Lecture-16

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Lecture 16: Numerical Analysis (UMA011)

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Multiple roots

Modified Newton's method (if multiplicity is given)

let $f(x)=0$ be an equation, $f(x)$ has a zero at $x=p$ i.e. $f(p)=0$ with multiplicity m i.e. $f(x) = (x-p)^m q(x)$ $\quad q(p)\neq 0$

Modified Newton method is

$$\boxed{p_{n+1} = p_n - m\frac{f(p_n)}{f'(p_n)}}$$, m is the multiplicity of the required root.

To check the order of convergence of Newton's method in case of multiple roots

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)} = g(p_n)$$

$\frac{d}{dx}(uvw)$

$$g(x) = x - m\frac{f(x)}{f'(x)} = x - \frac{(x-p)^m q(x)}{(x-p)^m q'(x) + q(x)\, m(x-p)^{m-1}}$$

To find the order of convergence for m.N.M. i.e. quadratic

$$g(x) = x - m\frac{(x-p)\, q(x)}{(x-p)q'(x) + m\, q(x)}$$

Let $x=p$

$$g(p) = p - m\frac{0}{0+mq(p)} = p - 0 = p \checkmark$$

$$g'(x) = 1 - m\left[\frac{(x-p)\; q'(x)}{(x-p)q'(x)+mq(x)} + \frac{q(x) \quad (1)}{(x-p)q'(x)+mq(x)} + (x-p)\, q(x)\, \frac{d}{dx}\frac{1}{(x-p)q'(x)+mq(x)}\right]$$

If $g(p)=p$

$g'(p)\neq 0 \Rightarrow$ linear

If $g'(p)=0 \Rightarrow$ atleast quadratic

Put $x=p$

$$g'(p) = 1 - m\left[0 + \frac{\cancel{q(p)}}{0+m\cancel{q(p)}} + 0\right] = 1 - \frac{1}{m} \neq 0 \quad = 1-1 = 0$$

$\Rightarrow$ N.M. (in case of multiple roots) generates linear type but not quad.

Multiple roots

Modified Newton's method (if multiplicity is not given)

Define a function $\mu(x) = \frac{f(x)}{f'(x)}$, $f(x)$ has a root p with multiplicity m.

i.e. $f(p)=0$, $f'(p)=0$

$- - \; f^{m-1}(p)=0$

but $f^m(p)\neq 0$

$f(x) = (x-p)^m q(x)$, $q(p)\neq 0$

$$\mu(x) = \frac{(x-p)^m q(x)}{(x-p)^m q'(x) + m(x-p)^{m-1} q(x)}$$

$$= \frac{(x-p)\, q(x)}{(x-p)q'(x) + mq(x)}$$

$$\mu(x) = (x-p)^1 Q(x), \qquad Q(x) = \frac{q(x)}{(x-p)q'(x)+mq(x)}$$

$$Q(p) = \frac{q(p)}{0+mq(p)} = \frac{1}{m} \neq 0$$

$Q(p)\neq 0$

$\Rightarrow$ $\mu(x)$ has a zero at $x=p$ with $m=1$

i.e. p is a simple zero of $\mu(x)$

Apply Newton's method on $\mu(x)$

$$p_{n+1} = p_n - \frac{\mu(p_n)}{\mu'(p_n)}$$

$$= p_n - \frac{\frac{f(p_n)}{f'(p_n)}}{\left(\frac{f(p_n)}{f'(p_n)}\right)'}$$

$$= p_n - \frac{f(p_n)}{f'(p_n)} \frac{1}{\frac{f'(p_n)f'(p_n) - f(p_n)f''(p_n)}{(f'(p_n))^2}}$$

$$= p_n - \frac{f(p_n)}{f'(p_n)} \frac{(f'(p_n))^2}{(f'(p_n))^2 - f(p_n)f''(p_n)}$$

$$\boxed{p_{n+1} = p_n - \frac{f(p_n)f'(p_n)}{(f'(p_n))^2 - f(p_n)f''(p_n)}}$$

$\longrightarrow$ Modified Newton's method

Multiple roots

Example:

Show that the modification of Newton's method improves the rate of convergence for $f(x) = e^x - x - 1$ at $x = 0$ with $p_0 = 1$.

$p=0$

Let $f(x) = e^x - x - 1$

$f(0) = e^0 - 0 - 1 = 0$

$f'(x) = e^x - 1$

$f'(0) = e^0 - 1 = 0$

$f''(x) = e^x$

$f''(0) = e^0 \neq 0$ $\Rightarrow$ $f(x)$ has a root at $x=0$ with $m=2$

Apply Modified Newton's method

$$p_{n+1} = p_n - 2\frac{f(p_n)}{f'(p_n)}$$

$$= p_n - 2\frac{e^{p_n} - p_n - 1}{e^{p_n} - 1}$$

Let $p_0 = 1$

$$p_1 = p_0 - 2\cdot\frac{e^{p_0} - p_0 - 1}{e^{p_0} - 1} = 1 - 2\frac{e^1 - 2}{e-1}$$

$$= \frac{3-e}{e-1} = 0.163953$$

$$p_2 = 0.16395 - 2\frac{e^{0.16395} - 0.163953 - 1}{e^{0.16395} - 1}$$

$$= 0.0044779$$

$p_3 = 0.000003341 9$

To check the convergence of $\langle p_n \rangle$

$$\frac{|p_2 - 0|}{|p_1 - 0|^2} = \frac{0.0044779}{(0.163953)^2} = 0.1665 < 1$$

$$\frac{p_3 - 0}{(p_2 - 0)^2} < 1$$

$\Rightarrow \langle p_n \rangle \to p = 0$ quadratically.

Multiple roots

Order of convergence of modified Newton's method:

$$g(x) = x - m\frac{f(x)}{f'(x)}, \quad f(x) = (x-p)^m q(x)$$

$g(p) = p$

To show $g'(p) = 0$

Multiple roots:

Exercise:

1. Use Newton's method and the modified Newton's method to find a solution of

$$f(x) = (1-x)\sin(1-x) = 0,$$

accurate to within 10^{-2}. Take initial approximation $x_0 = 0$.

2. Apply modified Newton's method with $m = 2$ and $x_0 = 0.8$ to the equation $f(x) = x^3 - x^2 - x + 1 = 0$, and verify that the convergence is of second-order.