Math 340 HW 4

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1. Appended to pdf.

2. **2.7.6**: An urn contains 8 white, 4 black, and 2 red balls. We win 2 euro for each black ball we draw and lose 1 euro for each white ball we draw. We choose three balls from the urn. Let X denote our winnings. Write down the probability mass function of X.

   We begin by noting that the random variable X ranges across integers in $[-3, 6]$. The minimum value of X occurs when we draw 3 white balls; the maximum when we draw 3 black balls. We now identify the outcomes that lead to the event where X takes on each value in its range. Using those outcomes, we derive the probability of each event (i.e. the PMF).
   $X = -3$: We draw 3 white balls.

$$\mathbb{P}(X = -3) = \frac{\binom{8}{3}}{\binom{14}{3}} = \frac{2}{13}$$

   $X = -2$: We draw 2 white balls and a red ball.

$$\mathbb{P}(X = -2) = \frac{\binom{8}{2}\binom{2}{1}}{\binom{14}{3}} = \frac{2}{13}$$

   $X = -1$: We draw a white ball and 2 red balls.

$$\mathbb{P}(X = -1) = \frac{\binom{8}{1}\binom{2}{2}}{\binom{14}{3}} = \frac{2}{91}$$

   $X = 0$: We draw a black ball and 2 white balls.

$$\mathbb{P}(X = 0) = \frac{\binom{4}{1}\binom{8}{2}}{\binom{14}{3}} = \frac{4}{13}$$

   $X = 1$: We draw a white ball, a black ball, and a red ball.

$$\mathbb{P}(X = 1) = \frac{\binom{4}{1}\binom{8}{1}\binom{2}{1}}{\binom{14}{3}} = \frac{16}{91}$$

   $X = 2$: We draw a black ball and 2 red balls.

$$\mathbb{P}(X = 2) = \frac{\binom{4}{1}\binom{2}{2}}{\binom{14}{3}} = \frac{1}{91}$$

   $X = 3$: We draw 2 black balls and a white ball.

$$\mathbb{P}(X = 3) = \frac{\binom{4}{2}\binom{8}{1}}{\binom{14}{3}} = \frac{12}{91}$$

   $X = 4$: We draw 2 black balls and a red ball.

$$\mathbb{P}(X = 4) = \frac{\binom{4}{2}\binom{2}{1}}{\binom{14}{3}} = \frac{3}{91}$$

   $X = 5$: This event cannot occur.

$$\mathbb{P}(X = 5) = 0$$

   $X = 6$: We draw 3 black balls.

$$\mathbb{P}(X = 6) = \frac{\binom{4}{3}}{\binom{14}{3}} = \frac{1}{91}$$

3. Let $Xn \sim$ Geometric(p) with $p = \lambda/n$. Let $\lambda > 0$ and $n \to \infty$. Let $Tn = \frac{1}{n}X_n$. Prove that for any $t > 0$,

$$\lim_{n\to\infty} \mathbb{P}(T_n > t) = e^{-\lambda t}$$

Proof. $\mathbb{P}(T_n > t) = \mathbb{P}(\frac{1}{n}X_n > t) = \mathbb{P}(X_n > nt)$. For a geometric distribution, this is

$$\begin{aligned}
\mathbb{P}(X_n > nt) &= \sum_{j=nt+1}^{\infty} p(1-p)^{j-1} && (1)\\
&= p(1-p)^{nt} + p(1-p)^{nt+1} + p(1-p)^{nt+2} + \dots && (2)\\
&= p(1-p)^{nt}\sum_{k=0}^{\infty}(1-p)^k && (3)\\
&= p(1-p)^{nt}\frac{1}{p} \qquad \text{geo. series convergence} && (4)\\
&= (1-p)^{nt} && (5)
\end{aligned}$$

$p = \lambda/n$, so (5) is equivalent to $\left(1 - \frac{\lambda}{n}\right)^{nt}$, which converges to $e^{-\lambda t}$ when $n \to \infty$. □

4. Let F_X be the CDF of X. Since X is the max of all the Y random variables, for $y \in R$, $X \le y$ iff the greatest of the Y_k random variables is less than y. This is guaranteed to occur when $Y_1 \le y, Y_2 \le y, \dots Y_n \le y$. Relating this to F_X,

$$F_X(y) = \mathbb{P}\left(\bigcap_{i=1}^{n} Y_i \le y\right) \tag{6}$$

Since $Y_1, \dots, Y_n$ are independent random variables, (6) is equivalent to

$$\prod_{i=1}^{n} \mathbb{P}(Y_i \le y)$$

Which is $\boxed{F^n}$.

5. Prove that Proposition 0.1.ii from the independence notes implies 0.1.i.

Proof. Let

$$a_1 \in R(X_1), a_2 \in R(x_2), \dots, a_n \in R(X_n) \tag{7}$$

Now let

$$I_1^k = (a_1 - \frac{1}{k}, a_1 + \frac{1}{k}), I_2^k = (a_2 - \frac{1}{k}, a_2 + \frac{1}{k}), \dots, I_n^k = (a_n - \frac{1}{k}, a_n + \frac{1}{k}) \tag{8}$$

Let $B_k = \{\omega | X_1(\omega) \in I_1^k, X_2(\omega) \in I_2^k, \dots, X_n(\omega) \in I_n^k\}$.

Now note that for $N > k$, $B_n \subset B_k$ since B_N represents the random variables falling into a smaller window than B_k (see (8)). So per lemma 2.1.14b,

$$\mathbb{P}\left(\bigcap_{i=1}^{n} B_i\right) = \mathbb{P}(B_n) \tag{9}$$

Proposition 0.1.ii states that

$$\mathbb{P}(X_1 \in I_1^k, X_2 \in I_2^k, \dots, X_n \in I_n^k) = \prod_{i=1}^{n} \mathbb{P}(X_i \in I_i^k) \tag{10}$$

Consequently,

$$\mathbb{P}(B_N) = \prod_{i=1}^{n} \mathbb{P}(X_i \in I_i^n) \tag{11}$$

Recalling (8), this means

$$\mathbb{P}(B_N) = \prod_{i=1}^{n} \mathbb{P}\left(X_i \in \left(a_i - \frac{1}{N}, a_i + \frac{1}{N}\right)\right) \tag{12}$$

As $N \to \infty$ (equivalent to $\varepsilon \to 0$ in the problem setup), this probability approaches

$$\prod_{i=1}^{n} \mathbb{P}\left(X_i \in (a_i - 0, a_i + 0)\right) = \prod_{i=1}^{n} \mathbb{P}(X_i = a_i) \tag{13}$$

□

6. Suppose that $X_1, \dots, X_n$ are independent random variables, each having the Geometric(p) distribution for some fixed $p \in (0, 1)$. Define a new random variable:

$$Y(\omega) = min(X_1(\omega), X_2(\omega), \dots, X_n(\omega)) \tag{14}$$

Show that Y has the Geometric(α) distribution for some α. Commpute $\mathbb{E}[Y]$ in terms of p and n.

First we show that $Y \sim$ Geo(α) for some α
Since Y is min of X's, we can express the CDF of Y as

$$\mathbb{P}(Y \leq k) = 1 - \mathbb{P}(Y > k) = 1 - \mathbb{P}(X_1 > k, \dots, X_n > k) = 1 - \mathbb{P}\left(\bigcap_{i=1}^{n} X_i > k\right) \tag{15}$$

$X_1, \dots, X_n$ are independent random variables, so continuing from (15),

$$1 - \mathbb{P}\left(\bigcap_{i=1}^{n} X_i > k\right) = 1 - \prod_{i=1}^{n} \mathbb{P}(X_i > k) \tag{16}$$

$X_1, \dots, X_n$ are indentical random variables, so the RHS of (16) is equivalent to

$$1 - \mathbb{P}(X_1 > k)^n \tag{17}$$

For $X \sim \text{Geo}(p)$, $\mathbb{P}(X > k) = (1-p)^k$, so (17) is equivalent to

$$1 - ((1-p)^k)^n = 1 - ((1-p)^n)^k \tag{18}$$

Generally, the CDF of an r.v. $Y \sim \text{Geo}(\alpha)$ has the form $(1 - (1-\alpha)^k)$, so we can see in our case that $Y \sim \text{Geo}(\alpha)$ for $\boxed{\alpha = 1 - (1-p)^n}$

We now find $\mathbb{E}[Y]$:
For $Y \sim \text{Geo}(\alpha)$, $\mathbb{E}[Y] = 1/\alpha$. In our case, this is

$$\mathbb{E}[Y] = \frac{1}{1 - (1-p)^n}$$

7. Consider the following game: Roll a standard six-sided die. If the number rolled is $1, 2, 3$, you win nothing. If the number rolled is $4, 5$, or 6, you win \$1 plus twice the value rolled. What is the expected amount you win in a single roll?

Let the random variable X represent winnings from playing a round of the game. We can directly calculate $\mathbb{E}[X]$ by taking the sum of its range of values by the probability those values occur.

$$\mathbb{E}[X] = \sum_{i=1}^{6} x\mathbb{P}(X = x) = \frac{1}{6}(0 + 0 + 0 + 9 + 11 + 13) = \frac{33}{6} = \boxed{\$5.50}$$

8. Suppose X is a random variable, uniformly distributed on $\{1, \dots, n\}$. Compute $\mathbb{E}[X^2]$ in terms of n.

Given an r.v. X and some $g : \mathbb{R} \mapsto \mathbb{R}$, $\mathbb{E}[g(X)] = \sum_{x \in R(X)} \mathbb{P}(X = x)g(X)$. In this case, $g(X) = X^2$, so

$$\begin{aligned}
\mathbb{E}[X^2] &= \mathbb{E}[g(X)] \\
&= \sum_{x \in R(X)} \left(\frac{1}{n}\right) x^2 \\
&= \frac{1}{n} \sum_{i=1}^{n} i^2 \\
&= \frac{1}{n} \frac{n(n+1)(2n+1)}{6} && \text{per series convergence rules} \\
&= \frac{(n+1)(2n+1)}{6}
\end{aligned}$$

9. .

Compute $\mathbb{E}[X]$

Let H_k be the event that we first see heads on our kth toss. Let W be the event that we win \$1. Since W represents winning a single dollar, $\mathbb{P}(W) = E[X]$. We only win if H_k occurs, so applying the partition rule,

$$\mathbb{P}(W) = \sum_{k=1}^{\infty} \mathbb{P}(W|H_k) * \mathbb{P}(H_k)$$

$\mathbb{P}(W|H_k)$ is either 1 or 0. If we've thrown heads on toss k, we win if $a_k = 1$ and lsoe if $a_k = 0$. Thus $\mathbb{P}(W|H_K) = a_k$. $\mathbb{P}(H_k)$ is the probability that we failed to toss heads on our first $k-1$ tosses, then tossed heads on our kth toss. We are tossing the fair coin, so $\mathbb{P}(H_k)$ is $(1/2)^{k-1} * (1/2) = (1/2)^k$. Thus,

$$\mathbb{P}(W) = \sum_{k=1}^{\infty} a_k \left(\frac{1}{2}\right)^k$$

And since W is the event that we win a dollar, it is also true that

$$E[X] = \sum_{k=1}^{\infty} a_k \left(\frac{1}{2}\right)^k$$

How to set $\mathbb{E}[X]$ **to any arbitrary** $\alpha \in (0,1)$
$E[X]$ is a binary expansion function. Given a sequence $a = b_1 b_2 b_3 \ldots$ (representing a number in base 2), $\mathbb{E}[X]$ will expand it to a base ten decimal. Given that there exists a bijection between binary sequences and decimal sequences, this allows us to tune $\mathbb{E}[X]$ to any decimal number.

10. .

(i) What is the probability that you have not drawn red after n attempts.

Let NR_k represent the event of not drawing red by the kth attempt. Then $\mathbb{P}(NR_k) = \mathbb{P}(NR_k|NR_{k-1}) * \mathbb{P}(NR_{k-1})$. This is a recursive definition. Observe for $k = 2$, $\mathbb{P}(NR_2) = \mathbb{P}(NR_2|NR_1) * \mathbb{P}(NR_1)$. $\mathbb{P}(NR_2|NR_1)$ is the chance of not drawing a red on the second draw given (obviously) we haven't drawn a red on the first draw. On the second draw, there are 2 blue marbles and 1 red marble in the box, so $\mathbb{P}(NR_2|NR_1) = 2/3$. $\mathbb{P}(NR_1)$ is the chance that we didn't draw a red on the first trial: $1/2$. Thus $\mathbb{P}(NR_2) = (2/3) * (1/2)$. More generally,

$$\begin{aligned}\mathbb{P}(NR_n) &= \prod_{i=1}^{n} \frac{n}{n+1} \\ &= \frac{1}{n+1}\end{aligned}$$

(ii) What is the probability that you never draw the red marble?

This is

$$\begin{aligned}\lim_{n\to\infty} \mathbb{P}(NR_n) &= \lim_{n\to\infty} \frac{1}{n+1} \\ &= 0\end{aligned}$$

(iii) Let T be the number of draws until you draw a red. What is the distribution of T? Is $\mathbb{E}[T]$ finite?

Distribution of T: The random variable T is akin to an r.v. from the geometric distribution, but with a varying p. E.g., $\mathbb{P}(T = 2)$ is the probability that the second draw is red $(1/3)$ times the probability that the previous draw(s) were not red: $NR_1 = 1/2$. This product is $\mathbb{P}(T = 2) = (1/3)(1/2) = 1/6$. More generally, the distribution of T is given by:

$$\begin{aligned}\mathbb{P}(T = k) &= \left(\frac{1}{k+1}\right) NR_{k-1} \\ &= \frac{1}{k(k+1)}\end{aligned}$$

Expected value of T: Generally,

$$\mathbb{E}[T] = \sum_{k \in R(T)} (k * T(k))$$

In this case, the range of T is the natural numbers, so

$$\begin{aligned}
\mathbb{E}[T] &= \sum_{i=1}^{\infty} k \left(\frac{1}{k(k+1)} \right) \\
&= \sum_{i=1}^{\infty} \frac{1}{k+1}
\end{aligned}$$

$\boxed{\text{This series does not converge, so } \mathbb{E}[T] \text{ is not finite.}}$

(iv) Instead of adding just one more blue marble each time you draw a blue, suppose you double the number of blue. Is the answer to parts ii and iii different in this case?

If we double the number of blue marbles each time we pick one, the probability that we haven't drawn red after n attempts is

$$\mathbb{P}(NR_n) = \prod_{i=1}^{n} \frac{2^{i-1}}{2^{i-1}+1}$$

So

$$\begin{aligned}
\log \mathbb{P}(NR_n) &= \log \left(\prod_{i=1}^{n} \frac{2^{i-1}}{2^{i-1}+1} \right) \\
&= \sum_{i=1}^{n} \log \left(\frac{2^{i-1}}{2^{i-1}+1} \right) \\
&= \sum_{i=1}^{n} \left(\log(2^{i-1}) - \log(2^{i-1}+1) \right)
\end{aligned}$$

I wasn't sure where to proceed from here. I've clearly messed something up. The $\log(1+x) \approx x$ approximation can't be used here given 2^{i-1} is a large quantity. If I do apply the approximation, I find that $\mathbb{P}(NR_n) = 1$, which doesn't make sense.