Lecture-5

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Lecture 5: Numerical Analysis (UMA011)

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Error Analysis

Exercise:

1. Evaluate $f(x) = x^3 - 3x^2 + 4x + 0.21$ at $x = 2.73$ using 3-digit arithmetic directly and with nesting. Also, find the absolute error and relative error.
2. Use four-digit rounding arithmetic and the formula to find the most accurate approximations to the roots of the following quadratic equations. Compute the absolute errors and relative errors.

$$\frac{1}{3}x^2 + \frac{123}{4}x - \frac{1}{6} = 0.$$

Error Analysis: Algorithms and Stability

Condition number:

$$K = \frac{\text{relative change in the output}}{\text{relative change in the input}}$$

$x \to x^*$

x
$f(x)$
$x + \Delta x$
$f(x) + \Delta f(x)$

$$= \frac{\dfrac{|f(x) - f(x^*)|}{|f(x)|}}{\dfrac{|x - x^*|}{(x)}}$$

$$= \frac{|f(x) - f(x^*)|}{|f(x)|} \frac{|x|}{|x - x^*|}$$

$$= \frac{|x|}{|f(x)|} \frac{|f(x) - f(x^*)|}{|x - x^*|}$$

$$\approx \frac{|x|}{|f(x)|} |f'(x)|$$

$\because \left\{ \lim_{y \to x} \frac{f(x) - f(y)}{x - y} = f'(x) \qquad \lim_{h \to 0} \frac{f(x+h) - f(x)}{x + h - x} = f'(x) \right\}$

$$\boxed{C.N. = \frac{|x f'(x)|}{|f(x)|} = K}$$

Remark 1) If C-N of a problem is less than or equal or near to 1 (≈ 1), then that problem is well-conditioned.
2) If C.N. >>> 1, then the problem is ill-conditioned.

for e.g. find C.N. of $f(x) = \dfrac{10}{1-x^2}$ $\qquad x \approx \pm 1$

$$C.N. = \left|\frac{x f'(x)}{f(x)}\right| = \left|\frac{x * (-1)(1-x^2)^{-2}(-2x) * 10}{\frac{10}{1-x^2}}\right|$$

$$= \left|\frac{2x^2}{1-x^2}\right|$$

The problem is ill-conditioned for $x \approx \pm 1$ otherwise, it is well-conditioned.

$f(x) = x^2$ $\qquad x^{100000}$

$$\left|\frac{x f'(x)}{f(x)}\right| = \left|\frac{x \, 2x}{x^2}\right| = 2$$

$f(x) = x^{100}$ $\qquad \left|\frac{x \, 100 x^{99}}{x^{100}}\right| = 100$

Error Analysis: Algorithms and Stability

Example:

Write an algorithm to calculate the expression $f(x) = \sqrt{x+1} - \sqrt{x}$ when x is quite large. By considering the condition number of the subproblem of evaluating the function. Show that such a function evaluation is not stable. Suggest a modification which makes it stable.

algorithm $\quad f(x) = \sqrt{x+1} - \sqrt{x}$, x is quite large.

$x_0 : x = 10000 = f_1(x_0)$
$x_1 : x_0 + 1 = f_2(x_0) = 10001$
$x_2 : \sqrt{x_1} = f_3(x_1) = 100.0050$
$x_3 : \sqrt{x_0} = f_4(x_0) = 100$
$x_4 : x_2 - x_3 = f_5(x_2)$ or $f_6(x_3)$

$f_1(x_0) = x_0 = x$

$$C.N. = \left|\frac{x_0 f_1'(x_0)}{f(x_0)}\right| = \left|\frac{x_0 (1)}{x_0}\right| = 1$$

$f_2(x_0) = x_0 + 1$

$$C.N. = \left|\frac{x_0 f_2'(x_0)}{f_2(x_0)}\right| = \left|\frac{x_0 (1)}{x_0 + 1}\right| \leq 1$$

$f_3(x_1) = \sqrt{x_1}$

$$C.N. = \left|\frac{x_1 f_3'(x_1)}{f_3(x_1)}\right| = \left|\frac{x_1 \frac{1}{2\sqrt{x_1}}}{\sqrt{x_1}}\right| = \left|\frac{x_1}{2x_1}\right| = \frac{1}{2} < 1$$

$f_4(x_0) = \sqrt{x_0}$

$$C.N. = \left|\frac{x_0 f_4'(x_0)}{f_4(x_0)}\right| = \frac{1}{2} < 1$$

$f_5(x_2) = x_2 - x_3$

$$C.N. = \left|\frac{x_2 f_5'(x_2)}{f_5(x_2)}\right| = \left|\frac{x_2 (1)}{x_2 - x_3}\right|$$

$$= \frac{100.0050}{100.0050 - 100} = 20001 >> 1$$

$\Rightarrow$ The algorithm is not stable.

To make the expression stable

$$f(x) = (\sqrt{x+1} - \sqrt{x}) * \frac{\sqrt{x+1} + \sqrt{x}}{\sqrt{x+1} + \sqrt{x}}$$

$$= \frac{x + 1 - x}{\sqrt{x+1} + \sqrt{x}} = \frac{1}{\sqrt{x+1} + \sqrt{x}}$$

Algorithm $\quad x_4 : x_2 + x_3 = g_1(x_2)$
$\qquad x_5 : 1/x_4 = g_2(x_4)$

$x_4 = x_2 + x_3$, $\quad C.N. = \left|\frac{x_2 \, g_1'(x_2)}{g_1(x_2)}\right| = \left|\frac{x_2 (1)}{x_2 + x_3}\right| \lessapprox 1$

$x_5 = 1/x_4$, $\quad C.N. = \left|\frac{x_4 \, g_2'(x_4)}{g_2(x_4)}\right| = \left|\frac{x_4 (-1/x_4^2)}{1/x_4}\right| = 1$

$\Rightarrow$ The modified expression is Stable

Error Analysis: Algorithms and Stability

Example:

Write an algorithm to calculate the expression $e^x - \cos x$ when x is near 0 and rewrite it to be stable.

$x_0 : x = 0.001 = f_1(x_0)$
$x_1 : e^{x_0} = f_2(x_0)$
$x_2 : \cos x_0 = f_3(x_0)$
$x_3 : x_1 - x_2 = f_4(x_1)$ or $f_5(x_2)$

find C.N. yourself!

$\to$ You will get unstability in last step

$$e^x - \cos x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} - - -\right) - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} - -\right)$$

$$= \left(x + x^2 + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{2x^6}{6!} - - -\right)$$

$$= x\left(1 + x + \frac{x^2}{3!} + \frac{x^4}{5!} + \frac{2x^5}{6!} - - -\right)$$

$$= x\left(1 + x\left(1 + \frac{x}{3!} + \frac{x^3}{5!} + \frac{2x^4}{6!} - - -\right)\right)$$

Error Analysis

Exercise:

1. Compute and interpret the condition number for:
   (i) $f(x) = \sin(x)$ for $x = 0.51\pi$
   and (ii) $f(x) = \tan(x)$ for $x = 1.7$.
2. Consider the stability (by calculating the condition number) of $\sqrt{x+1} - 1$ when x is near 0. Rewrite the expression to rid it of subtractive cancellation.
3. Write an algorithm to calculate the expression $f(x) = \ln(x+1) - \ln(x)$, for large values of x using six digit rounding arithmetic. By considering the condition number of the subproblem of evaluating the function, show that such a function evaluation is not stable. Also propose the modification of function evaluation so that algorithm will become stable.