Math 221 Lec 16

3.4: Dimension

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Definition 1 (linear transformation)**.** A function $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ is called a **linear transformation** if it satisfies

1. $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y}) \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$
2. $T(c\mathbf{x}) = cT(\mathbf{x}) \forall \mathbf{x} in \mathbb{R}^n$ and scalars c

Remark. If $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ is a linear transformation, then we can find a matrix A, the so-called standard matrix of A so that $T = \mu_A$. THe jth column of A is given by $T(\mathbf{e})j$, where $\mathbf{e}_j$ is the jth standard basis vector.

Proposition 2. Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation and let $\mathcal{E} = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be the stanadard basis for $\mathbb{R}^n$. LEt A be the matrix whose column vectors are the vectors $T(\mathbf{e}_1), \dots, T(\mathbf{e})_n \in \mathbb{R}^m$ (that is, the coordintae vectors of $T(\mathbf{e}_j)$ with respect to the standard basis of $\mathbb{R}^m$:

$$A = \begin{bmatrix} | & | & & | \\ T(\mathbf{e})_1 & T(\mathbf{e})_2 & \dots & T(\mathbf{e})_n \\ | & | & & | \end{bmatrix}$$

Then $T = \mu_A$ and we call A the standard matrix for T, denoted $[T]_{\text{stand}}$.

Proof. □

Definition 3 (matrix with respect to basis)**.** Let $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a linear transformation and let $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be an ordered basis for $\mathbb{R}^n$. For each $j = 1, \dots, n$, let $a_{1j}, a_{2j}, \dots a_{nj}$ denote the coordinates of $T(\mathbf{v}_j)$ with respect to the basis $\mathcal{B}$. We denote this matrix by $[T]_{\mathcal{B}}$.

Remark. The coefficients of $T(\mathbf{v}_j)$ will be entered as the jth column of the matrix $[T]_{\mathcal{B}}$. I.e. given a vector $\mathbf{x} \in \mathbb{R}^n$, we let $C_{\mathcal{B}(\mathbf{x})}$ denote the colum vector whose entires are the coordinates of $\mathbf{x}$ w.r.t. the basis of $\mathcal{B}$. That is, if $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis for $\mathcal{B}$,

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots c_n\mathbf{v}_n$$

then

$$C_{\mathcal{B}(\mathbf{x})} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

So

$$[T]_{\mathcal{B}} = \begin{bmatrix} | & | & & | \\ C_{\mathcal{B}}(T(\mathbf{v}_1)) & C_{\mathcal{B}}(T(\mathbf{v}_2)) & \dots & C_{\mathcal{B}}(T(\mathbf{v}_n)) \\ | & | & & | \end{bmatrix}$$