School of Mathematics, Thapar Institute of Engineering & Technology, Patiala

UMA007 : Numerical Analysis
Assignment 2
Roots of Non-linear Equations-A

1. Use the bisection method to find solutions accurate to within 10^{-3} for the following problems.

(a) $x - 2^{-x} = 0$ for $0 \le x \le 1$.

(b) $e^x - x^2 + 3x - 2 = 0$ for $0 \le x \le 1$.

(c) $x + 1 - 2\sin(\pi x) = 0$. for $0 \le x \le 0.5$ and $0.5 \le x \le 1$.

2. Find an approximation to $\sqrt[3]{25}$ correct to within 10^{-3} using the bisection algorithm.

3. Find a bound for the number of iterations needed to achieve an approximation by bisection method with an accuracy 10^{-2} to the solution of $x^3 - x - 1 = 0$ lying in the interval $[1, 2]$. Find an approximation to the root with this degree of accuracy.

4. Sketch the graphs of $y = x$ and $y = 2\sin x$. Use the bisection method to find an approximation to within 10^{-3} to the first positive value of x with $x = 2\sin x$.

5. The function defined by $f(x) = \sin(\pi x)$ has zeros at every integer. Show that when $-1 < a < 0$ and $2 < b < 3$, the bisection method converges to

(a) 0, if $a + b < 2$

(b) 2, if $a + b > 2$

(c) 1, if $a + b = 2$.

6. For each of the following equations, use the given interval or determine an interval $[a, b]$ on which fixed-point iteration will converge. Estimate the number of iterations necessary to obtain approximations accurate to within 10^{-2}, and perform the calculations.

(a) $x = \dfrac{5}{x^2} + 2$.

(b) $2 + \sin x - x = 0$ in interval $[2, 3]$.

(c) $3x^2 - e^x = 0$.

7. Use the fixed-point iteration method to find smallest and second smallest positive roots of the equation $\tan x = 4x$, correct to 4 decimal places.

8. Show that $g(x) = \pi + 0.5\sin(x/2)$ has a unique fixed point on $[0, 2\pi]$. Use fixed-point iteration to find an approximation to the fixed point that is accurate to within 10^{-2}. Also estimate the number of iterations required to achieve 10^{-2} accuracy, and compare this theoretical estimate to the number actually needed.

9. Find all the zeros of $f(x) = x^2 + 10\cos x$ by using the fixed-point iteration method for an appropriate iteration function g. Find the zeros accurate to within 10^{-2}.

10. The iterates $x_{n+1} = 2 - (1 + c)x_n + cx_n^3$ will converge to $\alpha = 1$ for some values of constant c (provided that x_0 is sufficiently close to α). Find the values of c for which convergence occurs? For what values of c, if any, convergence is quadratic.

11. Let A be a given positive constant and $g(x) = 2x - Ax^2$.

(a) Show that if fixed-point iteration converges to a nonzero limit, then the limit is $\alpha = 1/A$, so the inverse of a number can be found using only multiplications and subtractions.

(b) Find an interval about $1/A$ for which fixed-point iteration converges, provided x_0 is in that interval.

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12. Consider the root-finding problem $f(x) = 0$ with root α, with $f'(x) \neq 0$. Convert it to the fixed-point problem

$$x = x + cf(x) = g(x)$$

with c a nonzero constant. How should c be chosen to ensure rapid convergence of

$$x_{n+1} = x_n + cf(x_n)$$

to α (provided that x_0 is chosen sufficiently close to α)? Apply your way of choosing c to the root-finding problem $x^3 - 5 = 0$.

13. Show that if A is any positive number, then the sequence defined by

$$x_n = \frac{1}{2}x_{n-1} + \frac{A}{2x_{n-1}}, \quad \text{for } n \geq 1,$$

converges to $\sqrt{A}$ whenever $x_0 > 0$. What happens if $x_0 < 0$?

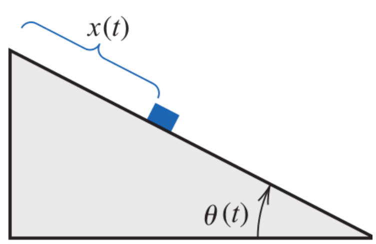
14. A particle starts at rest on a smooth inclined plane whose angle θ is changing at a constant rate

$$\frac{d\theta}{dt} = \omega < 0.$$

At the end of t seconds, the position of the object is given by

$$x(t) = -\frac{g}{2\omega^2}\left(\frac{e^{\omega t} - e^{-\omega t}}{2} - \sin \omega t\right).$$

Suppose the particle has moved 1.7 ft in 1 s. Find, to within 10^{-5}, the rate ω at which θ changes. Assume that $g = 32.17$ ft/s^2.



15. An object falling vertically through the air is subjected to viscous resistance as well as to the force of gravity. Assume that an object with mass m is dropped from a height s_0 and that the height of the object after t seconds is

$$s(t) = s_0 - \frac{mg}{k}t + \frac{m^2 g}{k^2}(1 - e^{-kt/m}),$$

where $g = 32.17$ ft/s^2 and k represents the coefficient of air resistance in lb-s/ft. Suppose $s_0 = 300$ ft, $m = 0.25$ lb, and $k = 0.1$ lb-s/ft. Find, to within 0.01 s, the time it takes this quarter-pounder to hit the ground.