Lecture-7

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Lecture 7: Numerical Analysis (UMA011)

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$f(x)=0$

from IVT, find the interval in which root of $f(x)=0$ lies

$f(0) = -ve$ $\quad [0,1]$

$f(1) = +ve$

$f(2)$

$\overset{-ve \quad -ve}{[0, 0.5]}$ or $\overset{-ve \quad +ve}{[0.5, 1]}$

$f(0.5)$

$x_1 = \frac{0+1}{2} = 0.5$

$x_2 = \frac{0.5+1}{2} = 0.75$

$x_1, x_2, x_3 \text{ ----- } x_n$

Root-finding problem

Bisection method: Stopping Criteria

x_1, x_2 $\quad |x_1 - x_2| \nless 10^{-2}$

x_3 $\quad |x_3 - x_2| \nless 10^{-2}$

x_4

$|x_n - x_{n-1}| < 10^{-2} = tol$

$|x_n - x_{n-1}| < tol$

Root-finding problem

Bisection method: Example

Show that $f(x) = x^3 + 2x^2 - 3x - 1 = 0$ has a root in $[1, 2]$ and use the bisection method to determine an approximation to the root i.e. is accurate to at least within 10^{-2}

Solution: $f(x) = x^3 + 2x^2 - 3x - 1$

Since $f(x)$ is a cubic poly; so it is cont. on $[1,2]$

$f(1) = 1 + 2 - 3 - 1 = -1 = -ve$

$f(2) = 8 + 8 - 6 - 1 = +ve$

from IVT, $\Rightarrow f(x)$ has a root in $[1,2]$

Apply B.M. on the interval $[1,2]$

$x_1 = \frac{1+2}{2} = 1.5$

Root lies in either $\overset{-ve \quad +ve}{[1, 1.5]}$ or $\overset{+ve \quad +ve}{[1.5, 2]}$

check the sign of $f(1.5) = (1.5)^3 + 2(1.5)^2 - 3(1.5) - 1 = +ve$

By IVT, the root lies in $[1, 1.5]$

Apply B.M.

$x_2 = \frac{1+1.5}{2} = 1.25$ $\qquad |x_1 - x_2| = |1.5 - 1.25| \nless 10^{-2}$

2+2

0.00

10^{-2}

Root lies in either $[1, 1.25]$ or $[1.25, 1.5]$

Check the sign of $f(1.25) = (1.25)^3 + 2(1.25)^2 - 3(1.25) - 1 = +ve$

By IVT, the root lies in $[1, 1.25]$

Table for B.M

n	a (−ve)	b (+ve)	x_n	$\lvert x_n - x_{n-1}\rvert < tol = 10^{-2}$	$f(x_n)$
1	1	2	$\frac{1+2}{2} = 1.5$		$f(1.5) = +ve$
2	1	1.5	$\frac{1+1.5}{2} = 1.25$	$\lvert 1.5 - 1.25\rvert \nless 10^{-2}$	$f(1.25) = +ve$
3	1	1.25	$\frac{1+1.25}{2} = 1.125$	$\lvert 1.25 - 1.125\rvert \nless 10^{-2}$	$f(1.125) = -ve$
4	1.125	1.25	$\frac{1.125+1.25}{2} = 1.1875$	$\lvert 1.125 - 1.1875\rvert \nless 10^{-2}$	$f(1.1875) = -ve$
5.	1.1875	1.25	1.21875 1.2188	$\lvert 1.1875 - 1.2188\rvert \nless 10^{-2}$	$f(1.2188) = +ve$
6.	1.1875	1.2188	1.203125	$\lvert - - \rvert \nless 10^{-2}$	$f(1.203125) = -ve$
7	1.203125	1.2188	1.2109375	$\lvert - - \rvert < 10^{-2}$	

1.21 ---

$x_7 = 1.2109375$ is the app. root.

$x_1, x_2, x_3, x_4 \text{ --- } x_n \rightarrow p$ (exact root)

$\frac{1}{n} \rightarrow 0 \qquad \frac{1}{n^2} \rightarrow 0 \qquad \frac{1}{n^3} \rightarrow 0 \qquad \frac{1}{n^{10}} \rightarrow 0$

Bisection method

space of continuous functions on $[a,b]$

$p_1, p_2 \text{ -- } p_n \rightarrow$ approximat

$p \rightarrow$ exact root

$|p_n - p|$

Maximum error bound

Suppose that $f \in C[a, b]$ and $f(a) * f(b) < 0$. The Bisection method generates a sequence $\{p_n\}_{n=1}^{\infty}$ approximating a zero p of f with $|p_n - p| \leq \frac{b-a}{2^n}$, when $n \geq 1$.

Proof: from IVT, Let $f(x)$ has a root in $[a, b]$

$a = a_1$, b_2, $b = b_1 = b_2$

length of the interval is $|b - a| = |b_1 - a_1|$

After applying bisection method and IVT, we get the root lie in $[a_2, b_2]$ (say)

length of the interval $[a_2, b_2] = |b_2 - a_2| = \frac{|b_1 - a_1|}{2^1}$

Again applying B.M and IVT, we get the root lie in $[a_3, b_3]$

length of the interval $[a_3, b_3] = |b_3 - a_3| = \frac{|b_1 - a_1|}{2^2}$

Continue the process up to n- iterations, we get the root lie in $[a_n, b_n]$

The length of the interval $[a_n, b_n] = |b_n - a_n| = \frac{|b_1 - a_1|}{2^{n-1}}$ — (*)

Now, $|p_n - p| = \left|\frac{a_n + b_n}{2} - p\right|$

$\leq \left|\frac{a_n + b_n}{2} - a_n\right|$

$= \left|\frac{a_n + b_n - 2a_n}{2}\right|$

$= \left|\frac{b_n - a_n}{2}\right| = \left|\frac{\frac{b_1 - a_1}{2^{n-1}}}{2}\right| = \frac{|b - a|}{2^n}$

$p \rightarrow$ exact root

$p \in [a_n, b_n]$

$\Downarrow$

$a_n \leq p \leq b_n$

$-p \leq a_n$

$\Rightarrow |p_n - p| \leq \frac{|b - a|}{2^n}, \quad n \geq 1$

from prev eg.

$|p_7 - p| \leq \frac{2-1}{2^7} \leq 10^{-2}$

$\frac{1}{2^7}$