Math 222: (Green's Theorem)

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March 8, 2024

Remark. Green's theorem lets us calculate the line integral of a vector field without needing to parameterize the path.

0.1 Using Green's Theorem to calculate circulation

Proposition 1. The circulation around a tiny recangle with vertices $(x, y), (x + \Delta x, y), (x, y + \Delta y), (x + \Delta x, y + \Delta y)$ is $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$

Proof. https://www.youtube.com/watch?v=JB99RbQAilI&list=PLHXZ9OQGMqxfW0GMqeUE1bLKaYor6kbHa&index=18 □

Remark. Can conceptualize Green's Thm as allowing us to calculate the circulation around a path by summing up the circulation around all infinitesmial internal areas within the region bounded by the path. $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is the circulation density (flux) at a tiny rectangle with length/width $dxdy$. So...

Theorem 2 (Green's Theorem)**.** Let D be a simple region and C be its boundary. Let $\vec{F} = P(x, y)\hat{\imath} + Q(x, y)\hat{\jmath}$

$$\int_C \vec{F} \cdot ds = \int \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial Y} \right) dxdy$$

0.2 Using Green's Theorem to calculate flux

Remark. We can similarly calculate the flux of $\vec{F}$ across some path c by summing up the flux across (i.e. divergence in) all the infinitesimal areas within the region bounded by c.

Proposition 3. The flux around a tiny recangle with vertices $(x, y), (x+\Delta x, y), (x, y+\Delta y), (x+\Delta x, y+\Delta y)$ is $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$, which is divergence!

Proof. trefor 19 □

Theorem 4 (Green's theorem (divergence form))**.** Let D be a simple region and C be its boundary. Let $\vec{F} = P(x, y)\hat{\imath} + Q(x, y)\hat{\jmath}$

$$\begin{aligned} \int_C \vec{F} \cdot \vec{n} ds &= \int \int_D \text{div } \vec{F} dA \\ &= \int \int_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial Y} \right) dxdy \\ &= \int \int_D \vec{\nabla} \cdot \vec{F} dxdy \end{aligned}$$

0.3 Stoke's Theorem

Remark. The circulation around a surface is the sum of the curl on infinitesimal areas of the surface

Theorem 5 (Stoke's THeorem).

$$\int_C = \int\int_D \mathrm{curl}\vec{F} \cdot d\vec{s}$$