Math 221 HW1

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1.1

6. Find a parametric equation of each of the following lines

   (a) $3x_1 + 4x_2 = 6$

   First, we rearrange the Cartesian equation to express x_1 in terms of x_2.

$$\begin{aligned} 3x_1 + 4x_2 &= 6 \\ x_1 &= -\frac{4}{3}x_2 + 2 \end{aligned}$$

   In parametric form,

$$\begin{aligned} \mathbf{x} &= (-\frac{4}{3}x_2 + 2, x_2) \\ &= (2,0) + x_2(-\frac{4}{3}, 1) \\ &= (2,0) + t(-\frac{4}{3}, 1) \end{aligned}$$

   (c) The line with slope 2/5 that passes through $A = (3, 1)$

   To parameterize a line, we find the span of the vector with the same direction running through the origin, then then translate it as appropriate.

   The span of that vector is $x_1(5, 2)$. We translate it by (3,1), so

$$\mathbf{x} = (3,1) + t(5,2)$$

   (g) The line through $A = (1, -2, 1)$ and $B = (2, 1, -1)$

   To parameterize a line, we find the span of the vector with the same direction running through the origin, then then translate it as appropriate.

The slope of the line we want to parameterize is $B - A = (1, 3, -2)$, which means its span is $t(1, 3, -2)$. We translate that by (1,-2,1), so

$$\mathbf{x} = (1, -2, 1) + t(1, 3, -2)$$

7. Suppose $\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$ and $\mathbf{y} = \mathbf{y}_0 + s\mathbf{w}$ are two parametric representations of the same line in $\mathbb{R}^n$.

   (a) Show that there is a scalar t_0 so that $\mathbf{y}_0 = \mathbf{x}_0 + t_0\mathbf{v}$.

   Proof. Since $\mathbf{x}$ and $\mathbf{y}$ represent the same line ℓ, they have the same span and contain the same vectors. Accordingly, since $\mathbf{y}$ contains the vector $\mathbf{y}_0$ (when $s = 0$), $\mathbf{x}$, which represents the same line, must also contain $\mathbf{y}_0$.

   That is, for some t_0, $\mathbf{y}_0 = \mathbf{x}_0 + t_0\mathbf{v}$ (y_0 can be expressed in terms of $\mathbf{x}$). □

   (b) Show that $\mathbf{v}$ and $\mathbf{w}$ are parallel

   Proof. To show that $\mathbf{v}$ and $\mathbf{w}$ are parallel, we must show that $\mathbf{v} = c\mathbf{w}$ (or equivalently $\mathbf{w} = c\mathbf{v}$) for some $c \in \mathbb{R}$. Since $\mathbf{x}$ and $\mathbf{y}$ represent the same line, there are some scalars $s, t \in \mathbb{R}$ and constant vectors $\mathbf{x}_0, \mathbf{y}_0$ such that $\mathbf{x} = \mathbf{y}$. We can express this relationship by setting their parametric expressions equal to each other.

$$x_0 + t\mathbf{v} = y_0 + s\mathbf{w}$$

   We know from 7a that there exists a scalar t_0 that lets us express $y_0 = x_0 + t_0\mathbf{v}$. We can plug the RHS of that equation into the equation above.

$$\begin{aligned} x_0 + t\mathbf{v} &= x_0 + t_0\mathbf{v} + s\mathbf{w} \\ t\mathbf{v} - t_0\mathbf{v} &= s\mathbf{w} \\ \frac{t - t_0}{s}\mathbf{v} &= \mathbf{w} \\ \mathbf{w} &= c\mathbf{v} \end{aligned}$$

   We know that $c \in \mathbb{R}$ since all of the variables in the expression for c are scalars. Thus, since $\mathbf{w} = c\mathbf{v}$, $\mathbf{v}$ and $\mathbf{w}$ are parallel. □

8. Decide whether each of the following vectors is a linear combination of $\mathbf{u} = (1, 0, 1)$ and $\mathbf{v} = (-2, 1, 0)$.

   (a) $\mathbf{x} = (1, 0, 0)$

   We want to see if $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$. In the context of this problem, we verify that

$$\begin{aligned} (1, 0, 0) &= s(1, 0, 1) + t(-2, 1, 0) \\ &= (s, 0, s) + (-2t, t, 0) \end{aligned}$$

   by ensuring the corresponding system of equations is consistent:

$$s - 2t = 1 \quad (1)$$
$$0 + t = 0 \quad (2)$$
$$s + 0 = 0 \quad (3)$$

From (2) and (3) we see that $s = t = 0$, but that is inconsistent with (1).

Therefore $\mathbf{x}$ is not a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

(b) $\mathbf{x} = (3, -1, -1)$

If $\mathbf{x}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$, $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$, which means the following set of linear equations must be consistent:

$$s - 2t = 3 \quad (1)$$
$$0 + t = -1 \quad (2)$$
$$s + 0 = 1 \quad (3)$$

From (2) and (3), we know $s = 1, t = -1$. This is consistent with (1).

Therefore $\mathbf{x}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

(c) $\mathbf{x} = (0, 1, 2)$

If $\mathbf{x}$ is a linear combination of $\mathbf{u}$gand $\mathbf{v}$, $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$, which means the following set of linear equations must be consistent:

$$s - 2t = 0 \quad (1)$$
$$0 + t = 1 \quad (2)$$
$$s + 0 = 2 \quad (3)$$

From (2) and (3), we know that $s = 2, t = 1$. This is consistent with (1).

Therefore $\mathbf{x}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

10. Find the parametric equation of the following planes:

(a) The plane containing the point $(-1, 0, 1)$ and the line $x = (1, 1, 1) + t(1, 7, -1)$

A plane is defined as the span of two non- scalar multiple vectors. In this case, one of those vectors is (1,7,-1). Another can be constructed from the line segment between two points on the plane: $(1, 1, 1) - (-1, 0, 1) = (2, 1, 0)$. A parametric equation including both is:

$$(-1, 0, 1) + s(2, 1, 0) + t(1, 7, -1)$$

(b) The plane parallel to the vector $(1, 3, 1)$ and containing the points $(1, 1, 1)$ and $(-2, 1, 2)$.

One vector in the plane is $(1, 3, 1)$. Another is $(-2, 1, 2) - (1, 1, 1) = (-3, 0, 1)$. Thus a parametric equation for the plane is

$$(1, 1, 1) + s(1, 3, 1) + t(-3, 0, 1)$$

(c) The plane containing the points $(1,1,2),(2,3,4)$, and $(0,-1,2)$.

One vector in the plane is $(2,3,4)-(1,1,2)=(1,2,2)$. Another vector in the plane is $(0,-1,2)-(1,1,2)=(-1,-2,0)$. Thus a parametric equation for the plane is

$$(1,1,2)+s(1,2,2)+t(-1,-2,0)$$

(d) The plane in $\mathbb{R}^4$ containing the points $(1,1,-1,2),(2,3,0,1)$, and $(1,2,2,3)$.

One vector in the plane is $(2,3,0,1)-(1,1,-1,2)=(1,2,1,-1)$. Another vector in the plane is $(1,2,2,3)-(1,1,-1,2)=(0,1,3,1)$. A parametric equation for the plane is

$$(1,1,-1,2)+t(1,2,1,-1)+u(0,1,3,1)$$

21. Suppose $\mathbf{v},\ \mathbf{w}\in\mathbb{R}^n$ and c is a scalar. Prove that $\text{span}(\mathbf{v}+c\mathbf{w},\mathbf{w})=\text{span}(\mathbf{v},\ \mathbf{w})$

Proof.

$$\begin{aligned}
\text{span}(\mathbf{v}+c\mathbf{w},\mathbf{w}) &= d_1(\mathbf{v}+c\mathbf{w})+d_2\mathbf{w} && \text{for } \forall d_1,d_2\in\mathbb{R} \text{ (by def of span)}\\
&= d_1\mathbf{v}+d_1c\mathbf{w}+d_2\mathbf{w} &&\\
&= d_1\mathbf{v}+d_3\mathbf{w}+d_2\mathbf{w} && (d_3=d_1c)\in\mathbb{R}\\
&= d_1\mathbf{v}+(d_3+d_2)\mathbf{w} &&\\
&= d_1\mathbf{v}+d_4\mathbf{w} && (d_4=d_3+d_2)\in\mathbb{R}\\
&= \text{span}(\mathbf{v},\ \mathbf{w}) && \text{by def. of span}
\end{aligned}$$

□

22. Suppose vectors $\mathbf{v}$ and $\mathbf{w}$ are both linear combinations of $\mathbf{v}_1,\dots,\mathbf{v}_k$.

(a) Prove that for any scalar c, $c\mathbf{v}$ is a linear combination of $\mathbf{v}_1,\dots,\mathbf{v}_k$.

Proof.

$$\begin{aligned}
c\mathbf{v} &= c(d_1\mathbf{v}_1+d_2\mathbf{v}_2+\ldots+d_k\mathbf{v}_k) && \text{for } d_1,\dots,d_k\in\mathbb{R}\\
&= (cd_1)\mathbf{v}_1+(cd_2)\mathbf{v}_2+\ldots+(cd_k)\mathbf{v}_k &&\\
&= e_1\mathbf{v}_1+e_2\mathbf{v}_2+\ldots+e_k\mathbf{v}_k && e\in\mathbb{R}
\end{aligned}$$

This is, by definition, a linear combination of $\mathbf{v}_1,\dots,\mathbf{v}_k$. □

(b) Prove that $\mathbf{v}+\mathbf{w}$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$.

Proof.

$$\begin{aligned}
\mathbf{v}+\mathbf{w} &= (c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots c_k\mathbf{v}_k) + (d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \ldots d_k) && \text{(by def. of linear combo)} \\
&= (c_1+d_1)\mathbf{v}_1 + (c_2+d_2)\mathbf{v}_2 + \ldots (c_k+d_k)\mathbf{v}_k \\
&= e_1\mathbf{v}_1 + e_2\mathbf{v}_2 + \ldots + e_k\mathbf{v}_k && e \in \mathbb{R}
\end{aligned}$$

This is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$. □

23. Consider the line $\ell : \mathbf{x} = x_0 + r\mathbf{v}$ $(r \in \mathbb{R})$ and the plane $P : x = s\mathbf{u} + t\mathbf{v}$ $(s, t \in \mathbb{R})$. Show that if ℓ and P intersect, $x_0 \in P$.

Proof. If ℓ and P intersect, $\ell = P$ for some $r, s, t \in \mathbb{R}$. That is, at some point,

$$x_0 + r\mathbf{v} = s\mathbf{u} + t\mathbf{v}$$

We can solve for x_0 to show that it will lie within P:

$$\begin{aligned}
x_0 &= s\mathbf{u} + t\mathbf{v} = r\mathbf{v} \\
&= s\mathbf{u} + (t-r)\mathbf{v} \\
&= s\mathbf{u} + t_1\mathbf{v}
\end{aligned}$$

This matches the equation of the plane P. We can thus say that if ℓ and P intersect, $x_0 \in P$.

□

24. (a) Using only the properties listed in Excercise 28, prove that for any $\mathbf{x} \in \mathbb{R}^n$, we have $0\mathbf{x} = \mathbf{0}$.

Proof.

$$\begin{aligned}
1\mathbf{x} &= \mathbf{x} && (h) \\
(0+1)\mathbf{x} &= \mathbf{x} && \text{by arithmetic} \\
0\mathbf{x} + 1\mathbf{x} &= \mathbf{x} && (g) \\
0\mathbf{x} + \mathbf{x} &= \mathbf{x} && (h) \\
0\mathbf{x} + \mathbf{x} + (-\mathbf{x}) &= \mathbf{x} + (-\mathbf{x}) && \text{adding } -\mathbf{x} \text{ to both sides} \\
0\mathbf{x} &= \mathbf{x} + (-\mathbf{x}) && (d) \\
0\mathbf{x} &= 0 && (d)
\end{aligned}$$

□

(b) Using the result of part a, prove that $(-1)\mathbf{x} = -\mathbf{x}$.

Proof.

$$
\begin{aligned}
(-1)\mathbf{x} &= (-1+0)\mathbf{x} && \text{additive identity}\\
&= (-1)\mathbf{x} + 0\mathbf{x} && (g)\\
&= (-1)\mathbf{x} + 0 && \text{From 29a above}\\
&= (-1)\mathbf{x} + \mathbf{x} + (-\mathbf{x}) && (d)\\
&= (-1)\mathbf{x} + 1\mathbf{x} + (-\mathbf{x}) && (h)\\
&= (-1)\mathbf{x} + (1)\mathbf{x} + (-\mathbf{x}) && \text{adding parens for clarity}\\
&= (-1+1)\mathbf{x} + (-\mathbf{x}) && (g)\\
&= 0x + (-\mathbf{x}) && \text{arithmetic}\\
&= 0 + (-\mathbf{x}) && \text{From 29a above}\\
&= (-\mathbf{x}) && \text{additive identity}\\
&= -\mathbf{x} && \text{remove parens for clarity}
\end{aligned}
$$

□

1.2

1. For each of the following pairs of vectors $\mathbf{x}$ and $\mathbf{y}$, calculate $\mathbf{x}\cdot\mathbf{y}$ and the angle θ between the vectors.

(b) $\mathbf{x} = (2,1), \mathbf{y} = (-1,1)$

$$
\begin{aligned}
\mathbf{x}\cdot\mathbf{y} &= -2+1 = -1\\
\cos(\theta) &= \frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|} = \frac{-1}{\sqrt{4+1}\sqrt{1+1}} = -\frac{1}{\sqrt{10}}\\
\theta &= \cos^{-1}\left(-\frac{1}{\sqrt{10}}\right) =\approx 108.4\,\mathrm{deg}
\end{aligned}
$$

(d) $\mathbf{x} = (1,4,-3), \mathbf{y} = (5,1,3)$

$$
\begin{aligned}
\mathbf{x}\cdot\mathbf{y} &= 5+4+(-9) = 0\\
\cos\theta &= \frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|} = \frac{0}{\|\mathbf{x}\|\|\mathbf{y}\|} = 0\\
\theta &= \cos^{-1}(0) = \frac{\pi}{2}
\end{aligned}
$$

(g) $\mathbf{x} = (1,1,1,1), \mathbf{y} = (1,-3,-1,5)$

$$
\begin{aligned}
\mathbf{x}\cdot\mathbf{y} &= 1+(-3)+(-1)+5 = 2\\
\cos\theta &= \frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|} = \frac{2}{\sqrt{1+1+1+1}\sqrt{1+9+1+25}} = \frac{2}{2*6} = \frac{1}{6}\\
\theta &= \cos^{-1}\left(\frac{1}{6}\right) = 80.4\,\mathrm{deg}
\end{aligned}
$$

2. For each pair of vectors in exercise 1, calculate $\text{proj}_{\mathbf{y}}\mathbf{x}$ and $\text{proj}_{\mathbf{x}}\mathbf{y}$

(b)

$$\begin{aligned}\text{proj}_{\mathbf{y}}\mathbf{x} &= \frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{y}\|}\frac{\mathbf{y}}{\|\mathbf{y}\|}\\ &= -\frac{1}{\sqrt{2}}\frac{(-1,1)}{\sqrt{2}} = -\frac{1}{2}(-1,1)\end{aligned}$$

$$\begin{aligned}\text{proj}_{\mathbf{x}}\mathbf{y} &= \frac{\mathbf{y}\cdot\mathbf{x}}{\|\mathbf{x}\|}\frac{\mathbf{x}}{\|\mathbf{x}\|}\\ &= -\frac{1}{\sqrt{5}^2}\mathbf{x} = -\frac{1}{5}(2,1)\end{aligned}$$

(d)

$$\begin{aligned}\text{proj}_{\mathbf{y}}\mathbf{x} &= \frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{y}\|}\frac{\mathbf{y}}{\|\mathbf{y}\|}\\ &= \frac{0}{\|\mathbf{y}\|^2}\mathbf{y} = \mathbf{0}\end{aligned}$$

$$\begin{aligned}\text{proj}_{\mathbf{x}}\mathbf{y} &= \frac{\mathbf{y}\cdot\mathbf{x}}{\|\mathbf{x}\|}\frac{\mathbf{x}}{\|\mathbf{x}\|}\\ &= \frac{0}{\|\mathbf{x}\|^2}\mathbf{x} = \mathbf{0}\end{aligned}$$

(g)

$$\begin{aligned}\text{proj}_{\mathbf{y}}\mathbf{x} &= \frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{y}\|^2}\mathbf{y}\\ &= \frac{2}{36}(1,-3,-1,5) = \frac{1}{18}(1,-3,-1,5)\end{aligned}$$

$$\begin{aligned}\text{proj}_{\mathbf{x}}\mathbf{y} &= \frac{\mathbf{y}\cdot\mathbf{x}}{\|\mathbf{x}\|^2}\mathbf{x}\\ &= \frac{2}{4}(1,1,1,1) = \frac{1}{2}(1,1,1,1)\end{aligned}$$

13. Prove $\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}\text{-}\mathbf{y}\|^2 = 2(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$.

Proof.

$$\begin{aligned}\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}\text{-}\mathbf{y}\|^2 &= (\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}+\mathbf{y}) + (\mathbf{x}-\mathbf{y})\cdot(\mathbf{x}-\mathbf{y}) && \text{(by } \mathbf{v}\cdot\mathbf{v} = \|\mathbf{v}\|^2\text{)}\\ &= \mathbf{x}\cdot(\mathbf{x}+\mathbf{y}) + \mathbf{y}\cdot(\mathbf{x}+\mathbf{y}) + \mathbf{x}\cdot(\mathbf{x}\text{-}\mathbf{y}) - \mathbf{y}\cdot(\mathbf{x}\text{-}\mathbf{y}) && \text{(by distrib. prop. of dot product)}\\ &= (\mathbf{x}\cdot\mathbf{x}) + (\mathbf{x}\cdot\mathbf{y}) + (\mathbf{y}\cdot\mathbf{x}) + (\mathbf{y}\cdot\mathbf{y}) && \text{(same)}\\ &+ (\mathbf{x}\cdot\mathbf{x}) - (\mathbf{x}\cdot\mathbf{y}) - (\mathbf{y}\cdot\mathbf{x}) + (\mathbf{y}\cdot\mathbf{y}) &&\\ &= 2(\mathbf{x}\cdot\mathbf{x}) + 2(\mathbf{y}\cdot\mathbf{y}) && \text{(cancellation)}\\ &= 2(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2) && \text{(by } \mathbf{v}\cdot\mathbf{v} = \|\mathbf{v}\|^2\text{)}\end{aligned}$$

$\square$

16. (a) If $\mathbf{x} \cdot \mathbf{y} = 0$ for all $\mathbf{x} \in \mathbb{R}^n$, prove $\mathbf{y} = 0$.

Proof. Assume $\mathbf{x} \cdot \mathbf{y} = 0$ for all $\mathbf{x} \in \mathbb{R}^n$. Let $\mathbf{x} = \mathbf{y}$. Then $\mathbf{y} \cdot \mathbf{y} = 0$. Since $\mathbf{y} \cdot \mathbf{y} = \|\mathbf{y}\|^2$, this means $\|\mathbf{y}\|^2 = 0$ and $\|\mathbf{y}\| = 0$, which is only true if $\mathbf{y} = \mathbf{0}$. $\square$

(b) Suppose $\mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ and $\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot \mathbf{z}$ for all $\mathbf{x} \in \mathbb{R}^n$. What can we conclude?
If $\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot \mathbf{z}$,

$$(\mathbf{x} \cdot \mathbf{y}) - (\mathbf{x} \cdot \mathbf{z}) = 0 \tag{1}$$
$$\mathbf{x} \cdot (\mathbf{y} - \mathbf{z}) = 0 \tag{2}$$

Since $\mathbf{x}$ can take on any value, $\mathbf{y} - \mathbf{z}$ must equal $\mathbf{0}$ to satisfy (2). Therefore, we know $\mathbf{y} = \mathbf{z}$.

18. Prove the triangle inequality: $\|\mathbf{x} + \mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|$.

To begin, we can square both sides of the equation, which will allow us to express the LHS as a dot product.

$$\begin{aligned}
\|\mathbf{x}+\mathbf{y}\|^2 &\le \|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\| + \|\mathbf{y}\|^2 && \\
(\mathbf{x}+\mathbf{y}) \cdot (\mathbf{x}+\mathbf{y}) &\le \text{" "} && (\|\mathbf{x}\|^2 = \mathbf{x}\cdot\mathbf{x}) \\
\mathbf{x} \cdot (\mathbf{x}+\mathbf{y}) + \mathbf{y} \cdot (\mathbf{x}+\mathbf{y}) &\le \text{" "} && \text{(distrib. prop. of dot product)} \\
\mathbf{x}\cdot\mathbf{x} + \mathbf{x}\cdot\mathbf{y} + \mathbf{y}\cdot\mathbf{x} + \mathbf{y}\cdot\mathbf{y} &\le \text{" "} && \text{(distrib. prop. of dot product)} \\
\|\mathbf{x}\|^2 + 2(\mathbf{x}\cdot\mathbf{y}) + \|\mathbf{y}\|^2 &\le \|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\| + \|\mathbf{y}\|^2 && (\|\mathbf{x}\|^2 = (\mathbf{x}\cdot\mathbf{x}) \\
2(\mathbf{x}\cdot\mathbf{y}) &\le 2\|\mathbf{x}\|\|\mathbf{y}\| && \text{(subtract factors from both sides)} \\
\mathbf{x}\cdot\mathbf{y} &\le \|\mathbf{x}\|\|\mathbf{y}\| && \text{(divide both sides by 2)} \\
\|\mathbf{x}\|\|\mathbf{y}\|\cos\theta &\le \|\mathbf{x}\|\|\mathbf{y}\| && (\textbf{def. } \text{angle between vectors}) \\
\cos\theta &\le 1 && (\text{divide both sides by } \|\mathbf{x}\|\|\mathbf{y}\|)
\end{aligned}$$