3.3: Local Extrema (Lec 8)

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Critical points

Definition 1 (local minimum/maximum). Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$. $\vec{x_0}$ is a **local minimum** of f if $\exists$ a neighborhood V of $\vec{x_0}$ s.t. $\forall \vec{x} \in V, f(\vec{x}) > f(\vec{x_0})$. Similar definition for **local maximum**.

Definition 2 (global maximum/minimum). $\vec{x_0}$ is a global minimum if $\forall \vec{x} \in V, f(\vec{x} > f(\vec{x_0}$

Definition 3 (critical point). $\vec{x_0}$ is a **critical point** of f if $Df(\vec{x_0}) = \vec{0}$ or if f is not differentiable at $\vec{x_0}$. Note: if f is real-valued, $Df(x_0) = 0$ is equivalent to saying $\nabla f_{x_0} = 0$.

Remark. What does the not differentiable at $\vec{x_0}$ imply?)

Theorem 4 (First derivative text). If $U \subseteq \mathbb{R}^n$ is open and f is differentiable then any local extrema is a critical point.

Proof. Suppose $\vec{x}$ is a local extrema of f. Now let $g(t) = f(x_0 + t\vec{v})$. $g(t)$ clearly has a local extrema when $t = 0$ for **any** $\vec{v} \in \mathbb{R}^n$.

To visualize this, imagine we're at the top of a hill (the local max of f). Now imagine the set of all vectors starting from the point we're at. This is g. The maximum of that set is our current position! i.e. $f(x) + 0\vec{v}$.

Per one variable calculus, since x_0 is a local maximum of g, $g'(t) = 0$. But $g(t) = f(x_0 + t\vec{v})$, so per multivariate rules, $g'(t) = D[f(\vec{x_0})]\vec{v} = 0$. Since this is true for any $\vec{v}$, $D[f(\vec{x_0})] = 0$, so $\vec{x_0}$ is a critical point. $\square$

Corollary 5. If $\vec{x_0}$ is not a critical point then $\vec{x_0}$ cannot be an extrema.

Determining type of maxima

Remark. Not all critical points are extrema. For example, $x = 0$ for $f(x) = x^3$ or $f(x, y) = y^2 - x^2$ at $(0, 0)$. The latter is an example of a **saddle point**. Though $Df = \nabla f = \vec{0}$ at $(0, 0)$, $(0, 0)$ is a local minimum in the x direction and a local maximum in the y direction.

Theorem 6 (Second derivative test for local extrema). If $f : U \subset \mathbb{R}^n \mapsto \mathbb{R}, x_0 \in U$ is a critical point of f, and...

- The Hessian $Hf(x_0)$ is **positive-definite**, x_0 is a **relative minimum** of f.
- $Hf(x_0)$ is **negative-definite**, x_0 is a **relative maximum** of f.
- $Hf(x_0)$ is **neither positive- nor negative-definite** but $\det Hf(x_0) \neq 0$, x_0 is a **saddle point**.
- $Hf(x_0)$ is **neither positive- nor negative-definite** and $\det(Hf(x_0)) = 0$, we **can't categorize critical point using the second derivative test**.

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Finding global extrema

If we wish wish to find teh global extrema of a function $f : U \mapsto R$, we need to find the critical points for all $x_0 \in U$, as well as the critical points of f as a function only on the boundary ∂U.