Math 221 Lec 9 (2.3: Inverse Matrices)

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1 Inverse matrices

Remark. If a matrix A is invertible, we can find A^{-1} by performing Gaussian elimination on the augmented matrix $[A|I]$. This augmented matrix represents multiple sets of simultaneous equations, wherein we solve for multiple $\mathbf{x}$ vectors, s.t. $A\mathbf{x}_1 = i_1$ (the first column of I), $A\mathbf{x}_2 = i_2 \ldots$

Example. Find the inverse of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Fill out...

Proposition 1. Nonsingular matrices are invertible.

Why do singular matrices not have inverses?

Proof. Assume A is a singular matrix. Then $A\mathbf{x} = \mathbf{0}$ has a nontrivial (non-zero) solution. But if A^{-1} existed, $A^{-1}A\mathbf{x} = A^{-1}\mathbf{0}$, which means $x = \mathbf{0}$. This is a contradiction! □

Proof. Alternatively, given singular matrix A, try solving the multiple systems of equations represented by $AB = I$, where B is the hypothetical A^{-1}. A will row reduce to a matrix with a row of zeros in the bottom, which cannot possibly equal the bottom row of I. □

2 Determinants

Definition 2 (determinant)**.** The **determinant** of a linear transformation tells us how much a unit of area changes after the transformation is applied.

2.1 Geometric interpretation of the determinant

Example (geometric interp of determinant in 2 dimensions)**.** If we apply the linear transformation represented by $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$, $\hat{\imath}$ is stretched by 3 and $\hat{\jmath}$ is stretched by 2. The area covered by $\begin{bmatrix} \hat{\imath} & \hat{\jmath} \end{bmatrix}$ thus increases from $1 * 1 = 1$ to $2 * 3 = 6$, so the determinant of the transformation is 6.

Remark. In two-dimensional space, a matrix with $\det(A) = 0$ represents a linear transformation that reduces the **area** of a unit square to 0, collapsing space onto a line (or point).

Remark. In three-dimensional space, a matrix with $\det(A) = 0$ represents a linear transformation that reduces the **volume** of a unit cube to 0, collapsing space onto a plane, line, or point.

Remark. The **sign of a determinant** tells us whether the orientation of space has changed. For example, in two-dimensional space, if $\det(A) = -1$, $\hat{\imath}$ might go from being to the right of $\hat{\jmath}$ to being to the left. We can use the right-hand rule to figure out whether the orientation of 3-space has changed.