Math 221 HW3

Asa Royal (ajr74)

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1.5

1. By solving a system of equations, find the linear combination of vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ that gives $\mathbf{b} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$.

   We want $\mathbf{x} = (x_1, x_2, x_3)$ s.t. $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{b}$, so we solve $A\mathbf{x} = \mathbf{b}$ where $A = \begin{bmatrix} \mathbf{v1} & \mathbf{v2} & \mathbf{v3} \end{bmatrix}$.

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 0 \\ -1 & 2 & 1 & -2 \end{array}\right] \xrightarrow{A_3 = A_3 + A_1} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 3 & 1 \end{array}\right] \xrightarrow{A_3 = A_3 - 2A_2} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow[A_2 = A_2 - A_3]{A_1 = A_1 - 2A_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

   We see that $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, which means the linear combo of columns that gives $\mathbf{b}$ is $\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3$

13. Suppose A is an $m \times n$ matrix with rank m and $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ are vectors with $\text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \mathbb{R}^n$. Prove that $\text{Span}(A\mathbf{v}_1, \ldots, A\mathbf{v}_k) = \mathbb{R}^m$

    Proof. Since A has rank m, there is an $A\mathbf{x} = \mathbf{b}$ solution for all $b \in \mathbb{R}^m$. We want to show that $\text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$ can form that vector $\mathbf{x}$ (i.e., any vector in $\mathbb{R}^n$). We let $\mathbf{x} = \text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_k) = c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k$, which can form any vector in $\mathbb{R}^n$. Then $A\mathbf{x}$ can form any vector in $\mathbb{R}^m$. $A\mathbf{x} = A(c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k) = c_1A\mathbf{v}_1 + \ldots + c_kA\mathbf{v}_k = \text{Span}(A\mathbf{v}_1, \ldots, A\mathbf{v}_k) = \mathbb{R}^m$ □

14. Let A be an $m \times n$ matrix with row vectors $\mathbf{A}_1, \ldots, \mathbf{A}_m \in \mathbb{R}^n$.

    (a) Suppose $\mathbf{A}_1 + \ldots + \mathbf{A}_m = \mathbf{0}$. Deduce that $\text{rank}(A) < m$.

    Proof. During Gaussian elimination, scalar multiples of rows $\mathbf{A}_1, \ldots, \mathbf{A}_{m-1}$ are added to $\mathbf{A}_m$ to cancel coefficients below pivot variables. Since $\mathbf{A}_m = -(\mathbf{A}_1 + \ldots + \mathbf{A}_{m-1})$, the row operations that cancel the coefficients of $\mathbf{A}_m$ are the addition of $\mathbf{A}_1, \ldots, \mathbf{A}_{m-1}$. This yields $\mathbf{A}_m = -(\mathbf{A}_1 + \ldots + \mathbf{A}_{m-1}) + (\mathbf{A}_1 + \ldots + \mathbf{A}_{m-1}) = \mathbf{0}$. Since $\mathbf{A}_m = \mathbf{0}$, it has no pivot variables, which means there can be at most **m-1** pivots. Therefore $\text{rank}(A) < m$. □

(b) More generally, suppose there is some linear combination $c_1\mathbf{A}_1 + \ldots + c_m\mathbf{A}_m = \mathbf{0}$, where some $c_i \neq 0$. Show that $\text{rank}(A) < m$.

Proof. Per the equation above,

$$c_m\mathbf{A}_m = -(c_1\mathbf{A}_1 + \ldots + c_{m-1}\mathbf{A}_{m-1})$$

which means

$$\mathbf{A}_m = -(c_1'\mathbf{A}_1 + \ldots + c_{m-1}'\mathbf{A}_{m-1})$$

During Gaussian elimination, scalar multiples of rows $\mathbf{A}_1, \ldots, \mathbf{A}_{m-1}$ are added to $\mathbf{A}_m$ to cancel coefficients below pivot variables. We can see that when scaled rows $c_1'\mathbf{A}_1, \ldots, c_{m-1}'\mathbf{A}_{m-1}$ are added to $\mathbf{A}_m$, the modified row will be $\mathbf{0}$. Since $\mathbf{A})_m = \mathbf{0}$, it has no pivot variables, which means there can be at most $m - 1$ pivots. Therefore $\text{rank}(A) < m$. $\square$