Math 340: Lec 12 Big Ideas Journal (Variance)

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Definition 1 (Variance)**.** For a random variable X, the variance of X is

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 \tag{1}$$

or alternatively, if $\mathbb{E}[X] = \mu$,

$$\mathrm{Var}(X) = \mathbb{E}[(X - \mu)^2] \tag{2}$$

Remark. We can think of variance as quantifying deviation from or closeness to the mean.

Properties of variance

1. $\mathrm{Var}(x) \geq 0$
2. $\mathrm{Var}(X)$ can be ∞ even if $\exists[X] < \infty$
3. $\mathrm{Var}(X) < \infty \iff \mathbb{E}[X^2] < \infty$
4. Scaling: For any $\alpha, \beta \in \mathbb{R}, \mathrm{Var}(\alpha X + \beta) = \alpha^2 \mathrm{Var}(X)$

Variance of an indicator variable

Proposition 2. Let $A \subset \Omega$ be any event. Consider $X = \chi_A$. $\mathrm{Var}(X) \leq 1/4$

Proof. Generally, $\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$. Because X is an indicator variable, $X^2 = X$ and $\mathbb{E}[X] = \mathbb{P}(X)$. Thus $\mathrm{Var}(X) = \mathbb{P}(X) - \mathbb{P}(X)^2 = \mathbb{P}(X)[1 - \mathbb{P}(X)]$. $\mathrm{Var}(X)$ is clearly maximized when $\mathbb{P}P(X) = 1/2$, so $\mathrm{Var}(X) \leq 1/4$. □

Covariance

Definition 3 (Covariance)**.** The covariance of two random variables X and Y is defined as

$$\mathrm{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] \tag{3}$$

Correlation is a normalized measure of covariance.

Proposition 4. Cov is bilinear because it is an inner product on certain vector spaces (see endof notes). Thus

1. $\mathrm{Cov}(cX, Y) = c\mathrm{Cov}(X, Y)$
2. $\mathrm{Cov}(X, Y + Z) = \mathrm{Cov}(X, Y) + \mathrm{Cov}(X, Z)$
3. $\mathrm{Cov}(X, X) = \mathrm{Var}(X)$
4. $\mathrm{Cov}(\sum_{i=1}^m a_i X_i, \sum_{j=1}^n b_j Y_j) = \sum_{i,j} a_i b_j \mathrm{Cov}(X_i, Y_j)$

Proof. $\mathrm{Cov}(X, Y+Z) = \mathbb{E}[X(Y+Z)] - \mathbb{E}[X]\mathbb{E}[Y+Z] = \mathbb{E}[XY] + \mathbb{E}[XZ] - \mathbb{E}[X]\mathbb{E}[Y] - \mathbb{E}[X]\mathbb{E}[Z] = \mathrm{Cov}(X, Y) + \mathrm{Cov}(X, Z)$ □

Variance of sums

Proposition 5.
$$\mathrm{Var}(X_1 + X_2 + \ldots + X_n) = \sum_{i=1}^{n} \mathrm{Var}(X_i) + 2\sum_{j<k} \mathrm{Cov}(X_j, X_k) \tag{4}$$

Remark. Note that

$$\begin{aligned}
\mathrm{Var}(X_1 + X_2) &= \mathrm{Cov}(X_1 + X_2, X_1 + X_2) \\
&= \mathrm{Cov}(X_1, X_1) + \mathrm{Cov}(X_1, X_2) + \mathrm{Cov}(X_2, X_1) + \mathrm{Cov}(X_2, X_2) \\
&= \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + 2\mathrm{Cov}(X_1, X_2)
\end{aligned}$$

Proof. Applying prop 4.4,

$$\begin{aligned}
Var(X_1 + \ldots + X_n) &= \mathrm{Cov}(X_1 + \ldots + X_n, X_1 + \ldots + X_n) \\
&= \mathrm{Cov}(X_1 + X_1) + \mathrm{Cov}(X_2 + X_2) + \ldots + \mathrm{Cov}(X_n + X_n) + \mathrm{Cov}(X_1, X_2) + \mathrm{Cov}(X_2, X_1) + \ldots \\
&= \sum_{i=1}^{n} \mathrm{Var}(X_i) + 2\sum_{j<k} \mathrm{Cov}(j, k)
\end{aligned}$$

□

Corollary 6. If $X_1, \ldots, X_n$ are independent (or just uncorrelated)

$$\mathrm{Var}(X_1 + \ldots + X_n) = \sum_{i=1}^{n} \mathrm{Var}(X_i)$$

Vector spaces of random variables

Remark. Given $\Omega, \mathbb{P}$, let S be the set of all random variables X on Ω s.t. $\mathbb{E}[X] = 0$ and $\mathbb{E}[X^2] < \infty$ (which means $\mathrm{Var}(X) < \infty$). Then S is a vector space. Thus, $X_1, X_2 \in S \implies \alpha X_1 + \beta X_2 \in S$ for any $\alpha, \beta \in R$.

Remark. We can think of Cov as an inner product on S. $\|X_1\| = \mathrm{Cov}(X_1, X_1) = \sqrt{\mathrm{Var}(X)}$, so $\|X_1\|^2 = \mathrm{Var}(X)$.

Remark. Independent vectors in S are orthogonal to each other, so for independent vectors X_1, X_2, $\|X_1 + X_2\| = \|X_1\| + \|X_2\|$. This makes sense, because the $\cos\theta$ term we'd see when calculating out $\|X_1 + X_2\|$ would be obliterated for orthogonal vectors.