Math 340: Lec 15 (Convolutions, random walks)

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0.1 Convolutions

Theorem 1. If we know the marginal distributions of two discrete random variables, X and Y, we can find the marginal distribution of their sum:

$$\mathbb{P}(X+Y=k) = \sum_{y \in R(Y)} \mathbb{P}(X=k-y)\mathbb{P}(Y=y)$$

We refer to that quantity as

$$\rho_X * \rho_Y(k)$$

0.2 Random walks

Definition 2 (Random walk)**.** Intuition: Consider a path on Z with $S_n =$ position at time n and $a \in \mathbb{N}$ as the path's starting point. At each point, fli pa coin. If heads, move in the positive direction. If tails, move in the negative. Mathematically, fix n. $\Omega = \{\omega - (\omega_1, \dots, \omega_n) | \omega_k \in \{-1, 1\}\}$. ω_k is the step taken at time k. $S_k(\omega)$ is the position after the kth step.

$$S_k(\omega) = \alpha + \sum_{j=1}^{k} \omega_j$$

Remark. If coin tosses are fair, $\mathbb{P}(\{\omega\} = 2^{-n}$. All paths are equally likely.

Example (Probability of ending at b if starting at a)**.**

$$\mathbb{P}^a(S_n = b) = \frac{\#\text{ paths from a to b}}{\text{total }\#\text{ paths}} = \frac{\binom{n}{\frac{n+b-a}{2}}}{2^n}$$

Theorem 3 (Reflection principle)**.** Let ${N_n}^0(a,b)$ be the number of paths $a \mapsto b$ of length n that touch or cross 0.

$$\begin{aligned} {N_n}^0(a,b) &= N_n(-a,b) \\ &= \binom{n}{\frac{n+b+a}{2}} \end{aligned}$$

Remark. Intuition for reflection principle: Imagine the set of paths $X : {S_n}^a = b$ that cross zero. Assume that the paths in X have $S_k = 0$. Now reflect each path in X over the x axis before step k. Keep the paths the same after step k. This partially-reflected set represents every path $Y : {S_n}^{-a} = b$; each path in Y is guaranteed to cross zero. Because there is a 1:1 correspondence between paths in X and Y, we can say ${N_n}^0(a,b) = N_n(-a,b)$

See the picture below for an example of the reflection principle. Note the 1:1 correspondence betwen the solid paths and the dashed paths before the x-axis crossing

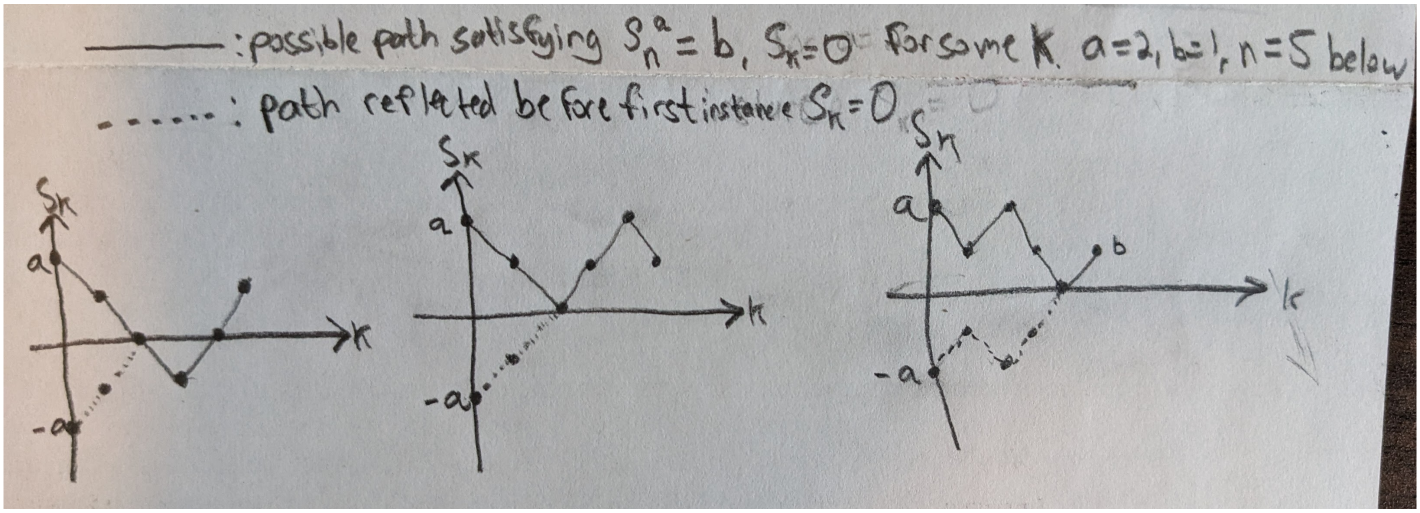

Figure 1: Demonstration of reflection principle. For each solid path that crosses the x-axis, there is a corresponding path reflected before crossing

Example (Applying reflection principle)**.** Compute the probability of going from a to b without hitting 0.

$$\mathbb{P}^a(S_n = b, S_k \neq 0 \text{ for all } k) = 1 - \frac{N_n{}^0(a,b)}{2^n} = 1 - \frac{\binom{n}{\frac{n+b+a}{2}}}{2^n}$$