Math 340 HW 9

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1. .

(i) Is the chain irreducible?

Yes. A Markov chain is irreducible if each of the states can communicate. That is the case for this markov chain. The following steps can be taken to navigate between states:

	1	2	3	4	5
1	$1 \to 3 \to 2 \to 1$	$1 \to 3 \to 2$	$1 \to 3$	$1 \to 3 \to 4$	$1 \to 3 \to 2 \to 5$
2	$2 \to 1$	$2 \to 1 \to 3 \to 2$	$2 \to 1 \to 3$	$2 \to 1 \to 3 \to 4$	$2 \to 5$
3	$3 \to 2 \to 1$	$3 \to 2$	$3 \to 2 \to 1 \to 3$	$3 \to 4$	$3 \to 2 \to 5$
4	$4 \to 2 \to 1$	$4 \to 2$	$4 \to 2 \to 1 \to 3$	$4 \to 2 \to 1 \to 3 \to 4$	$4 \to 2 \to 5$
5	$5 \to 3 \to 2 \to 1$	$5 \to 3 \to 2$	$5 \to 3$	$5 \to 3 \to 4$	$5 \to 3 \to 2 \to 5$

(ii) What is the period of each state?

(iii) Are there any transient states?

No. Note the filled diagonal entries in the matrix above. Also, in general, if a Markov chain is irreducible and has $|S| < \infty$, its states are all recurrent.

(iv) Make a graphical representation of the chain

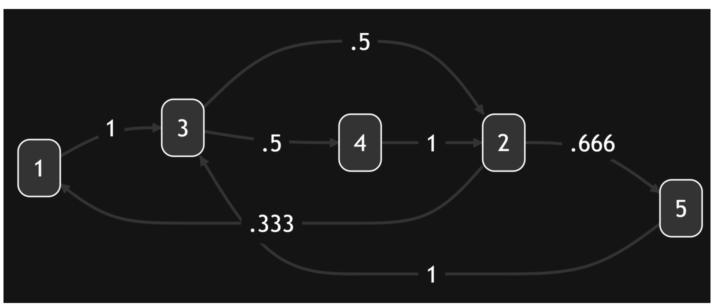

Figure 1: Markov Chain. Decimal values approx. Created with https://mikhad.github.io/graph-builder/

2. ..

The Markov chain in question has finite states and must be recurrent, since it spends only 1/4 of the time in state R. It thus has a stationary distribution. Per the problem, $\lim_{n\to\infty} \mathbb{P}(X_n = R) = 1/4$, so $\pi_R = 1/4$. Thus, $\pi_L = 3/4$

Since the Markov chain has a stationary distribution, its transition matrix P must satisfy $\pi P = \pi$. Additionally, $P_{x,y} \neq 0$ and $P_{x,y} \neq 1$ for $i, j \in S$, otherwise the periodicity assumption and/or long-run average assumption would be violated. We find $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ below.

$$\begin{bmatrix} 3/4 & 1/4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3/4 & 1/4 \end{bmatrix}$$

So

$$3a/4 + c/4 = 3/4$$
$$3b/4 + d/4 = 1/4$$

This set of linear equations satisfied by $a = \frac{3-c}{3}$ and $b = \frac{1-d}{3}$. More than one set of (a, b, c, d) satisfy these constraints, so $\boxed{P\text{, the choice of transition probabilities, is not unique}}$.

3. ..

(i) Given that $X_0 = 1$, what is the probability that $X_n = 1$ for all $n = 1, 2, 3, 4, 5$?
This is the probability that the chain makes the transition from state 1 to state 1 five times in a row, which is $(P_{1,1})^5 = (0.3)^5 = \boxed{.00243}$.

(ii) Given that $X_0 = 1$, what is the probability that $X_4 = 1$?

The probability that the Markov chain moves from state x to y in n steps is given by $(P^n)_{x,y}$

$$P^4 = \begin{bmatrix} 0.3803 & 0.2746 & 0.3451 \\ 0.3777 & 0.2814 & 0.3409 \\ 0.3672 & 0.2704 & 0.3624 \end{bmatrix}$$

So $\mathbb{P}(X_4 = 1 | X_0 = 1) = (P^4)_{1,1} = \boxed{.3803}$.

(iii) Given that $X_0 = 1$, what is the probability that $X_3 = 1$ and $X_5 = 1$?

$\mathbb{P}(X_3 = 1 | X_0 = 1)$ is given by $(P^3)_{1,1}$. And per the Markov assumption,

$$\mathbb{P}(X_5 = 1 | X_3 = 1, X_0 = 1) = \mathbb{P}(X_5 = 1 | X_3 = 1) = (P^2)_{1,1}$$

So given the independence of steps in a Markov chain,

$$\begin{aligned} \mathbb{P}(X_5 = 1, X_3 = 1 | X_0 = 1 &= (P^3)_{1,1} * (P^2)_{1,1} \\ &= (0.373)(0.43) \\ &\boxed{= .16039} \end{aligned}$$

(iv) In the long run, what fraction of time will X_n spend in state 1? Does the value depend on the starting value X_0?

Since the entries of P are strictly positive, the Markov Chain it represents is irreducible. The chain also clearly has a finite state space. As such, there exists a unique stationary distribution π for the Markov Chain that approximates the long-run average fraction of time the chain will spend in each state. Note that $\lim_{n\to\infty} tX_n \sim \pi$ for any vector t, so $\boxed{\text{the long-run average fraction of time } X_n = i \text{ for } i \in S \text{ does not depend on } X_0.}$

Per Wolfram Alpha,

$$\begin{bmatrix} 0.3 & 0.4 & 0.3 \\ 0.7 & 0.2 & 0.1 \\ 0.2 & 0.2 & 0.6 \end{bmatrix}^{100} = \begin{bmatrix} 0.3 & 0.4 & 0.3 \\ 0.7 & 0.2 & 0.1 \\ 0.2 & 0.2 & 0.6 \end{bmatrix}^{1000} = \begin{bmatrix} 0.375 & 0.275 & 0.35 \\ 0.375 & 0.275 & 0.35 \\ 0.375 & 0.275 & 0.35 \end{bmatrix}$$

Thus, $\boxed{\pi = \lim_{n\to\infty} X_n = \begin{bmatrix} 0.375 & 0.275 & 0.35 \end{bmatrix}}$. We know π is a left eigenvector of P with eigenvalue 1. Thus, π should satisfy $\pi P = \pi$. And indeed

$$\begin{bmatrix} 0.375 & 0.275 & 0.35 \end{bmatrix} \begin{bmatrix} 0.3 & 0.4 & 0.3 \\ 0.7 & 0.2 & 0.1 \\ 0.2 & 0.2 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.375 & 0.275 & 0.35 \end{bmatrix}$$