Math 222: (Line integrals)

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March 8, 2024

Definition 1 (Line integral of scalar field).

$$\int_c f(x,y)ds = \int_a^b f\left(x(t), y(t)\right) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Remark. An intuitive explanation of how we trasnsform ds::

$$\begin{aligned} ds &= \sqrt{(dx)^2 + (dy)^2} \text{ (by pythagorean)} \\ &= \frac{dt}{dt}\sqrt{(dx)^2 + (dy)^2} \\ &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \end{aligned}$$

Definition 2 (line integral of vector field). A line integral is the analog of a path integral but for vector fields. It helps us determine the work a vector field does moving a particle along a path. Imagine a particle travels along some path $c(t)$ from a to b and is influenced by field $\vec{F}$. The work done by the field is given by

$$\text{work} = \int_c \vec{F} \cdot d\vec{s} = \int_a^b \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t)dt$$

Remark. Why the dot product? It tells us how much the direction of the field $\vec{F}$ aligns with with the direction of the particle's path $d\vec{s}$, thus enabling us determine how much of the force applied translates to work.

Remark. Quick derivation of $d\vec{s} = \vec{c}\,'(t)$

$$\frac{ds}{dt} = x'(t)\hat{i} + y'(t)\hat{j} + z'(t)\hat{k}$$

So

$$ds = \left(x'(t)\hat{i} + y'(t)\hat{j} + z'(t)\hat{k}\right) dt = \vec{c}\,'(t)dt$$

0.1 Reparameterization

Definition 3 (Reparameterization). Let $h : I \mapsto I_1$ be a real-valued function that is a 1:1 map fo an interval $I = [a, b]$ onto another interval $I_1 = [a_1, b_1]$. Let $\vec{c}$ be a piecewise C_1 path. Then $\vec{p}(t) = \vec{c}(h(t))$ is called a **reparameterization** of $\vec{c}$.

A reparameterization is orientation-preserving if $p(a) = c(a), p(b) = c(b)$. It is non-orientation-preserving if $p(a) = c(b), p(b) = c(a)$.

Remark. A line integral of a scalar field over a path and its reparameterization is equivalent, even if the reparameterization is non-orientation-preserving. ds is always positive, and $f(x, y)$ is, too.

Theorem 4 (vector field line integral of reparameterization)**.** If a reparameterization, p, is orientation-preserving,

$$\int_p \vec{F} \cdot ds = \int_c \vec{F} \cdot ds$$

Otherwise,

$$\int_p \vec{F} \cdot ds = -\int_c \vec{F} \cdot ds$$

0.2 Fundamental theorem of line integrals

Remark. Recall that $\int_a^b f'(x)dx = f(b) - f(a)$

Theorem 5 (Fundamental theorem of line integrals)**.** Suppose $f : \mathbb{R}^3 \mapsto R$ is of class C^1 and that $\vec{c} : [a, b,] \mapsto \mathbb{R}^3$ is a piecewise C^1 path. Then for $\vec{F} = \nabla f$

$$\int_c F \cdot d\vec{s} = f(\vec{c}(b)) - f(\vec{c}(a))$$

Remark. The work done by a gradient vector field moving a particle along the path only depends on the endpoints of the path, not the path itself!

Remark. If the path integral of a vector field only depends on the endpoints of the path, we call the vector field conservative. Every conservative vector field is the gradient of some other vector field.