Homework 9

Reading: Markv Chain Notes.

Problems/Exercises: The following problems are due by **Friday, April 19, at 11:59pm.**

1. Suppose a Markov chain has the following transition probability matrix:

$$p = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1/3 & 0 & 0 & 0 & 2/3 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

   (i) Is the chain irreducible?

   (ii) What is the period of each state?

   (iii) Are there any transient states?

   (iv) Make a graphical representation of the chain, illustrating transition probabilities as weights on directed edges between nodes.

2. Define the transition probabilities for a Markov chain on two states, labeled L and R, which is aperiodic and for which the chain spends 1/4 of the time (on average) in state R. Is there a unique choice for the transition probabilities?

3. Consider a three-state markov chain with transition probability

$$P = \begin{pmatrix} 0.3 & 0.4 & 0.3 \\ 0.7 & 0.2 & 0.1 \\ 0.2 & 0.2 & 0.6 \end{pmatrix}$$

   Let's label the states $1, 2, 3$.

   (i) Given that $X_0 = 1$, what is the probability that $X_n = 1$ for all $n = 1, 2, 3, 4, 5$ (i.e. the chain stays at 1 at least through time $n = 5$)?

   (ii) Given that $X_0 = 1$, what is the probability that $X_4 = 1$?

   (iii) Given that $X_0 = 1$, what is the probability that $X_3 = 1$ and $X_5 = 1$?

   (iv) In the long run, what fraction of time will X_n spend in state 1? Does the answer depend on the starting value X_0?

   You can use matlab or similar program to assist with your calculations.

4. Consider a game with the following structure: There are 3 levels. To advance to level $n + 1$, you have to win at level n. If you win at level 3, you win the game and get a fabulous prize. If you fail at level 1, the game ends completely. If you are at level 2 or 3 and fail, then you move to the beginning of preceding level (i.e. level 1 or

2, respectively). There is no skill involved: winning a level is determined by chance. Starting at level 1, the probability of winning level 1 is 1/2. Starting at level 2, the probability of winning at level 2 is 1/4. Starting at level 3, the probability of winning at level 3 (and getting the prize) is 1/8. If you have to repeat a level, the probability of winning at that level is unchanged (nothing is gained from your previous experience with the level).

(i) Identify this game with a Markov chain. Find the transition probability matrix and represent the chain graphically.

(ii) What is the probability that a person at level 1 will eventually win the prize? Note: this is different from winning three levels in a row. For example, it is possible to fail several times at level 2 or 3 still later win the prize. Hint: let h_k be the probability of winning the grand prize starting from level k, write a system of equations satisfied by h_1, h_2, h_3. Then solve for h_1.

5. Suppose that X_n is a markov chain on a finite state space $\mathcal{S}$, and that the transition probablity matrix satisfies $P(x, y) > 0$ for all $x, y \in \mathcal{S}$. In particular, the chain is irreducible and aperiodic. This exercise walks you through a proof of uniqueness of an invariant probability distribution π, given that there is at least one invariant distribution. Suppose π and ν are two invariant probability distributions; we'd like to show that $\pi = \nu$ (meaning that $\pi(x) = \nu(x)$ for all $x \in \mathcal{S}$).

(i) Show that irreducibility of the chain implies that π and ν must satisfy

$$\min_{x \in \mathcal{S}} \pi(x) > 0, \qquad \min_{x \in \mathcal{S}} \nu(x) > 0$$

(ii) Explain why (i) implies that

$$\nu(x) = \epsilon \pi(x) + r(x), \quad \text{for all } x \in \mathcal{S}$$

for some constant $\epsilon > 0$ and some vector r with *non-negative* entries $r(x) \geq 0$. We may assume ϵ is chosen as large as possible.

(iii) Apply P to ν: using the relation from (ii) and the fact that both ν and π are invariant, we see that $\nu = \nu P = \epsilon \pi P + rP = \epsilon \pi + rP$.

(iv) If at least one of the entries of r is positive, why must the vector rP have *all positive* entries?

(v) Explain why (iii) and (iv) implies that $\pi = \nu$, assuming ϵ was chosen to be as large as possible. Argue that we must have $\epsilon = 1$ and $r(x) = 0$ for all x.

Extra Practice Problems; you are not required to turn these in:

1. This problem is about shuffling cards, but instead of working with a normal deck, we will consider a deck having only three cards. The cards are labeled: 1, 2, 3. When the deck is new, they are arranged in numerical order: (123), meaning that 1 is on top, 2 is in the middle, and 3 is on the bottom. We shuffle the deck using the following random procedure: Select one of the cards (i.e. top, middle, or bottom card) uniformly at random, and swap the positions other two cards, without moving the selected card. Thus, if the current order is (231) and the middle card is chosen, then the resulting ordering is (132). Similarly, if the current order is (231) and the bottom card is chosen, then the resulting ordering is (321). Let X_n denote the order of the cards after n such shuffles.

   (i) Describe this chain graphically by filling in the diagram below with appropriate arrows and weights:

$$A = (123)$$

$$F = (213) \qquad\qquad\qquad B = (132)$$

$$E = (312) \qquad\qquad\qquad C = (231)$$

$$D = (321)$$

   (ii) Write the transition matrix for this chain.

   (iii) Starting from a new deck (i.e. in the state $A = (123)$), let T_C be the number of random shuffles until the deck reaches the state $C = (231)$. Compute $\mathbb{E}[T_C \mid X_0 = A]$. Do this by identifying a system of equations that one must solve; then solve the system. You may use a computer if you wish, although solving it by hand is also easy if you make use of a certain symmetry in the system.

2. Recall the urn example from the notes (see equation (1.4)). There are a total of N marbles; some are in the red urn and some are in the blue urn. At each step you pick a marble at random (chosing uniformly from the N marbles), and move it to the other urn. Let X_n be the number of marbles in the red urn at time n.

   (i) Suppose $N = 4$. Write the transition probability matrix for this chain (write it in matrix form, not formulas).

(ii) Is the chain irreducible?

(iii) What is the period of the chain?

(iv) Verify that $\pi(k) = \binom{N}{k} 2^{-N}$, for $k = 0, \ldots, N$, is a stationary distribution for this chain.

(v) For this chain, X_n is an integer, so it makes sense to ask about $\mathbb{E}[X_n]$. If the chain starts with initial distribution π, given in part (iv), what is the mean an variance of X_n?

(vi) Consider the Markov chain defined by $Y_n = X_{2n}$ for $n = 0, 1, 2, 3, \ldots$. How many communication classes are there for the Y_n chain? For the Y_n chain, what is there period $d(i)$ for each state i?

3. For each of the following, give an example of a Markov chain having the stated property, or explain why no such example exists. Specify examples by defining the state space and the transition probabilities, and (if feasible) illustrating with a graphical representation.

(i) An irreducible chain with period 3.

(ii) A chain on 4 states that has a unique invariant distribution, but no self-loops.

(iii) A chain with exactly two invariant distributions.

(iv) A chain having infinitely many invariant probability distributions.

4. Consider a standard 8x8 chessboard. In the game of chess, a king can move one step at a time, to any of the neighboring squares (one step up, down, left, right, or diagonal). Suppose the king is initially placed on one of the squares, and makes random steps (choosing neighboring square uniformly at random). Let X_n be the position of the king after n steps.

(i) Describe the state space and transition probabilities for this Markov chain.

(ii) Is the chain irreducible?

(iii) What is the period of the chain?

(iv) Is there a unique stationary distribution? What is it?

(v) Suppose that the king starts in the lower left corner of the board (call this state x). Let T_x be the time until the king returns to the corner. What is $\mathbb{E}[T_x]$?

(vi) Define the sequence of random variables $Y_n = X_{2n}$, for $n = 1, 2, 3, \ldots$. Is this a Markov chain? Is it irreducible? Aperiodic?

5. At a major consulting firm, the career track for new consultants includes the ranks of consultant, project leader, principal, and partner. At the end of each year, employees at each rank either advance, stay in the same position, or they leave the firm (due to firing or other reasons). Let X_n be the state of an employee after n years (after n performance reviews). Let's suppose that X_n is a Markov chain such that each year

- 25% of consultants are promoted to project leader, 50% remain as consultant, 25% leave the firm.

- 15% of project leaders are promoted to principal, 65% remain as project leader, 20% leave the firm.
- 15% of principals are promoted to partner, 70% remain as principal, 15% leave the firm.
- 95% of partners remain as a partner, 5% leave the firm.

Explain clearly how you could answer each of the following questions. Some of the calculations would take a while to do by hand, but you can explain how to set up the appropriate system of equations which could be solved easily in Matlab (for example).

(i) What is the expected number of years that a new consultant will remain with the firm?

(ii) What is the probability that a consultant is eventually promoted to project leader?

(iii) What fraction of new consultants eventually achieve the rank of partner?

(iv) Conditioned on eventually making partner, what is the expected time (years) until a new consultant makes partner?

(v) What is the probability that the consultant works for the company for at least 5 years?

6. Consider the following betting game. In each round, you bet some integer number of dollars and then toss a p-coin. If you bet k dollars and the coin lands heads, then I pay you k dollars. If you bet k dollars and the coin lands tails, then you loose the k dollars. Suppose that you start with $X_0 = \$3$, and that $p = 0.4$. You keep playing until either you run out of money (i.e. $X_n = 0$), or you have \$8 (i.e. $X_n = 8$).

   Consider two different strategies for playing the game:

   - Strategy A: In each round, you bet exactly \$1.
   - Strategy B: In each round, you bet as much as you can without betting more than is necessary to get \$8. So, for example, if you happen to have \$6 at the beginning of some round, you would bet \$2 in that round.

   Which strategy maximizes the probability of getting \$8 (rather than ending up with \$0)? Or, are they the same? You choose one strategy for the entire game. Answer this question by describing and analyzing two different Markov chains, on $\mathcal{S} = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$, modeling these two different strategies.