Math 340: Lec 21 Poisson processes)

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0.1 Basics of Poisson arrival processes

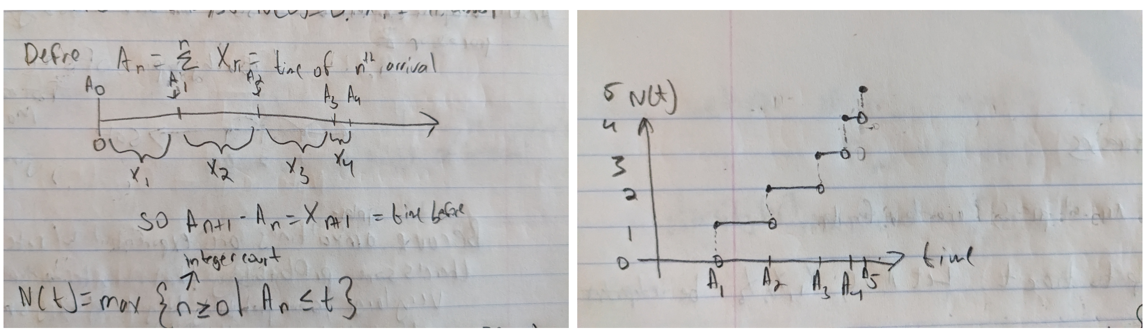
Definition 1 (Poisson arrival proces). $N(t)$ represents a Poisson arrival process, a random function denoting the number of arrivals that occur before some time t.

Let $\lambda > 0, X_1, X_2, X_3, \ldots$ be independent with $X_i \sim \text{Exp}(\lambda)$. X_i represents the ith inter-arrival time (aka waiting time). Since $X_i \sim \text{Exp}(\lambda)$,

$$\mathbb{P}(X_i > s) = e^{-\lambda s}, \forall s > 0$$

Moreover, if A_n is the nth arrival time, $A_n = \sum_{k=1}^{n} X_k$, so

$$\begin{aligned} N(t) &= \max\{n \geq 0 | A_n \leq t\} \\ &= \max\{\sum_{k=1}^{n} X_k \leq t\} \end{aligned}$$



(a) Inter-arrival and arrival times

(b) N(t) piecewise relationship w/ arrivals

Remark. $N(t)$ is:

1. Non-negative and non-decreasing
2. piecewise constant
3. has jumps of size 1 at arrival times

0.2 Distribution of $N(t)$

Definition 2 (Modified Poisson Arrival Process). For an interval $I = (a, b] \subset (0, \infty)$, define $N(I) = \#$ arrivals in interval I. i.e.

$$N(I) = N(b) - N(a)$$

Note that this means

$$N(t) = N((0, t]) = N(t) - N(0)$$

Theorem 3. Let N be a Poisson arrival process w/ parameter λ.

1. If $I = (a, b]$ is any interval then $N(I)$ has the Poisson($\lambda|I|$) distribution where $|I| = |b - a|$. Thus,

$$\mathbb{P}(N(I) = k) = \frac{(\lambda|I|)^k}{k!} e^{-\lambda t}$$

   And

$$\mathbb{E}[N(I)] = \lambda|I|$$

   (a) In particular, $N(t) \sim \text{Poisson}(\lambda t)$, so $\mathbb{E}[N(t)] = \lambda n$

2. For any disjoint intervals $I_j = (a_j, b_j], j = 1, \ldots, n$, the random variables $N(I_1), N(I_2), \ldots, N(I_n)$ are independent.

0.3 Distribution of nth arrival time

Definition 4 (Gamma distribution)**.** $G(n, \lambda)$ is a continuous distribution on $[0, \infty)$ with density:

$$g_n(t) = \begin{cases} \frac{(\lambda t)^{n-1}(n-1)!}{\lambda} e^{-\lambda t} & \text{if} t \geq 0 \\ 0 & \text{if} t < 0 \end{cases}$$

Remark. Gamma(n, λ) is a distribution of the sum of n independent Exp(λ) random variables.

Proposition 5. The nth arrival time A_n has the Gamma(n, λ) distribution