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Tuesday, July 26, 2022 5:16 PM

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Lecture 2: Numerical Analysis (UMA011)

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Error Analysis: Chopping and Rounding Errors

Maximum error bound in rounding

1. Let x be any real number we want to represent in a computer. Let $fl(x)$ be the representation by rounding with n-digits of x then the largest possible values of $\frac{|x-fl(x)|}{|x|}$ is $\frac{10^{1-n}}{2}$.

2. Let x be any real number we want to represent in a computer. Let $fl(x)$ be the representation by chopping with n-digits of x then the largest possible values of $\frac{|x-fl(x)|}{|x|}$ is 10^{1-n}.

Error Analysis: Maximum Rounding Errors

Proof:

Let $x = (0.a_1a_2 \cdots a_n a_{n+1} \cdots)_{10} \times 10^e$

$= \left(\sum_{i=1}^{\infty} \frac{a_i}{10^i}\right) \times 10^e$ $\quad 0 \le a_i \le 9$ or $0 \le a_i < 10$

$\left(\frac{a_1}{10} + \frac{a_2}{10^2} + \frac{a_3}{10^3} \cdots\right) \times 10^e$ $\quad 0 < a_1 \le 9$, $1 \le a_1 \le 9$, $a_i \in \mathbb{Z}$

Let us approximate x up to n-digits with rounding then

$$fl(x) = \begin{cases} (0.a_1a_2 \cdots a_n)_{10} \times 10^e; & a_{n+1} < 5 \\ \left((0.a_1a_2 \cdots a_n)_{10} + (0.00\cdots 1)_{10}\right) \times 10^e; & 5 \le a_{n+1} \le 9 \end{cases}$$

n^{th} place

0.635987

0.63599

$$= \begin{cases} \left(\sum_{i=1}^{n} \frac{a_i}{10^i}\right) \times 10^e; & a_{n+1} < 5 \\ \left(\sum_{i=1}^{n} \frac{a_i}{10^i} + \frac{1}{10^n}\right) \times 10^e; & 5 \le a_{n+1} \le 9 \end{cases}$$

A.E.

Case I $\quad a_{n+1} < 5$

$|x - fl(x)| = \left|\left(\sum_{i=n+1}^{\infty} \frac{a_i}{10^i}\right) \times 10^e\right|$

$a < b$
$a + c < b + c$

$= \left|\frac{a_{n+1}}{10^{n+1}} + \sum_{i=n+2}^{\infty} \frac{a_i}{10^i}\right| \times 10^e$

$x < 5$
$a + c < 5 + c$
$\le 4 + c$

$< \left|\frac{5}{10^{n+1}} + \sum_{i=n+2}^{\infty} \frac{a_i}{10^i}\right| \times 10^e$

$a_{n+1} < 5$
or
$a_{n+1} \le 4$

$\le \left|\frac{4}{10^{n+1}} + \sum_{i=n+2}^{\infty} \frac{9}{10^i}\right| \times 10^e$

$= \left|\frac{4}{10^{n+1}} + 9 \sum_{i=n+2}^{\infty} \frac{1}{10^i}\right| \times 10^e$

$\frac{a}{1-r}$

$= \left|\frac{4}{10^{n+1}} + 9 \frac{\frac{1}{10^{n+2}}}{1 - \frac{1}{10}}\right| \times 10^e$

$= \left|\frac{4}{10^{n+1}} + 9 * \frac{1}{10^{n+2}} \times \frac{10}{9}\right| \times 10^e$

$\frac{10}{2} * \frac{1}{10^{n+1}} * 10^e$

$= \left|\frac{4}{10^{n+1}} + \frac{10}{10^{n+2}}\right| \times 10^e = \left|\frac{5}{10^{n+1}}\right| \times 10^e$

$= \frac{1}{2} * 10^{e-n}$

Case II $\quad 5 \le a_{n+1} \le 9$

A.E. $= |x - fl(x)| = \left|\sum_{i=n+1}^{\infty} \frac{a_i}{10^i} - \frac{1}{10^n}\right| \times 10^e$

$|a-b| = |b-a|$

$= \left|\frac{1}{10^n} - \sum_{i=n+1}^{\infty} \frac{a_i}{10^i}\right| \times 10^e$

$5 \le a_{n+1}$
$-5 \ge -a_{n+1}$
$\Rightarrow -a_{n+1} \le -5$

$= \left|\frac{1}{10^n} - \frac{a_{n+1}}{10^{n+1}} - \sum_{i=n+2}^{\infty} \frac{a_i}{10^i}\right| \times 10^e$

$\le \left|\frac{1}{10^n} - \frac{5}{10^{n+1}} - \sum_{i=n+2}^{\infty} \frac{a_i}{10^i}\right| \times 10^e$

$|a-b-c| \le |a-b|$

$\le \left|\frac{1}{10^n} - \frac{5}{10^{n+1}}\right| \times 10^e$

$= \left|\frac{10-5}{10^{n+1}}\right| \times 10^e = \frac{5}{10^{n+1}} \times 10^e$

$= \frac{10^{e-n}}{2}$

R.E. $= \frac{|x - fl(x)|}{|x|} \le \frac{10^{e-n}}{2|x|}$

$|x| = |(0.a_1a_2 \cdots a_n a_{n+1} \cdots)_{10} \times 10^e|$

$\ge |(0.100 \cdots 0)_{10} \times 10^e|$

$= (0.1) \times 10^e = \frac{1}{10} \times 10^e$

$|x| \ge 10^{e-1} \Rightarrow \frac{1}{|x|} \le \frac{1}{10^{e-1}}$

R.E. $\frac{|x - fl(x)|}{|x|} \le \frac{10^{e-n}}{2} \left(\frac{1}{|x|}\right)$

$= \frac{10^{e-n}}{2} * \frac{1}{10^{e-1}}$

$= \frac{10^{1-n}}{2}$

$$\boxed{\text{R.E.} \le \frac{10^{1-n}}{2}}$$

Error Analysis: Significant figures

Significant figures

Significant figures of a no. are those digits that carry meaningful contributing to its precision.

for e.g. $\quad 3293 \quad 393$

$100\cancel{0}$ $\quad$ PPP1

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Rules :- 1) All the non-zeros digits of a no. are S.D.

e.g. $\quad 1000.334 \quad$ 7 S.D.

$1.349 \rightarrow$ 4 S.D.

2) zeros between non-zeros digits are S.D

$1000009 \quad$ 7 S.D.

3) Leading zeros to the left of the first non-zero digits are not S.D.

$000001234 \rightarrow$ 4 S.D.

4) Trailing zeros that are also to the right of a decimal pt. in a no. are S.D.

for e.g. $\quad 100000 \rightarrow$ 6 S.D.

100
$= 1 \times 10^2$

$1.00 \times 10^2 \quad$ 3 S.D

$1.0 \times 10^2 \quad$ 2 S.D

$1000 = 1 \times 10^3 \rightarrow$ 1 S.D $\quad 1 \times 10^2 \rightarrow$ 1 S.D.