Math 340 HW 4

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1. p-coin graph

2. **2.7.6**: An urn contains 8 white, 4 black, and 2 red balls. We win 2 euro for each black ball we draw and lose 1 euro for each white ball we draw. We choose three balls from the urn. Let X denote our winnings. Write down the probability mass function of X.

   We begin by noting that the random variable X ranges across integers in $[-3, 6]$. The minimum value of X occurs when we draw 3 white balls; the maximum when we draw 3 black balls. We now identify the outcomes that lead to the event where X takes on each value in its range. Using those outcomes, we derive the probability of each event (i.e. the PMF).
   $X = -3$: We draw 3 white balls.

$$\mathbb{P}(X = -3) = \frac{\binom{8}{3}}{\binom{14}{3}} = \frac{2}{13}$$

   $X = -2$: We draw 2 white balls and a red ball.

$$\mathbb{P}(X = -2) = \frac{\binom{8}{2}\binom{2}{1}}{\binom{14}{3}} = \frac{2}{13}$$

   $X = -1$: We draw a white ball and 2 red balls.

$$\mathbb{P}(X = -1) = \frac{\binom{8}{1}\binom{2}{2}}{\binom{14}{3}} = \frac{2}{91}$$

   $X = 0$: We draw a black ball and 2 white balls.

$$\mathbb{P}(X = 0) = \frac{\binom{4}{1}\binom{8}{2}}{\binom{14}{3}} = \frac{4}{13}$$

   $X = 1$: We draw a white ball, a black ball, and a red ball.

$$\mathbb{P}(X = 1) = \frac{\binom{4}{1}\binom{8}{1}\binom{2}{1}}{\binom{14}{3}} = \frac{16}{91}$$

   $X = 2$: We draw a black ball and 2 red balls.

$$\mathbb{P}(X = 2) = \frac{\binom{4}{1}\binom{2}{2}}{\binom{14}{3}} = \frac{1}{91}$$

   $X = 3$: We draw 2 black balls and a white ball.

$$\mathbb{P}(X = 3) = \frac{\binom{4}{2}\binom{8}{1}}{\binom{14}{3}} = \frac{12}{91}$$

   $X = 4$: We draw 2 black balls and a red ball.

$$\mathbb{P}(X = 4) = \frac{\binom{4}{2}\binom{2}{1}}{\binom{14}{3}} = \frac{3}{91}$$

   $X = 5$: This event cannot occur.

$$\mathbb{P}(X = 5) = 0$$

   $X = 6$: We draw 3 black balls.

$$\mathbb{P}(X = 6) = \frac{\binom{4}{3}}{\binom{14}{3}} = \frac{1}{91}$$

3. Let $Xn \sim$ Geometric(p) with $p = \lambda/n$. Let $\lambda > 0$ and $n \to \infty$. Let $Tn = \frac{1}{n}X_n$. Prove that for any $t > 0$,

$$\lim_{n\to\infty} \mathbb{P}(T_n > t) = e^{-\lambda t}$$

Proof. $\mathbb{P}(T_n > t) = \mathbb{P}(\frac{1}{n}X_n > t) = \mathbb{P}(X_n > nt)$. For a geometric distribution, this is

$$\begin{aligned}
\mathbb{P}(X_n > nt) &= \sum_{j=nt+1}^{\infty} p(1-p)^{j-1} && (1)\\
&= p(1-p)^{nt} + p(1-p)^{nt+1} + p(1-p)^{nt+2} + \dots && (2)\\
&= p(1-p)^{nt} \sum_{k=0}^{\infty}(1-p)^k && (3)\\
&= p(1-p)^{nt}\frac{1}{p} \qquad \text{geo. series convergence} && (4)\\
&= (1-p)^{nt} && (5)
\end{aligned}$$

$p = \lambda/n$, so (5) is equivalent to $\left(1 - \frac{\lambda}{n}\right)^{nt}$, which converges to $e^{-\lambda t}$ when $n \to \infty$. □

4. Let F_X be the CDF of X. Since X is the max of all the Y random variables, for $y \in R$, $X \leq y$ iff the greatest of the Y_k random variables is less than y. This is guaranteed to occur when $Y_1 \leq y, Y_2 \leq y, \dots Y_n \leq y$. Relating this to F_X,

$$F_X(y) = \mathbb{P}\left(\bigcap_{i=1}^{n} Y_i \leq y\right) \tag{6}$$

Since $Y_1, \dots, Y_n$ are independent random variables, (6) is equivalent to

$$\prod_{i=1}^{n} \mathbb{P}(Y_i \leq y)$$

Which is $\boxed{F^n}$.

5. Prove that Proposition 0.1.ii from the independence notes implies 0.1.i.

Proof. Let

$$a_1 \in R(X_1), a_2 \in R(x_2), \dots, a_n \in R(X_n) \tag{7}$$

Now let

$$I_1^k = (a_1 - \frac{1}{k}, a_1 + \frac{1}{k}), I_2^k = (a_2 - \frac{1}{k}, a_2 + \frac{1}{k}), \dots, I_n^k = (a_n - \frac{1}{k}, a_n + \frac{1}{k}) \tag{8}$$

Let $B_k = \{\omega | X_1(\omega) \in I_1^k, X_2(\omega) \in I_2^k, \dots, X_n(\omega) \in I_n^k\}$.

Now note that for $n > k$, $B_n \subset B_k$ since B_n represents the random variables falling into a smaller window than B_k (see (8)). So per lemma 2.1.14b,

$$\mathbb{P}\left(\bigcap_{i=1}^{n} B_i\right) = \mathbb{P}(B_n) \tag{9}$$

Proposition 0.1.ii states that

$$\mathbb{P}(X_1 \in I_1^k, X_2 \in I_2^k, \dots, X_n \in I_n^k) = \prod_{i=1}^{n} \mathbb{P}(X_i \in I_i^k) \tag{10}$$

Consequently,

$$\mathbb{P}(B_n) = \prod_{i=1}^{n} \mathbb{P}(X_i \in I_i^n) = \prod_{i=1}^{n} \mathbb{P}(X_i \in I_i^n) \tag{11}$$

Since the X_i variables are all independent, this is the same as

$$\begin{aligned}
&\mathbb{P}(X_i \in I_1^n, X_2 \in I_2^n, \dots, X_n \in I_n^n) && (1)\\
=&\mathbb{P}\left(X_1 \in \left(a_1 - \frac{1}{n}, a_1 + \frac{1}{n}\right), X_2 \in \left(a_2 - \frac{1}{n}, a_1 + \frac{1}{n}\right), \dots, X_n \in \left(a_n - \frac{1}{n}, a_n + \frac{1}{n}\right)\right. && (2)
\end{aligned}$$

As $n \to \infty$, this probability approaches

$$\mathbb{P}(X_1 = a_1, X_2 = a_2, \dots, X_n = a_n) = \prod_{i=1}^{n} \mathbb{P}(X_i = a_i) \tag{3}$$

□

6. .

7. Consider the following game: Roll a standard six-sided die. If the number rolled is $1, 2, 3$, you win nothing. If the number rolled is 4, 5, or 6, you win \$1 plus twice the value rolled. What is the expected amount you win in a single roll?

   Let the random variable X represent winnings from playing a round of the game. We can directly calculate $\mathbb{E}[X]$ by taking the sum of its range of values by the probability those values occur.

$$\mathbb{E}[X] = \sum_{i=1}^{6} x\mathbb{P}(X = x) = \frac{1}{6}(0 + 0 + 0 + 9 + 11 + 13) = \frac{33}{6} = \boxed{\$5.50}$$

8. Suppose X is a random variable, uniformly distributed on $\{1, \ldots, n\}$. Compute $\mathbb{E}[X^2]$ in terms of n.

   Given an r.v. X and some $g : \mathbb{R} \mapsto \mathbb{R}$, $\mathbb{E}[g(X)] = \sum_{x \in R(X)} \mathbb{P}(X = x)g(X)$. In this case, $g(X) = X^2$, so

$$\begin{aligned}
\mathbb{E}[X^2] &= \mathbb{E}[g(X)] \\
&= \sum_{x \in R(X)} \left(\frac{1}{n}\right) x^2 \\
&= \frac{1}{n} \sum_{i=1}^{n} i^2 \\
&= \frac{1}{n} \frac{n(n+1)(2n+1)}{6} && \text{per series convergence rules} \\
&= \frac{(n+1)(2n+1)}{6}
\end{aligned}$$

9. .

10. .

    (i) What is the probability that you have not drawn red after n attempts.

    Let NR_k represent the event of not drawing red by the kth attempt. Then $\mathbb{P}(NR_k) = \mathbb{P}(NR_k | NR_{k-1}) * \mathbb{P}(NR_{k-1})$. This is a recursive definition. Observe for $k = 2$, $\mathbb{P}(NR_2) = \mathbb{P}(NR_2 | NR_1) * \mathbb{P}(NR_1)$. $\mathbb{P}(NR_2 | NR_1)$ is the chance of not drawing a red on the second draw given (obviously) we haven't drawn a red on the first draw. On the second draw, there are 2 blue marbles and 1 red marble in the box, so $\mathbb{P}(NR_2 | NR_1) = 2/3$. $\mathbb{P}(NR_1)$ is the chance that we didn't draw a red on the first trial: $1/2$. Thus $\mathbb{P}(NR_2) = (2/3) * (1/2)$. More generally,

$$\begin{aligned}
\mathbb{P}(NR_n) &= \prod_{i=1}^{n} \frac{n}{n+1} \\
&= \frac{1}{n+1}
\end{aligned}$$

    (ii) What is the probability that you never draw the red marble?

    This is

$$\begin{aligned}
\lim_{n \to \infty} \mathbb{P}(NR_n) &= \lim_{n \to \infty} \frac{1}{n+1} \\
&= 0
\end{aligned}$$

    (iii) Let T be the number of draws until you draw a red. What is the distribution of T? Is $\mathbb{E}[T]$ finite?

    Distribution of T: The random variable T is akin to an r.v. from the geometric distribution, but with a varying p. E.g., $\mathbb{P}(T = 2)$ is the probability that the second draw is red $(1/3)$ times the probability that the previous draw(s) were not red: $NR_1 = 1/2$. This product is $\mathbb{P}(T = 2) = (1/3)(1/2) = 1/6$. More generally, the distribution of T is given by:

$$\begin{aligned}
\mathbb{P}(T = k) &= \left(\frac{1}{k+1}\right) NR_{k-1} \\
&= \frac{1}{k(k+1)}
\end{aligned}$$

Expected value of T: Generally,

$$\mathbb{E}[T] = \sum_{k \in R(T)} (k * T(k))$$

In this case, the range of T is the natural numbers, so

$$\begin{aligned}
\mathbb{E}[T] &= \sum_{i=1}^{\infty} k \left(\frac{1}{k(k+1)} \right) \\
&= \sum_{i=1}^{\infty} \frac{1}{k+1}
\end{aligned}$$

$\boxed{\text{This series does not converge, so } \mathbb{E}[T] \text{ is not finite.}}$