Math 221 Lec 13

3.3: Linear independence and basis

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1 Linear independence

Definition 1 (linear dependence/independence). A set of vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is **linearly independent** if none of them can be expressed as a linear combination of the others. The set is **linearly dependent** if at least one of the vectors can be expressed as a linear combination of the others.

Proposition 2. Linear indepenedence requires that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k = \mathbf{0} \Rightarrow c_1, \ldots, c_k \in \mathbb{R}$ are all 0. If that were not the case and some $c_i \neq 0$, we could write $\mathbf{v}_i = \frac{c_1}{c_i}\mathbf{v}_1 + \frac{c_2}{c_{i-1}}\mathbf{v}_2 + \frac{c_1}{c_{i+1}}\mathbf{v}_1 + \ldots + \frac{c_k}{c_i}\mathbf{v}_k$. I.e., at least one vector could be expressed as a linear combination of the others.

Remark. Let A be the matrix whose columsn are $\mathbf{v}_1, \ldots, \mathbf{v}_k$ (TODO: copy matrix from notes). $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent iff $A\mathbf{x} = \mathbf{0}$ has only the trivial solution in $\mathbb{R}^k$. That is, $N(A) = \{\mathbf{0}\}$.

Proposition 3. Suppose $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a linearly independent set in $\mathbb{R}^n$. Let $\mathbf{v} \in \mathbb{R}^n$. Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}\}$ is linearly independent iff $\mathbf{v} \notin \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$.

Proof. We prove the contrapositive statement: $\mathbf{v} \in \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k) \Leftrightarrow \{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}\}$ is linearly dependent.
Forwards conditional: if $\mathbf{v} \in \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k), \mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k$, which means $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k + (-1)\mathbf{v} = \mathbf{0}$. Note that the coefficient in front of $\mathbf{v}$ is not 0, which shows that $\mathbf{0}$ can be formed by a nontrivial combination of the vectors in the set. Thus, the set is linearly dependent.

Backwards conditional: If $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly dependent , $c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k + c_v\mathbf{v} = \mathbf{0}$. Then $c_v\mathbf{v} = -(c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k)$. There are two cases to explore.

1. If $c_v = 0$, some other $c_i \neq 0$, so $\mathbf{v}_i = -\frac{c_1}{c_i}\mathbf{v}_1 + \ldots + \frac{c_k}{c_i}\mathbf{v}_k$, but this violates the assumption that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent.
2. If $c_v \neq 0$, $\mathbf{v} = -\frac{c_1}{c_v}\mathbf{v}_1 + \ldots + \frac{c_k}{c_v}\mathbf{v}_k$, so $\mathbf{v} \in \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$.

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