Math 340: Lec 24 Markov Chains (2)

Asa Royal (ajr74)

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Remark.

0.1 N-step probability distributions with random start

Proposition 1. Assume $X_0 \sim \nu$ (i.e. $\mathbb{P}(X_0 = x_0) = \nu_{x_0}$). Then the distribution of X_n is given by

$$\mathbb{P}(X_n = y) = \nu(P^n)_y = \sum_{x \in S} \nu_x (P^n)_{x,y}$$

Note that ν is a $1 \times m$ row vector where $m = |S|$. P is obviously $m \times m$.

Proof.

$$\mathbb{P}(X_n = x_n) = \sum_{x_0 \in S} \mathbb{P}(X_n = x_n | X_0 = x_0) \mathbb{P}(X_0 = x_0)$$

$$\begin{aligned} &= \sum_{x_0 \in S} P^{(n)}(x_0, x_n)\nu(x_0) && \text{n-step prob} \\ &= \sum_{x_0 \in S} \nu(x_0) P^{(n)}(x_0, x_n) \\ &= \nu P^{(n)}(x_n) && \text{matrix-vec mult.} \end{aligned}$$

□

0.2 Stationary distributions

Definition 2 (Stationary distribution)**.** A distribution π on S is stationary if $\pi P = \pi$. This means

$$\mathbb{P}(X_n = y | X_0 \sim \pi) = \mathbb{P}(X_{n-1} = y | X_0 \sim \pi) = \ldots = \mathbb{P}(X_1 = y | X_0 \sim \pi) = \pi(y)$$

Or in English, the chance of hopping to state $y \in S$ is the same regardless of our current state. Also, note that P can be thought of as a linear transformation so

$$\pi P = \pi \implies \pi P^{(n)} = \pi$$

Remark. If the distribution π on S is stationary, π is a left eigenvector of P with eigenvalue 1.

Example (stationary distribution)**.** Consider

$$\pi = \begin{bmatrix} 0.54 & 0.41 & 0.05 \end{bmatrix}, P = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.4 & 0.6 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Think about $(\pi P)_1$. This is the probability that given $x_0 \sim \pi$, after one jump in the Markov chain, we end up at state 1. To find $(\pi P)_1$ we consider the probability of every path to $X_1 = 1$ (i.e. $P_{x,1} * \pi_x$ for $x \in S$).

$$(\pi P)_1 = \sum_{x \in S} \pi_x P_{x,1} = (0.54)(0.7) * (0.41)(0.4) * (0.05)(0) \approx 0.54$$