Math 221 Lec 13

6.2 Diagonlizability

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Definition 1 (Diagonlizability)**.** An $n \times n$ matrix is diagonalizable if

1. $\exists$ an invertible $n \times n$ matrix P s.t. $P^{-1}AP = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$
2. $\exists$ a basis for $\mathbb{R}^n$ consistsing of eigenvectors of A.

Definition 2 (eigenspace)**.** If λ is an eigenvalue of A, then the eigenspace associated with λ is $E(\lambda) = \{\mathbf{v} : A\mathbf{v} = \lambda\mathbf{v} = N(A - \lambda I_n)$

Lemma 3. If $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are eigenvectors of A whos associated eigenvalues are distinct, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent.

Proof. We will prove the lemma above by induction on k, the number of eigenvectors.
Base case P(1): If $\mathbf{v}_1$ is an eigenvector of A, $\{\mathbf{v}_1\}$ is trivially independent.
Inductive hypothesis P(k-1): Assume that $\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}$ are eigenvectors of A with distinct eigenvalues $\lambda_1, \ldots, \lambda_{k-1}$, and that $\{\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}\}$ is linearly independent.
Inductive step: Suppose $\mathbf{v}_k$ is an eigenvector and

$$c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k = 0 \tag{1}$$

We can multiply both sides of (1) by A, and since each term on the LHS contains an eigenvector, rewrite the equation as

$$c_1\lambda_1\mathbf{v}_1 + \ldots + c_k\lambda_k\mathbf{v}_k = 0 \tag{2}$$

Now suppose we multiply both sides of (1) by λ_k:

$$c_1\lambda_k\mathbf{v}_1 + \ldots + c_k\lambda_k\mathbf{v}_k = 0 \tag{3}$$

And subract (3) from (2):

$$c_1(\lambda_1 - \lambda_k)\mathbf{v}_1 + \ldots + c_{k-1}(\lambda_1 - \lambda_{k-1})\mathbf{v}_{k-1} = 0 \tag{4}$$

The inductive hypothesis states that $\{\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}\}$ are linearly independent, so $c_1 = \ldots = c_{k-1} = 0$. Plugging those values into (1), we see

$$c_k\mathbf{v}_k = 0 \tag{5}$$

And since $\mathbf{v}_k$ is an eigenvector and cannot equal $\mathbf{0}$, $c_k = 0$. Since (1) implies that $c_1 = \ldots = c_k = 0$, $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent. We have thus shown that $P(k-1) \Rightarrow P(k)$, so we conclude the proof. $\square$

Theorem 4. Let A be an $n \times n$ matrix with n distinct eigenvalues. Then A is diagonalizable.

Proof. If A has n distinct eigenvalues, it also has n distinct eigenvectors, which means its eigenvectors are a basis for $\mathbb{R}^n$. Thus, by definition, A is diagonalizabile. $\square$

Definition 5 (algebraic multiplicity)**.** The algebraic multiplicity of an eigenvalue c, denoted $a(c)$, is the multiplicity of c as a root of the characteristic polynomial of A. When we completely factor the characteristic polynomial, we will get a factor of $(c-\lambda)^a$, where a is the algebraic multiplicity.

Definition 6 (geometric multiplicity)**.** The geometric multiplicity of an eigenvalue λ is $\dim N(A - \lambda I_n)$.

Lemma 7. Let A be an $n \times n$ matrix with eigenvalue λ, associated algebraic multiplicity a, and associated geometric multiplicity g. Then $1 \leq a \leq g$.

Proposition 8. The determinant of a square matrix A equals the product of A's eigenvalues, counted with multiplicity.

Proof.

$$
\begin{aligned}
p_A(\lambda) &= \det(A - \lambda I) && (1)\\
&= \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 && \text{because } p_A(\lambda) \text{ is a polynomial of degree n} \quad (2)\\
&= (\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_n) && \text{(factor polynomial)} \quad (3)
\end{aligned}
$$

Per (3),

$$p_A(0) = (-\lambda_1)(-\lambda_2)\ldots(-\lambda_n) = (-1)^n\lambda_1\lambda_2\ldots\lambda_n \tag{4}$$

Per(1),

$$p_A(0) = \det(0 - A) = \det(-A) = (-1)^n \det A \tag{5}$$

Thus, $(-1)^n \det A = (-1)^n\lambda_1\lambda_2\ldots\lambda_n$, and $\det A = \lambda_1\lambda_2\ldots\lambda_n$.

□

Corollary 9. If A has eigenvalue 0, it has determinant 0 and is thus singular.