

Math 222: (Green's Theorem)

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Remark. Green's theorem lets us calculate the line integral of a vector field without needing to parameterize the path.

0.1 Using Green's Theorem to calculate circulation

Proposition 1. The circulation around a tiny recangle with vertices $(x, y), (x + \Delta x, y), (x, y + \Delta y), (x + \Delta x, y + \Delta y)$ is $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$

Proof. <https://www.youtube.com/watch?v=JB99RbQAiI&list=PLHXZ9OQGmQxfW0GMqeUE1bLKaYor6kbHa&index=18> □

Remark. Can conceptualize Green's Thm as allowing us to calculate the circulation around a path by summing up the circulation around all infinitesimal internal areas within the region bounded by the path. $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is the circulation density (flux) at a tiny rectangle with length/width $dx dy$. So...

Theorem 2 (Green's Theorem). Let D be a simple region and C be its boundary. Let $\vec{F} = P(x, y)\hat{i} + Q(x, y)\hat{j}$

$$\int_C \vec{F} \cdot ds = \int \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

0.2 Using Green's Theorem to calculate flux

Remark. We can similarly calculate the flux of \vec{F} across some path c by summing up the flux across all the infinitesimal areas within the region bounded by c .

Proposition 3. The flux around a tiny recangle with vertices $(x, y), (x + \Delta x, y), (x, y + \Delta y), (x + \Delta x, y + \Delta y)$ is $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$

Proof. trefor 19 □

Theorem 4 (Green's theorem (divergence form)). Let D be a simple region and C be its boundary. Let $\vec{F} = P(x, y)\hat{i} + Q(x, y)\hat{j}$

$$\begin{aligned} \int_C \vec{F} \cdot \vec{n} ds &= \int \int_D \operatorname{div} \vec{F} dA \\ &= \int \int_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dx dy \\ &= \int \int_D \vec{\nabla} \cdot \vec{F} dx dy \end{aligned}$$