Math 222 Lec 3

Asa Royal (ajr74)

January 19, 2024

1 Limits and continuity

Definition 1 (Limit).

$$\lim_{x \to x_0} = L \iff f(x) \in \text{nbd}(L)$$

In english, the **limit** of $f(x_0 = L)$ if all points x in the neighborhood of x_0 have $f(x)$ in the neighborhood of $f(x_0)$.

Remark. Tips for computing limts of multivariable functions

1. Apply l'hospital's to "simplify" limits.
   (a) $\lim_{x \to 0} \frac{\sin x}{x} = 1$
2. Replace complicated terms like xy in $\lim_{x \to 0} \frac{\cos xy}{xy}$ with a simple parameter t, yielding the simpler limit $\lim_{x \to 0} \frac{\cos t}{t}$, which is 1 by L'hospital's rule.
3. If you don't see a way to simplify the limit, the limit may not exist. Try pluging in approaches to x from various paths.

Theorem 2 (epsilon-delta def of limits). Let $f : A \subset \mathbb{R}^n \mapsto \mathbb{R}^m$ nad let $\vec{x}_0$ be a boundary point of A. THen $\lim_{x \to x0} f(x) = \vec{b}$ iff for every number $\varepsilon > 0$, there is a $\delta > 0$ such that for any $\vec{x} \in A$ satisfying $0 < \|\vec{x} - \vec{x}_0\| < \delta$, we have $\|f(\vec{x}) - \vec{b}\| < \varepsilon$.

Definition 3 (Alt epsilon-delta def of limits). $\lim_{x \to x_0} f(x) = L$ means $\forall \varepsilon > 0, \exists \delta > 0$ such that

$$0 < \|\vec{x} - \vec{x}_0\| < \delta \implies \|f(x) - L\| < \varepsilon$$

1.1 Epsilon-delta proofs of existence of limit

Remark. When carrying out an epsilon delta proof, we want to show that given any ε, we can find a δ such that

$$0 < \|\vec{x} - \vec{x}_0\| < \delta \implies \|f(x) - L\| < \varepsilon$$

i.e. Imagine there's some neighborhood U of radius ε around L. We want a radius δ that defines a neighborhood V around x_0 such that any $\vec{x} \in V$ has $f(\vec{x}) \in U$.

1. Write out "Assume $\|f(\vec{x}) - L\| < \varepsilon$ for some ε". "given $\varepsilon > 0$, choose $\delta = \dots$"
2. Continue "We will find a δ s.t. $0 < \|\vec{x} - \vec{x_0}\| < \delta$" that satisfies the assumption.
3. Find such a delta by showing that $\|\mathbf{f(x)} - L\| < c\delta = \varepsilon$ **
   (a) I.e. change the LHS to be some multiple of δ, showing that $\|\mathbf{f(x)} - L\| < c\delta$. Since $\|\mathbf{f(x)} - L\| < \varepsilon$, $c\delta = \varepsilon$ satisfies our hunt for a δ, thus showing that there's some way to limit the domain while ensuring that the output of f is in ε-neighborhood of f.
   (b) if in evaluating $\|$**f(x) - L**$\|$, we end up with some multiple of our δ bound times a factor involving x (call it β), we need to take an add'l step

(c) Note that $\delta = min(1, ValueWeSolveFor)$. Then $\|\mathbf{x - a}\| < 1$ and $-1 < \|\mathbf{x\text{-}a}\| < 1$. Manipulate this inequality so that the $x - a$ term resembles β. Then note that $\beta < 1$... Plug that back into **, thus concluding that $\delta = min(1, d\varepsilon$ for some d, which means $\delta \leq d\varepsilon$.

Remark. Tips for $\varepsilon - \delta$ proof: Remember:

1. Triangle inequality: $\|\mathbf{a}+\mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|$
2. Any term under a radical will always be positive. Applying this will often let us simplify inequalities by striking out radical terms.
3. Convert to polar to simplify fractions!
4. We can add positive terms at will to the RHS of a $\leq$ inequality.
5. Have a product of $\|xyz\|$? Remember that $|x| = \sqrt{x^2} \leq \sqrt{x^2 + y^2 + z^2}$. Symmetry for y and z. Thus if $(x, y, z) \to (0, 0, 0), \delta \geq \sqrt{x^2 + y^2 + z^2}$ and thus $\|xyz\| \leq \delta^3$

Example (epsilon-delta proof). Show $\lim_{(x,y)\to(0,0)} x = 0$. Steps for $\varepsilon - \delta$ proof:

2 Continuity

Definition 4 (continuous function). A function is **continuous** iff $\forall x \in \text{nbd}(x_0)$,

$$\lim_{x \to x_0} f(x) = f(x_0)$$

3 Differentiability

Definition 5 (Partial derivative). The partial derivative of a function $f(x_1, x_2, \ldots, x_n)$ at point $\vec{x} = (x_0, \ldots, x_n)$ is given by

$$\frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, x_2, \ldots x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n) - f(\vec{x})}{h}$$

Remark. A function can have partials at a point x, y even if the function is discontinuous at that point.

Definition 6 (Differentiable). $f : U \subset \mathbb{R}^n \mapsto \mathbb{R}^m$ is **differentiable** at $\vec{x_0}$ if

1. All partial derivatives of f exist at $\vec{x_0}$
2. The tangent plane at $\vec{x_0}$ provides a linear approximation of $f(\vec{x}) = f(x, y)$ near $\vec{x_0}$ In the 3d graph case, The linear approximation is

$$f(\vec{x}) \approx f(\vec{x_0}) + \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$$

So if the linear approximation is "good",

$$\lim_{x \to \vec{x_0}} \frac{\|f(\vec{x}) - [f(\vec{x_0}) + Df(\vec{x_0})(\vec{x} - \vec{x_0})]\|}{\|\vec{x} - \vec{x_0}\|} = 0$$

Definition 7 (C^1). Let $f : U \subset \mathbb{R}^n \mapsto \mathbb{R}^m$. $f \in C^1$ iff the partials of f exist and are **continuous**

Remark. $f \in C^1$ (is continuous and has continuous partials) $\implies$ f is differentiable $\implies$ $f \in C^0$ (is continuous)