Some Review Problems

1. On Halloween, children dressed in costumes arrive at your door and ask for candy. Suppose their arrivals are a Poisson arrival process with rate $\lambda = 20$ per hour, beginning at 5:00pm.

   (i) What is the expected number of children that arrive between 6:40 and 7:00?

   (ii) Suppose you start with 15 pieces of candy, and you give each child one piece. What is the probability that your candy lasts until at least 9:00pm? (i.e. the 16th child doesn't arrive before 9pm.)

   (iii) Strangely, it is already 5:30 and no children have arrived. Given this information, what is the probability that the first child arrives before 5:45?

   (iv) State the central limit theorem (including important assumptions). Then use the central limit theorem to answer the following: How many pieces of candy should you start with so that the probability of running out of candy before 10:00pm is less than 1% (approximately)? If n is the amount of candy you start with, then you should choose n to satisfy a certain equation – you don't have to solve that equation.

2. Suppose $X \sim \text{Exp}(2)$.

   (i) What is the **density** for the random variable $Z = 3X^2 - 2$?

   (ii) For what function $g : \mathbb{R} \to \mathbb{R}$ does the random variable $Y = g(X)$ have the $\text{Unif}(0,1)$ distribution?

   (iii) What is $\mathbb{P}(1 < X < 3)$?

3. Let $X_1, X_2, X_3, \ldots$ be a sequence of independent random variables, each begin distibuted uniformly on the discrete set $\{0, 1, 2, 4, 5\}$. Let N be a Geometric(p) random variable, independent of the X's. Let Y be the product
$$Y = X_1 \cdot X_2 \cdots X_N$$
   Compute $\mathbb{E}[Y]$, in terms of the parameter p.

4. Fix $p \in (0,1)$ and an integer $N > 1$. Let $Y \sim \text{Unif}(\{0, 1, 2, \ldots, N\})$. Given $Y = y$, toss a p-coin y times, and let X be the number of heads tossed (i.e. heads occurs with probability p).

   (i) What is the joint distribution of (X, Y)?

   (ii) What is $\text{Var}(X)$, in terms of N and p? Simplify your answer as much as possible. Hint: think about computing $\mathbb{E}[X^2]$ by conditioning.

   (iii) Suppose $N = 5$. What is $\mathbb{E}[Y \mid X = 2]$?

5. Let (X, Y) be a point chosen uniformly at random from the disc D of radius 1 centered at the origin in $\mathbb{R}^2$.

   (i) Compute the marginal density of X.

   (ii) Compute $f_Y(y \mid X = 1/2)$, the conditional density of Y given $X = 1/2$.

   (iii) Let $Z = \sqrt{X^2 + Y^2}$ be the radial coordinate of the point (X, Y). Compute $\mathbb{E}[Z]$.

   (iv) Suppose that random variables (θ, R) are independent with $\theta \sim \text{Unif}([0, 2\pi])$ and $R \sim \text{Unif}([0,1])$. Then define $(U, V) = (R\cos(\theta), R\sin(\theta))$. This also defines a random point in the disc D, with θ being the angular coordinate and R being the radial coordinate. What is the joint density of (U, V)? In particular, is (U, V) distributed uniformly on D?

6. Use Chebychev's inequality to obtain an upper bound on the probability that more than 400 heads occur out of 600 tosses of a fair coin.

7. Can you describe a joint density for a pair (X, Y) such that the following are all true (simultaneously):

   (i) the marginal distributions of X and Y are both uniform on $[0, 1]$,

   (ii) X and Y are uncorrelated, meaning that $\text{Cov}(X, Y) = 0$.

   (iii) X and Y are **not** independent.