3.5: Implicit Function Theorem

Asa Royal (ajr74)

2/21/24

Remark. Intuition of one-variable implicit function theorem: Imagine we have some C^1 function $y = f(x)$ and $f'(x_0) \neq 0$. Then locally near x_0, we can solve for x to find the inverse function $x = f^{-1}(y)$.

Why? If $f'(x_0) \neq 0$, the function is injective in the neighborhood of x_0. i.e. Given a value of $y = f(x)$, we can uniquely identify an x. i.e. f^{-1} exists!

Given an implicit curve $F(x, y) = c$, the ICT will tell us under what conditions the curve lets us eexplicitly define $y = f(x)$ and allows us to find $f'(x)$?

$f'(x) = -\frac{F_x}{F_y}$ if $F_y \neq 0$.

Remark. Assume we have some $F(x, y, z) = c$ level set. We can solve for z as a function of x, y at $(x_0, y_0, z_0$ iff $\frac{\partial F}{\partial z}(x_0, y_0, z_0) \neq 0$

Theorem 1 (IFT higher dimension)**.** Assume we want to solve m equations for m variables $z_1, \ldots, z_m$ at $(x_0, \ldots, x_n)$. Then we must have

$$\det \begin{bmatrix} \frac{\partial F_1}{\partial z_1} & \cdots & \frac{\partial F_1}{\partial z_m} \\ \vdots & & \vdots \\ \frac{\partial F_m}{\partial z_1} & \cdots & \frac{\partial F_m}{\partial z_m} \end{bmatrix} \neq 0$$

at $(x_0, \ldots, x_n)$. **The function must be full rank! Full rank = invertible!**

Remark. We can then find, e.g., $\frac{\partial u}{\partial x}$ by implicitly differentiating the m equations with respect to x and solving the set of linear equations.

Theorem 2 (inverse function theorem)**.** Say we have functions $f_1, \ldots, f_n$ with continuous partials. If near a given solution $\vec{x}_0, \vec{y}_0, J(f)(\vec{x}_0) \neq 0$, then the equations

$$\begin{cases} f_1(x_1, & \ldots, x_n) = y_1 \\ & \cdots \\ f_n(x_1, & \ldots, x_n) = y_n \end{cases}$$

can be uniquely solved as $\vec{x} = g(\vec{y})$