Math 221 HW 8

3.4

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7. Let $A = \mathbf{u}\mathbf{v}^\intercal$. Describe the four fundamental subspaces of A in terms of $\mathbf{u}$ and $\mathbf{v}$.
   $C(A)$: span($\mathbf{u}$)
   $N(A)$: $(\mathbf{v}^\intercal)^\perp$
   $R(A)$: span(($\mathbf{v}^\intercal$)
   $N(A^\intercal) = \mathbf{u}^\perp$

17. Let $V \subset \mathbb{R}^n$ be a subspace, and suppose you are given a linearly independent set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subset V$. Show that if $\dim V > k$, then there are vectors $\mathbf{v}_{k+1}, \ldots, \mathbf{v}_\ell \in V$ so that $\{\mathbf{v}_1, \ldots, \mathbf{v}_\ell$ forms a basis for V.

   Proof. Since $k < \dim V$, $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ cannot span V. Then $\exists \mathbf{v}_{k+1}$ s.t. $\mathbf{v}_{k+1} \in V$ but $\mathbf{v}_{k+1} \notin \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$. If we add it to the set of vectors, the set remains linearly independent since $\mathbf{v}_{k+1}$ was not in its members' span. If $\{\mathbf{v}_1, \ldots, \mathbf{v}_{k+1}\}$ spans V, we have found a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_\ell\}$ (of size one) that forms a basis for V. If the set does not span V, add some $\mathbf{v}_{k+2} \in V$ that is not $\in \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_{k+1})$. Continue this process until there are $\dim V$ vectors in the set. We now have a linearly independent set of vectors of size $\dim V$. Proposition 4.4 states that they span the $x = \dim V$ -dimensional subspace V. Therefore, they are a basis for V.

   We have thus found vectors $\mathbf{v}_{k+1}, \ldots, \mathbf{v}_\ell$ s.t. $\{\mathbf{v}_1, \ldots, \mathbf{v}_\ell\}$ form a basis for V. □

20. Let U and V be subspaces of $\mathbb{R}^n$. Prove that if $U \cap V = \{\mathbf{0}\}$, then $\dim(U + V) = \dim U + \dim V$.

   Proof. Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be a basis for U and $\{\mathbf{v}_1, \ldots, \mathbf{v}_\ell\}$ be a basis for V. No $\mathbf{u}_i \in \text{span}(v_1, \ldots, v_\ell)$, because if such a vector were, then $\mathbf{u}_i \in V$, and $U \cap V = \{\mathbf{u}_i\}$. The same argument applies to each $\mathbf{v}_i$– none of them lie in span($\mathbf{u}_1, \ldots, \mathbf{u}_k$). As we have proved before, if a set of vectors $\mathbf{w}_1, \ldots \mathbf{w}_z$ is linearly independent, and $\exists \mathbf{x} \notin \text{span}(\mathbf{w}_1, \ldots, \mathbf{w}_z)$, then $\mathbf{w}_1, \ldots, \mathbf{w}_z, \mathbf{x}$ are linearly independent. Thus, $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_\ell\}$ is a linearly independent set. Because it contains the basis vectors for U and V, it must span both subspaces, too. Consequently, it is a basis for $U + V$. $\dim(U + V)$, the number of vectors in the basis set above, is clearly $k + \ell$, which is $\dim U + \dim V$. □

22. Let A be an $m \times n$ matrix, and let B be an $n \times p$ matrix.

   (a) Prove that $\text{rank}(AB) \leq \text{rank}(A)$

   Proof. We know per exercise 3.2.10b that $C(AB) \subset C(A)$. THe dimension of a subspaces must be $\leq$ that of the space it is within, so $\dim C(AB) \leq \dim C(A)$. Since rank is the dimension of the column space, $\text{rank}(AB) \leq \text{rank}(A)$. □

   (b) Prove that if $n = p$ and B is nonsingular, then $\text{rank}(AB) = \text{rank}(A)$.

   Proof. We know per exercise 3.2.10 that $C(AB) = C(A)$ when B is nonsingular. In this case, $\dim C(AB) = \dim C(A)$, which is equaivalen to saying $\text{rank}(AB) = \text{rank}(A)$. □

   (c) Prove that $\text{rank}(AB) \leq \text{rank}(B)$

   Proof. There are p columns in AB and B. Thus, per the rank-nullity theorem,

   $$p = \text{null}(AB) + \text{rank}(AB) = \text{null}(B) + \text{rank}(B) \tag{1}$$

   so

   $$\text{rank}(B) = \text{rank}(AB) + \text{null}(AB) - \text{null}(B) \tag{2}$$

   We know that $\text{null}(AB) - \text{null}(B) \geq 0$ since $N(B) \subset N(AB)$ per exercise 3.2.10a. Plugging that into (2), we see that $\text{rank}(B) \geq \text{rank}(AB)$. □

(d) Prove that if $m = n$ and A is nonsingular, then $\text{rank}(AB) = \text{rank}(B)$.

Proof. Because A is nonsingular, when $(AB)\mathbf{x} = A(B\mathbf{x}) = \mathbf{0}$, $B\mathbf{x} = \mathbf{0}$, and when $B\mathbf{x} = \mathbf{0}, (AB)\mathbf{x} = \mathbf{0}$. Thus, $\mathbf{x} \in N(AB) \Leftrightarrow \mathbf{x} \in N(B)$. In other words, $N(AB) = N(B)$ and, of course, $\text{null}(AB) = \text{null}(B)$.

AB and B are both $n \times p$ matrices with p columns. Thus by the rank-nullity theorem,

$$p = \text{null}(AB) + \text{rank}(AB) = \text{null}(B) + \text{rank}(B) \tag{3}$$

We have shown that $\text{null}(AB) = \text{null}(B)$. Plugging that into (3), we see that $\text{rank}(AB) = \text{rank}(B)$. □

(e) Prove that if $\text{rank}(AB) = n$ then $\text{rank}(A) = \text{rank}(B) = n$.

Proof. We first prove that $\text{rank}(A) = n$. $C(AB)$ is defined as the subspace spanning all vectors of the form $AB(\mathbf{x})$. $AB(\mathbf{x}) = A(B\mathbf{x})$, so every vector in $C(AB)$ is in $C(A)$. Assume for contradiction that $\text{rank}(A) < n$. Then $C(AB)$ has higher dimension than $C(A)$, so there are vectors in $C(AB)$ that are not in $C(A)$. But this contradicts our finding that every vector in $C(AB)$ is in $C(A)$. We thus reject the false assumption that $\text{rank}(A) < n$ and accept that $\text{rank}(A) \geq n$. But since A has n columns and thus must have $\leq n$ pivots, $\text{rank}(A) \leq n$. Combining the two constraints, we deduce that $\text{rank}(A) = n$.

We next prove that $\text{rank}(B) = n$. Assume for contradiction that $\text{rank}(B) < n$. Then at least one row of B is a linear combination of the others, so $\dim R(B) \leq n$. Since AB is composed of a linear combination of the rows of B, $\dim R(AB)$ and $\dim C(AB)$ are $\leq n$. But this conflicts with the given statement that $\text{rank}(AB) = n$. We thus reject the assumption that $\text{rank}(B) < n$. So $\text{rank}(B) \geq n$. We know, however, that $\text{rank}(B) \leq n$ because B has n rows and, consequently, $\leq n$ pivots. Thus $\text{rank}(B) = n$.

We have shown that $\text{rank}(A) = \text{rank}(B) = n$ □

24. Let A be an $m \times n$ matrix.

(a) Prove $N(A^\intercal A) = N(A)$.

Proof. Per 3.2.10a, $N(A) \subset N(A^\intercal A)$. We now show that $N(A^\intercal A) \subset N(A)$. Let $\mathbf{x} \in N(A^\intercal A)$. Then $A^\intercal A(\mathbf{x}) = A^\intercal(A\mathbf{x}) = \mathbf{0}$. $A\mathbf{x} \in N(A^\intercal)$, and is, by definition, in $C(A)$. Since $C(A) \perp N(A^\intercal)$, $A\mathbf{x} = \mathbf{0}$, so $\mathbf{x} \in N(A)$. Since $\mathbf{x} \in N(A^\intercal A) \Rightarrow \mathbf{x} \in N(A)$, $N(A^\intercal A) \subset N(A)$. Having already shown that $N(A) \subset N(A^\intercal A)$, we conclude that $N(A) = N(A^\intercal A)$

□

(b) Prove that $\text{rank}(A) = \text{rank}(A^\intercal A)$

Proof. A and $A^\intercal A$ have the same number of columns, n. Per the rank-nullity theorem,

$$n = \text{null}(A) + \text{rank}(A) = \text{null}(A^\intercal A) + \text{rank}(A^\intercal A) \tag{4}$$

We showed in 23a that $\text{null}(A) = \text{null}(A^\intercal A)$. Subtracting away both of those terms from (4), we see that $\text{rank}(A) = \text{rank}(A^\intercal A)$. □

(c) Prove that $C(A^\intercal A) = C(A^\intercal)$

Proof. The columns of $A^\intercal A$ are composed of linear combinations of the columns of $A^\intercal$, so each column of $A^\intercal A$ is in the span of $C(A)$, and thus $C(A^\intercal) \subset C(A)$. We just showed in 24b that $\dim C(A) = \dim C(A^\intercal A)$. Since $C(A^\intercal A) \subset C(A)$ and $C(A^\intercal A)$ has the same dimension as $C(A)$, $C(A^\intercal A) = C(A)$. □