

# Math 221 HW1

Asa Royal (ajr74)

September 7, 2023

## 1.1

6. Find a parametric equation of each of the following lines

(a)  $3x_1 + 4x_2 = 6$

First, we rearrange the Cartesian equation to express  $x_1$  in terms of  $x_2$ .

$$\begin{aligned} 3x_1 + 4x_2 &= 6 \\ x_1 &= -\frac{4}{3}x_2 + 2 \end{aligned}$$

In parametric form,

$$\begin{aligned} \mathbf{x} &= \left(-\frac{4}{3}x_2 + 2, x_2\right) \\ &= (2, 0) + x_2\left(-\frac{4}{3}, 1\right) \\ &= (2, 0) + t\left(-\frac{4}{3}, 1\right) \end{aligned}$$

(c) The line with slope  $2/5$  that passes through  $A = (3, 1)$

To parameterize a line, we find the span of the vector with the same direction running through the origin, then then translate it as appropriate.

The span of that vector is  $x_1(5, 2)$ . We translate it by  $(3, 1)$ , so

$$\mathbf{x} = (3, 1) + t(5, 2)$$

(g) The line through  $A = (1, -2, 1)$  and  $B = (2, 1, -1)$

To parameterize a line, we find the span of the vector with the same direction running through the origin, then then translate it as appropriate.

The slope of the line we want to parameterize is  $B - A = (1, 3, -2)$ , which means its span is  $t(1, 3, -2)$ . We translate that by  $(1, -2, 1)$ , so

$$\mathbf{x} = (1, -2, 1) + t(1, 3, -2)$$

7. Suppose  $\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$  and  $\mathbf{y} = \mathbf{y}_0 + s\mathbf{w}$  are two parametric representations of the same line in  $\mathbb{R}^n$ .

(a) Show that there is a scalar  $t_0$  so that  $\mathbf{y}_0 = \mathbf{x}_0 + t_0\mathbf{v}$ .

*Proof.* Since  $\mathbf{x}$  and  $\mathbf{y}$  represent the same line  $\ell$ , they have the same span and contain the same vectors. Accordingly, since  $\mathbf{y}$  contains the vector  $\mathbf{y}_0$  (when  $s = 0$ ),  $\mathbf{x}$ , which represents the same line, must also contain  $\mathbf{y}_0$ .

That is, for some  $t_0$ ,  $\mathbf{y}_0 = \mathbf{x}_0 + t_0\mathbf{v}$  ( $\mathbf{y}_0$  can be expressed in terms of  $\mathbf{x}$ ). □

(b) Show that  $\mathbf{v}$  and  $\mathbf{w}$  are parallel

*Proof.* To show that  $\mathbf{v}$  and  $\mathbf{w}$  are parallel, we must show that  $\mathbf{v} = c\mathbf{w}$  (or equivalently  $\mathbf{w} = c\mathbf{v}$ ) for some  $c \in \mathbb{R}$ . Since  $\mathbf{x}$  and  $\mathbf{y}$  represent the same line, there are some scalars  $s, t \in \mathbb{R}$  and constant vectors  $\mathbf{x}_0, \mathbf{y}_0$  such that  $\mathbf{x} = \mathbf{y}$ . We can express this relationship by setting their parametric expressions equal to each other.

$$x_0 + t\mathbf{v} = y_0 + s\mathbf{w}$$

We know from 7a that there exists a scalar  $t_0$  that lets us express  $y_0 = x_0 + t_0\mathbf{v}$ . We can plug the RHS of that equation into the equation above.

$$\begin{aligned} x_0 + t\mathbf{v} &= x_0 + t_0\mathbf{v} + s\mathbf{w} \\ t\mathbf{v} - t_0\mathbf{v} &= s\mathbf{w} \\ \frac{t - t_0}{s}\mathbf{v} &= \mathbf{w} \\ \mathbf{w} &= c\mathbf{v} \end{aligned}$$

We know that  $c \in \mathbb{R}$  since all of the variables in the expression for  $c$  are scalars. Thus, since  $\mathbf{w} = c\mathbf{v}$ ,  $\mathbf{v}$  and  $\mathbf{w}$  are parallel. □

8. Decide whether each of the following vectors is a linear combination of  $\mathbf{u} = (1, 0, 1)$  and  $\mathbf{v} = (-2, 1, 0)$ .

(a)  $\mathbf{x} = (1, 0, 0)$

We want to see if  $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$ . In the context of this problem, we verify that

$$\begin{aligned} (1, 0, 0) &= s(1, 0, 1) + t(-2, 1, 0) \\ &= (s, 0, s) + (-2t, t, 0) \end{aligned}$$

by ensuring the corresponding system of equations is consistent:

$$\begin{aligned} s - 2t &= 1 & (1) \\ 0 + t &= 0 & (2) \\ s + 0 &= 0 & (3) \end{aligned}$$

From (2) and (3) we see that  $s = t = 0$ , but that is inconsistent with (1).

Therefore  $\mathbf{x}$  is not a linear combination of  $\mathbf{u}$  and  $\mathbf{v}$ .

(b)  $\mathbf{x} = (3, -1, -1)$

If  $\mathbf{x}$  is a linear combination of  $\mathbf{u}$  and  $\mathbf{v}$ ,  $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$ , which means the following set of linear equations must be consistent:

$$s - 2t = 3 \quad (1)$$

$$0 + t = -1 \quad (2)$$

$$s + 0 = 1 \quad (3)$$

From (2) and (3), we know  $s = 1, t = -1$ . This is consistent with (1).

Therefore  $\mathbf{x}$  is a linear combination of  $\mathbf{u}$  and  $\mathbf{v}$ .

(c)  $\mathbf{x} = (0, 1, 2)$

If  $\mathbf{x}$  is a linear combination of  $\mathbf{u}$  and  $\mathbf{v}$ ,  $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$ , which means the following set of linear equations must be consistent:

$$s - 2t = 0 \quad (1)$$

$$0 + t = 1 \quad (2)$$

$$s + 0 = 2 \quad (3)$$

From (2) and (3), we know that  $s = 2, t = 1$ . This is consistent with (1).

Therefore  $\mathbf{x}$  is a linear combination of  $\mathbf{u}$  and  $\mathbf{v}$ .

10. Find the parametric equation of the following planes:

(a) The plane containing the point  $(-1, 0, 1)$  and the line  $x = (1, 1, 1) + t(1, 7, -1)$

A plane is defined as the span of two non-scalar multiple vectors. In this case, one of those vectors is  $(1, 7, -1)$ . Another can be constructed from the line segment between two points on the plane:  $(1, 1, 1) - (-1, 0, 1) = (2, 1, 0)$ . A parametric equation including both is:

$$(-1, 0, 1) + s(2, 1, 0) + t(1, 7, -1)$$

(b) The plane parallel to the vector  $(1, 3, 1)$  and containing the points  $(1, 1, 1)$  and  $(-2, 1, 2)$ .

One vector in the plane is  $(1, 3, 1)$ . Another is  $(-2, 1, 2) - (1, 1, 1) = (-3, 0, 1)$ . Thus a parametric equation for the plane is

$$(1, 1, 1) + s(1, 3, 1) + t(-3, 0, 1)$$

- (c) The plane containing the points  $(1, 1, 2)$ ,  $(2, 3, 4)$ , and  $(0, -1, 2)$ .

One vector in the plane is  $(2, 3, 4) - (1, 1, 2) = (1, 2, 2)$ . Another vector in the plane is  $(0, -1, 2) - (1, 1, 2) = (-1, -2, 0)$ . Thus a parametric equation for the plane is

$$(1, 1, 2) + s(1, 2, 2) + t(-1, -2, 0)$$

- (d) The plane in  $\mathbb{R}^4$  containing the points  $(1, 1, -1, 2)$ ,  $(2, 3, 0, 1)$ , and  $(1, 2, 2, 3)$ .

One vector in the plane is  $(2, 3, 0, 1) - (1, 1, -1, 2) = (1, 2, 1, -1)$ . Another vector in the plane is  $(1, 2, 2, 3) - (1, 1, -1, 2) = (0, 1, 3, 1)$ . A parametric equation for the plane is

$$(1, 1, -1, 2) + t(1, 2, 1, -1) + u(0, 1, 3, 1)$$

21. Suppose  $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$  and  $c$  is a scalar. Prove that  $\text{span}(\mathbf{v} + c\mathbf{w}, \mathbf{w}) = \text{span}(\mathbf{v}, \mathbf{w})$

*Proof.*

$$\begin{aligned} \text{span}(\mathbf{v} + c\mathbf{w}, \mathbf{w}) &= d_1(\mathbf{v} + c\mathbf{w}) + d_2\mathbf{w} && \text{for } \forall d_1, d_2 \in \mathbb{R} \text{ (by def of span)} \\ &= d_1\mathbf{v} + d_1c\mathbf{w} + d_2\mathbf{w} \\ &= d_1\mathbf{v} + d_3\mathbf{w} + d_2\mathbf{w} && (d_3 = d_1c) \in \mathbb{R} \\ &= d_1\mathbf{v} + (d_3 + d_2)\mathbf{w} \\ &= d_1\mathbf{v} + d_4\mathbf{w} && (d_4 = d_3 + d_2) \in \mathbb{R} \\ &= \text{span}(\mathbf{v}, \mathbf{w}) && \text{by def. of span} \end{aligned}$$

□

22. Suppose vectors  $\mathbf{v}$  and  $\mathbf{w}$  are both linear combinations of  $\mathbf{v}_1, \dots, \mathbf{v}_k$ .

- (a) Prove that for any scalar  $c$ ,  $c\mathbf{v}$  is a linear combination of  $\mathbf{v}_1, \dots, \mathbf{v}_k$ .

*Proof.*

$$\begin{aligned} c\mathbf{v} &= c(d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \dots + d_k\mathbf{v}_k) && \text{for } d_1, \dots, d_k \in \mathbb{R} \\ &= (cd_1)\mathbf{v}_1 + (cd_2)\mathbf{v}_2 + \dots + (cd_k)\mathbf{v}_k \\ &= e_1\mathbf{v}_1 + e_2\mathbf{v}_2 + \dots + e_k\mathbf{v}_k && e \in \mathbb{R} \end{aligned}$$

This is, by definition, a linear combination of  $\mathbf{v}_1, \dots, \mathbf{v}_k$ .

□

(b) Prove that  $\mathbf{v} + \mathbf{w}$  is a linear combination of  $\mathbf{v}_1, \dots, \mathbf{v}_k$ .

*Proof.*

$$\begin{aligned}\mathbf{v} + \mathbf{w} &= (c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots c_k\mathbf{v}_k) + (d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \dots d_k) && \text{(by def. of linear combo)} \\ &= (c_1 + d_1)\mathbf{v}_1 + (c_2 + d_2)\mathbf{v}_2 + \dots (c_k + d_k)\mathbf{v}_k \\ &= e_1\mathbf{v}_1 + e_2\mathbf{v}_2 + \dots + e_k\mathbf{v}_k && e \in \mathbb{R}\end{aligned}$$

This is a linear combination of  $\mathbf{v}_1, \dots, \mathbf{v}_k$ . □

23. Consider the line  $\ell : \mathbf{x} = x_0 + r\mathbf{v}$  ( $r \in \mathbb{R}$ ) and the plane  $P : \mathbf{x} = s\mathbf{u} + t\mathbf{v}$  ( $s, t \in \mathbb{R}$ ). Show that if  $\ell$  and  $P$  intersect,  $x_0 \in P$ .

*Proof.* If  $\ell$  and  $P$  intersect,  $\ell = P$  for some  $r, s, t \in \mathbb{R}$ . That is, at some point,

$$x_0 + r\mathbf{v} = s\mathbf{u} + t\mathbf{v}$$

We can solve for  $x_0$  to show that it will lie within  $P$ :

$$\begin{aligned}x_0 &= s\mathbf{u} + t\mathbf{v} - r\mathbf{v} \\ &= s\mathbf{u} + (t - r)\mathbf{v} \\ &= s\mathbf{u} + t_1\mathbf{v}\end{aligned}$$

This matches the equation of the plane  $P$ . We can thus say that if  $\ell$  and  $P$  intersect,  $x_0 \in P$ . □

24. (a) Using only the properties listed in Exercise 28, prove that for any  $\mathbf{x} \in \mathbb{R}^n$ , we have  $0\mathbf{x} = \mathbf{0}$ .

*Proof.*

$$\begin{array}{ll}1\mathbf{x} = \mathbf{x} & (h) \\ (0 + 1)\mathbf{x} = \mathbf{x} & \text{by arithmetic} \\ 0\mathbf{x} + 1\mathbf{x} = \mathbf{x} & (g) \\ 0\mathbf{x} + \mathbf{x} = \mathbf{x} & (h) \\ 0\mathbf{x} + \mathbf{x} + (-\mathbf{x}) = \mathbf{x} + (-\mathbf{x}) & \text{adding } -\mathbf{x} \text{ to both sides} \\ 0\mathbf{x} = \mathbf{x} + (-\mathbf{x}) & (d) \\ 0\mathbf{x} = \mathbf{0} & (d)\end{array}$$

□

(b) Using the result of part a, prove that  $(-1)\mathbf{x} = -\mathbf{x}$ .

*Proof.*

$$\begin{aligned}
 (-1)\mathbf{x} &= (-1 + 0)\mathbf{x} && \text{additive identity} \\
 &= (-1)\mathbf{x} + 0\mathbf{x} && (g) \\
 &= (-1)\mathbf{x} + 0 && \text{From 29a above} \\
 &= (-1)\mathbf{x} + \mathbf{x} + (-\mathbf{x}) && (d) \\
 &= (-1)\mathbf{x} + 1\mathbf{x} + (-\mathbf{x}) && (h) \\
 &= (-1)\mathbf{x} + (1)\mathbf{x} + (-\mathbf{x}) && \text{adding parens for clarity} \\
 &= (-1 + 1)\mathbf{x} + (-\mathbf{x}) && (g) \\
 &= 0\mathbf{x} + (-\mathbf{x}) && \text{arithmetic} \\
 &= 0 + (-\mathbf{x}) && \text{From 29a above} \\
 &= (-\mathbf{x}) && \text{additive identity} \\
 &= -\mathbf{x} && \text{remove parens for clarity}
 \end{aligned}$$

□

## 1.2

1. For each of the following pairs of vectors  $\mathbf{x}$  and  $\mathbf{y}$ , calculate  $\mathbf{x} \cdot \mathbf{y}$  and the angle  $\theta$  between the vectors.

(b)  $\mathbf{x} = (2, 1), \mathbf{y} = (-1, 1)$

$$\begin{aligned}
 \mathbf{x} \cdot \mathbf{y} &= -2 + 1 = -1 \\
 \cos(\theta) &= \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|} = \frac{-1}{\sqrt{4+1}\sqrt{1+1}} = -\frac{1}{\sqrt{10}} \\
 \theta &= \cos^{-1}\left(-\frac{1}{\sqrt{10}}\right) \approx 108.4 \text{ deg}
 \end{aligned}$$

(d)  $\mathbf{x} = (1, 4, -3), \mathbf{y} = (5, 1, 3)$

$$\begin{aligned}
 \mathbf{x} \cdot \mathbf{y} &= 5 + 4 + (-9) = 0 \\
 \cos \theta &= \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|} = \frac{0}{\|\mathbf{x}\|\|\mathbf{y}\|} = 0 \\
 \theta &= \cos^{-1}(0) = \frac{\pi}{2}
 \end{aligned}$$

(g)  $\mathbf{x} = (1, 1, 1, 1), \mathbf{y} = (1, -3, -1, 5)$

$$\begin{aligned}
 \mathbf{x} \cdot \mathbf{y} &= 1 + (-3) + (-1) + 5 = 2 \\
 \cos \theta &= \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|} = \frac{2}{\sqrt{1+1+1+1}\sqrt{1+9+1+25}} = \frac{2}{2 * 6} = \frac{1}{6} \\
 \theta &= \cos^{-1}\left(\frac{1}{6}\right) \approx 80.4 \text{ deg}
 \end{aligned}$$

2. For each pair of vectors in exercise 1, calculate  $\text{proj}_{\mathbf{y}}\mathbf{x}$  and  $\text{proj}_{\mathbf{x}}\mathbf{y}$

(b)

$$\begin{aligned}\text{proj}_{\mathbf{y}}\mathbf{x} &= \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{y}\|^2} \mathbf{y} \\ &= -\frac{1}{\sqrt{2}} \frac{(-1, 1)}{\sqrt{2}} = -\frac{1}{2}(-1, 1)\end{aligned}$$

$$\begin{aligned}\text{proj}_{\mathbf{x}}\mathbf{y} &= \frac{\mathbf{y} \cdot \mathbf{x}}{\|\mathbf{x}\|^2} \mathbf{x} \\ &= -\frac{1}{\sqrt{5}^2} \mathbf{x} = -\frac{1}{5}(2, 1)\end{aligned}$$

(d)

$$\begin{aligned}\text{proj}_{\mathbf{y}}\mathbf{x} &= \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{y}\|^2} \mathbf{y} \\ &= \frac{0}{\|\mathbf{y}\|^2} \mathbf{y} = \mathbf{0}\end{aligned}$$

$$\begin{aligned}\text{proj}_{\mathbf{x}}\mathbf{y} &= \frac{\mathbf{y} \cdot \mathbf{x}}{\|\mathbf{x}\|^2} \mathbf{x} \\ &= \frac{0}{\|\mathbf{x}\|^2} \mathbf{x} = \mathbf{0}\end{aligned}$$

(g)

$$\begin{aligned}\text{proj}_{\mathbf{y}}\mathbf{x} &= \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{y}\|^2} \mathbf{y} \\ &= \frac{2}{36}(1, -3, -1, 5) = \frac{1}{18}(1, -3, -1, 5)\end{aligned}$$

$$\begin{aligned}\text{proj}_{\mathbf{x}}\mathbf{y} &= \frac{\mathbf{y} \cdot \mathbf{x}}{\|\mathbf{x}\|^2} \mathbf{x} \\ &= \frac{2}{4}(1, 1, 1, 1) = \frac{1}{2}(1, 1, 1, 1)\end{aligned}$$

13. Prove  $\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 = 2(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2)$ .

*Proof.*

$$\begin{aligned}
\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 &= (\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}) + (\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) && \text{(by } \mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2\text{)} \\
&= \mathbf{x} \cdot (\mathbf{x} + \mathbf{y}) + \mathbf{y} \cdot (\mathbf{x} + \mathbf{y}) + \mathbf{x} \cdot (\mathbf{x} - \mathbf{y}) - \mathbf{y} \cdot (\mathbf{x} - \mathbf{y}) && \text{(by distrib. prop. of dot product)} \\
&= (\mathbf{x} \cdot \mathbf{x}) + (\mathbf{x} \cdot \mathbf{y}) + (\mathbf{y} \cdot \mathbf{x}) + (\mathbf{y} \cdot \mathbf{y}) && \text{(same)} \\
&\quad + (\mathbf{x} \cdot \mathbf{x}) - (\mathbf{x} \cdot \mathbf{y}) - (\mathbf{y} \cdot \mathbf{x}) + (\mathbf{y} \cdot \mathbf{y}) \\
&= 2(\mathbf{x} \cdot \mathbf{x}) + 2(\mathbf{y} \cdot \mathbf{y}) && \text{(cancellation)} \\
&= 2(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2) && \text{(by } \mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2\text{)}
\end{aligned}$$

□

16. (a) If  $\mathbf{x} \cdot \mathbf{y} = 0$  for all  $\mathbf{x} \in \mathbb{R}^n$ , prove  $\mathbf{y} = \mathbf{0}$ .

*Proof.* Assume  $\mathbf{x} \cdot \mathbf{y} = 0$  for all  $\mathbf{x} \in \mathbb{R}^n$ . Let  $\mathbf{x} = \mathbf{y}$ . Then  $\mathbf{y} \cdot \mathbf{y} = 0$ . Since  $\mathbf{y} \cdot \mathbf{y} = \|\mathbf{y}\|^2$ , this means  $\|\mathbf{y}\|^2 = 0$  and  $\|\mathbf{y}\| = 0$ , which is only true if  $\mathbf{y} = \mathbf{0}$ . □

(b) Suppose  $\mathbf{y}, \mathbf{z} \in \mathbb{R}^n$  and  $\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot \mathbf{z}$  for all  $\mathbf{x} \in \mathbb{R}^n$ . What can we conclude?  
If  $\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot \mathbf{z}$ ,

$$(\mathbf{x} \cdot \mathbf{y}) - (\mathbf{x} \cdot \mathbf{z}) = 0 \tag{1}$$

$$\mathbf{x} \cdot (\mathbf{y} - \mathbf{z}) = 0 \tag{2}$$

Since  $\mathbf{x}$  can take on any value,  $\mathbf{y} - \mathbf{z}$  must equal  $\mathbf{0}$  to satisfy (2). Therefore, we know  $\mathbf{y} = \mathbf{z}$ .

18. Prove the triangle inequality:  $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ .

To begin, we can square both sides of the equation, which will allow us to express the LHS as a dot product.

*Proof.*

$$\begin{aligned}
\|\mathbf{x} + \mathbf{y}\|^2 &\leq \|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\| + \|\mathbf{y}\|^2 \\
(\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}) &\leq " " && (\|\mathbf{x}\|^2 = \mathbf{x} \cdot \mathbf{x}) \\
\mathbf{x} \cdot (\mathbf{x} + \mathbf{y}) + \mathbf{y} \cdot (\mathbf{x} + \mathbf{y}) &\leq " " && \text{(distrib. prop. of dot product)} \\
\mathbf{x} \cdot \mathbf{x} + \mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{x} + \mathbf{y} \cdot \mathbf{y} &\leq " " && \text{(distrib. prop. of dot product)} \\
\|\mathbf{x}\|^2 + 2(\mathbf{x} \cdot \mathbf{y}) + \|\mathbf{y}\|^2 &\leq \|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\| + \|\mathbf{y}\|^2 && (\|\mathbf{x}\|^2 = \mathbf{x} \cdot \mathbf{x}) \\
2(\mathbf{x} \cdot \mathbf{y}) &\leq 2\|\mathbf{x}\|\|\mathbf{y}\| && \text{(subtract factors from both sides)} \\
\mathbf{x} \cdot \mathbf{y} &\leq \|\mathbf{x}\|\|\mathbf{y}\| && \text{(divide both sides by 2)} \\
\|\mathbf{x}\|\|\mathbf{y}\| \cos \theta &\leq \|\mathbf{x}\|\|\mathbf{y}\| && \text{(\textbf{def.} angle between vectors)} \\
\cos \theta &\leq 1 && \text{(divide both sides by } \|\mathbf{x}\|\|\mathbf{y}\|\text{)}
\end{aligned}$$

□