Math 221 Lec 16

3.4: Dimension

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Definition 1 (ortohogonal set)**.** Let $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^m$. $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an **orthogonal set** of fectors iff $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ when $i \neq j$. $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthogonal basis for a subspace V if $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is both a basis for V and an orthogonal set. $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an **orthonormal basis** for V if it is an orthogonal basis consisting of unit vectors.

Proposition 2. Let $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^m$. If $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthogonal set of nonzero vectors, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent.

Proof. Fill this in. Think about dot product. □

Lemma 3. Suppose $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a basis for V. Then the equation

$$\mathbf{x} = \sum_{i=1}^{k} \text{proj}_{\mathbf{v}_i} \mathbf{x} = \sum_{i=1}^{k} \text{proj}_{\mathbf{v}_i} \frac{\mathbf{x} \cdot \mathbf{v}_i}{\|\mathbf{v}_i\|^2} \mathbf{v}_i$$

holds for all $\mathbf{x} \in V$ iff $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthogonal basis for V.

Proof. Suppose $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthogonal basis for V. Then there are scalars $c_1, \ldots, c_k$ s.t.

$$\mathbf{x} = c_1\mathbf{v}_1 + \ldots + c_i\mathbf{v}_i + \ldots + c_k\mathbf{v}_k$$

And since the $\mathbf{v}_j$'s are orthogonal, we can dot the equation above with $\mathbf{v}_i$.

$$\begin{aligned} \mathbf{x} \cdot \mathbf{v}_i &= c_1(\mathbf{v}_1 \cdot \mathbf{v}_i) + \ldots + c_i(\mathbf{v}_i \cdot \mathbf{v}_i) + \ldots + c_k(\mathbf{v}_k \cdot \mathbf{v}_i) \\ &= c_i\|\mathbf{v}_i\|^2 \end{aligned}$$

So

$$c_i = \frac{\mathbf{x} \cdot \mathbf{v}_i}{\|\mathbf{v}_i\|^2}$$

□

Proposition 4. Let $V \subset \mathbb{R}^m$ be a k-dimensional subspace. The equation

$$\text{proj}_V \mathbf{b} = \sum_{i=1}^{k} \text{proj}_{\mathbf{v}_i} \mathbf{b} = \sum_{i=1}^{k} \frac{\mathbf{b} \cdot \mathbf{v}_i}{\|\mathbf{v}_i\|^2} \mathbf{v}_i$$

holds for all $\mathbf{b} \in \mathbb{R}^m$ iff $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthogonal basis for V.

Proof. □

Theorem 5 (Gram-Schmidt process)**.** Given a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ for an innerproduct space V, we obtain an orthogonal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ for V as follows:

$$\begin{aligned}
\mathbf{w}_1 &= \mathbf{v}_1 \\
\mathbf{w}_2 &= \mathbf{v}_2 - \mathrm{proj}_{\mathbf{w}_1}\mathbf{v}_2 \\
&\vdots \\
\mathbf{w}_k &= \mathbf{v}_k - \mathrm{proj}_{\mathbf{w}_{k-1}}\mathbf{v}_k - \mathrm{proj}_{\mathbf{w}_{k-2}}\mathbf{v}_k - \ldots - \mathrm{proj}_{\mathbf{w}_1}\mathbf{v}_k
\end{aligned}$$

Definition 6 (QR Decomposition)**.** THe QR decomposition of a matrix expresses a matrix A as the product of its orthogonal basis, as determined by Graham-Schmidt, by an upper triangular $n \times n$ matrix R. The entries of R are computed by keeping track of the arithmetic during Gram-Schmidt, but because $Q^{-1} = Q^\intercal$, $Q^\intercal A = R$, so $r_{ij} = \mathbf{q}_i \cdot \mathbf{a}_j$

Definition 7 (orthogonal matrix)**.** An orthogonal matrix is a matrix with all columns orthogonal to each other (*cough*, the Q of QR). Note that for an orthogonal matrix Q, $Q^\intercal Q = I$, since the rows of $Q^\intercal$ are the columns of Q. Thus $Q^{-1} = Q^\intercal$.

Proposition 8. Using QR decomposition, we can find the least squares solution to a sys. of linear equations by solving $\bar{\mathbf{x}} = R^{-1}Q^\intercal\mathbf{b}$.

Proof. The normal equations for projection arithmetic are

$$(A^\intercal A)\bar{\mathbf{x}} = A^\intercal\mathbf{b}$$

We can express $A = QR$, so

$$\begin{aligned}
((QR)^\intercal(QR))\bar{\mathbf{x}} &= (QR)^\intercal\mathbf{b} \\
R^\intercal Q^\intercal QR\bar{\mathbf{x}} &= R^\intercal Q^\intercal\mathbf{b} \\
R^\intercal(Q^\intercal Q)R\bar{\mathbf{x}} &= R^\intercal Q^\intercal\mathbf{b}
\end{aligned}$$

and since $Q^\intercal Q = I_n$,

$$R^\intercal R\bar{\mathbf{x}} = R^\intercal Q^\intercal\mathbf{b}$$

And finally, because R is nonsingular and thus invertible,

$$\bar{x} = R^{-1}Q^\intercal\mathbf{b}$$

□