

Math 221 HW3

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1.5

1. By solving a system of equations, find the linear combination of vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ that gives

$$\mathbf{b} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}.$$

We want $\mathbf{x} = (x_1, x_2, x_3)$ s.t. $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{b}$, so we solve $A\mathbf{x} = \mathbf{b}$ where $A = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3]$.

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 0 \\ -1 & 2 & 1 & -2 \end{array} \right] \xrightarrow{A_3=A_3+A_1} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 3 & 1 \end{array} \right] \xrightarrow{A_3=A_3-2A_2} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array} \right] \xrightarrow[A_2=A_2-A_3]{A_1=A_1-2A_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

We see that $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, which means the linear combo of columns that gives \mathbf{b} is $\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3$

2. Let A be an $m \times n$ matrix with row vectors $\mathbf{A}_1, \dots, \mathbf{A}_m \in \mathbb{R}^n$.

- (a) Suppose $\mathbf{A}_1 + \dots + \mathbf{A}_m = \mathbf{0}$. Deduce that $\text{rank}(A) < m$.

Proof. During Gaussian elimination, scalar multiples of rows $\mathbf{A}_1, \dots, \mathbf{A}_{m-1}$ are added to \mathbf{A}_m to cancel coefficients below pivot variables. Since $\mathbf{A}_m = -(\mathbf{A}_1 + \dots + \mathbf{A}_{m-1})$, the row operations that cancel the coefficients of \mathbf{A}_m are the addition of $\mathbf{A}_1, \dots, \mathbf{A}_{m-1}$. This yields $\mathbf{A}_m = -(\mathbf{A}_1 + \dots + \mathbf{A}_{m-1}) + (\mathbf{A}_1 + \dots + \mathbf{A}_{m-1}) = \mathbf{0}$. Since $\mathbf{A}_m = \mathbf{0}$, it has no pivot variables, which means there can be at most $m-1$ pivots. Therefore $\text{rank}(A) < m$. \square

- (b) More generally, suppose there is some linear combination $c_1\mathbf{A}_1 + \dots + c_m\mathbf{A}_m = \mathbf{0}$, where some $c_i \neq 0$. Show that $\text{rank}(A) < m$.

Proof. Per the equation above,

$$c_m\mathbf{A}_m = -(c_1\mathbf{A}_1 + \dots + c_{m-1}\mathbf{A}_{m-1})$$

which means

$$\mathbf{A}_m = -(c'_1\mathbf{A}_1 + \dots + c'_{m-1}\mathbf{A}_{m-1})$$

During Gaussian elimination, scalar multiples of rows $\mathbf{A}_1, \dots, \mathbf{A}_{m-1}$ are added to \mathbf{A}_m to cancel coefficients below pivot variables. We can see that when scaled rows $c'_1 \mathbf{A}_1, \dots, c'_{m-1} \mathbf{A}_{m-1}$ are added to \mathbf{A}_m , the modified row will be $\mathbf{0}$. Since $\mathbf{A}_m = \mathbf{0}$, it has no pivot variables, which means there can be at most $m - 1$ pivots. Therefore $\text{rank}(A) < m$. \square