Math 340: Lec 20: Conditional Probability Distributions)

Asa Royal (ajr74)

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Remark. In general, marginal density alone does not determine joint density. Knowing how X is distributed and how Y are distributed is not enough to know how they are jointly distributed.

Conditional probability: discrete and continuous cases

Theorem 1 (Conditional density (discrete))**.** Note that in the discrete case,

$$\begin{aligned}\mathbb{P}(X = k|Y = j) &= \frac{\mathbb{P}(X = k, Y = j)}{\mathbb{P}(Y = j)} \\ &= \frac{\mathbb{P}(X = k, Y = j)}{\sum_{x \in R(X)} \mathbb{P}(Y = j|X = x)\mathbb{P}(X = x)} \\ &= \frac{\mathbb{P}(X = k, Y = j)}{\sum_{x \in R(X)} \mathbb{P}(X = x, Y = j)}\end{aligned}$$

Theorem 2 (Conditional density (continuous))**.** If X, Y have joint density $f(x, y)$ then the conditional density of X given $Y = y$ is

$$f(x|Y = k) = \frac{f(x, k)}{\int_{\mathbb{R}} f(\ell, k) d\ell} = \frac{f(x, k)}{f_Y(y)}$$

Remark. The conditional density is found by taking the joint density with one variable fixed and dividing that by the marginal of the non-fixed variable. Logically, this is very similar to how we find conditioanl probability in the discrete case!

Corollary 3.

$$\begin{aligned}f(x, y) &= f(x|Y = y) f_Y(y) \\ f_X(x) &= \int_{\mathbb{R}} f(x, y) dy = \int_{\mathbb{R}} f(x|Y = y) f_Y(y) dy\end{aligned}$$

Remark. Given iid exponentially-distributed RVs $T_1, \ldots, T_n$,

$$\min(T_1, \ldots, T_n) \sim \text{Exp}(n\lambda) \tag{1}$$