Math 340: Lec 17 Big Ideas Journal (Continuously distributed random variables)

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March 7, 2024

Proposition 1. For any continuously-distributed r.v. X, $\forall z, \mathbb{P}(X = z) = 0$.

Remark. Defining outcome spaces and probability measures for continuous random variables is a bit odd. Consider, e.g., the fact that because intervals for continuous r.v. uncountable unions,

$$1 = \mathbb{P}(X \in (\ell, r)) = \mathbb{P}\left(\bigcap_{z \in (\ell, r)} \{X = z\}\right) \neq \sum_{z \in (\ell, r)} \mathbb{P}(X = z) = 0$$

Definition 2 (Cummulative distribution function (CDF))**.** Let X be any real-valued r.v.. The CDF of X is

$$F(z) = \mathbb{P}(X \leq z) = \mathbb{P}(X \in (-\infty, z))$$

Note that $F : \mathbb{R} \mapsto [0, 1]$

Definition 3 (continuous r.v.)**.** Formally, a random variable is **continuous** if its CDF is continuous (and can thus be integrated to recover a density function).

Definition 4 (density function)**.** A continuously-distributed r.v. X has a density $f(x)$ if $\forall a < b \in R$,

$$\mathbb{P}(X \in (a, b)) = \int_a^b f(x)dx$$

Remark. Note that by the fundamental theorem of calculus, $F(b) - F(a) = \int_a^b f(x)dx$ and $F'(z) = f(z)$.

Examples of continuously-distributed r.v.s

1. Uniform distribution
   $X \sim \text{Unif}(\ell, r)$ if
   $$\mathbb{P}(X \in (a, b)) = \frac{b - a}{r - \ell}$$
   if $l < a < b < r$.

2. Exp(λ): exponential distribution w/ paramter λ.
   $$\mathbb{P}(X > t) = e^{-\lambda t}$$
   $$F(t) = \mathbb{P}(X \leq t) = \begin{cases} 1 - e^{-\lambda t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$
   $$f(t) = F'(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$
   The exponential distribution is used, e.g., to model time until an arrival, akin to how we use the geometric distribution with discretized time.

3. Gaussian (normal) dist
   Let $\mu \in R, \sigma^2 > 0$. $X \sim \text{Normal}(\mu, \sigma^2)$ if

$$f(x) = \frac{e^{\frac{-(x-\mu)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}$$
$$F(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-\left(\frac{x-\mu}{\sigma}\right)^2/2}$$

   Remark. $N(0,1)$ with $\mu = 0, \sigma^2 = 1$ is called the standard normal.

Proposition 5. Any random variable can be simulated as a function of a uniformly-distributed r.v.
Let $U \sim \text{Unif}(0,1)$. Let F be the CDF of a functino we wish to find an RV for. Assume that F is strictly increasing and continuous. Let $X = F^{-1}(U)$. Then $X \sim F$.

Proof.

$$\mathbb{P}(X \leq x) = \mathbb{P}(F^{-1}(U)) \leq x = \mathbb{P}(U \leq F(X)$$

So X has the same CDF as F. □