Math 221 HW 7

3.2 The 4 fundamental subspaces

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3.2

1. Show that if B is obtained from A by performing one or more row operations, then $R(B) = R(A)$.

   We first show that $R(B) \subset R(A)$. B is formed by some combination of three operations on the rows of A, swapping, scaling, and adding scalar multiples.

   (a) If we swap rows of A to form B, B still has the same rows as A, so $R(B) \subset R(A)$.

   (b) If we scale a row of A to form a row of B, $\mathbf{B}_i = c\mathbf{A}_i$ where $c \neq 0$. Then $R(B) = c_1\mathbf{A}_1 + \ldots c_i(c\mathbf{A}_i) + \ldots + c_m\mathbf{A}_m$, so $R(B) \subset R(A)$.

   (c) If we add a scalar multiple of some $\mathbf{A}_i$ to some $\mathbf{A}_j$ so that $\mathbf{B}_j = \mathbf{A}_j + c\mathbf{A}_i$, $R(B) = c_1\mathbf{B}_1 + \ldots c_i\mathbf{B}_i + \ldots c_j(\mathbf{A}_j + c\mathbf{A}_i) + \ldots c_m\mathbf{B}_m = c\mathbf{A}_1 + \ldots (c_jc + c_i)\mathbf{A}_i + \ldots c_j\mathbf{A}_j + \ldots c_m\mathbf{A}_m$. This in the span of the rows of A, so $R(B) \subset R(A)$.

   We now show that $R(A) \subset R(B)$. Since A can be formed from B by using the inverse of the row operations discussed above used to transform A into B, by the arguments above, $R(A) \subset R(B)$.

   Since $R(A) \subset R(B)$ and $R(B) \subset R(A)$, $R(A) = R(B)$.

10. Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. Prove that

    (a) $N(B) \subset N(AB)$

    Proof. $N(B)$ consists of $\mathbf{x} \in \mathbb{R}^n$ s.t. $B\mathbf{x} = \mathbf{0}$. $N(AB)$ consists of $\mathbf{x} \in \mathbb{R}^n$ s.t. $AB\mathbf{x} = \mathbf{0}$ or, equivalently, $A(B\mathbf{x}) = 0$. The set of vectors $\mathbf{x}$ that fulfill $B\mathbf{x} = 0$ also fulfill $(AB)\mathbf{x} = 0$, since when $B\mathbf{x} = \mathbf{0}$, $(AB)\mathbf{x} = A(B\mathbf{x}) = A(\mathbf{0}) = \mathbf{0}$. Otherwise stated, $\mathbf{x} \in N(B) \Rightarrow \mathbf{x} \in N(AB)$, so $N(B) \subset N(AB)$. □

    (b) $C(AB) \subset C(A)$

    Proof. $b \in C(AB) \Leftrightarrow \exists \mathbf{x} \in \mathbb{R}^p$ s.t. $b = AB(\mathbf{x}) = A(B\mathbf{x})$. The definition of matrix vector multiplication then requires that if $b = A(B\mathbf{x}), b = a_1(bx)_1 + a_2(bx)_2 + \ldots a_n(bx)_n$, where a_i is the i-th column of A and $(bx)_i$ is the i-th component of the vector $B\mathbf{x}$. We can see that b is expressed as a linear combination of the columns of A, which means that $b \in C(A)$. We have thus shown that $b \in C(AB) \Rightarrow b \in C(A)$, so we can conclude that $C(AB) \subset C(A)$. □

    (c) $N(B) = N(AB)$ when A is nonsingular.

    Proof. We begin by proving that $N(B) \subset N(AB)$. $N(AB)$ consists of vectors $\mathbf{x} \in \mathbb{R}^n$ s.t. $AB(\mathbf{x}) = A(B\mathbf{x}) = \mathbf{0}$. Since A is nonsingular, when $A(B\mathbf{x}) = \mathbf{0}$, $B\mathbf{x}$ must equal 0, given that $\forall \mathbf{y} \in \mathbb{R}^n, A\mathbf{y} = \mathbf{0}$ can have only the trivial solution $\mathbf{y} = \mathbf{0}$. The set of $\mathbf{x}$ s.t. $B\mathbf{x} = \mathbf{0} =$ N(B). Thus $\forall \mathbf{x} \in \mathbb{R}^n, \mathbf{x} \in N(B) \Rightarrow \mathbf{x} \in N(AB)$, so $N(B) \subset N(AB)$.

    We now prove that $N(AB) \subset N(B)$. The proof is quite similar to what we showed above. Since A is nonsingular, $N(AB)$, which is equal to $\{\mathbf{x} \in \mathbb{R}^p | A(B\mathbf{x}) = \mathbf{0}\}$, is (included in) the the set of vectors $\mathbf{x}$ that fulfills $B\mathbf{x} = \mathbf{0}$. Thus, $N(AB) \subset N(B)$.

    Because $N(AB) \subset N(B)$ and $N(B) \subset N(AB)$, $N(AB) = N(B)$. □

    (d) C(AB) = C(A) when B is nonsingular. We have already shown that $C(AB) \subset C(A)$. We know show that $C(A) \subset C(AB)$. Let b be an arbitrary vector in $C(A)$. By the definition of a column space, $\exists \mathbf{x} | A\mathbf{x} = \mathbf{b}$. Given that B is an $n \times n$ nonsingular matrix, we know it must be invertible, too. Using that fact, we can recover b from AB by multiplying it by $B^{-1}\mathbf{x}$. $AB(B^{-1}\mathbf{x}) = A\mathbf{x} = \mathbf{b}$. Thus $\mathbf{b} \in C(AB)$. Since

$b \in C(A) \Rightarrow \mathbf{b} \in C(AB), C(A) \subset C(AB)$.

We have now shown that $C(AB) \subset C(A)$ and that $C(A) \subset C(AB)$. Therefore, $C(AB) = C(A)$.

11. Let A be an $m \times n$ matrix. Prove that $N(A^\intercal A) = N(A)$.

Proof. We first show that $N(A) \subset N(A^\intercal A)$. $N(A^\intercal A) = \{\mathbf{x} | A^\intercal A\mathbf{x} = \mathbf{0}\}$. Per exercise 2.5.15, if $A^\intercal A\mathbf{x} = \mathbf{0}$, then $A\mathbf{x} = \mathbf{0}$. When $A\mathbf{x} = \mathbf{0}$ (i.e. $\mathbf{x} \in N(A)$), $A^\intercal A\mathbf{x} = A^\intercal \mathbf{0} = \mathbf{0}$, which means $\mathbf{x} \in N(A^\intercal A)$. Because $\mathbf{x} \in N(A) \Rightarrow \mathbf{x} \in N(A^\intercal A)$, we can conclude that $N(A) \subset N(A^\intercal A)$.

We then show that $N(A^\intercal A) \subset N(A)$. By the definition of matrix multiplication, $A^\intercal A$ is a linear combination of the rows of the rightmost matrix, A. Since $N(A^\intercal A)$ is the set of vectors orthogonal to the rows of $A^\intercal A$, and $A^\intercal A$ is a linear combination of the rows of A, a vector in $N(A^\intercal A)$ must also be orthogonal to the rows of A. In other words, $\mathbf{x} \in N(A^\intercal A) \Rightarrow \mathbf{x} \in N(A)$. Thus $N(A^\intercal A) \subset N(A)$.

Because $N(A) \subset N(A^\intercal A)$ and $N(A^\intercal A) \subset N(A)$, $N(A^\intercal A) = N(A)$. □

3.3

8. Supose $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and $\{\mathbf{v}, \mathbf{w}\}$ is linearly independent. Prove that $\{\mathbf{v} - \mathbf{w}, \mathbf{2v} + \mathbf{w}\}$ is linearly independent as well.

Proof. We will prove that $\{\mathbf{v} - \mathbf{w}, \mathbf{2v} + \mathbf{w}\}$ is indepedent by showing that if $c_1(\mathbf{v} - \mathbf{w}) + c_2(2\mathbf{v} + \mathbf{w}) = \mathbf{0}$, then $c_1 = c_2 = 0$. Multiplying out the LHS, we find that $c_1\mathbf{v} - c_1\mathbf{w} + 2c_2\mathbf{v} + c_2\mathbf{w} = 0$, which after factorization shows that $(c_1 + 2c_2)\mathbf{v} + (c_2 - c_1)\mathbf{w} = \mathbf{0}$. Since $\{\mathbf{v}, \mathbf{w}\}$ is linearly independent, $c_1 + 2c_2 = 0$ and $c_2 - c_1 = 0$. Solving that pair of linear equation shows that $c_1 = c_2 = 0$. Thus, $\{\mathbf{v} - \mathbf{w}, \mathbf{2v} + \mathbf{w}\}$ is linearly independent. □

10. Supose $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are nonzero vectors with the property that $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ whenever $i \neq j$. Prove that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent.

Proof. Suppose $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k = \mathbf{0}$. Take the dot product of both sides with the vector $\mathbf{v}_1$. Then we see that $c_1\|\mathbf{v}_1\| + c_2\mathbf{v}_2 \cdot \mathbf{v}_1 + \ldots c_k\mathbf{v}_k \cdot \mathbf{v}_1 = \mathbf{0}$. Since all of the vectors are orthogonal to each other, this simplifies to $c_1\|\mathbf{v}_1\| = \mathbf{0}$. $\|\mathbf{v}_1\| \neq 0$, so $c_1 = 0$. If we repeat this process for $\mathbf{v}_2 \ldots \mathbf{v}_k$, we find similarly that $c_2, \ldots c_k = 0$. Since only the trivial combination of $\mathbf{v}_1, \ldots \mathbf{v}_k$ produces the $\mathbf{0}$ vector, $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent. □

11. Supose $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are nonzero, mutually orthogonal vectors in $\mathbb{R}^n$.

   (a) Prove that they form a basis for $\mathbb{R}^n$

   Proof. To prove that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ form a basis for $\mathbb{R}^n$, we must show they are linearly independent and that they span $\mathbb{R}^n$. Per exercise 10, we know they are linearly independent. To show that they span $\mathbb{R}^n$, we assume for contradiction that there is some vector $\mathbf{v} \in \mathbb{R}^n$ s.t. $\mathbf{v} \notin \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$. Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{v}\}$ is linearly independent. But if we represent that set of vectors as a matrix like,

$$A = \begin{bmatrix} | & | & | & | \\ \mathbf{v}_1 & \ldots & \mathbf{v}_n & \mathbf{v} \\ | & | & | & | \end{bmatrix}$$

   A is an $n \times (n+1)$ matrix, which means rank(A) is constrained by the number of rows, n, and is thus less than the number of variables, $n+1$. This indicates that the equation $A\mathbf{x} = \mathbf{b}$ has free variables, and that there are multiple linear combinations of the columns of A, which again are $\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{v}$, that can form some arbitrary vector b. But that contradicts the assumption that $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent, since if that were the case, b could only be formed by precisely one linear combination of those vectors. We thus reject our false assumption and conclude that $\forall v \in \mathbb{R}^n, \mathbf{v} \in \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$. That is, $\mathbf{v}_1, \ldots, \mathbf{v}_n$ span $\mathbb{R}^n$.

   Because $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent and span $\mathbb{R}^n$, they are a basis for $\mathbb{R}^n$. □

14. Suppose $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ form a linearly independent set. Show that for any $1 \leq l < k$, the set$\{\mathbf{v}_1, \ldots, \mathbf{v}_l\}$ is linearly independent as well.

Proof. Assume that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ form a linearly independent set. We will prove by induction that for any $1 \leq \ell < k$, the set$\{\mathbf{v}_1, \ldots, \mathbf{v}_l\}$ is linearly independent as well.
Let $P(\ell)$ mean that the set of vectors with ℓ vectors from the original set is linearly independent.

Base case: P(1). The set with one vector is trivially indepedent. There are no other vectors it could be a linear combination of.
Inductive hypothesis: $P(\ell)$. Assume $\ell < k$ vectors from the set of k linearly independent are linearly independent, too.
Inductive step: $P(\ell) \Rightarrow P(\ell+1)$. Our inductive hypothesis is that $\ell < k$ vectors are linearly independent–none of them are in the span of the others. Now, when we add the $\ell+1$-th vector, assume for contradiction that it lies in the span of $\mathbf{v}_1, \ldots, \mathbf{v}_ell$, so that $\mathbf{v}_1, \ldots, \mathbf{v}_{\ell+1}$ is linearly dependent.
Then $\mathbf{v}_{\ell+1} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + 0\mathbf{v}_{\ell+2} + 0\mathbf{v}_{\ell+3} + \ldots + 0\mathbf{v}_k$. We can rearrange that to $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_\ell\mathbf{v}_\ell + (-1)\mathbf{v}_{\ell+1} + 0\mathbf{v}_{\ell+2} + 0\mathbf{v}_{\ell+3} + \ldots + 0\mathbf{v}_k = \mathbf{0}$. This violates our initial assumption that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent, because $c_{\ell+1} = -1$. We thus reject the ssumption that $\{\mathbf{v}_1, \ldots, \mathbf{v}_{\ell+1}\}$ is linearly dependent and conclude that it is linearly independent. We have now shown $P(\ell+1)$.
Induction conclusion: We have shown $P(1)$ and that $P(\ell) \Rightarrow P(\ell+1)$. We thus conclude $P(\ell)$, that a set of $\ell < k$ vectors chosen from the linearly independent set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent, too. ☐

15. Supose $k > n$. Prove that any k vectors in $\mathbb{R}^n$ must form a linearly dependent set.

Proof. Suppose we have $k > n$ vectors in $\mathbb{R}^n$. THe set of k vectors can be represented as the columns of a matrix A, where

$$A = \begin{bmatrix} | & | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_k \\ | & | & | & | \end{bmatrix}$$

Since the vectors that make up the columns of A have n components, there are at most n rows in A, and rank(A), which is limited by the number of rows, is $< k$, the number of columns. That means a solution to $A\mathbf{x} = \mathbf{b}$ has free variables and $A\mathbf{x} = \mathbf{0}$ therefore has a nontrivial solution, where $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_k\mathbf{v}_k = \mathbf{0}$ and some $x_i \neq 0$. Thus, the k vectors are, by definition, linearly dependent. ☐

19. Let A be an $n \times n$ matrix. Prove that if A is nonsingular and $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent, then $\{A\mathbf{v}_1, \ldots, A\mathbf{v}_k\}$ is likewise linearly independent. Give an example to show that the result is false if A is singular.

Proof. We can show that $\{A\mathbf{v}_1, \ldots, A\mathbf{v}_k\}$ is linearly independent by showing that $c_1A\mathbf{v}_1 + \ldots + c_kA\mathbf{v}_k = \mathbf{0}$ is only true if $c_1 = \ldots = c_k = 0$.

$c_1A\mathbf{v}_1 + \ldots + c_kA\mathbf{v}_k = \mathbf{0}$ can be factored to $A(c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k) = \mathbf{0}$. Because A is nonsingular, we know $c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k = \mathbf{0}$, and because $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent, we know that $c_1 = c_2 = \ldots = c_k = 0$. Thus, $\{A\mathbf{v}_1, \ldots, A\mathbf{v}_k\}$ is linearly independent.

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For an example showing that this results is false when A is singular, consider the linearly independent set of vector $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ and the matrix $A = \begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}$. $\{A\mathbf{v}_1, A\mathbf{v}_2\} = \{\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\}$, which cannot be linearly independent, given it contains the $\mathbf{0}$ vector.

21. Let A be an $m \times n$ matrix of rank n. Supose $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ and $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent. Prove that $\{A\mathbf{v}_1, \ldots, A\mathbf{v}_k \subset \mathbb{R}^m$ is likewise linearly independent.

Proof. We can show that $\{A\mathbf{v}_1, \ldots, A\mathbf{v}_k\}$ is linearly independent by showing that $c_1A\mathbf{v}_1 + \ldots + c_kA\mathbf{v}_k = \mathbf{0}$ is only true if $c_1 = \ldots = c_k = 0$.

$c_1A\mathbf{v}_1 + \ldots + c_kA\mathbf{v}_k = \mathbf{0}$ can be factored to $A(c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k) = \mathbf{0}$. Because rank$(A) = n$, $A\mathbf{x} = \mathbf{b}$ has precisely one solution when it is consistent (and $A\mathbf{x} = \mathbf{0}$ is always consistent, with solution $\mathbf{x} = \mathbf{0}$). Thus $c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k = \mathbf{0}$, and because $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent, by the definition of linear independence, we know that $c_1 = c_2 = \ldots = c_k = 0$. Thus, $\{A\mathbf{v}_1, \ldots, A\mathbf{v}_k\}$ is linearly independent.

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22. Let A be an $n \times n$ matrix and suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^n$ are nonzero vectors that satisfy

$$\begin{aligned} A\mathbf{v}_1 &= \mathbf{v}_1 \\ A\mathbf{v}_2 &= 2\mathbf{v}_2 \\ A\mathbf{v}_3 &= 3\mathbf{v}_3 \end{aligned}$$

Prove that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

Proof. We first show that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent. Assume for contradiction that they are not. Then $\mathbf{v}_2 = c\mathbf{v}_1$, which means $A\mathbf{v}_2 = 2\mathbf{v}_2 = 2c\mathbf{v}_1$, for some nonzero c. But $A\mathbf{v}_2 = Ac\mathbf{v}_1$, too. Thus, $2c\mathbf{v}_1 = c\mathbf{v}_1$, but this is impossible because $\mathbf{v}_1$ is a nonzero vector, and $2 \neq 1$. Thus, by contradiction, $\{\mathbf{v}_1, \mathbf{v}_2\}$ must be linearly independent.

We now proceed to show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent. Assume for contradiction that it is not. Then $\mathbf{v}_3 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, where at least one of c_1 or $c_2 \neq 0$. So $A\mathbf{v}_3 = 3\mathbf{v}_3 = 3c_1\mathbf{v}_1 + 3c_2\mathbf{v}_2$. We also know that $A\mathbf{v}_3 = A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2 = c_1\mathbf{v}_1 + 2c_2\mathbf{v}_2$. Then $3c_1\mathbf{v}_1 + 3c_2\mathbf{v}_2 = c_1\mathbf{v}_1 + 2c_2\mathbf{v}_2$, which simplifies to $2c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$. We have already shown that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent of each other, so the constants $2c_1 = c_2 = 0$, which means $c_1 = c_2 = 0$. This contradicts our asssuumption that either c_1 or $c_2 \neq 0$. We thus reject our initial assumpion and conclude that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

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