Math 222: (Area of surfaces, surface integrals)

Asa Royal (ajr74)

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0.1 Tangent vectors

Definition 1 (tangent vector to a parametrized surface). Per the definition of a partional derivative,

$$\frac{\partial \Phi}{\partial u} = \lim_{u \to \infty} \frac{\Phi(u + \Delta u, v) - \Phi(u, v)}{\Delta u}$$

Loosening rigor, we can kinda think of $\frac{\partial \Phi}{\partial U}$ as

$$\frac{\partial \Phi}{\partial u} = \frac{\Phi(u + du, v) - \Phi(u, v)}{du}$$

Which means

$$\frac{\partial \Phi}{\partial u} du = \vec{T}_u du = \Phi(u + du, v) - \Phi(u, v)$$

0.2 Calculating area of a parametrized surface

Remark. $\vec{T}_u \times \vec{T}_v = \frac{\partial \Phi}{\partial u} du \times \frac{\partial \Phi}{\partial v} dv$ forms a parallelogram ($d\sigma$)on the surface. The total area of the surface is the sum of all such parallelograms ($\int \int_\Sigma d\sigma$). But it's difficult to count how many parallelograms that would be. So instead, we count using u, v:

Theorem 2 (Surface of area). Imagine some surface is parametrized by $\Phi(u, v) = (x, y, z)$. We can calculate its surface area as

$$\begin{aligned} \text{Area}(s) &= \int \int_D \|\vec{T}_u \times \vec{T}_v\| du dv \\ &= \int \int_D \sqrt{\left(\frac{\partial(x, y)}{\partial(u, v)}\right)^2 + \left(\frac{\partial(y, z)}{\partial(u, v)}\right)^2 + \left(\frac{\partial(x, z)}{\partial(u, v)}\right)^2} \end{aligned}$$

Remark. Recall, the arc length of a path $c(t)$ is given by $\int_a^b \|\vec{c}'(t)\| dt$. When we calculate surface area of S, $\|\vec{T}_u \times \vec{T}_v\|$ is the analog fo $\|\vec{c}'(t)\|$

Corollary 3. Special case: The surface area of a graph $z = g(x, y)$ is given by

$$\begin{aligned} SA &= \int \int_D \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + \left(\frac{\partial g}{\partial z}\right)^2} dx dy \\ &= \int \int_D \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1} dx dy \end{aligned}$$

Parametrization: imagine $x = u, y = v, z = g(u, v)$

0.3 Surface integrals of scalar functions

Theorem 4 (Surface integral of scalar function)**.** Let f be a continuous scalar function defined on a parametrized surface S. The integral of f over S integrals

$$\int\int_S f dS = \int\int_\Phi f dS = \int\int_D f(\Phi(u,v))\|\vec{T}_u \times \vec{T}_v\| dudv$$

where $\Phi : D \mapsto S$.

Remark. If $f = \delta$ is a density function, $\int\int_S f ds =$ mass.

Theorem 5 (Surface integral of vector functions)**.** Let $\vec{F}$ be a vector field defined on S, the image of a parametrized surface Φ. The surface integral of $\vec{F}$ over S is:

$$\begin{aligned}\int\int_S \vec{F} \cdot dS &= \int\int_d \vec{F}(\Phi(u,v,)) \cdot (\vec{T}_u \times \vec{T}_v) dudv \\ &= \int\int_D \vec{F}(\Phi(u,v,)) \cdot \vec{N}\|\vec{T}_u \times \vec{T}_v\| dudv\end{aligned}$$

Where $\vec{N}$ is the unit normal vector to the surface.

Remark. The surface integral of a vector function lets us calculate flux: how much of a fluid (or vector field) flows through the surface. Each dS is a tiny bit of surface.