Math 221 Lec 13

3.2 The 4 fundamental subspaces

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October 3, 2023

1 Fundamental subspaces of linear transforms

Remark. There are really two fundamental subspaces of linear transformations

Definition 1 (Kernel and image)**.** Let $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation.
The **kernel** of f is $\ker(f) = \{\mathbf{v} \in \mathbb{R}^n : f(\mathbf{v}) = \mathbf{0}\}$. $\ker(f) \subset \mathbb{R}R^n$.
The **image** of f is $\text{Im}(f) = \{f(\mathbf{v}) : \mathbf{v} \in \mathbb{R}^n\}$. $\text{Im}(f) \subset \mathbb{R}^m$.

Theorem 2 (Kernel and image subspaces)**.** Let $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation.

(a) $\ker(f)$ is a subspace of $\mathbb{R}^n$.

(b) $\text{Im}(f)$ is a subspace of $\mathbb{R}^m$.

Proof. Normal subspace proof □

2 Fundamental subspaces of matrices

Definition 3 (column space)**.** The **column space** of a matrix A, $C(A)$, is the span of its columns.

Remark. The column space of A is the image of the linear transformation represented by A, μ_A.

Definition 4 (nullspace)**.** The **nullspace** of A is the set of vectors $\mathbf{x}$ s.t. $A\mathbf{x} = \mathbf{0}$, aka the homogoenous solution of A.

Remark. The null space of A is the kernel of the linear transformation represented by A.

Definition 5 (row space)**.** The row space of A, $R(A)$ is the span of its rows. Note that the book calls this $R(A^\intercal)$, because it treats every vector as a column.

Definition 6 (left nullspace)**.** The **left nullspace** of A, $N(A^\intercal)$consists of row vectors $\{\mathbf{x} \in \mathbb{R}^m_{row} | \mathbf{x}A = 0\}$. A linear combination of the rows of A using components of $\mathbf{x} \in N(A^\intercal)$ wil produce $\mathbf{0}$.

Theorem 7 (relationship between nullspace and row space)**.** $N(A) = R(A)^\perp$

Proof. Let $x \in N(A)$. Let $y_i \in R(A)$. We know that $A\mathbf{x} = \mathbf{0}$ since $x \in N(A)$.

$$A\mathbf{x} = \begin{bmatrix} — \mathbf{y}_1 — \\ — \mathbf{y}_2 — \\ \vdots \\ — \mathbf{y}_m — \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{y}_1 \cdot x \\ \mathbf{y}_2 \cdot x \\ \vdots \\ \mathbf{y}_m \cdot x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\mathbf{x} \cdot \mathbf{y}_i$ is always 0, so they are orthogonal. □