Math 221 Lec 9

2.3: Inverse Matrices

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1 Inverse functions

Definition 1 (inverse function)**.** Generally, for $f : S \mapsto T$,
We say that g is a **left inverse** of f if $g \circ f = I_s$. That is, $\forall x, g(f(x)) = x$
We say that g is a **right inverse** of f if $f \circ g = I_t$. That is, $\forall x, f(g(x)) = x$

Lemma 2. If f has a left inverse and a right inverse, they are equal. Is this true for non-square matrices???

Proof. Let l and r be the respective left and right inverses of f.
Pciture... □

2 Inverse matrices

Definition 3 (inverse matrices)**.** Let A be an $m \times n$ matrix.
The **left inverse** of A is an $n \times m$ matrix C s.t. $CA = I_n$.
The **right inverse** of A is an $n \times m$ matrix C s.t. $CA = I_m$.

Remark. Since A translates vectors of length n to length m, its left and right inverses must do the reverse (be $n \times m$ matrices that map from $m \mapsto n$.

Proposition 4. If A has a left inverse C, then a solution to $A\mathbf{x} = \mathbf{b}$, if it exists, must be unique.

Proof. If $A\mathbf{x} = \mathbf{b}$, then $C(A\mathbf{x}) = C\mathbf{b}$. We can rewrite this as $(CA)\mathbf{x} = C\mathbf{b}$. Since $C = A^{-1}$, $\mathbf{x} = C\mathbf{b}$. Thus, if $\mathbf{x}$ is a solution to $A\mathbf{x} = \mathbf{b}$, $\mathbf{x}$ is unique and equal to $C\mathbf{b}$. □

Remark. Let A be a left invertible matrix. Since $A\mathbf{x} = \mathbf{b}$ has only unique solutions, $\text{rank}(A) = n$.

Proposition 5. A has a right inverse precisely when $\text{rank}(A) = m$.

Proof. Let A be a matrix with a right inverse. If $\text{rank}(A) < m$, the reduced echelon form of A will have one or more rows of zeros at the bottom. When $[A|I]$ is solved to find the right inverse of A, these zero rows of A will be set equal to non-zero rows of the identity matrix, indicating there is no set of solutions that satisfies $AB = I$. Thus, by contradiction, we show $\text{rank}(A) = m$. $\square$

Lemma 6. Suppose A and B are inveritble $n \times n$ matrices. Then $(AB)^{-1} = B^{-1}A^{-1}$.

Proof. We can prove the above by showing $AB \cdot (B^{-1}A^{-1}) = I_n$

$$AB(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI_nA^{-1} = AA^{-1} = I_n$$

$\square$

Definition 7 (invertible matrix)**.** An **invertible** matrix has a left and right inverse. This requires that $\text{rank}(A) = n = m$. For an $n \times n$ matrix, the following are synonymous:

1. A is invertible
2. A has a right inverse
3. A has a left inverse
4. A is nonsingular. I.e., $\text{rank}(A) = n$

Proposition 8. Invertible matrices are nonsingular.

Proof. Let A be a an invertible matrix. Assume for contradiction that A is singular. Then $A\mathbf{x} = \mathbf{0}$ has a nontrivial (non-zero) solution. But since A is invertible and A^{-1} exists, $A^{-1}A\mathbf{x} = A^{-1}\mathbf{0}$, which means $x = \mathbf{0}$. This is a contradiction! $\square$

Proof. Alternatively, let A be an invertible matrix. Assume again for contradiction that A is singular. When solving $[A|I]$, A will row reduce to a matrix with a row of zeros in the bottom, which cannot possibly equal the bottom row of I. $\square$

2.1 Finding inverse matrices

Remark. If a matrix A is invertible, we can find A^{-1} by performing Gaussian elimination on the augmented matrix $[A|I]$. This augmented matrix represents multiple sets of simultaneous equations, wherein we solve for multiple $\mathbf{x}$ vectors, s.t. $(A\mathbf{x})_1 = i_1$ (the first column of I), $(A\mathbf{x})_2 = i_2 \ldots, (A\mathbf{x})_m = I_m$

Example. Find the inverse of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Fill out...

3 Determinants

Definition 9 (determinant)**.** The **determinant** of a linear transformation tells us how much a unit of area changes after the transformation is applied.

3.1 Geometric interpretation of the determinant

Example (geometric interp of determinant in 2 dimensions)**.** If we apply the linear transformation represented by $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$, $\hat{\imath}$ is stretched by 3 and $\hat{\jmath}$ is stretched by 2. The area covered by $\begin{bmatrix} \hat{\imath} & \hat{\jmath} \end{bmatrix}$ thus increases from $1 * 1 = 1$ to $2 * 3 = 6$, so the determinant of the transformation is 6.

Remark. In two-dimensional space, a matrix with $\det(A) = 0$ represents a linear transformation that reduces the **area** of a unit square to 0, collapsing space onto a line (or point).

Remark. In three-dimensional space, a matrix with $\det(A) = 0$ represents a linear transformation that reduces the **volume** of a unit cube to 0, collapsing space onto a plane, line, or point.

Remark. The **sign of a determinant** tells us whether the orientation of space has changed. For example, in two-dimensional space, if $\det(A) = -1$, $\hat{\imath}$ might go from being to the right of $\hat{\jmath}$ to being to the left. We can use the right-hand rule to figure out whether the orientation of 3-space has changed.