Math 340 HW 7

Asa Royal (ajr74) [collaborators: none]

March 22 2024

1. ..

2. Let Z be a standard normal random variable. Note that $\Phi(x) = F_Z$. Now let $X \sim N(2,2)$ be a random variable repesenting the weight of the 2,500 packages. Since the weight of each package (call this X_i) is independent and normally distributed, $\text{Var}(X) = \sum_{i=1}^{2500} \text{Var}(X_i) = 5000$. We can solve this problem by expressing F_X in terms of Φ, then subracting $\mathbb{P}(X \leq 4850)$ from $\mathbb{P}(X \leq 5150)$ to find $\mathbb{P}(4850 \leq X \leq 5150)$.

   To express F_x in terms of Φ, we observe that $X = \sigma Z + \mu$, so

$$\begin{aligned}\mathbb{P}(X \leq x) &= \mathbb{P}(\sigma Z + \mu \leq x) \\ &= \mathbb{P}(\sigma Z \leq x - \mu) \\ &= \mathbb{P}\left(Z \leq \frac{x-\mu}{\sigma}\right) \\ &= \Phi(x - \mu)\end{aligned}$$

   Therefore,

$$\begin{aligned}\mathbb{P}(X \leq 5150) = \Phi(5150 - 5000) = \Phi(150) \\ \mathbb{P}(X \leq 4850) = \Phi(4850 - 5000) = \Phi(-150)\end{aligned}$$

   So $\mathbb{P}(4850 \leq X \leq 5150) = \Phi(150) - \Phi(-150)$.

3. ...

4. **True**. If $f(x) < 0$ for some $x = a$, $F_x = \int f(x)dx$ would decrease along some interval $[a - \varepsilon, a + \varepsilon]$, violating the definition of a CDF. If $f(x) > 1$ for some $x = \beta$, then $\int_{\beta-\varepsilon}^{\beta+\varepsilon} f(x)dx > 1$. Thus, clearly $\int_{-\infty}^{\infty} f(x)dx \neq 1$, violaiting the definition of a CDF.

5. Suppose that X is uniformly distributed on the interval $[2, 4]$.

   (i) What is the density for X?

$$f(x) = \begin{cases} 1/2 & \text{if } x \in [2,4] \\ 0 & \text{otherwise} \end{cases}$$

   (ii) What is the CDF for X?

$$F_x = \begin{cases} 0 & \text{if } x \leq 2 \\ \frac{1}{2}(x-2) & \text{if } x \in [2,4] \\ 1 & \text{if } x \geq 4 \end{cases}$$

   (iii) What are the CDF and density for the random variable $Y = X^2 + 1$?

   CDF

$$\begin{aligned}\mathbb{P}(Y \leq b) &= \mathbb{P}(X^2 + 1 \leq b) \\ &= \mathbb{P}(X^2 \leq b - 1) \\ &= \mathbb{P}(X \leq \sqrt{b-1} \\ &= \mathbb{P}(X \in [2, \sqrt{b-1}]) \\ &= \frac{\sqrt{b-1} - 2}{2}\end{aligned}$$

But note that the Y has 0 density when $X \leq 2, Y \leq 5$. Thus,

$$F_Y = \begin{cases} 0 & \text{if } Y \leq 5 \\ \frac{\sqrt{b-1}-2}{2} & \text{if } 5 \leq b \leq 17 \\ 1 & \text{if } Y \geq 17 \end{cases}$$

PDF

$$\begin{aligned} f(b) &= \frac{d}{db}\left(\frac{\sqrt{b-1}-2}{2}\right) \\ &= \frac{1}{2}\frac{d}{db}(\sqrt{b-1}-2) \\ &= \frac{1}{2}\left(\frac{1}{2\sqrt{b-1}}\right) \\ &= \frac{1}{4\sqrt{b-1}} \end{aligned}$$

Taking into account the zero density regions,

$$\begin{cases} 0 & \text{if } Y \leq 5 \\ \frac{1}{4\sqrt{b-1}} & \text{if } 5 \leq Y \leq 17 \\ 0 & \text{if } Y \geq 17 \end{cases}$$

(iv) What are the mean and variance of Y?

Mean The mean of a random variable Y occurs at the "center" of the probability mass: where $F_Y = \mathbb{P}(Y \leq \mu) = 1/2$.

$$\begin{aligned} \frac{\sqrt{\mu-1}-2}{2} &= \frac{1}{2} \\ \sqrt{\mu-1} &= 3 \\ \mu - 1 &= 9 \\ \mu &= 10 \end{aligned}$$

variance

$\mathrm{Var}(Y) = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2$. We calculated $\mathbb{E}[Y]$ above. We can calculate $\mathbb{E}[Y^2]$ by noting $\mathbb{P}(Y^2 \leq k) = \mathbb{P}(Y \leq \sqrt{k})$. Applying the formula for F_Y found in 5iii, this is

$$\frac{\sqrt{\sqrt{k}-1}-2}{2} \tag{1}$$

$\mathbb{E}[Y^2]$ again occurs at the center of the probability mass of Y^2: where $F_{Y^2} = 1/2$. We find this value below:

$$\begin{aligned} \frac{\sqrt{\sqrt{k}-1}-2}{2} &= \frac{1}{2} \\ \sqrt{\sqrt{k-1}-2} &= 1 \\ \sqrt{\sqrt{k}-1} &= 3 \\ \sqrt{k}-1 &= 9 \\ \sqrt{k} &= 10 \\ k &= 100 \end{aligned}$$

So $\mathbb{E}[Y^2] = 100$. $\mathrm{Var}(Y) = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 = 100 - 10^2 = 0$??

6. ..

Find c:

For any pdf $f(x)$, $\int_{-\infty}^{\infty} f(x) = 1$, which means in this case, $\int_{-1}^{1} cx^2 = 1$. Solving for c:

$$\begin{aligned}
\int_{-1}^{1} cx^2 dx &= 1 \\
\frac{c}{3}x^3\Big|_{-1}^{1} &= 1 \\
\frac{2c}{3} &= 1 \\
c &= 3/2
\end{aligned}$$

Find $\mathbb{P}(X > 1/2)$

$$\begin{aligned}
\mathbb{P}(X > 1/2) &= 1 - \mathbb{P}(X \leq 1/2) \\
&= 1 - \int_{-\infty}^{1/2} f(x) \\
&= 1 - \left(\int_{-\infty}^{-1} f(x)dx + \int_{-1}^{1/2} f(x)dx\right) \\
&1 - \int_{-1}^{1/2} \frac{3}{2}x^2 dx \\
&= 1 - \frac{1}{2}x^3|_{-1}^{1/2} \\
&= 1 - \left(\frac{1}{2}\right)\left(\frac{9}{8}\right) \\
&= 7/16
\end{aligned}$$