Math 221 Lec 16

3.4: Dimension

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October 12, 2023

Proposition 1. $a_1, \ldots, a_n$ are dependent in R^m if $n > m$

Proof. $\text{rank}(A) \leq m < n$, and if $\text{rank}(A) < n$, the columns of A are dependent. □

Remark. The proof above shows that vectors are linearly independent iff they are a basis for their span.

Proposition 2. $A \in R^{n \times n}$ is nonsingular iff the columns of A form a basis of R^n

Proof. A is singular iff $N(A) = \{\mathbf{0}\}$ iff the columns of A are linearly independent. Since n linearly independent vectors span R^n, the columsn of A are both liearly independent and span R^n. They are thus a basis for R^n. □

Theorem 3 (Bases of subspaces)**.** Every subspace $V \subset R^n$ has a basis.

Proof. Every subspace can be expressed as a span of vectors. If $V = \{\mathbf{0}\}$, is a basis. now build upwards. Take a vector in it. if it spans v, we have a basis. If not, take a vector not in its span. Do those vectors span? Then we have a basis. If not...

terminates at or before $k = n$ by first prop on this page □

Theorem 4 (All bases of a subspace have the same size)**.** $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ and $\{\mathbf{w}_1, \ldots, \mathbf{w}_\ell\}$ are two bases for the subspace $V \subset \mathbb{R}^n \Rightarrow k = \ell$.

Proof. $w_i \in \text{span}(v_1, \ldots, v_k) \Rightarrow w_i = Ax_i$ for some $x_i \in \mathbb{R}^k$ where $A = \begin{bmatrix} | & & | \\ v_1 & \ldots & v_k \\ | & & | \end{bmatrix}$ and x_i is the set of coefficients for a linear combination of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$. We can express this in a single statement as an equation with a matrix on either side

$$\begin{bmatrix} | & & | \\ w_1 & \ldots & w_\ell \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ v_1 & \ldots & v_k \\ | & & | \end{bmatrix} \begin{bmatrix} | & & | \\ x_1 & \ldots & x_\ell \\ | & & | \end{bmatrix}$$

By the bye, W is an $n \times \ell$ matrix and A is a $n \times k$ matrix, which means X is an $k \times \ell$ matrix. We're attempting to show that $k = \ell$.

Imagine that $l > k$. Then the columns of X are linearly dependent and $N(X) \neq \{\mathbf{0}\}$. This implies that $\exists \mathbf{y} \neq \mathbf{0} \in N(X)$. If we multiply the matrix equation above by that vector y, we get $W\mathbf{y} = (AX)\mathbf{y} = A(X\mathbf{y}) = A\mathbf{0} = \mathbf{0}$. Then $N(W) \neq \{\mathbf{0}\}$, so $\mathbf{w}_1, \ldots, \mathbf{w}_l$ are linearly dependent. That is a contradiction, so $l \leq k$. But if we repeat the same argument above, noting that $\mathbf{v}_i \in \text{span}(\mathbf{w}_i, \ldots, \mathbf{w}_\ell)$, we see that $k \leq \ell$. Thus we conclude that $k = \ell$. □

Definition 5 (dimension)**.** $\dim V$ is the size of any basis of $V \subset \mathbb{R}^n$.

Proposition 6. Suppose V and W are subspaces of $\mathbb{R}^n$ with the property that $W \subset V$. If $\dim V = \dim W$, then $V = W$.

Proof. Since $W \subset V$, $V = W$ is true if $V \subset W$. Assume for contradiction that this is not true. Let $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ be a basis for W. Then $\exists v_i \in V$ s.t. $v_i \notin \text{span}(\mathbf{w}_1, \ldots, \mathbf{w}_k)$. Then $\{\mathbf{w}_1, \ldots, \mathbf{w}_k, \mathbf{v}_i\}$ is a linearly independent set of size with dimension $k+1$. But k linearly independent vectors span V, so those $k+1$ vectors mus tbe lienarly dependent, and each arbitrary $\mathbf{v}_i \in \text{span}(\mathbf{w}_1, \ldots, \mathbf{w}_k)$. A symmetrical proof shows that each $\mathbf{w}_i \in \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$. Therefore $V = W$. $\square$

Proposition 7. Let $V \subset \mathbb{R}^n$ be a k-dimensional subspace. Then any k vectors that span V must be linearly independent and any k linearly independent vectors in V must span V.

Proof. We first prove that any k vectors that span V must be linearly independent. Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ span V. Assume for contradiction that they are linearly dependent. Them some $\mathbf{v}_i \in \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_k)$, and at most $k-1$ vectors in $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent. But $k-1$ vectors cannot span a subspace with $\dim k$, so we have a contradiction. We thus conclude that $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent.

We now prove that any k linearly indepe3ndent vectors in V span $\mathbf{V}$. Let $A = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & & \mathbf{v}_k \\ | & | & & | \end{bmatrix}$. Because $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are lineraly independent, A is nonsingular, so $\forall \mathbf{b} \in \mathbb{R}^k, \exists \mathbf{x}$ s.t. $A\mathbf{x} = \mathbf{b}$. In otherwords, $\mathbf{b} \in C(A)$, which means any arbitrary $\mathbf{b} \in V$ is $\in \text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$. $\square$

Dimension of the fundamental subspaces

Theorem 8 (Dimension of the subspaces)**.** Let A be an $m \times n$ matrix. Let $U$and R denote teh echelon and reduced echelon form, respectively, of A, and let $EA = U$ represent the product of A with the product of elementary matrices to produce U.

1. The nonzero rows of U or R give a basis for $R(A)$
2. The vectors obtained by setting each freee variable equal to 1 and the remaining free variables equal to 0 in the general solution of $A\mathbf{x} = \mathbf{0}$ (read off from solutions to $R\mathbf{x} = \mathbf{0}$ give a basis for $NA(A)$.
3. The pivot columns of A give basis for $C(A)$.
4. The rows of E that correspond to the zero rows of U give a basis for $N(A^\intercal)$ (also true of E' where $E'A = R$).

Theorem 9 (Relation of subspaces to pivots)**.** Let A be an $m \times n$ matrix of rank r. Then

1. $\dim R(A) = \dim C(A) = r$
2. $\dim N(A) = n - r$
3. $\dim N(A^\intercal) = m - r$

Proof. Each basis vector for $R(A)$ and $C(A)$ contains a pivot, and there are r pivots.

The basis for the null space comes from free variables. There are $n - r$ free variables.

The number of zero rows in U is equal to the number of rows m minus the number of nonzero rows r. Thus $\dim N(A^\intercal) = m - r$. $\square$

Corollary 10 (Nullity-rank theorem)**.** Let A be an $m \times n$ matrix. Then $\text{null}(A) + \text{rank}(A) = n$

Proposition 11. Let $V \subset \mathbb{R}^n$ be a k-dimensional subspace. Then $\dim V^\perp = n - k$

Proof. Let the basis vectors of V be the rows of a matrix A. $\text{rank}(A) = k$, since each of the row vectors is linearly independent. The subspace perpendicular to $R(A)$, i.e. $V^{\perp}$, is $N(A)$, which must have $\dim n - k$ per rank-nullity theorem. $\square$

Theorem 12. Let $V \subset \mathbb{R}^n$ be a subspace. THen every vector in $\mathbb{R}^n$ can be written uniquely as teh sum of a vector in V and a vector in $V^{\perp}$. In particular, we have $\mathbb{R}^n = V + V^{\perp}$