Homework 4

Reading: Meester, Section 2.2, 2.3. Notes on independence (*indep.pdf*).

Idea Journal: Remember to submit the idea journal entries after class.

Problems/Exercises: All of these are due **Monday, February 12 by 11:00pm**.

1. Consider tossing a p-coin 3 times (independent tosses). Let X be the number of heads that occur.

   (i) Plot the distribution (as a bar graph) of X for $p = 1/2$ and for $p = 3/4$. For comparison, plot them on the same graph, with the bars side-by-side, slightly staggered or using different shading for the two cases. Label your axes and all salient features (i.e. heights of the bars).

   (ii) Plot the cumulative distributioin function (CDF) of X for $p = 1/2$ and for $p = 3/4$. For comparison, plot the CDFs on the same graph (but a different graph from what you created in part (i)).

2. Meester Exercise 2.7.6. Note: the three balls drawn **without** replacement.

3. Let $X_n \sim$ Geometric(p) with $p = \lambda/n$ (recall that $X_n \sim$ Geometric(p) means that X_n is a random variable with the has the Geometric(p) distribution). Here $\lambda > 0$ is a fixed parameter, and we will consider $n \to \infty$. Let $T_n = \frac{1}{n} X_n$. Prove that for any $t > 0$,

$$\lim_{n\to\infty} \mathbb{P}(T_n > t) = e^{-\lambda t}.$$

   A random variable T has the Exp(λ) distribution if $\mathbb{P}(T > t) = e^{-\lambda t}$ for all $t \geq 0$. This computation shows that the distribution of T_n is close to Exp(λ) when n is large.

4. Let $Y_1, \ldots, Y_n$ be independent random variables, having the same distribution. Since they have the same distribution, they have the same CDF: $F(y) = \mathbb{P}(Y_k \leq y)$. Define a new random variable:

$$X(\omega) = \max\left(Y_1(\omega), Y_2(\omega), \ldots, Y_n(\omega)\right). \tag{0.1}$$

   Compute the CDF of X in terms of F.

5. Referring to Proposition 0.1 in the notes on independence (*indep.pdf*) prove that (ii) implies (i). Hint: consider $a_1, \ldots, a_n \in \mathbb{R}$, $a_k \in \mathcal{R}(X_k)$ for $k = 1, \ldots, n$. Let $\epsilon > 0$ and define intervals $I_k^\epsilon = (a_k - \epsilon, a_k + \epsilon)$. Apply (ii) to these intervals, then take $\epsilon \to 0$ and apply Lemma 2.1.14(b) of Meester.

6. Suppose that $X_1, \ldots, X_n$ are independent random variables, each having the Geometric(p) distribution, for some fixed $p \in (0, 1)$. Define a new random variable:

$$Y(\omega) = \min\left(X_1(\omega), X_2(\omega), \ldots, X_n(\omega)\right). \tag{0.2}$$

Show that Y has the Geometric(α) distribution for some paramter α (compute α in terms of p and n). Compute $\mathbb{E}[Y]$ in terms of p and n. Hint: Think about $\mathbb{P}(Y > k)$. Comment: For an interpretation of Y, you might imagine that there are n people, each having a p-coin. They each toss their coin until they get heads. X_k is the number of tosses that the kth person must make until achieving heads. So, Y is the minimum of these (the lowest score among the group of n people).

7. Consider the following game: Roll a standard six-sided die. If the number rolled is 1,2,3, you win nothing. If the number rolled is 4,5, or 6, you win \$1 plus twice the value rolled. What is the expected amount you win in a single roll?

8. Suppose X is a random variable, uniformly distributed on $\{1, \dots, n\}$. Compute $\mathbb{E}[X^2]$ in terms of n.

9. Suppose you are given an infinite sequence of 1's and 0's: $a_1, a_2, a_3, \dots$ with $a_k \in \{0, 1\}$. Consider the following game: you toss a fair coin until heads occurs. If the first head occurs on the kth toss and $a_k = 1$ (the index k is not fixed; k is determined by the coin tosses) then you win \$1; otherwise you win nothing. Let X be the amount you win. Compute $\mathbb{E}[X]$. In particular, given any real number $\alpha \in (0, 1)$, can you choose the sequence $(a_1, a_2, \dots)$ so that $\mathbb{E}[X] = \alpha$?

10. A box contains two marbles: a red one, and a blue one. Pick a marble at random. If you pick red, you win and the process stops. If it is blue, replace it and add another blue marble (then there are 2 blue and 1 red). Continue in this way until you draw a red: each time you draw a blue, replace it and add one more blue. So, if you keep drawing blue, the proportion of blue grows and there are k blue marbles at the beginning of the k^{th} round. Thus, it becomes less likely to draw the red marble.

    (i) What is the probability that you have not drawn red after n attempts?

    (ii) What is the probability that you *never* draw the red marble?

    (iii) Let T be the number of draws until you draw a red. What is the distribution of T? Is $\mathbb{E}[T]$ finite?

    (iv) Instead of adding just one more blue marble each time you draw a blue, suppose you double the number of blue. Is the answer to parts (ii) and (iii) different in this case? Explain.

    Hint: when thinking about products of many numbers, it can be useful to think about the logarithm of the product and use the fact that $\log(1 + x) \approx x$ when $|x|$ is small.