Math 340 HW 9

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1. (i) Is the chain irreducible?

Yes. A Markov chain is irreducible if each of the states can communicate. That is the case for this markov chain. The following steps can be taken to navigate between states:

	1	2	3	4	5
1	$1 \to 3 \to 2 \to 1$	$1 \to 3 \to 2$	$1 \to 3$	$1 \to 3 \to 4$	$1 \to 3 \to 2 \to 5$
2	$2 \to 1$	$2 \to 1 \to 3 \to 2$	$2 \to 1 \to 3$	$2 \to 1 \to 3 \to 4$	$2 \to 5$
3	$3 \to 2 \to 1$	$3 \to 2$	$3 \to 2 \to 1 \to 3$	$3 \to 4$	$3 \to 2 \to 5$
4	$4 \to 2 \to 1$	$4 \to 2$	$4 \to 2 \to 1 \to 3$	$4 \to 2 \to 1 \to 3 \to 4$	$4 \to 2 \to 5$
5	$5 \to 3 \to 2 \to 1$	$5 \to 3 \to 2$	$5 \to 3$	$5 \to 3 \to 4$	$5 \to 3 \to 2 \to 5$

(ii) What is the period of each state?

1. Every state in an irreducible Markov chain has the same period, so to answer this question, we need only find the period of a single state. Note that we can navigate from state 1 back to state 1 by way of a 3-length path ($1 \to 3 \to 2 \to 1$) and a 4-length path ($1 \to 3 \to 4 \to 2 \to 1$). Since these lengths are relatively prime, **the period of state 1 (and all other states) is 1**.

(iii) Are there any transient states?

No. Note the filled diagonal entries in the matrix above. Also, in general, if a Markov chain is irreducible and has $|S| < \infty$, its states are all recurrent.

(iv) Make a graphical representation of the chain

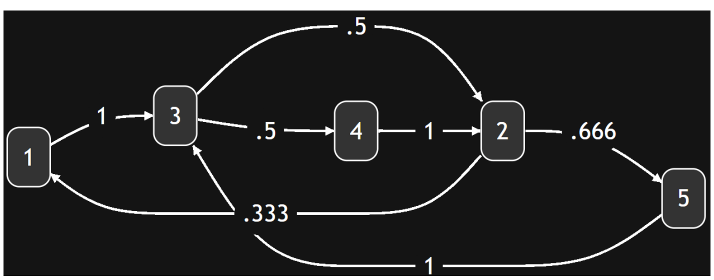

Figure 1: Markov Chain. Decimal values approx. Created with https://mikhad.github.io/graph-builder/

2. ..

The Markov chain in question has finite states and must be recurrent, since it spends only 1/4 of the time in state R. It thus has a stationary distribution. Per the problem, $\lim_{n\to\infty} \mathbb{P}(X_n = R) = 1/4$, so $\pi_R = 1/4$. Thus, $\pi_L = 3/4$

Since the Markov chain has a stationary distribution, its transition matrix P must satisfy $\pi P = \pi$. Additionally, $P_{x,y} \neq 0$ and $P_{x,y} \neq 1$ for $i, j \in S$, otherwise the periodicity assumption and/or long-run average assumption would be violated. We find $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ below.

$$\begin{bmatrix} 3/4 & 1/4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3/4 & 1/4 \end{bmatrix}$$

So

$$3a/4 + c/4 = 3/4$$
$$3b/4 + d/4 = 1/4$$

This set of linear equations satisfied by $a = \frac{3-c}{3}$ and $b = \frac{1-d}{3}$. More than one set of (a, b, c, d) satisfy these constraints, so $\boxed{P\text{, the choice of transition probabilities, is not unique}}$.

3. (i) Given that $X_0 = 1$, what is the probability that $X_n = 1$ for all $n = 1, 2, 3, 4, 5$?
This is the probability that the chain makes the transition from state 1 to state 1 five times in a row, which is $(P_{1,1})^5 = (0.3)^5 = \boxed{.00243}$.

(ii) Given that $X_0 = 1$, what is the probability that $X_4 = 1$?

The probability that the Markov chain moves from state x to y in n steps is given by $(P^n)_{x,y}$

$$P^4 = \begin{bmatrix} 0.3803 & 0.2746 & 0.3451 \\ 0.3777 & 0.2814 & 0.3409 \\ 0.3672 & 0.2704 & 0.3624 \end{bmatrix}$$

So $\mathbb{P}(X_4 = 1 | X_0 = 1) = (P^4)_{1,1} = \boxed{.3803}$.

(iii) Given that $X_0 = 1$, what is the probability that $X_3 = 1$ and $X_5 = 1$?

$\mathbb{P}(X_3 = 1 | X_0 = 1)$ is given by $(P^3)_{1,1}$. And per the Markov assumption,

$$\mathbb{P}(X_5 = 1 | X_3 = 1, X_0 = 1) = \mathbb{P}(X_5 = 1 | X_3 = 1) = (P^2)_{1,1}$$

So given the independence of steps in a Markov chain,

$$\begin{aligned} \mathbb{P}(X_5 = 1, X_3 = 1 | X_0 = 1 &= (P^3)_{1,1} * (P^2)_{1,1} \\ &= (0.373)(0.43) \\ &\boxed{= .16039} \end{aligned}$$

(iv) In the long run, what fraction of time will X_n spend in state 1? Does the value depend on the starting value X_0?

Since the entries of P are strictly positive, the Markov Chain it represents is irreducible. The chain also clearly has a finite state space. As such, there exists a unique stationary distribution π for the Markov Chain that approximates the long-run average fraction of time the chain will spend in each state. Note that $\lim_{n\to\infty} tX_n \sim \pi$ for any vector t, so $\boxed{\text{the long-run average fraction of time } X_n = i \text{ for } i \in S \text{ does not depend on } X_0.}$

Per Wolfram Alpha,

$$\begin{bmatrix} 0.3 & 0.4 & 0.3 \\ 0.7 & 0.2 & 0.1 \\ 0.2 & 0.2 & 0.6 \end{bmatrix}^{100} = \begin{bmatrix} 0.3 & 0.4 & 0.3 \\ 0.7 & 0.2 & 0.1 \\ 0.2 & 0.2 & 0.6 \end{bmatrix}^{1000} = \begin{bmatrix} 0.375 & 0.275 & 0.35 \\ 0.375 & 0.275 & 0.35 \\ 0.375 & 0.275 & 0.35 \end{bmatrix}$$

Thus, $\pi = \lim_{n\to\infty} X_n = \begin{bmatrix} 0.375 & 0.275 & 0.35 \end{bmatrix}$. $\boxed{\text{The long-run average fraction of time } X_n = 1 \text{ is given by } \pi_1 : 0.375}$.

Note: We know π is a left eigenvector of P with eigenvalue 1 so π should satisfy $\pi P = \pi$. And indeed

$$\begin{bmatrix} 0.375 & 0.275 & 0.35 \end{bmatrix} \begin{bmatrix} 0.3 & 0.4 & 0.3 \\ 0.7 & 0.2 & 0.1 \\ 0.2 & 0.2 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.375 & 0.275 & 0.35 \end{bmatrix}$$

4. (i) Identify this game with a Markov chain. Find the transition probability matrix and represent the chain graphically.

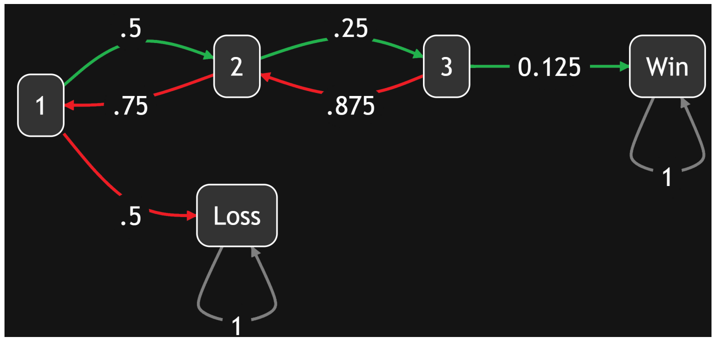

Figure 2: Graphical representation of Markov chain. Created with https://mikhad.github.io/graph-builder/

Transition matrix:

	Loss	1	2	3	Win
Loss	1	0	0	0	0
1	1/2	0	1/2	0	0
2	0	3/4	0	1/4	0
3	0	0	7/8	0	1/8
Win	0	0	0	0	1

(ii) What is the probability that a person at level 1 will eventually win the prize?

Let $h(x)$ be the probability that a Markov chain starting at x will reach b before a.

$$h(x) = \sum_{y \in S} h(y) P_{xy}$$

We are interested in $h(1)$, the probability of starting at state 1 and hitting the "Win" state before the "Loss" state. Note that the following system of equations holds:

$$h(1) = P_{1,L}\underbrace{h(L)}_{0} + \underbrace{P_{1,1}}_{0}h(1) + P_{1,2}h(2) + \underbrace{P_{1,3}}_{0}h(3) + \underbrace{P_{1,W}}_{0}h(L) = \frac{1}{2}h(2) \tag{1}$$

$$h(2) = \underbrace{P_{2,L}h(L)}_{0} + P_{2,1}h(1) + \underbrace{P_{2,2}}_{0}h(2) + P_{2,3}h(3) + \underbrace{P_{2,W}}_{0}h(L) = \frac{3}{4}h(1) + \frac{1}{4}h(3) \tag{2}$$

$$h(3) = \underbrace{P_{3,L}h(L)}_{0} + \underbrace{P_{3,1}}_{0}h(1) + P_{3,2}h(2) + \underbrace{P_{3,3}}_{0}h(3) + P_{3,W}\underbrace{h(W)}_{1} = \frac{7}{8}h(2) + \frac{1}{8} \tag{3}$$

We thus solve the system for $h(1)$:

$$h(2) = \left(\frac{3}{4}\right)\left(\frac{1}{2}\right)h(2) = \frac{3}{8}h(2) + \frac{1}{4}h(3) \qquad (1) \to (2) \tag{4}$$

$$h(2) = \frac{2}{5}h(3) \qquad \text{simplify (4)} \tag{5}$$

$$h(3) = \left(\frac{7}{8}\right)\left(\frac{2}{5}\right)h(3) + \frac{1}{8} = \frac{7}{20}h(3) + \frac{1}{8} \qquad (5) \to (3) \tag{6}$$

$$h(3) = \frac{5}{26} \qquad \text{simplify (6)} \tag{7}$$

$$h(2) = \left(\frac{2}{5}\right)\left(\frac{5}{26}\right) = \frac{1}{13} \qquad (7) \to (5) \tag{8}$$

$$h(1) = \left(\frac{1}{2}\right)\left(\frac{1}{13}\right) = \frac{1}{26} \qquad (8) \to (1) \tag{9}$$

So the probability that a person at level 1 will eventually win a prize is 1/26.

5. (i) ν and π are invariant distributions, so the two vectors' values represent long-term average fractions of time the Markov chain spends at given states. A negative value would be meaningless, because a negative fraction of time is nonsensical. A 0 value at some position $x \in S$ would indicate that in the long-run, it is impossible for a chain to reach given state x, a violation the irreducibility assumption. Since values of ν and π are never 0 or negative, the minimum value in both vectors must be positive.

(ii) Consider the two vectors ν and π. For non-negative $i < |S|$, let

$$\varepsilon = \min(\nu_i/\pi_i)$$

i.e., Let ε equal to the lowest ratio between elements of ν and π. Then for all non-negative $i < |S|$

$$\varepsilon\pi_i <= \nu_i \tag{10}$$

Since ν and π contain exclusively positive entries, $r = \nu - \varepsilon\pi_i$, applying (10), r has all non-negagtive entries, and, as desired,

$$\nu(x) = \varepsilon\pi(x) + r(x), \text{ for all } x \in S$$

(iii) Applying P to ν,

$$\begin{aligned}
\nu &= \nu P && \text{def of stationary dist} \\
&= (\varepsilon\pi + r)P && \text{per 5ii} \\
&= \varepsilon\pi + rP && \text{def of stationary dist}
\end{aligned}$$

(iv) The Markov Chain in question has $P_{x,y} > 0$ for all $x, y \in S$. That is, the chain can transition from any state to any other state in a single step. rP can be thought of as $\mathbb{P}(X_{n+1} = X)|X_n \sim r)$. Unless $r = \vec{0}$, there must be some positive chance we transition to any state, so rP must have all positive entries.

(v) Assume that at least one entry in r is positive. Then rP has all positive values. Per our construction of ε in 5ii, for some i, $\nu_i = \varepsilon\pi_i$. But $\nu = \varepsilon\pi + rP$, so rP must have at least one zero value. By 5iv, that means r is a vector with all 0 values!

We have shown $\nu = \varepsilon\pi$. But ν and π are both unit vectors by the definition of invariant distributions. So if one is a scalar multiple of the other, the scalar must be 1! Thus, $\nu = \pi$.