Anti Symmetric Relation

A relation R on a set A is called an anti-symmetric relation if aRb & $bRa \Rightarrow a = b$

i.e If $(a,b) \in R$ + $(b,a) \in R \Rightarrow a = b$

OR

A relation R on a set A is called anti-symmetric if $a, b \in A$ $(a \neq b)$ * & $(a,b) \in R \Rightarrow (b,a) \notin R$.

Example : Let $A = \{1,2,3\}$

Then $R = \{(1,1)\,(1,2), (2,1)\}$ is a relation on the set A.

Since, $(1,2) \in R$ & $(2,1) \in R$, but $1 \neq 2$

$\therefore$ R is not anti-symmetric relation.

If $R_1 = \{(3,3)\}$ $\Rightarrow$ it is an anti-symmetric relation on A.

Equivalence relation :

A relation 'R' is an equivalent relation on set 'A' if + only if.

(i) R is reflexive(R) $\forall$ $a \in R$, $(a,a) \in R$

(ii) R is Symmetric(S) $\forall$ $(a,b) \in R$ then $(b,a) \in R$.

(iii) R is transitive(T) $\forall$ $(a,b) \in R$ + $(b,c) \in R$ then $(a,c) \in R$

$\{R, S, T\}$ statisfied

Consider the set $A = \{a, b, c\}$ ②

Let $R = \{\}$, $\overset{R_2 = b}{\{(a,a), (b,b), (c,c)\}}$, $R_3 = \{(a,a), (b,b), (c,c), (b,a)\}$

$R_4 = \{(a,a), (a,c), (b,a), (c,a)\}$

$R_5 = \{(a,a), (b,b), (c,c), (a,b), (a,c), (b,a), (c,a)\}$

Sol^n. $R_1 = \{\ \}$

R is Not Reflexive $a \in A$, But $(a,a) \notin R$.

Now Consider $R_2 = \{(a,a), (b,b), (c,c)\}$

Clearly Reflexive property hold
$a \in A$ + $(a,a) \in R$

|||ly Symmetric property hold
$\because (a, a)$
$\downarrow \downarrow$
$(a, b) \in R$ then $(b,a) \in R$

In case of transitive
$(a,b) \in R$, $(b,c) \in R \Rightarrow (a,c) \in R$
this is also applicable on R $\because$
No such elements exist in R which violates the transitivity.

$\therefore$ Given R_2 is R, S, T
$\therefore$ it is an equivalence relation.

$R_3 = \{(a,a), (b,b), (c,c), (b,a)\}$

$\because (a,a), (b,b), (c,c)$ the Given relation R_3 is reflexive.

But in case of $(b,a) \in R \Rightarrow (a,b) \notin R$ in that case symmetric property doesn't hold true

$\therefore$ Not Equivalence relation.

$R_4 = \{(a,a), (a,c), (b,a), (c,a)\}$

$(b,b), (c,c) \notin R$

Reflexive property not hold true.

$\therefore$ Not an Equivalence relation.

$R_5 = \{(a,a), (b,b), (c,c), (a,b), (a,c), (b,a), (c,a)\}$

$\because (a,a), (b,b), (c,c) \in R$. Reflexive property hold true.

$(a,b) \in R \Rightarrow (b,a) \in R$

$(a,c) \in R \Rightarrow (c,a) \in R$

||ly for transitivity it holds true

$\therefore$ R, S, T ✓ it is an Equivalence relation

$R_6 = A \times A \Rightarrow$ it consist all the combination of all the elements present in A.

Hence R, S, T statisfied.

$\therefore R_6$ is an Equivalence relation.

Partial order Relation (POR)

Def:- A relation 'R' on a set 'A' is said to be partial order if following property statisfied

(i) 'R' Reflexive property holds true.

(ii) 'R' is Antisymmetric $\forall (a,b) \in R$ + $(a,b) \in R$ $a,b \in A$ then $a = b$.

(iii) 'R' is transitive

$\{R, A, T\}$ ✓

Example Let $A = \{1, 2, 3\}$, $a=1, b=2, c=3$

$R = \{\}$, $R_1 = \{(1,1), (2,2), (3,3)\}$ ✓
(X) $\{R, A, T\}$ hold true.

Symmetric follows

$R_2 = \{(1,1), (2,2), (3,3), (1,2), (2,1)\}$ ∴ No Antisymm. X
$1 \neq 2,\ 2 \neq 1$

$R_3 = \{(1,1), (2,2), (3,3), (1,3), (2,3)\}$ $\{R, A, T\}$ ✓

$R_4 = \{(1,1), (1,2), (2,3), (1,3)\}$ X No Reflexive

$R_5 = A \times A.$

$\{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3),$
$(3,1), (3,2), (3,3)\}$

$(1,2) \in R$, $(2,1) \in R$

$\Rightarrow 2 = 1$ which is absurd

Not Antisymmetric

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Give an example of a relation which is anti-symmetric & transitive but neither reflexive nor symmetric.

Sol^n. Let $A = \{1,2,3\}$, $A \times A = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

Let $R = \{(1,2), (2,3), (1,3)\}$

(i) Reflexive property is not holds true on R.

$1,2,3 \in A \Rightarrow \because (1,1), (2,2), (3,3) \notin R$.

(ii) Symmetric property also not holds true on R.

$\because (1,2) \in R \Rightarrow (2,1) \notin R$.

(iii) Transitive property holds

$\because (1,2) \in R, (2,3) \in R$

$\Rightarrow (1,3) \in R$

(iv) Anti symmetric property $(1,2) \in R \Rightarrow 1 \neq 2$

$\Rightarrow (2,1) \notin R$ which holds true

Illy for other elements as well.

Give an example of relation which is transitive but neither reflexive nor symmetric nor anti-symmetric.

Solⁿ. Let $A = \{1, 2, 3\}$

$A \times A = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

Let $R = \{(1,1), (2,2), (1,2), (2,1), (1,3), (2,3)\}$

Clearly R is transitive

$(1,2) \in R$ & $(2,1) \in R$

$\Rightarrow (1,1) \in R$

Reflexive $(3,3) \notin R$ as $3 \in A$.

$\therefore$ Not Reflexive

Symmetric $(1,3) \in R \Rightarrow (3,1) \notin R$

$\therefore$ Not symmetric

Antisymmetric $(1,2) \in R$ & $(2,1) \in R$

$\Rightarrow 2 = 1$

But $1 \neq 2$

$\therefore$ Not Antisymmetric.

Give an example of relation which is both an equivalence relation + partial order relation. 7

Soln. Let $A = \{1, 2, 3\}$

Define R on A by $R = \{(1,1), (2,2), (3,3)\}$

clearly Reflexive property holds true.

Also $\underset{a,b}{(1,1)} \in R \Rightarrow \underset{b,a}{(1,1)} \in R$

$\therefore$ Symmetric hold true as well property

Check for Antisymmetry

$\underset{a,b}{(1,1)} \in R, \ \underset{b,a}{(1,1)} \in R \Rightarrow 1 = 1$

which is true

Also R is transitive.

$\therefore$ R is Equivalence as well as partial order relation.

H.W

Consider the following five relation defined on set $A = \{1, 2, 3\}$

check it for $\left\{ \begin{matrix} R, S, T \\ R, A, T \end{matrix} \right\}$

$R = \{(1,1), (1,2), (1,3), (3,3)\}$

$S = \{(1,1), (1,2), (2,1), (2,2), (3,3)\}$

$T = \{(1,1), (2,2), (1,2), (2,3)\}$

ϕ = Empty relation

$A \times A$ = Universal relation

Equivalence relation

Let $X = \{1, 2, 3, 4, 5, 6, 7\}$
& $R = \{(x,y) : (x-y) \text{ is divisible by } 3\}$.
then Prove that 'R' is an Equivalence relation.

Solⁿ.

Equivalence relation
- Reflexive
- Symmetric
- Transitive.

let us check one by one.
first of all we don't need to form a universal set A×A it contains 49 elements out of which (elements) follows the property posses by R is taken into account.
Therefore, instead of create or make these type of sets, we check all the properties using the condition's given on R.

for example $R = \{ \overbrace{(1,4),(4,1)}^{-3,\ 3}$ ✓
$\underbrace{(3,6),(6,3)}_{-3,\ 3}$ ✓
$(1,7),(7,1)$ ✓
$\underbrace{(4,7),(7,4)}$ ✓

1. Reflexive :- $\forall x \in R$, check $(x,x) \in R$.

Now condition is (x,y): $(x-y)$ is divisible by 3

$x-x = 0$ & '0' is divisible by 3.

check $\underbrace{(1,1), (2,2), (3,3), \ldots, (7,7)}_{\text{remainder } 0}$

Clearly, it is reflexive.

2. Symmetric: $\forall (x,y) \in R \Rightarrow (y,x) \in R$

$$x - y = 3n$$
$$-(-x+y) = 3(+n)$$
$$(-x+y) = 3(-n)$$
$$(y-x) = 3(-n) \longrightarrow \text{Any Integer.}$$
$$\Rightarrow (y,x) \in R \quad \text{Hence Symmetric}$$

3. Transitive : $\forall$ $(x,y) \in R$, $(y,z) \in R \Rightarrow (x,z) \in R$

$$x - \cancel{y} = 3n_1$$
$$\cancel{y} - z = 3n_2$$

Add

$$x - z = 3n_1 + 3n_2$$
$$x - z = 3(n_1 + n_2)$$
$$x - z = 3N \quad \text{where } N = n_1 + n_2 \in I$$

Hence Condition statisfied

$(x,z) \in R$

Hence Transitive.

Equivalence class, Asymmetric, Irreflexive relation, Inverse + complementary relations, Partition & covering of a set, N-ary relations + database, Representation of relation using matrices + diagraph, closure of relations Warshall algorithm.

Equivalence class:- Let 'R' be an equivalence relation on a non-Empty set X. Let $a \in X$, then the equivalence class of a, denoted by [a] is defined as follows

$$[a] = \{ b \in X : aRb \}$$

Example. Let $A = \{1,2,3\}$

$$R \subseteq A \times A = \{(1,1),(2,1),(1,2),(2,2),(3,3)\}$$

$\downarrow$

Key point this should be Equivalence relation first.

then, we are able to define class on above R.

$[1] = \{1,2\}$

$[2] = \{1,2\} \,||\, \{2,1\}$

$[3] = \{3\}$

Note:-

Any two equivalence classes are either identical or disjoint.

$[1] \cap [2] = \{1,2\}$ clearly identical

Also $A = [1] \cup [2] \cup [3]$

$[1] \cap [3] = \phi$ disjoint.

Another definition :- Equivalence class of a is denoted by [a]. (11)

$\therefore [a] = \{ b \mid b \in A \text{ and } (a,b) \in R \}$

Another Example : $A = \{1,2,3,4,5\}$

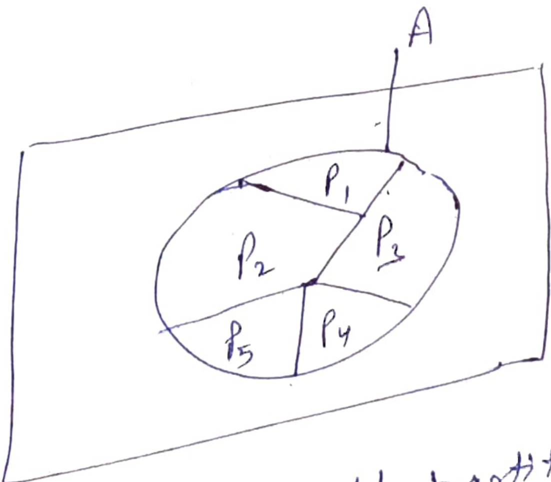
$\therefore R = \{ (1,1), (2,2), (3,3), (4,4), (5,5), (1,2), (2,1), (4,5), (5,4) \}$

$\left.\begin{array}{l} [1] = \{1,2\} \\ [2] = \{1,2\} \end{array}\right\}$ cluster 1

$\left.[3] = \{3\}\right\}$ cluster 2

$\left.\begin{array}{l} [4] = \{4,5\} \\ [5] = \{5,4\} \,\|\, \{4,5\} \end{array}\right\}$ cluster 3

Partition



Union of all partitions given by original set A.

Intersection b/w all partitions given as ϕ.

Now, if reverse is given to you.
i.e Partitions are given to you, & we have to find relation. R.

P_1	P_2	P_3
$\{1,2\}$	$\{3,4\}$	$\{5,6\}$
$\times$	$\times$	$\times$
$\{1,2\}$	$\{3,4\}$	$\{5,6\}$
$(1,1)(1,2),(2,1)(2,2)$	$(3,3),(3,4)$ $(4,3),(4,4)$	$(5,5),(5,6)$ $(6,5),(6,6)$

$$\therefore R = \{(1,1),(1,2),(2,1),(2,2),(3,3),(3,4),(4,3),(4,4),(5,5),(5,6),(6,5),(6,6)\}$$

Inverse & Complementary relations.

The inverse of a relation R, denoted by R^{-1}, is obtained from R by interchanging the first & second components of each ordered pair of R.

$$\therefore R^{-1} = \{(a,b) : (b,a) \in R\}$$

If R is a relation from a set A to set B, then R^{-1} is relation from set B to set A.

$\therefore$ Domain of R^{-1} = Range of R.
Range of R^{-1} = Domain of R.

Example: Let $A = \{1, 2, 3\}$

Let $R = \{(1,2), (1,3), (2,3), (3,2)\}$

$R^{-1} = \{(2,1), (3,1), (3,2), (2,3)\}$

Complement of a Relation ($\bar{R} \mid R^c$)

$\bar{R} = \{(a,b) \mid (a,b) \in A \times B \text{ \& } (a,b) \notin R\}$

$A \times B$: available in Cartesian product

$(a,b) \notin R$: but not available in R.

i.e $\boxed{\bar{R} = (A \times B) - R}$

for e.g $A = \{a, b\}$, $B = \{1, 2, 3\}$

$A \times B = \{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3)\}$

$R \subseteq A \times B = \{(a,2), (b,1), (b,3)\}$

$\therefore \bar{R} = (A \times B) - R = \{(a,1), (a,3), (b,2)\}$

NOTE $\boxed{R \cup \bar{R} = A \times B}$

$\boxed{R \cap \bar{R} = \phi}$

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Representation of Relation using matrices & Digraph.

Suppose, we have two sets
$A = \{1,2,3\}$, $B = \{1,2\}$

$R \subseteq A \times B = \{(a,b) \mid a \in A \& b \in B, a > b\}$

$= \{\underbrace{(2,1), (3,1), (3,2)}\}$

This Relation should be defined by matrices.

$[A] \xrightarrow{\text{elements}}$ become row, $[B] \xrightarrow{\text{elements}}$ become Column

i.e

$$\begin{array}{c|cc} & 1 & 2 \\ \hline 1 & & \\ 2 & & \\ 3 & & \end{array} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Matrix representation of given R.

Another Eg. $A = \{1,2,3,4\}$

$A \times A = \{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3)$
$(2,4),(3,1),(3,2),(3,3),(3,4),(4,1),$
$(4,2),(4,3),(4,4)\}$.

$R = \{(1,1), (2,2), (3,3), (4,4), (2,1) (1,2)$
$(3,2), (2,3), (3,1), (1,3), (4,4), (4,1)\}$

$$M_R = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 \\ 3 & 1 & 1 & 1 & 0 \\ 4 & 1 & 0 & 0 & 1 \end{array}$$

Now, In next example we will show how different properties of Relation (Reflexive, Symmetric, Antisymmetric, Asymmetric, transitive, Irreflexive) can be seen in matrix form.

Def:- Let $A = \{a_1, a_2, \ldots, a_m\}$ & $B = \{b_1, b_2, \ldots, b_n\}$ are finite sets containing m & n elements, respectively & Let 'R' be a relation defined from A to B. Then, 'R' can be represented by mn matrix.

$M_R = [M_{ij}]$, where $M_{ij} = \begin{cases} 1, & \text{If ordered pair } (a_i, b_j) \in R \text{ is given in R} \\ 0, & \text{If ordered pair is not given in R. } (a_i, b_j) \notin R. \end{cases}$

Reflexive property:- i.e $M_{ii} = 1$ i.e Diagonal componets are 1 ~~otherwise~~

If $A = \{1, 2, 3\}$

$$M_R = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 3 & 1 & 1 & 1 \end{array}$$

It should be verified from given R.

I reflexive property :- It means $M_{ii} = 0$
i.e all Diagonal elements are zero.

$$M_R = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Symmetric property :- It means $M_{ij} = M_{ji}$

$$\therefore M_R = \begin{array}{c|c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$

Check it in R
if M_{ij} have any value either 1 or 0.
then corresponding M_{ji} have same value i.e either 1 or 0.

$M_{1,2} = M_{2,1}$
$M_{1,3} = M_{3,1}$
$M_{2,3} = M_{3,2}$

(1, 2) (2, 1)
(1, 3), (3, 1)
(3, 2) (2, 3)

Antisymmetric property :- In this case if $M_{ij} = 1$, then its opposite pair i.e $M_{ji} = 0$ should not be there in R.

for e.g
$$M_R = \begin{array}{c|c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

Asymmetric property :- $M_{ii} = 0$, $M_{ij} = 1$, $M_{ji} = 0$ in R.

$$M_R = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

Transitive property :- $M_{ij} = 1$, $M_{jk} = 1 \Rightarrow M_{ik} = 1$

$$M_R = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

1-2
2-3
$\Rightarrow$ 1-3

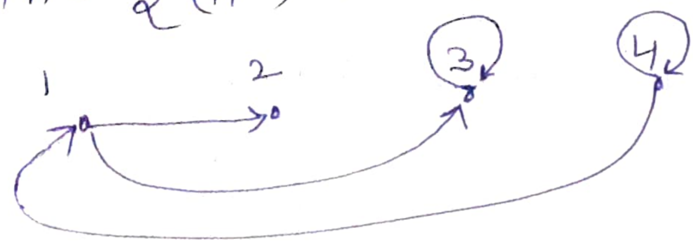
H.W = $A = \{1, 2, 3\}$
$B = \{1, 2, 3, 4, 5\}$

Construct 'R' which statisfies all the above properties (individual matrix).

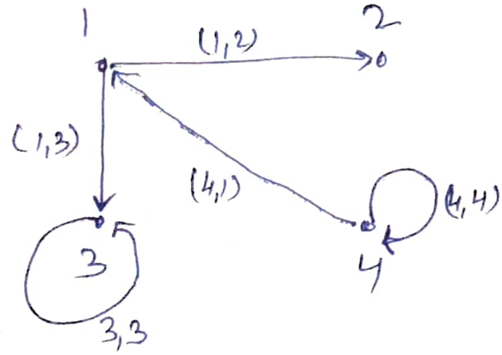
Representation of Relation using Digraph.

Consider set $A = \{1, 2, 3, 4\}$

$R \subseteq A \times A = \{(1,2)\ (1,3), (3,3), (4,1), (4,4)\}$



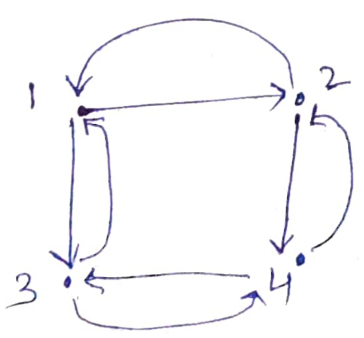
Diagraph.



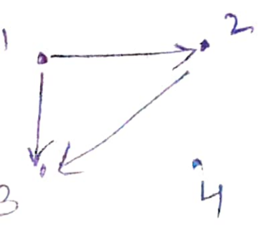
Here we can also check all the properties of Relations (Reflexive, Symmetric, transitive) using Digraph

Reflexive :- (1,1), (2,2), (3,3), . . .
i.e in diagraph, at each vertex we have a loop which defines the given relation is reflexive one.

Symmetric :- In this, if we have edge between the pair (a,b) then we also do have edge in the opposite direction between two adjacent vertex.



Transitive :- If we have edge between two vertex i.e (1,2) + (2,3) then this implies we also do have edge b/w (1,3)
i.e (a,b) has Edge (b,c) have edge
⟹ (a,c) have edge in diagraph



Partition & Covering of a set

Consider a set A, then the partition of a set A is a collection of non-empty subsets $A_1, A_2, A_3, \ldots, A_n$ of A called blocks, having following properties

(i) 'A' is the union of all the subsets

i.e $A_1 \cup A_2 \cup A_3 \ldots \cup A_n = A$.

or $\bigcup_{i=1}^{n} A_i = A$

(2) Subsets are pairwise disjoint i.e $A_i \cap A_j = \phi$ for $i \neq j$. Also $A_i \neq \phi \ \forall i$.

Example $S = \{1, 2, 3, 4, 5, 6\}$ and the subsets which are defined as follows are the valid partitions or not according to definition

$S_1 = \{1, 3, 5\}$, $S_2 = \{3, 5\}$, $S_3 = \{2, 4\}$,

$S_4 = \{1, 3\}$, $S_5 = \{2, 4, 6\}$, $S_6 = \{1, 2, 3\}$

$S_7 = \{4, 5\}$, $S_8 = \{6\}$.

$S_1, S_2, S_3, \ldots, S_8 \subseteq S$

$P_1 = \{S_6, S_7, S_8\}$, $P_2 = \{S_3, S_4, S_5\}$

Sol^n

$$\underbrace{S_6 = \{1, 2, 3\},\ S_7 = \{4, 5\},\ S_8 = \{6\}}_{\neq \phi}$$

$\therefore S_6, S_7, S_8 \neq 0$

Condition statisfied.

Also $S_6 \cap S_7 \cap S_8 = \phi$ This partition set $\{S_6, S_7, S_8\}$ are pairwise disjoint.

Checkit one ~~one~~ by one

$$[S_6, S_7] = \{S_6\} \cap \{S_7\}$$
$$= \phi$$

$$[S_7, S_8] = \{S_7\} \cap \{S_8\}$$
$$= \{4,5\} \cap \{6\} = \phi$$

Illy $[S_6, S_8] = \phi$.

(iii) $\{S_6, S_7, S_8\} = \{S_6\} \cup \{S_7\} \cup \{S_8\}$
$$= \{1,2,3\} \cup \{4,5\} \cup \{6\}$$
$$= \{1,2,3,4,5,6\}.$$

All conditions are statisfied.

∴ Required set $\{S_6, S_7, S_8\}$ are the valid partition.

Check it for $P_2 = \{S_3, S_4, S_5\}$

\# Consider the set $S = \{1,2,3\}$

$S_1 = \{1\}$, $S_2 = \{3\}$, $S_3 = \{1,2\}$, $S_4 = \{2,3\}$
$S_5 = \{1,3\}$

$$\boxed{S_1, S_2, S_3, \dots, S_5 \subseteq S}$$

(i) $P_1 = \{S_1, S_2, S_3\}$, (ii) $P_2 = \{S_2, S_5\}$