Numerical Analysis

Solution of Some Exercises : Chapter 7
Initial-Value Problems for Ordinary Differential Equations

1. Show that each of the following initial-value problems (IVP) has a unique solution, and find the solution.

(a) $y' = y\cos t,\ 0 \le t \le 1,\ y(0) = 1.$

(b) $y' = \dfrac{2}{t}y + t^2e^t,\ 1 \le t \le 2,\ y(1) = 0.$

Sol.

(a) We must show that $f(t,y) = y\cos t$ is continuous and satisfies a Lipschitz condition in the variable y on $\{(t,y) \mid 0 \le t \le 1,\ -\infty < y < \infty\}$.
Clearly f is continuous and we have

$$\begin{aligned} |f(t,y_1) - f(t,y_2)| &= |y_1\cos t - y_2\cos t| \\ &= |\cos t|\,|y_1 - y_2| \\ &\le 1\cdot|y_1 - y_2| = |y_1 - y_2|. \end{aligned}$$

Thus f satisfies a Lipschitz condition with Lipschitz constant $L = 1$.
The solution to the equation is given by

$$\begin{aligned} \frac{dy}{dt} &= y\cos t \\ \int \frac{dy}{y} &= \int \cos t\,dt \\ \ln y &= \sin t + \ln c \\ y &= ce^{\sin t} \\ 1 &= ce^{0} \\ \Longrightarrow\ y &= e^{\sin t}. \end{aligned}$$

(b) Clearly $f(t,y) = \dfrac{2}{t}y + t^2e^t$ is continuous.
We have

$$\begin{aligned} |f(t,y_1) - f(t,y_2)| &= \left|\frac{2}{t}y_1 + t^2e^t - \frac{2}{t}y_2 - t^2e^t\right| \\ &= \frac{2}{t}\,|y_1 - y_2| \\ &\le 2|y_1 - y_2|. \end{aligned}$$

Thus f satisfies a Lipschitz condition with Lipschitz constant $L = 2$.
Now we solve the given IVP. The given equation is first order linear and has integrating factor $e^{\int -\frac{2}{t}dt} = \frac{1}{t^2}$. Its solution is given by

$$\begin{aligned} \ln y\frac{1}{t^2} &= \frac{1}{t^2}t^2e^t + c \\ \Longrightarrow\ y &= t^2(e^t + c). \\ 1 &= 1^2(e^1 + c) \qquad \Longrightarrow\ c = -e. \\ \therefore\ y &= t^2(e^t - e). \end{aligned}$$

2. Generate $y_0(t),\ y_1(t),\ y_2(t)$, and $y_3(t)$ for the initial-value problem using Picard's method.

$$y' = -y + t + 1, \quad 0 \le t \le 1,\ y(0) = 1.$$

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Sol.

$$\begin{aligned}
t_0 = 0, y_0(t) &= y(0) = 1. \\
y_1(t) &= 1 + \int_0^t f(s, y_0(s))ds \\
&= 1 + \int_0^t (-1 + s + 1)ds \\
&= 1 + \frac{t^2}{2}. \\
y_2(t) &= 1 + \int_0^t f(s, y_1(s))ds \\
&= 1 + \int_0^t \left[-\left(1 + \frac{s^2}{2}\right) + s + 1\right] ds \\
&= 1 + \int_0^t \left(s - \frac{1}{2}s^2\right) ds \\
&= 1 + \frac{1}{2}t^2 - \frac{1}{6}t^3.
\end{aligned}$$

$$\begin{aligned}
y_3(t) &= 1 + \int_0^t f(s, y_2(s))ds \\
&= 1 + \int_0^t \left[-\left(1 + \frac{1}{2}s^2 - \frac{1}{6}s^3\right) + s + 1\right] ds \\
&= 1 + \frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{1}{24}t^4.
\end{aligned}$$

We can check these approximations are Maclaurin series expansion of $t + e^{-t}$ which is the exact solution of given IVP.

3. Use Taylor's method of order two and four to approximate the solution for the following initial-value problem.

$$y' = y/t - (y/t)^2, \ 1 \le t \le 1.2, \ y(1) = 1, \ h = 0.1.$$

Sol. For the second order Taylor's method we have

$$y(t_1) = y(t_0) + hy'(t_0) + \frac{h^2}{2}y''(t_0).$$

Thus

$$\begin{aligned}
t_0 = 1, \qquad & y_0 = 1 \\
y'(0) &= 0/1 - (0/1)^2 = 0 \\
y''(t) &= y'/t - y/t^2 - 2(y/t)(y'/t) + 2y^2/t^3 \\
y''(0) &= 1. \\
y_1 = y(1.1) &= 1 + h \cdot 0 + \frac{0.1^2}{2} \cdot 1 = 1.005.
\end{aligned}$$

Similarly we can compute values at other points.

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4. Use Euler's method to approximate the solutions for each of the following initial-value problems.

(a) $y' = te^{3t} - 2y,\ 0 \le t \le 1,\ y(0) = 0,\ h = 0.5$
(b) $y' = 1 + (t-y)^2,\ 2 \le t \le 3,\ y(2) = 1,\ h = 0.5.$

Sol.

(a) Here $f(t,y) = te^{3t} - 2y, t_0 = 0, y_0 = 0, h = 0.5.$

$$\begin{aligned} y_1 = y(0.5) &= y_0 + hf(t_0, y_0) \\ &= 0 + 0.5[0 \cdot e^{3(0)} - 2(0)] = 0. \\ t_1 &= 0.5 \\ y_2 = y(1) &= y_1 + hf(t_1, y_1) \\ &= 0 + 0.5[(0.5)e^{3(0.5)-2(0)} = 1.1204223. \\ t_2 &= 1. \end{aligned}$$

(b) Here $f(t,y) = 1 + (t-y)^2, t_0 = 2, y_0 = 1, h = 0.5.$

$$\begin{aligned} y_1 = y(0.5) &= 1 + 0.5[1 + (2-1)^2] = 2 \\ t_1 &= 2.5 \\ y_2 = y(1) &= 2 + 0.5[1 + (2.5-2)^2] = 2.625 \\ t_2 &= 3. \end{aligned}$$

5. Show that the following initial-value problem has a unique solution.

$$y' = t^{-2}\,(\sin 2t - 2ty), \quad 1 \le t \le 2,\ y(1) = 2.$$

Find $y(1.1)$ and $y(1.2)$ with step-size $h = 0.1$ using modified Euler's method.
Sol.

$$y' = t^{-2}\,(\sin 2t - 2ty) = f(t,y).$$

Holding t as a constant,

$$\begin{aligned} |f(t,y_1) - f(t,y_2)| &= |t^{-2}\,(\sin 2t - 2ty_1) - t^{-2}\,(\sin 2t - 2ty_2)| \\ &= \frac{2}{|t|}|y_1 - y_2| \\ &\le 2|y_1 - y_2|. \end{aligned}$$

Thus f satisfies a Lipschitz condition in the variable y with Lipschitz constant $L = 2$. Additionally, $f(t,y)$ is continuous when $1 \le t \le 2$, and $-\infty < y < \infty$, so Existence Theorem implies that a unique solution exists to this initial-value problem.
Now we apply Modified Euler's method to find the solution.

$$\begin{aligned} t_0 &= 1, y_0 = 2, h = 0.1, t_1 = 1.1. \\ K_1 &= hf(t_0, y_0) = hf(1,2) = -0.309072 \\ K_2 &= hf(t_1, y_0 + K_1) = hf(1, 1.6909298) = -0.24062 \\ y_1 &= y(1.1) = y_0 + 1/2(K_1 + K_2) = 1.725152. \\ \text{Now } y_1 &= 1.725152, h = 0.1, t_2 = 1.2. \\ \therefore K_1 &= -0.24684 \\ K_2 &= -0.19947 \\ y_2 &= y(0.2) = 1.50199. \end{aligned}$$

6. Given the initial-value problem

$$y' = \frac{1}{t^2} - \frac{y}{t} - y^2, \quad 1 \le t \le 2,\ y(1) = -1,$$

with exact solution $y(t) = -\dfrac{1}{t}$.

(a) Use modified Euler's method with $h = 0.05$ to approximate the solution, and compare it with the actual values of y.

(b) Use the answers generated in part (a) and linear interpolation to approximate the following values of y, and compare them to the actual values.
i. y(1.052) ii. y(1.555) iii. y(1.978).

Sol.

(a) We take $f(t, y) = \dfrac{1}{t^2} - \dfrac{y}{t} - y^2$, $t_0 = 1$, $= 0.05$, $y_0 = -1$. We calculate error at each step by $E_i = |y_i - (-1/t)|$.

$$\begin{aligned} t_1 &= t_0 + h = 1.05 \\ k_1 &= 0.05 \\ k_2 &= 0.045465 \\ y_1 &= y_0 + (k_1 + k_2)/2 = -0.952268 \\ E_1 &= 0.047732. \end{aligned}$$

In a similar manner

$$\begin{aligned} t_2 &= 1.1,\ y_2 = -0.908879,\ E_2 = 0.043502 \\ t_3 &= 1.15,\ y_3 = -0.869265,\ E_3 = 0.039826 \\ t_4 &= 1.2,\ y_4 = -0.832955,\ E_4 = 0.036610 \\ t_5 &= 1.25,\ y_5 = -0.799550,\ E_5 = 0.033783 \\ t_6 &= 1.3,\ y_6 = -0.768716,\ E_6 = 0.031284 \\ t_7 &= 1.35,\ y_7 = -0.740165,\ E_7 = 0.029065 \\ t_8 &= 1.4,\ y_8 = -0.713654,\ E_8 = 0.027087 \\ t_9 &= 1.45,\ y_9 = -0.688971,\ E_9 = 0.025315 \\ t_{10} &= 1.5,\ y_{10} = -0.665932,\ E_{10} = 0.023723 \\ t_{11} &= 1.55,\ y_{11} = -0.644380,\ E_{11} = 0.022287 \\ t_{12} &= 1.6,\ y_{12} = -0.624173,\ E_{12} = 0.020989 \\ t_{13} &= 1.65,\ y_{13} = -0.605190,\ E_{13} = 0.019810 \\ t_{14} &= 1.7,\ y_{14} = -0.587322,\ E_{14} = 0.018738 \\ t_{15} &= 1.75,\ y_{15} = -0.570474,\ E_{15} = 0.017761 \\ t_{16} &= 1.8,\ y_{16} = -0.554562,\ E_{16} = 0.016867 \\ t_{17} &= 1.85,\ y_{17} = -0.539508,\ E_{17} = 0.016048 \\ t_{18} &= 1.9,\ y_{18} = -0.525245,\ E_{18} = 0.015295 \\ t_{19} &= 1.95,\ y_{19} = -0.511713,\ E_{19} = 0.014603 \\ t_{20} &= 2.0,\ y_{20} = -0.498856,\ E_{20} = 0.013965. \end{aligned}$$

(b) We use the answers generated in part (a) and linear Lagrange interpolation to approximate the values of y, and compare them to the actual values.
(i) For $t = 1.052$ and using the points $(1.05, -0.952268)$, $(1.1, -0.908879)$, we have

$$y(1.052) = \frac{1.052 - 1.1}{1.05 - 1.1}(-0.952268) + \frac{1.052 - 1.05}{1.1 - 1.05}(-0.908879) = -0.95053.$$

$$E(1.052) = 0.000040342.$$

(ii) For $t = 1.555$ and using the points $(1.55, -0.644380)$, $(1.6, -0.624173)$, we have

$$y(1.555) = \frac{1.555 - 1.6}{1.55 - 1.6}(-0.644380) + \frac{1.555 - 1.55}{1.6 - 1.55}(-0.624173) = -0.64236.$$

$$E(1.555) = 0.000726817.$$

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(iii) For $t = 1.978$ and using the points $(1.95, -0.511731)$, $(2, -0.498856)$, we have

$$y(1.978) = \frac{1.978 - 2}{1.95 - 2}(-0.511731) + \frac{1.978 - 1.95}{2 - 1.95}(-0.498856) = -0.50452.$$

$$E(1.978) = 0.001041173.$$

7. Use the modified Euler's method to approximate the solution to the following initial-value problem

$$y' = te^{3t} - 2y,\ 0 \le t \le 1,\ y(0) = 0,\ h = 0.5.$$

Sol. We have $f(t, y) = te^{3t} - 2y, t_0 = 0, y_0 = 0, h = 0.5$
Using the modified Euler method, we have:

$$\begin{aligned}
K_1 &= hf(t_0, y_0) = 0.5[0 \cdot e^{3(0)} - 2 \cdot 0] = 0 \\
K_2 &= hf(t_0 + h, y_0 + K_1) = hf(0.5, 0) \\
&= 0.5[0.5 \cdot e^{3(0.5)} - 2 \cdot 0] = 1.120422268 \\
y_1 = y(0.5) &= y_0 + \frac{1}{2}(K_1 + K_2) = 0.560211134 \\
t_1 &= 0.5 \\
K_1 &= hf(t_1, y_1) = hf(0.5, 0.560211134) \\
&= 0.5[0.5e^{1.5} - 2(0.560211134)] = 0.560211134 \\
K_2 &= hf(t_1 + h, y_1 + K_1) = hf(1, 1.120422268) \\
&= 0.5[1.0e^3 - 2(1.120422268)] = 7.80192392559 \\
y_2 = y(1) &= y_1 + \frac{1}{2}(K_1 + K_2) = 4.18106753.
\end{aligned}$$

8. A projectile of mass $m = 0.11$ kg shot vertically upward with initial velocity $v(0) = 8\ m/s$ is slowed due to the force of gravity, $F_g = -mg$, and due to air resistance, $F_r = -kv|v|$, where $g = 9.8\ m/s^2$ and $k = 0.002\ kg/m$. The differential equation for the velocity v is given by

$$mv' = -mg - kv|v|.$$

(a) Find the velocity after $0.1, 0.2, \cdots, 1.0$ s.

(b) To the nearest tenth of a second, determine when the projectile reaches its maximum height and begins falling.

Sol. Here $v' = -g - \frac{k}{m}v|v| = f(t, v), t_0 = 0, v_0 = 8.$

(a) We take $h = 0.1$ and apply modified Euler's method (you can apply any method).

$$\begin{array}{rl}
t_0 = 0 & v_0 = v(0) = 8 \\
t_1 = 0.1 & v_1 = v(0.1) = 6.91849 \\
t_2 = 0.2 & v_2 = v(0.2) = 5.86385 \\
t_3 = 0.3 & v_3 = v(0.3) = 4.83146 \\
t_4 = 0.4 & v_4 = v(0.4) = 3.81705 \\
t_5 = 0.5 & v_5 = v(0.5) = 2.81662 \\
t_6 = 0.6 & v_6 = v(0.6) = 1.82639 \\
t_7 = 0.7 & v_7 = v(0.7) = 0.84272 \\
t_8 = 0.8 & v_8 = v(0.8) = -0.13791 \\
t_9 = 0.9 & v_9 = v(0.9) = -1.11676 \\
t_{10} = 1 & v_{10} = v(1) = -2.09164.
\end{array}$$

(b) Thus the projectile reaches its maximum height and begins falling at 0.8s to the nearest tenth of a second.

9. Using Runge-Kutta fourth-order method to solve the IVP in $[0,1]$ for

$$\frac{dy}{dx} = y - x^2 + 1,\ y(0) = 0.5$$

with mesh length $h = 0.5$.

Sol. With $h = 0.5$. From $x = 0$ to $x = 2$ with step size $h = 0.5$, it takes 4 steps.

$$\begin{aligned}
x_0 &= 0,\ y_0 = 0.5\\
K_1 &= hf(x_0, y_0) = 0.5f(0, 0.5) = 0.75\\
K_2 &= hf(x_0 + h/2, y_0 + K_1/2) = 0.5f(0.25, 0.875) = 0.90625\\
K_3 &= hf(x_0 + h/2, y_0 + K_2/2) = 0.5f(0.25, 0.953125) = 0.9453125\\
K_4 &= hf(x_0 + h, y_0 + K_3) = 0.5f(0.5, 1.4453125) = 1.09765625\\
y_1 &= y_0 + (K_1 + 2K_2 + 2K_3 + K_4)/6 = 1.425130208333333.
\end{aligned}$$

$$\begin{aligned}
x_1 &= 0.5,\ y_1 = 1.425130208333333\\
K_1 &= hf(x_1, y_1) = 0.5f(0.5, 1.425130208333333) = 1.087565104166667\\
K_2 &= hf(x_1 + h/2, y_1 + K_1/2) = 0.5f(0.75, 1.968912760416667) = 1.203206380208333\\
K_3 &= hf(x_1 + h/2, y_1 + K_2/2) = 0.5f(0.75, 2.0267333984375) = 1.23211669921875\\
K_4 &= hf(x_1 + h, y_1 + K_3) = 0.5f(1, 2.657246907552083) = 1.328623453776042\\
y_2 &= y_1 + (K_1 + 2K_2 + 2K_3 + K_4)/6 = 2.639602661132812.
\end{aligned}$$

10. Water flows from an inverted conical tank with circular orifice at the rate

$$\frac{dx}{dt} = -0.6\pi r^2\sqrt{2g}\frac{\sqrt{x}}{A(x)},$$

where r is the radius of the orifice, x is the height of the liquid level from the vertex of the cone, and $A(x)$ is the area of the cross section of the tank x units above the orifice. Suppose $r = 0.1$ ft, $g = 32.1$ ft$/s^2$, and the tank has an initial water level of 8 ft and initial volume of $512(\pi/3)$ ft^3 . Use the Runge-Kutta method of order four to find the following.

(a) The water level after 10 min with $h = 20$ s.

(b) When the tank will be empty, to within 1 min.

Sol. At $t = 0$, water level is $h = 8$ ft. Given initial volume is $512(\pi/3)$ ft^3. Thus

$$V = \frac{\pi R^2 h}{3} = \frac{512\pi}{3}$$

which gives $R = 8$ ft, where is the radius of initial level cross section.
Now two triangles OAB and OCD are similar, so we have

$$\frac{R}{r_1} = \frac{8}{x} \implies r_1 = x.$$

Now area of cross section is $A(x) = \pi r_1^2 = \pi x^2$. Thus we have

$$\frac{dx}{dt} = -0.6\pi r^2\sqrt{2g}\frac{\sqrt{x}}{\pi x^2} = -\frac{0.0480749415}{x^{3/2}}.$$

(a) To find the water level after 10 min with $h = 20$ s, we apply Runge-Kutta method of order four with $h = 20$, $x_0 = 8$. Using MATLAB code, we get $x = 6.52863210414819$ at time $t = 600$ sec=10 min.

(b) For this part, we take $h = 60$, that is 1 min, we get $x \approx 0$ in 1500 sec, that is 25 min.

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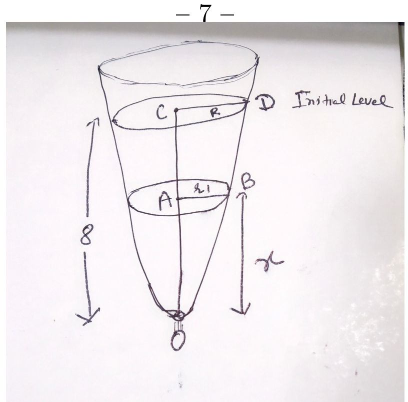

11. The following system represent a much simplified model of nerve cells

$$\begin{aligned} \frac{dx}{dt} &= x + y - x^3, \; x(0) = 0.5 \\ \frac{dy}{dt} &= -\frac{x}{2}, \; y(0) = 0.1 \end{aligned}$$

where $x(t)$ represents voltage across the boundary of nerve cell and $y(t)$ is the permeability of the cell wall at time t. Solve this system using Runge-Kutta fourth-order method to generate the profile up to $t = 0.2$ with step size 0.2.

Sol. We have

$$\begin{aligned} \frac{dx}{dt} &= x + y - x^3 = f(t, x, y) \\ \frac{dv}{dt} &= -\frac{x}{2} = g(t, x, y), \\ t_0 &= 0, \; x_0 = 0.5, \; y_0 = 0.1, \; h = 0.2, \; t_1 = t_0 + h = 0.2. \end{aligned}$$

In order to apply Runge-Kutta method of order four, we calculate the following values:

$$\begin{aligned} K_1 &= hf(t_0, x_0, y_0) = 0.2f(0, 0.5, 0.1) = 0.095 \\ L_1 &= hg(t_0, x_0, y_0) = 0.2g(0, 0.5, 0.1) = -0.05 \\ K_2 &= hf\left(t_0 + \frac{h}{2}, x_0 + \frac{K_1}{2}, y_0 + \frac{L_1}{2}\right) = 0.2f(0.1, 0.5475, 0.075) = 0.091677 \\ L_2 &= hg\left(t_0 + \frac{h}{2}, x_0 + \frac{K_1}{2}, y_0 + \frac{L_1}{2}\right) = 0.2g(0.1, 0.5475, 0.075) = -0.054750 \\ K_3 &= hf\left(t_0 + \frac{h}{2}, x_0 + \frac{K_2}{2}, y_0 + \frac{L_2}{2}\right) = 0.2f(0.1, 0.5458385, 0.072625) = 0.091167 \\ L_3 &= hg\left(t_0 + \frac{h}{2}, x_0 + \frac{K_2}{2}, y_0 + \frac{L_2}{2}\right) = 0.2g(0.1, 0.5458385, 0.072625) = -0.054584 \\ K_4 &= hf(t_0 + h, x_0 + K_3, y_0 + L_3) = 0.2f(0.2, 0.591167, 0.045416) = 0.085997 \\ L_4 &= hg(x_0 + h, x_0 + K_3, y_0 + L_3) = 0.2g(0.2, 0.591167, 0.045) = -0.059117. \end{aligned}$$

Therefore

$$\begin{aligned} x(0.2) \approx x_1 &= x_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 0.591114167. \\ y(0.2) \approx y_1 &= y_0 + \frac{1}{6}(L_1 + 2L_2 + 2L_3 + L_4) = 0.045369167. \end{aligned}$$

12. The motion of a swinging pendulum is described by the following second-order differential equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0, \quad \theta(0) = \frac{\pi}{6}, \quad \theta'(0) = 0,$$

where θ be the angle with vertical at time t, length of the pendulum $L = 2$ ft, and $g = 32.17$ ft/s^2. With $h = 0.1$ s, find the angle θ at $t = 0.1$ using Runge-Kutta fourth order method.

Sol. First of all we convert the given second order initial value problem into simultaneous first order initial value problems.

Assuming $\dfrac{d\theta}{dt} = y$, we obtain the following system:

$$\begin{aligned}
\frac{d\theta}{dt} &= y = f(t,\theta,y), \;\; \theta(0) = \pi/6 \\
\frac{dy}{dt} &= -\frac{g}{L}\sin\theta = g(t,\theta,y), \;\; y(0) = 0.
\end{aligned}$$

Here $t_0 = 0$, $\theta_0 = \pi/6$, and $y_0 = 0$. We have, by Runge-Kutta fourth order method, taking $h = 0.1$.

$$\begin{aligned}
K_1 &= hf(t_0,\theta_0,y_0) = 0.00000000 \\
L_1 &= hg(t_0,\theta_0,y_0) = -0.80425000 \\
K_2 &= hf(t_0+0.5h,\theta_0+0.5K_1,y_0+0.5L_1) = -0.04021250 \\
L_2 &= hg(t_0+0.5h,\theta_0+0.5K_1,y_0+0.5L_1) = -0.80425000 \\
K_3 &= hf(t_0+0.5h,\theta_0+0.5K_2,y_0+0.5L_2) = -0.04021250 \\
L_3 &= hg(t_0+0.5h,\theta_0+0.5K_2,y_0+0.5L_2) = -0.77608129 \\
K_4 &= hf(t_0+h,\theta_0+K_3,y_0+L_3) = -0.07760813 \\
L_4 &= hg(t_0+h,\theta_0+K_3,y_0+L_3) = -0.74759884. \\
\theta_1 &= \theta_0 + \frac{(K_1+2K_2+2K_3+K_4)}{6} = 0.48385575.
\end{aligned}$$

Therefore, $\theta(0.1) \approx \theta_1 = 0.48385575$.

13. Use Runge-Kutta method of order four to solve

$$t^2y'' - 2ty' + 2y = t^3\ln t, \;\; 1 \le t \le 1.2, \;\; y(1) = 1, \;\; y'(1) = 0$$

with $h = 0.1$.

Sol. Solving this higher-order differential equation is equivalent to solving the following system of differential equations.

$$\begin{aligned}
y' &= u = f(t,y,u) \\
t^2u' - 2tu + 2y &= t^3\ln t \\
\implies u' &= \frac{2}{t} - \frac{2}{t^2} + t\ln t = g(t,y,u) \\
y(1) &= 1, u(1) = 0.
\end{aligned}$$

Applying Runge-Kutta to this system we get the following.

$$\begin{aligned}
t_0 &= 1 \\
y_0 &= 1 \\
u_0 &= 0 \\
y_1 &= y(0.1) = 0.99017818 \\
u_1 &= u(0.1) = -0.19451307 \\
t_1 &= 1.1 \\
y_2 &= y(1.2) = 0.96152437 \\
u_2 &= u(1.2) = -0.37618726 \\
t_2 &= 1.2.
\end{aligned}$$