Example

If X and Y are two random independent variables with means 5 and 10 and standard deviation 2 and 3 respectively. Find the covariance between $3X+4Y$ and $3X-Y$?

Sol^n

Let $U = 3X+4Y$, $V = 3X-Y$

Goal

$Cov(U,V) = E(UV) - E(U)E(V)$

$Cov(U,V) = E((U-E(U))(V-E(V)))$

$E(U) = 3E(X) + 4E(Y)$
$= 15 + 40$
$= 55$

$E(V) = 15 - 10 = 5$

$E(UV) = E(9X^2 + 9XY - 4Y^2)$
$= 9E(X^2) + 9E(XY) - 4E(Y^2)$
$= 9 \cdot 29 + 9 \cdot 5 \cdot 10 - 4 \cdot 109$
$= 261 + 450 - 436 = 275$

$var(X) = E(X^2) - E(X)^2$
$\Rightarrow 4 = E(X^2) - 25$
$\Rightarrow E(X^2) = 29$

$var(Y) = E(Y^2) - E(Y)^2$

$cov(U,V) = 275 - 55 \cdot 5$

$= 2.75 - 2.75$ $\Rightarrow 9 = E(Y^2) - 100$

$= 0$ $E(Y^2) = 109$

correllation $\rho_{UV} = \dfrac{\mathrm{cov}(U,V)}{\sigma_U \sigma_V} = 0$

So U, and V are uncorrelated

Def if $\rho_{UV} > 0$ then U and V are said to be positively correlated

similarly, $\rho_{UV} < 0$ then they are said to be negatively correlated.

$$\boxed{-1 \le \rho_{UV} \le +1}$$

if $\rho_{UV} = 1 \Rightarrow$ U and V are linearly dependent

$|x \cdot y| \le \|x\| \|y\|$

Cauchy inequality

$|\mathrm{cov}(U,V)| \le \sigma_U \sigma_V$

equality holds if U and V are linearly dependent

Example X, Y, Z are uncorrelated random variables with zero means and standard deviations 5, 12 and 9 respectively. $U = X + Y$ and $V = Y + Z$, find the covariance between U and V

Soln

Given $\text{cov}(X, Y) = 0$; $\text{cov}(X, Z) = 0$, $\text{cov}(Y, Z) = 0$

$$E(X) = E(Y) = E(Z) = 0;$$

we compute $\text{var}(X) = 5^2 = 25$ $\quad \text{var}(Y) = 144$, $\text{var}(Z) = 8$

$\boxed{\text{var}(Y) = E(Y^2) - E(Y)^2}$

$$E(U) = 0 \quad , E(V) = 0$$

$$E(UV) = E((X+Y)(Y+Z))$$

$$= E(XY + Y^2 + XZ + YZ)$$

$$= E(XY) + E(Y^2) + E(XZ) + E(YZ)$$

$= E(XY) + E(Y\ldots) + E(X\ldots)$

$= \underbrace{E(X)E(Y)}_{0} + 144 + \underbrace{E(X)E(Z)}_{0} + \underbrace{E(Y)E(Z)}_{0}$

Since $\operatorname{cov}(X,Y)=0$

$= \underline{144}$

$\operatorname{cov}(U, V) = E(UV) - E(U)E(V)$

$= \underline{144}$