2.5: Properties of Derivatives (Lec 6)

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January 31, 2024

Theorem 1 (Chain rule)**.** Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open setse. Let $g : U \subset \mathbb{R}^n \mapsto \mathbb{R}^m$ and $f : V \subset \mathbb{R}^m \mapsto \mathbb{R}^p$ be functions such that $f \circ g$ is defined. Suppose g is differentiable at $\vec{x_0}$ and f is differentiable at $\vec{y_0} = g(\vec{x_0})$. Then $f \circ g$ is differentiable at $\vec{x_0}$ and

$$\mathbf{D}(f \circ g)(\vec{x_0}) = \mathbf{D}f(\vec{y_0})\mathbf{D}g(\vec{x_0})$$

Where

Theorem 2 (Special case of chain rule, with function of path)**.** Imagine we have

$$\begin{aligned} \mathbb{R} &\mapsto \mathbb{R}^3 \mapsto \mathbb{R} \\ t &\mapsto \vec{c}(t) \mapsto f(\vec{c}(t)) \end{aligned}$$

Where path $\vec{c}(t) = (x(t), y(t), z(t)$. Let $h(t) = f(\vec{c}(t))$. Then

$$\frac{dh}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dy}{dt}$$

Remark. We can intuitively think about this as describing the change in h as the change in f w.r.t x times how much x varies with t, and so on for each of the component functions of f. Note that another way to state the chain rule above is:

$$\frac{dh}{dt} = \nabla f(\vec{c}(t)) \cdot \vec{c}\,'(t)$$

Which is a special case of the chain rule as enumerated in theorem 1 above, where $\vec{c} = g$ and $m = 3$:

$$\nabla(f(\vec{c}(t)) \cdot \vec{c}\,'(t) = \mathbf{D}f(\vec{c}(t))\mathbf{D}\vec{c}(t)$$

Note that $\mathbf{D}f$ is a 1×3 matrix, because it has a single output and three inputs ($m = 3$), while $\mathbf{D}c$ is a 3×1 matrix, because it has one input ((t) and three outputs.