Math 340: Lec 18 Big Ideas Journal (Expectation of continuously-distributed random variabless)

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Theorem 1 (Expectation of continuously-distributed r.v.s). For a a continuously-distributed random variable X with density $f(z)$,

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} zf(z)dz$$

Assuming the integral converges

Properties of expectation for continuous r.v.s

Remark. Many of the properties of expectation of discrete r.v.s carry over to continuous r.v.s, including

1. Linearity: for $\alpha, \beta \in \mathbb{R}$,
$$\mathbb{E}[\alpha X + \beta Y] = \alpha\mathbb{E}[X] + \beta\mathbb{E}[Y]$$
2. Expectation of functions of r.v.s. In particular, if $g : \mathbb{R} \mapsto \mathbb{R}$,
$$\mathbb{E}[g(X)] = \int_{\mathbb{R}} g(z)f(z)dz$$
   (a) In particular, $\mathbb{E}[X] = \int_{\mathbb{R}} zf(z)dz$ and $\mathbb{E}[X^2] = \int_{\mathbb{R}} z^2 f(z)dz$, and $\mathrm{Var}(X)$ is still $\mathbb{E}[X^2] - \mathbb{E}[X]^2$.
3. Tail sum formula. If X is a positive random variable,
$$\mathbb{E}[X] = \int_0^{\infty} \mathbb{P}(X \geq t)dt = \int_0^{\infty} (1 - F(t))\, dt$$
4. Markov's inequality and Chebyshev's inequality hold for positive random variables:
$$\mathbb{P}(X \geq r) \leq \frac{1}{r}\mathbb{E}[X]$$

Remark. When trying to compute the variance of symmetric continuously-distributed random variables, we can try to recenter them around 0 so that $\mathbb{E}[X]^2 = 0$. Then we only need to calculate $\mathbb{E}[X^2] = \int_{-\infty}^{\infty} z^2 f(z)dz$