Math 340: Lec 26 Markov Chains (4)

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0.1 Expected return time

Definition 1 (Return time). Fix $x \in S$. The **first arrival/return time** for S is defined as

$$T_x = min\{n \geq 1 | X_n = x\}$$

Corollary 2. State X is recurrent iff

$$\mathbb{P}(T_x < \infty | X_0 = x) = 1$$

Definition 3 (Expected time of firrst return to x). The epxected time to first return is

$$\mu_x = \mathbb{E}[T_x | X_0 = x]$$

Definition 4 (positive/null recurrence). We say that x is positive recurrent if $\mathbb{E}[T_x | X_0 = x] < \infty$. I.e. we expect to return to x in a finite amount of time. We say that x is null recurrent if $\mathbb{E}[T_x | X_0 = x] = \infty$.

Remark. Example of a null-recurrent markov chain: A simple random walk on $\mathbb{Z}$.

Theorem 5 (Stationary distribution relation to expected first return). An irreducible chhain has a stationary distribution iff all states are positive recorrent. Additionally,

$$\pi_x = \frac{1}{\mathbb{E}[T_x | X_0 = x]}$$

0.2 Q's about random walks

Example (Given two states A and B, what's the probability of reaching A before B?). Imagine we start at state X_0. Define $h(x)$ as the probability of reaching A before B when starting at x. Note that $h(A) = 1, h(B) = 0$ and additionally that $\forall x \in S \backslash \{A, B\}$,

$$h(X) = \sum_{y \in S} h(y) P_{x,y}$$

$h(A), h(B)$, and the $h(x)$ equations for a system of linear equations that can be solved for each $h(x)$.

Example (Expected return time for a simple random walk). For a simple random walk, $h(x) = x/B$. If we've hit B, it would take $2B + 1$ steps to return to zero, so note that

$$\mathbb{P}(T_0 > B | X_0 = 1) \geq \frac{1}{B}$$

And thus by the tail sum formula,

$$\mathbb{E}[T_0 | X_0 = 1] = \sum_{k=1}^{\infty} \mathbb{P}(T_0 \geq t | X_0 = 1)$$
$$\geq \sum_{k=1}^{\infty} \frac{1}{k} = \infty$$

So the expected time of return for a simple random walk is ∞. The Markov Chain is null recurrent!