Math 340 HW 2

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1. **1.7.10**: Supose we dial a random 6-digit number on my telephone. What is the probability that:
   (a) The number does not contain a 6
   $\approx$ **.531**. The probability that any given digit is not 6 is $9/10$. There are 6 digits, and each is dialed independently, so the probability that the number does not contain a 6 is $(9/10)^6$.
   (b) The number contains only even digits
   1/64. The probability that any given digit is even is $1/2$. There are 6 digits, and each is independently, so the probability that all of them are even is $(1/2)^6$.
   (c) The number contains the pattern 2345
   .0003. If the pattern 2345 appears in the phone number, it must begin at the first, second, or third index. The other two digits in the phone number can be chosen freely; there are 10 choices for each. There are thus $3 * 10 * 10$ 6-digit phone numbers containing the pattern 2345. There are 10^6 6-digit phone numbers, so the probability of 2345 appearing in a randomly dialed phone number is $(3 * 10 * 10)/(10^6) = 3/(10^4) = .0003$
   (d) The number contains the pattern 2222
   .0003. Per part c, there are $3 * 10 * 10$ 6-digit phone numbers that include a specific 4-digit pattern. It does not matter that this pattern contains a block of 2s. Dialing one 2 first and another second is no different than dialing the second first and the first second. $(3 * 10 * 10)/(10^6) = 3/(10^4) = .0003$

2. **1.7.21**: Supose that 20 rabbits live in a certain region. We catch 5 of them, mark these, and let them go again. After a while we catch 4 rabbits. Compute the probability that exactly 2 of these 4 are marked. Be very precise about the assumptions that you make when you compute this probability.

   Assume that our selection of rabbits in the second round is independent of our selection fo rabbits in the first round. There are $\binom{5}{2}$ combinations of marked rabbits we could pick in the second round and $\binom{15}{2}$ combinations of unmarked rabbits. The total number of ways to pick 4 rabbits from the 20 in the region is $\binom{20}{4}$. Thus, since each ω is equally likely, the probability of picking 4 rabbits and finding that exactly two of them are marked is

   $$\mathbb{P}(2 \text{ marked}) = \frac{\binom{5}{2}\binom{15}{2}}{\binom{20}{4}} \approx 0.217$$

3. A box contains five tickets labeled $1, 2, 3, 4, 5$ which are otherwise identical Ten people take turns drawing a ticket randomly from the box, replacing and mixing the tickets after each draw so that the draws are independent.
   (a) For $k = 1, \ldots, 5$, let A_k be the event that ticket k is not chosen by anyone. What is $\mathbb{P}(A_k)$?
   .107. There is a $4/5$ chance that ticket k is not picked on any given draw. Since the draws are independent, the chance of this occuring ten times is $(4/5)^{10} \approx .107$
   (b) Are the events A_k and A_j independent if $k \neq j$?
   No. $\mathbb{P}(A_j \cap A_k) = (3/5)^{10}$, because there is a $3/5$ chance that neither ticket j nor k is picked on any given draw. $\mathbb{P}(A_j) = \mathbb{P}(A_k) = (4/5)^{10}$. $\mathbb{P}(A_j \cap A_k) \neq \mathbb{P}(A_j) * \mathbb{P}(A_k)$, so A_j and A_k are not independent.
   (c) $B = ({A_1}^c \cap {A_2}^c \cap {A_3}^c \cap {A_4}^c \cap {A_5}^c) = (\bigcup A_i)^c$

4. There are 20 dice. 17 of them are fair dice, and 3 of them are trick dice. The "trick dice" are just like the fair dice except that the faces on the trick dice are labeled 1,1,2,3,4,5. Imagine you draw a die randomly from the box, and then you roll it. Let T be the event that you drew a trick die. Let R_k be the event that you rolled the number k. Compute and compare the quantities $\mathbb{P}(T), \mathbb{P}(T|R_5)$, and $\mathbb{P}(T|R_1)$.

   $\mathbb{P}(T) = 3/20$. Given that we draw dice randomly from the box, we are equally likely to any of the dice, so $\mathbb{P}(T) = |\text{Trick}|/|\Omega|$.

$\mathbb{P}(T|R_5) = 3/20$. We supect this because R_5 is equally likely whether we pick a trick or fair die. But we can confirm the probability by applying Bayes' rule with partitioning:

$$\mathbb{P}(T|R_5) = \frac{\mathbb{P}(R_5|T)\mathbb{P}(T)}{\mathbb{P}(R_5)} = \frac{(1/6)(3/20)}{(3/20)(1/6)+(17/20)(1/6)} = \frac{(1/6)(3/20)}{1/6} = 3/20$$

$\mathbb{P}(T|R_1) = 6/23$. We find this by applying Bayes' rule with partitioning.

$$\mathbb{P}(T|R_1) = \frac{\mathbb{P}(R_1|T)\mathbb{P}(T)}{\mathbb{P}(R_1)} = \frac{(1/3)(3/20)}{(3/20)(1/3)+(17/20)(1/6)} = \frac{1/20}{1/20+17/120} = (6/120)(120/23) = 6/23$$

5. Out of a group of 20 people, 5 are chosen randomly to receieve a prize, and the rest get nothing. Let A be the event that Alonzo gets a prize. Let B be the event that Biji gets a prize. Are the events A and B independent?

   No. There are $\binom{18}{3}$ combinations of winners that include Biji and Alonzo, and $\binom{20}{5}$ combinations of winners overall. Thus $\mathbb{P}(A \cap B) = \frac{\binom{18}{3}}{\binom{20}{5}}$. If A and B are independent, $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$. $\mathbb{P}(A) = \mathbb{P}(B) = 5/20$. $\mathbb{P}(A) * \mathbb{P}(B) = 1/6 \neq \mathbb{P}(A \cap B)$. Thus A and B are not independent.

6. A box contains 10 cards: 5 cards are red, 5 cards are blue. The red cards are labeled $1, 2, 3, 4, 5$. The blue cards are also labeled $1, 2, 3, 4, 5$. Consider the following game:

   Joe draws a card at random, **without** replacement. Then Kamala chooses randomly from among the remaining cards. Kamala wins if either (a) her card has a higher numerical value than Joe's or (b) the color of her cards is different from Joe's card. If neither (a) nor (b) happen, then Joe wins.

   (a) What is the probability that Kamala wins?

   (b) Suppose we are told that Kamala won and that she drew a red 4. Given this information, what is the probability that Joe's card was blue?
   Let A be the event that Joe loses. Let B be the event that Joe draws a card with value < 4. Let C be the event that Joe draws a blue card. $A = B \cup C$. And thus,

   $$\mathbb{P}(A) = \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(B \cap C)$$

   Since Joe lost, $\mathbb{P}(A) = 1$. We know that $B \cup C$ happened. Thus,

   $$1 = \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(B \cap C) = 6/9 + \mathbb{P}(C) - 1/9$$

   Since there are 6 of the 9 remaining cards Joe could have chosen with value < 4 and $1/9$ that have value < 4 and are blue. And therefore,

   $$\mathbb{P}(C) = 1 - 6/9 + 1/9 = 4/9$$