# Math 340: Lec 13: Variance, Weak LLN, Chebyshev/Markov)

Asa Royal (ajr74)

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## 0.1 Variance of sum of iids

**Remark.** Recall that generally, $\text{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \text{Var}(X_i) + 2\sum_{i<j} \text{Cov}(X_i, X_j)$

**Proposition 1.** If random variables $X_1, \ldots, X_n$ have equal variance,

$$\text{Var}(X_1 + \ldots + X_n) = \sum_{i=1}^{n} \sigma^2 = n\sigma^2$$

and

$$\text{SD}(X_1 + \ldots + X_n) = \sigma\sqrt{n}$$

**Remark.** The standard deviation of a sum of iid random variables grows more slowly than the sum of the variables

**Example** (Compute the vriance of $X \sim \text{Bin}(n,p)$)**.** We can express $X$ as a sum of indicators. Each has teh same variance, $p(1-p)$, so $\text{Var}(X) = np(1-p)$

## 0.2 Inequalities

**Theorem 2** (Markov's inequality)**.** Intuition: Extreme values of a random variable are unlikely, because we need to balance probability mass around the mean. For a positive random variable $Y$,

$$\mathbb{P}(Y > t) \leq \frac{1}{t}\mathbb{E}[Y]$$

The tail probability of a positive random variable is bounded by its expected value.

**Theorem 3** (Chebyshev's inequality)**.** Intuition: If the $\text{Var}(X)$ is small, then $X$ is unikely to be far from its mean. The amount a random variable $X$ with $\text{Var}(X) < \infty$ can vary from its mean, $\mu$ is bounded:

$$\mathbb{P}(|X - \mu| \geq t) \leq \frac{\text{Var}(x)}{t^2}, \forall t > 0$$

**Remark.** Markov's inequality leads to the weak law of large numbers

**Theorem 4** (Weak law of large numbers)**.** Let $X$ be the sum of $n$ iid random variables. When $\text{Var}(X) < \infty$, we can bound the probability that the average of the $X_1, \ldots, X_n$ deviates from the mean of the random variables:

$$\mathbb{P}\left(\left|\frac{X_1 + \ldots + X_n}{n} - \mu\right| < \varepsilon\right) \leq \frac{\sigma^2}{\varepsilon^2 n}$$

And even if $\text{Var}(X)$ is infinite, if $\mathbb{E}[X] < \infty$,

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\frac{X_1 + \ldots + X_n}{n} - \mu\right| > \varepsilon\right) = 0$$