Lecture-4

Tuesday, August 2, 2022 3:15 PM

Lecture-4

Lecture 4: Numerical Analysis (UMA011)

Dr. Meenu Rani

School of Mathematics
TIET, Patiala
Punjab-India

Evaluate $f(x)$ at $x = 1.79$

Exact value $\rightarrow f(1.79) = X$

n-digit arithmetic

$X^* = f(1.79) \approx$ A.E $= |X - X^*|$

R.E. $\frac{|X - X^*|}{|X|}$

Nested Arithmetic

$f(1.79) = X^{**}$

A.E. $|X - X^{**}|$

R.E $= \frac{|X - X^{**}|}{|X|}$

$|X - X^*| > |X - X^{**}|$

$\frac{|X - X^*|}{|X|} > \frac{|X - X^{**}|}{|X|}$

Error Analysis: Nested Arithmetic

Example:
Evaluate $y \approx x - \sin(x)$, when x is small.

$x = 0.01$

$y \approx 0.01 - \sin(0.01)$

$$y \approx x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} - \cdots\right)$$

$$= \left(\frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots\right)$$

$$= \frac{x^3}{3!}\left(1 - \frac{x^2\, 3!}{5!} + \frac{x^4\, 3!}{7!} - \cdots\right)$$

$$= \frac{x^3}{3!}\left(1 - \frac{x^2\, 3!}{5!}\left(1 - \frac{x^2\, 5!}{7!} - \cdots\right)\right)$$

Error Analysis

Exercise:

1. Evaluate $f(x) = x^3 - 3x^2 + 4x + 0.21$ at $x = 2.73$ using 3-digit arithmetic directly and with nesting. Also, find the absolute error and relative error.

Exact value $= X$

3-digit Arithmetic directly $= X^*$

3-digit Nested Arithmetic $= X^{**}$

$|X - X^*|$, $|X - X^{**}|$

Error Analysis: Loss of Significance

Example:
Use four-digit rounding arithmetic and the formula for the roots of a quadratic equation, to find the most accurate approximations to the roots of the following quadratic equation. Compute the absolute and relative errors.

$$\overset{a}{1.002}x^2 + \overset{b}{11.01}x + \overset{c}{0.01265} = 0.$$

Solution: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$= \frac{-11.01 + \sqrt{(11.01)^2 - 4(1.002)(0.01265)}}{2(1.002)}$$

Exact Roots
$x_1 = -0.00114907565991$
$x_2 = -10.98687487643590$

Using 4-digit rounding arithmetic

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-11.01 \pm \sqrt{(11.01)^2 - 4(1.002)(0.01265)}}{2(1.002)}$$

$$= \frac{-11.01 \pm \sqrt{121.2 - (4.008)(0.01265)}}{2.004}$$

$$= \frac{-11.01 \pm \sqrt{121.2 - 0.05070}}{2.004} = \frac{-11.01 \pm \sqrt{121.1}}{2.004}$$

$$= \frac{-11.01 \pm 11.00}{2.004}$$

$x_1 = \frac{-11.01 + 11.00}{2.004}$, $x_2 = \frac{-11.01 - 11.00}{2.004}$

$x_1^* = -0.004990$ $x_2^* = \frac{-22.01}{2.004} = -10.98$

already most accurate app.

$x = -b + \sqrt{b^2 - 4ac}$

$-11.01 + 11.00$

To find the most accurate approximation in x_1

$$x_1^{**} = \frac{-b + \sqrt{b^2 - 4ac}}{2a} * \frac{-b - \sqrt{b^2 - 4ac}}{-b - \sqrt{b^2 - 4ac}}$$

$$= \frac{b^2 - (b^2 - 4ac)}{-2a\,(b + \sqrt{b^2 - 4ac})}$$

$$= \frac{-4ac}{2a\,(b + \sqrt{b^2 - 4ac})}$$

$$= \frac{-2c}{b + \sqrt{b^2 - 4ac}} = \frac{-2(0.01265)}{11.01 + 11.00}$$

$$= \frac{-0.02530}{22.01} = -0.001149$$

A.E. $= |x_1 - x_1^*|$, $|x_1 - x_1^{**}|$, $|x_2 - x_2^*|$

R.E. $\frac{|x_1 - x_1^*|}{|x_1|}$, $\frac{|x_1 - x_1^{**}|}{|x_1|}$, $\frac{|x_2 - x_2^*|}{|x_2|}$

$>$

Error Analysis: Algorithms and Stability

Algorithms:

$f(x) = x^2 - 2 * \cos x + e^x$ $x = 0.1$

$x_1 \rightarrow$ 1st $(0.1)^2$

x_2 2nd $\cos(0.1)$

3rd $2 * \cos(0.1)$

$2 * x_2$

$x_1 = (0.1)^2$

$x_2 = \cos(0.1)$

$x_3 = 2 * x_2$

$x_4 = e^{0.1}$

$x_5 = x_1 - x_3 + x_4$

An algorithm is a procedure that describes a finite sequence of steps to be performed in a specified order.

$f(x) = \boxed{\cos 2x + \sin x/2}$ $x = 0.01$

$\downarrow$

$0.01 + 0.001$

$x + \Delta x$

$f(x) \rightarrow f + \Delta f$

$x_0 :$ $x = 0.01$

$x_1 :$ $2 * x_0$

$x_2 :$ $\cos(x_1)$

$x_3 :$ $\frac{x_0}{2}$

$x_4 :$ $\sin(x_3)$

$x_5 :$ $x_2 + x_4$

Criteria :- small changes in the initial data produce correspondingly small changes in the output

\# Stable or unstable :-

An algorithm satisfies the above criteria is called stable, otherwise it is unstable.

\# Well-conditioned or Ill conditioned :-

A problem is well-conditioned if small changes in the input data can produce only small changes in the output otherwise it is ill-conditioned.