# Lecture-18

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## Lecture 18: Numerical Analysis (UMA011)

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### System of linear equations

**Example**

Apply Gaussian elimination to the system:

$$0.003000x_1 + 59.14x_2 = 59.17$$
$$5.291x_1 - 6.130x_2 = 46.78$$

using four-digit arithmetic with rounding, and compare the results to the exact solution $x_1 = 10.00$ and $x_2 = 1.000$.

**Solution:** $[A:b] = \begin{array}{c} E_1 \\ E_2 \end{array}\left[\begin{array}{cc|c} 0.003000 & 59.14 & 59.17 \\ 5.291 & -6.130 & 46.78 \end{array}\right]$ $a_{11} \neq 0$

$$\boxed{E_2 \to E_2 - \frac{5.291}{0.003000} E_1}$$

$E_2 \to E_2 - 1764 E_1$

$46.78 - 1764 \times 59.17$
$= 46.78 - 104375.88$
$= 46.78 - 1.044 \times 10^5$
$= 0.0004678 \times 10^5 - 1.044 \times 10^5$
$= -1.044 \times 10^5$

$\sim \begin{array}{c} E_1 \\ E_2 \end{array}\left[\begin{array}{cc|c} 0.003 & 59.14 & 59.17 \\ 0 & -1.043 \times 10^5 & -1.044 \times 10^5 \end{array}\right]$

$\downarrow$
Upper triangular Matrix.

$-6.130 - 1764 * 59.14$
$-6.130 - 104322.96$
$= -6.130 - 1.0432296 \times 10^5$
$= -0.00006130 \times 10^5 - 1.043 \times 10^5$
$= -1.043 \times 10^5$

Use backward substitution,

$-1.043 \times 10^5 x_2 = -1.044 \times 10^5$

$x_2 = 1.001$

$0.003x_1 + 59.14x_2 = 59.17$
$0.003x_1 + 59.14 * 1.001 = 59.17$
$0.003x_1 = 59.17 - 59.20$
$= -0.03$
$\Rightarrow x_1 = -\frac{0.03}{0.003} = -10$

$x_1 = -10$

$X = \begin{bmatrix} -10 \\ 1.001 \end{bmatrix}$



### System of linear equations

$E_j \to E_j - \frac{a_{ji}}{a_{ii}} E_i$

**Pivot element**

In the elimination process, we divide with diagonal element $a_{ii}$ at each stage and assume that $a_{ii} \neq 0$. These elements are known as pivot element.

**Pivot Strategies**

Partial Pivoting



### System of linear equations

**Partial Pivoting**

If at any stage of elimination, one of the pivot becomes small (or zero) then we bring other element as pivot by interchanging the rows. This process is called Gauss elimination with partial pivoting.

$A = \begin{array}{c} E_1 \\ E_2 \\ \vdots \\ E_n \end{array}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

$\max\{|a_{11}|, |a_{21}|, |a_{31}| \cdots |a_{n1}|\} = a_{p1}$

$E_1 \leftrightarrow E_p$

$A = \begin{array}{c} E_1 \\ E_2 \\ E_n \end{array}\begin{bmatrix} a_{p1}^* & a_{p2}^* & \cdots & a_{pn}^* \\ a_{21} & \cdots & a_{2n} \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$ $E_j \to E_j - \frac{a_{j1}}{a_{11}} E_1$ $j = 2, \ldots n$

$A \sim \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22}^* & \cdots & a_{2n}^* \\ \vdots & a_{32} & & \\ 0 & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ $\max\{|a_{22}|, |a_{32}| \cdots |a_{n2}|\} = a_{q2}$

$E_2 \leftrightarrow E_q$



### System of linear equations

**Example**

Apply Gaussian elimination to the system:

$$0.003000x_1 + 59.14x_2 = 59.17$$
$$5.291x_1 - 6.130x_2 = 46.78$$

using partial pivoting and four-digit arithmetic with rounding, and compare the results to the exact solution $x_1 = 10.00$ and $x_2 = 1.000$.

**Solution:** $[A:b] = \begin{array}{c} E_1 \\ E_2 \end{array}\left[\begin{array}{cc|c} 0.003 & 59.14 & 59.17 \\ 5.291 & -6.130 & 46.78 \end{array}\right]$

$\max\{|a_{11}|, |a_{21}|\} = \max\{|0.003|, |5.291|\}$
$= |5.291| = |a_{21}|$
$E_1 \leftrightarrow E_2$

$[A:b] = \begin{array}{c} E_1 \\ E_2 \end{array}\left[\begin{array}{cc|c} 5.291 & -6.130 & 46.78 \\ 0.003 & 59.14 & 59.17 \end{array}\right]$

$E_2 \to E_2 - \frac{0.003}{5.291} E_1$ $E_2 \to E_2 - 0.0005670 E_1$

$\sim \left[\begin{array}{cc|c} 5.291 & -6.130 & 46.78 \\ 0 & 59.14 & 59.14 \end{array}\right]$

$59.14 - 0.0005670 * (-6.130)$
$= 59.14 + 0.003476$
$= 59.14$

$\downarrow$
Upper triangular Matrix

$59.17 - 0.0005670 * 46.78$
$= 59.17 - 0.02652$
$= 59.14$

Use backward sub.

$x_2 = \frac{59.14}{59.14} = 1$

$5.291x_1 - 6.130x_2 = 46.78$
$5.291x_1 - 6.130(1) = 46.78$
$5.291x_1 = 46.78 + 6.130$
$\Rightarrow 5.291x_1 = 52.91$
$x_1 = 10$

$X = \begin{bmatrix} 10 \\ 1 \end{bmatrix}$ Ans



### System of linear equations

**Example:**

Using four-digit arithmetic operations, solve the following system of equations by Gaussian elimination with partial pivoting

$$0.729x_1 + 0.81x_2 + 0.9x_3 = 0.6867$$
$$x_1 + x_2 + x_3 = 0.8338$$
$$1.331x_1 + 1.21x_2 + 1.1x_3 = 1.000.$$

This system has exact solution, rounded to four places $x_1 = 0.2245$, $x_2 = 0.2814$, $x_3 = 0.3279$. Compare your answers!

After using P.P.

$[A:b] = \begin{array}{c} E_1 \\ E_2 \\ E_3 \end{array}\left[\begin{array}{ccc|c} 1.331 & 1.21 & 1.1 & 1.000 \\ 1 & 1 & 1 & 0.8338 \\ 0.729 & 0.81 & 0.9 & 0.6867 \end{array}\right]$ $E_2 \to E_2 - \frac{1}{1.331} E_1$

$E_3 \to E_3 - \frac{0.729}{1.331} E_1$

After using P.P. (for 2nd time)

$\sim \left[\begin{array}{ccc|c} 1.331 & 1.21 & 1.1 & 1.000 \\ 0 & 0.1473 & 0.2975 & 0.1390 \\ 0 & 0.0909 & 0.1736 & 0.0825 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1.331 & 1.21 & 1.1 & 1.000 \\ 0 & 0.1473 & 0.2975 & 0.1390 \\ 0 & 0 & -0.01 & -0.00328 \end{array}\right]$

$x_3 = 0.328$
$x_2 = 0.2812$
$x_1 = 0.2246$



### System of linear equations:

**Exercise:**

1. Use Gaussian elimination with partial pivoting and three-digit chopping arithmetics to solve the following linear system, and compare the approximations with the actual solution $[0, 10, 1/7]^T$.

$$3.03x_1 - 12.1x_2 + 14x_3 = -119$$
$$-3.03x_1 + 12.1x_2 - 7x_3 = 120$$
$$6.11x_1 - 14.2x_2 + 21x_3 = -139.$$



### System of linear equations

**LU Factorization:**



### System of linear equations

**Example:**

Determine the $LU$ factorization for matrix $A$ in the linear system $Ax = b$, where $A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \\ -3 \\ 4 \end{bmatrix}$. Then use the factorization to solve the system.



### System of linear equations

**Solution:**



### System of linear equations:

**Exercise:**

1. Modify the $LU$ Factorization Algorithm so that it can be used to solve a linear system, and then solve the following linear system:

$$2x_1 - x_2 + x_3 = -1$$
$$3x_1 + 3x_2 + 9x_3 = 0$$
$$3x_1 + 3x_2 + 5x_3 = 4.$$



### System of linear equations:

**Exercise:**

1. Use Gaussian elimination with scaled partial pivoting and three-digit chopping arithmetics to solve the following linear system, and compare the approximations with the actual solution $[0, 10, 1/7]^T$.

$$3.03x_1 - 12.1x_2 + 14x_3 = -119$$
$$-3.03x_1 + 12.1x_2 - 7x_3 = 120$$
$$6.11x_1 - 14.2x_2 + 21x_3 = -139.$$