Math 222 Lec 4

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epsilon-delta proofs of limits

Definition 1 (vector-valued function)**.** Multivariate functions involving real numbers are generally $\mathbb{R}^n \mapsto \mathbb{R}^m$. **Vector-valued functions** are a special case where n=1. I.e. vector valued functions are $f : \mathbb{R}- > \mathbb{R}^m$. E.g. $f : t \mapsto f(t_0 = (x(t), y(t), z(t_)))$. They take real numbers as an input (parameter) and output a vector.

Definition 2 (path/curve)**.** Consider

$$\begin{aligned} \vec{c} : A \subset \mathbb{R} &\mapsto \mathbb{R}^m \\ t &\mapsto \vec{c}(t) \end{aligned}$$

If $\vec{c} = [a, b] \mapsto \mathbb{R}^3$, $C = \{\vec{c}(t) | a \leq t \leq b\}$ is the curve traced out by $\vec{c}$.

Definition 3 (velocity/speed/acceleration)**.** If $\vec{c}$ is a differentiable path, then the **velocity** of $\vec{c}$ at time t is

$$\vec{c}\,'(t) = \lim_{h \to 0} \frac{\vec{c}(t + h) - \vec{c}(h)}{h}$$

The speed of of $\vec{c}$ is $\|\vec{c}\,'(t)\|$, and the acceleration of $\vec{c}$ is $\vec{c}\,''(t)$

Remark. The derivative of a differntiable path at time t is an $n \times 1$ matrix containig the derivative of each component of the path.

$$\vec{c}\,'(t) = \begin{bmatrix} dx_1/dt \\ dx_2/dt \\ \vdots \\ dx_n/dt \end{bmatrix}$$

Circular orbits

Remark. Consider a particle of mass m moving at constant speed s in a circular path of radius r_0. Assuming it moves in the xy plane, we can ignore the z component of its path and write its location as

$$\vec{c}(t) = \left(r_0 \cos \frac{st}{r_0}, r_0 \sin \frac{st}{r_0} \right)$$

Note that θ in radians is $\frac{arclength}{radius}$. The particle's **frequency** is defined as

$$\omega = \frac{s}{r_0}$$

which means we can also express its location as

$$\vec{c}(t) = r_0 \cos \omega t, r_0 \sin \omega t$$

and its acceleratino as

$$\vec{a}(t) = \vec{c}\,''(t) = \left(-\frac{s^2}{r_0} \cos \frac{st}{r_0}, -\frac{s^2}{r_0} \sin \frac{st}{r_0} \right) = -\frac{s^2}{r_0^2} \vec{c}(t) = -\omega^2 \vec{c}(t)$$

Remark. When a particle is moving at a constant speed in a circle, acceleration is in a direction opposite to $\vec{c}(t)$: towards the center of the circle. (This is perpendicular to $\vec{c}\,'(t)$). Acceleration times mass here = **centripetal force**.

Definition 4 (frequency).