Notes

Application of integration

Washer method: $V = \int_a^b \pi f^2(x)\, dx$ (rotation about x-axis).

Cylindrical method: $V = \int_a^b 2\pi x f(x)\, dx$ (rotation about y-axis)

Arc length: $L = \int_a^b \sqrt{1+(f'(x))^2}\, dx$

Surface area of revolution: $S = \int_a^b 2\pi f(x) \sqrt{1+(f'(x))^2}\, dx$.

Sequences and Series

Squeeze theorem: If $a_n \le b_n \le c_n$ for all n, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Important limits:

$\lim_{n\to\infty} (1+\frac{v}{n})^n = e^v$

$\lim_{n\to\infty} v^{\frac{1}{n}} = 1$

$\lim_{n\to\infty} \frac{v^n}{n!} = 0$

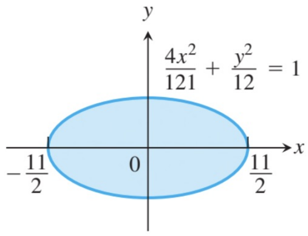
$\lim_{n\to\infty} v^n = 0$, $|v|<1$.

Geometric series: $\sum_{n=1}^{\infty} ar^{n-1}$ $(a \neq 0)$ converges to $\frac{a}{1-r}$ when $|r|<1$, divergent when $|r|>1$.

Telescopic sum: $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \to \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$

Application of Integration   60 mins

16. Volume of a football The profile of a football resembles the ellipse shown here. Find the football's volume to the nearest cubic inch.



$$\frac{4x^2}{121} + \frac{y^2}{12} = 1$$

$$\frac{y^2}{12} = 1 - \frac{4x^2}{121}$$

$$y^2 = 12 - \frac{48}{121} x^2.$$

$$y = \pm\sqrt{12 - \frac{48}{121}x^2}.$$

$$\therefore \text{Volume} = 2\pi \int_0^{\frac{11}{2}} \left(\sqrt{12 - \frac{48}{121}x^2}\right)^2 dx.$$

$$= 2\pi \int_0^{\frac{11}{2}} 12 - \frac{48}{121}x^2 \, dx$$

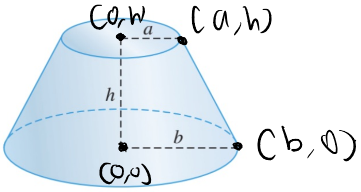
$$= 2\left(12\pi x \Big|_0^{\frac{11}{2}} - \left(\frac{48}{121}\pi \frac{x^3}{3}\Big|_0^{\frac{11}{2}}\right)\right)$$

$$= 24\pi \cdot \frac{11}{2} - 2 \cdot \frac{48}{121}\pi \frac{\left(\frac{11}{2}\right)^3}{3}$$

$$= 132\pi - 44\pi$$

$$= 88\pi \,//$$

17. Find the volume of the given circular frustum of height h and radii a and b.



eqn of line:

$$(y-0) = \frac{h-0}{a-b}(x-b).$$

$$y = \frac{h}{a-b}(x-b)$$

Using cylindrical method

$$\text{Volume} = \int_0^h 2\pi x\left(\frac{h}{a-b}(x-b)\right)dx$$

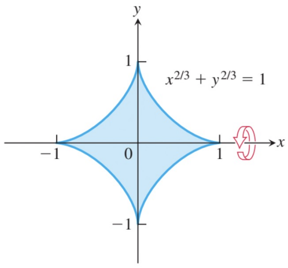
$$= 2\pi\frac{h}{a-b}\int_0^h x^2 - bx \, dx.$$

$$= 2\pi\frac{h}{a-b}\left(\frac{x^3}{3} - \frac{bx^2}{2}\right)\Big|_0^h$$

$$= 2\pi\frac{h}{a-b}\left(\frac{h^3}{3} - \frac{bh^2}{2}\right).$$

//

18. The graph of $x^{2/3} + y^{2/3} = 1$ is called an astroid and is given below. Find the volume of the solid formed by revolving the region enclosed by the astroid about the x-axis.



$$y^{2/3} = 1 - x^{2/3}$$

$$y = \left(1 - x^{2/3}\right)^{3/2}$$

$$\text{Volume} = 2\int_0^1 \pi \left(1 - x^{2/3}\right)^3 dx$$

$$(1-x^{2/3})^3$$
$$= (1-x^{2/3})(1-x^{2/3}-x^{2/3}+x^{2/3}x^{2/3})$$
$$= (1-2x^{2/3}+x^{4/3})(1-x^{2/3})$$
$$= (1-2x^{2/3}+x^{4/3} - x^{2/3}+2x^{4/3}-x^2)$$

$$= 2\pi \int_0^1 1 - 3x^{2/3} + 3x^{4/3} - x^2 \, dx \qquad = (1 - 3x^{2/3} + 3x^{4/3} - x^2)$$

$$= 2\pi \left(x\Big|_0^1 - 3\,\frac{x^{\frac{2}{3}+1}}{\frac{2}{3}+1}\Big|_0^1 + 3\,\frac{x^{\frac{4}{3}+1}}{\frac{4}{3}+1}\Big|_0^1 - \frac{x^3}{3}\Big|_0^1 \right)$$

$$= 2\pi - \frac{9}{5}\cdot 2\pi + \frac{9}{7}\cdot 2\pi - \frac{1}{3}\cdot 2\pi.$$

$$= \frac{32}{105}\pi. \;//$$

Lengths of Curves

Find the lengths of the curves in Exercises 19–22.

19. $y = x^{1/2} - (1/3)x^{3/2}, \quad 1 \le x \le 4$

20. $x = y^{2/3}, \quad 1 \le y \le 8$

21. $y = (5/12)x^{6/5} - (5/8)x^{4/5}, \quad 1 \le x \le 32$

22. $x = (y^3/12) + (1/y), \quad 1 \le y \le 2$

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

19. $y = x^{1/2} - (1/3)x^{3/2}, \quad 1 \le x \le 4.$

$$y = x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{3}{2}}$$

$$y' = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{\cancel{3}}\cdot\frac{\cancel{3}}{2}x^{\frac{1}{2}}$$

$$= \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{\frac{1}{2}}.$$

$$y'^2 = \left(\frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{\frac{1}{2}}\right)^2.$$

$$= \frac{1}{4}\left(\frac{1}{x} - 2 + x\right)$$

$$\int_1^4 \sqrt{1 + \frac{1}{4}\left(\frac{1}{x} - 2 + x\right)}\, dx$$

$$= \int_1^4 \sqrt{\frac{1}{4}\left(\frac{1}{x} + 2 + x\right)}\, dx$$

$$= \int_1^4 \sqrt{\frac{1}{4}\left(x^{-\frac{1}{2}} + x^{\frac{1}{2}}\right)^2}\, dx$$

$$= \int_1^4 \frac{1}{2}\left(x^{-\frac{1}{2}} + x^{\frac{1}{2}}\right) dx$$

$$= \frac{1}{2}\int_1^4 x^{-\frac{1}{2}} + x^{\frac{1}{2}}\, dx$$

$$= \frac{1}{2}\left(\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right)\Bigg|_1^4$$

$$= \frac{10}{3}.$$

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20.

Range for x: $1 \le x \le 4$

$$x = y^{2/3}$$

$$x^{\frac{3}{2}} = y$$

$$y = x^{3/2}$$

$$y' = \frac{3}{2}\sqrt{x}$$

$$y'^2 = \frac{9}{4}x$$

$$\therefore \int_1^4 \sqrt{1 - \frac{9}{4}x}\, dx$$

$$= \frac{2}{3}\left(1 - \frac{9}{4}x\right)^{\frac{3}{2}} \cdot \frac{4}{9}\Big|_1^4$$

$$x = \frac{y^3}{12} + \frac{1}{y} \quad 1 \le y \le 2$$

$$x' = \frac{3}{12}y^2 - \frac{1}{y^2}$$

$$x'^2 = \left(\frac{3}{12}y^2\right)^2 - 2\left(\frac{3}{12}y^2\right)\left(\frac{1}{y^2}\right) + \left(\frac{1}{y^2}\right)^2$$

$$= \frac{1}{16}y^4 - \frac{1}{2} + \frac{1}{y^4}$$

$$L = \int_1^2 \sqrt{1 + \left(\frac{1}{16}y^4 - \frac{1}{2} + \frac{1}{y^4}\right)}\, dy$$

$$= \int_1^2 \sqrt{\frac{1}{16}y^4 + \frac{1}{2} + \frac{1}{y^4}}\, dy$$

$$= \int_1^2 \sqrt{\left(\frac{3}{12}y^2 + \frac{1}{y^2}\right)^2}\, dy$$

$$= \int_1^2 \frac{3}{12}y^2 + \frac{1}{y^2}\, dy$$

$$= \frac{3}{12}\frac{y^3}{3}\Big|_1^2 + \frac{y^{-2+1}}{-2+1}\Big|_1^2 \quad //$$

21.

$$y = \frac{5}{12}x^{6/5} - \frac{5}{8}x^{\frac{4}{5}}$$

$$y' = \frac{1}{2}x^{\frac{1}{5}} - \frac{1}{2}x^{-\frac{1}{5}}$$

$$y'^2 = \left(\frac{1}{2}x^{\frac{1}{5}} - \frac{1}{2}x^{-\frac{1}{5}}\right)^2$$

$$= \left(\left(\frac{1}{2}x^{\frac{1}{5}}\right)^2 - \cancel{2\left(\frac{1}{2}x^{\frac{1}{5}}\right)\left(\frac{1}{2}x^{-\frac{1}{5}}\right)} + \left(\frac{1}{2}x^{-\frac{1}{5}}\right)^2\right)$$

$$= \frac{1}{4}x^{\frac{2}{5}} + \frac{1}{4}x^{-\frac{2}{5}} - 1$$

$$\int_1^{32} \sqrt{\cancel{1} + \left(\frac{1}{4}x^{\frac{2}{5}} + \frac{1}{4}x^{-\frac{2}{5}} \cancel{-1}\right)}\, dx$$

$$= \int_1^{32} \sqrt{\frac{1}{4}x^{\frac{2}{5}} + \frac{1}{4}x^{-\frac{2}{5}}}\, dx$$

$$= \frac{1}{2}\int_1^{32} \sqrt{\underbrace{x^{\frac{2}{5}} + x^{-\frac{2}{5}}}}\, dx$$

$$x^{\frac{2}{5}}\left(1 + \frac{x^{-\frac{2}{5}}}{x^{\frac{2}{5}}}\right)$$

$$x^{\frac{2}{5}}\left(1 + x^{\frac{4}{5}}\right)$$

$$y = \frac{5}{12}x^{\frac{6}{5}} - \frac{5}{8}x^{\frac{4}{5}}$$

$$y' = \frac{1}{2}\left(x^{\frac{1}{5}} - x^{-\frac{1}{5}}\right)$$

$$y'^2 = \frac{1}{4}\left(x^{\frac{2}{5}} - 2 + x^{-\frac{2}{5}}\right)$$

$$\int_1^{32} \sqrt{1 + \frac{1}{4}\left(x^{\frac{2}{5}} - 2 + x^{-\frac{2}{5}}\right)}\, dx$$

$$\int_1^{32} \sqrt{1 + \frac{1}{4}x^{\frac{2}{5}} - \frac{1}{2} + \frac{1}{4}x^{-\frac{2}{5}}}\, dx$$

$$\int_1^{32} \sqrt{\frac{1}{4}x^{\frac{2}{5}} + \frac{1}{2} + \frac{1}{4}x^{-\frac{2}{5}}}\, dx$$

$$\int_1^{32} \sqrt{\left(\frac{1}{4}x^{\frac{1}{5}} + \frac{1}{4}x^{-\frac{1}{5}}\right)^2}\, dx$$

$$\int_1^{32} \frac{1}{4}x^{\frac{1}{5}} + \frac{1}{4}x^{-\frac{1}{5}}\, dx$$

$$\frac{1}{4} \cdot \frac{x^{\frac{1}{5}+1}}{\frac{1}{5}+1}\Big|_1^{32} + \frac{1}{4} \cdot \frac{x^{-\frac{1}{5}+1}}{-\frac{1}{5}+1}\Big|_1^{32}$$

Areas of Surfaces of Revolution

In Exercises 23–26, find the areas of the surfaces generated by revolving the curves about the given axes.

23. $y = \sqrt{2x+1}, \quad 0 \le x \le 3; \quad x\text{-axis}$

24. $y = x^3/3, \quad 0 \le x \le 1; \quad x\text{-axis}$

25. $x = \sqrt{4y - y^2}, \quad 1 \le y \le 2; \quad y\text{-axis}$

26. $x = \sqrt{y}, \quad 2 \le y \le 6; \quad y\text{-axis}$

23. $S = \int_a^b 2\pi f(x) \sqrt{1+(f'(x))^2}\, dx$

$y = \sqrt{2x+1}$

$y' = \frac{2}{2\sqrt{2x+1}} = \frac{1}{\sqrt{2x+1}}$

$y'^2 = \frac{1}{2x+1}$

$S = \int_0^3 2\pi (\sqrt{2x+1}) \cdot \underbrace{\sqrt{1+\frac{1}{2x+1}}}\, dx$

$\sqrt{1+\frac{1}{2x+1}} = \frac{\sqrt{2x+2}}{\sqrt{2x+1}}$

$\therefore S = 2\pi \int_0^3 \sqrt{2x+2}\, dx$

$S = 2\sqrt{2}\pi \int_0^3 \sqrt{x+1}\, dx$

$= 2\sqrt{2}\pi \left.\frac{(x+1)^{\frac{3}{2}}}{\frac{3}{2}}\right|_0^3$

24. $y = \frac{x^3}{3}$

$y' = 3\frac{x^2}{3}$

$y' = x^2$

$y'^2 = x^4$

$S = \int_0^1 2\pi \frac{x^3}{3} \cdot \sqrt{1+x^4}\, dx$

$\frac{2}{3}\pi \int_0^1 x^3 \sqrt{1+x^4}\, dx$

$u = 1+x^4$

$du = 4x^3$

$\frac{1}{4} \cdot \frac{2}{3}\pi \int_0^1 4x^3 \sqrt{1+x^4}\, dx$

$= \frac{2}{9}\pi \int_1^2 \sqrt{u}\, du$

$= \frac{2}{9}\pi \left.\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_1^2$

25. $x = \sqrt{4y-y^2} \qquad 1 \le y \le 2.$

$x' = \frac{\overset{2}{\cancel{4}} - \cancel{2}y}{\cancel{2}\sqrt{4y-y^2}}$

$= \frac{1-y}{\sqrt{4y-y^2}}$

$x'^2 = \frac{(1-y)^2}{4y-y^2}$

$\int_1^2 2\pi \cancel{\sqrt{4y-y^2}} \cdot \sqrt{1+\frac{(1-y)^2}{4y-y^2}}$

$\frac{(1-y)^2+4y-y^2}{\cancel{4y-y^2}}$

$= 2\pi \int_1^2 \sqrt{1-2y+\cancel{y^2}+4y-\cancel{y^2}}\, dy$

$= 2\pi \int_1^2 \sqrt{1+2y}\, dy.$

$= 2\pi \left.\frac{(1+2y)^{\frac{3}{2}}}{\frac{3}{2}} \cdot \frac{1}{2}\right|_1^2$

26. $x = \sqrt{y} \qquad 2-6$

$x' = \frac{1}{2\sqrt{y}}$

$x'^2 = \frac{1}{4y}$

$S = \int_2^6 2\pi \cancel{\sqrt{y}} \underbrace{\sqrt{1+\frac{1}{4y}}}$

$\frac{4y+1}{4y}$

$= \pi \int_2^6 \sqrt{4y+1}\, dy.$

$= \pi \left.\frac{(4y+1)^{\frac{3}{2}}}{\frac{3}{2}} \cdot \frac{1}{4}\right|_2^6$

Test For Convergence & Divergence.

n^{th} term test: If a_n does not tends to 0 if $n\to\infty$, then $\sum_{n=1}^{\infty} a_n$ is divergent (only test for divergence)

Integral test: If $\int_1^{\infty} f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

If $\int_1^{\infty} f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

p-series: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent iff $p>1$

Comparison test: If $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent

If $\sum_{n=1}^{\infty} a_n$ is divergent, then $\sum_{n=1}^{\infty} b_n$ is divergent.

$a_n \le b_n$ for all n.

Limit Comparison test: If $\lim_{n\to\infty} \frac{a_n}{b_n} = C$, $C\neq 0$ then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.

Absolute convergence: $\sum_{n=1}^{\infty} a_n$ is abs. conv. if $\sum_{n=1}^{\infty} |a_n|$ converges.

If $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges, then $\sum_{n=1}^{\infty} a_n$ converges conditionally.

Alternating Series Test: $\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \cdots$ converges if

1. u_n are positive
2. $u_{n+1} \le u_n$ for all n
3. $u_n \to 0$ as $n\to\infty$.

Absolute convergence test: If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Ratio test: Suppose $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$ | Root test: Suppose $\lim_{n\to\infty} \sqrt[n]{a_n} = L$, originally a_n

1. If $0 \le L < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
2. If $L>1$ or infinite, then $\sum_{n=1}^{\infty} a_n$ is divergent.
3. If $L=1$, then the test is inconclusive

Which of the sequences whose nth terms appear in Exercises 1–18 converge, and which diverge? Find the limit of each convergent sequence.

1. $a_n = 1 + \frac{(-1)^n}{n}$ **2.** $a_n = \frac{1 - (-1)^n}{\sqrt{n}}$

3. $a_n = \frac{1 - 2^n}{2^n}$ **4.** $a_n = 1 + (0.9)^n$

5. $a_n = \sin \frac{n\pi}{2}$ **6.** $a_n = \sin n\pi$

7. $a_n = \frac{\ln (n^2)}{n}$ **8.** $a_n = \frac{\ln (2n + 1)}{n}$

9. $a_n = \frac{n + \ln n}{n}$ **10.** $a_n = \frac{\ln (2n^3 + 1)}{n}$

11. $a_n = \left(\frac{n - 5}{n}\right)^n$ **12.** $a_n = \left(1 + \frac{1}{n}\right)^{-n}$

13. $a_n = \sqrt[n]{\frac{3^n}{n}}$ **14.** $a_n = \left(\frac{3}{n}\right)^{1/n}$

15. $a_n = n(2^{1/n} - 1)$ **16.** $a_n = \sqrt[n]{2n + 1}$

17. $a_n = \frac{(n + 1)!}{n!}$ **18.** $a_n = \frac{(-4)^n}{n!}$

1. $\lim_{n\to\infty} \left(1 + \frac{(-1)^n}{n}\right)$ ($\frac{(-1)^n}{n} \to 0$)

$= 1$ ∴ converge.

4. $\lim_{n\to\infty} 1 + 0.9^n$ ($0.9^n \to 0$)

$= 1$ ∴ converge.

2. $\lim_{n\to\infty} \frac{1 - (-1)^n}{\sqrt{n}}$

$= \frac{1}{\sqrt{n}} - \frac{(-1)^n}{\sqrt{n}}$ ($\to 0$, $\to 0$)

$= 0$ converges.

5. $\lim_{n\to\infty} \sin \frac{n\pi}{2}$

$= ?$ diverges.

3. $\lim_{n\to\infty} \frac{1 - 2^n}{2^n}$

$= \lim_{n\to\infty} \frac{1}{2^n} - \frac{2^n}{2^n}$ ($\to 0$, 1)

$= -1$ converges

6. $\lim_{n\to\infty} \sin n\pi$

diverges.

13. $\sqrt[n]{\frac{3^n}{n}} = 3\sqrt[n]{\frac{1}{n}}$

$= 3 \frac{1}{\sqrt[n]{n}}$

$= 3.$

14. $\left(\frac{3}{n}\right)^{1/n}$

$= \sqrt[n]{\frac{3}{n}}$

$= \sqrt[n]{3} \cdot \frac{1}{\sqrt[n]{n}}$

$= 3^{\frac{1}{n}} \cdot 1$ ($\frac{1}{n} \to 0$)

$= 1$

7. $\lim_{n\to\infty} \frac{\ln(n^2)}{n}$

$= \frac{2\ln n}{n}$

$2\frac{\frac{1}{n}}{1} = 0$ converges ($\frac{1}{n} \to 0$)

8. $\frac{\ln(2n+1)}{n}$

$= \frac{\frac{1}{2n+1}}{1}$ converges to 0. ($\to 0$)

9. $\frac{n + \ln n}{n}$

$= \frac{n}{n} + \frac{\ln n}{n} = 1 + \frac{\frac{1}{n}}{1}$ converges. ($\to 0$)

10. $\frac{\ln(2n^3+1)}{n}$

$= \frac{\frac{1}{2n^3+1}}{4}$ converges ($\to 0$)

11. $\left(\frac{n-5}{n}\right)^n$

$= \left(\frac{n}{n} - \frac{5}{n}\right)^n$

$= \left(1 - \frac{5}{n}\right)^n$

$= e^{-5}.$ (special)

12. $\left(1 + \frac{1}{n}\right)^{-n}$

$= \frac{1}{\left(1 + \frac{1}{n}\right)^n}$

$= e^{-1}$

15. $n\left(2^{\frac{1}{n}} - 1\right)$

$= \frac{2^{\frac{1}{n}} - 1}{\frac{1}{n}}$

$= \frac{2^{\frac{1}{n}} \cdot \ln 2 \cancel{\left(-\frac{1}{n^2}\right)}}{\cancel{-\frac{1}{n^2}}} = \ln 2$

16. $a_n = \sqrt[n]{2n+1}$

$n^{\frac{1}{n}} \le (2n+1)^{\frac{1}{n}} \le (3n)^{\frac{1}{n}}$

$n^{\frac{1}{n}} \to 1 \le (2n+1)^{\frac{1}{n}} \le 3^{\frac{1}{n}} n^{\frac{1}{n}} \to 1$

$\therefore \to 1.$

17. $\dfrac{(n+1)!}{n!} = \dfrac{\cancel{n!}(n+1)}{\cancel{n!}}$

$= \infty \therefore$ diverges

18. $\dfrac{(-4)^n}{n!}$

$= 0$ converge.

Convergent Series

Find the sums of the series in Exercises 19–24.

19. $\sum_{n=3}^{\infty} \frac{1}{(2n-3)(2n-1)}$ **20.** $\sum_{n=2}^{\infty} \frac{-2}{n(n+1)}$

21. $\sum_{n=1}^{\infty} \frac{9}{(3n-1)(3n+2)}$ **22.** $\sum_{n=3}^{\infty} \frac{-8}{(4n-3)(4n+1)}$

23. $\sum_{n=0}^{\infty} e^{-n}$ **24.** $\sum_{n=1}^{\infty} (-1)^n \frac{3}{4^n}$

19 $\sum_{n=3}^{\infty} \frac{1}{(2n-3)(2n-1)}$

$\frac{1}{(2n-3)(2n-1)} = \frac{A}{2n-3} + \frac{B}{2n-1}$

$1 = A(2n-1) + B(2n-3)$

$A = +\frac{1}{2}$:

$-2B = 1$

$B = -\frac{1}{2}$

$A = \frac{1}{2}$.

$\therefore \sum_{n=3}^{\infty} \frac{1}{2(2n-3)} - \frac{1}{2(2n-1)}$

$= \frac{1}{6} - \cancel{\frac{1}{10}} + \cancel{\frac{1}{10}} - \cancel{\frac{1}{14}} + \ldots$

$+ \cancel{\frac{1}{2(2n-3)}} - \frac{1}{2(2n-1)}$

$= \frac{1}{6} - \frac{1}{2(2n-1)}$

$\lim_{n\to\infty} \frac{1}{6} - \cancel{\frac{1}{2(2n-1)}}$ 0.

$= \frac{1}{6}$

23. $\sum_{n=0}^{\infty} e^{-n}$

$= e^{-1} + e^{-2} + e^{-3} + \ldots + e^{-n}$

$= e^{-1} + e^{-1}e^{-1} + e^{-1}e^{-1}e^{-1} + \ldots$

$\therefore$ GP.

$\frac{1}{1-e^{-1}}$ //

24. $\sum_{n=1}^{\infty} (-1)^n \frac{3}{4^n}$

$\frac{3}{4} - \frac{3}{4^2} + \frac{3}{4^3} - \frac{3}{4^4}$

gp: $r = -\frac{1}{4}$.

$\frac{-\frac{3}{4}}{1-(-\frac{1}{4})}$

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Integral test: If $\int_1^\infty f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

If $\int_1^\infty f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

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If $\sum_{n=1}^{\infty} a_n$ is divergent, then $\sum_{n=1}^{\infty} b_n$ is divergent.

$a_n \le b_n$ for all n.

Limit Comparison test: If $\lim_{n\to\infty} \frac{a_n}{b_n} = C$, $C \neq 0$ then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.

Absolute convergence: $\sum_{n=1}^{\infty} a_n$ is abs. conv. if $\sum_{n=1}^{\infty} |a_n|$ converges.

If $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverge, then $\sum_{n=1}^{\infty} a_n$ converges conditionally.

Alternating Series Test: $\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \cdots$ converges if

1. u_n are positive
2. $u_{n+1} \le u_n$ for all n
3. $u_n \to 0$ as $n \to \infty$.

Absolute convergence test: If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Ratio test: Suppose $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$ | Root test: Suppose $\lim_{n\to\infty} \sqrt[n]{a_n} = L$, originally a_n

1. If $0 \le L < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
2. If $L > 1$ or infinite, then $\sum_{n=1}^{\infty} a_n$ is divergent.
3. If $L = 1$, then the test is inconclusive

Determining Convergence of Series

Which of the series in Exercises 25–44 converge absolutely, which converge conditionally, and which diverge? Give reasons for your answers.

25. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

26. $\sum_{n=1}^{\infty} \frac{-5}{n}$

27. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

28. $\sum_{n=1}^{\infty} \frac{1}{2n^3}$

29. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$

30. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

31. $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ $\frac{1}{n^2}$ Compare.

32. $\sum_{n=3}^{\infty} \frac{\ln n}{\ln(\ln n)}$ $\ln(n - \ln n)$

33. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt{n^2+1}}$

34. $\sum_{n=1}^{\infty} \frac{(-1)^n 3n^2}{n^3+1}$

35. $\sum_{n=1}^{\infty} \frac{n+1}{n!}$

36. $\sum_{n=1}^{\infty} \frac{(-1)^n(n^2+1)}{2n^2+n-1}$

37. $\sum_{n=1}^{\infty} \frac{(-3)^n}{n!}$

38. $\sum_{n=1}^{\infty} \frac{2^n 3^n}{n^n}$

39. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)(n+2)}}$

40. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$

41. $1 - \left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^4 - \left(\frac{1}{\sqrt{3}}\right)^6 + \left(\frac{1}{\sqrt{3}}\right)^8 - \cdots$

42. $\sum_{n=0}^{\infty} \frac{(-1)^n}{e^{-n}+1}$

43. $\sum_{n=0}^{\infty} \frac{1}{1+r+r^2+\cdots+r^n}$, for $-1 < r < 1$

44. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+100}-\sqrt{n}}$

25. $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ diverges by p-series.

26. $-5\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-series.

$$\lim_{n\to\infty} \frac{\cancel{\frac{1}{n}}}{\frac{1}{\cancel{n}(\ln n)^2}} = (\ln n)^2 \to \infty.$$

32. $\sum_{n=3}^{\infty} \ln(n - \ln n)$

Power Series

Power Series about 0:

$$\sum_{k=0}^{\infty} C_n x^k = C_0 + C_1 x + \dots + C_k x^k + \dots$$

Power Series about a:

$$\sum_{k=0}^{\infty} C_k (x-a)^k = C_0 + C_1(x-a) + \dots + C_k(x-a)^k + \dots$$

Radius of Convergence:

- If power series is convergent for $x = c \neq 0$, the it converges absolutely for all x with $|x| < |c|$
- If series diverges for $x = d$, then it diverges for all x with $|x| > |d|$.
- Convergence of $\sum_{k=0}^{\infty} C_k (x-a)^k$:

1. Ratio test: $\sum_{n=0}^{\infty} C_k (x-a)^k$ is convergent if $|x-a| < \underbrace{\lim_{x\to\infty} \frac{|C_n|}{|C_{n+1}|}}_{\text{Radius.}}$

2. Root test: $\sum_{n=0}^{\infty} C_n (x-a)^k$ is convergent if $|x-a| < \overbrace{\lim_{x\to\infty} \frac{1}{|C_n|^{1/n}}}$

Term by Term Diffrentiation Theorem:

Let $f(x) = \sum_{k=0}^{\infty} C_k (x-a)^k$ which converges for $|x-a| < R$.

Then $f'(x) = \sum_{k=0}^{\infty} C_n k (x-a)^{k-1}$

$$\int f(x)\,dx = \sum_{n=0}^{\infty} C_k \frac{(x-a)^{k+1}}{k+1} + C \quad \text{for } |x-a| < R.$$

Taylor Series: $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{k!} (x-a)^k$ Maclaurin Series: $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$

Power Series

In Exercises 45–54, **(a)** find the series' radius and interval of convergence. Then identify the values of x for which the series converges **(b)** absolutely and **(c)** conditionally.

45. $\sum_{n=1}^{\infty} \frac{(x+4)^n}{n3^n}$

46. $\sum_{n=1}^{\infty} \frac{(x-1)^{2n-2}}{(2n-1)!}$

47. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(3x-1)^n}{n^2}$

48. $\sum_{n=0}^{\infty} \frac{(n+1)(2x+1)^n}{(2n+1)2^n}$

49. $\sum_{n=1}^{\infty} \frac{x^n}{n^n}$

50. $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$

51. $\sum_{n=0}^{\infty} \frac{(n+1)x^{2n-1}}{3^n}$

52. $\sum_{n=0}^{\infty} \frac{(-1)^n(x-1)^{2n+1}}{2n+1}$

53. $\sum_{n=1}^{\infty} (\operatorname{csch} n)x^n$

54. $\sum_{n=1}^{\infty} (\coth n)x^n$

47. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(3x-1)^n}{n^2} = \frac{(-1)^{n-1}(\frac{1}{3})^n(x-\frac{1}{3})^n}{n^2}$

$C_k = \frac{(-1)^{n-1}}{n^2}(\frac{1}{3})^n \qquad a = \frac{1}{3}$

$|x - \frac{1}{3}| < \lim_{n\to\infty} \frac{\left|\frac{(-1)^{n-1}}{n^2}(\frac{1}{3})^n\right|}{\left|\frac{(-1)^n}{(n+1)^2}(\frac{1}{3})^{n+1}\right|}$

$|x-\frac{1}{3}| < 1$

$-1 < x - \frac{1}{3} < 1$

$-\frac{1}{3} < x < \frac{4}{3}$

$(-\frac{1}{3}, \frac{4}{3})$

$\frac{\frac{(-1)^{n-1}}{n^2}}{\frac{(-1)^n}{(n+1)^2}(\frac{1}{3})}$

$\frac{(-1)^{n-1}}{(-1)^n}$

$(-1)^{n-1-n}$

$(-1)^{-1}$

$-\frac{\frac{1}{n^2}}{\frac{1}{(n+1)^2}}$

$-\frac{(n+1)^2}{n^2}$

$-\frac{n^2+2n+1}{n^2}$

$|-1 + \frac{2}{n} + \frac{1}{n^2}|$

$= 1$

Maclaurin Series

Each of the series in Exercises 55–60 is the value of the Taylor series at $x = 0$ of a function $f(x)$ at a particular point. What function and what point? What is the sum of the series?

55. $1 - \frac{1}{4} + \frac{1}{16} - \cdots + (-1)^n \frac{1}{4^n} + \cdots$ $\quad \frac{1}{1+x}. \quad x = \frac{1}{4}.$

56. $\frac{2}{3} - \frac{4}{18} + \frac{8}{81} - \cdots + (-1)^{n-1} \frac{2^n}{n3^n} + \cdots$ $\quad \ln(1+x) \quad x = \frac{2}{3}$

57. $\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \cdots + (-1)^n \frac{\pi^{2n+1}}{(2n+1)!} + \cdots$ $\quad \sin x \quad x = \pi.$

58. $1 - \frac{\pi^2}{9 \cdot 2!} + \frac{\pi^4}{81 \cdot 4!} - \cdots + (-1)^n \frac{\pi^{2n}}{3^{2n}(2n)!} + \cdots$ $\quad \cos x \quad x = \frac{\pi}{3}$

59. $1 + \ln 2 + \frac{(\ln 2)^2}{2!} + \cdots + \frac{(\ln 2)^n}{n!} + \cdots$ $\quad e^x \quad x = \ln 2.$

60. $\frac{1}{\sqrt{3}} - \frac{1}{9\sqrt{3}} + \frac{1}{45\sqrt{3}} - \cdots$

$+ (-1)^{n-1} \frac{1}{(2n-1)(\sqrt{3})^{2n-1}} + \cdots$ $\quad \tan^{-1} x \quad x = \frac{1}{\sqrt{3}}$

Find Taylor series at $x = 0$ for the functions in Exercises 61–68.

61. $\frac{1}{1 - 2x}$

62. $\frac{1}{1 + x^3}$

63. $\sin \pi x$

64. $\sin \frac{2x}{3}$

65. $\cos(x^{5/3})$

66. $\cos \frac{x^3}{\sqrt{5}}$

67. $e^{(\pi x/2)}$

68. e^{-x^2}

Taylor Series

In Exercises 69–72, find the first four nonzero terms of the Taylor series generated by f at $x = a$.

69. $f(x) = \sqrt{3 + x^2}$ at $x = -1$

70. $f(x) = 1/(1 - x)$ at $x = 2$

71. $f(x) = 1/(x + 1)$ at $x = 3$

72. $f(x) = 1/x$ at $x = a > 0$

69. Taylor series:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$f(x) = \sqrt{3+x^2}$$

$$f'(x) = \frac{2x}{2\sqrt{3+x^2}} = \frac{x}{\sqrt{3+x^2}}$$

$$f''(x) = \frac{\sqrt{3+x^2} - x\frac{2x}{2\sqrt{3+x^2}}}{3+x^2} = \frac{\sqrt{3+x^2} - \frac{x^2}{\sqrt{3+x^2}}}{3+x^2} = \frac{\frac{3+\cancel{x^2}-\cancel{x^2}}{\sqrt{3+x^2}}}{3+x^2}$$

$a = -1$.

$$= \frac{3}{\sqrt{3+x^2}} \cdot \frac{1}{3+x^2} = \frac{3}{(3+x^2)^{3/2}}$$

$$f'''(x) = \frac{(3+x^2)^{3/2} - 3\frac{3}{\cancel{2}}\cdot\cancel{2}x(3+x^2)^{1/2}}{\left((3+x^2)^{3/2}\right)^2} = \frac{(3+x^2)^{3/2} - 9x\sqrt{3+x^2}}{\left((3+x^2)^{3/2}\right)^2}.$$

$$\frac{f^{(0)}(a)}{0!}(x-a)^0 = f(a) = \sqrt{3+1} = 2$$

$$\frac{f^{(1)}(a)}{1!}(x+1)^1 = -\frac{1}{2}(x+1)$$

$$\frac{f^{2}(a)}{2!}(x+1)^2 = \frac{\frac{2-\frac{1}{2}}{2}}{2}(x+1)^2 = \frac{3}{8}(x+1)^2$$

$$\frac{f^{3}(a)}{2!}(x+1)^2 = \frac{\frac{8+9(2)}{64}}{3!}(x+1)^3 = \frac{9}{32}(x+1)^3.$$

$$\therefore \sqrt{3+x^2} = 2 - \frac{(x+1)}{2\cdot 1!} + \frac{3(x+1)^2}{2^3\cdot 2!} + \frac{9(x+1)^3}{2^5\cdot 3!} + \cdots$$

$a = 2.$

70. $\frac{1}{1-x} = (1-x)^{-1}$ $\quad f(2) = (1-2)^{-1} = -1$

$f'(x) = (1-x)^{-2}$ $\quad f'(2) = (1-2)^{-2} = 1$

$f''(x) = 2(1-x)^{-3}$ $\quad f''(2) = 2(1-2)^{-3} = -2$

$f'''(x) = 6(1-x)^{-4}$ $\quad f'''(2) = 6$

$$\therefore \frac{1}{1-x} = -1 + (x-2) - (x-2)^2 + (x-2)^3 + \cdots$$

71. $\frac{1}{1+x} = (1+x)^{-1}$ $\quad f(3) = 4^{-1} = \frac{1}{4}$

$f'(x) = -(1+x)^{-2}$ $\quad f'(3) = -4^{-2} = -\frac{1}{16}$

$f''(x) = 2(1+x)^{-3}$ $\quad f''(3) = 2(4)^{-3} = \frac{2}{64} = \frac{1}{32}$

$f'''(x) = -6(1+x)^{-4}$ $\quad f'''(3) = -6(4)^{-4} = -\frac{6}{256} = -\frac{3}{128}$

$$\frac{1}{1+x} = \frac{1}{4} - \frac{1}{4^2}(x-3) + \frac{1}{4^3}(x-3)^2 + \cdots$$

72. $\frac{1}{x} = \frac{1}{x-a+a}$

$$= \frac{1}{a}\cdot\frac{1}{\frac{x-a}{a}+1}$$

$$= \frac{1}{a}\cdot\frac{1}{\frac{x-a}{a}+1}$$

$$= \frac{1}{a}\left(1 - \frac{x-a}{a} + \left(\frac{x-a}{a}\right)^2 - \left(\frac{x-a}{a}\right)^3 + \cdots\right)$$

$$= \frac{1}{a} - \frac{1}{a^2}(x-a) + \frac{1}{a^3}(x-a)^2 - \frac{1}{a^4}(x-a)^3 + \cdots$$

Max / Min of 2 variables.

If $f(x,y)$ has a local minimum at (a,b), then $f_x(a,b)=0$ and $f_y(a,b)=0$.
(provided that f_x and f_y exist).

2^{nd} derivative test for local extreme values:

(Suppose $f_x(a,b) = f_y(a,b) = 0$)

Then 1. f has a local max at (a,b) if
$f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$

2. f has a local min at (a,b) if
$f_{xy} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$

3. f has a saddle point at (a,b) if
$f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a,b).

Inconclusive if $f_{xx} - f_{yy} - f_{xy}^2 = 0$.

Lagrange Multiplier

If max/min value of $f(x,y)$ is subjected to a constraint $g(x,y)=0$ at point (a,b)

and $(g_x(a,b), g_y(a,b)) \neq (0,0)$

then $f_x = \lambda g_x$ and $f_y = \lambda g_y$.

Local Extrema

Test the functions in Exercises 65–70 for local maxima and minima and saddle points. Find each function's value at these points.

65. $f(x, y) = x^2 - xy + y^2 + 2x + 2y - 4$

66. $f(x, y) = 5x^2 + 4xy - 2y^2 + 4x - 4y$

67. $f(x, y) = 2x^3 + 3xy + 2y^3$

68. $f(x, y) = x^3 + y^3 - 3xy + 15$

69. $f(x, y) = x^3 + y^3 + 3x^2 - 3y^2$

70. $f(x, y) = x^4 - 8x^2 + 3y^2 - 6y$

65. $f_x = 2x - y + 2 = 0 \Rightarrow 2x - y = -2 \quad y - 2x = 2$

$f_y: -x + 2y + 2 = 0 \qquad -x + 2y + y - 2x = 0$

$3y - 3x = 0$

Critical pts: $y = x \qquad y = x$

$y = 2 \quad x = 2$

$f_{xx} = 2$

$f_{yy}: 2$

$f_{xy}: -1 < 0$

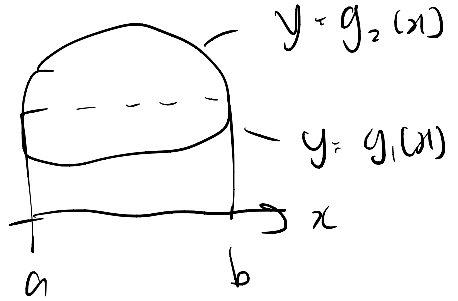
$f_{xx} f_{yy} - f_{xy}^2 = 2 \cdot 2 - (-1)^2$

$= 3 > 0$

$(2, 2)$ is max.

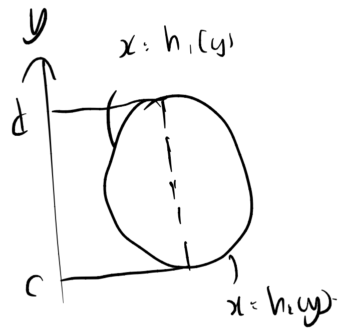
Double Integral

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dx\, dy$$



$\Downarrow$

$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy$$



(Interchangeable if all constants)

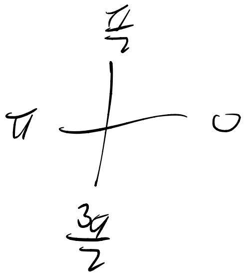
polar forms:

Given $f(x,y)$,

$$= f(r\cos\theta, r\sin\theta)$$
$$= F(r,\theta)$$

$$\iint_R F(r,\theta)\, dA = \lim_{n\to\infty} \sum_{k=1}^{n} F(r,\theta)\, r\, \Delta r\, \Delta\theta$$

↳ find out what the question mean first before using polar



Evaluating Double Iterated Integrals

In Exercises 1–4, sketch the region of integration and evaluate the double integral.

1. $\displaystyle\int_1^{10}\int_0^{1/y} ye^{xy}\,dx\,dy$ **2.** ~~$\displaystyle\int_0^1\int_0^{x^3} e^{y/x}\,dy\,dx$~~

3. $\displaystyle\int_0^{3/2}\int_{-\sqrt{9-4t^2}}^{\sqrt{9-4t^2}} t\,ds\,dt$ **4.** $\displaystyle\int_0^1\int_{\sqrt{y}}^{2-\sqrt{y}} xy\,dx\,dy$

1. $\displaystyle\int_1^{10}\int_0^{1/y} ye^{xy}\,dx\,dy.$

$\displaystyle\int_1^{10} ye^{xy}\frac{1}{y}\Big|_0^{1/y}\,dy.$

$\displaystyle\int_1^{10} e^1 - e^0$

$\displaystyle\int_1^{10} e-1\ dy.$

$ey - y\Big|_1^{10}$

$10e - 10 - e + 1$

$\underline{9e - 9}$

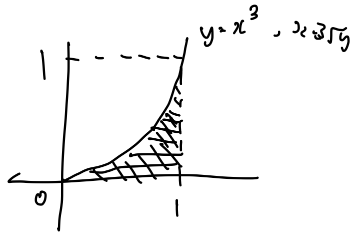
2. $\displaystyle\int_0^1\int_0^{x^3} e^{\frac{y}{x}}\,dy\,dx.$

$\displaystyle = \int_0^1\int_0^{x^3} e^{\frac{y}{x}}\,dy\,dx.$

$\displaystyle = e^{\frac{1}{x}y}\cdot x\Big|_0^{x^3}$

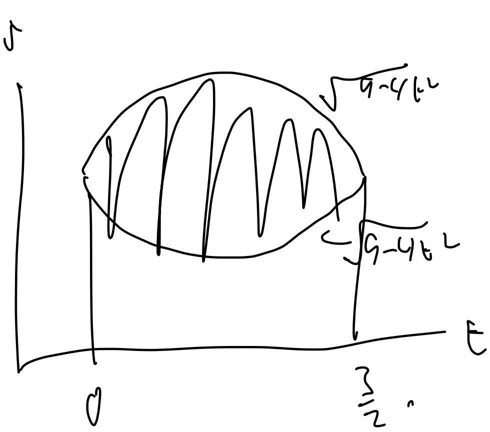
$\displaystyle = \int_0^1 xe^{x^2} - x\cancel{e}^{0}\,dx$

$\displaystyle = \int_0^1 x(e^{x^2}-1)\,dx$



$\displaystyle\int_0^1\int_{\sqrt[3]{y}}^1 e^{\frac{y}{x}}\,dx\,dy.$

3. $\displaystyle\int_0^{3/2}\int_{-\sqrt{9-4t^2}}^{\sqrt{9-4t^2}} t\,ds\,dt$



$\displaystyle st\Big|_{-\sqrt{9-4t^2}}^{\sqrt{9-4t^2}}$

$\displaystyle\int_0^{3/2} 2t\sqrt{9-4t^2}\,dt.$

$u = 9-4t^2$

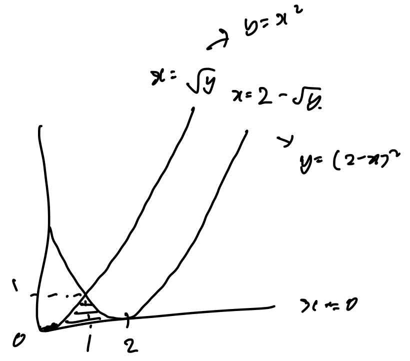
$du = -8t$

$x = 2-\sqrt{y}$

4. $\int_0^1 \int_{\sqrt{y}}^{2-\sqrt{y}} xy\,dx\,dy$

$x-2 = -\sqrt{y}$

$y = (2-x)^2$



$u = x \quad v' = (2-x)^4$

$u' = 1 \quad v = -\frac{(2-x)^5}{5}$

$$\int_0^1\int_0^{x^2} xy\,dy\,dx + \int_1^2\int_0^{(2-x)^2} xy\,dy\,dx$$

$$\int_0^1 \frac{xy^2}{2}\Big|_0^{x^2} dx + \int_1^2 \frac{xy^2}{2}\Big|_0^{(2-x)^2} dx$$

$$\int_0^1 \frac{x^5}{2}\,dx \qquad \frac{1}{2}\int_1^2 x(2-x)^4\,dx$$

$$\frac{1}{2}\,\frac{x^6}{6}\Big|_0^1 \qquad \frac{1}{2}\left(x - \frac{(2-x)^5}{5}\Big|_1^2 - \int_1^2 \frac{(2-x)^5}{5}\,dx\right)$$

$$= \frac{1}{6}\cdot\frac{1}{2} +$$

$$= \frac{1}{12}$$

$$= -\frac{x(2-x)^5}{5} + \frac{1}{5}\int (2-x)^5\,dx$$

$$\frac{7}{60}$$

$$\frac{1}{2}\left(-\frac{x(2-x)^5}{5} + \frac{1}{5}\cdot\frac{(2-x)^6}{6}\right)\Big|_1^2$$

$$= \frac{1}{2}\left(\frac{1}{5} + \frac{1}{5}\cdot\frac{1}{6}\right)$$

In Exercises 5–8, sketch the region of integration and write an equivalent integral with the order of integration reversed. Then evaluate both integrals.

5. $\int_0^4 \int_{-\sqrt{4-y}}^{(y-4)/2} dx\, dy$

6. $\int_0^1 \int_{x^2}^{x} \sqrt{x}\, dy\, dx$

7. $\int_0^{3/2} \int_{-\sqrt{9-4y^2}}^{\sqrt{9-4y^2}} y\, dx\, dy$

8. $\int_0^2 \int_0^{4-x^2} 2x\, dy\, dx$

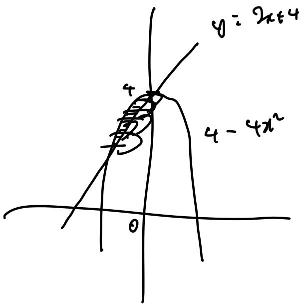
$= \int_0^4 \int_{-\sqrt{4-y}}^{(y-4)/2} 1\, dx\, dy.$

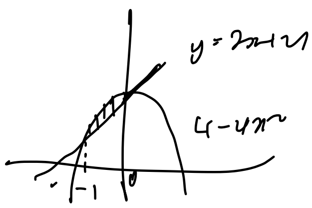
$x = \frac{(y-4)}{2} \rightarrow y = 2x+4$

$x = \frac{-\sqrt{4-y}}{2} \rightarrow (-2x)^2 = 4-y$

$y = 4-(-2x)^2$

$y = 4-4x^2$



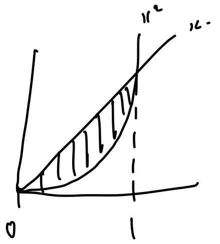


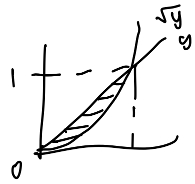
$\int_{-1}^{0} \int_{2x+2}^{4-4x^2} dy\, dx$

$\int_0^1 \int_{x^2}^{x} \sqrt{x}\, dy\, dx$

$y = x \rightarrow x = y$

$y = x^2 \rightarrow x = \sqrt{y}$





$\int_0^1 \int_y^{\sqrt{y}} \sqrt{x}\, dx\, dy.$

Evaluate the integrals in Exercises 9–12.

9. $\displaystyle\int_0^1 \int_{2y}^{2} 4\cos(x^2)\, dx\, dy$

10. $\displaystyle\int_0^2 \int_{y/2}^{1} e^{x^2}\, dx\, dy$

11. $\displaystyle\int_0^8 \int_{\sqrt[3]{x}}^{2} \frac{dy\, dx}{y^4 + 1}$

12. $\displaystyle\int_0^1 \int_{\sqrt[3]{y}}^{1} \frac{2\pi \sin \pi x^2}{x^2}\, dx\, dy$

Areas and Volumes Using Double Integrals

13. Area between line and parabola Find the area of the region enclosed by the line $y = 2x + 4$ and the parabola $y = 4 - x^2$ in the xy-plane.

14. Area bounded by lines and parabola Find the area of the "triangular" region in the xy-plane that is bounded on the right by the parabola $y = x^2$, on the left by the line $x + y = 2$, and above by the line $y = 4$.

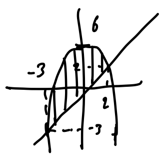
15. Volume of the region under a paraboloid Find the volume under the paraboloid $z = x^2 + y^2$ above the triangle enclosed by the lines $y = x$, $x = 0$, and $x + y = 2$ in the xy-plane.

16. Volume of the region under a parabolic cylinder Find the volume under the parabolic cylinder $z = x^2$ above the region enclosed by the parabola $y = 6 - x^2$ and the line $y = x$ in the xy-plane.

16.

$y = 6 - x^2$

$y = x$



let $6 - x^2 = x$

$6 - x - x^2 = 0$

$x^2 + x - 6 = 0.$

$(x+3)(x-2) = 0.$

$x = -3$ or $x = 2$

$$\int_{-3}^{2} \int_{x}^{6-x^2} x^2 \, dy\, dx.$$

$$\int_{-3}^{2} yx^2 \Big|_{x}^{6-x^2} dx.$$

$$\int_{-3}^{2} x^2(6-x^2) - x^3 \, dx.$$

$$\int_{-3}^{2} 6x^2 - x^4 - x^3 \, dx$$

$$6\frac{x^3}{3} - \frac{x^5}{5} - \frac{x^4}{4} \Big|_{-3}^{2}$$