Lecture-19

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Lecture 19: Numerical Analysis (UMA011)

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System of linear equations

$L \to$ Lower triangular Matrix
$U \to$ upper " "

→ LU Decomposition

LU Factorization:

$AX = b$

$A = L * U$

$$A = \begin{matrix} E_1 \\ E_2 \\ \vdots \\ E_n \end{matrix}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & & \cdots & a_{nn} \end{bmatrix} \sim \begin{bmatrix} a_{11} & \cdots & & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & 0 & & \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} = U \text{ (say)}$$

$$E_j \to E_j - \underbrace{\left(\frac{a_{ji}}{a_{ii}}\right)}_{= l_{ji}} E_i$$

$$\left\{ E_2 \to E_2 - \underbrace{\frac{a_{21}}{a_{11}}}_{= l_{21}} E_1,\ E_3 \to E_3 - \underbrace{\frac{a_{31}}{a_{11}}}_{= l_{31}} E_1 \ \cdots\ E_n \to E_n - \underbrace{\frac{a_{n1}}{a_{11}}}_{= l_{n1}} E_1 \right\}$$

$$L = \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ l_{21} & 1 & & & \\ l_{31} & l_{32} & 1 & & \\ l_{41} & l_{42} & & \ddots & \\ l_{n1} & 1 & & l_{n,n-1} & 1 \end{bmatrix}$$

$l_{32} = \frac{a^*_{32}}{a^*_{22}}$

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$AX = b$

$(L * U) X = b$ $\qquad (AB)C = A(BC)$

$L * (UX) = b$ $\qquad$ Let $UX = Y$ (say)

$L * Y = b$

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ l_{21} & 1 & 0 & \cdots & 0 \\ l_{31} & l_{32} & 1 & \cdots & 0 \\ l_{n1} & \cdots & & & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

find Y by using forward sub.
$y_1 = b_1,\quad l_{21} y_1 + y_2 = b_2$ ——

$Y = !$

$UX = Y$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ 0 & & 0 & a_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$ known

use backward sub. to get X

System of linear equations

Example:
Determine the LU factorization for matrix A in the linear system $Ax = b$, where $A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \\ -3 \\ 4 \end{bmatrix}$. Then use the factorization to solve the system.

System of linear equations

Solution: $A = \begin{matrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{matrix}\begin{bmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{bmatrix}$

$E_2 \to E_2 - 2E_1,\quad E_3 \to E_3 - 3E_1,\quad E_4 \to E_4 + E_1$

$$\sim \begin{matrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{matrix}\begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & -4 & -1 & -7 \\ 0 & 3 & 3 & 2 \end{bmatrix}$$

$E_3 \to E_3 - 4E_2,\quad E_4 \to E_4 + 3E_2$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{bmatrix} = U \text{ (say)}$$

$E_4 \to E_4 - 0E_3$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{bmatrix}$$

$\because (l_{11} * u_{11} = a_{11})$, $u_{11} = a_{11}$

$AX = b$
$(LU)X = b$
$L(UX) = b$ (let $UX = Y$)
$LY = b$,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -3 \\ 4 \end{bmatrix}$$

Use forward sub.

$\boxed{y_1 = 1}$

$2y_1 + y_2 = 1$
$2 + y_2 = 1$
$\boxed{y_2 = -1}$

$3y_1 + 4y_2 + y_3 = -3$
$3 - 4 + y_3 = -3$
$-1 + y_3 = -3$
$\boxed{y_3 = -2}$

$-y_1 - 3y_2 + y_4 = 4$
$-1 + 3 + y_4 = 4$
$\boxed{y_4 = 2}$

$\Rightarrow Y = \begin{bmatrix} 1 \\ -1 \\ -2 \\ 2 \end{bmatrix}$

Now $UX = Y$

$$\begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -2 \\ 2 \end{bmatrix}$$

Use backward sub.

$-13x_4 = 2$
$\boxed{x_4 = \frac{-2}{13}}$

$3x_3 + 13x_4 = -2$
$3x_3 - 2 = -2$
$\boxed{x_3 = 0}$

$-x_2 - x_3 - 5x_4 = -1$
$\boxed{x_2 = \frac{23}{13}}$

$x_1 + x_2 + 3x_4 = 1$
$\boxed{x_1 = \frac{-4}{13}}$

$$\Rightarrow X = \begin{bmatrix} -4/13 \\ 22/13 \\ 0 \\ -2/13 \end{bmatrix} \text{ Ans.}$$

System of linear equations:

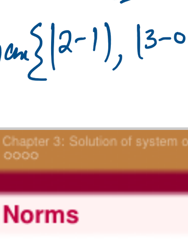
Exercise:

1. Modify the LU Factorization Algorithm so that it can be used to solve a linear system, and then solve the following linear system:

$$2x_1 - x_2 + x_3 = -1$$
$$3x_1 + 3x_2 + 9x_3 = 0$$
$$3x_1 + 3x_2 + 5x_3 = 4.$$

Iterative methods to solve System of linear equations:

Distance between $n-$dimensional vectors
To discuss iterative methods for solving linear systems, we first need to determine a way to measure the distance between $n-$dimensional column vectors.



$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$ $\qquad X^{(0)} = \begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \\ \vdots \\ x_n^{(0)} \end{bmatrix} \to$

$X = (x_1, x_2)^t$

$\sqrt{(2-1)^2 + (3-0)^2} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
$= \sqrt{1^2 + 9}$
$= \sqrt{10}$
$= l_2$-norm

$\boxed{\text{Jacobi}}$
Gauss-seidel $\qquad$ tol

$X^{(1)} = \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ \vdots \\ x_n^{(1)} \end{bmatrix},\quad X^{(2)} = \begin{bmatrix} x_1^{(2)} \\ \vdots \\ x_n^{(2)} \end{bmatrix}$ $\qquad \| X^{(1)} - X^{(2)} \|$ ↳ norm

$\max\{|2-1|, |3-0|\} = l_\infty$-norm $\qquad \|$

Norms

$X \in \mathbb{R}^n,\quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad x_i \in \mathbb{R}.$

Vector norms
Let $\mathbb{R}^n$ denote the set of all $n-$dimensional column vectors with real-number components.
To define a distance in $\mathbb{R}^n$ we use the notion of a norm, which is the generalization of the absolute value on $\mathbb{R}$, the set of real numbers.

$X^{(1)} = \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \end{bmatrix} \begin{matrix} =1 \\ =0 \end{matrix} \qquad X^{(2)} = \begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \end{bmatrix} \begin{matrix} =2 \\ =3 \end{matrix}$

$\| \cdot \| : \mathbb{R}^n \to \mathbb{R}$