Lecture-9

Wednesday, August 10, 2022 10:51 AM

Lecture 9: Numerical Analysis (UMA011)

Dr. Meenu Rani

School of Mathematics
TIET, Patiala
Punjab-India

Newton's method:

$\boxed{10^{-2}} \to$ tol.

Example:
Find the root of an equation $f(x) = \cos(x) - x = 0$ by N.M.

By IVT $f(x) = \cos x - x$ $[a,b]$

$f(0) = \cos 0 - 0 = 1 = +ve$

$f(1) = \cos 1 - 1 < 0$ $f(\pi/2) = \cos\frac{\pi}{2} - \pi/2 = 0 - \pi/2 = -ve$

$\Rightarrow$ Root lies in $[0, \pi/2]$

By N.M. $p_{n+1} = p_n - \dfrac{f(p_n)}{f'(p_n)}$, $n \geq 0, n \in \mathbb{N}$

$= p_n + \dfrac{\cos p_n - p_n}{\sin p_n + 1}$

Let $p_0 = \pi/4$

$p_1 = p_0 - \dfrac{f(p_0)}{f'(p_0)} = \dfrac{\pi}{4} - \dfrac{f(\pi/4)}{f'(\pi/4)}$

$p_1 = \dfrac{\pi}{4} + \dfrac{(\cos\frac{\pi}{4} - \pi/4)}{\sin\frac{\pi}{4} + 1} = 0.7854$

radian value

$p_2 = p_1 - \dfrac{f(p_1)}{f'(p_1)} = 0.7854 + \dfrac{\cos(0.7854) - 0.7854}{\sin(0.7854) + 1}$

$p_2 = 0.7395$ $|p_1 - p_2| \not< 10^{-2}$

$p_3 = 0.7395 + \dfrac{\cos(0.7395) - 0.7395}{\sin(0.7395) + 1}$ $|p_2 - p_3| < 10^{-2}$

$= 0.7391$

Newton's method:

Convergence result for Newton's method:
Let $f \in C^2[a,b]$. If $p \in (a,b)$ is such that $f(p) = 0$ and $f'(p) \neq 0$, then there exists a $\delta > 0$ (small) such that Newton's method generates a sequence $\{p_n\}_{n=1}^{\infty}$ converging to p for any initial approximation $p_0 \in [p - \delta, p + \delta]$.

$\{p_n\} \to p$ $p_0 \in [p-\delta, p+\delta]$

i.e. $|p - p_0|$ is small

$p - \delta$ p $p + \delta$ p_0

2.5 3.5 $\delta = 0.25$

3

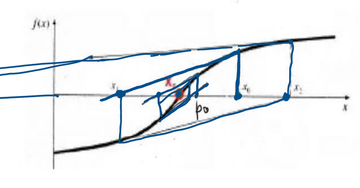
$3 - \delta$ $3 + \delta$

$3 - 0.25$ $3 + 0.25$

Newton's method:

Case of failure:
(i) When the initial guess is on the inflection of the function i.e $f''(p_0) = 0$.

for e.g. $y = x^{1/3}$

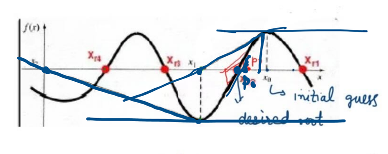


In this case, the sequence generated by N.M. diverges.

Newton's method:

Case of failure:
(ii) When there is an another slope near to the initial guess.

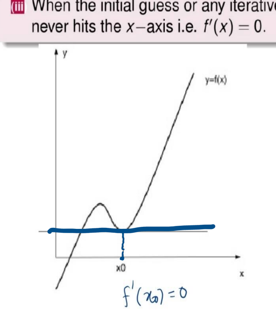
for e.g. $y = \sin x$ or $y = \cos x$



In this case, the sequence generated by N.M. converges to undesire root

Newton's method:

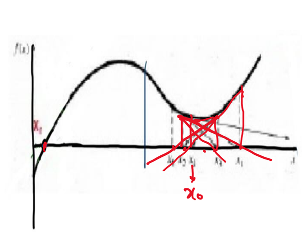
Case of failure:
(iii) When the initial guess or any iterative value of function never hits the x-axis i.e. $f'(x) = 0$.



In this case, N.M. does not generate any sequence

Case of failure:
(iv) When the initial guess is between local maximum or local minimum.

In this case, the sequence generated by N.M oscillates



Newton's method:

Exercise:
1. Find the root of an equation $x - e^{-x} = 0$ by using Newton's method with the accuracy of 10^{-2}.
2. The function $f(x) = \sin x$ has a zero on the interval $(3, 4)$ namely, $x = \pi$. Perform three iterations of Newton's method to approximate this zero, using $x_0 = 4$.

Secant method:

Importance:
requires only evaluation of function

Secant method:

Derivation:

To find the root of $f(x) = 0$

By. N.M $p_{n+1} = p_n - \dfrac{f(p_n)}{f'(p_n)}$, $f'(p_n) = \lim_{p_{n-1} \to p_n} \dfrac{f(p_n) - f(p_{n-1})}{p_n - p_{n-1}}$

$$p_{n+1} \approx p_n - \frac{f(p_n)}{\dfrac{f(p_n) - f(p_{n-1})}{(p_n - p_{n-1})}}$$

$$p_{n+1} = p_n - \frac{f(p_n)(p_n - p_{n-1})}{f(p_n) - f(p_{n-1})}$$

$$= \frac{\cancel{p_n f(p_n)} - p_n f(p_{n-1}) - \cancel{p_n f(p_n)} + p_{n-1} f(p_n)}{f(p_n) - f(p_{n-1})}$$

$$\boxed{p_{n+1} = \frac{p_{n-1} f(p_n) - p_n f(p_{n-1})}{f(p_n) - f(p_{n-1})}} \to \text{Secant Method.}$$

Secant method:

Graphical representation:

Eqn of secant line

$y - f(p_1) = \dfrac{f(p_1) - f(p_0)}{p_1 - p_0}(x - p_1)$

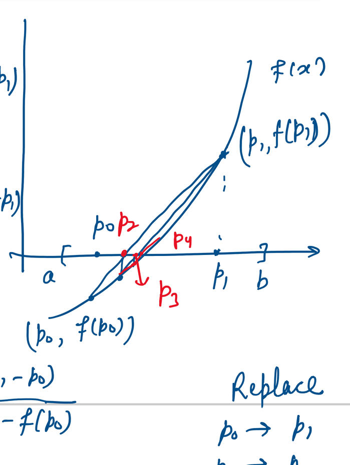
at axis $y = 0$

$-f(p_1) = \dfrac{f(p_1) - f(p_0)}{p_1 - p_0}(x - p_1)$

$\dfrac{-f(p_1)(p_1 - p_0)}{f(p_1) - f(p_0)} = x - p_1$

$x = p_1 - \dfrac{f(p_1)(p_1 - p_0)}{f(p_1) - f(p_0)}$

$= p_2$



Replace
$p_0 \to p_1$
$p_1 \to p_2$

$p_3 = p_2 - \dfrac{f(p_2)(p_2 - p_1)}{f(p_2) - f(p_1)}$

$p_4 = p_3 - \dfrac{f(p_3)(p_3 - p_2)}{f(p_3) - f(p_2)}$

$$\boxed{p_{n+1} = p_n - \frac{f(p_n)(p_n - p_{n-1})}{f(p_n) - f(p_{n-1})}} \to \text{Secant method.}$$

$f(p_n) \neq f(p_{n-1})$ $\forall n \geq 0$