r)\big) \rightarrow_b (p \rightarrow_b r)$
T	T	F	T	F	F	F	F

Therefore, $\big((p \rightarrow_b q) \wedge (q \rightarrow_b r)\big) \rightarrow_b (p \rightarrow_b r)$ is not a tautology.

Alternative 3: $\rightarrow_c$

p	q	r	$p \rightarrow_c q$	$q \rightarrow_c r$	$p \rightarrow_c r$	$(p \rightarrow_c q) \wedge (q \rightarrow_c r)$	$\big((p \rightarrow_c q) \wedge (q \rightarrow_c r)\big) \rightarrow_c (p \rightarrow_c r)$
F	T	F	F	F	T	F	F

Therefore, $\big((p \rightarrow_c q) \wedge (q \rightarrow_c r)\big) \rightarrow_c (p \rightarrow_c r)$ is not a tautology.

6a. It is intuitively clear that $(p \leftrightarrow q) \rightarrow (p \rightarrow q)$ is true, i.e. the statement is a tautology. Can you prove this just by citing a rule of inference covered in class?

b. Prove that biconditional ("if and only if") is transitive, that is,

$$\big((p \leftrightarrow q) \wedge (q \leftrightarrow r)\big) \rightarrow (p \leftrightarrow r)$$

Answers

a. Since by definition of biconditional, $(p \leftrightarrow q) \equiv (p \rightarrow q) \wedge (q \rightarrow p)$, we may use specialization:

$(p \rightarrow q) \wedge (q \rightarrow p)$
$\therefore p \rightarrow q$ (by specialization)

b. $(p \leftrightarrow q) \wedge (q \leftrightarrow r)$

$\equiv (p \rightarrow q) \wedge (q \rightarrow p) \wedge (q \rightarrow r) \wedge (r \rightarrow q)$	by definition of $\leftrightarrow$
$\equiv (p \rightarrow q) \wedge (q \rightarrow r) \wedge (r \rightarrow q) \wedge (q \rightarrow p)$	by associativity and commutativity
$\rightarrow (p \rightarrow r) \wedge (r \rightarrow p)$	by D5 and transitivity of $\rightarrow$ in Q5
$\equiv p \leftrightarrow r$	by definition of $\leftrightarrow$

7a. Given the following argument:

$p \vee (q \wedge r)$
$\sim p$
$\therefore q \wedge r$

<u>Without</u> actually drawing the truth table, determine the values of p, q and r in the critical row(s) of the truth table. Is the argument valid?

b. Give a counterexample to show that the following argument is invalid.

$p \vee (q \wedge r)$
$\sim (p \wedge q)$
$\therefore r$

c. Determine whether the following argument is valid or invalid.

If I go to the beach, I will take my shades or my sunscreen.
I am taking my shades but not my sunscreen.
$\therefore$ I will go to the beach.

d. Determine whether the following argument is valid or invalid.

I will buy a new goat or a used Yugo.

If I buy both a new goat and a used Yugo, I will need a loan.
I bought a used Yugo but I don't need a loan.
∴ I didn't buy a new goat.

Answers

a. Critical rows are those where the premises are true. Therefore, we can use deduction here instead of drawing the whole truth table. Here, the two premises are $p \vee (q \wedge r)$ and $\sim p$. So, p must be false and hence both q and r must be true. There is one critical row and the conclusion $q \wedge r$ in that row is true. Therefore, the argument is **valid**.

b. Counterexample: $p \equiv true, q \equiv r \equiv false$.
Both premises $p \vee (q \wedge r)$ and $\sim(p \wedge q)$ are true, but the conclusion r is false.

c. Let p = "I go to the beach"; q = "I take my shades"; and r = "I take my sunscreen".

$$p \rightarrow q \vee r$$
$$q \wedge \sim r$$
$$\therefore p$$

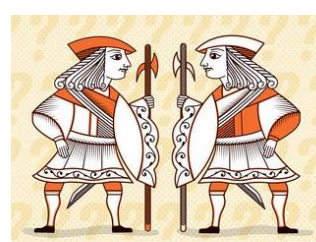
Critical row: $p \equiv false$; $q \equiv true$; $r \equiv false$. Conclusion: false.
Therefore, the argument is **invalid**.

d. Let p = "I buy a new goat"; q = "I buy a used Yugo"; and r = "I need a loan".

$$p \vee q$$
$$(p \wedge q) \rightarrow r$$
$$q \wedge \sim r$$
$$\therefore \sim p$$

Critical row: $p \equiv false$; $q \equiv true$; $r \equiv false$. Conclusion: true.
Therefore, the argument is **valid**.

8. The island of Wantuutreewan is inhabited by exactly two types of people: **knights** who always tell the truth and **knaves** who always lie. Every native is a knight or a knave, <u>but not both</u>. You visit the island and have the following encounters with a few natives.



a. You meet two natives *A* and *B*.

A says: Both of us are knights.
B says: *A* is a knave.

What are *A* and *B*?

b. You meet two natives *C* and *D*.

C says: Both of us are knaves.
D says nothing.

What are *C* and *D*?

Part (a) has been solved for you (see below). Study the solution, and use the same format in answering part (b).

Answer for part (a):

Proof (by contradiction).

1. If *A* is a knight, then:
   - 1.1 What *A* says is true. (by definition of knight)
   - 1.2 $\therefore$ *B* is a knight too. (that's what *A* says)
   - 1.3 $\therefore$ What *B* says is true. (by definition of knight)
   - 1.4 $\therefore$ *A* is a knave. (that's what *B* says)
   - 1.5 $\therefore$ *A* is not a knight. (one is a knight or a knave, but not both)
   - 1.6 $\therefore$ Contradiction to 1.
2. $\therefore$ *A* is not a knight.
3. $\therefore$ *A* is a knave. (one is a knight or a knave, but not both)
4. $\therefore$ What *B* says is true.
5. $\therefore$ *B* cannot be a knave. (as *B* has said something true)
6. $\therefore$ *B* is a knight. (one is a knight or a knave)
7. Conclusion: *A* is a knave and *B* is a knight.

Notes:

- It is tempting to say "Contradiction" right after line 1.4. However, this is not valid because contradiction requires $p \wedge \sim p$, but 'knave' is <u>not</u> the negation of 'knight'. Hence line 1.5 is required before we arrive at the contradiction in 1.6.

Answer

b. Proof

1. If *C* is a knight,
   - 1.1 What *C* says is true. (by definition of knight)
   - 1.2 $\therefore$ *C* is a knave. (that's what *C* says)
   - 1.3 $\therefore$ *C* is a not a knight. (one is a knight or a knave, but not both)
   - 1.4 This contradicts that *C* is a knight.
2. Therefore, *C* is not a knight.
   - 2.1 Then *C* is a knave. (one is a knight or a knave)
   - 2.2 $\therefore$ what *C* says is false. (by definition of knave)
   - 2.3 $\therefore$ not both of *C* and *D* are knaves. (*C* says both are knaves)
   - 2.4 So *D* cannot be a knave. (otherwise, both of them are knaves)
   - 2.5 $\therefore$ *D* must be a knight. (one is a knight or a knave)
3. Therefore, *C* is a knave and *D* is a knight.

9. Recall the definitions of even and odd integers in Lecture #1 slide 27:

> If n is an integer, then
> n is even if and only if $\exists k \in \mathbb{Z}$ s.t. $n = 2k$;
> n is odd if and only if $\exists k \in \mathbb{Z}$ s.t. $n = 2k + 1$.

Prove the following:

The product of any two odd integers is an odd integer.

Answer

Proof (direct proof).

1. Take any two odd integers n, m.
2. Then $n = 2k + 1$ and $m = 2p + 1$ for $k, p \in \mathbb{Z}$ (by definition of an odd number).
3. Hence $nm = (2k+1)(2p+1) = \big(2k(2p+1)\big) + (2p+1)$
   $= (4kp + 2k) + (2p + 1) = 2(2kp + k + p) + 1$ (by basic algebra)
4. Let $q = 2kp + k + p$ which is an integer (by closure of integers under $+$ and $\times$).
5. Then $nm = 2q + 1$ which is odd (by definition of an odd number).
6. Therefore, the product of any two odd integers is an odd integer.

10. Your classmate Smart came across this claim:

Let n be an integer. Then n^2 is odd if and only if n is odd.

a. Smart attempts to prove the above claim as follows:

Proof (by contradiction).

1. Suppose n is an even integer.
2. Then $\exists k \in \mathbb{Z}$ s.t. $n = 2k$.
3. Squaring both sides, we get $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$.
4. Since k is an integer, so is $2k^2$.
5. Hence $n^2 = 2p$, with $p = 2k^2 \in \mathbb{Z}$.
6. Therefore, n^2 is even.
7. So, if n is even, then n^2 is even, which is the same as saying, if n^2 is odd, then n is odd.
8. Therefore, n^2 is odd if and only if n is odd.

Comment on Smart's proof.

b. Write your own proof.

Answers

a. There are several things that Smart did not do right:
   - He is proving the contrapositive of "if n^2 is odd, then n is odd", so it is only one direction.
   - He wrote "proof (by contradiction)", which is wrong. It should be proof by contraposition.
   - No justification at a few places. For example, step 2 (by definition of an even integer), step 4 (by closure of integers under multiplication), and step 6 (by definition of an even integer).

b. Proof:

1. ($\Rightarrow$) Proving the contraposition of "if n^2 is odd, then n is odd".
   1.1. Suppose n is even.
   1.2. Then $\exists k \in \mathbb{Z}$ s.t. $n = 2k$. (by definition of an even integer)
   1.3. Squaring both sides, we get $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$. (by basic algebra)
   1.4. Hence $n^2 = 2p$, with $p = 2k^2 \in \mathbb{Z}$. (by closure of integers under $\times$)
   1.5. Therefore, n^2 is even.
   1.6. This proves that if n^2 is odd, then n is odd.
2. ($\Leftarrow$) If n is odd, then $n \times n = n^2$ is odd. (by question 9)
3. Therefore, n^2 is odd if and only if n is odd. (by lines 1.6 and 2)