Math 221 Lec 5 (1.5)

Asa Royal (ajr74)

September 14, 2023

We can interpret A through a row lens or a column lens. The row lens is covered in the previous section. It states that $A\mathbf{x} = \mathbf{b}$ identifies the intersection of the hyperplanes defined by $A_i\mathbf{x} = \mathbf{b}_i$. We calculate it by finding $A_i \cdot \mathbf{x}$ for each row A_i. We can interpret $A\mathbf{x} = \mathbf{b}$ as a linear combination of the columns of $\mathbf{A}$. This is equivalent to saying:

1. $\mathbf{b} \in C(A)$ ($\mathbf{b}$ is in the column space of $\mathbf{A}$.
2. $\mathbf{b} \in \text{span}(\mathbf{A})$

Definition 1 (rank)**.** The **rank** of an $m \times n$ matrix is the number of pivots it has in echelon form.

Theorem 2 (rank/consistency)**.** $A\mathbf{x} = \mathbf{b}$ is consistent iff $\text{rank}(A) = \text{rank}(A|\mathbf{b})$.

Corollary 3. $A\mathbf{x} = \mathbf{b}$ for an $m \times n$ matrix A is consistent for all $\mathbf{b} \in \mathbb{R}^m$ iff $\text{rank}(A) = m$. This ensures that every row has a pivot variable, and that a zero-row isn't set equal to a non-zero $\mathbf{b}_k$.

Definition 4 (Inconsistent)**.** A system is **inconsistent** precisely when theree is an equation that reads

$$0x_1 + 0x_2 + \ldots + 0x_n = c$$

for non-zero c.

Uniqueness and non-uniqueness of solutions

Definition 5 (homogeneous/non-homogeneous systems)**.** A system of equations with form $A\mathbf{x} = 0$ is a **homogeneous linear system**. $A\mathbf{x} = \mathbf{b}$ with a non-zero $\mathbf{b}$ is called **non-homogeneous**.

Proposition 6. 0 Let $\mathbf{x}_0 \in \mathbb{R}^n$ be a solution to $A\mathbf{x} = \mathbf{b}$ (i.e. $A\mathbf{x}_0 = \mathbf{b}$). Then all solutions to $A\mathbf{x} = \mathbf{b}$ are given by $\mathbf{x} = \mathbf{x}_0 + \mathbf{v}$ where $\mathbf{v}$ is a solution for the homogeneous equation $A\mathbf{x} = \mathbf{0}$

Corollary 7. As proposition 6 indicates, a particular solution to $A\mathbf{x} = \mathbf{b}$ shifts the solution set from the set defined by $A\mathbf{x} = \mathbf{0}$ along the particular solution vector $\mathbf{x}_0$. Thus, $A\mathbf{x} = \mathbf{b}$ has a unique solution iff $A\mathbf{x} = \mathbf{0}$ has one solution (which must be the trivial solution).

Proposition 8. The homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x}$=$\mathbf{0}$ iff $r = n$ (i.e. rank = # variables).

Proof. A system of equations has a unique solution iff it has no free vars, which requires that all variable be pivot variables. By definition, this means $r = n$. □

Definition 9 (singular/nonsingular matrix)**.** An $n \times n$ matrix of rank $r = n$ is called **nonsingular**. An $n \times n$ matrix of rank $r < n$ is called **singular**.

Theorem 10 (Non-singular matrix theorem)**.** If A be an $n \times n$ matrix, the following statements are equivalent.

1. A is a nonsingular matrix.
2. $r = n$
3. $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b}$ and in fact has a unique solution for each $\mathbf{b}$.
4. The reduced echelon form of A is

$$\begin{bmatrix} 1,0,0,0 \\ 0,1,0,0 \\ 0,0,1,0 \\ 0,0,0,1 \end{bmatrix}$$

Proposition 11. *note*: **This should probably be a part of 2.1/2.2 notes**
The function $\mathbb{R}^n \mapsto \mathbb{R}^m$ represented by matrix A is a linear transformation, and

$$A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} \text{ for } \mathbf{x},\mathbf{y} \in \mathbb{R}^n$$

$$A(c\mathbf{x}) = c(A\mathbf{x}) \text{ for } \mathbf{x} \in \mathbb{R}^n, c \in \mathbb{R}.$$