Math 222: Conservative fields etc

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Definition 1 (conservative vector field). Let $\vec{F}$ be a C^1 vector field. $\vec{F}$ is conservative if it is the gradient of a continuously differentiable function f. I.e.

$$\vec{F} \text{ is conservative} \iff F = \nabla f$$

for some "potential" function f.

Theorem 2. Let $W \subset \mathbb{R}^3$ and $\vec{F}$ be a vector field on W. TFAE:

1. $\vec{F}$ is conservative
2. For any closed $C \subset W$,
$$\oint \vec{F} \cdot d\vec{s} = 0 \text{ by Fundamental theorem of line integrals}$$
3. For any two oriented simple curves C_1, C_2 with the same endpoints:
$$\oint_{C_1} \vec{F} \cdot d\vec{s} = \oint_{C_2} \vec{F} \cdot d\vec{s}$$

Theorem 3 (Conservative fields <-> curl-free field?). Relationship between curl and conservative fields:

1. If $\vec{F}$ is conservative, it is the gradient of some potential function f, so it must be curl-free.
2. If $\vec{F}$ is curl-free and W is "simply connected", then $\vec{F}$ is a conservative field.

Definition 4 (Divergence free field). $\vec{F}$ is a divergence free field if $\vec{\nabla} \cdot \vec{F} = 0$.

Theorem 5 (divergence free field <-> curl field>). Relationship between curl and conservative fields:

1. If $\vec{F}$ is a curl field, it is divergence free since $\vec{\nabla} \cdot \nabla f = 0$
2. If $\vec{F}$ is divergence free and and every surface in W is the boundary of a bounded solid in W, $\vec{F}$ is a curl field.

Theorem 6 (curl field stuff). TFAE:

1. G is a curl field
2. For any closed surface $S \subset W$, $\oint \vec{G} \cdot d\vec{s} = 0$
3. For any $S_1, S_2 \subset W$ with $\partial S_1 = \partial S_2$,
$$\int\int_{S_1} \vec{G} \cdot d\vec{S_1} = \int\int_{S_2} \vec{G} \cdot d\vec{S_2}$$