Lecture-20

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Lecture 20: Numerical Analysis (UMA011)

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initial guess x_0
Apply Method
$x_i \in \mathbb{R}$
$x_1, x_2, x_3, \ldots, x_n \to p$
$f(x)=0$, $x \in \mathbb{R}$
$|x_1 - x_2| < tol$
$|3 - 3.1| < tol$

$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $\quad X_0 = \begin{bmatrix} x_1^{(0)} \\ \vdots \\ x_n^{(0)} \end{bmatrix} \in \mathbb{R}^n$

$\|X_1 - X_2\| < tol$ $\quad \langle X_1, X_2, X_3, \ldots, X_n \rangle \to X$

Iterative methods to solve System of linear equations:

Distance between n−dimensional vectors

To discuss iterative methods for solving linear systems, we first need to determine a way to measure the distance between n−dimensional column vectors.

$\|X_1 - X_2\|$

Norms

$\mathbb{R}$ $\quad \mathbb{R}^2 \quad \mathbb{R}^3 \quad \mathbb{R}^4$

Vector norms

Let $\mathbb{R}^n$ denote the set of all n−dimensional column vectors with real-number components.
To define a distance in $\mathbb{R}^n$ we use the notion of a norm, which is the generalization of the absolute value on $\mathbb{R}$, the set of real numbers.

$|\ |$

$X \in \mathbb{R}$, $\quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $\quad x_i \in \mathbb{R}$ $\quad \|\cdot\|$

A **vector norm** on $\mathbb{R}^n$ is a function, $\|\cdot\|$, from $\mathbb{R}^n$ into $\mathbb{R}$ with the following properties:

(i) $\|\mathbf{x}\| \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$, $\quad |x| \geq 0$

(ii) $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$, $\quad |x| = 0$ iff $x = 0$

(iii) $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$ for all $\alpha \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$, $\quad |2x| = 2|x|$

(iv) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. $\quad |x+y| \leq |x| + |y|$

Norms

Vector norms

We will need only two specific norms on $\mathbb{R}^n$,

$\max\{|2-1|, |3-1|\} = 2$

$\sqrt{(2-1)^2 + (3-1)^2}$

(a,b,c) $\quad (l,m,n)$

$\sqrt{(a-l)^2 + (b-m)^2 + (c-n)^2}$

The l_2 and l_∞ norms for the vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^t$ are defined by

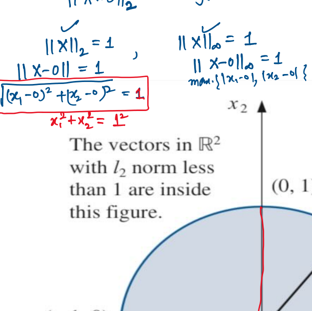
$$\|\mathbf{x}\|_2 = \left\{\sum_{i=1}^{n} x_i^2\right\}^{1/2} \quad \text{and} \quad \|\mathbf{x}\|_\infty = \max_{1 \leq i \leq n} |x_i|.$$

$\|X-0\|_\infty$ → by default; Maximum Norm or Uniform n.

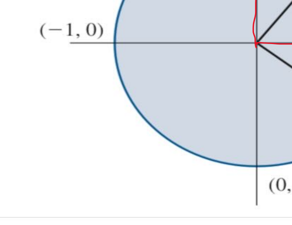
$\|X-0\|_2 = \sqrt{(x_1-0)^2 + (x_2-0)^2 - \cdots (x_n-0)^2}$

$\|X\|_2 = 1$, $\quad \|X\|_\infty = 1$
$\|X-0\| = 1$ $\quad \|X-0\|_\infty = 1$
$\sqrt{(x_1-0)^2 + (x_2-0)^2} = 1$ $\quad \max\{|x_1-0|, |x_2-0|\} = 1$
$x_1^2 + x_2^2 = 1^2$

$\max\{|x_1|, |x_2|\}$
$|x_1| = 1$
$|x_2| = 1$



The vectors in $\mathbb{R}^2$ with l_2 norm less than 1 are inside this figure.



The vectors in $\mathbb{R}^2$ with l_∞ norm less than 1 are inside this figure.

Norms

Example:

Determine the l_2 norm and the l_∞ norm of the vector $x = (-1, 1, -2)^t$.

$X = (-1, 1, -2)^t$

$\|X\|_2 = \sqrt{(-1)^2 + (1)^2 + (-2)^2}$
$= \sqrt{1+1+4} = \sqrt{6} = 2.45$

$\|X\|_\infty = \max\{|-1|, |1|, |-2|\} = 2$

Norms

$\|X_1 - X_2\|_\infty < tol$

Distance between Vectors in $\mathbb{R}^n$:

If $x = (x_1, x_2, \cdots, x_n)^t$ and $y = (y_1, y_2, \cdots, y_n)^t$ are vectors in $\mathbb{R}^n$, then l_2 and l_∞ distances between x and y are defined by

$$\|x - y\|_2 = \left\{\sum_{i=1}^{n} (x_i - y_i)^2\right\}^{1/2} \quad \text{and} \quad \|x - y\|_\infty = \max_{1 \leq i \leq n} |x_i - y_i|.$$

$\|X-Y\|_2 = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 - \cdots (x_n-y_n)^2}$

$\|X-Y\|_\infty = \max\{|x_1-y_1|, |x_2-y_2| \cdots |x_n-y_n|\}$

Norms

$a_n \to a$ $\quad |a_n - a| < \epsilon \ \forall n > N$

Convergence of a sequence in $\mathbb{R}^n$:

The sequence of vectors $\{x^{(k)}\}_{k=1}^{\infty}$ converges to x in $\mathbb{R}^n$ with respect to the l_∞ norm if and only if $\lim_{k\to\infty} x_i^{(k)} = x_i$, for each $i = 1, 2, \cdots, n$.

X

$X^{(n)} = \begin{bmatrix} x_1^{(n)} \\ x_2^{(n)} \\ \vdots \\ x_n^{(n)} \end{bmatrix} \to \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = X$

$X^{(n)} \to X$

Convergence of a sequence in $\mathbb{R}^n$

$X \in \mathbb{R}^4$

Example:

Show that
$x^{(k)} = (x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, x_4^{(k)})^t = (1, 2 + \frac{1}{k}, \frac{3}{k^2}, e^{-k}\sin(k))^t$
converges to $x = (1, 2, 0, 0)^t$ with respect to l_∞ norm.

$X^{(k)} = \begin{bmatrix} 1 \\ 2+\frac{1}{k} \\ \frac{3}{k^2} \\ e^{-k}\sin(k) \end{bmatrix} \to \begin{matrix} 1 \\ 2 \\ 0 \\ 0 \end{matrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$

$k \to \infty$, $a_n \leq b_n \to a$

$\lim_{k\to\infty} \frac{\sin k}{e^k} \leq \lim_{k\to\infty} \frac{1}{e^k} = 0$

System of linear equations:

Exercise:

1. Find l_∞ and l_2 norms of the vectors.
   a) $x = (3, -4, 0, \frac{3}{2})^t$.
   b) $x = (\sin k, \cos k, 2^k)^t$ for a fixed positive integer k.
2. Find the limit of the sequence
   $x^{(k)} = (x_1^{(k)}, x_2^{(k)}, x_3^{(k)})^t = \left(\frac{4}{k+1}, \frac{2}{k^2}, k^2 e^{-k}\right)^t$ with respect to l_∞ norm.

Iterative methods to solve System of linear equations

Jacobi Method

Consider the system of linear equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \quad \text{— (1)} \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \quad \text{— (2)} \\ \vdots \qquad \vdots \qquad \vdots &= \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \quad \text{— (n)} \end{aligned}$$

from (1) $\quad x_1^{(k+1)} = \dfrac{b_1 - (a_{12}x_2^{(k)} + a_{13}x_3^{(k)} \cdots a_{1n}x_n^{(k)})}{a_{11}} = \dfrac{b_1 - \sum_{j=2}^{n} a_{1j}x_j^{(k)}}{a_{11}}$

from (2) $\quad x_2^{(k+1)} = \dfrac{b_2 - (a_{21}x_1^{(k)} + a_{23}x_3^{(k)} \cdots a_{2n}x_n^{(k)})}{a_{22}} = \left(b_2 - \sum_{j=1, j\neq 2}^{n} a_{2j}x_j^{(k)}\right)/a_{22}$

$x_3^{(k+1)} = \dfrac{b_3 - (a_{31}x_1^{(k)} \cdots a_{3n}x_n^{(k)})}{a_{33}} = \dfrac{b_3 - \sum_{j=1, j\neq 3}^{n} a_{3j}x_j^{(k)}}{a_{33}}$

$x_n^{(k+1)} = \dfrac{b_n - (a_{n1}x_1^{(k)} + a_{n2}x_2^{(k)} \cdots a_{n\,n-1}x_{n-1}^{(k)})}{a_{nn}} = \dfrac{b_n - \sum_{j=1}^{n-1} a_{nj}x_j^{(k)}}{a_{nn}}$

Take $X^0 = \begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \\ \vdots \\ x_n^{(0)} \end{bmatrix}$, $\quad X^1 = \begin{bmatrix} x_1^{(1)} \\ \vdots \\ x_n^{(1)} \end{bmatrix}$, $\quad X^{(2)} = \begin{bmatrix} x_1^{(2)} \\ \vdots \end{bmatrix}$

The **Jacobi iterative method** is obtained by solving the ith equation in $A\mathbf{x} = \mathbf{b}$ for x_i to obtain (provided $a_{ii} \neq 0$)

$$x_i = \sum_{\substack{j=1 \\ j\neq i}}^{n} \left(-\frac{a_{ij}x_j}{a_{ii}}\right) + \frac{b_i}{a_{ii}}, \qquad \text{for } i = 1, 2, \ldots, n.$$

For each $k \geq 1$, generate the components $x_i^{(k)}$ of $\mathbf{x}^{(k)}$ from the components of $\mathbf{x}^{(k-1)}$ by

$$x_i^{(k)} = \frac{1}{a_{ii}}\left[\sum_{\substack{j=1 \\ j\neq i}}^{n} \left(-a_{ij}x_j^{(k-1)}\right) + b_i\right], \qquad \text{for } i = 1, 2, \ldots, n.$$

Iterative methods to solve System of linear equations

Gauss-Seidel Method

Consider the system of linear equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \quad \text{— (1)} \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \quad \text{— (2)} \\ \vdots \qquad \vdots \qquad \vdots &= \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \quad \text{— (n)} \end{aligned}$$

from (1) $\quad x_1^{(k+1)} = \dfrac{b_1 - (a_{12}x_2^{(k)} \cdots a_{1n}x_n^{(k)})}{a_{11}} = \dfrac{b_1 - \sum_{j=2}^{n} a_{1j}x_j^{(k)}}{a_{22}}$

$x_2^{(k+1)} = \dfrac{b_2 - (a_{21}x_1^{(k+1)} + a_{23}x_3^{(k)} \cdots a_{2n}x_n^{(k)})}{a_{22}} = \dfrac{1}{a_{22}}\left(b_2 - \sum_{j=1}^{1} a_{2j}x_j^{(k+1)} - \sum_{j=3}^{n} a_{2j}x_j^{(k)}\right)$

$x_3^{(k+1)} = \dfrac{b_3 - (a_{31}x_1^{(k+1)} + a_{32}x_2^{(k+1)} + a_{34}x_4^{(k)} \cdots a_{3n}x_n^{(k)})}{a_{33}} = \dfrac{1}{a_{33}}\left(b_3 - \sum_{j=1}^{2} a_{3j}x_j^{(k+1)} - \sum_{j=4}^{n} a_{3j}x_j^{(k)}\right)$

$x_n^{(k+1)} = \dfrac{b_n - (a_{n1}x_1^{(k+1)} \cdots a_{n\,n-1}x_{n-1}^{(k+1)})}{a_{nn}} = \dfrac{1}{a_{nn}}\left(b_n - \sum_{j=1}^{n-1} a_{nj}x_j^{(k+1)}\right)$

$X^0 = \begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \\ \vdots \\ x_n^{(0)} \end{bmatrix}$ $\quad X^{(1)} = \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ \vdots \\ x_n^{(1)} \end{bmatrix}$

$$x_i^{(k)} = \frac{1}{a_{ii}}\left[-\sum_{j=1}^{i-1}(a_{ij}x_j^{(k)}) - \sum_{j=i+1}^{n}(a_{ij}x_j^{(k-1)}) + b_i\right]$$

Iterative methods to solve System of linear equations

Strictly diagonally dominant matrix:

A square matrix A is called diagonally dominant if

$$|A_{ii}| \geq \sum_{j\neq i} |A_{ij}| \ \forall\, i. \qquad |A_{11}| \geq \sum_{j\neq 1} |A_{1j}|$$

S.D.D.

A is called strictly diagonally dominant if

$$|A_{ii}| > \sum_{j\neq i} |A_{ij}| \ \forall\, i.$$

A_{11}, A_{22}, A_{33}

$A = $ a_{11}, a_{22}, a_{33}

Iterative methods to solve System of linear equations

Example:

Check whether the following matrices are strictly diagonal dominant or not: $A = \begin{bmatrix} 4 & -2 & 1 \\ 1 & -6 & 3 \\ -2 & 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & -2 & 0 \end{bmatrix}$

$A = \begin{bmatrix} 4 & -2 & 1 \\ 1 & -6 & 3 \\ -2 & 2 & 5 \end{bmatrix}$

$|4| \geq |-2| + |1|$ ✓
$|-6| \geq |1| + |3|$ ✓
$|5| \geq |-2| + |2|$ ✓
$\Rightarrow A$ is S.D.D.

$|-2| > |2| + |1|$ ✗
$\Rightarrow B$ is not S.D.

Iterative methods to solve System of linear equations

Result:

If A is strictly diagonally dominant, then for any choice of $x^{(0)}$, [redacted] the [redacted] Gauss-Seidel methods give sequences $\{x^{(k)}\}_{k=0}^{\infty}$ that converge to the unique solution of $Ax = b$.

$x^{(0)}$

$Ax = b$

Iterative methods to solve System of linear equations

Example:

Use Jacobi and Gauss-Seidel iterative technique to find approximate solutions to

$$\begin{aligned} 10x_1 - x_2 + 2x_3 &= 6 \quad \text{— (1)} \\ -x_1 + 11x_2 - x_3 + 3x_4 &= 25 \quad \text{— (2)} \\ 2x_1 - x_2 + 10x_3 - x_4 &= -11 \quad \text{— (3)} \\ 3x_2 - x_3 + 8x_4 &= 15 \quad \text{— (4)} \end{aligned}$$

starting with $x = (0, 0, 0, 0)^t$ and iterating until $\|x^{(k)} - x^{(k-1)}\|_\infty < 10^{-3} = tol$

Jacobi method.

from (1)
$x_1^{(k+1)} = \frac{1}{10}(6 + x_2^{(k)} - 2x_3^{(k)})$
$x_2^{(k+1)} = \frac{1}{11}(25 + x_1^{(k)} + x_3^{(k)} - 3x_4^{(k)})$
$x_3^{(k+1)} = \frac{1}{10}(-11 - 2x_1^{(k)} + x_2^{(k)} + x_4^{(k)})$
$x_4^{(k+1)} = \frac{1}{8}(15 - 3x_2^{(k)} + x_3^{(k)})$

Gauss-Seidel method

$x_1^{(k+1)} = \frac{1}{10}(6 + x_2^{(k)} - 2x_3^{(k)})$
$x_2^{(k+1)} = \frac{1}{11}(25 + x_1^{(k+1)} + x_3^{(k)} - 3x_4^{(k)})$
$x_3^{(k+1)} = \frac{1}{10}(-11 - 2x_1^{(k+1)} + x_2^{(k+1)} + x_4^{(k)})$
$x_4^{(k+1)} = \frac{1}{8}(15 - 3x_2^{(k+1)} + x_3^{(k+1)})$

Jacobi Table

k	0	1	2	3	4	5	6	7	8	9	10
$x_1^{(k)}$	0.0000	0.6000	1.0473	0.9326	1.0152	0.9890	1.0032	0.9981	1.0006	0.9997	1.0001
$x_2^{(k)}$	0.0000	2.2727	1.7159	2.053	1.9537	2.0114	1.9922	2.0023	1.9987	2.0004	1.9998
$x_3^{(k)}$	0.0000	−1.1000	−0.8052	−1.0493	−0.9681	−1.0103	−0.9945	−1.0020	−0.9990	−1.0004	−0.9998
$x_4^{(k)}$	0.0000	1.8750	0.8852	1.1309	0.9739	1.0214	0.9944	1.0036	0.9989	1.0006	0.9998

Gauss-Seidel Table

k	0	1	2	3	4	5
$x_1^{(k)}$	0.0000	0.6000	1.030	1.0065	1.0009	1.0001
$x_2^{(k)}$	0.0000	2.3272	2.037	2.0036	2.0003	2.0000
$x_3^{(k)}$	0.0000	−0.9873	−1.014	−1.0025	−1.0003	−1.0000
$x_4^{(k)}$	0.0000	0.8789	0.9844	0.9983	0.9999	1.0000

System of linear equations:

Exercise:

1. The linear system

$$x_1 - x_3 = 0.2$$
$$-\frac{1}{2}x_1 + x_2 - \frac{1}{4}x_3 = -1.425$$
$$x_1 - \frac{1}{2}x_2 + x_3 = 2$$

has the solution $(0.9, -0.8, 0.7)^T$.

a. Is the coefficient matrix strictly diagonally dominant?
b. Perform four iterations of the Jacobi and Gauss-Seidel iterative method to approximate the solution. Take $x^{(0)} = 0$.