MA2001 LINEAR ALGEBRA

Matrices

National University of Singapore
Department of Mathematics

Definition of Matrix

- **Definition.** A **matrix** (plural **matrices**) is a rectangular array of numbers. finite
  - $\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$
    - m is the number of **rows** in the matrix.
    - n is the number of **columns** in the matrix. row × column
    - The **size** of the matrix is given by $m \times n$. 'by'
    - The **(i, j)-entry** is the entry in ith row & jth column.
  - In the given matrix, the (i, j)-entry is a_{ij}.
- **Remark.** Some books use $[\cdots]$ instead of $(\cdots)$.
  - Notations $|\cdots|$ and $\|\cdots\|$ are reserved to use later.

determinants / norm

Examples

1. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 0 & -1 \end{pmatrix}$ is a 3×2 matrix.
   - The $(1, 2)$-entry is 2 and the $(3, 1)$-entry is 0.
2. $\begin{pmatrix} \sqrt{2} & 3.1 & -2 \\ 3 & \frac{1}{2} & 0 \\ 0 & \pi & 0 \end{pmatrix}$ is a 3×3 matrix.
3. $\begin{pmatrix} 2 & 1 & 0 \end{pmatrix}$ is a 1×3 matrix.
4. $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ is a 3×1 matrix.
5. (4) is a 1×1 matrix.
   - A 1×1 matrix is usually treated as a real number in computation. For instance, $(4) = 4$.

ring system?

Notation of Matrices

- A matrix is usually denoted by capital letters $A, B, C, \ldots$. Capital letters
  - $m \times n$ matrix $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$.
    - a_{ij} is the (i,j)-entry of A.
    - It is denoted by $\boxed{A = (a_{ij})_{m\times n}}$. size. entry

    Sometimes, if the size of A is known (or not important)
    - simple notation: $\boxed{A = (a_{ij})}$. drop size.

Examples

- Write down the following matrices explicitly.
  1. $A = (a_{ij})_{2\times 3}$, where $a_{ij} = i + j$.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} 1+1 & 1+2 & 1+3 \\ 2+1 & 2+2 & 2+3 \end{pmatrix}$$

2rows, 3columns

$$= \begin{pmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$$

  2. $B = (b_{ij})_{3\times 2}$, where $b_{ij} = \begin{cases} 1 & \text{if } i+j \text{ is even} \\ -1 & \text{if } i+j \text{ is odd} \end{cases}$

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ 1 & -1 \end{pmatrix}$$

3row, 2columns

Special Matrices

- A **row matrix** (**row vector**) is a matrix with only one row. $1 \times n$ matrix.
  - $\begin{pmatrix} 2 & 1 & 0 \end{pmatrix}$ is a row matrix (row vector).
- A **column matrix** (**column vector**) is a matrix with only one column.
  - $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ is a column matrix (column vector).
- A **square matrix** is a matrix with the same number of rows and columns. pivot points $\Rightarrow$ REF.
  - An $n \times n$ matrix is called a **square matrix** of **order** n. only refers to this
  - $\begin{pmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{pmatrix}$ is a square matrix of order 3.

Special Matrices

- Let $\boldsymbol{A} = (a_{ij})$ be a **square matrix** of order n. $\sim (a_{ij})_{n \times n}$.
  - The **diagonal** of $\boldsymbol{A}$ is the sequence of entries
    - $\boxed{a_{11}, a_{22}, \ldots, a_{nn}}$
  - $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$. only write this if square matrix $\Rightarrow$ rep. diagonal.
  - $a_{ii}, i = 1, \ldots, n$, are the **diagonal entries**.
  - $a_{ij}, i \neq j$, are called the **non-diagonal entries**.
    - a_{ij} is $\begin{cases} \text{a diagonal entry} & \text{if } i = j, \\ \text{a non-diagonal entry} & \text{if } i \neq j. \end{cases}$

Special Matrices

- Let $\boldsymbol{A} = (a_{ij})$ be a **square matrix** of order n.
  - The **diagonal** of $\boldsymbol{A}$ is the sequence of entries
    - $\boxed{a_{11}, a_{22}, \ldots, a_{nn}}$
  - $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$.

anti-diagonal

diagonal *.

- **Remark.** In some textbook,
  - The **diagonal** is also called the **principle diagonal** or **major diagonal**.
  - The **anti-diagonal** or **minor diagonal** refers to the diagonal from the right top to the left bottom.
    - $a_{1n}, a_{2,n-1}, \ldots, a_{n1}$.

also a square matrix.

Special Matrices

- A **square matrix** is called a **diagonal matrix** if all its non-diagonal entries are zero.
  - $\boldsymbol{A} = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$.

everything else except diagonal is 0.

diagonal entries may or may not be 0.

  - $\boldsymbol{A} = (a_{ij})_{n\times n}$ is diagonal $\Leftrightarrow a_{ij} = 0$ for all $i \neq j$.

Example.

  - $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is a diagonal matrix. ✓
  - $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$, a, b, c are (possibly zero) numbers. ✓

∴ A zero matrix is diagonal. eg. $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

A 1x1 matrix is always a diagonal matrix.

Special Matrices

- A **diagonal matrix** is called a **scalar matrix** if all its diagonal entries are the same.
  - $A = \begin{pmatrix} c & 0 & \cdots & 0 \\ 0 & c & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c \end{pmatrix}$, where c is a constant.
  - $A = (a_{ij})_{n\times n}$ is scalar $\Leftrightarrow a_{ij} = \begin{cases} c & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$

  Example.
  - $\begin{pmatrix} 1101 & 0 \\ 0 & 1101 \end{pmatrix}$ is a scalar matrix. same
  - $\begin{pmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix}$, where c is a (possibly zero) number.

Same: diagonal + scalar type matrix.

Special Matrices

- A **scalar matrix** is called an **identity matrix** if all its diagonal entries ~~matrix~~ are 1.
  - Denote the identity matrix of order n (size $n \times n$) by I_n. → for each n, there is only 1 identity matrix. don't need write $I_{n,n}$.
    - If no confusion in order, write I instead of I_n.
  - $A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$.
  - $A = (a_{ij})_{n\times n}$ is identity $\Leftrightarrow a_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$
- **Note**: There is exactly one identity matrix in order n.
  - $I_1 = (1) = 1; \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Scalar → constant.

identity → number 1

Special Matrices

- A matrix with all entries equal to zero is a **zero matrix**.
  - Denote the zero matrix of size $m \times n$ by $\mathbf{0}_{m\times n}$.
    - If no confusion in size, write $\mathbf{0}$ instead of $\mathbf{0}_{m\times n}$. if know the size.
  - $\boldsymbol{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$. may not be a square matrix.

    ∴ no diagonal dot.
  - $\boldsymbol{A} = (a_{ij})_{m\times n}$ is zero $\Leftrightarrow a_{ij} = 0$ for all i, j.

  Note: There is exactly one zero matrix in size $m \times n$.
  - $\mathbf{0}_{1\times 1} = (0) = 0$; $\mathbf{0}_{1\times 3} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$.
  - $\mathbf{0}_{3\times 2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$; $\mathbf{0}_{3\times 4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. may diff

Special Matrices

- A **square matrix** is called **symmetric** if it is symmetric with respect to the diagonal.
  - $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & a_{34} & \cdots & a_{3n} \\ a_{41} & a_{42} & a_{43} & a_{44} & \cdots & a_{4n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & \cdots & a_{nn} \end{pmatrix}$. square & symmetric w.r.t. diagonal ⟹ symmetric
  - $\boldsymbol{A} = (a_{ij})_{n\times n}$ is symmetric $\Leftrightarrow a_{ij} = a_{ji}$ for all i, j. $(i \neq j)$
    - (There is no restriction to the diagonal entries.) same ∴ $a_{ii} = a_{ii}$

      no restriction to diagonal entries.
- **Examples.**
  - (2); $\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$; $\begin{pmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & j \end{pmatrix}$

Special Matrices

- A **square matrix** is called **upper triangular** if all the entries **below** the diagonal are zero.

  i > j

  - $$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & a_{24} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & a_{34} & \cdots & a_{3n} \\ 0 & 0 & 0 & a_{44} & \cdots & a_{4n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix}.$$

    upper triangle

  - $A = (a_{ij})_{n\times n}$ is upper triangular $\Leftrightarrow a_{ij} = 0$ if $i > j$. if lower triangle is 0.
    - (There is no restriction to the diagonal entries.)
- **Examples.**
  - (2); $\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$; $\begin{pmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \\ 0 & 0 & 0 & j \end{pmatrix}$

Special Matrices

- A **square matrix** is called **lower triangular** if all the entries **above** the diagonal are zero.
  - $$A = \begin{pmatrix} a_{11} & 0 & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & 0 & \cdots & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & \cdots & a_{nn} \end{pmatrix}.$$

    upper triangle all 0.

  - $A = (a_{ij})_{n\times n}$ is lower triangular $\Leftrightarrow a_{ij} = 0$ if $i < j$. whenever $i < j$, $a_{ij} = 0$.
    - (There is no restriction to the diagonal entries.)
- **Examples.**
  - (2); $\begin{pmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & f \end{pmatrix}$; $\begin{pmatrix} a & 0 & 0 & 0 \\ b & e & 0 & 0 \\ c & f & h & 0 \\ d & g & i & j \end{pmatrix}$

Special Matrices

- Both upper triangular matrices and lower triangular matrices are called **triangular matrices**.
  - (A matrix is both upper and lower triangular

    $\Leftrightarrow$ it is diagonal.)
- **More Examples.**
  - Square matrices:
    - $\begin{pmatrix} 4 \end{pmatrix} \begin{pmatrix} 0 & 4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ -1 & 3 & 2 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 & 6 & 2 \\ 0 & 3 & 9 & -1 \\ 1 & 0 & 0 & 0 \\ -2 & 3 & 0 & 1 \end{pmatrix}$
  - Diagonal matrices:
    - $\begin{pmatrix} 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Special Matrices

- **More Examples.**
  - Scalar matrices: diagonal same
    - $\begin{pmatrix} 4 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$
  - Identity matrices: diagonal 1
    - $\begin{pmatrix} 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
  - Zero matrices: all zero
    - $\begin{pmatrix} 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Special Matrices

- **More Examples.**
  - Symmetric matrices: square, symmetric w.r.t. diagonal
    - $\begin{pmatrix} 4 \end{pmatrix}$ $\begin{pmatrix} 0 & 4 \\ 4 & 2 \end{pmatrix}$ $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 3 & 2 \\ 0 & 2 & 2 \end{pmatrix}$ $\begin{pmatrix} 2 & 1 & 6 & -2 \\ 1 & 3 & 0 & -1 \\ 6 & 0 & 0 & 0 \\ -2 & -1 & 0 & 1 \end{pmatrix}$
  - Upper triangular matrices:
    - $\begin{pmatrix} 4 \end{pmatrix}$ $\begin{pmatrix} 0 & 4 \\ 0 & 2 \end{pmatrix}$ $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{pmatrix}$
  - Lower triangular matrix:
    - $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 6 & 0 & 0 & 0 \\ -2 & -1 & 0 & 1 \end{pmatrix}$

Matrix Operations (long chapter).

Identical Matrices

- A matrix is completely determined by its size and entries.
- **Definition.** Two matrices are **equal** if
  - they have the same size (same number of rows, same number of columns), and
  - all the corresponding entries are the same.

  Let $\boldsymbol{A} = (a_{ij})_{m\times n}$ and $\boldsymbol{B} = (b_{ij})_{p\times q}$. Then
  - $\boldsymbol{A} = \boldsymbol{B} \Leftrightarrow$ $\boxed{m = p \text{ \& } n = q \text{ \& } a_{ij} = b_{ij} \text{ for all } i, j}$

    same rows same columns no. all corresponding entries are same.
- **Examples.**
  - $\mathbf{0}_{2\times 2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$; $\mathbf{0}_{2\times 3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.
    - $\mathbf{0}_{2\times 2}$ and $\mathbf{0}_{2\times 3}$ have different size $\Rightarrow \mathbf{0}_{2\times 2} \neq \mathbf{0}_{2\times 3}$.

Identical Matrices

- A matrix is completely determined by its size and entries.
- **Definition.** Two matrices are **equal** if
  - they have the same size (same number of rows, same number of columns), and
  - all the corresponding entries are the same.

  Let $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{p \times q}$. Then
  - $A = B \Leftrightarrow \boxed{m = p \;\&\; n = q \;\&\; a_{ij} = b_{ij} \text{ for all } i, j}$
- **Examples.**
  - $A = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$; $B = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$.
    - $A = B \Leftrightarrow \begin{cases} x = 1 \\ y = -1 \\ z = 2 \\ w = 4 \end{cases}$ all entries must be same

Addition, Subtraction & Scalar Multiplication

- Let $A = (a_{ij})_{m \times n}$ and $B = (b_{ij})_{m \times n}$ be matrices. (same size).
  - **Addition**: $A + B = (a_{ij} + b_{ij})_{m \times n}$. add them correspondingly. + component wise.
  - **Subtraction**: $A - B = (a_{ij} - b_{ij})_{m \times n}$. ↑ similarly.
- **Example.**
  - Let $A = \begin{pmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 3 \\ -1 & -1 & -1 \end{pmatrix}$.

$$A + B = \begin{pmatrix} 2+1 & 3+2 & 4+3 \\ 4+(-1) & 5+(-1) & 6+(-1) \end{pmatrix} = \begin{pmatrix} 3 & 5 & 7 \\ 3 & 4 & 5 \end{pmatrix}$$

$$A - B = \begin{pmatrix} 2-1 & 3-2 & 4-3 \\ 4-(-1) & 5-(-1) & 6-(-1) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 5 & 6 & 7 \end{pmatrix}$$

Addition, Subtraction & Scalar Multiplication

- Let $\boldsymbol{A} = (a_{ij})_{m\times n}$ be a matrix, and c a constant.
  - **Scalar multiplication**: $c\boldsymbol{A} = (ca_{ij})_{m\times n}$. multiply component wise by c.
- **Example.**
  - Let $\boldsymbol{A} = \begin{pmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \end{pmatrix}$. Then

$$4\boldsymbol{A} = \begin{pmatrix} 4\cdot 2 & 4\cdot 3 & 4\cdot 4 \\ 4\cdot 4 & 4\cdot 5 & 4\cdot 6 \end{pmatrix} = \begin{pmatrix} 8 & 12 & 16 \\ 16 & 20 & 24 \end{pmatrix}$$

- **Remarks.**
  - $(-1)\boldsymbol{A}$ is usually denoted by $-\boldsymbol{A}$. (negation).
  - It can be proved that $\boldsymbol{A} - \boldsymbol{B} = \boldsymbol{A} + (-\boldsymbol{B})$.
    - In the discussion we usually only consider addition and scalar multiplication.

Properties

- **Theorem.** Let $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ be matrices of the same size.
  - $\boldsymbol{A} - \boldsymbol{B} = \boldsymbol{A} + (-\boldsymbol{B})$. generalization of numbers
  - Commutative Law for Matrix Addition:
    - $\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$.
  - Associative Law for Matrix Addition:
    - $(\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C} = \boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C})$. group A+B and B+C.
  - Let $\boldsymbol{0}$ be the zero matrix of the same size as $\boldsymbol{A}$. any number.
    - additive identity $\boldsymbol{0} + \boldsymbol{A} = \boldsymbol{A}$; $\boldsymbol{A} - \boldsymbol{A} = \boldsymbol{0}$; $0\boldsymbol{A} = \boldsymbol{0}$; $c\boldsymbol{0} = \boldsymbol{0}$. zero matrix.
  - Distributive Law for Scalar Multiplication over Addition: not the same.
    - $c(\boldsymbol{A} + \boldsymbol{B}) = c\boldsymbol{A} + c\boldsymbol{B}$. addition.
    - multiplication $(c + d)\boldsymbol{A} = c\boldsymbol{A} + d\boldsymbol{A}$. First 0 is a number, second is zero matrix.
  - $c(d\boldsymbol{A}) = (cd)\boldsymbol{A}$, $1\boldsymbol{A} = \boldsymbol{A}$.

Constant can put anywhere ; scalar multiplication: number × matrix ; distributes A to c and d

Properties

- Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be matrices of the same size.
  - Prove that $\boldsymbol{A}-\boldsymbol{B}=\boldsymbol{A}+(-\boldsymbol{B})$.

Proof. Suppose that $\boldsymbol{A}=(a_{ij})_{m\times n}$ and $\boldsymbol{B}=(b_{ij})_{m\times n}$.

1. Verify that LHS and RHS have the same size.
   - $\boldsymbol{A}$ is $m\times n$ & $\boldsymbol{B}$ is $m\times n \Rightarrow \boldsymbol{A}-\boldsymbol{B}$ is $m\times n$. by defn.
   - $\boldsymbol{B}$ is $m\times n \Rightarrow -\boldsymbol{B}$ is $m\times n$.
     - $\boldsymbol{A}, -\boldsymbol{B}$ are $m\times n \Rightarrow \boldsymbol{A}+(-\boldsymbol{B})$ is $m\times n$.
2. Verify: "(i,j)-entry of LHS" = "(i,j)-entry of RHS".

$$\begin{aligned}(i,j)\text{-entry of } (\boldsymbol{A}-\boldsymbol{B}) &= a_{ij}-b_{ij}\\ &= a_{ij}+(-b_{ij})\\ &= a_{ij}+(i,j)\text{-entry of } (-\boldsymbol{B})\\ &= (i,j)\text{-entry of } [\boldsymbol{A}+(-\boldsymbol{B})]\end{aligned}$$

A

Properties

- Let $\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}$ be matrices of the same size.
  - Prove that $\boldsymbol{A}+(\boldsymbol{B}+\boldsymbol{C})=(\boldsymbol{A}+\boldsymbol{B})+\boldsymbol{C}$.

Proof

Proof. Let $\boldsymbol{A}=(a_{ij})_{m\times n}$, $\boldsymbol{B}=(b_{ij})_{m\times n}$, $\boldsymbol{C}=(c_{ij})_{m\times n}$.

1. It is clear that LHS & RHS are both $m\times n$. (Why?) Same size.
2. Verify: "(i,j)-entry of LHS" = "(i,j)-entry of RHS".

$$\begin{aligned}(i,j)\text{-entry of } \boldsymbol{A}+(\boldsymbol{B}+\boldsymbol{C}) &= a_{ij}+[(i,j)\text{-entry of } \boldsymbol{B}+\boldsymbol{C}]\\ &= a_{ij}+(b_{ij}+c_{ij})\\ &= (a_{ij}+b_{ij})+c_{ij} \quad \text{associative law.}\\ &= [(i,j)\text{-entry of } \boldsymbol{A}+\boldsymbol{B}]+c_{ij}\\ &= (i,j)\text{-entry of } (\boldsymbol{A}+\boldsymbol{B})+\boldsymbol{C}.\end{aligned}$$

$\therefore\ \boldsymbol{A}+(\boldsymbol{B}+\boldsymbol{C})=(\boldsymbol{A}+\boldsymbol{B})+\boldsymbol{C}$.

- **Exercise**: Prove the remaining properties. (Question 2.18)

Matrix Multiplication

want to solve linear system.

- Consider the following linear systems:
  - $\begin{cases} a_{11}y_1 + a_{12}y_2 = z_1 \\ a_{21}y_1 + a_{22}y_2 = z_2 \\ a_{31}y_1 + a_{32}y_2 = z_3 \end{cases}$ $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$ coefficients only.
  - $\begin{cases} b_{11}x_1 + b_{12}x_2 + b_{13}x_3 = y_1 \\ b_{21}x_1 + b_{22}x_2 + b_{23}x_3 = y_2 \end{cases}$ $\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$

$z_1 = a_{11}(\cdots) + a_{12}(\cdots)$

$z_1 = a_{11}(b_{11}x_1 + b_{12}x_2 + b_{13}x_3) + a_{12}(b_{21}x_1 + \cdots)$

how to compose them together?

$$\begin{cases} (a_{11}b_{11} + a_{12}b_{21})x_1 + (a_{11}b_{12} + a_{12}b_{22})x_2 + (a_{11}b_{13} + a_{12}b_{23})x_3 = z_1 \\ (a_{21}b_{11} + a_{22}b_{21})x_1 + (a_{21}b_{12} + a_{22}b_{22})x_2 + (a_{21}b_{13} + a_{22}b_{23})x_3 = z_2 \\ (a_{31}b_{11} + a_{32}b_{21})x_1 + (a_{31}b_{12} + a_{32}b_{22})x_2 + (a_{31}b_{13} + a_{32}b_{23})x_3 = z_3 \end{cases}$$

idea is composite of 2 linear systems.

Can we use the coefficient matrices of the first two linear systems to obtain the coefficient matrix of their composite?

  - $\begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{12}b_{22} & a_{21}b_{13} + a_{22}b_{23} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} & a_{31}b_{13} + a_{32}b_{23} \end{pmatrix}$ composite

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Matrix Multiplication

- Define matrix multiplication so that the **product** of

no. of column — no. of rows

  - $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$ and $\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$ is given by

row × column.

  - $\begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{12}b_{22} & a_{21}b_{13} + a_{22}b_{23} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} & a_{31}b_{13} + a_{32}b_{23} \end{pmatrix}$

$a_{row} \times b_{column}$ (dist.).

The $(3, 2)$-entry of the product is $a_{31}b_{12} + a_{32}b_{22}$.

3rd row 2nd column.

  - a_{31} and a_{32} are from the **3rd row** of the first matrix.
  - b_{12} and b_{22} are from the **2nd column** of the second.

In order to get the (i, j)-**entry** of the **product** matrix:

1. Find the ith **row** of the first matrix;
2. Find the jth **column** of the second matrix;
3. Multiply the corresponding entries.
4. Add the products together.

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Matrix Multiplication

same

- **Definition.** Let $\boldsymbol{A} = (a_{ij})_{m\times p}$ and $\boldsymbol{B} = (b_{ij})_{p\times n}$.
  - $\boldsymbol{AB}$ is the $m \times n$ matrix such that its (i, j)-entry is
    - $\boxed{a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj} = \sum_{k=1}^{p} a_{ik}b_{kj}}$

  Note: No. of columns of $\boldsymbol{A}$ = the no. of rows of $\boldsymbol{B}$. * must.

  1. ith row of $\boldsymbol{A}$: $\begin{pmatrix} & \vdots & \\ a_{i1} \quad a_{i2} & \cdots & a_{ip} \\ & \vdots & \end{pmatrix}$
  2. jth column of $\boldsymbol{B}$: $\begin{pmatrix} & b_{1j} & \\ & b_{2j} & \\ \cdots & \vdots & \cdots \\ & b_{pj} & \end{pmatrix}$
  3. Multiply componentwise and add the products.

Examples

- $\underset{2\times 3}{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}} \underset{3\times 2}{\begin{pmatrix} 1 & 1 \\ 2 & 3 \\ -1 & -2 \end{pmatrix}} = \begin{pmatrix} 2 & 1 \\ 8 & 7 \end{pmatrix}$
  - $\begin{pmatrix} 1\cdot 1 + 2\cdot 2 + 3\cdot(-1) & 1\cdot 1 + 2\cdot 3 + 3\cdot(-2) \\ 4\cdot 1 + 5\cdot 2 + 6\cdot(-1) & 4\cdot 1 + 5\cdot 3 + 6\cdot(-2) \end{pmatrix}$
- $\underset{3\times 2}{\begin{pmatrix} 1 & 1 \\ 2 & 3 \\ -1 & -2 \end{pmatrix}} \underset{2\times 3}{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}} = \begin{pmatrix} 5 & 7 & 9 \\ 14 & 19 & 24 \\ -9 & -12 & -15 \end{pmatrix}$
  - $\begin{pmatrix} 1\cdot 1 + 1\cdot 4 & 1\cdot 2 + 1\cdot 5 & 1\cdot 3 + 1\cdot 6 \\ 2\cdot 1 + 3\cdot 4 & 2\cdot 2 + 3\cdot 5 & 2\cdot 3 + 3\cdot 6 \\ (-1)\cdot 1 + (-2)\cdot 4 & (-1)\cdot 2 + (-2)\cdot 5 & (-1)\cdot 3 + (-2)\cdot 6 \end{pmatrix}$

More complicated.

- **Remark.** Matrix multiplication is **NOT commutative**.
  - $\boldsymbol{AB}$ is the **pre-multiplication** of $\boldsymbol{A}$ to $\boldsymbol{B}$ (to $\boldsymbol{B}$ by $\boldsymbol{A}$).
  - $\boldsymbol{BA}$ is the **post-multiplication** of $\boldsymbol{A}$ to $\boldsymbol{B}$ (to $\boldsymbol{B}$ by $\boldsymbol{A}$).

AB ≠ BA

* not the same!

Emphasize the order!

Properties

- **Theorem.**
  - Let $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ be $m \times p, p \times q, q \times n$ matrices, resp.
    - Associative Law: $\boldsymbol{A}(\boldsymbol{BC}) = (\boldsymbol{AB})\boldsymbol{C}$. ABC – no need order.
  - Let $\boldsymbol{A}$ be $m \times p$ matrix, $\boldsymbol{B}_1, \boldsymbol{B}_2$ be $p \times n$ matrices.
    - Distributive Law: $\boldsymbol{A}(\boldsymbol{B}_1 + \boldsymbol{B}_2) = \boldsymbol{AB}_1 + \boldsymbol{AB}_2$.
  - Let $\boldsymbol{A}_1, \boldsymbol{A}_2$ be $m \times p$ matrices, $\boldsymbol{B}$ be $p \times n$ matrix.
    - Distributive Law: $(\boldsymbol{A}_1 + \boldsymbol{A}_2)\boldsymbol{B} = \boldsymbol{A}_1\boldsymbol{B} + \boldsymbol{A}_2\boldsymbol{B}$. reverse.
  - Let $\boldsymbol{A}$ be $m \times p$ and $\boldsymbol{B}$ be $p \times n$. For constant c,
    - $c(\boldsymbol{AB}) = (c\boldsymbol{A})\boldsymbol{B} = \boldsymbol{A}(c\boldsymbol{B})$. doesn't matter.
  - Let $\boldsymbol{A}$ be an $m \times n$ matrix.
    - $\boldsymbol{A0}_{n\times p} = \boldsymbol{0}_{m\times p}$; $\quad \boldsymbol{0}_{p\times m}\boldsymbol{A} = \boldsymbol{0}_{p\times n}$. – still 0   * size diff.
    - $\boldsymbol{AI}_n = \boldsymbol{A}$; $\quad \boldsymbol{I}_m\boldsymbol{A} = \boldsymbol{A}$. * order.

harder to proof.

m×n   m×n   m×n

role of identity matrix.

Properties

- Let $\boldsymbol{A}$ be an $m \times p$ matrix and $\boldsymbol{B}_1, \boldsymbol{B}_2$ be $p \times n$ matrices.
  - Prove that $\boldsymbol{A}(\boldsymbol{B}_1 + \boldsymbol{B}_2) = \boldsymbol{AB}_1 + \boldsymbol{AB}_2$.

  Proof. Verify that LHS and RHS have the same size.
  - $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ are $p \times n \Rightarrow \boldsymbol{B}_1 + \boldsymbol{B}_2$ is $p \times n$.

    $\Rightarrow \boldsymbol{A}(\boldsymbol{B}_1 + \boldsymbol{B}_2)$ is $m \times n$.
  - $\boldsymbol{AB}_1$ is $m \times n$, and $\boldsymbol{AB}_2$ is $m \times n$

    $\Rightarrow \boldsymbol{AB}_1 + \boldsymbol{AB}_2$ is $m \times n$.

  Let $\boldsymbol{A} = (a_{ij})_{m\times p}$, $\boldsymbol{B}_1 = (b_{ij})_{p\times n}$ and $\boldsymbol{B}_2 = (b'_{ij})_{p\times n}$. can use a, b, c as notation
  - We shall verify that the following are equal:
    - "the (i, j)-entry of $\boldsymbol{A}(\boldsymbol{B}_1 + \boldsymbol{B}_2)$" and
    - "the (i, j)-entry of $\boldsymbol{AB}_1 + \boldsymbol{AB}_2$",

    for all $i = 1, \ldots, m, j = 1, \ldots, n$.

(i, j) entry of left = (i, j) entry of right proving ~ 30%   understand + use idea of proof to do something else.

Properties

- Let $\boldsymbol{A}$ be an $m \times p$ matrix and $\boldsymbol{B}_1, \boldsymbol{B}_2$ be $p \times n$ matrices.
  - Prove that $\boldsymbol{A}(\boldsymbol{B}_1 + \boldsymbol{B}_2) = \boldsymbol{A}\boldsymbol{B}_1 + \boldsymbol{A}\boldsymbol{B}_2$.

Proof. Let $\boldsymbol{A} = (a_{ij})$, $\boldsymbol{B}_1 = (b_{ij})$, $\boldsymbol{B}_2 = (b'_{ij})$.

- $\boldsymbol{B}_1 + \boldsymbol{B}_2 = (b_{ij} + b'_{ij})$. Let $b''_{ij} = b_{ij} + b'_{ij}$. Suppose.

$$
\begin{aligned}
& (i,j)\text{-entry of } \boldsymbol{A}(\boldsymbol{B}_1 + \boldsymbol{B}_2) \quad \text{(}i\text{ row; } j\text{th column of } B_1 + B_2\text{.)} \\
&= a_{i1}b''_{1j} + a_{i2}b''_{2j} + \cdots + a_{ip}b''_{pj} \\
&= a_{i1}(b_{1j} + b'_{1j}) + a_{i2}(b_{2j} + b'_{2j}) + \cdots + a_{ip}(b_{pj} + b'_{pj}) \\
&= (a_{i1}b_{1j} + a_{i1}b'_{1j}) + (a_{i2}b_{2j} + a_{i2}b'_{2j}) + \cdots + (a_{ip}b_{pj} + a_{ip}b'_{pj}) \\
&= (a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj}) + (a_{i1}b'_{1j} + a_{i2}b'_{2j} + \cdots + a_{ip}b'_{pj}) \\
&= (i,j)\text{-entry of } \boldsymbol{A}\boldsymbol{B}_1 + (i,j)\text{-entry of } \boldsymbol{A}\boldsymbol{B}_2 \\
&= (i,j)\text{-entry of } (\boldsymbol{A}\boldsymbol{B}_1 + \boldsymbol{A}\boldsymbol{B}_2).
\end{aligned}
$$

j col of B_1 ith row A AB_1

ith row A   j col of B_2. AB_2

$\therefore$ $\boldsymbol{A}(\boldsymbol{B}_1 + \boldsymbol{B}_2) = \boldsymbol{A}\boldsymbol{B}_1 + \boldsymbol{A}\boldsymbol{B}_2$.

Same matrices.

Get entries and do one by one.

Powers of Square Matrices

- Let $\boldsymbol{A}$ be an $m \times n$ matrix.
  - $\boldsymbol{A}\boldsymbol{A}$ is well-defined $\Leftrightarrow m = n \Leftrightarrow \boldsymbol{A}$ is square.

Definition. Let $\boldsymbol{A}$ be a square matrix of order n. For nonnegative integers k, the **powers** of $\boldsymbol{A}$ are defined as

$$
\boldsymbol{A}^k = \begin{cases} \boldsymbol{I}_n & \text{if } k = 0, \\ \underbrace{\boldsymbol{A}\boldsymbol{A}\cdots\boldsymbol{A}}_{k \text{ times}} & \text{if } k \geq 1. \end{cases}
$$

$A^0 = I_n$ replace "1"

- **Example.** Let $\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$. Then
  - $\boldsymbol{A}^2 = \boldsymbol{A}\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 8 \\ 4 & 11 \end{pmatrix}$.
  - $\boldsymbol{A}^3 = \boldsymbol{A}\boldsymbol{A}\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 11 & 30 \\ 15 & 41 \end{pmatrix}$.

Same ans.

(i,j)

ith row of A

$a_{i1}, a_{i2}, \ldots a_{in}$

jth col of A

a_{1j}
a_{2j}
$\vdots$
a_{nj}

$n + (n-1)$

$\Rightarrow 2n-1$ operation for 1 entry of the product

A^2 $n \times n$ $n^2(2n-1) \approx n^2 \cdot 2n$ $(= 2n^3)$

almost 2000 computation! for a small matrix.

Properties

proof by induction.

- Let $\boldsymbol{A}$ be a square matrix, and m, n nonnegative integers.
  - $\boldsymbol{A}^m\boldsymbol{A}^n = \boldsymbol{A}^{m+n}$, $(\boldsymbol{A}^m)^n = \boldsymbol{A}^{mn}$.
- Recall that matrix multiplication is NOT commutative.
  - In general, $(\boldsymbol{AB})^n \neq \boldsymbol{A}^n\boldsymbol{B}^n$ for $n = 2, 3, \ldots$.
  - For example, let $\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
    - $(\boldsymbol{AB})^2 = (\boldsymbol{AB})(\boldsymbol{AB}) = \boldsymbol{ABAB} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.
    - $\boldsymbol{A}^2\boldsymbol{B}^2 = (\boldsymbol{AA})(\boldsymbol{BB}) = \boldsymbol{AABB} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$.

      not the same.
- **Exercise.** Let $\boldsymbol{A}, \boldsymbol{B}$ be square matrices of the same size.
  - Suppose that $\boldsymbol{AB} = \boldsymbol{BA}$. Prove that
    - $(\boldsymbol{AB})^n = \boldsymbol{A}^n\boldsymbol{B}^n$ for all nonnegative integers n.

$(A+B)^2$
$= (A+B)(A+B)$
$= AA + AB + BA + BB$
$= A^2 + (AB+BA) + B^2$
$AB+BA \neq 2AB$.

Matrix Representation

entry not number.

- Let $\boldsymbol{A} = (a_{ij})_{m\times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$.
  - Let $\boldsymbol{a}_i$ denote the ith row of $\boldsymbol{A}$, $i = 1, \ldots, m$.
    - $\boldsymbol{a}_1 = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{pmatrix}$
    - $\boldsymbol{a}_2 = \begin{pmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{pmatrix}$
    - ....
    - $\boldsymbol{a}_m = \begin{pmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$

    Then each $\boldsymbol{a}_i$ is a $1 \times n$ matrix (row vector).
    - $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_m \end{pmatrix}$.

Matrix Representation

- Let $\boldsymbol{A} = (a_{ij})_{m\times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$.
  - Let $\boldsymbol{b}_j$ denote the jth column of $\boldsymbol{A}$, $j = 1, \dots, n$.
    - $\boldsymbol{b}_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \boldsymbol{b}_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \dots, \boldsymbol{b}_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$.

    Then each $\boldsymbol{b}_j$ is an $m \times 1$ matrix (column vector).
    - $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{b}_1 & \boldsymbol{b}_2 & \cdots & \boldsymbol{b}_n \end{pmatrix}$.

Matrix Representation

- Let $\boldsymbol{a} = \underbrace{\begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}}_{1\times n}$ and $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$. $n\times 1$.
  - Then $\boldsymbol{ab}$ is a 1×1 matrix, i.e., a real number.
    - $\boldsymbol{ab} = a_1b_1 + a_2b_2 + \cdots + a_nb_n$.

Note: $\boldsymbol{ba}$ is an $n \times n$ matrix. * diff from ab.
  - (i, j)-entry is the ith entry of $\boldsymbol{b}$ times the jth entry of $\boldsymbol{a}$.
    - $\boldsymbol{ba} = \begin{pmatrix} b_1a_1 & b_1a_2 & \cdots & b_1a_n \\ b_2a_1 & b_2a_2 & \cdots & b_2a_n \\ \vdots & \vdots & \ddots & \vdots \\ b_na_1 & b_na_2 & \cdots & b_na_n \end{pmatrix}$.

Matrix Representation

- Suppose $\boldsymbol{A} = (a_{ij})_{m\times p}$.
  - Let $\boldsymbol{a}_i = \begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{ip} \end{pmatrix}$ be the ith row of $\boldsymbol{A}$.

  row vec

  Suppose $\boldsymbol{B} = (b_{ij})_{p\times n}$.
  - Let $\boldsymbol{b}_j = \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{pj} \end{pmatrix}$ be the jth column of $\boldsymbol{B}$.

  col vec

$$\begin{aligned}\boldsymbol{a}_i\boldsymbol{b}_j &= a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj} \\ &= (i,j)\text{-entry of } \boldsymbol{AB}.\end{aligned}$$

  - $$\boldsymbol{AB} = \begin{pmatrix} \boldsymbol{a}_1\boldsymbol{b}_1 & \boldsymbol{a}_1\boldsymbol{b}_2 & \cdots & \boldsymbol{a}_1\boldsymbol{b}_n \\ \boldsymbol{a}_2\boldsymbol{b}_1 & \boldsymbol{a}_2\boldsymbol{b}_2 & \cdots & \boldsymbol{a}_2\boldsymbol{b}_n \\ \vdots & \vdots & & \vdots \\ \boldsymbol{a}_m\boldsymbol{b}_1 & \boldsymbol{a}_m\boldsymbol{b}_2 & \cdots & \boldsymbol{a}_m\boldsymbol{b}_n \end{pmatrix}.$$

  matrix product
  vec ith row of A
  and jth col of B.

Matrix Representation

- The (i,j)-entry of $\boldsymbol{AB}$ is
  - The ith row of $\boldsymbol{A}$ times the jth column of $\boldsymbol{B}$.

matrix multiplication

$$\begin{aligned}\boldsymbol{a}_i\boldsymbol{B} &= \boldsymbol{a}_i \begin{pmatrix} \boldsymbol{b}_1 & \boldsymbol{b}_2 & \cdots & \boldsymbol{b}_n \end{pmatrix} \\ &= \begin{pmatrix} \boldsymbol{a}_i\boldsymbol{b}_1 & \boldsymbol{a}_i\boldsymbol{b}_2 & \cdots & \boldsymbol{a}_i\boldsymbol{b}_n \end{pmatrix} \\ &= i\text{th row of } \boldsymbol{AB}\end{aligned}$$

nth entry.

1 row.

$$\boldsymbol{AB} = \begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_m \end{pmatrix} \boldsymbol{B} = \begin{pmatrix} \boldsymbol{a}_1\boldsymbol{B} \\ \boldsymbol{a}_2\boldsymbol{B} \\ \vdots \\ \boldsymbol{a}_m\boldsymbol{B} \end{pmatrix}.$$

Matrix Representation

row col.

- The (i, j)-entry of AB is
  - The ith row of A times the jth column of B.

m rows

$$Ab_j = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} b_j = \begin{pmatrix} a_1 b_j \\ a_2 b_j \\ \vdots \\ a_m b_j \end{pmatrix} = j\text{th column of } AB.$$

$$\begin{aligned} AB &= A \begin{pmatrix} b_1 & b_2 & \cdots & b_n \end{pmatrix} \\ &= \begin{pmatrix} Ab_1 & Ab_2 & \cdots & Ab_n \end{pmatrix} \end{aligned}$$

— Can multiply column wise

- **Remark.** Matrices can be multiplied in blocks (provided that the sizes are matched).
  - Reference: Question 2.23.

Example

- Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ -1 & -2 \end{pmatrix}$.
  - Let $a_1 = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$, $a_2 = \begin{pmatrix} 4 & 5 & 6 \end{pmatrix}$. Then

$$\begin{aligned} AB &= \underbrace{\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} B}_{} = \underbrace{\begin{pmatrix} a_1 B \\ a_2 B \end{pmatrix}}_{} \\ &= \begin{pmatrix} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ -1 & -2 \end{pmatrix} \\ \begin{pmatrix} 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ -1 & -2 \end{pmatrix} \end{pmatrix} \\ &= \begin{pmatrix} \begin{pmatrix} 2 & 1 \end{pmatrix} \\ \begin{pmatrix} 8 & 7 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 8 & 7 \end{pmatrix} \end{aligned}$$

much faster!

Example

split column wise

- Let $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ -1 & -2 \end{pmatrix}$.
  - Let $\boldsymbol{b}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\boldsymbol{b}_2 = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$. Then

$$\begin{aligned} \boldsymbol{AB} &= \boldsymbol{A}\begin{pmatrix} \boldsymbol{b}_1 & \boldsymbol{b}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}\boldsymbol{b}_1 & \boldsymbol{A}\boldsymbol{b}_2 \end{pmatrix} \\ &= \left(\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \right) \\ &= \left(\begin{pmatrix} 2 \\ 8 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 7 \end{pmatrix} \right) = \begin{pmatrix} 2 & 1 \\ 8 & 7 \end{pmatrix} \end{aligned}$$

Example

- Let $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ -1 & -2 \end{pmatrix}$.

$$\begin{aligned} \boldsymbol{AB} &= \begin{pmatrix} \boldsymbol{a}_1\boldsymbol{b}_1 & \boldsymbol{a}_1\boldsymbol{b}_2 \\ \boldsymbol{a}_2\boldsymbol{b}_1 & \boldsymbol{a}_2\boldsymbol{b}_2 \end{pmatrix} \\ &= \begin{pmatrix} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \\ \begin{pmatrix} 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} & \begin{pmatrix} 4 & 5 & 7 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \end{pmatrix} \\ &= \begin{pmatrix} 2 & 1 \\ 8 & 7 \end{pmatrix} \end{aligned}$$

similar

Representation of Linear System in terms of matrix.

- Linear System of m equations in n variables $x_1, \ldots, x_n$:
  - $$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

$$b_i = \boxed{a_{i1}}x_1 + \boxed{a_{i2}}x_2 + \cdots + \boxed{a_{in}}x_n$$

$$= \begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Representation of Linear System

- ~~Linear System of m equations in n variables~~ $x_1, \ldots, x_n$.
  - $$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$
  - $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$, **coefficient matrix**. → as.
- $\boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, **variable matrix**. → xs.
- $\boldsymbol{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$, **constant matrix**. Then $\boxed{\boldsymbol{Ax} = \boldsymbol{b}}$

→ bs.

Representation of Linear System

- Let $\boldsymbol{A} = (a_{ij})_{m\times n}$, $\boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$.
  - Then $\boldsymbol{Ax} = \boldsymbol{b}$ is the linear system of
    - m linear equations in n variables $x_1, \dots, x_n$,
    - a_{ij} are the coefficients, and b_i are the constants.
  - Let $\boldsymbol{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$.
    - $x_1 = u_1, \dots, x_n = u_n$ is a solution to the system

      $\Leftrightarrow \boldsymbol{Au} = \boldsymbol{b}$

      $\Leftrightarrow \boldsymbol{u}$ is a solution to $\boldsymbol{Ax} = \boldsymbol{b}$.

Form is shorter

x's soln

Representation of Linear System

Direct

- **Problem.** Suppose that a linear system has more than one solutions. Then it has infinitely many solutions.

  Proof. Let the system be represented by $\boldsymbol{Ax} = \boldsymbol{b}$.
  - Suppose that it has two solutions $\boldsymbol{u}_1 \neq \boldsymbol{u}_2$.
    - $\boldsymbol{Au}_1 = \boldsymbol{b}$ and $\boldsymbol{Au}_2 = \boldsymbol{b}$. same. ①−②

      generate more soln from u_1 and u_2
    - $\boldsymbol{0} = \boldsymbol{b} - \boldsymbol{b} = \boldsymbol{Au}_1 - \boldsymbol{Au}_2 = \boldsymbol{A}(\boldsymbol{u}_1 - \boldsymbol{u}_2)$.

      by distribution

    Then for any real number t,

    u_1 u_2 $u_1 + t(u_2 - u_1)$
    - $\boldsymbol{0} = t\boldsymbol{0} = t\boldsymbol{A}(\boldsymbol{u}_1 - \boldsymbol{u}_2) = \boxed{\boldsymbol{A}(t(\boldsymbol{u}_1 - \boldsymbol{u}_2))}$.

    $$\begin{aligned} \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{0} &= \boldsymbol{Au}_2 + \boldsymbol{A}(t(\boldsymbol{u}_1 - \boldsymbol{u}_2)) \\ &= \boldsymbol{A}(\boldsymbol{u}_2 + t(\boldsymbol{u}_1 - \boldsymbol{u}_2)). \end{aligned}$$

    always a solution

    all vectors along line passing thru u_1 & u_2.

    So $\boldsymbol{u}_2 + t(\boldsymbol{u}_1 - \boldsymbol{u}_2)$ is a solution to $\boldsymbol{Ax} = \boldsymbol{b}$ for every real number t. for $t \in \mathbb{R}$.
  - Therefore, the system has infinitely many solutions. since

Representation of Linear System

- Consider $\begin{cases} 4x + 5y + 6z = 5 \\ x - y = 2 \\ y - z = 3. \end{cases}$

$$
\begin{aligned}
\begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} &= \begin{pmatrix} 4 & 5 & 6 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\
&= \begin{pmatrix} 4x + 5y + 6z \\ x - y \\ y - z \end{pmatrix} \\
&= \begin{pmatrix} 4x \\ x \\ 0 \end{pmatrix} + \begin{pmatrix} 5y \\ -y \\ y \end{pmatrix} + \begin{pmatrix} 6z \\ 0 \\ -z \end{pmatrix} \\
&= x \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix} + z \begin{pmatrix} 6 \\ 0 \\ -1 \end{pmatrix}
\end{aligned}
$$

1st col   2nd col   last col.

Col. are more impt than row.

Linear system is consistent iff can be written

Representation of Linear System

- $\boldsymbol{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$, $\boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$.

$$
\begin{aligned}
\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} &= \begin{pmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{pmatrix} \\
&= x_1 \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix}
\end{aligned}
$$

in general

How to rep. constant matrix as col. of coeff matrix?

  - Let $\boldsymbol{a}_j$ denote the jth column of $\boldsymbol{A}$. Then
    - $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x} = x_1\boldsymbol{a}_1 + \cdots + x_n\boldsymbol{a}_n = \sum_{j=1}^{n} x_j \boldsymbol{a}_j$.

Transpose

- Let $\boldsymbol{A} = (a_{ij})_{m\times n}$ be a matrix.
  - The **transpose** of $\boldsymbol{A}$ is the $n \times m$ matrix $\boldsymbol{A}^{\mathrm{T}}$ (or $\boldsymbol{A}^{\mathrm{t}}$)
    - whose (i,j)-entry is a_{ji}.

rows $\Leftrightarrow$ columns

- **Example.** Let $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}$.
  - $\boldsymbol{A}^{\mathrm{T}} = \begin{pmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{pmatrix}$ and $(\boldsymbol{A}^{\mathrm{T}})^{\mathrm{T}} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}$.

$A = A^T$?

$\rightarrow$ Symmetric.

- **Remarks.**
  - The ith row of $\boldsymbol{A}^{\mathrm{T}}$ is the ith column of $\boldsymbol{A}$.
  - The jth column of $\boldsymbol{A}^{\mathrm{T}}$ is the jth row of $\boldsymbol{A}$.

Properties

- **Theorem.** Let $\boldsymbol{A}$ be an $m \times n$ matrix.
  - $(\boldsymbol{A}^{\mathrm{T}})^{\mathrm{T}} = \boldsymbol{A}$.
  - $\boldsymbol{A}$ is symmetric $\Leftrightarrow$ $\boldsymbol{A} = \boldsymbol{A}^{\mathrm{T}}$.
  - Let c be a scalar. Then $(c\boldsymbol{A})^{\mathrm{T}} = c\boldsymbol{A}^{\mathrm{T}}$.
  - Let $\boldsymbol{B}$ be $m \times n$. Then $(\boldsymbol{A}+\boldsymbol{B})^{\mathrm{T}} = \boldsymbol{A}^{\mathrm{T}} + \boldsymbol{B}^{\mathrm{T}}$.
  - Let $\boldsymbol{B}$ be $n \times p$. Then $(\boldsymbol{AB})^{\mathrm{T}} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}$.

$A^T B^T$?

AB : $m \times p$.

$(AB)^T = p \times m$

A^T : $n \times m$ $\quad B^T$: $p \times n$

$B^T_{p \times n} A^T_{n \times m}$

- **Proof.** We only prove the last statement.
  - Left-hand side:
    - $\boldsymbol{AB}$ is $m \times p \Rightarrow (\boldsymbol{AB})^{\mathrm{T}}$ is $p \times m$.
  - Right-hand side:
    - $\boldsymbol{B}^{\mathrm{T}}$ is $p \times n$ & $\boldsymbol{A}^{\mathrm{T}}$ is $n \times m \Rightarrow \boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}$ is $p \times m$.

    So $(\boldsymbol{AB})^{\mathrm{T}}$ and $\boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}$ have the same size.

Properties

- **Proof.** (Cont'd) Let $\boldsymbol{A} = (a_{ij})_{m\times n}$ and $\boldsymbol{B} = (b_{ij})_{n\times p}$.

$$(i,j)\text{-entry of } (\boldsymbol{AB})^{\mathrm{T}} = (j,i)\text{-entry of } \boldsymbol{AB}$$ by definition //
$$= a_{j1}b_{1i} + a_{j2}b_{2i} + \cdots + a_{jn}b_{ni}.$$

jth row of A — $a_{j1}\ a_{j2} \cdots a_{jn}$ ith col of B. — $b_{1i}\ b_{2i} \cdots b_{ni}$

  ◦ Let $\boldsymbol{A}^{\mathrm{T}} = (a'_{ij})_{n\times m}$ and $\boldsymbol{B}^{\mathrm{T}} = (b'_{ij})_{p\times n}$.
    - $a'_{ij} = a_{ji}$ and $b'_{ij} = b_{ji}$. Since its transpose

$$(i,j)\text{-entry of } \boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}} = b'_{i1}a'_{1j} + b'_{i2}a'_{2j} + \cdots + b'_{in}a'_{nj}$$
ith row of B^T — $b_{i1}' \cdots b_{in}'$ jth col. of A^T — $a_{1j}'\ a_{2j}' \cdots a_{nj}'$
$$= b_{1i}a_{j1} + b_{2i}a_{j2} + \cdots + b_{ni}a_{jn}$$ same (substitution).
$$= a_{j1}b_{1i} + a_{j2}b_{2i} + \cdots + a_{jn}b_{ni}.$$

$\therefore\quad (\boldsymbol{AB})^{\mathrm{T}} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}$.

Note: In general, $(\boldsymbol{AB})^{\mathrm{T}} \neq \boldsymbol{A}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}$. ※.

Inverses of Square Matrices

Inverses of Numbers

- Let a and b be real numbers.
  ◦ $a + x = b \Rightarrow x = b - a = b + (-a)$.

  The number $-a$ is the **additive inverse** of a.

  ◦ $ax = b \Rightarrow x = b/a = a^{-1}b$, provided that $a \neq 0$.

  The number a^{-1} is the **multiplicative inverse** of a $(\neq 0)$.

- Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be matrices of the same size.
  ◦ $\boldsymbol{A} + \boldsymbol{X} = \boldsymbol{B} \Rightarrow \boldsymbol{X} = \boldsymbol{B} - \boldsymbol{A} = \boldsymbol{B} + (-\boldsymbol{A})$.

  So $-\boldsymbol{A}$ is the **additive inverse** of $\boldsymbol{A}$.

- Let $\boldsymbol{A}$ be an $m \times n$ matrix and $\boldsymbol{B}$ be an $m \times p$ matrix.
  ◦ $\boldsymbol{AX} = \boldsymbol{B} \Rightarrow \boldsymbol{X} = \cdots$. Solve for X.

  It is expected to have a matrix $\boldsymbol{A}^{-1}$ so that $\boldsymbol{X} = \boldsymbol{A}^{-1}\boldsymbol{B}$.

similar to number,
∴ matrix shouldn't be 0.

Inverses of Square Matrices

only for

- **Definition.** Let $\boldsymbol{A}$ be a square matrix of order n.
  - If there exists a square matrix $\boldsymbol{B}$ of order n so that
    - $\boxed{\boldsymbol{AB} = \boldsymbol{I}_n}$ and $\boxed{\boldsymbol{BA} = \boldsymbol{I}_n}$, 2 sides*.

    then $\boldsymbol{A}$ is called **invertible**, and $\boldsymbol{B}$ is an **inverse** of $\boldsymbol{A}$. or non-singular / non-invertible.
  - If $\boldsymbol{A}$ is not **invertible**, $\boldsymbol{A}$ is called **singular**.

  Note: Non-square matrix is neither invertible nor singular.

- **Example.** Suppose that $\boldsymbol{A}$ is invertible with inverse $\boldsymbol{B}$.

$$\begin{aligned}\boldsymbol{AX} = \boldsymbol{C} &\Rightarrow \boldsymbol{B}(\boldsymbol{AX}) = \boldsymbol{BC}\\ &\Rightarrow (\boldsymbol{BA})\boldsymbol{X} = \boldsymbol{BC}\\ &\Rightarrow \boldsymbol{IX} = \boldsymbol{BC}\\ &\Rightarrow \boldsymbol{X} = \boldsymbol{BC}.\end{aligned}$$

($\boldsymbol{A}$ and $\boldsymbol{C}$ must have the same number of rows.) matrix multiplication.

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how matrix 'invertible'?

how get inverse?

Examples

- Let $\boldsymbol{A} = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$.
  - $\boldsymbol{AB} = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \boldsymbol{I}.$
  - $\boldsymbol{BA} = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \boldsymbol{I}.$

  Therefore, $\boldsymbol{A}$ is invertible and $\boldsymbol{B}$ is an inverse of $\boldsymbol{A}$. or B is invertible, A is inverse of B.

- Solve the equation $\underbrace{\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}}_{A}\boldsymbol{X} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$.
  - Pre-multiply the equation by $\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$.

$$\begin{pmatrix} 12 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}\boldsymbol{X} = \boldsymbol{IX} = \boldsymbol{X}.$$

where they get this?

know the inverse.

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Examples

- Prove that $\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is invertible, and find its inverse.
  - Let $\boldsymbol{B} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ 2x2. such that $\boldsymbol{AB} = \boldsymbol{BA} = \boldsymbol{I}$.
    - $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \boldsymbol{AB} = \begin{pmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{pmatrix}$.

    Solve a linear system in a, b, c, d:
    - $\begin{cases} 1 = a+2c \\ 0 = b+2d \\ 0 = 3a+4c \\ 1 = 3d+4d \end{cases} \cdots \Rightarrow \cdots \begin{cases} a = -2 \\ b = 1 \\ c = 3/2 \\ d = -1/2. \end{cases}$ 2 linear systems in 2 variables.

    Moreover, one must verify that $\boldsymbol{BA} = \cdots = \boldsymbol{I}$.
  - $\boldsymbol{A}$ is invertible with an inverse $\begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$. Inverse is unique.

Examples

- Prove that $\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ is a singular matrix.
  - Assume that $\boldsymbol{A}$ is invertible (prove by contradiction).
    - Then $\boldsymbol{A}$ has an inverse $\boldsymbol{B}$: $\boldsymbol{AB} = \boldsymbol{BA} \cancel{=} \boldsymbol{I}$. doesn't matter.

    Let $\boldsymbol{B} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then
    - $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ a & b \end{pmatrix}$.
      - $1 = a, 0 = b, 0 = a, 1 = b$, a contradiction!

    So such $\boldsymbol{B}$ does not exist.
  - Therefore, $\boldsymbol{A}$ is not invertible, i.e., it is singular. ✓

Remark: One also gets a contradiction by checking $\boldsymbol{BA} = \boldsymbol{I}$.

Properties

- **Theorem.** Let $\boldsymbol{A}$ be a square matrix.
  - If $\boldsymbol{A}$ is invertible, then its inverse is unique.
- **Proof.** Suppose that $\boldsymbol{B}, \boldsymbol{C}$ are both inverses of $\boldsymbol{A}$. — verify $B = C$.
  - $\boldsymbol{AB} = \boldsymbol{BA} = \boldsymbol{I}$ and $\boldsymbol{AC} = \boldsymbol{CA} = \boldsymbol{I}$.

  We need to verify that $\boldsymbol{B} = \boldsymbol{C}$:
  - $\boldsymbol{B} = \boldsymbol{BI} = \boldsymbol{B}(\boldsymbol{AC}) = (\boldsymbol{BA})\boldsymbol{C} = \boldsymbol{IC} = \boldsymbol{C}$.
- **Notation.** The unique inverse of $\boldsymbol{A}$, is denoted by $\boldsymbol{A}^{-1}$.
  - $\boldsymbol{AA}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$.
- Suppose that $\boldsymbol{A}$ is an invertible matrix. Then
  - If $\boldsymbol{AX} = \boldsymbol{B}$, ($\boldsymbol{A}$ and $\boldsymbol{B}$ have the same no. of rows)
    - $\boldsymbol{X} = \boldsymbol{IX} = (\boldsymbol{A}^{-1}\boldsymbol{A})\boldsymbol{X} = \boldsymbol{A}^{-1}(\boldsymbol{AX}) = \boldsymbol{A}^{-1}\boldsymbol{B}$.

$\underbrace{A^{-1}Ax}_{Ix} = A^{-1}B$

$Ix = x.$

Properties

- **Cancellation Law.** Let $\boldsymbol{A}$ be an invertible matrix.
  - $\cancel{\boldsymbol{A}}\boldsymbol{B}_1 = \cancel{\boldsymbol{A}}\boldsymbol{B}_2 \Rightarrow \boldsymbol{B}_1 = \boldsymbol{B}_2$.
  - $\boldsymbol{C}_1\cancel{\boldsymbol{A}} = \boldsymbol{C}_2\cancel{\boldsymbol{A}} \Rightarrow \boldsymbol{C}_1 = \boldsymbol{C}_2$. can cancel the invertible matrix.
- **Proof.** Suppose that $\boldsymbol{AB}_1 = \boldsymbol{AB}_2$. Then — can pre-multiply.

  a soln to
  - $\boldsymbol{B}_1$ is the solution to $\boldsymbol{AX} = \boldsymbol{AB}_2$.
    - $\boldsymbol{B}_1 = \boldsymbol{A}^{-1}(\boldsymbol{AB}_2) = (\boldsymbol{A}^{-1}\boldsymbol{A})\boldsymbol{B}_2 = \boldsymbol{IB}_2 = \boldsymbol{B}_2$.

  The other statement is left as an exercise.
- **Remark.** The cancellation law fails if $\boldsymbol{A}$ is singular.
  - Recall that $\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ is singular.
    - $\boldsymbol{A}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$ & $\boldsymbol{A}\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$

$a b_1 = a b_2$

$\Downarrow$

$b_1 = b_2$?

not same

$\therefore$ cancellation law fails

A: $AB = BA = I$.

invertible $\searrow$ $B = A^{-1}$

$x = A^{-1}b$

Example

- Let $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}_{2\times 2}$. Find the condition when $\boldsymbol{A}$ is invertible.

  Let $\boldsymbol{B} = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$. Suppose that $\boldsymbol{AB} = \boldsymbol{BA} = \boldsymbol{I}$.

  ○ $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \boldsymbol{AB} = \begin{pmatrix} ax+bz & ay+bw \\ cx+dz & cy+dw \end{pmatrix}$. not commutable

  Solve a linear system in x, y, z, w:

  ○ $\begin{cases} ax \quad + bz \quad = 1 \\ \quad ay \quad + bw = 0 \\ cx \quad + dz \quad = 0 \\ \quad cy \quad + dw = 1 \end{cases} \Rightarrow \begin{cases} ax + bz = 1 \\ cx + dz = 0 \end{cases} \quad \begin{cases} ay + bw = 0 \\ cy + dw = 1 \end{cases}$

  - They are inconsistent $\Leftrightarrow a : c = b : d \Leftrightarrow ad = bc$.
  - They are consistent $\Leftrightarrow ad \neq bc$.

Example

- Let $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Find the condition when $\boldsymbol{A}$ is invertible.

  inconsistent. not invertible

  ○ If $ad = bc$, then $\boldsymbol{A}$ is singular. Suppose that $ad \neq bc$.

  - $\begin{cases} ax + bz = 1 \\ cx + dz = 0 \end{cases} \Rightarrow x = \dfrac{d}{ad-bc},\ z = \dfrac{-c}{ad-bc}$.
  - $\begin{cases} ay + bw = 0 \\ cy + dw = 1 \end{cases} \Rightarrow y = \dfrac{-b}{ad-bc},\ w = \dfrac{a}{ad-bc}$.

  Let $\boldsymbol{B} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

  - One verifies that $\boldsymbol{AB} = \boldsymbol{I}$ and $\boldsymbol{BA} = \boldsymbol{I}$. — either way gives you I.

    else is singular.

  Conclusion: $\boldsymbol{A}$ is invertible $\Leftrightarrow ad - bc \neq 0$.

  ○ If $\boldsymbol{A}$ is invertible, then $\boldsymbol{A}^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}_{2\times 2}$.

Inverse:

Properties

- **Theorem.** Let $\boldsymbol{A}, \boldsymbol{B}$ be invertible matrices of same size.
  - Let $c \neq 0$. $c\boldsymbol{A}$ is invertible, and $(c\boldsymbol{A})^{-1} = \frac{1}{c}\boldsymbol{A}^{-1}$.
  - $\boldsymbol{A}^{\mathrm{T}}$ is invertible, and $(\boldsymbol{A}^{\mathrm{T}})^{-1} = (\boldsymbol{A}^{-1})^{\mathrm{T}}$. interchangeable.
  - $\boldsymbol{A}^{-1}$ is invertible, and $(\boldsymbol{A}^{-1})^{-1} = \boldsymbol{A}$.
  - $\boldsymbol{AB}$ is invertible, and $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$. may not be $A^{-1}B^{-1}$
- **Proof.** To prove that $\boldsymbol{A}$ is invertible with $\boldsymbol{A}^{-1} = \boldsymbol{M}$,
  - Verify that $\boldsymbol{AM} = \boldsymbol{MA} = \boldsymbol{I}$.

  To prove that $\boldsymbol{A}^{\mathrm{T}}$ is invertible with inverse $(\boldsymbol{A}^{-1})^{\mathrm{T}}$. M.
  - We shall verify that $\boldsymbol{A}^{\mathrm{T}}(\boldsymbol{A}^{-1})^{\mathrm{T}} = (\boldsymbol{A}^{-1})^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}} = \boldsymbol{I}$.
    - $\boldsymbol{A}^{\mathrm{T}}(\boldsymbol{A}^{-1})^{\mathrm{T}} = (\boldsymbol{A}^{-1}\boldsymbol{A})^{\mathrm{T}} = \boldsymbol{I}^{\mathrm{T}} = \boldsymbol{I}$. symmetric
    - $(\boldsymbol{A}^{-1})^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}} = (\boldsymbol{A}\boldsymbol{A}^{-1})^{\mathrm{T}} = \boldsymbol{I}^{\mathrm{T}} = \boldsymbol{I}$. (reverse order).

  $A^T B^T = (BA)^T$.
  $B^T A^T = (AB)^T$.

  Other properties are left as exercises.

Properties

only when A is invertible

- Let $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_k$ be invertible matrices of same size.
  - $(\boldsymbol{A}_1\boldsymbol{A}_2\cdots\boldsymbol{A}_k)^{-1} = \boldsymbol{A}_k^{-1}\cdots\boldsymbol{A}_2^{-1}\boldsymbol{A}_1^{-1}$. reverse order

  In particular, $(\underbrace{\boldsymbol{AA}\cdots\boldsymbol{A}}_{k \text{ times}})^{-1} = \underbrace{\boldsymbol{A}^{-1}\cdots\boldsymbol{A}^{-1}\boldsymbol{A}^{-1}}_{k \text{ times}}$.
  - $(\boldsymbol{A}^k)^{-1} = (\boldsymbol{A}^{-1})^k$.

eg. $(ABC)^{-1}$
$= ((AB)C)^{-1}$
$= C^{-1}(AB)^{-1}$
$= C^{-1}B^{-1}A^{-1}$.

- **Definition.** Let $\boldsymbol{A}$ be an invertible matrix.
  - For any positive integer k, $\boxed{\boldsymbol{A}^{-k} = (\boldsymbol{A}^{-1})^k}$
- **Exercise.** Let $\boldsymbol{A}$ be an invertible matrix.
  - For any integers m and n,
    - $\boldsymbol{A}^{m+n} = \boldsymbol{A}^m\boldsymbol{A}^n$ and $(\boldsymbol{A}^m)^n = \boldsymbol{A}^{mn}$.
- **Note.** If $\boldsymbol{A}$ is singular, then $\boldsymbol{A}^{-1}$ is undefined. = no negative integer power.

Elementary Operations

$\boxed{Ax = B}$

- Recall the **elementary row operations** of matrices.
  - Multiply a row by a nonzero constant.
  - Interchange two rows.
  - Add a constant multiple of a row to another row.
- What is the resulting matrix by applying an elementary row operation to the identity matrix $\boldsymbol{I}$?
  - $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{cR_3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
  - cR_i, where $c \neq 0$:
    - Replace the ith diagonal entry by c:
      - $a_{ii} = 1 \mapsto c$.

Elementary Operations

- Recall the **elementary row operations** of matrices.
  - Multiply a row by a nonzero constant.
  - Interchange two rows.
  - Add a constant multiple of a row to another row.
- What is the resulting matrix by applying an elementary row operation to the identity matrix $\boldsymbol{I}$?
  - $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ Still symmetric.

Interchange.

  - $R_i \leftrightarrow R_j$, where $i \neq j$:
    - The ith and jth diagonal entries become 0.
    - The (i, j) and (j, i)-entries become 1.
      - $a_{ii} = a_{jj} = 1 \mapsto 0$, $a_{ij} = a_{ji} = 0 \mapsto 1$.

Elementary Operations

- Recall the **elementary row operations** of matrices.
  - Multiply a row by a nonzero constant.
  - Interchange two rows.
  - Add a constant multiple of a row to another row.
- What is the resulting matrix by applying an elementary row operation to the identity matrix $\boldsymbol{I}$?
  - $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 + cR_4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & c \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ (2,4) c
  - $R_i + cR_j$, where $i \neq j$:
    - The (i, j)-entry becomes c.
      - $a_{ij} = 0 \mapsto c$.

Elementary Matrices

- **Definition.** A square matrix is called an **elementary matrix** if it can be obtained from the identity matrix by performing a single elementary row operation.
  - cR_i, where $c \neq 0$: $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
  - $R_i \leftrightarrow R_j$, where $i \neq j$, $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$
  - $R_i + cR_j$, where $i \neq j$, $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & c \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Connection to Matrix Multiplication

- Let $\boldsymbol{E}$ be an elementary matrix.
  - Suppose that it is obtained from $\boldsymbol{I}$ by
    - Multiplying the ith row by a nonzero number c.

Let $\boldsymbol{E} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix}$.

- $\boldsymbol{EA} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ ca_{31} & ca_{32} & ca_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix}$ by number c.

Pre-multiplication by $\boldsymbol{E}$ to $\boldsymbol{A}$

$\Leftrightarrow$ Multiplying the ith row of $\boldsymbol{A}$ by number c.

Connection to Matrix Multiplication

- Let $\boldsymbol{E}$ be obtained from $\boldsymbol{I}_m$ by multiplying kth row by c.
  - $e_{ij} = 0$ if $i \neq j$, 1 if $i = j \neq k$, c if $i = j = k$.

Let $\boldsymbol{A} = (a_{ij})_{m \times n}$. Then $\boldsymbol{EA}$ is well-defined.

- Let $i \neq k$. Then (i, j)-entry of $\boldsymbol{EA}$ is

$e_{ij} = 0$ if $p \neq i$ non-zero

$$\begin{aligned} & e_{i1}a_{1j} + \cdots + \boxed{e_{ii}}a_{ij} + \cdots + e_{im}a_{mj} \\ = \; & \underbrace{0 + \cdots +}_{0} 1 \cdot a_{ij} \underbrace{+ \cdots + 0}_{0} = a_{ij}. \end{aligned}$$

- Let $i = k$. Then (i, j)-entry of $\boldsymbol{EA}$ is

$$\begin{aligned} & \overbrace{e_{i1}a_{1j} + \cdots +}^{0} e_{ii}a_{ij} \overbrace{+ \cdots + e_{im}a_{mj}}^{0} \\ = \; & 0 + \cdots + c \cdot a_{ij} + \cdots + 0 = ca_{ij}. \end{aligned}$$

Therefore, $\boldsymbol{EA}$ is the matrix obtained from $\boldsymbol{A}$ by multiplying the kth row by c.

Connection to Matrix Multiplication

- Let $\boldsymbol{E}$ be an elementary matrix.
  - Suppose that it is obtained from $\boldsymbol{I}$ by
    - Interchanging the ith row and the jth row.

Let $\boldsymbol{E} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$, $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix}$

- $\boldsymbol{EA} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{41} & a_{42} & a_{43} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$ Interchanged.

Pre-multiplication by $\boldsymbol{E}$ to $\boldsymbol{A}$

$\Leftrightarrow$ Interchanging the ith row and jth row of $\boldsymbol{A}$.

Connection to Matrix Multiplication

- Let $\boldsymbol{E}$ be an elementary matrix.
  - Suppose that it is obtained from $\boldsymbol{I}$ by
    - Adding c times of the jth row to the ith row.

Let $\boldsymbol{E} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & c \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix}$

- $\boldsymbol{EA} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} + \underline{ca_{41}} & a_{22} + \underline{ca_{42}} & a_{23} + \underline{ca_{43}} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{33} \end{pmatrix}$ $c * e_4 + e_2$

Pre-multiplication by $\boldsymbol{E}$ to $\boldsymbol{A}$

$\Leftrightarrow$ Adding c times the jth row to the ith row of $\boldsymbol{A}$.

Connection to Matrix Multiplication

- Let $\boldsymbol{E}$ be from $\boldsymbol{I}_m$ by adding c times of ℓth row to kth row.
  - $e_{ij} = 1$ if $i = j$, c if $(i, j) = (k, \ell)$, 0 otherwise.

  Let $\boldsymbol{A} = (a_{ij})_{m \times n}$. Then $\boldsymbol{EA}$ is well-defined.
  - Let $i \neq k$. Then (i, j)-entry of $\boldsymbol{EA}$ is

$$
\begin{aligned}
&\overset{0}{e_{i1}a_{1j}} + \cdots + \overset{1}{e_{ii}}a_{ij} + \cdots + \overset{0}{e_{im}}a_{mj} \\
&= 0 + \cdots + 1 \cdot a_{ij} + \cdots + 0 = a_{ij}.
\end{aligned}
$$

  - Let $i = k$. Then (i, j)-entry of $\boldsymbol{EA}$ is

row = k

$$
\begin{aligned}
&\overset{0}{e_{i1}a_{1j} + \cdots} + e_{ii}a_{ij} + \overset{0}{\cdots} + e_{i\ell}a_{\ell j} + \overset{0}{\cdots + e_{im}a_{mj}} \\
&= 0 + \cdots + 1 \cdot a_{ij} + \cdots + c \cdot a_{\ell j} + \cdots + 0 \\
&= a_{ij} + ca_{\ell j}.
\end{aligned}
$$

kth row   ℓth row

  Therefore, $\boldsymbol{EA}$ is the matrix obtained from $\boldsymbol{A}$ by adding c times of ℓth row to the kth row.

Connection to Multiplication

- **Theorem.**
  - Let $\boldsymbol{E}$ be the elementary matrix obtained
    - by performing an elementary row operation to $\boldsymbol{I}_m$.

    Then for any $m \times n$ matrix $\boldsymbol{A}$, $\boldsymbol{EA}$ can be obtained
    - by performing same elementary row operation to $\boldsymbol{A}$.

matrix multiplication

- Let $\boldsymbol{A}$ be an $m \times n$ matrix.
  - $\boldsymbol{I}_m \xrightarrow{cR_i} \boldsymbol{E} \Rightarrow \boxed{\boldsymbol{A} \xrightarrow{cR_i}} \boldsymbol{EA}$.
  - $\boldsymbol{I}_m \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{E} \Rightarrow \boxed{\boldsymbol{A} \xrightarrow{R_i \leftrightarrow R_j}} \boldsymbol{EA}$.
  - $\boldsymbol{I}_m \xrightarrow{R_i + cR_j} \boldsymbol{E} \Rightarrow \boxed{\boldsymbol{A} \xrightarrow{R_i + cR_j}} \boldsymbol{EA}$.

Same o.r.

Example

- Let $\boldsymbol{A} = \begin{pmatrix} 0 & 4 & 2 \\ -2 & 1 & -3 \\ 1 & 0 & 2 \end{pmatrix}$.
  - $\boldsymbol{A} = \begin{pmatrix} 0 & 4 & 2 \\ -2 & 1 & -3 \\ 1 & 0 & 2 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} 1 & 0 & 2 \\ -2 & 1 & -3 \\ 0 & 4 & 2 \end{pmatrix} = \boldsymbol{A}_1$.
    - $\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \boldsymbol{E}_1$.

$$\begin{aligned} \boldsymbol{E}_1\boldsymbol{A} &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 4 & 2 \\ -2 & 1 & -3 \\ 1 & 0 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 2 \\ -2 & 1 & -3 \\ 0 & 4 & 2 \end{pmatrix} = \boldsymbol{A}_1. \end{aligned}$$

Example

- Let $\boldsymbol{A} = \begin{pmatrix} 0 & 4 & 2 \\ -2 & 1 & -3 \\ 1 & 0 & 2 \end{pmatrix}$.
  - $\boldsymbol{A}_1 = \begin{pmatrix} 1 & 0 & 2 \\ -2 & 1 & -3 \\ 0 & 4 & 2 \end{pmatrix} \xrightarrow{R_2 + 2R_1} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 4 & 2 \end{pmatrix} = \boldsymbol{A}_2$.
    - $\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 + 2R_1} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \boldsymbol{E}_2$.

$$\begin{aligned} \boldsymbol{E}_2\boldsymbol{A}_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ -2 & 1 & -3 \\ 0 & 4 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 4 & 2 \end{pmatrix} = \boldsymbol{A}_2. \end{aligned}$$

Example

- Let $\boldsymbol{A} = \begin{pmatrix} 0 & 4 & 2 \\ -2 & 1 & -3 \\ 1 & 0 & 2 \end{pmatrix}$.

REF.

  - $\boldsymbol{A}_2 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 4 & 2 \end{pmatrix} \xrightarrow{R_3+(-4)R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{pmatrix} = \boldsymbol{A}_3.$
    - $\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_3+(-4)R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{pmatrix} = \boldsymbol{E}_3.$

-4.

$$\boldsymbol{E}_3\boldsymbol{A}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 4 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{pmatrix} = \boldsymbol{A}_3.$$

Example

- Let $\boldsymbol{A} = \begin{pmatrix} 0 & 4 & 2 \\ -2 & 1 & -3 \\ 1 & 0 & 2 \end{pmatrix}$.
  - $\boldsymbol{A}_3 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{pmatrix} \xrightarrow{-\frac{1}{2}R_3} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \boldsymbol{A}_4.$
    - $\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{-\frac{1}{2}R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} = \boldsymbol{E}_4.$

$$\boldsymbol{E}_4\boldsymbol{A}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \boldsymbol{A}_4.$$

Example

- Let $\boldsymbol{A} = \begin{pmatrix} 0 & 4 & 2 \\ -2 & 1 & -3 \\ 1 & 0 & 2 \end{pmatrix}$.
  - $\boldsymbol{A}_4 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1+(-2)R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \boldsymbol{A}_5$.
    - $\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1+(-2)R_3} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \boldsymbol{E}_5$

$$\begin{aligned} \boldsymbol{E}_5\boldsymbol{A}_4 &= \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \boldsymbol{A}_5. \end{aligned}$$

Example

- Let $\boldsymbol{A} = \begin{pmatrix} 0 & 4 & 2 \\ -2 & 1 & -3 \\ 1 & 0 & 2 \end{pmatrix}$.
  - $\boldsymbol{A}_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2+(-1)R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \boldsymbol{A}_6 = \boldsymbol{I}$.
    - $\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2+(-1)R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \boldsymbol{E}_6$.

$$\begin{aligned} \boldsymbol{E}_6\boldsymbol{A}_5 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \boldsymbol{A}_6 = \boldsymbol{I}. \end{aligned}$$

Invertibility

- **Theorem.** Every elementary matrix is invertible. we know the inverse
  - The inverse of an elementary matrix is elementary. — of the same type.

Proof. There are three types of elementary matrices.

- Suppose that $I \xrightarrow{cR_i} E$, where $c \neq 0$.
  - Then $E \xrightarrow{\frac{1}{c}R_i} I$.

  Let D denote the elementary matrix $I \xrightarrow{\frac{1}{c}R_i} D$.
  - $I \underset{E}{\xrightarrow{cR_i}} E \underset{D}{\xrightarrow{\frac{1}{c}R_i}} I$. So $DE = I$.
  - $I \underset{D}{\xrightarrow{\frac{1}{c}R_i}} D \underset{E}{\xrightarrow{cR_i}} I$. So $ED = I$.

  DE=ED=I

  It follows that E is invertible and $E^{-1} = D$.

The other two cases are similar and left as exercises.

Invertibility

- Let E be an elementary matrix.
  - $I \xrightarrow{cR_i} E \Rightarrow I \xrightarrow{\frac{1}{c}R_i} E^{-1}$.
  - $I \xrightarrow{R_i \leftrightarrow R_j} E \Rightarrow I \xrightarrow{R_i \leftrightarrow R_j} E^{-1}$. (So $E = E^{-1}$)
  - $I \xrightarrow{R_i + cR_j} E \Rightarrow I \xrightarrow{R_i + (-c)R_j} E^{-1}$.
- Suppose that matrices A and B are row equivalent.
  - $A = A_0 \xrightarrow{\text{ero}1} A_1 \xrightarrow{\text{ero}2} A_2 \cdots \to A_{k-1} \xrightarrow{\text{ero}k} A_k = B$.

  Let E_i be the elementary matrix corresponding to the ith elementary row operation: $I \xrightarrow{\text{ero}i} E_i$.
  - $A = A_0 \underset{E_1}{\xrightarrow{\text{ero}1}} A_1 \underset{E_2}{\xrightarrow{\text{ero}2}} A_2 \cdots \to A_{k-1} \underset{E_k}{\xrightarrow{\text{ero}k}} A_k = B$.

  Then $B = E_k E_{k-1} \cdots E_2 E_1 A$. $E_k A_{k-1} = A_k$.

row equivalent matrixes.

Invertibility

- **Theorem.** Two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are row equivalent

$\Leftrightarrow$ there exist elementary matrices $\boldsymbol{E}_1, \boldsymbol{E}_2, \ldots, \boldsymbol{E}_k$

such that $\boldsymbol{B} = \boldsymbol{E}_k \boldsymbol{E}_{k-1} \cdots \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{A}$. aka they are the same no matter how you manipulate it

- **Remarks.** Suppose for elementary matrices $\boldsymbol{E}_i$,
  - $\boldsymbol{B} = \boldsymbol{E}_k \boldsymbol{E}_{k-1} \cdots \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{A}$.
    - $\boldsymbol{A} \xrightarrow{\boldsymbol{E}_1} \bullet \xrightarrow{\boldsymbol{E}_2} \bullet \to \cdots \to \bullet \xrightarrow{\boldsymbol{E}_{k-1}} \bullet \xrightarrow{\boldsymbol{E}_k} \boldsymbol{B}$.
    - $\boldsymbol{A} \xleftarrow{\boldsymbol{E}_1^{-1}} \bullet \xleftarrow{\boldsymbol{E}_2^{-1}} \bullet \leftarrow \cdots \leftarrow \bullet \xleftarrow{\boldsymbol{E}_{k-1}^{-1}} \bullet \xleftarrow{\boldsymbol{E}_k^{-1}} \boldsymbol{B}$. to go back just pre-multiply inverse E to B.

$\therefore\ \boldsymbol{A} = \boldsymbol{E}_1^{-1} \boldsymbol{E}_2^{-1} \cdots \boldsymbol{E}_{k-1}^{-1} \boldsymbol{E}_k^{-1} \boldsymbol{B}$.

We can now prove the theorem stated in Chapter 1:

- **Theorem.** Suppose that the augmented matrices of two linear systems are row equivalent.
  - Then the two systems have the same solution set.

Invertibility

- **Proof.** Let the two linear systems be
  - $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{C}\boldsymbol{x} = \boldsymbol{d}$.

Then the associated augmented matrices are

- $(\boldsymbol{A} \mid \boldsymbol{b})$ and $(\boldsymbol{C} \mid \boldsymbol{d})$.

There exist elementary matrices $\boldsymbol{E}_1, \boldsymbol{E}_2, \ldots, \boldsymbol{E}_k$ so that

- $\boldsymbol{E}_k \cdots \boldsymbol{E}_1 (\boldsymbol{A} \mid \boldsymbol{b}) = (\boldsymbol{C} \mid \boldsymbol{d})$.
  - $\boldsymbol{E}_k \cdots \boldsymbol{E}_1 \boldsymbol{A} = \boldsymbol{C}$ and $\boldsymbol{E}_k \cdots \boldsymbol{E}_1 \boldsymbol{b} = \boldsymbol{d}$. multiplication by blocks

Let $\boldsymbol{u}$ be a solution to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, i.e., $\boldsymbol{A}\boldsymbol{u} = \boldsymbol{b}$.

- $\boldsymbol{E}_k \cdots \boldsymbol{E}_1 \boldsymbol{A}\boldsymbol{u} = \boxed{\boldsymbol{E}_k \cdots \boldsymbol{E}_1}\boldsymbol{b} \Rightarrow \boldsymbol{C}\boldsymbol{u} = \boldsymbol{d}$.

So $\boldsymbol{u}$ is also a solution to $\boldsymbol{C}\boldsymbol{x} = \boldsymbol{d}$.

Invertibility

- **Proof.** Let the two linear systems be
  - $\boldsymbol{Ax} = \boldsymbol{b}$ and $\boldsymbol{Cx} = \boldsymbol{d}$.

  Then the associated augmented matrices are
  - $(\boldsymbol{A} \mid \boldsymbol{b})$ and $(\boldsymbol{C} \mid \boldsymbol{d})$.

  There exist elementary matrices $\boldsymbol{E}_1, \boldsymbol{E}_2, \ldots, \boldsymbol{E}_k$ so that
  - $\boldsymbol{E}_k \cdots \boldsymbol{E}_1 (\boldsymbol{A} \mid \boldsymbol{b}) = (\boldsymbol{C} \mid \boldsymbol{d})$.
    - $\boldsymbol{E}_k \cdots \boldsymbol{E}_1 \boldsymbol{A} = \boldsymbol{C}$ and $\boldsymbol{E}_k \cdots \boldsymbol{E}_1 \boldsymbol{b} = \boldsymbol{d}$.   reverse
    - $\boldsymbol{A} = \boldsymbol{E}_1^{-1} \cdots \boldsymbol{E}_k^{-1} \boldsymbol{C}$ and $\boldsymbol{b} = \boldsymbol{E}_1^{-1} \cdots \boldsymbol{E}_k^{-1} \boldsymbol{d}$.

  Let $\boldsymbol{v}$ be a solution to $\boldsymbol{Cx} = \boldsymbol{d}$, i.e., $\boldsymbol{Cv} = \boldsymbol{d}$.
  - $\boldsymbol{E}_1^{-1} \cdots \boldsymbol{E}_k^{-1} \boldsymbol{Cv} = \boldsymbol{E}_1^{-1} \cdots \boldsymbol{E}_k^{-1} \boldsymbol{d} \Rightarrow \boldsymbol{Av} = \boldsymbol{b}$.

  So $\boldsymbol{v}$ is also a solution to $\boldsymbol{Ax} = \boldsymbol{b}$. $\therefore$ same solution sets.

$A(B_1, B_2) = [AB_1 \ AB_2]$

Main Theorem for Invertible Matrices   (and 10).

- **Theorem.** Let $\boldsymbol{A}$ be a square matrix. Then the followings are equivalent:
  1. $\boldsymbol{A}$ is an invertible matrix.
  2. Linear system $\boldsymbol{Ax} = \boldsymbol{b}$ has a unique solution.
  3. Linear system $\boldsymbol{Ax} = \boldsymbol{0}$ has only the trivial solution.
  4. The reduced row-echelon form of $\boldsymbol{A}$ is $\boldsymbol{I}$.
  5. $\boldsymbol{A}$ is the product of elementary matrices.

Constructs invertible matrices

can be splitted

Main Theorem for Invertible Matrices

- **Proof.** We prove 1 $\Rightarrow$ 2 $\Rightarrow$ 3 $\Rightarrow$ 4 $\Rightarrow$ 5 $\Rightarrow$ 1: cycle.

1 $\Rightarrow$ 2: Suppose $\boldsymbol{A}$ is invertible.

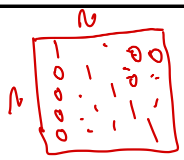
- $\boldsymbol{Ax} = \boldsymbol{b} \Rightarrow \boldsymbol{x} = \boldsymbol{Ix} = \boldsymbol{A}^{-1}\boldsymbol{Ax} = \boldsymbol{A}^{-1}\boldsymbol{b}$ unique.

2 $\Rightarrow$ 3: Suppose $\boldsymbol{Ax} = \boldsymbol{b}$ has a unique solution $\boldsymbol{u}$. $Ax=0$ has only trivial.

- If $\boldsymbol{Ax} = \boldsymbol{0}$ has a solution $\boldsymbol{v}$, then $\boldsymbol{Av} = \boldsymbol{0}$. ← show this. $v=0$
  - $\boldsymbol{A}(\boldsymbol{u} - \boldsymbol{v}) = \boldsymbol{Au} - \boldsymbol{Av} = \boldsymbol{b} - \boldsymbol{0} = \boldsymbol{b}$. (can also add) $Au=b$
  - $\boldsymbol{u} - \boldsymbol{v}$ is also a solution to $\boldsymbol{Ax} = \boldsymbol{b}$.
- By uniqueness, $\boldsymbol{u} = \boldsymbol{u} - \boldsymbol{v}$; so $\boldsymbol{v} = \boldsymbol{0}$.

3 $\Rightarrow$ 4: Suppose $\boldsymbol{Ax} = \boldsymbol{0}$ has only the trivial solution.

- Let $\boldsymbol{R}$ be the reduced row-echelon form of $\boldsymbol{A}$. → is identity matrix.
  - Except the last column, all other columns of $(\boldsymbol{R} \mid \boldsymbol{0})$ are pivot columns. augmented matrix
- Note that $\boldsymbol{R}$ is a square matrix. So $\boldsymbol{R} = \boldsymbol{I}$.



Main Theorem for Invertible Matrices

- **Proof.** We prove 1 $\Rightarrow$ 2 $\Rightarrow$ 3 $\Rightarrow$ 4 $\Rightarrow$ 5 $\Rightarrow$ 1:

4 $\Rightarrow$ 5: Suppose that the RREF of $\boldsymbol{A}$ is $\boldsymbol{I}$. — A is product of E matrices.

- $\boldsymbol{I}$ can be obtained from $\boldsymbol{A}$ by ele. row operations.
  - $\boldsymbol{A} \xrightarrow{\text{ero}1} \bullet \cdots \rightarrow \cdots \bullet \xrightarrow{\text{ero}k} \boldsymbol{I}$. — row equivalent.
- Let $\boldsymbol{E}_i$ be the elementary matrix corresponding to the ith elementary row operation.
  - $\boldsymbol{I} = \boldsymbol{E}_k\boldsymbol{E}_{k-1}\cdots\boldsymbol{E}_2\boldsymbol{E}_1\boldsymbol{A}$. product of elementary matrices. $A = \overbrace{E_1^{-1}\cdots E_{k-1}^{-1}E_k^{-1}}^{A}I$
- Then $\boldsymbol{A} = \boldsymbol{E}_1^{-1}\boldsymbol{E}_2^{-1}\cdots\boldsymbol{E}_{k-1}^{-1}\boldsymbol{E}_k^{-1}$. can use to find more of A. $A \xleftarrow{E_1^{-1}} \cdots \leftarrow \xleftarrow{E_{k-1}^{-1}} \xleftarrow{E_k^{-1}} I$
  - (Recall: each $\boldsymbol{E}_i^{-1}$ is also an elementary matrix.)

5 $\Rightarrow$ 1: Suppose that $\boldsymbol{A} = \boldsymbol{E}_1\boldsymbol{E}_2\cdots\boldsymbol{E}_k$,

- where $\boldsymbol{E}_1, \ldots, \boldsymbol{E}_k$ are elementary matrices.

Each $\boldsymbol{E}_i$ is invertible $\Rightarrow$ $\boldsymbol{A}$ is invertible.

Find Inverse

inverse is unique

- Let $\boldsymbol{A}$ be an invertible matrix of order n. Its RREF is $\boldsymbol{I}_n$.
  - There exist elementary matrices $\boldsymbol{E}_i$ such that
    - $\boxed{\boldsymbol{E}_k \cdots \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{A} = \boldsymbol{I}_n}$. BA = I_n.

    Then $\boldsymbol{E}_k \cdots \boldsymbol{E}_2 \boldsymbol{E}_1 = \boldsymbol{A}^{-1}$.

  Consider the $n \times 2n$ matrix $(\boldsymbol{A} \mid \boldsymbol{I})$. $A \mid I$.
  - Apply the ele. row oper. corresponding to $\boldsymbol{E}_1, \dots, \boldsymbol{E}_k$:

$$
\begin{aligned}
(\boldsymbol{A} \mid \boldsymbol{I}_n) &\xrightarrow{\boldsymbol{E}_1} (\boldsymbol{E}_1\boldsymbol{A} \mid \boldsymbol{E}_1) \\
&\xrightarrow{\boldsymbol{E}_2} (\boldsymbol{E}_2\boldsymbol{E}_1\boldsymbol{A} \mid \boldsymbol{E}_2\boldsymbol{E}_1) \\
&\to \cdots \to \cdots \\
&\xrightarrow{\boldsymbol{E}_k} (\boldsymbol{E}_k \cdots \boldsymbol{E}_2\boldsymbol{E}_1\boldsymbol{A} \mid \boldsymbol{E}_k \cdots \boldsymbol{E}_2\boldsymbol{E}_1) \\
&= (\boldsymbol{I}_n \mid \boldsymbol{A}^{-1}).
\end{aligned}
$$

Simplest way to find Inverse of an invertible matrix.

Find Inverse

- **Theorem.** Let $\boldsymbol{A}$ be an invertible matrix.
  - The RREF of $(\boldsymbol{A} \mid \boldsymbol{I})$ is $\boxed{(\boldsymbol{I} \mid \boldsymbol{A}^{-1})}$
- **Example.** Let $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$. Find $\boldsymbol{A}^{-1}$.

$$
\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right)
\xrightarrow[R_3+(-1)R_1]{R_2+(-2)R_1}
\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right)
$$

$$
\xrightarrow{R_3+2R_2}
\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right)
$$

$$
\xrightarrow{(-1)R_2}
\left(\begin{array}{ccc|ccc} 1 & 2 & \boxed{3} & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)
$$

Find Inverse

- **Theorem.** Let A be an invertible matrix.
  - The RREF of $(A \mid I)$ is $\boxed{(I \mid A^{-1})}$
- **Example.** Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$. Find A^{-1}.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right) \xrightarrow[R_2+3R_3]{R_1+(-3)R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)$$

$$\xrightarrow{R_1+(-2)R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)$$

RREF

$I_n \mid A^{-1}$

Therefore, $A^{-1} = \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}$.

Find Inverse

- A square matrix is **invertible**

$\Leftrightarrow$ Its reduced row-echelon form is I

$\Leftrightarrow$ All the columns in its row-echelon form are pivot.

$\Leftrightarrow$ All the rows in its row-echelon form are nonzero.

Algorithm to find inverse.

A square matrix is **singular**

inverse ∄ this.

$\Leftrightarrow$ Its reduced row-echelon form is not I.

$\Leftrightarrow$ Some columns in its row-echelon form are non-pivot.

$\Leftrightarrow$ Some rows in its row-echelon form are zero.

Example. Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 4 \\ 3 & 6 & 3 \end{pmatrix}$. Then A is singular.

REF. non-pivot. column.

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 4 \\ 3 & 6 & 3 \end{pmatrix} \xrightarrow[R_3+(-3)R_1]{R_2+(-2)R_1} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 3 \end{pmatrix} \xrightarrow{R_3+(-\frac{3}{4})R_2} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

Find Inverse

- **Theorem.** Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be square matrices of the same size. If $\boldsymbol{AB} = \boldsymbol{I}$, then
  - $\boldsymbol{A}$ and $\boldsymbol{B}$ are invertible, and $\boldsymbol{A}^{-1} = \boldsymbol{B}$, $\boldsymbol{B}^{-1} = \boldsymbol{A}$.

  Proof. Consider the linear system $\boldsymbol{Bx} = \boldsymbol{0}$. — show B is invertible.
  - $\boldsymbol{Bx} = \boldsymbol{0} \Rightarrow \boldsymbol{ABx} = \boldsymbol{A0} \Rightarrow \boldsymbol{x} = \boldsymbol{0}$.
  - $\boldsymbol{Bx} = \boldsymbol{0}$ has only the trivial solution $\Rightarrow$ $\boldsymbol{B}$ is invertible. — B^{-1} exist.

  $\boldsymbol{B}^{-1}$ exists such that $\boldsymbol{BB}^{-1} = \boldsymbol{B}^{-1}\boldsymbol{B} = \boldsymbol{I}$.
  - $\boldsymbol{AB} = \boldsymbol{I} \Rightarrow \boldsymbol{ABB}^{-1} = \boldsymbol{IB}^{-1} \Rightarrow \boldsymbol{A} = \boldsymbol{B}^{-1}$. — do not need verify on both side.

  Therefore, $\boldsymbol{A}$ is invertible, and $\boldsymbol{A}^{-1} = (\boldsymbol{B}^{-1})^{-1} = \boldsymbol{B}$.
- **Corollary and Exercise.** Let $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_k$ be square matrices of the same size.
  - $\boldsymbol{A}_1\boldsymbol{A}_2\cdots\boldsymbol{A}_k$ is invertible $\Leftrightarrow$ all $\boldsymbol{A}_i$ are invertible.
  - $\boldsymbol{A}_1\boldsymbol{A}_2\cdots\boldsymbol{A}_k$ is singular $\Leftrightarrow$ some $\boldsymbol{A}_i$ are singular.

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Find Inverse

- Let $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\boldsymbol{B} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, $ad \neq bc$.
  - One verifies that $\boldsymbol{AB} = \boldsymbol{I}$.
  - Using the theorem, $\boldsymbol{A}^{-1} = \boldsymbol{B}$. — must be true.
    - The verification that $\boldsymbol{BA} = \boldsymbol{I}$ is not necessary.
- Let $\boldsymbol{A}$ be a square matrix such that $\boldsymbol{A}^2 - 3\boldsymbol{A} - 4\boldsymbol{I} = \boldsymbol{0}$.
  - Prove that $\boldsymbol{A}$ is invertible, and find $\boldsymbol{A}^{-1}$.

  Wrong Proof. $\boldsymbol{0} = (\boldsymbol{A} - 4\boldsymbol{I})(\boldsymbol{A} + \boldsymbol{I})$.
  - $\boldsymbol{A} - 4\boldsymbol{I} = \boldsymbol{0}$ or $\boldsymbol{A} + \boldsymbol{I} = \boldsymbol{0} \Rightarrow \boldsymbol{A} = 4\boldsymbol{I}$ or $\boldsymbol{A} = -\boldsymbol{I}$.

  Proof. $4\boldsymbol{I} = \boldsymbol{A}^2 - 3\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{A} - 3\boldsymbol{I})$.
  - $\boldsymbol{I} = \frac{1}{4}\boldsymbol{A}(\boldsymbol{A} - 3\boldsymbol{I}) = \boldsymbol{A}\left[\frac{1}{4}(\boldsymbol{A} - 3\boldsymbol{I})\right]$.
  - So $\boldsymbol{A}$ is invertible with $\boldsymbol{A}^{-1} = \frac{1}{4}(\boldsymbol{A} - 3\boldsymbol{I})$.

Coh

row A A^{-1}

com A^{-1} A

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Column Operations

- Recall that the pre-multiplication of an elementary matrix $\Leftrightarrow$ corresponding elementary row operation.

  Question. What is the effect of **post-multiplication** of an elementary matrix?

  - Answer: Corresponding elementary column operation.

- **Elementary column operations**:

  - kC_i: multiply ith column by a nonzero constant k.
  - $C_i \leftrightarrow C_j$: interchange ith and jth columns.
  - $C_i + kC_j$: add k times jth column to ith column.

  Let $\boldsymbol{E}$ be the matrix obtained from $\boldsymbol{I}$ by a single elementary column operation.

  - Then $\boldsymbol{E}$ is an **elementary matrix**.
    - (i.e., $\boldsymbol{E}$ can be obtained from $\boldsymbol{I}$ by a single elementary row operation.)

Column Operations

- If $\boldsymbol{E}$ is obtained from $\boldsymbol{I}_n$ by a single elementary column operation, then $\boldsymbol{E}$ is an elementary matrix.

  - $\boldsymbol{I} \xrightarrow{kC_i} \boldsymbol{E} \Leftrightarrow \boldsymbol{I} \xrightarrow{kR_i} \boldsymbol{E}$.
  - $\boldsymbol{I} \xrightarrow{C_i \leftrightarrow C_j} \boldsymbol{E} \Leftrightarrow \boldsymbol{I} \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{E}$.
  - $\boldsymbol{I} \xrightarrow{C_i + kC_j} \boldsymbol{E} \Leftrightarrow \boldsymbol{I} \xrightarrow{R_j + kR_i} \boldsymbol{E}$.

- Let $\boldsymbol{A}$ be an $m \times n$ matrix. Then $\boldsymbol{A}^{\mathrm{T}}$ is $n \times m$.

  - Suppose that $\boldsymbol{I}_n \xrightarrow{kC_i} \boldsymbol{E}$. Note that $\boldsymbol{E} = \boldsymbol{E}^{\mathrm{T}}$.
    - $\boldsymbol{I}_n \xrightarrow{kR_i} \boldsymbol{E}^{\mathrm{T}} \Rightarrow \boldsymbol{A}^{\mathrm{T}} \xrightarrow{kR_i} \boldsymbol{E}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}} = (\boldsymbol{A}\boldsymbol{E})^{\mathrm{T}}$.

    Then $\boldsymbol{A} \xrightarrow{kC_i} \boldsymbol{A}\boldsymbol{E}$.

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- Let $\boldsymbol{A}$ be an $m \times n$ matrix. Then $\boldsymbol{A}^{\mathrm{T}}$ is $n \times m$.
  - Suppose that $\boldsymbol{I}_n \xrightarrow{C_i \leftrightarrow C_j} \boldsymbol{E}$. Note that $\boldsymbol{E} = \boldsymbol{E}^{\mathrm{T}}$.
    - $\boldsymbol{I} \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{E}^{\mathrm{T}} \Rightarrow \boldsymbol{A}^{\mathrm{T}} \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{E}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}} = (\boldsymbol{AE})^{\mathrm{T}}$.

    Then $\boldsymbol{A} \xrightarrow{C_i \leftrightarrow C_j} \boldsymbol{AE}$.

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  - $\boldsymbol{I} \xrightarrow{C_i + kC_j} \boldsymbol{E} \Leftrightarrow \boldsymbol{I} \xrightarrow{R_j + kR_i} \boldsymbol{E}$.

- Let $\boldsymbol{A}$ be an $m \times n$ matrix. Then $\boldsymbol{A}^{\mathrm{T}}$ is $n \times m$.
  - Suppose that $\boldsymbol{I} \xrightarrow{C_i + kC_j} \boldsymbol{E}$.
    - Then $\boldsymbol{I} \xrightarrow{R_i + kR_j} \boldsymbol{E}^{\mathrm{T}}$.
    - $\boldsymbol{A}^{\mathrm{T}} \xrightarrow{R_i + kR_j} \boldsymbol{E}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}} = (\boldsymbol{AE})^{\mathrm{T}}$.

    Then $\boldsymbol{A} \xrightarrow{C_i + kC_j} \boldsymbol{AE}$.

Column Operations

- If $\boldsymbol{E}$ is obtained from $\boldsymbol{I}_n$ by a single elementary column operation, then $\boldsymbol{E}$ is an elementary matrix.
  - $\boldsymbol{I} \xrightarrow{kC_i} \boldsymbol{E} \Leftrightarrow \boldsymbol{I} \xrightarrow{kR_i} \boldsymbol{E}$.
  - $\boldsymbol{I} \xrightarrow{C_i \leftrightarrow C_j} \boldsymbol{E} \Leftrightarrow \boldsymbol{I} \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{E}$.
  - $\boldsymbol{I} \xrightarrow{C_i + kC_j} \boldsymbol{E} \Leftrightarrow \boldsymbol{I} \xrightarrow{R_j + kR_i} \boldsymbol{E}$.

- Let $\boldsymbol{A}$ be an $m \times n$ matrix, and $\boldsymbol{E}$ an $n \times n$ elementary matrix. Then
  - The post-multiplication of $\boldsymbol{E}$ to $\boldsymbol{A}$

  $\Leftrightarrow$ Corresponding elementary column operation to $\boldsymbol{A}$.
    - $\boldsymbol{I} \xrightarrow{kC_i} \boldsymbol{E} \Rightarrow \boldsymbol{A} \xrightarrow{kC_i} \boldsymbol{AE}$.
    - $\boldsymbol{I} \xrightarrow{C_i \leftrightarrow C_j} \boldsymbol{E} \Rightarrow \boldsymbol{A} \xrightarrow{C_i \leftrightarrow C_j} \boldsymbol{AE}$.
    - $\boldsymbol{I} \xrightarrow{C_i + kC_j} \boldsymbol{E} \Rightarrow \boldsymbol{A} \xrightarrow{C_i + kC_j} \boldsymbol{AE}$.

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Examples

- Let $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{pmatrix}$.
  - $\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{2C_3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \boldsymbol{E}_1$.
  - $\boldsymbol{AE}_1 = \cdots = \begin{pmatrix} 1 & 0 & 4 & 3 \\ 2 & -1 & 6 & 6 \\ 1 & 4 & 8 & 0 \end{pmatrix}$
    - $\boldsymbol{A} \xrightarrow{2C_3} \boldsymbol{AE}_1$.

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Examples

- Let $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{pmatrix}$.
  - $\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{C_2 \leftrightarrow C_4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \boldsymbol{E}_2$.
  - $\boldsymbol{A}\boldsymbol{E}_2 = \cdots = \begin{pmatrix} 1 & 3 & 2 & 0 \\ 2 & 6 & 3 & -1 \\ 1 & 0 & 4 & 4 \end{pmatrix}$
    - $\boldsymbol{A} \xrightarrow{C_2 \leftrightarrow C_4} \boldsymbol{A}\boldsymbol{E}_2$.

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Examples

- Let $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{pmatrix}$.
  - $\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{C_1 + 2C_3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \boldsymbol{E}_3$.
  - $\boldsymbol{A}\boldsymbol{E}_3 = \cdots = \begin{pmatrix} 5 & 0 & 2 & 3 \\ 8 & -1 & 3 & 6 \\ 9 & 4 & 4 & 0 \end{pmatrix}$.
    - $\boldsymbol{A} \xrightarrow{C_1 + 2C_3} \boldsymbol{A}\boldsymbol{E}_3$.

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Determinant of 2×2 Matrix

- Recall that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible $\Leftrightarrow ad - bc \neq 0$.
- **Definition.** Let $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
  - The **determinant** of $\boldsymbol{A}$ is $\boxed{\det(\boldsymbol{A}) = |\boldsymbol{A}| = ad - bc}$

  Therefore, $\boldsymbol{A}$ is invertible $\Leftrightarrow \det(\boldsymbol{A}) \neq 0$.
- **Definition.** If $\boldsymbol{A} = (a)$, it is natural to set $\det(\boldsymbol{A}) = a$.
- **Properties & Exercises.** Let $\boldsymbol{A}, \boldsymbol{B}$ be 2×2 matrices.
  - $\det(\boldsymbol{I}_2) = 1$.
  - $\boldsymbol{A} \xrightarrow{cR_i} \boldsymbol{B} \Rightarrow \det(\boldsymbol{B}) = \underline{c}\det(\boldsymbol{A})$. Scalar multiple
  - $\boldsymbol{A} \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{B} \Rightarrow \det(\boldsymbol{B}) = \underline{-}\det(\boldsymbol{A})$. Interchange
  - $\boldsymbol{A} \xrightarrow{R_i + cR_j} \boldsymbol{B} \Rightarrow \det(\boldsymbol{B}) = \det(\boldsymbol{A})$. Constant multiple.

Determinant of 2×2 Matrix

- Consider linear system $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.
  - Suppose $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is invertible.
    - $\boldsymbol{A}^{-1} = \dfrac{1}{\det(\boldsymbol{A})} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \boldsymbol{A}^{-1} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \cdots = \frac{1}{\det(\boldsymbol{A})} \begin{pmatrix} a_{22}b_1 - a_{12}b_2 \\ -a_{21}b_1 + a_{11}b_2 \end{pmatrix}$$

- $x_1 = \dfrac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$ and $x_2 = \dfrac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$.

Determinant of 2×2 Matrix

- One can verify that

$$\begin{aligned}
\begin{vmatrix} a_{11}+a'_{11} & a_{12}+a'_{12} \\ a_{21} & a_{22} \end{vmatrix} &= (a_{11}+a'_{11})a_{22} - (a_{12}+a'_{12})a_{21} \\
&= (a_{11}a_{22} - a_{12}a_{21}) + (a'_{11}a_{22} - a'_{12}a_{21}) \\
&= \underbrace{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}_{\text{det}} + \underbrace{\begin{vmatrix} a'_{11} & a'_{12} \\ a_{21} & a_{22} \end{vmatrix}}_{\text{det.}}
\end{aligned}$$

In particular,

$$\begin{aligned}
\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} &= \begin{vmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \\
&= a_{11}\begin{vmatrix} 1 & 0 \\ a_{21} & a_{22} \end{vmatrix} + a_{12}\begin{vmatrix} 0 & 1 \\ a_{21} & a_{22} \end{vmatrix} \\
&= a_{11} \cdot a_{22} - a_{12} \cdot a_{22} \\
&= a_{11} \cdot \det(a_{22}) - a_{12} \cdot \det(a_{21}).
\end{aligned}$$

(×)

$$\begin{pmatrix} 1 & 0 \\ a_{22} & a_{21} \end{pmatrix} \xrightarrow{a_{11}R_1} \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix}.$$

Determinant of 3×3 Matrix

- Let $\boldsymbol{A}$ be a square matrix. It is expected that
  - $\det(\boldsymbol{I}) = 1$.
  - $\boldsymbol{A}$ is invertible $\Leftrightarrow \det(\boldsymbol{A}) \neq 0$. — aka A is non-singular.
    - Equivalently, $\boldsymbol{A}$ is singular $\Leftrightarrow \det(\boldsymbol{A}) = 0$.
  - $\boldsymbol{A} \xrightarrow{cR_i} \boldsymbol{B} \Rightarrow \det(\boldsymbol{B}) = c\det(\boldsymbol{A})$. ERO
  - $\boldsymbol{A} \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{B} \Rightarrow \det(\boldsymbol{B}) = -\det(\boldsymbol{A})$.
  - $\boldsymbol{A} \xrightarrow{R_i + cR_j} \boldsymbol{B} \Rightarrow \det(\boldsymbol{B}) = \det(\boldsymbol{A})$. no change
- Suppose $\boldsymbol{A}$ is invertible. Then there exist elementary row operations
  - $\boldsymbol{A} \xrightarrow{\text{ero1}} \boldsymbol{A}_1 \xrightarrow{\text{ero2}} \boldsymbol{A}_2 \to \cdots \to \boldsymbol{A}_{k-1} \xrightarrow{\text{ero}k} \boldsymbol{A}_k = \boldsymbol{I}$.

  Then $\det(\boldsymbol{A})$ can be evaluated backwards.
- **Example.** $\underset{-\frac{1}{3}}{\boldsymbol{A}} \xrightarrow{R_1 \leftrightarrow R_3} \underset{\frac{1}{3}}{} \xrightarrow{3R_2} \underset{1}{} \xrightarrow{R_2 + 2R_4} \underset{1}{\boldsymbol{I}} \Rightarrow \det(\boldsymbol{A}) = -\frac{1}{3}$.

Determinant of 3×3 Matrix

- It is also expected that
  - $$\det\begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{pmatrix} + \det\begin{pmatrix} R_1' \\ R_2 \\ \vdots \\ R_n \end{pmatrix} = \det\begin{pmatrix} R_1 + R_1' \\ R_2 \\ \vdots \\ R_n \end{pmatrix}$$
- Consider 3×3 matrix $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$
  - It is expected to have $\det(\boldsymbol{A})$:
    - $$\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

Determinant of 3×3 Matrix

determinant

invertibility

- Assume $a_{11} \neq 0$; o/w the determinant is supposed to be 0.

$$\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} 1 & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \xrightarrow{\frac{1}{a_{11}} R_1} \begin{pmatrix} 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\xrightarrow[R_3 + (-a_{31})R_1]{R_2 + (-a_{21})R_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}.$$

Same determinant.

We use the same elementary row operations:

- $\begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix} \to$ RREF and $\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} \to$ RREF.

0 0

Determinant of 3×3 Matrix

- Assume $a_{12} \neq 0$; o/w the determinant is supposed to be 0.

$$\begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -a_{12} \begin{vmatrix} 1 & 0 & 0 \\ 0 & a_{21} & a_{23} \\ 0 & a_{31} & a_{33} \end{vmatrix} = -a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$\begin{pmatrix} 0 & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \xrightarrow{\frac{1}{a_{12}} R_1} \begin{pmatrix} 0 & 1 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\xrightarrow[R_3 + (-a_{32})R_1]{R_2 + (-a_{22})R_1} \begin{pmatrix} 0 & 1 & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{pmatrix} \xrightarrow{C_1 \leftrightarrow C_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{21} & a_{23} \\ 0 & a_{31} & a_{33} \end{pmatrix}$$

a

We use the same elementary row operations:

- $\begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{21} & a_{23} \\ 0 & a_{31} & a_{33} \end{pmatrix} \rightarrow$ RREF and $\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} \rightarrow$ RREF.

Determinant of 3×3 Matrix

- Assume $a_{13} \neq 0$; o/w the determinant is supposed to be 0.

$$\begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{13} \begin{vmatrix} 1 & 0 & 0 \\ 0 & a_{21} & a_{22} \\ 0 & a_{31} & a_{32} \end{vmatrix} = a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{33} \end{vmatrix}$$

$$\begin{pmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \xrightarrow{\frac{1}{a_{13}} R_1} \begin{pmatrix} 0 & 0 & 1 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\xrightarrow[R_3 + (-a_{33})R_1]{R_2 + (-a_{23})R_1} \begin{pmatrix} 0 & 0 & 1 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{pmatrix} \xrightarrow{C_1 \leftrightarrow C_3} \xrightarrow{C_2 \leftrightarrow C_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{21} & a_{22} \\ 0 & a_{31} & a_{32} \end{pmatrix}$$

2 operations.

keep order.

We use the same elementary row operations:

- $\begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{21} & a_{22} \\ 0 & a_{31} & a_{32} \end{pmatrix} \rightarrow$ RREF and $\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \rightarrow$ RREF.

1 2 3   1 3 2   3 1 2.

Determinant of 3×3 Matrix

- **Definition.** Let $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.
  - $\boxed{\det(\boldsymbol{A}) = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}}$

alternating.

- **Notation.** Let $\boldsymbol{A} = (a_{ij})_{n \times n}$.
  - Let $\boldsymbol{M}_{ij}$ be the square submatrix obtained from $\boldsymbol{A}$ by deleting the ith row and jth column.
  - If $\boldsymbol{A} = (a_{ij})_{3\times 3}$, then $\det(\boldsymbol{A})$ is given by
    - $a_{11} \underbrace{\det(\boldsymbol{M}_{11})}_{A_{11}} - a_{12} \underbrace{\det(\boldsymbol{M}_{12})}_{A_{12}} + a_{13} \underbrace{\det(\boldsymbol{M}_{13})}_{A_{13}}$
  - Let $A_{ij} = (-1)^{i+j} \det(\boldsymbol{M}_{ij})$, the (i,j)-**cofactor** of $\boldsymbol{A}$.
  - If $\boldsymbol{A} = (a_{ij})_{3\times 3}$, then
    - $\boxed{\det(\boldsymbol{A}) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}}$

Example

- Let $\boldsymbol{B} = (b_{ij})_{3\times 3} = \begin{pmatrix} -3 & -2 & 4 \\ 4 & 3 & 1 \\ 0 & 2 & 4 \end{pmatrix}$.
  - $(1,1)$-cofactor: $(-1)^{1+1} \begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix} = 3 \cdot 4 - 1 \cdot 2 = 10.$
  - $(1,2)$-cofactor:
    - $(-1)^{1+2} \begin{vmatrix} 4 & 1 \\ 0 & 4 \end{vmatrix} = -(4 \cdot 4 - 1 \cdot 0) = -16.$
  - $(1,3)$-cofactor: $(-1)^{1+3} \begin{vmatrix} 4 & 3 \\ 0 & 2 \end{vmatrix} = 4 \cdot 2 - 3 \cdot 0 = 8.$

$$\begin{aligned} \det(\boldsymbol{B}) &= (-3) \cdot 10 + (-2) \cdot (-16) + 4 \cdot 8 \\ &= 34. \end{aligned}$$

An Alternative Formula

- Let $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.

$$
\begin{aligned}
\det(\boldsymbol{A}) &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\
&= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) \\
&\quad + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\
&= (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32})^{\text{positive}} \\
&\quad -(a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} + a_{13}a_{22}a_{31}).^{\text{negative.}}
\end{aligned}
$$

  - The positive terms come from the
    - 3 (broken) diagonals from the top left to bottom right.

    The negative terms come from the
    - 3 (broken) diagonals from the top right to bottom left.

Example

- Let $\boldsymbol{B} = (b_{ij})_{3\times 3} = \begin{pmatrix} -3 & -2 & 4 \\ 4 & 3 & 1 \\ 0 & 2 & 4 \end{pmatrix}$.

$$
\begin{aligned}
\det(\boldsymbol{B}) &= [(-3)\cdot 3\cdot 4 + (-2)\cdot 1\cdot 0 + 4\cdot 4\cdot 2] \\
&\quad - [(-3)\cdot 1\cdot 2 + (-2)\cdot 4\cdot 4 + 4\cdot 3\cdot 0] \\
&= (-36 + 0 + 32) - (-6 - 32 + 0) = \underline{34}.
\end{aligned}
$$

- Find the determinant of $\boldsymbol{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

$$
\begin{aligned}
\det(\boldsymbol{I}_3) &= (1\cdot 1\cdot 1 + 0\cdot 0\cdot 0 + 0\cdot 0\cdot 0) \\
&\quad - (1\cdot 0\cdot 0 + 0\cdot 0\cdot 1 + 0\cdot 1\cdot 0) \\
&= 1 - 0 = \underline{1}.
\end{aligned}
$$

Elementary Row Operation

- Let $\boldsymbol{A} = (a_{ij})_{3\times 3} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.
  - $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \xrightarrow{cR_2} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ ca_{21} & ca_{21} & ca_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \boldsymbol{B}.$

$$\begin{aligned} \det(\boldsymbol{B}) &= (a_{11}ca_{22}a_{33} + a_{12}ca_{23}a_{31} + a_{13}ca_{21}a_{31}) \\ &\qquad - (a_{11}ca_{23}a_{32} + a_{12}ca_{21}a_{33} + a_{13}ca_{22}a_{31}) \\ &= c \cdot [(a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{31}) \\ &\qquad - (a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} + a_{13}a_{22}a_{31})] \\ &= c\det(\boldsymbol{A}). \end{aligned}$$

Using broken diagonal.

In particular, $\boldsymbol{I} \xrightarrow{cR_i} \boldsymbol{E} \Rightarrow \det(\boldsymbol{E}) = c \cdot \det(\boldsymbol{I}) = c.$

  - $\boldsymbol{A} \xrightarrow{cR_i} \boldsymbol{EA} \Rightarrow \det(\boldsymbol{EA}) = c\det(\boldsymbol{A}) = \det(\boldsymbol{E})\det(\boldsymbol{A}).$

Elementary Row Operation

- Let $\boldsymbol{A} = (a_{ij})_{3\times 3} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.
  - $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{pmatrix} = \boldsymbol{B}.$

$$\begin{aligned} \det(\boldsymbol{B}) &= (a_{31}a_{22}a_{13} + a_{32}a_{23}a_{11} + a_{33}a_{21}a_{11}) \\ &\qquad - (a_{31}a_{23}a_{12} + a_{32}a_{21}a_{13} + a_{33}a_{22}a_{11}) \\ &= (-1) \cdot [(a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}) \\ &\qquad - (a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} + a_{13}a_{22}a_{31})] \\ &= -\det(\boldsymbol{A}). \end{aligned}$$

In particular, $\boldsymbol{I} \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{E} \Rightarrow \det(\boldsymbol{E}) = -\det(\boldsymbol{I}) = -1.$

  - $\boldsymbol{A} \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{EA} \Rightarrow \det(\boldsymbol{EA}) = \det(\boldsymbol{E})\det(\boldsymbol{A}).$

Elementary Row Operation

- Let $\boldsymbol{A} = (a_{ij})_{3\times 3} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.
  - $\boldsymbol{A} \xrightarrow{R_2 + cR_1} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} + ca_{11} & a_{22} + ca_{12} & a_{23} + ca_{13} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \boldsymbol{B}$.

$$\begin{aligned} \det(\boldsymbol{B}) = &[a_{11}(a_{22} + ca_{12})a_{33} + a_{12}(a_{23} + ca_{13})a_{31} \\ &+ a_{13}(a_{21} + ca_{11})a_{32}] \\ &- [a_{11}(a_{23} + ca_{13})a_{32} + a_{12}(a_{21} + ca_{11})a_{33} \\ &+ a_{13}(a_{22} + ca_{12})a_{31}] \\ \overset{\text{Exercise}}{=\cdots\cdots\cdots=} &\det(\boldsymbol{A}). \end{aligned}$$

term involving c are cancelled

In particular, $\boldsymbol{I} \xrightarrow{R_i + cR_j} \boldsymbol{E} \Rightarrow \det(\boldsymbol{E}) = \det(\boldsymbol{I}) = 1$.

  - $\boldsymbol{A} \xrightarrow{R_i + cR_j} \boldsymbol{EA} \Rightarrow \det(\boldsymbol{EA}) = \det(\boldsymbol{E})\det(\boldsymbol{A})$.

Elementary Row Operation

- Let $\boldsymbol{A}$ be a square matrix of order 3.
  - For any elementary matrix $\boldsymbol{E}$ of order 3,
    - $\det(\boldsymbol{EA}) = \det(\boldsymbol{E})\det(\boldsymbol{A})$. → 0, -1 or 1.

  This property can be used to find $\det(\boldsymbol{A})$.
- Let $\boldsymbol{R}$ be the reduced row-echelon form of $\boldsymbol{A}$.
  - Then $\boldsymbol{R} = \boldsymbol{E}_k(\boldsymbol{E}_{k-1}\cdots\boldsymbol{E}_2\boldsymbol{E}_1\boldsymbol{A})$, $\boldsymbol{E}_i$ elementary.

$$\begin{aligned} \det(\boldsymbol{R}) &= \det(\boldsymbol{E}_k)\det(\boldsymbol{E}_{k-1}\cdots\boldsymbol{E}_2\boldsymbol{E}_1\boldsymbol{A}) \\ &= \cdots\cdots\cdots\cdots\cdots\cdots \\ &= \det(\boldsymbol{E}_k)\det(\boldsymbol{E}_{k-1})\cdots\det(\boldsymbol{E}_2)\det(\boldsymbol{E}_1\boldsymbol{A}) \\ &= \det(\boldsymbol{E}_k)\det(\boldsymbol{E}_{k-1})\cdots\det(\boldsymbol{E}_2)\det(\boldsymbol{E}_1)\det(\boldsymbol{A}). \end{aligned}$$

split.

If $\boldsymbol{A}$ is **invertible**, $\boldsymbol{R} = \boldsymbol{I}$, and $\det(\boldsymbol{R}) = 1$.

  - $\det(\boldsymbol{A}) = [\det(\boldsymbol{E}_k)\det(\boldsymbol{E}_{k-1})\cdots\det(\boldsymbol{E}_2)\det(\boldsymbol{E}_1)]^{-1}$

Elementary Row Operation

- Let $\boldsymbol{A}$ be a square matrix of order 3.
  - For any elementary matrix $\boldsymbol{E}$ of order 3,
    - $\det(\boldsymbol{EA}) = \det(\boldsymbol{E})\det(\boldsymbol{A})$.

  This property can be used to find $\det(\boldsymbol{A})$.
- Let $\boldsymbol{R}$ be the reduced row-echelon form of $\boldsymbol{A}$.

  $c \neq 0$

  -1
  - Then $\boldsymbol{R} = \boldsymbol{E}_k \boldsymbol{E}_{k-1} \cdots \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{A}$, $\boldsymbol{E}_i$ elementary.

$$0 = \det(\boldsymbol{R}) = \overset{\neq 0}{\det(\boldsymbol{E}_k)}\,\overset{\neq 0}{\det(\boldsymbol{E}_{k-1})} \cdots \overset{\neq 0}{\det(\boldsymbol{E}_2)}\,\overset{\neq 0}{\det(\boldsymbol{E}_1)}\,\overset{= 0}{\det(\boldsymbol{A})}.$$

  If $\boldsymbol{A}$ is **singular**, then the last row of $\boldsymbol{R}$ is zero.
  - $\boldsymbol{R} \xrightarrow{2R_3} \boldsymbol{R} \Rightarrow 2\det(\boldsymbol{R}) = \det(\boldsymbol{R}) \Rightarrow \det(\boldsymbol{R}) = 0$.

  Note that $\det(\boldsymbol{E}) \neq 0$ for any elementary matrix $\boldsymbol{E}$.
  - We must have $\det(\boldsymbol{A}) = 0$.

Example

- Let $\boldsymbol{A} = \begin{pmatrix} -3 & -2 & 4 \\ 4 & 3 & 1 \\ 0 & 2 & 4 \end{pmatrix}$.

$$\boldsymbol{A} \xrightarrow[\boldsymbol{E}_1]{R_2+\frac{4}{3}R_1} \bullet \xrightarrow[\boldsymbol{E}_2]{R_3+(-6)R_2} \bullet \xrightarrow[\boldsymbol{E}_3]{-\frac{1}{3}R_1} \bullet \xrightarrow[\boldsymbol{E}_4]{3R_2} \bullet$$
$$\xrightarrow[\boldsymbol{E}_5]{-\frac{1}{34}R_3} \bullet \xrightarrow[\boldsymbol{E}_6]{R_1+\frac{4}{3}R_3} \bullet \xrightarrow[\boldsymbol{E}_7]{R_2+(-19)R_3} \bullet \xrightarrow[\boldsymbol{E}_8]{R_1+(-\frac{2}{3}R_2)} \boldsymbol{I}.$$

  - $\det(\boldsymbol{E}_i) = 1$ if $i = 1, 2, 6, 7, 8$.
  - $\det(\boldsymbol{E}_3) = -\frac{1}{3}$, $\det(\boldsymbol{E}_4) = 3$, $\det(\boldsymbol{E}_5) = -\frac{1}{34}$.

  $\det(\boldsymbol{A}) = [\det(\boldsymbol{E}_1) \cdots \det(\boldsymbol{E}_8)]^{-1} = (\frac{1}{34})^{-1} = 34.$
- We will show that in order to find $\det(\boldsymbol{A})$ it suffices to use the Gaussian elimination to get a row-echelon form of $\boldsymbol{A}$.

Determinant

- **Definition.** Let $\boldsymbol{A} = (a_{ij})_{n\times n}$. Its **determinant** is:
  - If $n = 1$, define $\det(\boldsymbol{A}) = a_{11}$.
  - If $n > 1$, let A_{ij} be its (i, j)-cofactor, define ← rewrite defn.
    - $\underbrace{\det(\boldsymbol{A})}_{n} = a_{11}\underbrace{A_{11}}_{(-1)^{1+1}\det(\underbrace{M_{11}}_{n-1})} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}$.

- **Example.** Let $\boldsymbol{A} = (a_{ij})_{4\times 4} = \begin{pmatrix} a_{11} & \cancel{a_{12}} & \cancel{a_{13}} & \cancel{a_{14}} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$.

$(-1)^{1+1}$

$$\det(\boldsymbol{A}) = a_{11}\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix}$$
$$+ a_{13}\begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} - a_{14}\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}.$$

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Determinant

- **Example.** Find $\det(\boldsymbol{C})$ if $\boldsymbol{C} = \begin{pmatrix} 0 & -1 & 2 & 0 \\ 2 & -3 & 3 & -2 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 2 & -1 \end{pmatrix}$.

$n\times n \Rightarrow$ sum of $\underline{\alpha_1 \ldots \alpha_n}$ $n!$

we need to calculate

$$\det(\boldsymbol{C}) = 0\cdot\begin{vmatrix} -3 & 3 & -2 \\ 2 & 4 & 0 \\ 0 & 2 & -1 \end{vmatrix} - (-1)\cdot\begin{vmatrix} 2 & 3 & -2 \\ 0 & 4 & 0 \\ 0 & 2 & -1 \end{vmatrix}$$
$$+ 2\cdot\begin{vmatrix} 2 & -3 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{vmatrix} - 0\cdot\begin{vmatrix} 2 & -3 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 2 \end{vmatrix}$$
$$= 0\cdot 10 - (-1)\cdot(-8) + 2\cdot(-4) - 0\cdot 8 = -16.$$

- **Warning**: The "diagonal expansion" of $\det(\boldsymbol{A})$ for 2×2 or 3×3 matrices is no longer valid if the order ≥ 4.

aka broken diagonal.

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Properties

- **Theorem.** $\det(\boldsymbol{I}) = 1$. For any square matrices,
  - If $\boldsymbol{A} \xrightarrow{cR_i} \boldsymbol{B}$, then $\det(\boldsymbol{B}) = c\det(\boldsymbol{A})$.
    - In particular, If $\boldsymbol{I} \xrightarrow{cR_i} \boldsymbol{E}$, then $\det(\boldsymbol{E}) = c$.
  - If $\boldsymbol{A} \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{B}$, then $\det(\boldsymbol{B}) = -\det(\boldsymbol{A})$.
    - In particular, if $\boldsymbol{I} \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{E}$, then $\det(\boldsymbol{E}) = -1$.
  - If $\boldsymbol{A} \xrightarrow{R_i + cR_j} \boldsymbol{B}$, then $\det(\boldsymbol{B}) = \det(\boldsymbol{A})$.
    - In particular, if $\boldsymbol{I} \xrightarrow{R_i + cR_j} \boldsymbol{E}$, then $\det(\boldsymbol{E}) = 1$.

proven by Induction

- **Theorem.** Let $\boldsymbol{A}$ be a square matrix.
  - For any elementary matrix $\boldsymbol{E}$ of the same order,
    - $\det(\boldsymbol{EA}) = \det(\boldsymbol{E})\det(\boldsymbol{A})$.

Can use this one to determine invertibility of sqr matrix.

Properties

- **Theorem.** Suppose a square matrix $\boldsymbol{A}$ has a zero row.
  - Then $\det(\boldsymbol{A}) = 0$.

  Proof. Suppose the ith row of square matrix $\boldsymbol{A}$ is $\boldsymbol{0}$.
  - $\boldsymbol{A} \xrightarrow{2R_i} \boldsymbol{A} \Rightarrow \det(\boldsymbol{A}) = 2\det(\boldsymbol{A}) \Rightarrow \det(\boldsymbol{A}) = 0$.

- Suppose square matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are row equivalent.
  - There exist elementary matrices $\boldsymbol{E}_1, \boldsymbol{E}_2, \dots, \boldsymbol{E}_k$ s.t.
    - $\boldsymbol{B} = \boldsymbol{E}_k \cdots \boldsymbol{E}_2\boldsymbol{E}_1\boldsymbol{A}$.

    $\det(\boldsymbol{B}) = \det(\boldsymbol{E}_k) \cdots \det(\boldsymbol{E}_2)\det(\boldsymbol{E}_1)\det(\boldsymbol{A})$.

  Note that $\det(\boldsymbol{E}) \neq 0$ for every elementary matrix $\boldsymbol{E}$.

  c, -1, 1 never 0.
  - $\det(\boldsymbol{A}) = 0 \Leftrightarrow \det(\boldsymbol{B}) = 0$.
  - Equivalently, $\det(\boldsymbol{A}) \neq 0 \Leftrightarrow \det(\boldsymbol{B}) \neq 0$.

Properties

- **Theorem.** Suppose a square matrix $\boldsymbol{A}$ has a **zero row**.
  - Then $\det(\boldsymbol{A}) = 0$.

Proof. Suppose the ith row of square matrix $\boldsymbol{A}$ is $\mathbf{0}$.

- $\boldsymbol{A} \xrightarrow{2R_i} \boldsymbol{A} \Rightarrow \det(\boldsymbol{A}) = 2\det(\boldsymbol{A}) \Rightarrow \det(\boldsymbol{A}) = 0$.

- **Theorem.** $\det(\boldsymbol{A}) = 0 \Leftrightarrow \boldsymbol{A}$ is **singular**. **.
  - Equivalently, $\det(\boldsymbol{A}) \neq 0 \Leftrightarrow \boldsymbol{A}$ is **invertible**. **

Proof. Suppose $\boldsymbol{A}$ is invertible. Then

- $\boldsymbol{A}$ is row equivalent to $\boldsymbol{I}$.
  - $\det(\boldsymbol{I}) = 1 \neq 0 \Rightarrow \det(\boldsymbol{A}) \neq 0$.

Suppose $\boldsymbol{A}$ is singular. Then the RREF of $\boldsymbol{A}$ is not $\boldsymbol{I}$.

- The RREF of $\boldsymbol{A}$ has a zero row $\Rightarrow \det(\boldsymbol{A}) = 0$.

Properties

- **Theorem.** Let $\boldsymbol{A}, \boldsymbol{B}$ be square matrices of the same size.
  - Then $\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B})$.

Proof. Suppose that $\boldsymbol{A}$ is invertible. Then

- $\boldsymbol{A} = \boldsymbol{E}_1\boldsymbol{E}_2\cdots\boldsymbol{E}_k$ for elementary matrices $\boldsymbol{E}_i$.

$$
\begin{aligned}
\det(\boldsymbol{AB}) &= \det(\boldsymbol{E}_1\boldsymbol{E}_2\cdots\boldsymbol{E}_k\boldsymbol{B}) \\
&= \det(\boldsymbol{E}_1)\det(\boldsymbol{E}_2)\cdots\det(\boldsymbol{E}_k)\det(\boldsymbol{B}) \\
&= \det(\boldsymbol{E}_1\boldsymbol{E}_2\cdots\boldsymbol{E}_k)\det(\boldsymbol{B}) \\
&= \det(\boldsymbol{A})\det(\boldsymbol{B}).
\end{aligned}
$$

operation apply to I.

Suppose that $\boldsymbol{A}$ is singular. Then $\boldsymbol{AB}$ is singular.

show that

Suppose AB is not singular, AB is invertible then $(AB)C = I$

- $\det(\boldsymbol{A}) = 0$ and $\det(\boldsymbol{AB}) = 0$.

Then $\det(\boldsymbol{AB}) = 0 = \det(\boldsymbol{A})\det(\boldsymbol{B})$.

$A(BC) = I \Rightarrow A^{-1} = B$.

$\therefore$ Contradiction

Properties

- **Theorem.** For any square $\boldsymbol{A}$, $\det(\boldsymbol{A}) = \det(\boldsymbol{A}^{\mathrm{T}})$.

  Proof. Suppose $\boldsymbol{A}$ is singular. A is invertible then A^T is invertible $\Rightarrow (A^T)^{-1} = (A^{-1})^T$.

  ○ Then $\boldsymbol{A}^{\mathrm{T}}$ is also singular, because

  - $\boldsymbol{A}^{\mathrm{T}}$ is invertible $\Rightarrow \boldsymbol{A} = (\boldsymbol{A}^{\mathrm{T}})^{\mathrm{T}}$ is invertible.

  For this case, $\det(\boldsymbol{A}) = 0 = \det(\boldsymbol{A}^{\mathrm{T}})$. Contradiction.

  Suppose $\boldsymbol{A}$ is invertible. Then

  ○ $\boldsymbol{A} = \boxed{\boldsymbol{E}_1\boldsymbol{E}_2\cdots\boldsymbol{E}_k}$ for elementary matrices $\boldsymbol{E}_i$.

  Note that $\det(\boldsymbol{E}) = \det(\boldsymbol{E}^{\mathrm{T}})$ for elementary matrix $\boldsymbol{E}$.

  Type 1 det ≠ 0 $E = E^T$. Type 2 $R_i \leftrightarrow R_j$ det = −1 $E = E^T$. Type 3 (i,j) $R_i + cR_j$, (j,i) $R_j + cR_i$ det = 1.

  reversed order

$$
\begin{aligned}
\det(\boldsymbol{A}^{\mathrm{T}}) &= \det(\boldsymbol{E}_k^{\mathrm{T}}\cdots\boldsymbol{E}_2^{\mathrm{T}}\boldsymbol{E}_1^{\mathrm{T}}) \\
&= \det(\boldsymbol{E}_k^{\mathrm{T}})\cdots\det(\boldsymbol{E}_2^{\mathrm{T}})\det(\boldsymbol{E}_1^{\mathrm{T}}) \quad \text{multiplicative} \\
&= \det(\boldsymbol{E}_k)\cdots\det(\boldsymbol{E}_2)\det(\boldsymbol{E}_1) \\
&= \det(\boldsymbol{E}_1)\det(\boldsymbol{E}_2)\cdots\det(\boldsymbol{E}_k) \\
&= \det(\boldsymbol{E}_1\boldsymbol{E}_2\cdots\boldsymbol{E}_k) = \det(\boldsymbol{A}). \quad \square
\end{aligned}
$$

  each of the det's is a order. ∴ can rearrange

Properties

- **Theorem.** Suppose $\boldsymbol{A} = (a_{ij})_{n\times n}$ is **upper triangular**. lower triangular

  ○ Then $\det(\boldsymbol{A}) = a_{11}a_{22}\cdots a_{nn}$. product of diagonal,

- **Example.**

  ○ $\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 6 & 7 & 8 & 9 \\ 0 & 0 & 10 & 11 & 12 \\ 0 & 0 & 0 & 13 & 14 \\ 0 & 0 & 0 & 0 & 15 \end{vmatrix} = 1\cdot 6\cdot 10\cdot 13\cdot 15 = 11700.$

- **Remarks.**

  ○ The result is also true for **lower triangular matrices**. take transpose.

  ○ Note that a row-echelon form of a square matrix is always upper triangular.

  - To find the determinant using elementary row operation, it suffices to use Gaussian elimination to get a row-echelon form.

Properties

- **Theorem.** Suppose $\boldsymbol{A} = (a_{ij})_{n\times n}$ is **upper triangular**.
  - Then $\det(\boldsymbol{A}) = a_{11}a_{22}\cdots a_{nn}$. non-zero

Proof. Suppose that $a_{ii} \neq 0$ for all $i = 1, \dots, n$.

- Then $\boldsymbol{A}$ is in row-echelon form. To get its RREF,

G-J

1. Multiply the ith row by a_{ii}^{-1}
2. Starting from the last row, add a suitable multiple of the row to every row above so that every entry above the pivot point is 0.

- $\boldsymbol{A} \xrightarrow[i=1,\dots,n]{a_{ii}^{-1}R_i} \boldsymbol{B} \xrightarrow{R_i + c_{ij}R_j} \boldsymbol{R} = \boldsymbol{I}$.

divide all these

  - $1 = \det(\boldsymbol{I}) = \det(\boldsymbol{B}) = a_{11}^{-1}\cdots a_{nn}^{-1}\det(\boldsymbol{A})$.

  Therefore, $\det(\boldsymbol{A}) = a_{11}a_{22}\cdots a_{nn}$.

Properties

- **Theorem.** Suppose $\boldsymbol{A} = (a_{ij})_{n\times n}$ is **upper triangular**.
  - Then $\det(\boldsymbol{A}) = a_{11}a_{22}\cdots a_{nn}$. some entry is 0. (otherwise).

Proof. Suppose that $a_{ii} = 0$ for some i.

- Illustration: Assume that $a_{22} = 0$ but $a_{33} \neq 0$, $a_{44} \neq 0$.

$$\begin{pmatrix} * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & \otimes & * \\ 0 & 0 & 0 & \otimes \end{pmatrix} \xrightarrow[\text{to the 3rd and 4th rows}]{\text{row operations}} \begin{pmatrix} * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[\text{to the 2nd row}]{\text{add multiples of 3rd and 4th rows}} \begin{pmatrix} * & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$\boldsymbol{A}$ is row equivalent to a singular matrix; $\det(\boldsymbol{A}) = 0$.

Properties

- **Theorem.** Suppose $\boldsymbol{A} = (a_{ij})_{n\times n}$ is **upper triangular**.
  - Then $\det(\boldsymbol{A}) = a_{11}a_{22}\cdots a_{nn}$.

Proof. Suppose that $a_{ii} = 0$ for some i. $a_{ii} = 0.$

- Aim: $\boldsymbol{A}$ is row equivalent to a matrix with a zero row: non-zero
  1. Suppose that $a_{jj} \neq 0$ for all $j = i+1, \ldots, n$.
  2. Then the submatrix $\boldsymbol{B}$ by taking the $(i+1)$th to nth rows and $(i+1)$th to nth columns is an upper triangular matrix with nonzero diagonal entries. B $n-i$
  3. Apply row operations to only the last $n-i$ rows of $\boldsymbol{A}$ to reduce $\boldsymbol{B}$ to its RREF $\boldsymbol{I}_{n-i}$.
  4. Add suitable multiples of the last $n-i$ rows to the ith row, so that the ith row becomes 0.
- I Therefore, $\det(\boldsymbol{A}) = 0 = a_{11}\cdots a_{ii}\cdots a_{nn}$.

Cofactor Expansion

- **Theorem.** Let $\boldsymbol{A}$ be a square matrix of order n.
  - Let A_{ij} denote the (i,j)-cofactor of $\boldsymbol{A}$.

  Then for any i and j,
  - $\det(\boldsymbol{A}) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$.
  - $\det(\boldsymbol{A}) = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}$. jth col.

Proof. Fix i. Let $\boldsymbol{B} = (b_{ij})$ be obtained from $\boldsymbol{A}$ by

- Moving the ith row to the top. Then $b_{1j} = a_{ij}$.
  - $$\boldsymbol{A} = \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_{i-1} \\ R_i \\ \vdots \end{pmatrix} \xrightarrow[\ldots, R_2 \leftrightarrow R_1]{R_i \leftrightarrow R_{i-1}, R_{i-1} \leftrightarrow R_{i-2},} \begin{pmatrix} R_i \\ R_1 \\ \vdots \\ R_{i-2} \\ R_{i-1} \\ \vdots \end{pmatrix} = \boldsymbol{B}$$

    b_{1j}

    interchange $i-1$ times

Cofactor Expansion

- **Proof.** Fix i. Let $\boldsymbol{B} = (b_{ij})$ be obtained from $\boldsymbol{A}$ by   ith row expansion
  - Moving the ith row to the top. Then $b_{1j} = a_{ij}$.
    - $\boldsymbol{A} = \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_{i-1} \\ R_i \\ \vdots \end{pmatrix} \xrightarrow[\ldots, R_2 \leftrightarrow R_1]{R_i \leftrightarrow R_{i-1}, R_{i-1} \leftrightarrow R_{i-2},} \begin{pmatrix} R_i \\ R_1 \\ \vdots \\ R_{i-2} \\ R_{i-1} \\ \vdots \end{pmatrix} = \boldsymbol{B}$

      ith row   i - 1 times   1st row
  - Then $\det(\boldsymbol{A}) = (-1)^{i-1} \det(\boldsymbol{B})$.

  The submatrix obtained from $\boldsymbol{B}$ by deleting its 1st row and jth column is the same as that obtained from $\boldsymbol{A}$ by deleting its ith row and jth column, say $\boldsymbol{M}_{ij}$.
  - Let B_{ij} be the (i, j)-cofactor of $\boldsymbol{B}$.
    - $B_{1j} = (-1)^{1+j} \det(\boldsymbol{M}_{ij})$, $A_{ij} = (-1)^{i+j} \det(\boldsymbol{M}_{ij})$.

    So $B_{1j} = (-1)^{i-1} A_{ij}$, $j = 1, \ldots, n$.

Cofactor Expansion

- **Proof.** Fix i. Let $\boldsymbol{B} = (b_{ij})$ be obtained from $\boldsymbol{A}$ by
  - Moving the ith row to the top.

  Then $b_{1j} = a_{ij}$, $B_{1j} = (-1)^{i-1} A_{ij}$, $j = 1, \ldots, n$.

$$
\begin{aligned}
\det(\boldsymbol{A}) &= (-1)^{i-1} \det(\boldsymbol{B}) \\
&= (-1)^{i-1} \cdot (b_{11} B_{11} + b_{12} B_{12} + \cdots + b_{1n} B_{1n}) \\
&= (-1)^{i-1} \cdot [a_{i1} (-1)^{i-1} A_{i1} + a_{i2} (-1)^{i-1} A_{i2} \\
&\qquad + \cdots + a_{in} (-1)^{i-1} A_{in}] \\
&= (-1)^{2i-2} (a_{i1} A_{i1} + a_{i2} A_{i2} + \cdots + a_{in} A_{in}) \\
&= a_{i1} A_{i1} + a_{i2} A_{i2} + \cdots + a_{in} A_{in}.
\end{aligned}
$$

expansion of det(A) along ith row

  The is called the **cofactor expansion along the ith row**.
- The proof for the **cofactor expansion along the jth column** is left as exercises. (*Hint*: Consider $\boldsymbol{A}^{\mathrm{T}}$.)

det.
find det. — def. expansion along any row/col
effect of apply E upper triangular via REF.

Cofactor Expansion

- In evaluating the determinant using cofactor expansion,
  - expand along the row or column with the most zeros.
- **Example.** $A = \begin{pmatrix} 0 & -1 & 2 & 0 \\ \boxed{2} & -3 & 3 & -2 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 2 & -1 \end{pmatrix}$.

M_{ij} — del ith row jth col.

$A_{ij} = (-1)^{i+j} \det(M_{ij})$

$\det(A) = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}$. } expand along row

$\det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$. } or col

← most num of zeros.

$$\det(A) = \overset{0}{a_{11}}A_{11} + a_{21}A_{21} + \overset{0}{a_{31}}A_{31} + \overset{0}{a_{41}}A_{41}$$

$$= 2 \cdot (-1)^{2+1} \begin{vmatrix} -1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 2 & -1 \end{vmatrix}$$

$$= -2 \cdot (-1) \cdot (-1)^{3+3} \boxed{\begin{vmatrix} -1 & 2 \\ 2 & 4 \end{vmatrix}}$$

$$= \boxed{-16.}$$

$b_{13}B_{13} + b_{23}B_{23} + b_{33}B_{33}$.
0 0

Finding Determinant

- Find $\det(A)$ if A is a square matrix of order n.
  - If A has a zero row/column, then $\det(A) = 0$.
  - If A is triangular, $\det(A) = a_{11} \cdots a_{nn}$. product of diagonal entries.
  - Suppose that A is not triangular. others.
    - Use REF. If $n = 2$, use formula $\det(A) = a_{11}a_{22} - a_{12}a_{21}$.
    - If a row/coln has many 0, use cofactor expansion. → along that row/col.
    - Otherwise, use ele. row operations to get REF:
      - $\det(EA) = \det(E)\det(A)$. via gaussian elimination
- Note the following formulas (exercises for the last two): → either same or negative (depends on how many time ⟺ interchanging row is used).
  - $\det(A) = \det(A^{\mathrm{T}})$.
  - $\det(AB) = \det(A)\det(B)$.
  - $\det(cA) = c^n \det(A)$, where A is $n \times n$. * $n=4$ $\det(-A) = (-1)^4 \det(A)$ need multiply each row by c.
  - $\det(A^{-1}) = \det(A)^{-1}$ if A is invertible.

$A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \xrightarrow{cR_1} \begin{pmatrix} ca_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \xrightarrow{cR_2} \begin{pmatrix} ca_1 \\ ca_2 \\ \vdots \\ a_n \end{pmatrix} \to \cdots \xrightarrow{cR_n} cA.$

Examples

- Find $\det(\boldsymbol{A})$, where $\boldsymbol{A} = \begin{pmatrix} 3 & -1 & 1 & 1 \\ 3 & -1 & 2 & 1 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 2 & -1 \end{pmatrix}$.

$$\begin{pmatrix} 3 & -1 & 1 & 1 \\ 3 & -1 & 2 & 1 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 2 & -1 \end{pmatrix} \xrightarrow[\boldsymbol{E}_1]{R_2+(-1)R_1} \begin{pmatrix} 3 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 2 & -1 \end{pmatrix}$$

det 1

$$\xrightarrow[\boldsymbol{E}_2]{R_2 \leftrightarrow R_3} \begin{pmatrix} 3 & -1 & 1 & 1 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 \end{pmatrix} \xrightarrow[\boldsymbol{E}_3]{R_4+(-2)R_3} \begin{pmatrix} 3 & -1 & 1 & 1 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \boldsymbol{B}. = \text{REF}(A)$$

det −1   det 1

- $\det(\boldsymbol{B}) = 3 \cdot 2 \cdot 1 \cdot (-1) = -6$.
  - $\det(\boldsymbol{A}) = [\det(\boldsymbol{E}_1)\det(\boldsymbol{E}_2)\det(\boldsymbol{E}_3)]^{-1}\det(\boldsymbol{B}) = 6$. $= -\det(B)$

how many interchanging row?

Examples

- Find $\det(\boldsymbol{A})$, where $\boldsymbol{A} = \begin{pmatrix} 3 & -1 & 1 & 1 \\ 3 & -1 & 2 & 1 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 2 & -1 \end{pmatrix}$.
  - Expand along the 4th row: $\det(\boldsymbol{A}) = (-1)(-6) = 6$.
    - $2 \cdot (-1)^{4+3} \begin{vmatrix} 3 & -1 & 1 \\ 3 & -1 & 1 \\ 0 & 2 & 0 \end{vmatrix} + (-1) \cdot (-1)^{4+4} \begin{vmatrix} 3 & -1 & 1 \\ 3 & -1 & 2 \\ 0 & 2 & 4 \end{vmatrix}$

      0 row   Cofactor $\therefore$ det = 0.
    - $\begin{vmatrix} 3 & -1 & 1 \\ 3 & -1 & 1 \\ 0 & 2 & 0 \end{vmatrix} \overset{R_2+(-1)R_1}{=\!=\!=\!=} \begin{vmatrix} 3 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{vmatrix} = 0$
    - $\begin{vmatrix} 3 & -1 & 1 \\ 3 & -1 & 2 \\ 0 & 2 & 4 \end{vmatrix} = 3 \cdot \begin{vmatrix} -1 & 2 \\ 2 & 4 \end{vmatrix} - 3 \begin{vmatrix} -1 & 1 \\ 2 & 4 \end{vmatrix} = -6$.

Examples

- Let $\boldsymbol{A} = \begin{pmatrix} -3 & -2 & 4 \\ 4 & 3 & 1 \\ 0 & 2 & 4 \end{pmatrix}$ 3x3 and $\boldsymbol{B} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$.
  - It is given that $\det(\boldsymbol{A}) = 34$ and $\det(\boldsymbol{B}) = -1$.

1. $\det(\boldsymbol{A}^{\mathrm{T}}) = \det(\boldsymbol{A}) = 34$.
2. $\det(2\boldsymbol{A}) = 2^3 \det(\boldsymbol{A}) = 8 \cdot 34 = 272$.
3. $\det(\boldsymbol{A}^{-1}) = [\det(\boldsymbol{A})]^{-1} = \frac{1}{34}$.
4. $\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B}) = 34 \cdot (-1) = -34$.
5. $\det(\boldsymbol{BA}) = \det(\boldsymbol{B})\det(\boldsymbol{A}) = (-1) \cdot 34 = -34$.

number is commutative

Adjoint Matrix

- **Definition.** Let $\boldsymbol{A}$ be a square matrix of order n. The **(classical) adjoint** (or **adjugate**, or **adjunct**) of $\boldsymbol{A}$ is
  - $\boxed{\mathbf{adj}(\boldsymbol{A}) = (A_{ji})_{n\times n}}$

  where A_{ij} is the (i, j)-cofactor of $\boldsymbol{A}$.

$(-1)^{i+j} \det(M_{ij})$ (number).

$(A_{ij})^T = (A_{ji})$

- **Example.** Let $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. Then
  - $A_{11} = a_{22}$, $A_{12} = -a_{21}$, $A_{21} = -a_{12}$, $A_{22} = a_{11}$.
  - $\mathbf{adj}(\boldsymbol{A}) = \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix} = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$.

  transpose of $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$.

  Recall that if $\boldsymbol{A}$ is invertible, then
  - $\boldsymbol{A}^{-1} = \dfrac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$.

  It is conjectured that $\boldsymbol{A}^{-1} = [\det(\boldsymbol{A})]^{-1}\, \mathbf{adj}(\boldsymbol{A})$.

Adjoint Matrix

- **Theorem.** Let $\boldsymbol{A}$ be a square matrix. Then
  - $\boldsymbol{A}[\mathbf{adj}(\boldsymbol{A})] = \det(\boldsymbol{A})\boldsymbol{I}$. $C = \det(A)$.

Proof. Let $\boldsymbol{A}[\mathbf{adj}(\boldsymbol{A})] = (c_{ij})$. Then $\begin{pmatrix} c & & 0 \\ & \ddots & \\ 0 & & c \end{pmatrix}$ scalar

$\mathrm{adj}(A) = (A_{ji}) = (b_{ij})$

- $c_{ij} = a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn}$. expression

Let $i = j$. Then $c_{ii} = a_{i1}A_{i1} + \cdots + a_{in}A_{in} = \det(\boldsymbol{A})$. expansion of det along ith row.

Suppose that $i \neq j$. Let $\boldsymbol{B}$ be the matrix obtained from $\boldsymbol{A}$ by replacing jth row by the ith row.

ith row = jth row $\quad b_{ij} = a_{ij}$

- $\boldsymbol{B} \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{B} \Rightarrow \det(\boldsymbol{B}) = -\det(\boldsymbol{B}) \Rightarrow \det(\boldsymbol{B}) = 0.$

$A: \begin{matrix} a_1 \\ \vdots \\ a_i \\ \vdots \\ a_j \\ \vdots \\ a_n \end{matrix} \longrightarrow \begin{matrix} a_1 \\ \vdots \\ a_i \\ \vdots \\ a_i \\ \vdots \\ a_n \end{matrix} = B$ same row $\therefore \det(B) = 0$

$$\begin{aligned} c_{ij} &= a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} \\ &= b_{j1}B_{j1} + b_{j2}B_{j2} + \cdots + b_{jn}B_{jn} \\ &= \det(\boldsymbol{B}) = 0. \end{aligned}$$

Therefore, $\boldsymbol{A}[\mathbf{adj}(\boldsymbol{A})] = \det(\boldsymbol{A})\boldsymbol{I}$.

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$A \quad i \quad j \qquad B \quad i \quad j \qquad$ same after delete.

Adjoint Matrix

- **Theorem.** Let $\boldsymbol{A}$ be a square matrix. Then
  - $\boldsymbol{A}[\mathbf{adj}(\boldsymbol{A})] = \det(\boldsymbol{A})\boldsymbol{I}$.
  - $[\mathbf{adj}(\boldsymbol{A})]\boldsymbol{A} = \det(\boldsymbol{A})\boldsymbol{I}$. (Exercise!)

  If $\boldsymbol{A}$ is invertible, then $\boldsymbol{A}^{-1} = \dfrac{1}{\det(\boldsymbol{A})}\mathbf{adj}(\boldsymbol{A})$.
- **Exercises.** Let $\boldsymbol{A}, \boldsymbol{B}$ be invertible matrices of order n.
  - Find $[\mathbf{adj}(\boldsymbol{A})]^{-1}$ and $\mathbf{adj}(\boldsymbol{A}^{-1})$.
  - Find $\det(\mathbf{adj}(\boldsymbol{A}))$ and $\mathbf{adj}(\mathbf{adj}(\boldsymbol{A}))$.
  - Prove that $\mathbf{adj}(\boldsymbol{AB}) = \mathbf{adj}(\boldsymbol{B})\,\mathbf{adj}(\boldsymbol{A})$.
- **Challenging Problems.** Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are not necessarily invertible.
  - Find $\det(\mathbf{adj}(\boldsymbol{A}))$ and $\mathbf{adj}(\mathbf{adj}(\boldsymbol{A}))$.
  - Is it true that $\mathbf{adj}(\boldsymbol{AB}) = \mathbf{adj}(\boldsymbol{B})\,\mathbf{adj}(\boldsymbol{A})$?

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Example

- Let $\boldsymbol{A} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 3 \end{pmatrix}$. $\boxed{\det(\boldsymbol{A})} = (-1)\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = -2.$

$$\mathbf{adj}(\boldsymbol{A}) = \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}$$

$$= \begin{pmatrix} \begin{vmatrix} -1 & 0 \\ 0 & 3 \end{vmatrix} & -\begin{vmatrix} -1 & 1 \\ 0 & 3 \end{vmatrix} & \begin{vmatrix} -1 & 1 \\ -1 & 0 \end{vmatrix} \\ -\begin{vmatrix} 0 & 0 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} \\ \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 0 & -1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} -3 & 3 & 1 \\ 0 & 2 & 0 \\ 1 & -1 & -1 \end{pmatrix}$$

$$\boldsymbol{A}^{-1} = \boxed{\frac{1}{\det(\boldsymbol{A})}}\mathbf{adj}(\boldsymbol{A}) = \begin{pmatrix} \frac{3}{2} & -\frac{3}{2} & -\frac{1}{2} \\ 0 & -1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Cramer's Rule

$x = \left[\frac{1}{\det(A)}\right] \cdot [\underline{\operatorname{adj}(A)b}].$

$B = \operatorname{adj}(A)$

- Let $\boldsymbol{A} = (a_{ij})_{n\times n}$ be an invertible matrix.
  - The linear system $\boldsymbol{Ax} = \boldsymbol{b}$ has a unique solution
    - $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$.

  Recall that $\boldsymbol{A}^{-1} = \boxed{\frac{1}{\det(\boldsymbol{A})}}[\mathbf{adj}(\boldsymbol{A})]$. Let $\boldsymbol{b} = (b_i)_{n\times 1}$.

  - $x_j = \frac{1}{\det(\boldsymbol{A})}(A_{1j}b_1 + A_{2j}b_2 + \cdots + A_{nj}b_n)$.

Co factor expansion of A_j along j column.

$x_j = b_{j1}b_1 + b_{j2}b_2 + \cdots + b_{jn}b_n$

$b_{ij} = A_{ji}$ (j, i) cofactor

  Fix j, and let $\boldsymbol{A}_j$ be the matrix obtained by replacing the jth column of $\boldsymbol{A}$ by $\boldsymbol{b}$. Then

  - b_i is the (i, j)-entry of $\boldsymbol{A}_j$.
  - A_{ij} is the (i, j)-cofactor of $\boldsymbol{A}_j$.

$A = (a_1, \ldots a_j \ldots a_n)$

$A_j = (a_1 \ldots b \ldots a_n)$

  Therefore, $x_j = \frac{\det(\boldsymbol{A}_j)}{\det(\boldsymbol{A})}$, $j = 1, \ldots, n$.

Cramer's Rule

- **Cramer's Rule.** Let $\boldsymbol{A}$ be an invertible matrix of order n.
  - For every column matrix $\boldsymbol{b}$ of size $n \times 1$, the linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ has a unique solution
    - $\boldsymbol{x} = \dfrac{1}{\det(\boldsymbol{A})}\begin{pmatrix}\det(\boldsymbol{A}_1)\\ \vdots \\ \det(\boldsymbol{A}_n)\end{pmatrix}$, column matrix.

    $\boldsymbol{A}_j$ is obtained from $\boldsymbol{A}$ by replacing its jth coln by $\boldsymbol{b}$.
- **Example.** Let $\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.
  - Suppose that $\boldsymbol{A}$ is invertible. $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ implies
    - $\boldsymbol{x} = \dfrac{1}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{vmatrix}}\begin{pmatrix}\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22}\end{vmatrix} A_1 \\ \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2\end{vmatrix} A_2 .\end{pmatrix}$

    may not need all x_1 and x_2, ...

Example

must be invertible.

- $\begin{pmatrix} 1 & 1 & 3 \\ 2 & -2 & 2 \\ 3 & 9 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix}$. $\begin{vmatrix} 1 & 1 & 3 \\ 2 & -2 & 2 \\ 3 & 9 & 0 \end{vmatrix} = 60.$
  - $x = \dfrac{\begin{vmatrix} 0 & 1 & 3 \\ 4 & -2 & 2 \\ 3 & 9 & 0 \end{vmatrix}}{60} = \dfrac{132}{60} = 2.2$
  - $y = \dfrac{\begin{vmatrix} 1 & 0 & 3 \\ 2 & 4 & 2 \\ 3 & 3 & 0 \end{vmatrix}}{60} = \dfrac{-24}{60} = -0.4$
  - $z = \dfrac{\begin{vmatrix} 1 & 1 & 0 \\ 2 & -2 & 4 \\ 3 & 9 & 3 \end{vmatrix}}{60} = \dfrac{-36}{60} = -0.6$