PROBABILITY AND STATISTICS (UCS401)

Lecture-20

(Uniform or Rectangular distribution with illustrations)

Random Variables and their Special Distributions(Unit –III & IV)

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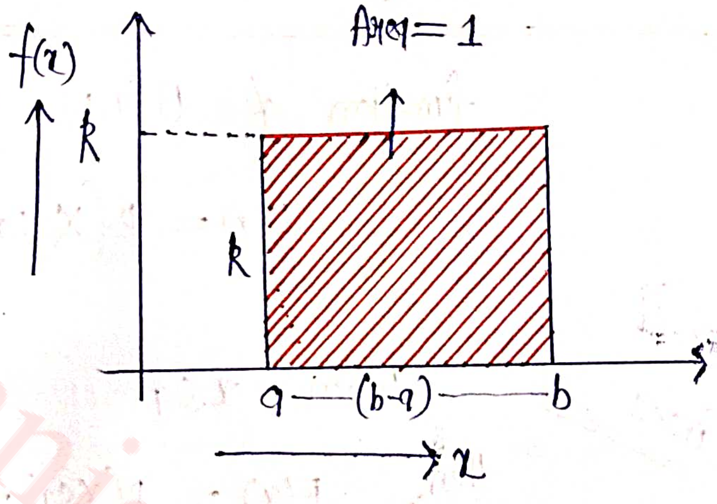
Uniform distribution with solved examples

A random variable X is said to uniform or rectangular distribution over the finite interval (a,b) if its probability density function (p.d.f.) is given by

$$f(x) = \begin{cases} k & ; \quad a < x < b \\ 0 & ; \quad \text{elsewhere.} \end{cases}$$

In order to find k, we draw the figure as:



$$\text{Area} = 1$$

$$\Rightarrow \quad k(b-a) = 1$$

$$\Rightarrow \quad \boxed{k = \frac{1}{b-a}}$$

or

Since $f(x)$ is a p.d.f., so

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{-\infty}^{a} \overset{0}{f(x)}\,dx + \int_{a}^{b} \overset{k}{f(x)}\,dx + \int_{b}^{\infty} \overset{0}{f(x)}\,dx = 1$$

$$\Rightarrow \int_{a}^{b} k\,dx = 1$$

$$\Rightarrow \quad k(b-a) = 1 \quad \Rightarrow \quad \boxed{k = \frac{1}{b-a}}$$

Hence, p.d.f. for uniform distribution is

$$f(x) = \begin{cases} \dfrac{1}{b-a} & ; \quad a < x < b \\ 0 & ; \quad \text{elsewhere.} \end{cases}$$

And we write it as $X \sim U(a, b)$.

Distribution function (c.d.f.) –: Let F be the distribution function of $U(a,b)$, i.e.,

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx$$

Case-I

When $x \leq a$

then $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{x} 0\,dx$.

$\boxed{F(x) = 0}$ when $x \leq a$

Case-II

When $a < x < b$

$$F(x) = \int_{-\infty}^{a} \overset{0}{f(x)}\,dx + \int_{a}^{x} \overset{1/(b-a)}{f(x)}\,dx$$

$\boxed{F(x) = \dfrac{x-a}{b-a}}$; when $a < x < b$

Case-III

when $x \geq b$

$$F(x) = \left[\int_{-\infty}^{a} f(x)\,dx + \int_{a}^{b} f(x)\,dx\right] + \int_{b}^{x} \overset{0}{f(x)}\,dx$$

$$= \left(\frac{x-a}{b-a}\right)_{at\ x=b} + 0 = 1$$

$\boxed{F(x) = 1}$ when $x \geq b$.

• Thus, required cummulative distribution function (c.d.f.) is given by

$$F(x) = \begin{cases} 0 & ; \text{ if } x \leq a \\ \frac{x-a}{b-a} & ; \text{ if } a < x < b \\ 1 & ; \text{ if } x \geq b \end{cases}$$

Question:- If a random variable X has the p.d.f.

$$f(x) = \begin{cases} 1/4 & ; |x| < 2 \\ 0 & ; o/w \end{cases}$$

Find (i) $P(X < 1)$ (ii) $P(|X| > 1)$

(iii) $P(2X + 3 > 5)$.

Solution:- As the p.d.f. is given as

$$f(x) = \begin{cases} 1/4 & ; -2 < x < 2 \\ 0 & ; o/w \end{cases}$$

this clearly a uniform distribution.

Thus c.d.f. is given by

$$F(x) = P(X \leq x) = \begin{cases} 0 & ; x \leq -2 \\ \frac{x+2}{4} & ; -2 < x < 2 \\ 1 & ; x \geq 2. \end{cases}$$

In order to find the required probabilities, we can follow either p.d.f. or c.d.f. approach.

<u>By PDF approach :-</u>

As the p.d.f. is given as

$$f(x) = \begin{cases} 1/4 & ; \ -2 < x < 2 \\ 0 & ; \ o/w \end{cases}$$

$$P(X<1) = \int_{-\infty}^{1} f(x)\,dx$$

$$= \int_{-\infty}^{-2} \cancel{f(x)}^{\,0}\,dx + \int_{-2}^{1} f(x)^{\,1/4}\,dx$$

$$= \frac{1}{4}(1+2) = 3/4$$

$$\boxed{P(X<1) = 3/4}$$

<u>By CDF approach :-</u> As

$$F(x) = P(X \le x) = \begin{cases} 0 & ; \ x \le -2 \\ \frac{x+2}{4} & ; \ -2 < x < 2 \\ 1 & ; \ x \geq 2 \end{cases}$$

$$P(X<1) = P(X \le 1) = F(1)$$

$$= \left[\frac{x+2}{4}\right]_{\text{at } x=1}$$

$$\boxed{P(X<1) = 3/4}$$

Ans

(ii) To find $P(|X|>1)$

By PDF

$$P(|X|>1) = 1 - P(|X| \le 1)$$

$$= 1 - P(-1 \le X \le 1)$$

$$= 1 - \int_{-1}^{1} f(x)\,dx$$

$$= 1 - \int_{-1}^{1} \frac{1}{4}\,dx$$

$$= 1 - \frac{1}{2} = \frac{1}{2}$$

$$\boxed{P(|X|>1) = \frac{1}{2}}$$

By CDF :- $\because$ Rule $P(c<X<d) = F(d) - F(c)$

$$P(|X|>1) = 1 - P(|X| \le 1)$$

$$= 1 - P(-1 \le X \le 1)$$

$$= 1 - [F(1) - F(-1)]$$

$$= 1 - \left[\left.\frac{x+2}{4}\right|_{x=1} - \left.\frac{x+2}{4}\right|_{x=-1} \right]$$

$$= 1 - \left[\frac{3}{4} - \frac{1}{4}\right] = 1 - \frac{1}{2}$$

$$\boxed{P(|X|>1) = \frac{1}{2}}$$

Ans

(iii) $P(2X+3>5) = P(2X>2) = P(X>1)$

By PDF approach –:

$$P(X>1) = \int_1^\infty f(x)\,dx \qquad \text{as} \quad f(x) = \begin{cases} 1/4 ; & -2<x<2 \\ 0 ; & o/w \end{cases}$$

$$= \int_1^2 f(x)dx + \int_2^\infty f(x)\,dx$$

$$= \int_1^2 \frac{1}{4}dx + \int_2^\infty 0\,dx$$

$$= \frac{1}{4}(2-1) = \frac{1}{4}$$

$$\boxed{P(X>1) = 1/4}$$

By CDF approach –:

$$P(2X+3>5) = P(X>1) \qquad \text{As} \quad F(x) = P(X\le x) = \begin{cases} 0 & x<-2 \\ \frac{x+2}{4} & -2<x \end{cases}$$

$$= 1 - P(X \le 1)$$

$$= 1 - F(1)$$

$$= 1 - 3/4 = 1/4$$

$$\boxed{P(2X+3>5) = 1/4}$$

Ans

Mean and varience of uniform distribution :

$$\text{Mean} = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx \qquad f(x) = \begin{cases} \frac{1}{b-a} & a<x<b \\ 0 & o/w \end{cases}$$

$$= \int_a^b x \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$$

$$= \frac{1}{b-a}\left[\frac{b^2-a^2}{2}\right]$$

$$= \frac{a+b}{2}$$

$$\boxed{\text{Mean} = E(X) = \frac{a+b}{2}}$$

Varience : The varience is given by

$$Var(X) = E(X^2) - (E(X))^2$$

Now, $$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \int_a^b x^2 \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a}\left(\frac{x^3}{3}\right)_a^b$$

$$= \frac{(b^3-a^3)}{3(b-a)}$$

$$= \frac{\cancel{(b-a)}(b^2+ab+a^2)}{3\cancel{(b-a)}}$$

$$\boxed{E(x^2) = \frac{b^2+ab+a^2}{3}}$$

$$Var(x) = E(x^2) - (E(x))^2 = \frac{b^2+ab+a^2}{3} - \frac{(a+b)^2}{4}$$

$$= \frac{4(b^2+ab+a^2) - 3(a^2+2ba+b^2)}{12}$$

$$= \frac{a^2-2ab+b^2}{12}$$

$$= \frac{(a-b)^2}{12}$$

$$\boxed{Var(x) = \frac{(a-b)^2}{12}}$$

Moment generating function of uniform distribution —:

The moment generating function is given by

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_a^b e^{tx} \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a}\left(\frac{e^{tx}}{t}\right)_a^b$$

$$= \frac{1}{b-a}\left(\frac{e^{tb}-e^{ta}}{t}\right)$$

$$\boxed{M_X(t) = \frac{e^{bt}-e^{at}}{t(b-a)}}$$

Ans

Mean and varience ~~for~~ by moment generating function -:

For rectangular distribution :

$$M_X(t) = \frac{e^{bt}-e^{at}}{t(b-a)}$$

We know that $E(X^r) = \frac{d^r}{dt^r}\left(M_X(t)\right)\Big|_{t=0}$

$$\text{Mean} = E(X) = \frac{d}{dt}\left[\frac{e^{bt}-e^{at}}{t(b-a)}\right]\Bigg|_{t=0}$$

$$= \frac{1}{b-a}\left[\frac{t(be^{bt}-ae^{at})-(e^{bt}-e^{at})}{t^2}\right]_{t=0}$$

$$= \frac{1}{b-a}\left[\frac{\cancel{(be^{bt}-ae^{at})}+t(b^2e^{bt}-a^2e^{at}) \cancel{-(be^{bt}-ae^{at})}}{2t}\right]_{t=0}$$

$$= \frac{1}{b-a}\left[\frac{\cancel{t}(b^2e^{bt}-a^2e^{at})}{2\cancel{t}}\right]_{t=0}.$$

$$= \frac{1}{b-a}\left(\frac{b^2-a^2}{2}\right) = \frac{a+b}{2}$$

$$\Rightarrow \boxed{\text{Mean} = E(X) = \frac{a+b}{2}}$$ #

Varience—:

We know that

$$Var(X) = E(X^2) - (E(X))^2$$

$$E(X^2) = \frac{d^2}{dt^2}\left(M_X(t)\right)\Big|_{t=0}$$

$$= \frac{1}{(b-a)}\frac{d}{dt}\left[\frac{t(be^{bt}-ae^{at})-(e^{bt}-e^{at})}{t^2}\right]_{t=0}$$

$$= \frac{1}{b-a}\;\frac{t^2\left[t(b^2e^{bt}-a^2e^{at})+\cancel{(be^{bt}-ae^{at})}-\cancel{(be^{bt}-ae^{at})}\right]-2t\left[t(be^{bt}-ae^{at})-(e^{bt}-e^{at})\right]}{t^4}\Bigg|_{t=0}$$

$$= \frac{1}{(b-a)}\;\frac{t^2(b^2e^{bt}-a^2e^{at})-2\left[t(be^{bt}-ae^{at})-(e^{bt}-e^{at})\right]}{t^3}\Bigg|_{t=0}$$

$$= \frac{2t(b^2e^{at}-a^2e^{at})+t^2(b^3e^{bt}-a^3e^{at})-2\left[\cancel{(be^{bt}-ae^{at})}+t(b^2e^{bt}-a^2e^{at})-\cancel{(be^{bt}-ae^{at})}\right]}{(b-a)\;3t^2}\Bigg|_{t=0}$$

$$= \frac{1}{(b-a)} \left. \frac{\cancel{2t(b^2e^{bt}-a^2e^{at})} + t^2(b^3e^{bt}-a^3e^{at}) \cancel{-2t(b^2e^{bt}-a^2e^{at})}}{3t^2} \right|_{t=0}$$

$$= \left. \frac{\cancel{t^2}(b^3e^{bt}-a^3e^{at})}{(b-a)\, 3\cancel{t^2}} \right|_{t=0}$$

$$= \frac{b^3-a^3}{3(b-a)}$$

$$= \frac{\cancel{(b-a)}(b^2+ab+a^2)}{3\cancel{(b-a)}}$$

$$\boxed{E(X^2) = \frac{b^2+ab+a^2}{3}}$$

$$Var(X) = \frac{b^2+ab+a^2}{3} - \frac{(a+b)^2}{4}$$

$$= \frac{4(b^2+ab+a^2) - 3(a+b)^2}{12}$$

$$= \frac{4b^2+4ab+4a^2-3a^2-6ab-3b^2}{12}$$

$$= \frac{a^2-2ab+b^2}{12} = \frac{(a-b)^2}{12}$$

$$\boxed{Var(X) = \frac{(a-b)^2}{12}}$$

Question:- If X is uniformly distributed with mean $=1$ and variance $=1/3$ then find $P(X<0)$.

Solution:- We know that for uniform distribution

$$E(X) = \text{mean} = \frac{a+b}{2} = 1$$

$$Var(X) = \frac{(a-b)^2}{12} = \frac{1}{3}$$

$$\Rightarrow \quad a+b = 2$$

$$(a-b)^2 = 16 \quad \Rightarrow \quad a-b = \pm 4$$

$a-b = 4$	$a-b = -4$
$a+b = 2$	$a+b = 2$
$\Downarrow$	$\Downarrow$
$a=3 \quad b=-1$	$a=-1 \quad \& \quad b=3$

Given

$$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & o/w \end{cases}$$

$$= \begin{cases} \frac{1}{3-(-1)} & -1 < x < 3 \\ 0 & o/w \end{cases}$$

$$f(x) = \begin{cases} 1/4 & -1 < x < 3 \\ 0 & o/w \end{cases}$$

Now

$$P(X<0) = \int_{-\infty}^{0} f(x)\,dx$$

$$= \int_{-\infty}^{-1} 0\,dx + \int_{-1}^{0} \frac{1}{4}\,dx$$

$$\boxed{P(X<0) = \frac{1}{4}}$$

Questions: The number of personal Computers (PC's) sold daily at alpha computer is uniformly distributed with a minimum of 2000 PC's and a maximum of 5000 PC's.

(i) Find the probability that the daily sales will fall between 2500 and 3000 PC's.

(ii) What is the probability that alpha Computer will sell at least 4000 PC's.

(iii) What is the probability that alpha Computer will exactly sell 2500 PC's.

Solution :- Let X be the number of PC's sold daily at Alpha Computer, then X follow uniform distribution over the $(2000, 5000)$.

Thus its p.d.f. is given by

$$f(x) = \begin{cases} \frac{1}{3000} & ; \quad 2000 < x < 5000 \\ 0 & ; \quad o/w \end{cases}$$

and its c.d.f. is given by

$$F(x) = P(X \leq x) = \begin{cases} 0 & ; \quad x \leq 2000 \\ \frac{x-2000}{3000} & ; \quad 2000 < x < 5000 \\ 1 & ; \quad x \geq 5000. \end{cases}$$

(i) In between 2500 to 3000, the required probability $= P(2500 < X < 3000)$

By PDF :-

$$= \int_{2500}^{3000} f(x)\,dx = \frac{1}{3000}\int_{2500}^{3000} dx$$

$$= \frac{500}{3000}$$

$$P(2500 < X < 3000) = \frac{1}{6} \quad \text{Ans}$$

By CDF approach :- $\because P(c<X<d) = F(d) - F(c)$

$$P(2500<X<3000) = F(3000) - F(2500)$$

$$= \left(\frac{3000-2000}{3000}\right) - \left(\frac{2500-2000}{3000}\right)$$

$$= \frac{500}{3000} = \frac{1}{6}$$

$$\boxed{P(2500<X<3000) = \frac{1}{6}}$$ Ans

(ii) At least 4000 PC's, the required probability

$$P(X \geq 4000) = \int_{4000}^{\infty} f(x)\,dx$$

By PDF :-

$$= \int_{4000}^{5000} \frac{1}{3000}\,dx + \int_{5000}^{\infty} 0\,dx$$

$$= \frac{1000}{3000} = \frac{1}{3}$$

$$\boxed{P(X \geq 4000) = \frac{1}{3}}$$

By CDF :- $P(X \geq 4000) = 1 - P(X < 4000)$

$$= 1 - P(X \leq 4000)$$

$$= 1 - F(4000)$$

$$= 1 - \frac{4000-2000}{3000} = 1 - \frac{2}{3}$$

$$\Rightarrow \boxed{P(X \geq 4000) = \frac{1}{3}}$$ Ans

(iii) The required probability

$$P(X = 2500) = 0.$$

Because X is a continuous probability distribution and in it, probability at a single point is always ZERO.

Moreover,

$$P(X = 2500) = P(2500 \leq X \leq 2500)$$

$$= \int_{2500}^{2500} f(x)\,dx = 0$$