Math 340: Lec 19 Big Ideas Journal (Joint distributions of cont. r.v.s)

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Joint density

Definition 1 (joint density). X and Y have a joint density $f(x, y)$ if $\mathbb{P}((x, y) \in B) = \int\int_B f(x, y)dxdy$ for all open $B \subset \mathbb{R}^2$.

Remark. $\int\int_{\mathbb{R}^2} f(x, y)dxdy = 1$

Theorem 2 (marginal density from joint density). The marginal density of X is

$$f_x(x) = \int_{\mathbb{R}} f(x, y)dy$$

This calculation of the marginal density from the joint density is similar to the one we did in the discrete case. Here, instead of summing over all the discrete values Y can take, we take an infinite sum across the domain of Y, which is $\mathbb{R}$.

Independence

Definition 3 (independence of continuous random variables). Suppose X and Y have densities f_X, f_Y respectively. They are **independent** iff $\forall x, y$, their joint density is

$$f(x, y) = f_X(x)f_Y(y)$$

Remark. Even if X and Y have a constant joint density on some region B, their marginal densities need not be constant.

Examples: moving between joint and marginal densities

Example (Obtain marginal density from joint). Consider the right triangle with vertices at $(0, 0), (1, 0), (1, 1)$. Assume density is uniformly distributed across the triangle.

Note that the marginal density of X is not constant! the density between $(0, \varepsilon)$ and $(1 - \varepsilon, 1)$ is not equivalent! The latter is clearly larger.

We can find $\mathbb{P}(X < a)$ for some $0 < a < 1$ by integrating $f(x, y)$ across the triangle formed by the existing hypotenuse, the x axis, and the line $x = a$.

Example (Obtain joint density from marginals). Imagine $T_1 \sim \text{Unif}(1, 4)$ and $T_2 \sim \text{Unif}(2, 5)$ are independent and represent arrival times. What is $\mathbb{P}(T_1 < T_2)$?

Because the random variables are independent, we can derive their joint distribution from their marginals:

$$f(t_1, t_2) = f_1(t_1)f_2(t_2)$$

Both marginals have density 1/3 in their respective non-zero areas, and thus 1/9 on their overlapping non-zero areas. The overlapping area (a square) has corners $(1, 2), (4, 2), (1, 5), (4, 5)$ and area 9. The joint density is thus 1/9 on the overlapping region, and the $\mathbb{P}(T_1 < T_2 =$

$$\int\int_B \frac{1}{9}dt_1dt_2 = \frac{1}{9}B$$

where B is the region in the overlapping areas above the lime $t_1 = t_2$.