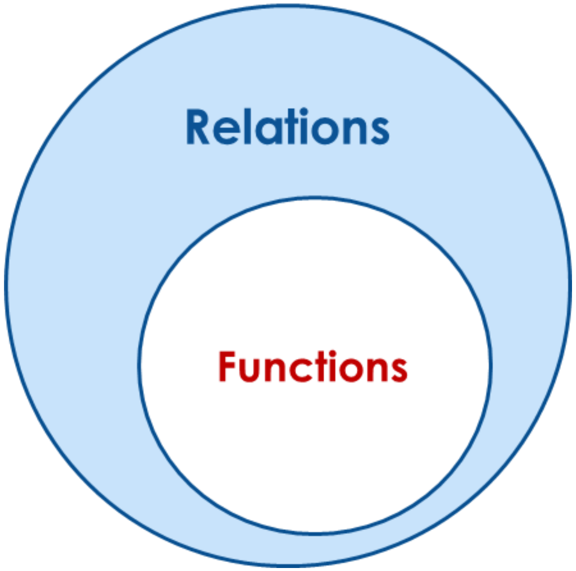


Relations

Contents

Partial Ordering

- A relation R on a set P is called a ***partial ordering***, or ***partial order***, if it is:

- Reflexive
- Antisymmetric
- Transitive

- A set $\boldsymbol{P}$ together with a partial order relation $\boldsymbol{R}$, defined on it, is called a ***partially ordered set***, or ***poset***, and is denoted by $(\boldsymbol{P}, \boldsymbol{R})$. Members of *P*are called *elements* of the poset.

Examples

1. Consider the set of integers. Is the relation "less than or equal" ($\leq$), a partial ordering on the given set of Integers?

- Reflexive?
- Antisymmetric?
- Transitive?

Yes

Examples (Cont..)

2. Consider the set of integers. Is the relation "divisibility" (|), a partial ordering on the given set of Integers?

- ❑ Reflexive?
- ❑ Antisymmetric?
- ❑ Transitive?

No

$(Z^+, |)$ is a POSET.

Examples (Cont..)

3. Show that the inclusion relation ($\subseteq$) is a partial ordering on the power set of a set S.

- Reflexive?
- Antisymmetric?
- Transitive?

Comparability

- The elements a and b of a poset $(P,\preccurlyeq)$ are *comparable* if either $a \preccurlyeq b$ or $b \preccurlyeq a$.
- When a and b are elements of P so that neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, then a and b are called *incomparable*.

The symbol $\preccurlyeq$ is used to denote the relation in any poset.

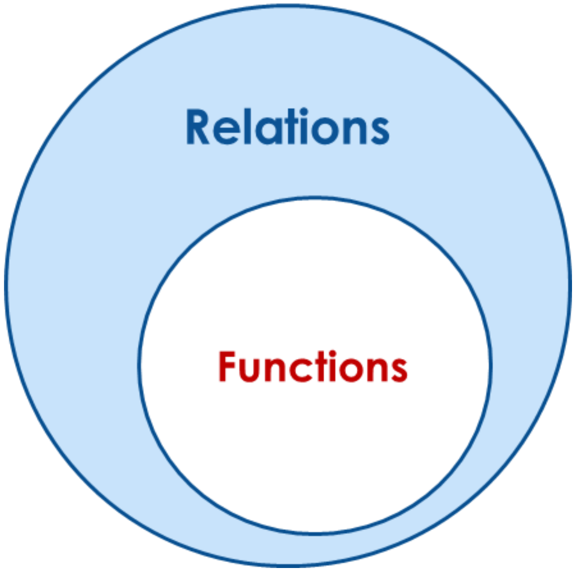
Comparability (Cont..)

- If $(P,\preccurlyeq)$ is a poset and every two elements of P are comparable, P is called a ***totally ordered*** or ***linearly ordered set***, and $\preccurlyeq$ is called a ***total order*** or a ***linear order***.

- A totally ordered set is also called a *chain*.

- $(P,\preccurlyeq)$ is a well-ordered set if it is a poset such that $\preccurlyeq$ is a total ordering and every nonempty subset of P has a least element.


Thank you!!!



Relations

Contents

Lexicographic Ordering

- Given two posets $(A_1, \preccurlyeq_1)$ and $(A_2, \preccurlyeq_2)$, the *lexicographic ordering* on $A_1 \times A_2$ is defined by specifying that (a_1, a_2) is less than (b_1, b_2), that is,

$$(a_1, a_2) \prec (b_1, b_2),$$

either if $a_1 \prec_1 b_1$ or if $a_1 = b_1$ and $a_2 \prec_2 b_2$.

- This definition can be easily extended to a lexicographic ordering on strings.

Examples

1. Consider strings of lowercase English letters.

A lexicographic ordering can be defined using the ordering of the letters in the alphabet.

This is the same ordering as that used in dictionaries.

- *discreet* $\prec$ *discrete*, because these strings differ in the seventh position and $e \prec t$.
- *discreet* $\prec$ *discretion*, because the first six letters agree, but the strings differ in the seventh position and $e \prec t$.

Examples (Cont..)

2. Determine whether $(3,5) \prec (4,8)$, whether $(3,8) \prec (4,5)$ and whether $(4,9) \prec (4,11)$ in the poset $(Z \times Z, \preccurlyeq)$, where $\preccurlyeq$ is the lexicographic ordering constructed from the usual $\leq$ relation on Z.


Thank you!!!
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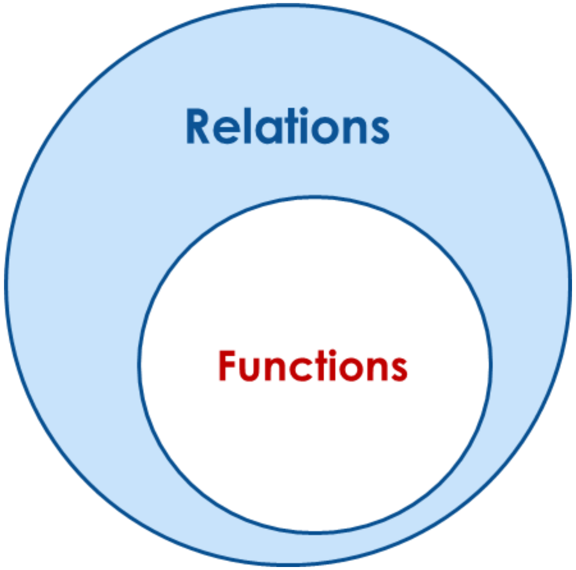


Relations

Contents

Hasse Diagram

- A **Hasse diagram** is a visual representation of a partial ordering that leaves out edges that must be present because of the reflexive and transitive properties.

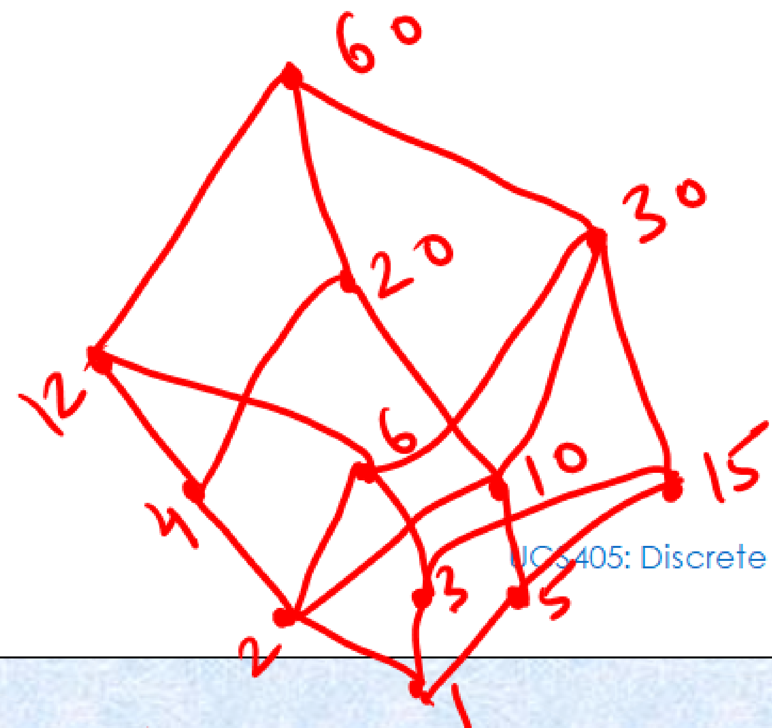
Procedure for drawing a Hasse Diagram

- To represent a finite poset $(S, \preccurlyeq)$ using a Hasse diagram, start with the directed graph of the relation:

  - Remove the loops (a, a) present at every vertex due to the reflexive property.
  - Remove all edges (x, y) for which there is an element $z \in S$ such that $x \prec z$ and $z \prec y$. These are the edges that must be present due to the transitive property.
  - Arrange each edge so that its initial vertex is below the terminal vertex. Remove all the arrows, because all edges point upwards toward their terminal vertex.

Relations

Draw a Hasse diagram for the partial ordering
$\{(a,b) \mid a / b\}$
on $\{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$
these are the divisors of 60.

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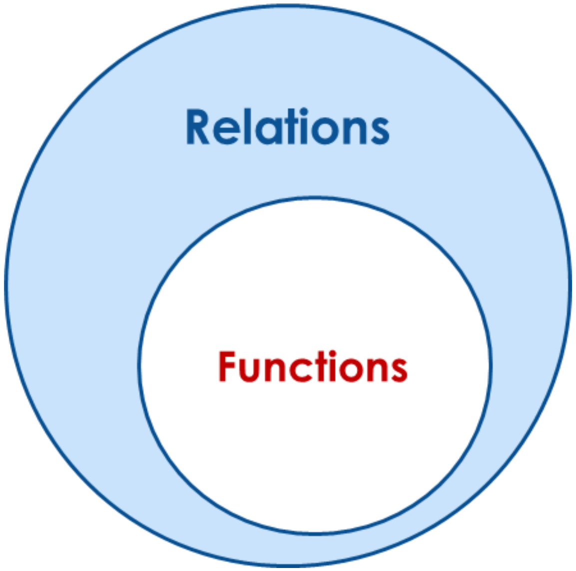


Relations

Contents

Topological Sorting

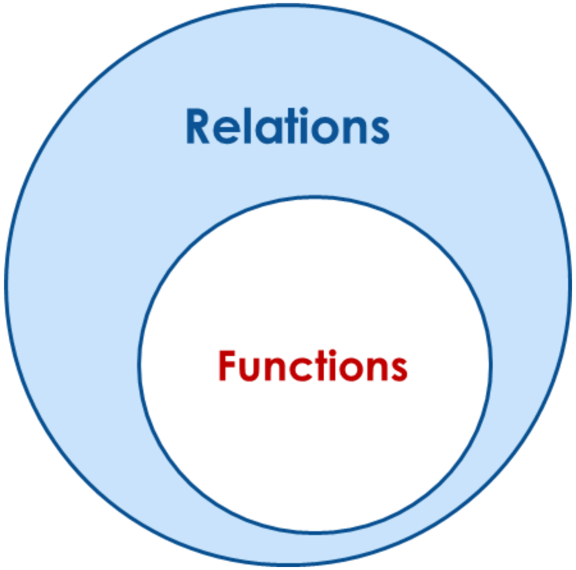
- If A is a poset with partial order $\preccurlyeq$, we sometimes need to find a linear order $\prec$ for the set A in the sense that if $a \preccurlyeq b$ then $a \prec b$.
- The process of constructing a linear order is called Topological Sorting.

Linear Order corresponding to partial ordering

Examples

Relations


Thank you!!!
...



Relations

Contents

Extremal Elements of POSET

- Maximal Element
- Minimal element

Maximal and Minimal Elements

- Let (A,$\preccurlyeq$) be a POSET.

- An element $M \in A$ is called a maximal element of A if there is no element x in A such that $M < x$.

- An element $m \in A$ is called a minimal element of A if there is no element x in A such that $x < m$.

Examples

1. Let A be the poset of all non –ve real numbers with the usual partial order ≤.

   Minimal Element?

   0

   Maximal Element?

   No maximal element

   Infinite series

2. Let us consider the poset $(Z, \leq)$.

   No Minimal Element

   No Maximal Element

Greatest Element and Least Element

- An element $a \in A$ is called a greatest element of A if $x \leq a,\ \forall x \in A$.

I

- An element $a \in A$ is called a least element of A if $a \leq x$, $\forall x \in A$.

0 - Zero Element

Examples

1. Let A be the POSET of all non –ve real numbers with the usual partial order $\leq$.

   Least Element?

   0

   Greatest Element?

   No greatest element

2. Let us consider the POSET $(P(S), \subseteq)$.

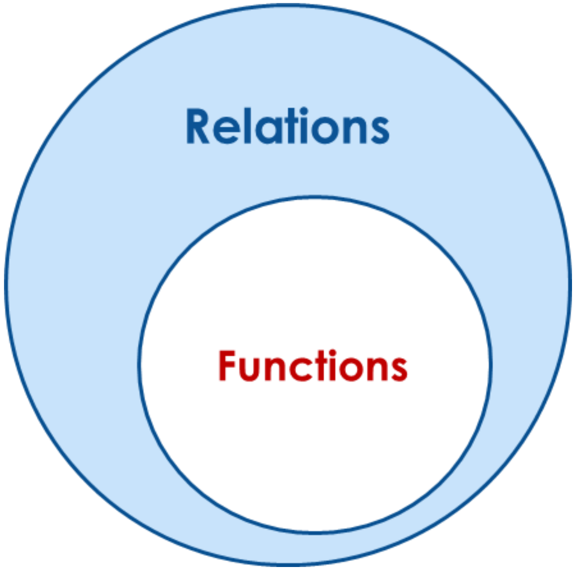
   Least Element?

   $\emptyset$

   Greatest Element?

   Set S


Thank you!!!
...



Relations

Contents

Upper Bound and Lower Bound

- Let A be a POSET and $B \subseteq A$.

- An element $a \in A$ is called an upper bound of B if
$$b \leq a, \forall\, b \in B$$

- An element $a \in A$ is called a lower bound of B if
$$a \leq b, \forall\, b \in B$$

Greatest Lower Bound (GLB)

- Let A be a POSET and $B \subseteq A$.

- An element $a \in A$ is called **a Greatest Lower Bound (GLB) of B** if a is a lower bound of B and $a' \leq a$, whenever a' is a lower bound of B.

- Thus, $a = \text{GLB}(B)$, if $a \leq b, \forall b \in B$ and if whenever $a' \in A$ is also a lower bound of B $(a' \leq b, \forall b \in B)$ then $a' \leq a$.

Least Upper Bound (LUB)

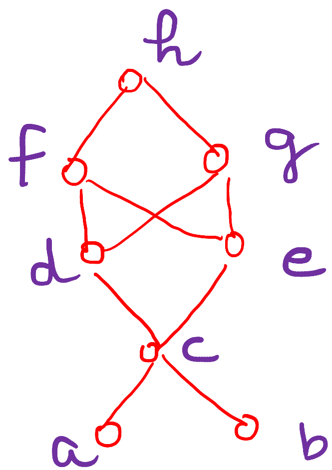
- Let A be a POSET and $B \subseteq A$.

- An element $a \in A$ is called **a Least Upper Bound (LUB) of** B if a is an upper bound of B and $a \leq a'$, whenever a' is an upper bound of B.

- Thus, $a = \text{LUB}(B)$, if $b \leq a, \forall b \in B$ and if whenever $a' \in A$ is also an upper bound of B $(b \leq a', \forall b \in B)$ then $a \leq a'$.

Examples

- Let$(A, \preccurlyeq)$ be a POSET on $A = \{a, b, c, d, e, f, g, h\}$.
  Hasse Diagram is shown Below:



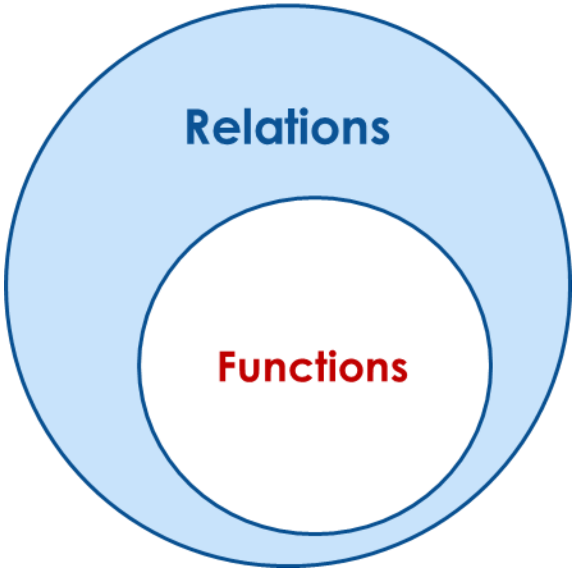
Find all **upper and lower bounds** of the following subsets of A:

a) $B_1 = \{a, b\}$

b) $B_2 = \{c, d, e\}$

Also find **Least Upper Bound (LUB)** and **Greatest Lower Bound (GLB)** for above subsets of A.


Thank you!!!
...



Relations

Contents

Lattice

- A lattice is a POSET $(L, \preccurlyeq)$ in which every subset $\{a, b\}$ consisting of 2 elements has a Least Upper Bound (LUB) and a Greatest Lower Bound (GLB).

- $LUB\ (\{a, b\}) = a \vee b$ (Join of a and b)
- $GLB\ (\{a, b\}) = a \wedge b$ (Meet of a and b)

Examples

1. Let us consider the POSET $(P(S), \subseteq)$.

- Is this a lattice?

Yes

- Let A and B are 2 elements of $P(S)$.
- Then the join of A and B is their union $A \cup B$,
- and the meet of A and B is their $A \cap B$.
- Hence, L is lattice.

Examples

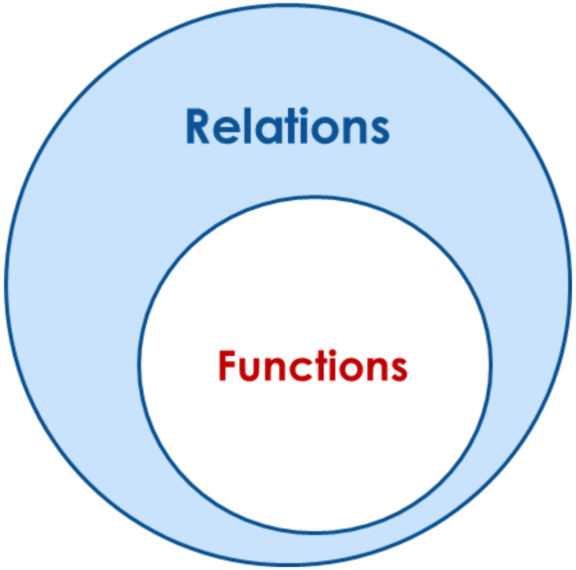
2. Let us consider the POSET $(z^{+}, \preccurlyeq)$, where $a \preccurlyeq b$ iff a/b.

- Is this a lattice?

Yes

$$a \vee b = LCM(a, b)$$
$$a \wedge b = \mathrm{GCD}(a, b)$$


Thank you!!! ...



Relations

Contents

Special Types of Lattices

- Bounded Lattice
- Distributive Lattice
- Complemented Lattice
- Boolean Lattice

Bounded Lattice

- A **lattice** L is said to be **bounded** if it has a greatest element I and a least element 0.

Examples

1. Let us consider lattice Z^+ under the partial order of divisibility.

Is it a bounded lattice?

No

Only least element is there with the value 1. No greatest element.

Examples (Cont..)

2. Let us consider lattice Z under the partial order of $\leq$.

   Is it a bounded lattice?

   No

   Neither least element nor greatest element.

3. Let us consider the lattice $P(S)$ of all subsets of a set S with partial order subset.

   Is it a bounded lattice?

   Yes

   least element is ∅ and greatest element is the set S itself.

Theorem: If L is a finite lattice then L is bounded.

Distributive Lattice

- A **lattice** L is called distributive if for any elements a, b and c in L, we have the following distributive laws:

1) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
2) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

- If L is not distributive, then L is called nondistributive lattice.

Examples

1. Let us consider Lattice $P(S)$ with partial order of subset.

   Is it Distributive?

   Yes

   Only two operations are there i.e. union and intersection. These both operations are distributive.

2. Let us consider lattice Z^+ under the partial order of $\leq$.

   Is it Distributive?

   Yes

   How?

Complemented Lattice

- Let L be a bounded lattice with greatest element I and least element 0, and let $a \in L$.
- An element $a' \in L$ is called a complement of a if

$$a \vee a' = I$$

and, $$a \wedge a' = 0$$

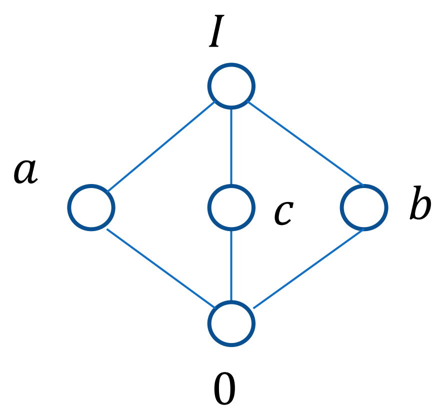
$$0' = I \text{ and } I' = 0$$

A lattice L is called complemented if it is bounded and if every element in L has a complement.

Examples

1. Lattice $L = P(S)$ with subset partial order is Complemented Lattice.

2. Let us consider following Lattice:



Is it a complimented Lattice?

Yes

Boolean Lattice

- A **lattice** L is called **Boolean Lattice** if it is

- Bounded
- Distributive
- Complemented

Example

- Lattice $L = P(S)$ with subset partial order is a Boolean Lattice.


Thank you!!!
...