Math 340: Lec 25 Markov Chains (3)

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0.1 Chain properties

Definition 1 (irreducible/reducible). A Markov chain is **irreducible** if it is possible with positive probability to get from any state to any other state. If a chain is not irreducible, it is **reducible**. An irreducible Markov chain is kind of like a connected graph.

Two states $x, y \in S$ **communicate** ($x \longleftrightarrow y$ if it is possible to navigate from either state to the other. i.e. $P^n_{x,y} > 0$ and $P^m_{y,x} > 0$ for $m, n > 0$. If all states communicate, a graph is irreducible.

0.2 State properties

Definition 2 (recurrent/transient states). A state is **recurrent** if $\mathbb{P}(X_n = x \text{ for some } n \geq 1 | X_0 = x) = 1$. That is, a state is recurrent if we are guaranteed to eventually return to it.
If a state is not recurrent, it is **transient**. That means that there is some probability that after visiting it, we may never return to it: $\mathbb{P}(X_n = x \text{ for some } n \geq 1 | X_0 = x) < 1$

Definition 3 (Absorbing state). An **absorbing state** $x \in S$ is a state with $P_{x,x} = 1$. Once the Markov chain reaches an absorbing state, it never moves from it (think of species extinction in a ecosystem population model).

0.2.1 Periodicity

Definition 4 (periodicity). The **period** of a state $x \in S$ is

$$d(x) = \gcd\{n \geq 1 | (P^n)_{x,x} > 0\}$$

This is the gcd of length of all paths that loop from x to x.

Corollary 5. If a chain is irreducible and $P_{x,x} > 0$, then $d(x) = 1$ because we can go from x to x in one step. Thus, any irreducible chain with a self-loop is aperiodic.

Proposition 6. If a chain is irreducible, all its states have the same period. We then define the common period to be the period of the chain. We call a chain **aperiodic** if the period is 1. To show aperiodicity, we can show that the lengths of two return paths to a node are relatively prime.

0.3 Connection between state and chain properties

Theorem 7 (Markov chain <-> state properties). If a Markov chain is irreducible, either

1. All of its states are transient
2. All of its states are recurrent

Corollary 8. If a Markov chain is irreducible and $|S| < \infty$, there must be at least 1 recurrent state, which means all states are recurrent.

0.4 stationary distributions

Theorem 9 (limit converges to stationary)**.** Assume $|S| < \infty$. If a chain is irreducible, then there is a unique invariant (stationary) probability distribution π. Furthermore, if the chain is aperiodic, for any initial distribution ν,

$$\lim_{n\to\infty} \nu P^n = \pi$$

i.e.

$$\lim_{n\to\infty} \mathbb{P}(X_n = y | X_0 \sim \nu) = \pi(y)$$

The distribution of the nth step of the Markov Chain is given by π, no matter our starting place.

Theorem 10 (Ergodic theorem)**.** If a Markov chain is aperiodic and irreducible, for any function $F : S \mapsto \mathbb{R}$ (function on a state), the following holds with probability 1:

$$\underbrace{\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} F(X_k)}_{\text{temporal avg}} = \underbrace{\sum_{x\in S} F(x)\pi(x)}_{\text{spacial average}}$$

Remark. We can think of F as a cost or reward function that tells us how much it costs or how much we're rewarded for being at some state k. Over time, the average cost will be $F(\mathbb{E}[\pi])$, where π is the stationary distribution. Note that the RHS of theorem 11 looks like an expected value of F on the state space (because it is one...)