# Lecture-11

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## Lecture 11: Numerical Analysis (UMA011)

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**Procedure of F.P.I method**

To find the root of $f(x)=0$

# find Fixed point form $g(x)=x$ — with the help of $f(x)=0$

find the Appropriate $g(x)$?

# $x=g(x)$

Take an initial guess $p_0 \in [a,b]$ ↳ given

$p_1 = g(p_0) \approx p_0$

$p_2 = g(p_1) \approx p_1$

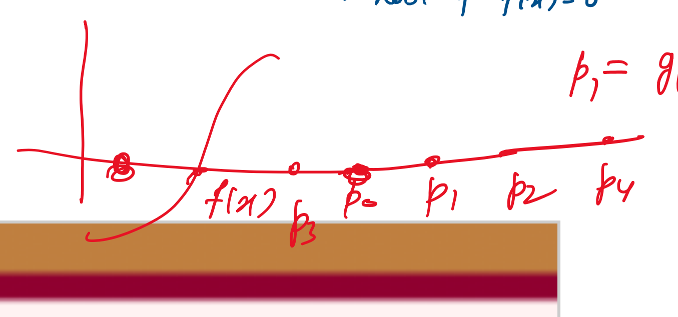
$p_3 = g(p_2)$

— —

$p_{n+1} = g(p_n) \longrightarrow p \to$ fixed pt for $g(x) = x$ ↳ Root of $f(x)=0$

stopping Criterian $|p_n - p_{n-1}| < tol$ (tol given.)

$p_1 = g(p_0)$






### Fixed point iteration

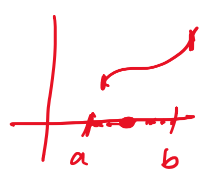
**Convergence conditions satisfied by $g(x)$:**

(i) (existence) If $g \in C[a,b]$ and $g(x) \in [a,b]$, $\forall x \in [a,b]$, then $g(x)$ has at least one fixed point in $[a,b]$. ↳ g maps into itself.

$g(x)=x$

✓ $f(x) = g(x) - x = 0$

$f(x) \in C[a,b] \because g \in C[a,b]$




$f(a) = g(a) - a \geq 0$

$f(b) = g(b) - b \leq 0$

By IVT, $f(x)=0$ has a root in $[a,b]$

$\exists\ c \in (a,b)$ s.t $f(c)=0 \Rightarrow g(c)=c \rightarrow g$ has a fixed pt in $[a,b]$.



### Fixed point iteration

L.M.V.T $\quad \frac{|f(a)-f(b)|}{|a-b|} = |f'(c)|, \quad c \in (a,b)$

**Convergence conditions satisfied by $g(x)$:**

(ii) (uniqueness) If, in addition, $g'(x)$ exists in $(a,b)$ and a positive constant $k<1$ exists with $|g'(x)| \leq k$, for all $x \in (a,b)$, then there is exactly one fixed point in $[a,b]$.

Let $p$ & $q$ be two fixed pt in $[a,b]$ $\qquad |g'(x)| \leq k < 1$

i.e. $g(p)=p, \quad g(q)=q$

Now $|p-q| = |g(p)-g(q)| = |g'(c)|\,|p-q| \qquad c \in (p,q)$

$|p-q| < 1 * |p-q|$

which is not true, our supposition is wrong. $\qquad p<q$, $p+a<q+a$

There is only one fixed pt in $[a,b]$.



### Fixed point iteration

**Convergence conditions satisfied by $g(x)$:**

(iii) (convergence) If conditions of (i) and (ii) are satisfied, then for any number $p_0 \in [a,b]$, the sequence defined by $p_n = g(p_{n-1})$, $n \geq 1$ converges to the unique fixed point $p$ in $[a,b]$.

By L.M.V.T.

$|p_n - p| = |g(p_{n-1}) - g(p)| = |g'(c_n)|\,|p_{n-1}-p|$, $\quad |g'(c_n)| \leq k$

$|p_n - p| \leq k\,|p_{n-1}-p| \leq k * k\,|p_{n-2}-p|$ (same)

$|p_n - p| \leq k^2\,|p_{n-2}-p| \leq k^3\,|p_{n-3}-p| \leq k^4\,|p_{n-4}-p|$

$|p_n - p| \leq k^n\,|p_0 - p|$

$\lim_{n\to\infty} |p_n - p| \leq \lim_{n\to\infty} k^n\,|p_n - p| = 0 \quad \because (k<1)$

$\Rightarrow \lim_{n\to\infty} |p_n - p| \leq 0$

$\Rightarrow \lim_{n\to\infty} p_n = p.$



### Fixed point iteration

**Example:**

Show that $g(x) = \frac{x^2-1}{3}$ has a unique fixed point on the interval $[-1,1]$.

**Solution:** $g(x) = \frac{x^2-1}{3}$ is continuous on $[-1,1]$

To show $g(x)$ maps into itself.

$g'(x) = \frac{2x}{3} \geq 0$ on $[0,1]$

$< 0$ on $[-1,0)$

$\Rightarrow g(x)$ is increasing function on $[0,1]$

& " decreasing function on $[-1,0)$



### Fixed point iteration

**Solution(continued):**

$g(0) = \frac{-1}{3}$ is the minimum value

$g(1) = g(-1) = 0$ is the max. value.

& $-1 \leq \frac{-1}{3} \leq 1$ (min. value) and $-1 \leq 0 \leq 1$ (max. value)

$\Rightarrow g(x) \in [-1,1]\ \forall x \in [-1,1]$

To show $|g'(x)| < 1 \quad \forall x \in (-1,1)$

$|g'(x)| = \left|\frac{2x}{3}\right| < 1 \quad \forall x \in [-1,1]$

$\Rightarrow g(x)$ satisfies all the convergence conditions on $[-1,1]$

$\Rightarrow g(x)$ has a unique fixed pt in $[-1,1]$.



### Root finding problem

**Exercise:**

1. Show that $g(x) = 2^{-x}$ has a unique fixed point on the interval $[\frac{1}{3}, 1]$.