Math 340: Lec 23 Markov Chains (1)

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Remark. Markov chains are useful because they reduce conditional probability calculations to matrix operations

0.1 Overview of Markov Chains

Definition 1 (Markov chain). A **Markov Chain** is a sequence of rnadom variables $X_1, \dots, X_n$ that takes values in some "state space" S and satisfy the Markov property. The Markov Property states that the future is independent of the past but conditioned on the present. Formally,

$$\mathbb{P}(X_{n+1} = x_{n+1} | X_n = x_n, X_{n-1} = x_{n-1}, \dots, X_0 = x_0) = \mathbb{P}(X_{n+1} = x_{n+1} | X_n = x_n)$$

Definition 2 (time-homogenous Markov Chain). If we assume that $\mathbb{P}(X_{n+1} = y | X_n = x)$ does not depend on n, we say that the chain is time-homogenous. If a Markov Chain is time-homogenous, we can define a transition matrix, P, which includes the transition probability between all states in the state space.

Theorem 3 (Properties of the transition matrix P). Let P be a transition matrix. Then

1. $0 \leq P(x, y) \leq 1$
2. $\forall x, \sum_{y \in S} P(x, y) = 1$. (Interpretation: If we're at state x, we must move somewhere)

0.2 Examples of markov chains

Example (Simple random walk). A simple random walk on $\mathbb{Z}$ is a Markov Chain with $S = \mathbb{Z}$ and

$$P(x, y) = \begin{cases} 0 & \text{if } |x - y| \neq 1 \\ p & \text{if } |x - y| = 1 \end{cases}$$

Example (Other examples). Simple walk on graph, random process with urn of red/blue marbles where num of given color of marbles change when we pick one of its kind.

0.3 n-step transitions

Motivating question: What is the distribution of X_n given we're at some current state X_0?

Proposition 4.

$$\mathbb{P}(X_n = y | X_0 = x) = P^{(n)}(x, y)$$

Where $P^{(n)}$ is the nth power of the transition matrix P.

Example (Finding two step transition probabilities).

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\begin{aligned}
\mathbb{P}(X_2 = x_2 | X_0 = x_0) &= \sum_{x_1 \in S} (X_2 = x_2, X_1 = x_1 | X_0 = x_0) && \text{partitioning} \\
&= \sum_{x_1 \in S} \frac{\mathbb{P}(X_2 = x_2, X_1 = x_1, X_0 = x_0)}{\mathbb{P}(X_0 = 0)} && \text{cond. prob} \\
&= \sum_{x_1 \in S} \frac{\mathbb{P}(X_2 = x_2 | X_1 = x_1, X_0 = x_0)\mathbb{P}(X_1 = x_1, X_0 = x_0)}{\mathbb{P}(X_0 = 0)} && \text{cond. prob} \\
&= \sum_{x_1 \in S} \frac{\mathbb{P}(X_2 = x_2 | X_1 = x_1)\mathbb{P}(X_1 = x_1, X_0 = x_0)}{\mathbb{P}(X_0 = 0)} && \text{Markov property} \\
&= \sum_{x_1 \in S} \frac{\mathbb{P}(X_2 = x_2 | X_1 = x_1)\mathbb{P}(X_1 = x_1 | X_0 = x_0)\mathbb{P}(X_0 = 0)}{\mathbb{P}(X_0 = 0)} && \text{cond. prob} \\
&= \sum_{x_1 \in S} \mathbb{P}(X_2 = x_2 | X_1 = x_1)\mathbb{P}(X_1 = x_1 | X_0 = x_0) && \text{cond. prob} \\
&= \sum_{x_1 \in S} P(x_0, x_1)P(x_1, x_2)\text{trans. matrix} \\
&= P^{(2)}(x, y) && \text{def. matrix mult}
\end{aligned}
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