Homework 8

Reading: Meester: Chapter 5, Chapter 7.

Idea Journal: Remember to submit the idea journal entries after each class.

Problems/Exercises: All of these are due **Friday, March 29 by 5:00pm**.

1. Suppose that the pair of random variables (X, Y) have the joint density

$$f(x,y) = \begin{cases} c(x^2+3y), & x, y \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

   (i) What is c?
   (ii) What is the marginal density for X?
   (iii) What is the conditional density $f_Y(y \mid X = 1/2)$?
   (iv) Are X and Y independent? Justify your answer.

2. Suppose that (X, Y) has joint density

$$f(x,y) = \begin{cases} 3x, & \text{if } (x,y) \in \mathcal{T} \\ 0, & \text{otherwise} \end{cases}$$

   where $\mathcal{T}$ is the triangular region

$$\mathcal{T} = \{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le x \le 1\}.$$

   (i) What is the marginal density $f_X(x)$ of X?
   (ii) What is the conditional density $f_Y(y \mid X = \frac{1}{2})$?
   (iii) What is $\mathbb{E}[X \mid Y]$?

3. Let $X \sim N(0,1)$ (normal with mean 0 and variance 1). Let $\sigma > 0$ and $\mu \in \mathbb{R}$. Define the random variable. $Y = \mu + \sigma X$. What is the distribution (named) of Y? What is the density of Y? What is the CDF of Y in terms of μ, σ and the function

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\, dy$$

4. Compute the function $\phi(\lambda) = \mathbb{E}[e^{\lambda X}]$ for $\lambda \in \mathbb{R}$ in the following cases:

   (i) $X \sim \mathrm{Unif}([0,1])$.
   (ii) $X \sim \mathrm{N}(\mu, \sigma^2)$. (Hint: complete the square and change variables in the integral.)

5. Suppose $U_1 \sim \mathrm{Unif}([0,1])$ and $U_2 \sim \mathrm{Unif}([0,1])$ are independent random variables. Define

$$X = \min(U_1, U_2), \qquad Y = \max(U_1, U_2)$$

(i) Compute $f_X(x)$, the marginal density for X.

(ii) Compute $\mathbb{P}(X < 1/4 \mid Y \geq 1/2)$.

6. Imagine a wooden floor where the strips of wood are 2 inches appart. You have a needle that is one inch long, and you accidentally drop it on the floor. When it lands, the needle may intersect one of the lines that are defined by the cracks between the wooden strips, or maybe the needle will lie entirely between the lines. See the figure below. In the left diagram, the circled needles indicate intersection with a crack. The diagram on the right is an enlarged version of the picture near one of the needles.

   (i) What is the probability that the needle intersects a line between the strips of wood? In this computation, let Y be the distance between the center of the needle and the nearest edge; let $\theta \in [0, \pi/2]$ denote the angle relative to the parallel lines. Assume that Y and θ are independent random variables with marginal distributions $Y \sim \text{Unif}([0, 1])$ and $\theta \sim \text{Unif}([0, \pi/2])$. You shoul find the answer to be $1/\pi$; explain clearly how you get this.

   (ii) So, in view of part (i), to estimate the irrational number π you can do the following: toss the needle randomly on the floor until it crosses a line. Let T be the number of tosses until the needle crosses a line. Since $1/\pi$ is the probability that the needle crosses a line, we might model this scenario mathematically by $T \sim \text{Geo}(1/\pi)$ so that $\mathbb{E}[T] = \pi$. You repeat this experiment many times, recording the number of tosses required to hit a line. Let T_k be the number of tosses until you hit a line in the k^{th} experiment. Then estimate $\pi \approx H_N = \frac{1}{N}\sum_{k=1}^{N} T_k$, which is just an average of a bunch of random integers. What is $\mathbb{E}[H_N]$ and $\text{Var}(H_N)$?

   (iii) Suppose you want to estimate the number π to within 10^{-5}, with 90% confidence. How many rounds do you need (i.e. what should you choose for N)? Use normal approximation in this computation. So, how many total times should you expect to toss the needle to achieve this accuracy in your estimate? (note: the k^{th} "round" involves T_k tosses).

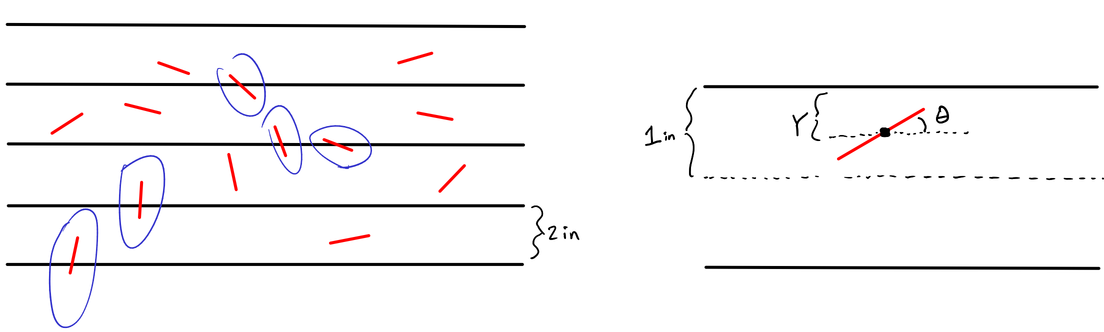

Figure 1: Needles scattered randomly on a wooden floor.