Math 340: Lec 24 Markov Chains (2)

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Remark.

0.1 State properties

Definition 1 (recurrent/transient states). A state is **recurrent** if $\mathbb{P}(X_n = x \text{ for some } n \geq 1 | X_0 = x) = 1$. That is, a state is recurrent if we are guaranteed to eventually return to it.
If a state is not recurrent, it is **transient**. That means that there is some probability that after visiting it, we may never return to it.

Definition 2 (Absorbing state). An **absorbing state** $x \in S$ is a state with $P_{x,x} = 1$. Once the Markov chain reaches an absorbing state, it never moves from it (think of species extinction in a ecosystem population model).

Definition 3 (irreducible/reducible). A Markov chain is **irreducible** if it is possible with positive probability to get from any state to any other state. If a chain is not irreducible, it is **reducible**. An irreducible Markov chain is kind of like a connected graph.

Two states $x, y \in S$ **communicate** ($x \longleftrightarrow y$ if it is possible to navigate from either state to the other. i.e. $P^n_{x,y} > 0$ and $P^m_{y,x} > 0$ for $m, n > 0$. If all states communicate, a graph is irreducible.

0.2 Connection between state and chain properties

Theorem 4 (Markov chain <-> state properties). If a Markov chain is irreducible, either

1. All of its states are transient (and $|S| = \infty$)
2. All of its states are recurrent (and $|S| < \infty$)

Theorem 5 (Convergence to stationary distribution). If a Markov chain is irreducible and has finitely many states,

1. A unique stationary distribution π exists for the Markov Chain s.t.

$$\pi_i = \frac{1}{r_i}$$

   where r_i is the average return time to state i.

2. If P^m is strictly positive for some m, then $\mathbb{P}(X_n = i) \to \pi_i$ as $n \to \infty$. (i.e. the distribution of X_n converges to the stationary distribution). As such, for irreducible chains with finitely many states, the stationary distribution can be thought of as giving the long-run average % of time a chain will spend at its states.