2.6: Gradients and directional derivatives (Lec 7)

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Definition 1 (gradient). If $f : U \subset \mathbb{R}^3 \mapsto \mathbb{R}$ is differentiable, the **gradient** of f at (x, y, z) is the vector in space given by

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

The gradient is denoted $\nabla f(x, y, z)$. It is really just the matrix of the derivative $\mathbf{D}f$ written as a vector (f is a real-valued function).

Definition 2 (directional derivative). $D_{\vec{v}}$ is the directional derivative in a direction $\vec{v}$. It quantifies how the output of f changes with a nudge in the direction of $\vec{v} = (v_1, v_2)$.

We can express the same idea using limits. Let $\vec{x_0} = (x_0, y_0)$. Additionially, assume we are operating on a 3d graph, so $\vec{v} = (v_1, v_2)$, where v_1 is the x-component of the vector, v_2 is the y-component. Then

$$D_{\vec{v}} f(\vec{x}) = \lim_{h \to 0} \frac{f(x_0 + hv_1, y_0 + hv_2) - f(x_0, y_0)}{h} = \lim_{h \to 0} \frac{f(\vec{x} + h\vec{v}) - f(\vec{a})}{h}$$

Note that the directional derivative is the slope of the secant line between (x_0, y_0, z_0) and $(x_0 + hv_1, y_0 + hv_2, f(x_0 + hv_1, y_0 + hv_2))$ We can easily calculate the directional derivative by noting that

$$D_{\vec{v}}(f) = \left(\frac{\partial f}{\partial x} \right) v_1 + \left(\frac{\partial f}{\partial y} \right) v_2 = \nabla f \cdot \vec{v}$$

The directional derivative is sometimes written as $\nabla_{\vec{v}} f$ or $\frac{\partial f}{\partial \vec{v}}$

Remark. Note that because scaling $\vec{v}$ by c changes the output of our formula for $D_{\vec{v}} f$ by c, to calculate the precise amount f changes with a nudge in $\vec{v}$, we must ensure $\vec{v}$ is a unit vector.

Remark. For some real-valued function $f : \mathbb{R}^2 \mapsto \mathbb{R}$, we can visualize the directional derivative by slicing the surface of the function along the plane containing v, then looking at the slope of the resulting slice function.

Remark. For some $f(x_1, x_2, \ldots, x_n)$, $\frac{\partial f}{\partial x_1} = D_{\vec{e_1}}$. This tells us how much a nudge in the x_1 direction moves f.

Theorem 3 (Direction of gradient). The gradient points in the direction of steepest ascent.

Proof. Consider some unit vector $\vec{v}$ and real-valued function f.

$$\begin{aligned} D_{\vec{v}} f &= \nabla f \cdot \vec{v} \\ &= \|\nabla f\| \|\vec{v}\| \cos\theta \\ &= \|\nabla f\| \cos\theta \end{aligned}$$

$D_{\vec{v}}$ is maximized when $\cos\theta = 1$, which occurs when $\vec{v}$ points in the same direction as ∇f. $D_{\vec{v}}$ is minimized when $\vec{v}$ points in the negative gradient direction. □

Proposition 4. The magnitude of the gradient tells us how quickly the function increases in the direction of steepest ascent.

Proof. Let f be a real valued function, and let $\vec{w} = \nabla f$.

$$\begin{aligned} D_{\vec{w}} f &= \nabla f \cdot \vec{w} \\ &= \frac{\nabla f \cdot \nabla f}{\|\nabla f\|} \\ &= \frac{\|\nabla f\|^2}{\|\nabla f\|} \\ &= \|\nabla f\| \end{aligned}$$

$\square$

Theorem 5 (Gradient is normal to level curve/surface)**.** Let $f : \mathbb{R}^3 \mapsto \mathbb{R}$ be a map. Consider some x_0, y_0, z_0 on a level surface of f. The tangent vector to the path of the level surface, $\vec{v}$ tells us which direction to go in to follow the level surface. But all points on the level surface have the same value of $f(x, y, z) = c$, which means $D_{\vec{v}} = 0 = \nabla f \cdot v$, so $\vec{v}$ is orthogonal to ∇f, and thus ∇f is normal to the level surface.

Remark. Think of the graph of $f(x, y) = x^2 + y^2$. $\nabla f = \begin{bmatrix} 2x & 2y \end{bmatrix}$. Vectors in the direction of the gradient will always be orthogonal to the tangent vectors of level curves $f(x, y) = c$, circles centered at the origin.

Remark. Or think of hiking with a contour map. If you want to reach the highest elevation as quickly as possible, you walk in the direction perpendicular to the contours.

Definition 6 (tangent planes to level surfaces)**.** Let S be the surfacer consisting of (x, y, z) s.t. $f(x, y, z) = k$ for some constant k. The **tangent plane** of S at a point $(x_0, y_0, z_0$ of S is defined by the equation

$$\nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

Remark. Resources:

1. Khan gradient/directional derivative videos: explanation of limit definition, illustration of "steepest" ascent.