Math 340: Lec 9 Big Ideas Journal (Expectation)

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Definition 1 (Expectation of a random variable)**.** If X is a discrete random variable, its expectation (mean) is

$$\mathbb{E}(X) = \sum_{x \in R(x)} x\mathbb{P}(X = x)$$

so long as the series converges absolutely.

Remark. There are a few ways to interpret $\mathbb{E}(X)$.

1. $\mathbb{E}(X)$ is a **weighted average** of the outcomes of X or the "center of probability mass". $\mathbb{E}(X)$ would be the the position of the fulcrum if you graphed the distribution of X) with the x-axis as a lever

2. $\mathbb{E}(X)$ is a **long-run average**. Suppose $X_1, X_2, \ldots, X_n$ are independent random variables with the same distribution as some random variable X. For large n, $\frac{1}{n}(X_1 + \ldots + X_n) \approx \mathbb{E}(X)$. Imagine you play n slot machines. Let X_j be your winnings from a slot machine j. $\frac{1}{n}(X_1 + \ldots + X_n)$ is your average slot machine winnings over n plays. We expect this sum will $\approx \mathbb{E}(X)$.

   Remark. The above follows from the weak law of large numbers, which essentially says that the probability that the difference between the average of the X_js and $\mathbb{E}(X)$ is greater than any $\varepsilon > 0$ approaches zero as $n \to \infty$.

3. $\mathbb{E}(X)$ is the **"fair price"** for a random prize.

Properties of expectation

Remark. Consider using the below properties of expectation to calculate expectation when direct calcuation is tricky.

1. **Linearity:** for any two random variables X and Y and any $\alpha, \beta \in \mathbb{R}$,

$$\mathbb{E}(\alpha X + \beta Y) = \alpha\mathbb{E}(X) + \beta\mathbb{E}(Y)$$

2. **Method of indicators:** Let $A \subset \Omega$ be any event. Consider $X(\omega) = \chi_A(\omega)$.

$$\mathbb{E}(X) = \mathbb{E}(\chi_A) = \sum_{x \in R(\chi_a)} x\mathbb{P}(\chi_a = x) = (0)\mathbb{P}(A^c) + (1)\mathbb{P}(A) = \mathbb{P}(A)$$

   Remark. $\mathbb{E}(\chi_A)$ for any indicator χ_A is always $\mathbb{P}(A)$.

3. **Functions:** Let X be a random variable and $g(x) : \mathbb{R} \mapsto \mathbb{R}$. Consider the random variable $g(X(\omega))$.

$$\mathbb{E}(g(X)) = \sum_{x \in R(X)} g(x)\mathbb{P}(X = x)$$

4. **Tail sum formula:** Suppose X has range $R(X) = \{0, 1, 2, \ldots\}$. Then

   Theorem 2 (tail sum formula)**.**

$$\mathbb{E}(X) = \sum_{k=1}^{\infty} \mathbb{P}(X \geq k) = \sum_{j=0}^{\infty} \mathbb{P}(X > j)$$