Optimal Binary Search Tree
(without dummy keys)

Optimal Binary Search Trees

- Problem
  - » Given sequence $K = k_1 < k_2 < \cdots < k_n$ of n sorted keys, with a search probability p_i for each key k_i.
  - » Want to build a binary search tree (BST) with minimum expected search cost.
  - » Actual cost = # of items examined.
  - » For key k_i, $cost(k_i) = \text{depth}_T(k_i) + 1$, where $\text{depth}_T(k_i)$ = depth of k_i in BST T .

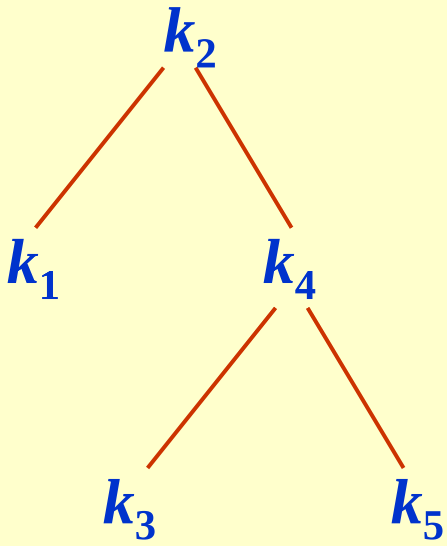
Expected Search Cost

$$
\begin{aligned}
&E[\text{search cost in } T] \\
&= \sum_{i=1}^{n} cost(k_i) \cdot p_i \\
&= \sum_{i=1}^{n} (\text{depth}_T(k_i) + 1) \cdot p_i \\
&= \sum_{i=1}^{n} \text{depth}_T(k_i) \cdot p_i + \sum_{i=1}^{n} p_i \\
&= 1 + \sum_{i=1}^{n} \text{depth}_T(k_i) \cdot p_i \qquad (15.16)
\end{aligned}
$$

Sum of probabilities is 1.

Example

- Consider 5 keys with these search probabilities:
$p_1 = 0.25$, $p_2 = 0.2$, $p_3 = 0.05$, $p_4 = 0.2$, $p_5 = 0.3$.

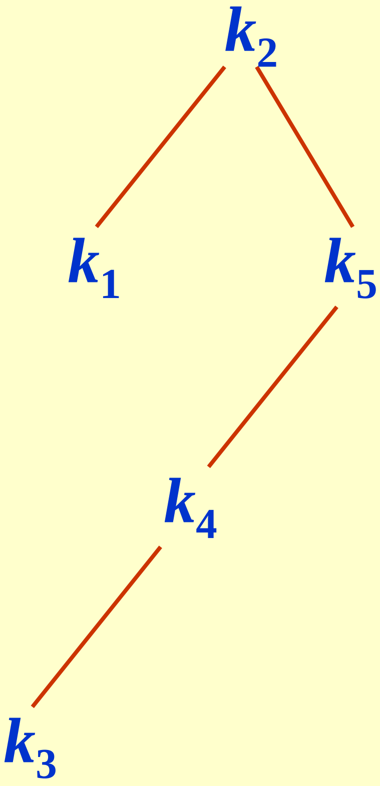


i	$\text{depth}_T(k_i)$	$\text{depth}_T(k_i)\cdot p_i$
1	1	0.25
2	0	0
3	2	0.1
4	1	0.2
5	2	0.6
		1.15

Therefore, E[search cost] = 2.15.

Example

- $p_1 = 0.25$, $p_2 = 0.2$, $p_3 = 0.05$, $p_4 = 0.2$, $p_5 = 0.3$.



i	$\text{depth}_T(k_i)$	$\text{depth}_T(k_i) \cdot p_i$
1	1	0.25
2	0	0
3	3	0.15
4	2	0.4
5	1	0.3
		1.10

Therefore, E[search cost] = 2.10.

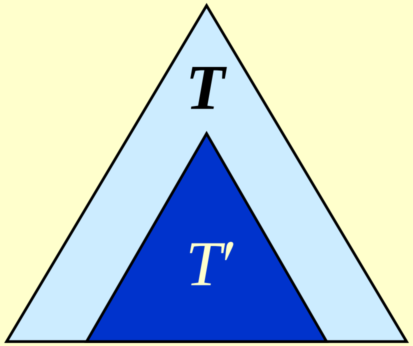
This tree turns out to be optimal for this set of keys.

Observation

- **Observations:**
  - » Optimal BST may not have smallest height.
  - » Optimal BST may not have highest-probability key at root.
- Build by exhaustive checking?
  - » Construct each n-node BST.
  - » For each,
    assign keys and compute expected search cost.
  - » But there are $\Omega(4^n/n^{3/2})$ different BSTs with n nodes.

Optimal Substructure

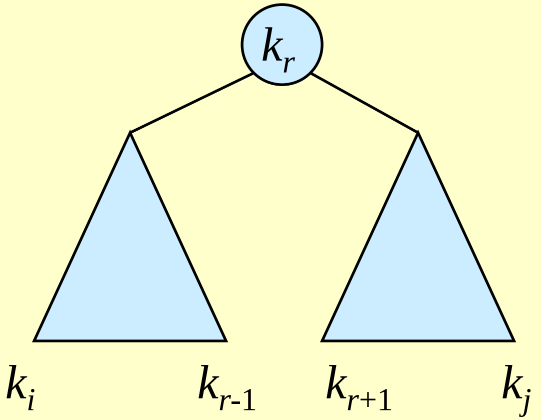
- Any subtree of a BST contains keys in a contiguous range $k_i, \ldots, k_j$ for some $1 \le i \le j \le n$.



- If T is an optimal BST and
  T contains subtree T' with keys $k_i, \ldots, k_j$,
  then T' must be an optimal BST for keys $k_i, \ldots, k_j$.
- **Proof:** … … .

Optimal Substructure

- One of the keys in $k_i, \ldots, k_j$, say k_r, where $i \le r \le j$, must be the root of an optimal subtree for these keys.
- Left subtree of k_r contains $k_i, \ldots, k_{r-1}$.
- Right subtree of k_r contains $k_{r+1}, \ldots, k_j$.



- To find an optimal BST:
  - » Examine all candidate roots k_r, for $i \le r \le j$
  - » Determine all optimal BSTs containing $k_i, \ldots, k_{r-1}$ and containing $k_{r+1}, \ldots, k_j$

Recursive Solution

- Find optimal BST for $k_i, \ldots, k_j$, where $i \geq 1, j \leq n, j \geq i-1$. When $j = i - 1$, the tree is empty.
- Define $e[i, j]$ = expected search cost of optimal BST for $k_i, \ldots, k_j$.

- If $j = i - 1$, then $e[i, j] = 0$.
- If $j \geq i$,
  - » Select a root k_r, for some $i \leq r \leq j$.
  - » Recursively make an optimal BSTs
    - for $k_i, .., k_{r-1}$ as the left subtree, $e[i, r - 1]$ and
    - for $k_{r+1}, \ldots, k_j$ as the right subtree, $e[r + 1, j]$.

Recursive Solution

- When the OPT subtree becomes a subtree of a node:
  - » Depth of every node in OPT subtree goes up by 1.
  - » Expected search cost increases by

$$\boxed{w(i, j) = \sum_{l=i}^{j} p_l}$$ from (15.16)

- If k_r is the root of an optimal BST for $k_i, \ldots, k_j$:
  - » $e[i, j] = p_r + (e[i, r-1] + w(i, r-1)) + (e[r+1, j] + w(r+1, j))$

$= e[i, r-1] + e[r+1, j] + w(i, j)$. (because $w(i, j) = w(i, r-1) + p_r + w(r+1, j)$)

- But, we don't know k_r. Hence,

$$e[i, j] = \begin{cases} 0 & \text{if } j = i-1 \\ \min_{i \le r \le j} \{e[i, r-1] + e[r+1, j] + w(i, j)\} & \text{if } i \le j \end{cases}$$

Computing an Optimal Solution

For each subproblem (i, j), store:

- expected search cost in a table $e[1..n+1, 0..n]$
  - » Will use only entries $e[i, j\ ]$, where $j \geq i-1$.
- root$[i, j]$ = root of subtree with keys $k_i, \ldots, k_j$, for $1 \leq i \leq j \leq n$.
- $w[1..n+1, 0..n]$ = sum of probabilities
  - » $w[i, i-1] = 0$ for $1 \leq i \leq n$.
  - » $w[i, j] = w[i, j-1] + p_j$ for $1 \leq i \leq j \leq n$.

Pseudo-code

OPTIMAL-BST(*p*, *n*)

```
for i ← 1 to n + 1
    do e[i, i– 1] ← 0
       w[i, i – 1] ← 0
for l ← 1 to n
    do for i ← 1 to n – l + 1
        do j ← i + l – 1
       w[i, j] ← w[i, j – 1] + p_j
       e[i, j]← ∞
       for r ←i to j
           do t ← e[i, r– 1] + e[r + 1, j] + w[i, j ]
            if t < e[i, j]
            then e[i, j] ← t
            root[i, j] ←r
return e and root
```

Consider all trees with *l* keys.

Fix the first key.

Fix the last key

Determine the root of the optimal (sub)tree

Time: $O(n^3)$

$$e[i, j] = \begin{cases} 0 & \text{if } j = i-1 \\ \min\limits_{i \le r \le j}\{e[i, r-1] + e[r+1, j] + w(i, j)\} & \text{if } i \le j \end{cases}$$

Example

Construct an optimal binary search tree over five key values k1 < k2 < k3 < k4 < k5 with access probability 0.3, 0.2, 0.1, 0.15, and 0.25, respectively.

1. w[i,j]

	j=0	1	2	3	4	5
i=1	0	0.3				
2		0	0.2			
3			0	0.1		
4				0	0.15	
5					0	0.25
6						0

1. e[i,j]

	j=0	1	2	3	4	5
i=1	0	0.3				
2		0	0.2			
3			0	0.1		
4				0	0.15	
5					0	0.25
6						0

1. r[i,j]

	j=0	1	2	3	4	5
i=1		1				
2			2			
3				3		
4					4	
5						5
6						

2. w[i,j]

	j=0	1	2	3	4	5
i=1	0	0.3	0.5			
2		0	0.2	0.3		
3			0	0.1	0.25	
4				0	0.15	0.4
5					0	0.25
6						0

2. e[i,j]

	j=0	1	2	3	4	5
i=1	0	0.3	0.7			
2		0	0.2	0.4		
3			0	0.1	0.35	
4				0	0.15	0.55
5					0	0.25
6						0

2. r[i,j]

	j=0	1	2	3	4	5
i=1		1	1			
2			2	2		
3				3	4	
4					4	5
5						5
6						

3. w[i,j]

	j=0	1	2	3	4	5
i=1	0	0.3	0.5	0.6		
2		0	0.2	0.3	0.45	
3			0	0.1	0.25	0.5
4				0	0.15	0.4
5					0	0.25
6						0

3. e[i,j]

	j=0	1	2	3	4	5
i=1	0	0.3	0.7	1		
2		0	0.2	0.4	0.8	
3			0	0.1	0.35	0.85
4				0	0.15	0.55
5					0	0.25
6						0

3. r[i,j]

	j=0	1	2	3	4	5
i=1		1	1	2		
2			2	2	3	
3				3	4	5
4					4	5
5						5
6						

4. w[i,j]

	j=0	1	2	3	4	5
i=1	0	0.3	0.5	0.6	0.75	
2		0	0.2	0.3	0.45	0.7
3			0	0.1	0.25	0.5
4				0	0.15	0.4
5					0	0.25
6						0

4. e[i,j]

	j=0	1	2	3	4	5
i=1	0	0.3	0.7	1	1.4	
2		0	0.2	0.4	0.8	1.35
3			0	0.1	0.35	0.85
4				0	0.15	0.55
5					0	0.25
6						0

4. r[i,j]

	j=0	1	2	3	4	5
i=1		1	1	2	2	
2			2	2	3	4
3				3	4	5
4					4	5
5						5
6						

5. w[i,j]

	j=0	1	2	3	4	5
i=1	0	0.3	0.5	0.6	0.75	1
2		0	0.2	0.3	0.45	0.7
3			0	0.1	0.25	0.5
4				0	0.15	0.4
5					0	0.25
6						0

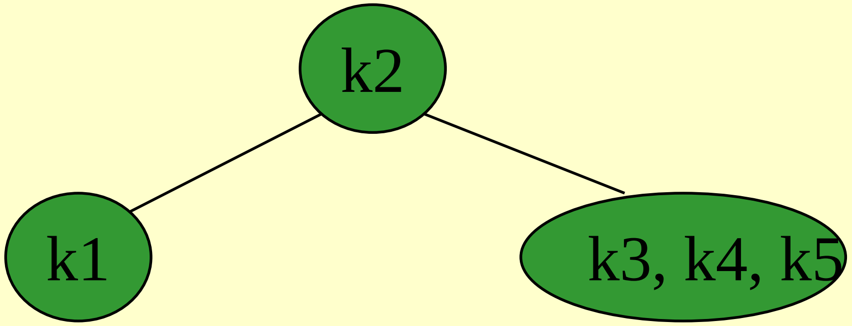
5. e[i,j]

	j=0	1	2	3	4	5
i=1	0	0.3	0.7	1	1.4	2.15
2		0	0.2	0.4	0.8	1.35
3			0	0.1	0.35	0.85
4				0	0.15	0.55
5					0	0.25
6						0

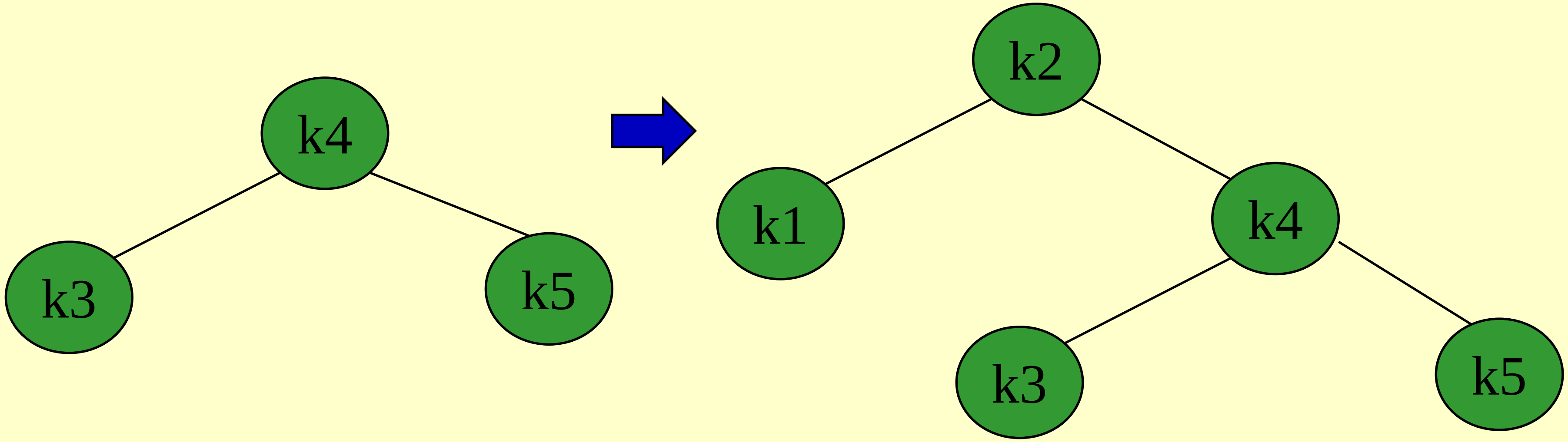
5. r[i,j]

	j=0	1	2	3	4	5
i=1		1	1	1	2	2
2			2	2	2	4
3				3	4	4
4					4	5
5						5
6						

r[1, 5] = 2 shows that the root of the tree over k1, k2, k3, k4, k5 is k2.



r[3, 5] = 4 shows that the root of the subtree over k3, k4, k5 is k4.



Elements of Dynamic Programming

- Optimal substructure
- Overlapping subproblems

Optimal Substructure

- Show that a solution to a problem consists of making a choice, which leaves one or more subproblems to solve.
- Suppose that you are given this last choice that leads to an optimal solution.
- Given this choice, determine which subproblems arise and how to characterize the resulting space of subproblems.
- Show that the solutions to the subproblems used within the optimal solution must themselves be optimal. Usually use cut-and-paste.
- Need to ensure that a wide enough range of choices and subproblems are considered.

Optimal Substructure

- Optimal substructure varies across problem domains:
  - » 1. *How many subproblems* are used in an optimal solution.
  - » 2. *How many choices* in determining which subproblem(s) to use.
- Informally, running time depends on (# of subproblems overall) × (# of choices).
- How many subproblems and choices do the examples considered contain?
- Dynamic programming uses optimal substructure **bottom up**.
  - » *First* find optimal solutions to subproblems.
  - » *Then* choose which to use in optimal solution to the problem.

Optimal Substucture

- Does optimal substructure apply to all optimization problems? No.
- Applies to determining the shortest path but NOT the longest simple path of an unweighted directed graph.
- Why?
  - » Shortest path has independent subproblems.
  - » Solution to one subproblem does not affect solution to another subproblem of the same problem.
  - » Subproblems are not independent in longest simple path.
    - Solution to one subproblem affects the solutions to other subproblems.
  - » Example:

Overlapping Subproblems

- The space of subproblems must be “small”.
- The total number of distinct subproblems is a polynomial in the input size.
  - » A recursive algorithm is exponential because it solves the same problems repeatedly.
  - » If divide-and-conquer is applicable, then each problem solved will be brand new.