# Lecture-15

Tuesday, August 30, 2022 10:25 PM

Lecture-15



## Lecture 15: Numerical Analysis (UMA011)

Dr. Meenu Rani

School of Mathematics
TIET, Patiala
Punjab-India



### Multiple roots

**Definition:**
An equation $f(x) = 0$ has a root $p$ with multiplicity $m$ if for $x \neq p$, we can write $f(x) = (x-p)^m q(x)$, $q(p) \neq 0$. $q(p)=0$

If $m = 1$, then equation $f(x) = 0$ has a simple root at $p$.

$f(x)=0$ $f(x) = (x-2)^2(x-3)$
$2, 2, 3$

$= (x-7)^2(x-5)$
$f(x) = (x^2-14x+49)(x-5)$
$7, 7, 5$

$f(x) = (x-p)^m$ (1)
$f(x) = (x-p)(x-p) \overset{m \text{ times}}{---} (x-p)\,(x-q)^3(x-r)^4$

$f(x) = (x-2)^3(x-5)$
$f(2) = 0$
$f'(x) = (x-2)^3 (1) + (x-5)\, 3(x-2)^2$
$f'(2) = 0$
$f''(x) = 3(x-2)^2 + 6(x-5)(x-2) + 3(x-2)^2$
$= 6(x-2)^2 + 6(x-5)(x-2)$
$f'''(x) = 12(x-2) + 6[(x-5)1 + (x-2)(1)]$
$f'''(2) \neq 0$

$f''(x) = 3(x-2)^2 + 3[(x-5)\,2(x-2) + (x-2)^2]$
$f''(2) = 0 + 3[0+0] = 0$



### Multiple roots

$f, f' \in C[a,b]$

**Result:**
The function $f \in C^1[a,b]$ has a simple zero at $p$ in $[a,b]$ iff $f(p) = 0$ but $f'(p) \neq 0$.

$f(x) = (x-2)^2(x-3) = 0$ $f(x) = (x-p)^1 q(x)$, $q(p) \neq 0$
$\Rightarrow$ 2 is multiple
3 is simple

$f, f', f'', f''' \cdots f^m \in C[a,b]$

**Generalized result:**
The function $f \in C^m[a,b]$ has a zero of multiplicity $m$ at $p$ in $[a,b]$ iff $f(p) = 0$, $f'(p) = 0, \cdots, f^{m-1}(p) = 0$, but $f^m(p) \neq 0$.

$f(x) = (x-p)^2(x-q)(x-r)$ — —
$f(p) = 0$
$\boxed{f'(p)=0}$ but $f''(p) \neq 0$



### Multiple roots

**Remarks:**
(i) The Newton's method (in which $f'(p) \neq 0$ is required) works for those functions which has a simple zero not for multiplicity.
(ii) Newton's method gives quadratically convergent sequence but quadratic convergence might not occur if the zero is not simple.



### Multiple roots

**Example:**
Let $f(x) = e^x - x - 1 = 0$
a) Show that $f$ has a zero of multiplicity 2 at $x = 0$. $m=2$
b) Show that Newton's method with $p_0 = 1$ converges to $x = 0$ but not quadratically.

a) $f(x) = e^x - x - 1$
$f(0) = e^0 - 0 - 1 = 1 - 1 = 0$
$f'(x) = e^x - 1$
$f'(0) = e^0 - 1 = 1 - 1 = 0$
$f''(x) = e^x$
$f''(0) = e^0 \neq 0$ $\Rightarrow$ $x=0$ is a multiple root of $f(x)=0$ with multiplicity $m=2$

b) Apply Newton's method

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}$$

$$p_{n+1} = p_n - \frac{e^{p_n} - p_n - 1}{e^{p_n} - 1}$$

Take $n = 0$, $p_0 = 1$

$$p_1 = p_0 - \frac{e^{p_0} - p_0 - 1}{e^{p_0} - 1} = 1 - \frac{e^1 - 2}{e - 1} = \frac{e-1-e+2}{e-1} = \frac{1}{e-1} = 0.68198$$

$$p_2 = p_1 - \frac{e^{p_1} - p_1 - 1}{e^{p_1} - 1} = 0.68198 - \frac{e^{0.68198} - 0.68198 - 1}{e^{0.68198} - 1} = 0.31906$$

$$p_3 = 0.31906 - \frac{e^{0.31906} - 0.31906 - 1}{e^{0.31906} - 1} = 0.16800$$

$p_4 = 0.08635$, $p_5 = 0.04380$, $p_6 = 0.02206$

To check the order of convergence (by defn)
$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} = \lambda$$

\# To check the linear order convergence ie $\alpha = 1$

Take $n=0$
Here $p=0$ $\left|\frac{p_1 - 0}{(p_0 - 0)^1}\right| = \left|\frac{0.68198}{1-0}\right| = \lambda < 1$

Take $n=1$ $\left|\frac{p_2 - 0}{p_1 - 0}\right| = \left|\frac{0.31906 - 0}{0.68198}\right| < 1$

Take $n=2$ $\left|\frac{p_3 - 0}{p_2 - 0}\right| = \left|\frac{0.16800}{0.31906}\right| < 1$

$\Rightarrow \left|\frac{p_{n+1} - 0}{(p_n - 0)^1}\right| < 1 \quad \forall n \geq 0$

$\Rightarrow \{p_n\} \to p$ linearly.

\# To check the quadratic convergence ie $\alpha = 2$

Take $n=0$
$p=0$ $\left|\frac{p_1 - 0}{(p_0 - 0)^2}\right| = \left|\frac{0.68198}{1}\right| < 1$

$n=1$ $\frac{|p_2 - 0|}{(p_1 - 0)^2} = \frac{|0.31906|}{(0.68198)^2} = 0.68601 < 1$

$n=2$ $\frac{|p_3 - 0|}{(p_2 - 0)^2} = \frac{|0.16800|}{(0.31906)^2} = 1.65031 > 1$

$n=3$ $\frac{|p_4 - 0|}{(p_3 - 0)^2} = \frac{0.08635}{(0.16800)^2} \geq 1$

$\Rightarrow \{p_n\} \to p$ linearly but not quadratically.



### Multiple roots:

**Exercise:**
1 (a) Apply the Newton's method with $x_0 = 0.8$ to the equation $f(x) = x^3 - x^2 - x + 1 = 0$, and verify that the convergence is only of first-order. Further show that root (b) $\alpha = 1$ has multiplicity 2.