Math 340: Lec 6 Big Ideas Journal (Law of large numbers)

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Remark. The law of small numbers and law of large numbers are both applied in the context of coin tossing when we study A_k, the event where k heads are tossed. **The law of small numbers is applied when $n \to \infty$ and p is very small**. (p equals λn for a constant λ, so it actually approaches 0).

Remark. The law of large numbers is applied when p is fixed and $n \to \infty$. It stems from the intitution is that we expect that when a fair coin is tossed n times, the average proportion of heads will be $1/2$, and the average number of heads will be $n/2$.

Theorem 1 (Law of large numbers (weak))**.** For any $\varepsilon > 0$ and any $n \geq 1$, the probability that more than $(1/2+\varepsilon)n$ heads are tossed (or that the proportion of heads is greater than $1/2+\varepsilon$ is bounded by:

$$\mathbb{P}\left(\bigcup_{k \geq (1/2+\varepsilon)n} A_k\right) \leq e^{-\varepsilon^2 n}$$

Corollary 2. We can use the law of large numbers to apply a lower bound, too. The probability that fewer than $(1/2 = \varepsilon)n$ heads are tossed (or that the proportion of heads is less than $1/2-\varepsilon$ is bounded by:

$$\mathbb{P}\left(\bigcup_{k \leq (1/2-\varepsilon)n} A_k\right) \leq e^{-\varepsilon^2 n}$$

Corollary 3. Unifying the two and thinking about complements, we can bound the probability that we see a proportion or number of heads inside a given range:

$$\mathbb{P}\left(\bigcup_{(1/2-\varepsilon)n < k < (1/2-\varepsilon)n} A_k\right) \geq 1 - 2e^{-\varepsilon^2 n}$$

Corollary 4. Consider $\mathbb{P}(\frac{1}{2} - \delta \leq f \leq \frac{1}{2} + \beta)$ for $\delta \neq \beta$.

$$\mathbb{P}(\frac{1}{2} - \delta \leq f \leq \frac{1}{2} + \beta) = 1 - \mathbb{P}\left(\bigcup_{k \geq (1/2+\beta)n} A_k \cup \bigcup_{k \leq (1/2-\delta)n} A_k\right)$$

Since both sets of events on either side of the union are disjoin, the probability of their union (call it H) is the sum of their probabilities, and is thus, per the law of large numbers, bounded by $H \leq e^{-\beta^2 n} + e^{-\delta^2 n}$. Then

$$\begin{aligned}\mathbb{P}(\frac{1}{2} - \delta \leq f \leq \frac{1}{2} + \beta) &= 1 - H \\ &\geq 1 - e^{-\beta^2 n} + e^{-\delta^2 n}\end{aligned}$$