Math 340: Lec 8 Big Ideas Journal (Independence of random variables)

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Definition 1 (Independence of random variables)**.** Two discrete random variables X and Y are independent if $\forall a \in \text{range}(X), \forall b \in \text{Range}(Y)$,

$$\mathbb{P}(X = a, Y = b) = \mathbb{P}(X = a)\mathbb{P}(Y = b)$$

Or, assuming $\mathbb{P}(Y = b) \neq 0$,

$$\mathbb{P}(X = a | Y = b) = \mathbb{P}(X = a)$$

Remark. If X and Y are independent random variables, the event that X takes on some value a should give us no information about the value Y takes on.

Theorem 2 (Independence of many random variables)**.** Discrete random variables $X_1, X_2, \ldots, X_n$ are independent iff

$$\mathbb{P}(X_1 = a_1, X_2 = a_2, \ldots, X_n = a_n) = \mathbb{P}(X_1 = a_1) * \mathbb{P}(X_2 = a_2) * \ldots * \mathbb{P}(X_n = a_n)$$

where $a_k \in \text{Range}(X_k)$.

Corollary 3. TFAE:

1. $X_1, X_2, \ldots, X_n$ are independent random variables
2. For any intervals $I_1, \ldots, I_n \subset R$,

$$\mathbb{P}(X_1 \in I_1, X_2 \in I_2, \ldots, X_n \in I_n) = \mathbb{P}(X_1 \in I_1) * \mathbb{P}(X_2 \in I_2) * \ldots * \mathbb{P}(X_n \in I_n)$$

3. For any $a_1, \ldots, a_n$

$$\mathbb{P}(X_1 \leq a_1, X_2 \leq a_2, \ldots, X_n \leq a_n) = \mathbb{P}(X_1 \leq a_1) * \mathbb{P}(X_2 \leq a_2) * \ldots * \mathbb{P}(X_n \leq a_n)$$

Proof. 2 $\iff$ 3 is trivial. Proved 2 $\implies$ 1 in hw. □

Independence when N is chosen randomly

Example (Trivial non-independence of t/f in N flips)**.** Imagine we toss N p-coins. Let $X = \#$ heads, $Y = \#$ tails. Clearly $X + Y = N$, so we imagine the two r.v.s are not independent. Indeed, observe $\mathbb{P}(X = N, Y = N) = 0$, since we cannot flip n heads and n tails in n tosses. But $\mathbb{P}(X = n) * \mathbb{P}(Y = n) \neq 0$.

Example (Independence of t/f counts when N is Poisson random)**.** Let $N \sim \text{Poisson}(\lambda)$. If we toss N p-coins and let $X = \#$ heads, $Y = \#$ tails, then:

1. X and Y are independent
2. $X \sim \text{Poisson}(\lambda p), Y \sim \text{Poisson}(\lambda(1 - p))$

Proof. Proofs are a bitch. Check notes. □

Remark. The difference between the two examples above where X, Y are dependent/independent is that in example 2, N is randomly chosen, so knowing $X = a$ doesn't give us information about Y.