1.i. The joint density of $f(x,y)$ must equal one.

i $\int_0^1 \int_0^1 c(x^2+3y)\,dx\,dy = 1$

$\int\int_0 f(x) = c\int_0^1 \left(\left[\frac{x^3}{3}+3xy\right]_{x=0}^{x=1}\right)dy$

$= c\int_0^1 \left(\frac{1}{3}+3y\right)dy$

$= c\left[\frac{1}{3}y+\frac{3}{2}y^2\right]_{y=0}^{y=1}$

$= c\left(\frac{1}{3}+\frac{3}{2}\right)$

$= 11c/6$

$11c/6 = 1$ so $\boxed{c = 6/11}$

ii. We can find the marginal density of X by integrating the joint density function across all values $\in R(Y)$

$$\rho_X x = \int_D f(x,y)\,dy = \int_0^1 c(x^2+3y)\,dy = c\int_0^1 (x^2+3y)\,dy$$

$$= c\left[x^2y + \frac{3y^2}{2}\right]_{y=0}^{y=1} = c(x^2 + 3/2)$$

$$\boxed{\rho_X x = \begin{cases} \frac{6}{11}(x^2+3/2) & \text{if } x\in[0,1] \\ 0 & \text{otherwise} \end{cases}}$$

iii. $$f_Y(y \mid X=\tfrac{1}{2}) = \frac{f(\frac{1}{2},y)}{\rho_X \frac{1}{2}} = \frac{\frac{6}{11}(\frac{1}{4}+3y)}{\frac{6}{11}(\frac{1}{4}+\frac{3}{2})} = \frac{\frac{1}{4}+3y}{\frac{1}{4}+\frac{3}{2}} = \frac{4}{7}(\tfrac{1}{4}+3y) = \frac{1}{7}+\frac{12}{7}y$$

$$\boxed{f_Y(y \mid X=\tfrac{1}{2}) = \begin{cases} \frac{1}{7}+\frac{12}{7}y & \text{if } y\in[0,1] \\ 0 & \text{otherwise} \end{cases}}$$

iv. X and Y are independent iff $f_Y(y \mid X=\frac{1}{2}) = f_Y(y)$

$$f_Y(y) = \int_D f(x,y)\,dx = \int_0^1 c(x^2+3y)\,dx = c\left[\frac{x^3}{3}+3xy\right]_{x=0}^{x=1}$$

$$= c(\tfrac{1}{3}+3y) = \frac{6}{11}(\tfrac{1}{3}+3y) = \frac{2}{11}+\frac{18}{11}y$$

$f_Y(y) \neq f_Y(y \mid X=x)$ so the marginal distribution of Y depends on the value of X, and thus X and Y are dependent

2 i. The marginal density $f_X(x)$ is given by integrating the joint density function $f(x,y)$ across all y in the domain.

$$f_X(x) = \int_0^x 3x \, dy \quad \text{for } x \in [0,1]$$

$$= 3xy \Big|_{y=0}^{y=x} = 3x^2$$

$$\boxed{f_X(x) = \begin{cases} 3x^2 & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}}$$

ii. The conditional density $f_Y(y|X=1/2)$ is given by

$$f_Y(y|X=\tfrac{1}{2}) = \frac{f(\frac{1}{2}, y)}{f_X(\frac{1}{2})} = \frac{3/2}{3(\frac{1}{2})^2} = \begin{cases} 2 & \text{for } y \in [0,\frac{1}{2}] \\ 0 & \text{otherwise} \end{cases}$$

Compute $\mathbb{E}[Y|X=x]$ for practice

iii. ① $\mathbb{E}[X|Y=y] = \int x f_X(X|Y=y) \, dx$

We begin by finding $f_X(x|Y=y)$

$$f_X(x|Y=y) = \frac{f(x,y)}{\int_{\mathbb{R}} f(t,y) dt} \rightarrow \text{marginal of } Y=y$$

$$= \frac{3x}{\int_y^1 3x \, dx} = \frac{3x}{\frac{3}{2}x^2 \Big|_y^1} = \frac{3x}{\frac{3}{2} - \frac{3}{2}y^2}$$

We then plug $f_X(x|Y=y)$ into ①

$$\mathbb{E}[X|Y=y] = \int_y^1 x\left(\frac{3x}{\frac{3}{2}-\frac{3}{2}y^2}\right)dx = \frac{1}{\frac{3}{2}-\frac{3}{2}y^2}\int_y^1 3x^2 dx$$

$$= \frac{1}{\frac{3}{2}-\frac{3}{2}y^2} x^3 \Big|_{x=y}^{x=1} = \frac{1-y^3}{\frac{3}{2}-\frac{3}{2}y^2}$$

3.

Named distribution

$Y \sim N(\mu, \sigma^2)$ (normal w/ mean μ, variance σ^2

CDF

$\mathbb{P}(Y \leq y) = \mathbb{P}(\mu + \sigma X \leq y)$

$= \mathbb{P}\left(X \leq \frac{y-\mu}{\sigma}\right)$

$= \Phi\left(\frac{y-\mu}{\sigma}\right)$

PDF: $f_Y(y) = \frac{d}{dy} \Phi\left(\frac{y-\mu}{\sigma}\right)$ derivative of CDF

$= \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{y-\mu}{\sigma}\right)^2/2}$

4i. If $X \sim \text{Unif}(0,1)$,

$\mathbb{E}[e^{\lambda X}] = \int_0^1 f_X(x) e^{\lambda x} dx$ by Law of Unconscious Statistician

$= \int_0^1 e^{\lambda x} dx$ Since $f_X(x) = 1$ for $X \sim \text{Unif}(0,1)$ on $[0,1]$

$= \frac{1}{\lambda} e^{\lambda x} \Big|_0^1$

$= \frac{1}{\lambda}(e^{\lambda} - 1)$

4ii. $\mathbb{E}[e^{\lambda x}] = \int_{-\infty}^{\infty} e^{-\left(\frac{x-\mu}{\sigma}\right)^2/2} e^{\lambda x}\, dx$

$= \int_{-\infty}^{\infty} e^{-\left[\left(\frac{x}{\sqrt{2}\sigma}\right)^2 - \frac{x\mu}{\sigma^2} + \left(\frac{\mu}{\sqrt{2}\sigma}\right)^2 - \lambda x\right]} dx \quad (1)$

We will now complete the square in the exponent

The integrand above $= -\left[\left(\frac{x}{\sqrt{2}\sigma}\right)^2 - \left(\frac{\mu}{\sigma^2}+\lambda\right)x + \left(\frac{\mu}{\sqrt{2}\sigma}\right)^2\right] \quad (2)$

To complete the square, we wish to find some term, y

s.t. $2\left(\frac{x}{\sqrt{2}\sigma}\right)(y) = \left(\frac{\mu}{\sigma^2}+\lambda\right)x$

Solving for y, $\frac{2}{\sigma\sqrt{2}} y = \left(\frac{\mu}{\sigma^2}+\lambda\right)$, so $y = \frac{1}{\sqrt{2}}\left(\frac{\mu}{\sigma}+\lambda\sigma\right) = \frac{1}{\sqrt{2}}\left(\frac{\mu+\sigma^2\lambda}{\sigma}\right)$

So $(2) = -\left[\left(\left(\frac{x}{\sqrt{2}\sigma}\right) - \left(\frac{\mu+\sigma^2\lambda}{\sqrt{2}\sigma}\right)\right)^2 - \frac{\mu+\sigma^2\lambda}{\sqrt{2}\sigma} + \left(\frac{\mu}{\sqrt{2}\sigma}\right)^2\right] \quad (3)$

Let $u = \frac{x}{\sqrt{2}\sigma} - \frac{\mu+\sigma^2\lambda}{\sqrt{2}\sigma}$. Then $du = \frac{1}{\sqrt{2}\sigma}dx$

Substituting u and (3) into (1),

$e^{\left[\left(\frac{\mu}{\sqrt{2}\sigma}\right)^2 - \frac{\mu+\sigma^2\lambda}{\sqrt{2}\sigma}\right]} \int_{-\infty}^{\infty} e^{u^2} \sigma\sqrt{2}\, du = \sigma\sqrt{2}\, e^{\left[\left(\frac{\mu}{\sqrt{2}\sigma}\right)^2 - \frac{\mu+\sigma^2\lambda}{\sqrt{2}\sigma}\right]} \int_{-\infty}^{\infty} e^{u^2} du$

I believe I failed to complete the square properly, because my solution terminates in an integral I can't solve.

5i. Compute $f_X(x)$

We begin by computing $F_X(x)$

$$F_X(x) = \mathbb{P}(X \leq x) = 1 - \mathbb{P}(X > x)$$
$$= 1 - \mathbb{P}((U_1 > x) \cap (U_2 > x))$$
$$= 1 - \mathbb{P}(U_1 > x)^2 \quad \text{since } U_1, U_2 \text{ are iid}$$
$$= 1 - (1-x)^2 \quad \text{since } U_1 \sim \text{Uniform}(0,1)$$

Now $f_X(x) = \frac{d}{dx}(1-(1-x)^2) = -2(1-x)(-1)$

$$\boxed{= \begin{cases} 2-2x & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}}$$

5ii. Compute $\mathbb{P}(X < 1/4 \mid Y \geq 1/2)$

Per the definition of conditional probability, this is $\dfrac{\mathbb{P}(X < 1/4, Y \geq 1/2)}{\mathbb{P}(Y \geq 1/2)}$ (1)

To compute $\mathbb{P}(Y \geq 1/2)$, we must find $F_Y(y)$

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(U_1 < y, U_2 < y)$$
$$= \mathbb{P}(U_1 < y)^2 \quad \text{since } U_1, U_2 \text{ are iid}$$
$$= y^2 \text{ for } y \in [0,1] \quad \text{since } U_1 \sim \text{Uniform}(0,1)$$

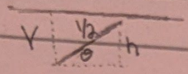
$$\mathbb{P}(Y \geq \tfrac{1}{2}) = 1 - F_Y(\tfrac{1}{2}) = 1 - (\tfrac{1}{2})^2 = 3/4$$

To compute $\mathbb{P}(X \leq \frac{1}{4}, Y \geq 1/2)$, we note that to satisfy both conditions, one of the uniform r.v.s $< \frac{1}{4}$ and the other $\geq \frac{1}{2}$. Since they are iid $\sim$ Unif$(0,1)$, the probability of this happening is $(\frac{1}{4})(\frac{1}{2})(2 \text{ vars}) = \frac{1}{4}$

Plugging everything back into (1), $\mathbb{P}(X < \frac{1}{4} \mid Y \geq 1/2) = \dfrac{1/4}{3/4} = 1/3$

6i.

diagram:

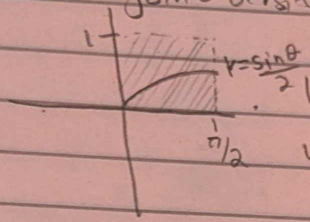


For a needle to touch the crack, we need $h \geq Y$, which means

$$\left(\tfrac{1}{2}\right)\sin\theta \geq Y$$

$$Y \leq \tfrac{1}{2}\sin\theta$$

$\theta \sim \text{Unif}(0, \pi/2)$, $Y \sim \text{Unif}(0,1)$ and they are independent, so the following graph reflects their joint density:



We wish to find $\mathbb{P}(Y \leq \tfrac{1}{2}\sin\theta)$, which is equivalent to the area under $Y = \tfrac{1}{2}\sin\theta$ from $\theta = 0$ to $\theta = \pi/2$ divided by the total area of the box

So, $\mathbb{P}(\text{Needle Intersection}) = \mathbb{P}(Y \leq \tfrac{1}{2}\sin\theta)$

$$= \frac{\int_0^{\pi/2} \tfrac{1}{2}\sin\theta \, d\theta}{(1)(\pi/2)} = \frac{1}{\pi}\int_0^{\pi/2} \sin\theta$$

$$= 1/\pi \left(-\cos\theta\right)\Big|_0^{\pi/2} = 1/\pi$$

6ii. $Var(H_n) = Var\left(\frac{1}{N}\sum_{k=1}^{N} T_k\right) = \frac{1}{N^2} Var\left(\sum_{k=1}^{N} T_k\right)$ Since Cov is inner product

$= \frac{1}{N^2}\sum_{k=1}^{N} Var(T_k)$ because the T_k are independent

$= \frac{1}{N^2}\sum_{k=1}^{N}\frac{1-p}{p^2}$ Variance of geo r.v.

$= \frac{1}{N^2}\sum_{k=1}^{N}\frac{(1-1/\pi)}{(1/\pi)^2}$

$= \frac{1}{N^2}\sum_{k=1}^{N}\left(\frac{\pi-1}{\pi}\right)(\pi)^2$

$= \frac{1}{N^2}\sum_{k=1}^{N}\pi(\pi-1)$

$= \frac{N\pi(\pi-1)}{N^2}$

$= \frac{\pi(\pi-1)}{N}$

And $\mathbb{E}[H_n] = \pi$

6iii. We can set this problem up using the CLT.

We want $\mathbb{P}(3.14158 \leq H_n \leq 3.14160) = \mathbb{P}\left('' \leq \sum_{k=1}^{N}\frac{T_k}{N} \leq ''\right)$ def of H_n

$= \mathbb{P}\left(N{*}3.14158 \leq \sum_{k=1}^{N} T_k \leq N{*}3.14160\right)$

$* = \mathbb{P}\left(\frac{3.14158N - \mu N}{\sigma\sqrt{N}} \leq \frac{\sum_{k=1}^{N} T_k - \mu_n}{\sigma\sqrt{n}} \leq \frac{3.1416N - \mu N}{\sigma\sqrt{N}}\right)$

We found in 6ii that $\sigma^2 = \frac{\pi-1}{N}$, so $\sigma = \sqrt{\frac{\pi-1}{N}} = \frac{\sqrt{\pi-1}}{\sqrt{N}}$

$\mu = \mathbb{E}[H_n] = \pi$

So $* = \mathbb{P}\left(\frac{(-10^{-5})(N)}{\frac{\sqrt{\pi(\pi-1)}}{\sqrt{N}}\sqrt{N}} \leq \frac{\sum_{k=1}^{n} T_n - \mu n}{\sigma\sqrt{n}} \leq \frac{(10^{-5})(N)}{\frac{\sqrt{\pi(\pi-1)}}{\sqrt{N}}\sqrt{N}}\right)$

$+ = \mathbb{P}\left(\frac{(-10^{-5})(N)}{\sqrt{\pi(\pi-1)}} \leq \frac{\sum_{k=1}^{N} T_n - \mu N}{\sigma\sqrt{N}} \leq \frac{(10^{-5})(N)}{\sqrt{\pi(\pi-1)}}\right)$

Let $a = \frac{(10^{-5})}{\sqrt{\pi(\pi-1)}}$

$+ = \frac{1}{\sqrt{2\pi}}\int_{-aN}^{aN} e^{-u^2/2} = \Phi(aN) - \Phi(-aN)$.

Using an online calculator, I find this expression ≈ 0.90 when $N \approx 428{,}000$

Note: This makes $aN \approx 1.65$, which is the crit value for a 2-tailed T test at 90% confidence