Math 221 Lec 16

3.4: Dimension

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Proposition 1. $a_1, \ldots, a_n$ are dependent in R^m if $n > m$

Proof. $\text{rank}(A) \leq m < n$, and if $\text{rank}(A) < n$, the columns of A are dependent. $\square$

Remark. The proof above shows that vectors are linearly independent iff they are a basis for their span.

Proposition 2. $A \in R^{n\times n}$ is nonsingular iff the columns of A form a basis of R^n

Proof. A is singular iff $N(A) = \{\mathbf{0}\}$ iff the columns of A are linearly independent. Since n linearly independent vectors span R^n, the columsn of A are both liearly independent and span R^n. They are thus a basis for R^n. $\square$

Theorem 3 (Bases of subspaces)**.** Every subspace $V \subset R^n$ has a basis.

Proof. Every subspace can be expressed as a span of vectors. If $V = \{\mathbf{0}\}$, is a basis. now build upwards. Take a vector in it. if it spans v, we have a basis. If not, take a vector not in its span. Do those vectors span? Then we have a basis. If not...

terminates at or before $k = n$ by first prop on this page $\square$

Theorem 4 (All bases of a subspace have the same size)**.** $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ and $\{\mathbf{w}_1, \ldots, \mathbf{w}_\ell\}$ are two bases for the subspace $V \subset \mathbb{R}^n \Rightarrow k = \ell$.

Proof. $w_i \in \text{span}(v_1, \ldots, v_k) \Rightarrow w_i = Ax_i$ for some $x_i \in \mathbb{R}^k$ where $A = \begin{bmatrix} | & & | \\ v_1 & \ldots & v_k \\ | & & | \end{bmatrix}$ and x_i is the set of coefficients for a linear combination of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$. We can express this in a single statement as an equation with a matrix on either side

$$\begin{bmatrix} | & & | \\ w_1 & \ldots & w_\ell \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ v_1 & \ldots & v_k \\ | & & | \end{bmatrix} \begin{bmatrix} | & & | \\ x_1 & \ldots & x_\ell \\ | & & | \end{bmatrix}$$

By the bye, W is an $n \times \ell$ matrix and A is a $n \times k$ matrix, which means X is an $k \times \ell$ matrix. We're attempting to show that $k = \ell$.

Imagine that $l > k$. Then the columns of X are linearly dependent and $N(X) \neq \{\mathbf{0}\}$. This implies that $\exists \mathbf{y} \neq \mathbf{0} \in N(X)$. If we multiply the matrix equation above by that vector y, we get $W\mathbf{y} = (AX)\mathbf{y} = A(X\mathbf{y}) = A\mathbf{0} = \mathbf{0}$. Then $N(W) \neq \{\mathbf{0}\}$, so $\mathbf{w}_1, \ldots, \mathbf{w}_l$ are linearly dependent. That is a contradiction, so $l \leq k$. But if we repeat the same argument above, noting that $\mathbf{v}_i \in \text{span}(\mathbf{w}_i, \ldots, \mathbf{w}_\ell)$, we see that $k \leq \ell$. Thus we conclude that $k = \ell$. $\square$

Definition 5 (dimension)**.** $\dim V$ is the size of a(ny) basis of $V \subset \mathbb{R}^n$.

Proposition 6. Suppose V and W are subspaces of $\mathbb{R}^n$ with the property that $W \subset V$. If $\dim V = \dim W$, then $V = W$.

Proof. need help with this proof Since $W \subset V$, $V = W$ is true if $V \subset W$. Assume for contradiction that this is not true. Let $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ be a basis for W. Then $\exists v_1 \in V$ s.t. $v_1 \notin \text{span}(\mathbf{w}_1, \ldots, \mathbf{w}_k)$. Then $\{\mathbf{w}_1, \ldots, \mathbf{w}_k, \mathbf{v}_1\}$ is a linearly independent set of size with dimension $k+1$. But given that $W \subset V$, $\square$

Proposition 7. Let $V \subset \mathbb{R}^n$ be a k-dimensional subspace. Then any k vectors that span V must b elinearly independent and any k linearly independent vectors in V must span V.

$$\begin{aligned}(Treated_{after} - Treated_{before}) - (Control_{after} - Controlbefore) &= (.139 - .170) - (.142 - .163) \\ &= (-.031) - (-.021) \\ &= -.01\end{aligned}$$