6. Find a parametric equation of each of the following lines

   (a) $3x_1 + 4x_2 = 6$

   First, we rearrange the Cartesian equation to express x_1 in terms of x_2.

$$3x_1 + 4x_2 = 6$$
$$x_1 = \frac{-4}{3}x_2 + 6$$

   In parametric form,

$$\begin{aligned}\mathbf{x} &= (-\frac{4}{3}x_2 + 6, x_2) \\ &= (6,0) + x_2(-\frac{4}{3}, 1) \\ &= (6,0) + t(-\frac{4}{3}, 1)\end{aligned}$$

   (c) The line with slope $2/5$ that passes through $A = (3,1)$

   To parameterize a line, we find the span of the vector with the same direction running through the origin, then then translate it as appropriate.

   The span of that vector is $x_1(5,2)$. We translate it by (3,1), so

$$\mathbf{x} = (3,1) + t(5,2)$$

   (g) To parameterize a line, we find the span of the vector with the same direction running through the origin, then then translate it as appropriate.

   The slope of the line we want to parameterize is $B - A = (1,3,-2)$, which means its span is $t(1,3,-2)$. We translate that by (1,-2,1), so

$$\mathbf{x} = (1,-2,1) + t(1,3,-2)$$

7. Suppose

   (a) Show that there is a scalar t_0 so that $\mathbf{y}_0 = \mathbf{x}_0 + t_0\mathbf{v}$.

   Since $\mathbf{x}$ and $\mathbf{y}$ represent the same line ℓ, they have the same span and contain the same vectors. Thus, since $\mathbf{y}$ contains the vector $\mathbf{y}_0$ (when $s = 0$), $\mathbf{x}$ must also contain $\mathbf{y}_0$.

   Thus there is some parameterization of $\mathbf{x}$ such that it equals $\mathbf{y}_0$. That is, for some t_0, $\mathbf{y}_0 = \mathbf{x}_0 + t_0\mathbf{v}$.

(b) Show that $\mathbf{v}$ and $\mathbf{w}$ are parallel ?????????????????

We want to show that $\mathbf{v}$ and $\mathbf{w}$ are scalar multiples of each other (get some equation like $\mathbf{v} = a\mathbf{w}$

8. Decide whether each of the following vectors is a linear combination of $\mathbf{u} = (1, 0, 1)$ and $\mathbf{v} = (-2, 1, 0)$.

(a) $\mathbf{x} = (1, 0, 0)$

We want to see if $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$. In the context of this problem, we verify that

$$\begin{aligned}(1, 0, 0) &= s(1, 0, 1) + t(-2, 1, 0)\\ &= (s, 0, s) + (-2t, t, 0)\end{aligned}$$

by ensuring the corresponding system of equations is consistent:

$$s - 2t = 1 \tag{1}$$
$$0 + t = 0 \tag{2}$$
$$s + 0 = 0 \tag{3}$$

From (2) and (3) we see that $s = t = 0$, but that is inconsistent with (1).

Therefore $\mathbf{x}$ is not a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

(b) $\mathbf{x} = (3, -1, -1)$

If $\mathbf{x}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$, $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$, which means the following set of linear equations must be consistent:

$$s - 2t = 3 \tag{1}$$
$$0 + t = -1 \tag{2}$$
$$s + 0 = 1 \tag{3}$$

From (2) and (3), we know $s = 1, t = -1$. This is consistent with (1).

Therefore $\mathbf{x}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

(c) $\mathbf{x} = (0, 1, 2)$

If $\mathbf{x}$ is a linear combination of $\mathbf{u}$and $\mathbf{v}$, $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$, which means the following set of linear equations must be consistent:

$$s - 2t = 0 \quad (1)$$
$$0 + t = 1 \quad (2)$$
$$s + 0 = 2 \quad (3)$$

From (2) and (3), we know that $s = 2, t = 1$. This is consistent with (1).

Therefore $\mathbf{x}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

10. Find the parametric equation of the following planes:

   (a) The plane containing the point $(-1, 0, 1)$ and the line $x = (1, 1, 1) + t(1, 7, -1)$

   A plane is defined as the span of two non- scalar multiple vectors. In this case, one of those vectors is (1,1,1). Another can be constructed from the line segment between two points on the plane: $(1, 1, 1) - (-1, 0, 1) = (2, 1, 0)$. A parametric equation including both is:

   $$(-1, 0, 1) + s(2, 1, 0) + t(1, 7, -1)$$

   (b) The plane parallel to the vector $(1, 3, 1)$ and containing the points $(1, 1, 1)$ and $(-2, 1, 2)$.

   One vector in the plane is $(1, 3, 1)$. Another is $(-2, 1, 2) - (1, 1, 1) = (-3, 0, 1)$. Thus a parametric equation for the plane is

   $$(1, 1, 1) + s(1, 3, 1) + t(-3, 0, 1)$$

   (c) The plane containing the points $(1, 1, 2), (2, 3, 4)$, and $(0, -1, 2)$.

   One vector in the plane is $(2, 3, 4) - (1, 1, 2) = (1, 2, 2)$. Another vector in the plane is $(0, -1, 2) - (1, 1, 2) = (-1, -2, 0)$. Thus a parametric equation for the plane is

   $$(1, 1, 2) + s(1, 2, 2) + t(-1, -2, 0)$$

   (d) The plane in $\mathbb{R}^4$ containing the points $(1, 1, -1, 2), (2, 3, 0, 1)$, and $(1, 2, 2, 3)$.

   One vector in the plane is $(2, 3, 0, 1) - (1, 1, -1, 2) = (1, 2, -1, -1)$. Another vector in the plane is $(1, 2, 2, 3) - (1, 1, -1, 2) = (0, 1, 3, 1)$. A parametric equation for the plane is

   $$(1, 1, -1, 2) + t(1, 2, -1, -1) + u(0, 1, 3, 1)$$

21. Suppose $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and c is a scala. Prove that $\text{span}(\mathbf{v} + c\mathbf{w}, \mathbf{w}) = \text{span}(\mathbf{v}, \mathbf{w}$

Proof.

$$\begin{aligned}
\text{span}(\mathbf{v} + c\mathbf{w}, \mathbf{w}) &= d_1(\mathbf{v} + c\mathbf{w}) + d_2\mathbf{w} && \text{for } \forall d_1, d_2 \in \mathbb{R} \text{ (by def of span)} \\
&= d_1\mathbf{v} + d_1 c\mathbf{w} + d_2\mathbf{w} \\
&= d_1\mathbf{v} + d_3\mathbf{w} + d_2\mathbf{w} && (d_3 = d_1 c) \in \mathbb{R} \\
&= d_1\mathbf{v} + (d_3 + d_2)\mathbf{w} \\
&= d_1\mathbf{v} + d_4\mathbf{w} && (d_4 = d_3 + d_2) \in \mathbb{R} \\
&= \text{span}(\mathbf{v}, \mathbf{w}) && \text{by def. of span}
\end{aligned}$$

□

22. Suppose vectors $\mathbf{v}$ and $\mathbf{w}$ are both linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_k$.

(a) Prove that for any scalar c, $c\mathbf{v}$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$.

Proof.

$$\begin{aligned}
c\mathbf{v} &= c(d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \ldots + d_k\mathbf{v}_k) && \text{for } d_1, \ldots, d_k \in \mathbb{R} \\
&= (cd_1)\mathbf{v}_1 + (cd_2)\mathbf{v}_2 + \ldots + (cd_k)\mathbf{v}_k \\
&= e_1\mathbf{v}_1 + e_2\mathbf{v}_2 + \ldots + e_k\mathbf{v}_k && e \in \mathbb{R}
\end{aligned}$$

This is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$. □

(b) Prove that $\mathbf{v}+\mathbf{w}$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$.

Proof.

$$\begin{aligned}
\mathbf{v} + \mathbf{w} &= (c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots c_k\mathbf{v}_k) + (d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \ldots d_k && \text{(by def. of linear combo)} \\
&= (c_1 + d_1)\mathbf{v}_1 + (c_2 + d_2)\mathbf{v}_2 + \ldots (c_k + d_k)\mathbf{v}_k \\
&= e_1\mathbf{v}_1 + e_2\mathbf{v}_2 + \ldots + e_k\mathbf{v}_k && e \in \mathbb{R}
\end{aligned}$$

This is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$. □

23. Consider the line $\ell : \mathbf{x} = x_0 + r\mathbf{v}(r \in \mathbb{R})$ and the plane $P : x = s\mathbf{u} + t\mathbf{v}(s, t \in \mathbb{R}$. Show that if ℓ and P intersect, $x_0 \in P$. If ℓ and P intersect, $\ell = P$ at some point. That is,

$$x_0 + r\mathbf{v} = s\mathbf{u} + t\mathbf{v}$$

We can solve for x_0 to show that it will lie within P:

$$\begin{aligned} x_0 &= s\mathbf{u} + t\mathbf{v} = r\mathbf{v} \\ &= s\mathbf{u} + (t-r)\mathbf{v} \\ &= s\mathbf{u} + t_1\mathbf{v} \end{aligned}$$

This matches the equation of the plane P. We can thus say that if ℓ and P intersect, $x_0 \in P$.

24. (a) Using only the properties listed in Excercise 28, prove that for any $\mathbf{x} \in \mathbb{R}^n$, we have $0\mathbf{x} = \mathbf{0}$.

Proof.

$$\begin{aligned} 1\mathbf{x} &= \mathbf{x} && (h) \\ (0+1)\mathbf{x} &= \mathbf{x} && \text{by arithmetic} \\ 0\mathbf{x} + 1\mathbf{x} &= \mathbf{x} && (g) \\ 0\mathbf{x} + \mathbf{x} &= \mathbf{x} && (h) \\ 0\mathbf{x} + \mathbf{x} + (-\mathbf{x}) &= \mathbf{x} + (-\mathbf{x}) && \text{adding } -\mathbf{x} \text{ to both sides} \\ 0\mathbf{x} &= \mathbf{x} + (-\mathbf{x}) && (d) \\ 0\mathbf{x} &= 0 && (d) \end{aligned}$$

□

(b) Using the result of part a, prove that $(-1)\mathbf{x} = -\mathbf{x}$.

Proof.

$$\begin{aligned} (-1)\mathbf{x} &= (-1+0)\mathbf{x} && \text{additive identity} \\ &= (-1)\mathbf{x} + 0\mathbf{x} && (g) \\ &= (-1)\mathbf{x} + 0 && \text{From 29a above} \\ &= (-1)\mathbf{x} + \mathbf{x} + (-\mathbf{x}) && (d) \\ &= (-1)\mathbf{x} + 1\mathbf{x} + (-\mathbf{x}) && (h) \\ &= (-1)\mathbf{x} + (1)\mathbf{x} + (-\mathbf{x}) && \text{adding parens for clarity} \\ &= (-1+1)\mathbf{x} + (-\mathbf{x}) && (g) \\ &= 0x + (-\mathbf{x}) && \text{arithmetic} \\ &= 0 + (-\mathbf{x}) && \text{From 29a above} \\ &= (-\mathbf{x}) && \text{additive identity} \\ &= -\mathbf{x} && \text{remove parens for clarity} \end{aligned}$$

□

1.2

1. For each of the following pairs of vectors $\mathbf{x}$ and $\mathbf{y}$, calculate $\mathbf{x} \cdot \mathbf{y}$ and the angle θ between the vectors.

   (b) $\mathbf{x} = (2,1), \mathbf{y} = (-1,1)$

$$\begin{aligned} \mathbf{x} \cdot \mathbf{y} &= -2 + 1 = -1 \\ \cos(\theta) &= \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|} = \frac{-1}{\sqrt{4+1}\sqrt{1+1}} = -\frac{1}{\sqrt{10}} \end{aligned}$$

   (d) $\mathbf{x} = (1,4,-3), \mathbf{y} = (5,1,3)$

$$\begin{aligned} \mathbf{x} \cdot \mathbf{y} &= 5 + 4 + (-9) = 0 \\ \cos\theta &= \frac{0}{\|\mathbf{x}\|\|\mathbf{y}\|} = 0 \end{aligned}$$

   (g) $\mathbf{x} = (1,1,1,1), \mathbf{y} = (1,-3,-1,5)$

$$\begin{aligned} \mathbf{x} \cdot \mathbf{y} &= 1 + (-3) + (-1) + 5 = \\ \cos\theta &= \frac{2}{\sqrt{1+1+1+1}\sqrt{1+9+1+25}} = \frac{2}{2*6} = \frac{1}{6} \end{aligned}$$

2. For each pair of vectors in exercise 1, calculate $\mathrm{proj}_{\mathbf{y}}\mathbf{x}$ and $\mathrm{proj}_{\mathbf{x}}\mathbf{y}$

   (b)

$$\begin{aligned} \mathrm{proj}_{\mathbf{y}}\mathbf{x} &= \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{y}\|} \frac{\mathbf{y}}{\|\mathbf{y}\|} \\ &= -\frac{1}{\sqrt{2}} \frac{(-1,1)}{\sqrt{2}} = -\frac{1}{2}(-1,1) \end{aligned}$$

$$\begin{aligned} \mathrm{proj}_{\mathbf{x}}\mathbf{y} &= \frac{\mathbf{y} \cdot \mathbf{x}}{\|\mathbf{x}\|} \frac{\mathbf{x}}{\|\mathbf{x}\|} \\ &= -\frac{1}{\sqrt{5}^2}\mathbf{x} = -\frac{1}{5}(2,1) \end{aligned}$$

   (d)

$$\begin{aligned} \mathrm{proj}_{\mathbf{y}}\mathbf{x} &= \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{y}\|} \frac{\mathbf{y}}{\|\mathbf{y}\|} \\ &= \frac{0}{\|\mathbf{y}\|^2}\mathbf{y} = 0 \end{aligned}$$

$$\begin{aligned} \mathrm{proj}_{\mathbf{x}}\mathbf{y} &= \frac{\mathbf{y} \cdot \mathbf{x}}{\|\mathbf{x}\|} \frac{\mathbf{x}}{\|\mathbf{x}\|} \\ &= \frac{0}{\|\mathbf{x}\|^2}\mathbf{x} = 0 \end{aligned}$$

(g)

$$
\begin{aligned}
\mathrm{proj}_{\mathbf{y}}\mathbf{x} &= \frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{y}\|^2}\mathbf{y} \\
&= \frac{2}{36}(1,-3,-1,5) = \frac{1}{18}(1,-3,-1,5)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{proj}_{\mathbf{x}}\mathbf{y} &= \frac{\mathbf{y}\cdot\mathbf{x}}{\|\mathbf{x}\|^2}\mathbf{x} \\
&= \frac{2}{4}(1,1,1,1) = \frac{1}{2}(1,1,1,1)
\end{aligned}
$$

13. Prove $\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}\text{-}\mathbf{y}\|^2 = 2(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$.

Proof.

$$
\begin{aligned}
\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}\text{-}\mathbf{y}\|^2 &= (\mathbf{x}+\mathbf{y}) + \cdot(\mathbf{x}+\mathbf{y}) + (\mathbf{x}-\mathbf{y})\cdot(\mathbf{x}-\mathbf{y}) \\
&= \mathbf{x}\cdot(\mathbf{x}+\mathbf{y}) + \mathbf{y}\cdot(\mathbf{x}+\mathbf{y}) + \mathbf{x}\cdot(\mathbf{x}\text{-}\mathbf{y}) - \mathbf{y}\cdot(\mathbf{x}\text{-}\mathbf{y}) \\
&= (\mathbf{x}\cdot\mathbf{y}) + (\mathbf{x}\cdot\mathbf{y}) + (\mathbf{x}\cdot\mathbf{y}) + (\mathbf{y}\cdot\mathbf{y}) + (\mathbf{x}\cdot\mathbf{x}) - (\mathbf{y}\cdot\mathbf{x}) - (\mathbf{x}\cdot\mathbf{y}) + (\mathbf{y}\cdot\mathbf{y}) \\
&= 2(\mathbf{x}\cdot\mathbf{x}) + 2(\mathbf{y}\cdot\mathbf{y}) \\
&= 2(\|\mathbf{x}\| + \|\mathbf{y}\|)
\end{aligned}
$$

□