Section 1.1

Linear Systems and their solutions

Objective

- What is a linear equation and a linear system?
- What is a general solution of a LE/LS?
- What is the geometrical interpretation?
- How to find a general solution of a LE?

What is a linear equation?

Discussion 1.1.1

A line in the xy-plane

is represented algebraically by
a linear equation
in the variables x and/or y

e.g. $x + y = 1$

$x = 2$

$y = -3$

General form $ax + by = c$

a, b, c represent some real numbers

a and b are not both zero

What is a linear equation?

Definition 1.1.2

A linear equation in 3 variables $ax + by + cz = d$

geometrical meaning: plane

A linear equation in 4 variables $ax + by + cz + dw = e$

geometrical meaning: none

A linear equation in n variables

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

variables: $x_1, x_2, \ldots, x_n$ also called the unknowns

constants: $a_1, a_2, \ldots, a_\mathrm{n}$ and b

What is a linear equation?

Example 1.1.3.1

The following are (specific) linear equations:

a) $x + 3y = 7$

b) $x_1 + 2x_2 + 2x_3 + x_4 = x_5$

c) $y = x - 0.5z + 4.5$

d) $x_1 + x_2 + \cdots + x_n = 1$

$$\boxed{a_1x_1 + a_2x_2 + \cdots + a_nx_n = b}$$

Example 1.1.3.2

The following are not linear equations:

a) $xy = 2$ cross term

b) $\sin(\theta) + \cos(\varphi) = 0.2$ not linear in θ in φ / linear in $\sin(\theta)$ and $\cos(\varphi)$

c) $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$ square terms

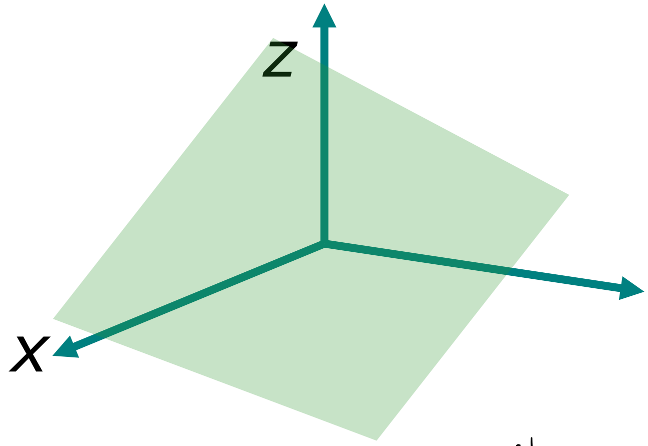
d) $x = e^y$ function of y

$$\boxed{a_1x_1 + a_2x_2 + \cdots + a_nx_n = b}$$

Example 1.1.3.3

$ax + by + cz = d$ not all a, b, c are zero

represents a plane in the three dimensional space



not parallel to any of the 3 coordinate planes

If a, b, c all non-zero, the plane is "slanting"

If some of a, b, c is zero, the plane is parallel to some axis

If d = 0, the plane passes through origin

What is a general solution of a LE?

Definition 1.1.4

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

variables: $x_1, x_2, \ldots, x_n$

constants: $a_1, a_2, \ldots, a_n, b$

real numbers $s_1, s_2, \ldots, s_n$

If the equation is satisfied,

$$\boxed{x_1 = s_1,\ x_2 = s_2,\ \ldots,\ x_n = s_n}$$

a solution of the linear equation

A linear equation has (infinitely) many solutions unless n = 1

The set of all solutions: solution set

An expression that represents all solutions: general solution

How to find a general solution of a LE?

Example 1.1.5.1

$4x - 2y = 1$

some solutions $\begin{cases} x = 1 \\ y = 1.5 \end{cases}$ $\begin{cases} x = 1.5 \\ y = 2.5 \end{cases}$ $\begin{cases} x = -1 \\ y = -2.5 \end{cases}$

infinitely many solutions

- pick a random value for x
- substitute this value into the equation
- solve the value of y

general solution

- set x = t (parameter)
- substitute t for x in the equation
- express y in terms of t

$$\boxed{\begin{cases} x = t \\ y = 2t - \frac{1}{2} \end{cases}}$$

How to find a general solution of a LE?

Example 1.1.5.1

$4x - 2y = 1$

some solutions $\begin{cases} x = 1 \\ y = 1.5 \end{cases}$ $\begin{cases} x = 1.5 \\ y = 2.5 \end{cases}$ $\begin{cases} x = -1 \\ y = -2.5 \end{cases}$

general solution

- set x = t (parameter)
- substitute t for x in the equation
- express y in terms of t

$$\begin{cases} x = t \\ y = 2t - \frac{1}{2} \end{cases}$$

general solution (alternative)

- set y = s (parameter)
- substitute s for y in the equation
- express x in terms of s

$$\begin{cases} x = \frac{1}{2}s + \frac{1}{4} \\ y = s \end{cases}$$

How to find a general solution of a LE?

Example 1.1.5.2

two variables = 1 parameter

three variables = 2 parameters

⋮

$$x_1 - 4x_2 + 7x_3 = 5$$

general solution

- set $x_2 = s$ and $x_3 = t$ (parameters)
- substitute s for x_2 and t for x_3 in the equation
- express x_1 in terms of s and t

$$\begin{cases} x_1 = 5 + 4s - 7t \\ x_2 = s \\ x_3 = t \end{cases}$$

Geometrical interpretation

Example 1.1.5.3(a)

equation

$x + y = 1$

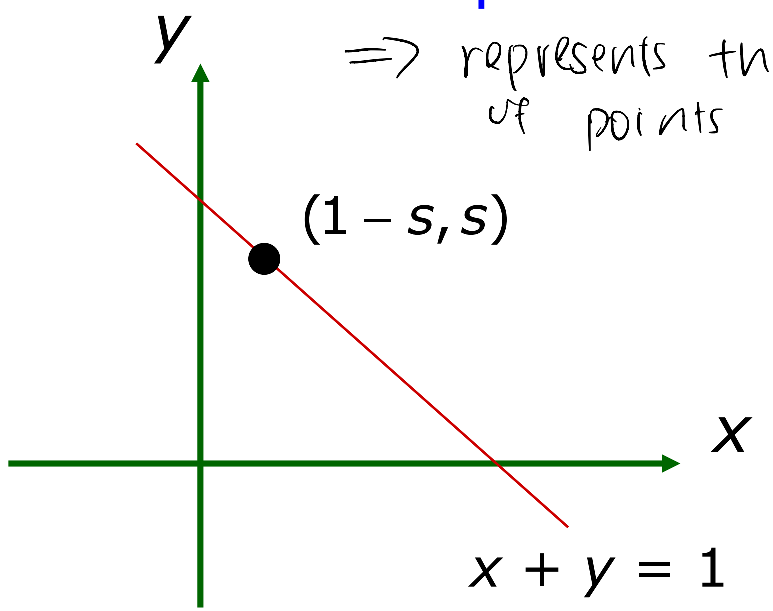
represents
a line in xy-plane

general solutions $\begin{cases} x = 1 - s \\ y = s \end{cases}$

Rewrite: $(x, y) = (1 - s, s)$

represents coordinates
of points on the line

$\Rightarrow$ represents the collections of points on the line



Geometrical interpretation

Example 1.1.5.3(b)

equation
$x + y = 1$

check whether it is 2D/3D

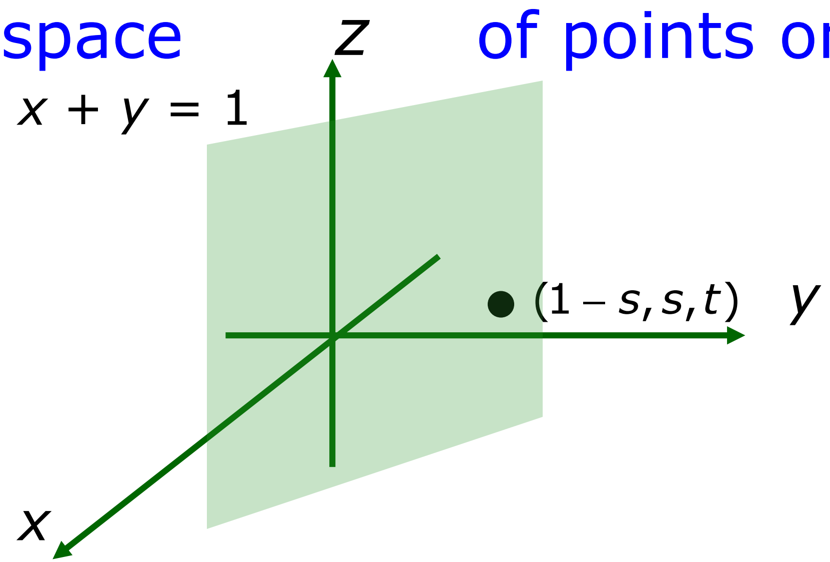
general solutions $\begin{cases} x = 1 - s \\ y = s \\ z = t \end{cases}$

regarded as
$x + y + 0z = 1$

Rewrite: $(x, y, z) = (1 - s, s, t)$

represents
a plane in 3D space

represents coordinates
of points on the plane



What is a linear system?

all a and b = 0 ⇒ zero system

otherwise ⇒ non-zero system

Definition 1.1.6

A system of linear equations (or a linear system)

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & & & & & & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{cases}$$

putting a few linear equations tgt

m linear equations

n variables $x_1, x_2, \ldots, x_n$

$a_{11}, a_{12}, \ldots, a_{mn}$ and $b_1, b_2, \ldots, b_m$ are real constants

double indices

What is a general solution of a LS?

Definition 1.1.6

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{cases}$$

real numbers $s_1, s_2, \ldots, s_n$

If all the equations are satisfied,

$$\boxed{x_1 = s_1, \ x_2 = s_2, \ \ldots, \ x_n = s_n}$$

a solution of the linear system

solution set and general solution of the system are defined similarly as before.

Solution of a LS

Example 1.1.7

$$\begin{cases} 4x_1 - x_2 + 3x_3 = -1 \\ 3x_1 + x_2 + 9x_3 = -4 \end{cases}$$

$x_1 = 1,\ x_2 = 2,\ x_3 = -1$ is a solution

$x_1 = 1,\ x_2 = 8,\ x_3 = 1$ is not a solution

solutions need to satisfy all the equations

Remark 1.1.8

Not all systems of linear equations have solutions.

$$\begin{cases} x + y = 4 \\ 2x + 2y = 6 \end{cases}$$

contradict

$\Rightarrow x + y = 3$

This system has no solution

What is a consistent/inconsistent LS?

Definition 1.1.9

A system of linear equations

no solution → **inconsistent system**

$$\begin{cases} x + y = 4 \\ 2x + 2y = 6 \end{cases}$$

at least one solution → **consistent system**

$$\begin{cases} 4x_1 - x_2 + 3x_3 = -1 \\ 3x_1 + x_2 + 9x_3 = -4 \end{cases}$$

Solution of a LS

Remark 1.1.10

Every system of linear equations has either

- no solution
- exactly one solution or
- infinitely many solutions

Discussion 1.1.11.1

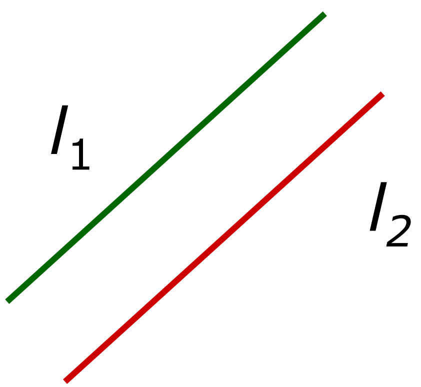
In the xy-plane, the system

$$\begin{cases} a_1x + b_1y = c_1 & (l_1) \\ a_2x + b_2y = c_2 & (l_2) \end{cases}$$

represent two straight lines.

a) l_1 and l_2 are parallel lines

The system has no solution



Geometrical interpretation

Discussion 1.1.11.1

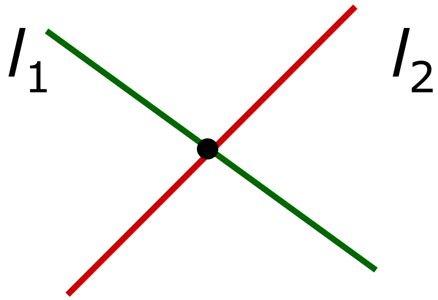
In the *xy*-plane, the system

$$\begin{cases} a_1x + b_1y = c_1 & (l_1) \\ a_2x + b_2y = c_2 & (l_2) \end{cases}$$

represent two straight lines.

b) l_1 and l_2 are not parallel lines.

The system has exactly one solution



Geometrical interpretation

Discussion 1.1.11.1

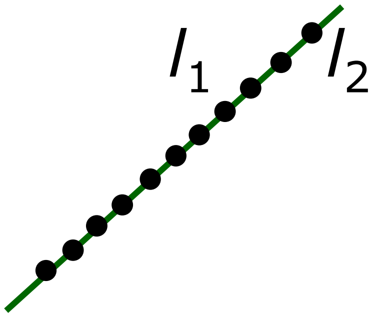
In the xy-plane, the system

$$\begin{cases} a_1x + b_1y = c_1 & (l_1) \\ a_2x + b_2y = c_2 & (l_2) \end{cases}$$

represent two straight lines.

c) l_1 and l_2 are the same lines.

The system has infinitely many solutions



Geometrical interpretation

Discussion 1.1.11.2

In the xyz-space, the system

2 equations
3 variables

$$\begin{cases} a_1x + b_1y + c_1z = d_1 & (p_1) \\ a_2x + b_2y + c_2z = d_2 & (p_2) \end{cases}$$

represents two planes.

The system has either
no solution or infinitely many solutions.

when variables > equations

either no / infinitely many solutions

Section 1.2

Elementary Row Operations

Objective

- What are the three elementary row operations?
- How to perform ERO on an augmented matrix?
- What is meant by row equivalence between two augmented matrices?

What is an augmented matrix of a LS?

Definition 1.2.1

linear system

m equations
n variables

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ \vdots & & & & & & & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m \end{cases}$$

variable not shown to facilitate row operations

augmented matrix

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right)$$

rectangular array

m rows
n+1 columns

Example 1.2.2

Consider the system of linear equations:

$$\begin{cases} x_1 + x_2 + 2x_3 = 9 \\ 2x_1 + 4x_2 - 3x_3 = 1 \\ 3x_1 + 6x_2 - 5x_3 = 0 \end{cases}$$

The augmented matrix of the system:

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right)$$

What are the three elementary row operations?

Definition 1.2.4

augmented matrix $$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

Consider the following three operations on the augmented matrix:

1. Multiply a row by a nonzero constant.
2. Interchange two rows.
3. Add a multiple of one row to another row.

These are called elementary row operations.

How to perform elementary row operations?

Definition 1.2.4

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right) \xrightarrow{\text{Multiply first row by 3}} \left(\begin{array}{ccc|c} 3 & 3 & 6 & 27 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right)$$

Add 2 times of first row to second row

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 4 & 6 & 1 & 19 \\ 3 & 6 & -5 & 0 \end{array}\right)$$

only second row is changed

Interchange second and third rows

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 3 & 6 & -5 & 0 \\ 2 & 4 & -3 & 1 \end{array}\right)$$

Why perform ERO ?

Discussion 1.2.3

Elementary row operations

1. Multiply a row by a nonzero constant.
2. Interchange two rows.
3. Add a multiple of one row to another row.

These are the basic steps for solving linear system.

Correspond to the following action on the system

1. Multiply an equation by a nonzero constant.
2. Interchange two equations.
3. Add a multiple of one equation to another equation.

Why perform ERO ?

Example 1.2.5

$$\begin{cases} x + y + 3z = 0 & (1) \\ 2x - 2y + 2z = 4 & (2) \\ 3x + 9y = 3 & (3) \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 2 & -2 & 2 & 4 \\ 3 & 9 & 0 & 3 \end{array}\right)$$

Add -2 times of Equation (1) to Equation (2) to obtain Equation (4).

$$\begin{cases} x + y + 3z = 0 & (1) \\ \quad - 4y - 4z = 4 & (4) \\ 3x + 9y = 3 & (3) \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 3 & 9 & 0 & 3 \end{array}\right)$$

This is equivalent to adding -2 times of the first row of the matrix to the second row.

Why perform ERO ?

Example 1.2.5

$$\begin{cases} x + y + 3z = 0 & (1) \\ -4y - 4z = 4 & (4) \\ 3x + 9y = 3 & (3) \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 3 & 9 & 0 & 3 \end{array}\right)$$

Add -3 times of Equation (1) to Equation (3) to obtain Equation (5).

$$\begin{cases} x + y + 3z = 0 & (1) \\ -4y - 4z = 4 & (4) \\ 6y - 9z = 3 & (5) \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 0 & 6 & -9 & 3 \end{array}\right)$$

This is equivalent to adding -3 times of the first row of the matrix to the third row.

Why perform ERO ?

Example 1.2.5

$$\begin{cases} x + y + 3z = 0 & (1) \\ \quad -4y - 4z = 4 & (4) \\ \quad 6y - 9z = 3 & (5) \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 0 & 6 & -9 & 3 \end{array}\right)$$

Add 6/4 times of Equation (4) to Equation (5) to obtain Equation (6).

$$\begin{cases} x + y + 3z = 0 & (1) \\ \quad -4y - 4z = 4 & (4) \\ \quad -15z = 9 & (6) \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 0 & 0 & -15 & 9 \end{array}\right)$$

This is equivalent to adding 6/4 times of the second row of the matrix to the third row.

Why perform ERO ?

Example 1.2.5

$$\begin{cases} x + y + 3z = 0 & (1) \\ \quad - 4y - 4z = 4 & (4) \\ \quad\quad - 15z = 9 & (6) \end{cases}$$

row echelon form (ref)

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 0 & 0 & -15 & 9 \end{array}\right)$$

By Equation (6), $z = -3/5$.

Substituting $z = -3/5$ into Equation (4),

$-4y - 4(-3/5) = 4 \quad \Leftrightarrow \quad y = -2/5$.

Substituting $y = -2/5$ and $z = -3/5$ into Equation (1)

$x + (-2/5) + 3(-3/5) = 0 \quad \Leftrightarrow \quad x = 11/5$.

Section 1.2

Elementary Row Operations

Objective

- What is meant by row equivalence between two augmented matrices?

What is row equivalence ?

Definition 1.2.6

Two augmented matrices are row equivalent (to each other)
if one can be obtained from the other
by a series of elementary row operations.

In example 1.2.5,

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 2 & -2 & 2 & 4 \\ 3 & 9 & 0 & 3 \end{array}\right) \leftrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 3 & 9 & 0 & 3 \end{array}\right) \leftrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 0 & 6 & -9 & 3 \end{array}\right) \leftrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 0 & 0 & -15 & 9 \end{array}\right)$$

Any 2 of the augmented matrices are row equivalent

What can we say about 2 row equivalent LS ?

Theorem 1.2.7

If augmented matrices of two linear systems are row equivalent, then the two systems have the same set of solutions.

$$
\begin{tikzcd}
\text{Linear System } \boldsymbol{X} \arrow[rr, leftrightarrow, "\text{same}", "\text{solutions}"'] \arrow[d] \arrow[drr, phantom, "\Uparrow"] & & \text{Linear System } \boldsymbol{Y} \arrow[d] \\
\text{Augmented Matrix of System } \boldsymbol{X} \arrow[rr, leftrightarrow, "\text{row}", "\text{equivalent}"'] & & \text{Augmented Matrix of System } \boldsymbol{Y}
\end{tikzcd}
$$

What can we say about 2 row equivalent LS ?

Example 1.2.8

All augmented matrices in Example 1.2.5 are row equivalent.

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 2 & -2 & 2 & 4 \\ 3 & 9 & 0 & 3 \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 3 & 9 & 0 & 3 \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 0 & 6 & -9 & 3 \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 0 & 0 & -15 & 9 \end{array}\right)$$

So all systems of linear equations in Example 1.2.5 have the same solution.

$$\begin{cases} x + y + 3z = 0 & (1) \\ 2x - 2y + 2z = 4 & (2) \\ 3x + 9y = 3 & (3) \end{cases} \qquad \begin{cases} x + y + 3z = 0 & (1) \\ -4y - 4z = 4 & (4) \\ 3x + 9y = 3 & (3) \end{cases}$$

$$\begin{cases} x + y + 3z = 0 & (1) \\ -4y - 4z = 4 & (4) \\ 6y - 9z = 3 & (5) \end{cases} \qquad \begin{cases} x + y + 3z = 0 & (1) \\ -4y - 4z = 4 & (4) \\ -15z = 9 & (6) \end{cases}$$

Remark 1.2.9

To see why Theorem 1.2.7 is true,
we only need to check that
every elementary row operation applied to an augmented matrix
will not change the solution set of the corresponding linear system.

1. Multiply a row by a nonzero constant
2. Interchange two rows
3. Add a multiple of one row to another

Section 1.3

Row-Echelon Forms

Objective

- How to identify a row-echelon form (REF) and a reduced row-echelon form (RREF)?
- How to use REF / RREF to get solutions of linear system?
- How to tell the number of solutions from REF?

How to identify REF?

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Definition 1.3.1

An augmented matrix is said to be in
row-echelon form
if it has the following 2 properties:

1. If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.

$$\left(\begin{array}{cccc|c} * & * & \cdots & \cdots & * \\ \vdots & \vdots & & & \vdots \\ * & * & \cdots & \cdots & * \\ \hline 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & 0 \end{array}\right) \begin{array}{l} \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} \text{nonzero rows} \\ \left.\begin{array}{c} \\ \\ \end{array}\right\} \text{zero rows (if any)} \end{array}$$

How to identify REF?

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 3 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Definition 1.3.1

2. In any two successive non-zero rows, the first nonzero number in the lower row occurs farther to the right than the first nonzero number in the higher row.

$$\left(\begin{array}{cccccc|c} 0 & 0 & \otimes & * & \cdots & \cdots & * \\ 0 & \cdots & \cdots & \otimes & * & \cdots & \cdots \end{array}\right) \left.\vphantom{\begin{array}{c} 0 \\ 0 \end{array}}\right\} \text{two successive rows}$$

leading entries

How to identify REF?

Definition 1.3.1

Combining properties 1 and 2:

$$
\left(\begin{array}{ccccccc|c}
0 & \otimes & \cdots & \cdots & * & \cdots & * & * \\
0 & \cdots & 0 & \otimes & * & \cdots & * & * \\
0 & \cdots & \cdots & \cdots & \ddots & \cdots & * & * \\
0 & \cdots & \cdots & \cdots & 0 & \otimes & * & * \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0
\end{array}\right)
\begin{array}{l}
\left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} \text{nonzero rows} \\
\left.\begin{array}{l} \\ \end{array}\right\} \text{zero rows (if any)}
\end{array}
$$

leading entries

everything below the "staircase" is 0

This is a row-echelon form (REF)

How to identify REF?

Definition 1.3.1

columns that contain pivot points called pivot columns

$$\left(\begin{array}{cccccc|c|c} 0 & \otimes & \cdots & \cdots & * & \cdots & * & * \\ 0 & \cdots & 0 & \otimes & * & \cdots & * & * \\ 0 & \cdots & \cdots & \cdots & \ddots & \cdots & * & * \\ 0 & \cdots & \cdots & \cdots & 0 & \otimes & * & * \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \end{array}\right)$$

nonzero rows

zero rows (if any)

non-pivot column

leading entries also called pivot point

How to identify RREF?

$$\begin{pmatrix} 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

×

$$\begin{pmatrix} 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

✓

Definition 1.3.1

An augmented matrix is said to be in reduced row-echelon form (RREF)

if it is in row-echelon form and has the following properties:

3. The leading entry of every nonzero row is 1.

$$\left(\begin{array}{ccccccc|c} 0 & \boxed{1} & \cdots & \cdots & * & \cdots & * & * \\ 0 & \cdots & 0 & \boxed{1} & * & \cdots & * & * \\ 0 & \cdots & \cdots & \cdots & \ddots & \cdots & * & * \\ 0 & \cdots & \cdots & \cdots & 0 & \boxed{1} & * & * \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \end{array}\right)$$

How to identify RREF?

$$\begin{pmatrix} 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \times \qquad\qquad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \checkmark$$

Definition 1.3.1

4. In each pivot column, except the pivot point, all other entries are zeros.

$$\left(\begin{array}{cccccccc|c} 0 & 1 & \cdots & 0 & * & 0 & \cdot & * & * \\ 0 & 0 & 0 & 1 & * & \vdots & \cdot & * & * \\ 0 & \vdots & \cdots & 0 & \ddots & 0 & \cdot & * & * \\ 0 & \vdots & \cdots & \vdots & 0 & 1 & \cdot & * & * \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdot & 0 & 0 \end{array}\right)$$

pivotal columns

Remark 1.3.2

In this module

Properties 1 + 2: REF

Properties 1 + 2 + 3 + 4: RREF

In some textbooks

Properties 1 + 2 + 3: REF

Properties 1 + 2 + 3 + 4: RREF

How to use REF / RREF to get solutions?

Discussion 1.3.4

If the augmented matrix of a linear system is in REF or RREF,

we can get the solutions of the system easily.

$$\left(\begin{array}{ccccccc|c} 0 & \otimes & \cdots & \cdots & * & \cdots & * & * \\ 0 & \cdots & 0 & \otimes & * & \cdots & * & * \\ 0 & \cdots & \cdots & \cdots & \ddots & \cdots & * & * \\ 0 & \cdots & \cdots & \cdots & 0 & \otimes & * & * \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{ccccccc|c} 0 & 1 & \cdots & 0 & * & 0 & * & * \\ 0 & \cdots & 0 & 1 & * & \cdots & * & * \\ 0 & \cdots & \cdots & \cdots & \ddots & 0 & * & * \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & * & * \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \end{array}\right)$$

REF RREF

How to use REF / RREF to get solutions?

Example 1.3.5.1

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right) \qquad \begin{cases} x_1 & & & = & 1 \\ & x_2 & & = & 2 \\ & & x_3 & = & 3 \end{cases}$$

The system has only one solution:

$$x_1 = 1, \; x_2 = 2, \; x_3 = 3.$$

How to use REF / RREF to get solutions?

Example 1.3.5.2

leading entries

more variables than equations

$$\left(\begin{array}{ccccc|c} 0 & 2 & 2 & 1 & -2 & 2 \\ 0 & 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 2 & 4 \end{array}\right) \qquad \left\{\begin{array}{rcrcrcrcl} 2x_2 & + & 2x_3 & + & x_4 & - & 2x_5 & = & 2 \\ & & x_3 & + & x_4 & + & x_5 & = & 3 \\ & & & & & & 2x_5 & = & 4 \end{array}\right.$$

3 2 1

non-pivot columns

variables - equations = number of parameters

use non-pivot column as parameters

x_1 = free parameter s

x_4 = free parameter t

$x_5 = 2$

$x_3 = 1 - t$

$x_2 = 2 + (1/2)t$

due to free parameters

The general solution is

$$\left\{\begin{array}{rcl} x_1 & = & s \\ x_2 & = & 2 + \frac{1}{2}t \\ x_3 & = & 1 - t \\ x_4 & = & t \\ x_5 & = & 2 \end{array}\right.$$

The system has infinitely many solutions.

How to use REF / RREF to get solutions?

Example 1.3.5.3

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 3 & -2 \\ 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{cases} x_1 - x_2 + 3x_4 = -2 \\ x_3 + 2x_4 = 5 \\ 0 = 0 \end{cases}$$

non-pivot columns

The general solution is

$$\begin{cases} x_1 = -2 + s - 3t \\ x_2 = s \\ x_3 = 5 - 2t \\ x_4 = t \end{cases}$$

The system has infinitely many solutions

Example 1.3.5.4

$$\left(\begin{array}{ccc|c} \boxed{0} & \boxed{0} & \boxed{0} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

non-pivot columns

zero system

The general solution is

$$\begin{cases} x_1 = r \\ x_2 = s \\ x_3 = t \end{cases}$$

The system has infinitely many solutions

How to use REF / RREF to get solutions?

Example 1.3.5.5

$$\left(\begin{array}{cc|c} 3 & 1 & 4 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{array}\right) \qquad \begin{cases} 3x_1 + x_2 = 4 \\ \quad\quad 2x_2 = 1 \\ \quad\quad\ \ 0 = 1 \end{cases}$$

This system is inconsistent, i.e. no solution.

Recall:
Any linear system has
- no solution
- exactly one solution
- infinitely many solutions

Section 1.4

Gaussian Elimination

Objective

- What are Gaussian elimination and Gauss-Jordan elimination?
- How to use GE / GJE to reduce an augmented matrix to a REF / RREF ?

Row echelon form of augmented matrix

Definition 1.4.1

Gaussian Elimination is an algorithm
to reduce an augmented matrix
to a row-echelon form
by using elementary row operations.

Carl Friedrich Gauss
(1777-1855)

$$\begin{pmatrix} \text{Augmented} \\ \text{matrix} \end{pmatrix} \xrightarrow{\text{e.r.o.}} \begin{pmatrix} \text{row - echelon} \\ \text{form} \end{pmatrix}$$

Systematic process

How to use GE to reduce a matrix to REF?

Algorithm 1.4.2

Step 1: Locate the leftmost column that does not consist entirely of zero.

Example A

$$\begin{pmatrix} 0 & 3 & \cdots & \cdots \\ 1 & -2 & \cdots & \cdots \\ 4 & 0 & \cdots & \cdots \end{pmatrix}$$

the first nonzero column

Example B

$$\begin{pmatrix} 0 & 3 & 1 & \cdots \\ 0 & 2 & -3 & \cdots \\ 0 & -1 & 6 & \cdots \end{pmatrix}$$

the first nonzero column

How to use GE to reduce a matrix to REF?

Algorithm 1.4.2

Step 2: Interchange the top row with another row, if necessary, to bring a nonzero entry to the top of the column found in Step 1.

Example A

$$\begin{pmatrix} 0 & 3 & \dots & \dots \\ 1 & -2 & \dots & \dots \\ 4 & 0 & \dots & \dots \end{pmatrix}$$

Interchange the 1st row with the 2nd row.

Example B

$$\begin{pmatrix} 0 & 3 & 1 & \dots \\ 0 & 2 & -3 & \dots \\ 0 & -1 & 6 & \dots \end{pmatrix}$$

No action is needed

How to use GE to reduce a matrix to REF?

Algorithm 1.4.2

Step 3: For each row below the top row, add a suitable multiple of the top row to it so that the entry below the leading entry of the top row becomes zero.

Example A

$$\begin{pmatrix} 1 & -2 & \dots & \dots \\ 0 & 3 & \dots & \dots \\ 4 & 0 & \dots & \dots \end{pmatrix}$$

Add -4 times of the 1st row to the 3rd row so that the entry marked by ● becomes 0.

Example B

$$\begin{pmatrix} 0 & 3 & 1 & \dots \\ 0 & 2 & -3 & \dots \\ 0 & -1 & 6 & \dots \end{pmatrix}$$

Add $-2/3$ times of the 1st row to the 2nd row so that the entry marked by ● becomes 0.

Add $1/3$ times of the 1st row to the 3rd row so that the entry marked by ● becomes 0.

Algorithm 1.4.2

Step 4: Now cover the top row in the matrix and begin again with Step 1 applied to the submatrix that remains.

Example A

$$\begin{pmatrix} 1 & -2 & \ldots & \ldots \\ \boxed{0} & 3 & \ldots & \ldots \\ 0 & 8 & \ldots & \ldots \end{pmatrix}$$

Cover the 1st row and work on the remaining rows.

Example B

$$\begin{pmatrix} 0 & 3 & 1 & \ldots \\ 0 & 0 & -11/3 & \ldots \\ 0 & 0 & 19/3 & \ldots \end{pmatrix}$$

Cover the 1st row and work on the remaining rows.

Continue in this way until the entire matrix is in row-echelon form.

What is Gauss-Jordan elimination?

Algorithm 1.4.3

Gauss-Jordan Elimination is an algorithm
to reduce an augmented matrix
to the reduce row-echelon form
by using elementary operations.

$$\overbrace{\underbrace{\begin{pmatrix}\text{Augmented}\\ \text{matrix}\end{pmatrix} \xrightarrow{\text{e.r.o.}} \begin{pmatrix}\text{row - echelon}\\ \text{form}\end{pmatrix}}_{\text{Gaussian Elimination}} \xrightarrow{\text{e.r.o.}} \begin{pmatrix}\text{reduce row}\\ \text{echelon form}\end{pmatrix}}^{\text{Gauss-Jordan Elimination}}$$

How to use GJE to reduce a matrix to RREF?

Algorithm 1.4.3

Given an augmented matrix, use Algorithm 1.4.1 to reduce it to a row-echelon form.

Example A

$$\begin{pmatrix} 0 & 3 & \cdots & \cdots \\ 1 & -2 & \cdots & \cdots \\ 4 & 0 & \cdots & \cdots \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 1 & -2 & \cdots & \cdots \\ 0 & 3 & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots \end{pmatrix}$$

Example B

$$\begin{pmatrix} 0 & 3 & 1 & \cdots \\ 0 & 2 & -3 & \cdots \\ 0 & -1 & 6 & \cdots \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 0 & 3 & 1 & \cdots \\ 0 & 0 & -11/3 & \cdots \\ 0 & 0 & 0 & \cdots \end{pmatrix}$$

How to use GJE to reduce a matrix to RREF?

Algorithm 1.4.3

Step 5: Multiply a suitable constant to each row so that all the leading entries becomes 1.

Example A

$$\begin{pmatrix} 1 & -2 & \dots & \dots \\ 0 & 3 & \dots & \dots \\ 0 & 0 & \dots & \dots \end{pmatrix}$$

No action is needed.

Multiply the 2nd row by 1/3 so that the entry marked by ● becomes 1.

Example B

$$\begin{pmatrix} 0 & 3 & 1 & \dots \\ 0 & 0 & -11/3 & \dots \\ 0 & 0 & 0 & \dots \end{pmatrix}$$

Multiply the 1st row by 1/3 so that the entry marked by ● becomes 1.

Multiply the 2nd row by – 3/11 so that the entry marked by ● becomes 1.

How to use GJE to reduce a matrix to RREF?

Algorithm 1.4.3

Step 6: Beginning with the last nonzero row and working upward, add a suitable multiples of each row to the rows above to introduce zeros above the leading entries.

Example C

$$\begin{pmatrix} 1 & -2 & \cdots & \cdots & -4 & \cdots \\ & 1 & \cdots & \cdots & 3 & \cdots \\ & & \ddots & \cdots & \cdots & \cdots \\ & & & 1 & 2 & \cdots \\ & & & & 1 & \cdots \end{pmatrix}$$

Add 4 times of the last row to the 1st row so the entry marked by becomes 0.

Add – 3 times of the last row to the 2nd row so the entry marked by becomes 0.

Add – 2 times of the last row to the next row so the entry marked by becomes 0.

How to use GJE to reduce a matrix to RREF?

Algorithm 1.4.3

Step 6: Beginning with the last nonzero row and working upward, add a suitable multiples of each row to the rows above to introduce zeros above the leading entries.

Example C

Apply the same process to the next pivot column on the left

$$\begin{pmatrix} 1 & -2 & \cdots & \cdots & 0 & \cdots \\ & 1 & \cdots & \cdots & 0 & \cdots \\ & & \ddots & \cdots & \cdots & \cdots \\ & & & 1 & 0 & \cdots \\ & & & & 1 & \cdots \end{pmatrix}$$

How to use GJE to reduce a matrix to RREF?

Example 1.4.4

$$\left(\begin{array}{ccccc|c} 0 & 0 & 2 & 4 & 2 & 8 \\ 1 & 2 & 4 & 5 & 3 & -9 \\ -2 & -4 & -5 & -4 & 3 & 6 \end{array}\right)$$

Gaussian Elimination

Gauss-Jordan Elimination

$$\left(\begin{array}{ccccc|c} 1 & 2 & 4 & 5 & 3 & -9 \\ 0 & 0 & 2 & 4 & 2 & 8 \\ 0 & 0 & 0 & 0 & 6 & -24 \end{array}\right) \longrightarrow \left(\begin{array}{ccccc|c} 1 & 2 & 0 & -3 & 0 & -29 \\ 0 & 0 & 1 & 2 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & -4 \end{array}\right)$$

Do we have to strictly follow the steps ?

Remark 1.4.5.2

In the actual implementation of the algorithms, the steps mentioned in Algorithm 1.4.2 and Algorithm 1.4.3 are usually modified to avoid the round-off errors during the computation

Ill-conditioned matrix

See Exercise 1 Q21

Do we have to strictly follow the steps ?

Additional remarks

Modification in GE

Example

Standard

$$\begin{pmatrix} 4 & 3 & \ldots & \ldots \\ 1 & -2 & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots \end{pmatrix} \longrightarrow \begin{pmatrix} 4 & 3 & \ldots & \ldots \\ 0 & -11/4 & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots \end{pmatrix}$$

Variation

$$\begin{pmatrix} 1 & -2 & \ldots & \ldots \\ 4 & 3 & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -2 & \ldots & \ldots \\ 0 & 11 & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots \end{pmatrix}$$

Remark 1.4.5.1

Every matrix can have many different row-echelon forms.

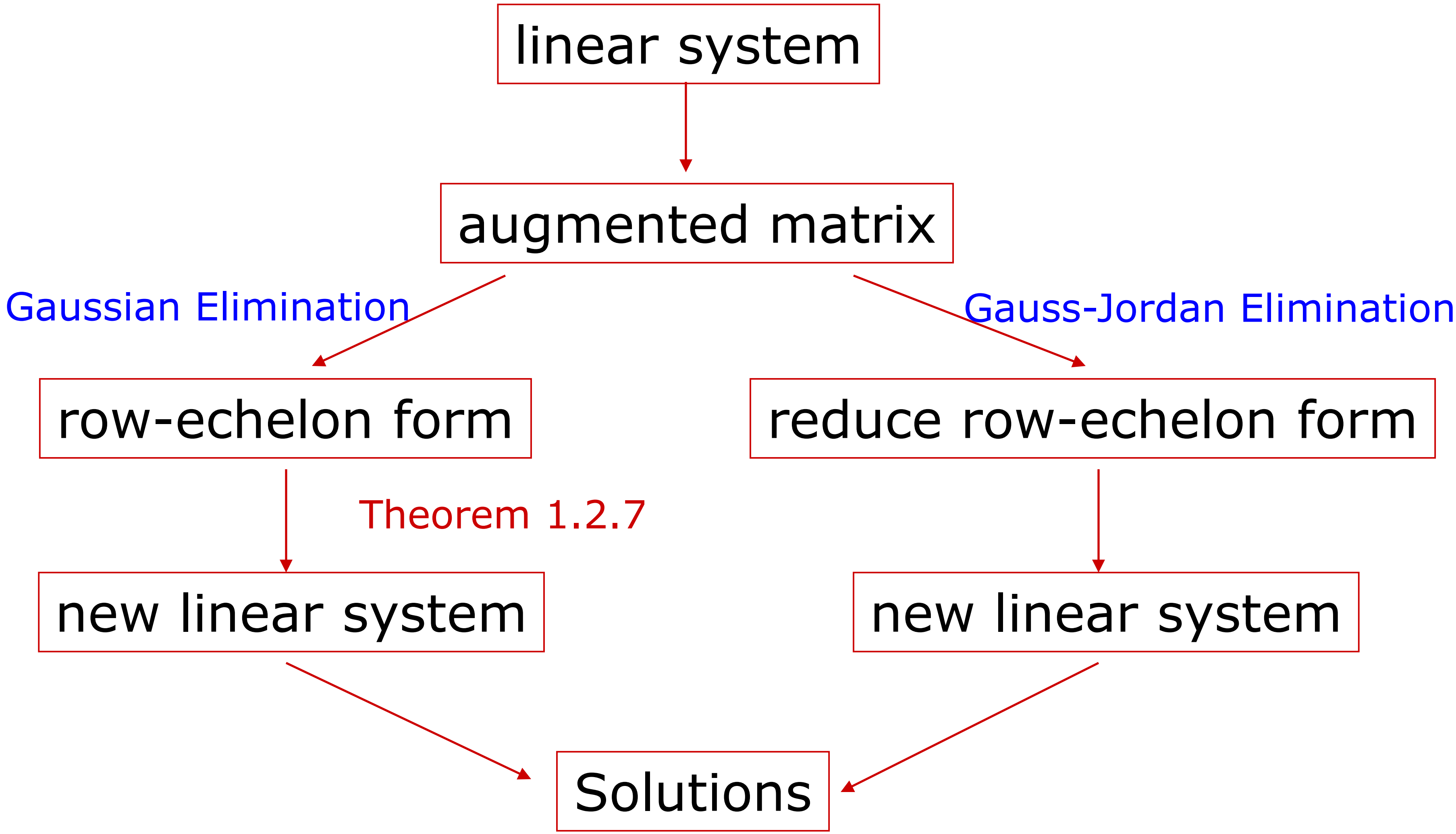
Every matrix has a unique reduced row-echelon form

$$\begin{tikzcd}
 & \left(\begin{matrix}\text{row - echelon}\\ \text{form}\end{matrix}\right) \arrow[dr] & \\
\left(\begin{matrix}\text{Augmented}\\ \text{matrix}\end{matrix}\right) \arrow[ur] \arrow[r, "\text{e.r.o.}"] \arrow[dr] & \left(\begin{matrix}\text{row - echelon}\\ \text{form}\end{matrix}\right) \arrow[r, "\text{e.r.o.}"] & \left(\begin{matrix}\text{reduce row}\\ \text{echelon form}\end{matrix}\right) \\
 & \left(\begin{matrix}\text{row - echelon}\\ \text{form}\end{matrix}\right) \arrow[ur] &
\end{tikzcd}$$

two augmented matrix are row equivalent if they give rise to the same rref

How to use GE/GJE to find solutions of LS ?

Discussion 1.4.6



How to tell the number of solutions from REF?

Remark 1.4.8.1

A linear system has no solution if:

REF has a row with nonzero last entry but zero elsewhere.

The last column of REF is a pivot column.

$$\left(\begin{array}{ccccccc|c} 0 & \otimes & \cdots & \cdots & * & \cdots & * & * \\ 0 & \cdots & 0 & \otimes & * & \cdots & * & * \\ 0 & \cdots & \cdots & \cdots & \ddots & \cdots & * & * \\ 0 & \cdots & \cdots & \cdots & 0 & \otimes & * & * \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \otimes \end{array}\right) \quad \text{e.g.} \quad \left(\begin{array}{ccc|c} 3 & 2 & 3 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right)$$

Remark 1.4.8.2

A linear system has exactly one solution if:

every column of REF is a pivot column, except the last column.

$$\left(\begin{array}{cccc|c} \otimes & \cdots & * & \cdots & * \\ \cdots & \otimes & * & \cdots & * \\ \cdots & \cdots & \ddots & \cdots & * \\ \cdots & \cdots & 0 & \otimes & * \\ \cdots & \cdots & \cdots & \cdots & 0 \end{array}\right)$$

How to tell the number of solutions from REF?

Remark 1.4.8.2

In other words, a consistent linear system has exactly one solution if:

# of variables in LS = # of nonzero rows in REF

e.g. $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & -1 & 2 \end{array}\right)$ $\left(\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 4 \\ 0 & 2 & 0 & 1 & -1 \\ 0 & 0 & 4 & -1 & 2 \\ 0 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

How to tell the number of solutions from REF?

Remark 1.4.8.3

A linear system has infinitely many solutions if:

there is a non-pivot column in the REF, other than the last column.

$$\left(\begin{array}{ccccccc|c} 0 & \otimes & \cdots & \cdots & * & \cdots & * & * \\ 0 & \cdots & 0 & \otimes & * & \cdots & * & * \\ 0 & \cdots & \cdots & \cdots & \ddots & \cdots & * & * \\ 0 & \cdots & \cdots & \cdots & 0 & \otimes & * & * \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \end{array}\right)$$

How to tell the number of solutions from REF?

Remark 1.4.8.3

In other words, a consistent linear system has infinitely many solutions if:

# of variables in LS > # of nonzero rows in REF

e.g. $\left(\begin{array}{cccc|c} 5 & 1 & 2 & 3 & 4 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)$ $\left(\begin{array}{cccc|c} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)$