Math 221 Lec 13

6.1 eigenvectors

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Definition 1 (Eigenvector, eigenvalue). Fix $T : V \mapsto V$. $v \in V$ is an eigenvector if $\mathbf{v} \neq 0$ and $T\mathbf{v} = \lambda\mathbf{v}$ for some scalar λ, called an eigenvalue of T

Remark. If $A \in \mathbb{R}^{n\times n}$ is singular, A has eigenvalue 0 since $\exists\mathbf{x}$ s.t. $\mathbf{x} \in N(A)$, so for that $\mathbf{x}$, $A\mathbf{x} = 0 = 0(\mathbf{x})$.

Proposition 2. $\mathcal{B}$ is a basis of eigenvectors if $T : V \mapsto V$ iff $[T]_{\mathcal{B}}$ is diagonal.

Proof. Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be the eigenvectors of T. Then

$$T \begin{bmatrix} \mathbf{v}_1 & \dots & \dots\mathbf{v}_n \end{bmatrix} = \begin{bmatrix} \lambda_1\mathbf{v}_1 & \dots & \lambda_n\mathbf{v}_n \end{bmatrix}$$

so

$$\begin{bmatrix} \lambda_1\mathbf{v}_1 & \dots & \lambda_n\mathbf{v}_n \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

The last term is $[T]_{\mathcal{B}}$, the change of basis matrix $\square$

Definition 3 (diaganolizable). $T : V \mapsto V$ is diagonalizable if $[T]_{\mathcal{B}}$ is diagonal for some basis $\mathcal{B}$ of V.

Lemma 4. A is diagonalizable iff A is similar to a diagonal matrix.

Proof. Since A is diagonalizable,

$$\mu_A \begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

Which means

$$A \begin{bmatrix} | & & | \\ \mathbf{v}_1 & \dots & \mathbf{v}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \mathbf{v}_1 & \dots & \mathbf{v}_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$\begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{bmatrix}$ is invertible, so we can write $AP = P\Lambda$ and thus $A = P\Lambda P^{-1}$. $\square$

Proposition 5. λ is an eigenvalue of $T : V \mapsto V$ (or an $n \times n$ matrix A) iff $\ker(T - \lambda I) \neq 0$ or $N(A - \lambda I_) \neq 0$.

Proof. If λ is an eigenvalue of T, $\exists\mathbf{v} \in V$ s.t. $T\mathbf{v} = \lambda\mathbf{v}$. Thus $(T - \lambda I)\mathbf{v} = 0$, so $\ker(T - \lambda I) \neq 0$. $\square$

Definition 6 (eigenspace)**.** $E(\lambda)$ is the λ-eigenspace of A. If $\mathbf{v} \in E(\lambda)$ and $\mathbf{v} \neq 0$, $T\mathbf{v} = \lambda\mathbf{v}$, so T has an eigenvector $\mathbf{v}$ with eigenvalue λ.

Definition 7 (characteristic polynomial)**.** Let A be a square matrix. Then $p_A(t) = \det(A - tI)$ is called the **characteristic polynomial** of A. The eigenvalues of A are the roots of p_A.

Proposition 8. If A and B are similar mnatrices, then $p_A(t) = p_B(t)$.

Proof. $p_A(t) = \det(A - tI)$. Since $A = P^{-1}BP$ for some invertible matrix P, $p_A(t) = \det(P^{-1}BP - tI) = \det(P^{-1}BP - tIP^{-1}P) = \det(P^{-1}P(B - tI)) = \det(P^{-1}P)\det(B - tI) = \det(B - tI) = p_B(t)$. $\square$