PROBABILITY AND STATISTICS (UCS401)

Lecture-23

(Gamma distribution with illustrations)

Random Variables and their Special Distributions(Unit –III & IV)

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Gamma distribution with illustration

Gamma function :

For $n>0$, the gamma function Γn is defined as

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx$$

Their most important properties are :

(i) For any $n>1$

$$\Gamma n = (n-1)\,\Gamma(n-1)$$

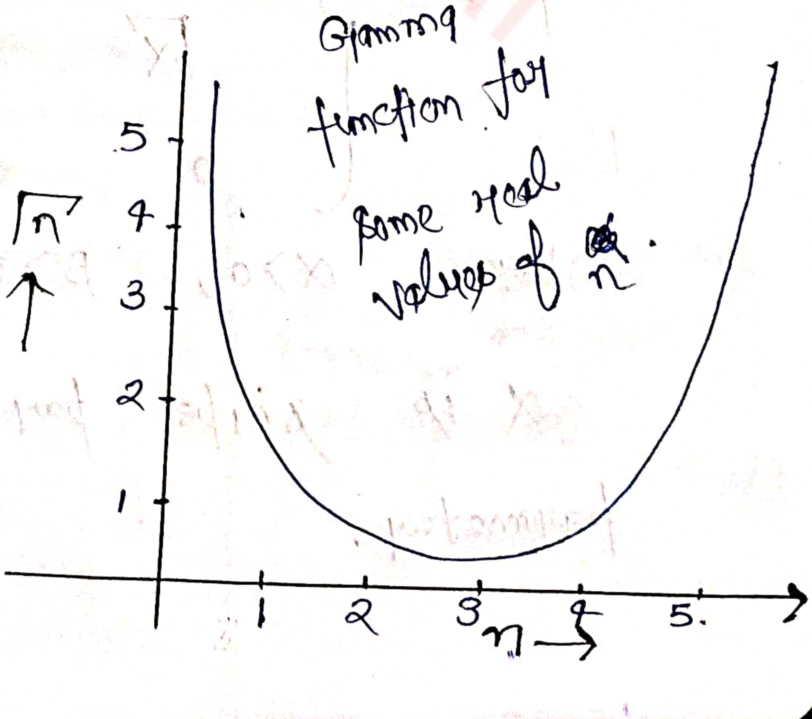
Easialy obtained by integration by part.

(ii) For any integer n,

$$\Gamma n = (n-1)!$$

i.e., $\Gamma(n+1) = n!$

(iii) $\Gamma \frac{1}{2} = \sqrt{\pi}$



Gamma distribution :

A Continuous random variable X is said to follow gamma distribution if its probability density function is defined as

Ist kind (standard gamma distribution) :

$$f(x) = \begin{cases} \dfrac{e^{-x} x^{\alpha-1}}{\Gamma\alpha} & ; \quad x > 0 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

where the parameter $\alpha > 0$.

IInd kind gamma distribution (two parameter gamma dist^n)

$$f(x) = \begin{cases} \dfrac{\beta^{\alpha} e^{-\beta x} x^{\alpha-1}}{\Gamma\alpha} & ; \quad x > 0 \\ 0 & ; \quad \text{elsewhere} \end{cases}$$

where $\alpha > 0$, $\beta > 0$

α is shape parameter, β is scale parameter.

* Clearly, when $\beta=1$, then it converts to standard gamma distribution.

Special cases:

If $\alpha=1$, $\beta=\theta$, then

$$\boxed{\text{Gamma distribution}} \xrightarrow[\alpha=1,\ \beta=\theta]{\text{reduces to}} \boxed{\text{Exponential distribution}}$$

$$f(x)=\begin{cases} \dfrac{\beta^{\alpha}e^{-\beta x}x^{\alpha-1}}{\overline{|\alpha}} & ; x>0 \\ 0 & ; o/w \end{cases} \Longrightarrow f(x)=\begin{cases} \theta e^{-\theta x} & ; x>0 \\ 0 & ; o/w \end{cases}$$

i.e., exponential distribution is special case of gamma distribution.

Cummulation distribution function (c.d.f.):

$$F(x)=P(X\leq x)=\int_{-\infty}^{x} f(x)\,dx$$

$$\because f(x)=\begin{cases} \dfrac{e^{-x}x^{\alpha-1}}{\overline{|\alpha}} & ; x>0 \\ 0 & ; o/w \end{cases}$$

$$F(x)=\int_{0}^{x}\frac{e^{-x}x^{\alpha-1}}{\overline{|\alpha}}\,dx$$

Special cases: For standard gamma distribution

$$f(x) = \begin{cases} \dfrac{e^{-x} x^{\alpha-1}}{\Gamma\alpha} & ; \quad x > 0 \\ 0 & ; \quad \text{elsewhere} \end{cases}$$

Erlang distribution

If $\alpha = 1$, $\beta = \theta$ → Exponential distribution

If α is positive integer → Chi-square distribution

$\Leftarrow$

$X \sim exp(\theta)$

$$f(x) = \begin{cases} \theta e^{-\theta x} & x > 0 \\ 0 & o/w \end{cases}$$

or

$X \sim exp(1/\beta)$

$$f(x) = \begin{cases} 1/\beta \, e^{-x/\beta} & x > 0 \\ 0 & o/w \end{cases}$$

If $\Gamma\alpha$ is replaced by $2^{\alpha}\Gamma\alpha$ or $2^{n/2}\Gamma n/2$

$$f(x) = \begin{cases} \dfrac{e^{-x} x^{\alpha-1}}{2^{\alpha}\Gamma\alpha} & x > 0 \\ 0 & o/w \end{cases}$$

or

$$f(x) = \begin{cases} \dfrac{e^{-x} x^{n/2-1}}{2^{n/2}\Gamma n/2} & x > 0 \\ 0 & o/w \end{cases}$$

Question-: The daily consumption of milk in a city in excess of 20,000 gallons is approximately distributed as a Gamma distribution with parameter $\beta = \frac{1}{10,000}$ and $\alpha = 2$. The city has a daily stock of 30,000 gallons. What is probability that the stock is insufficient on a particular day.

Solution: Let X denotes the daily consumption of milk (in liters) in a city.

Then the random variable

$$Y = X - 20,000$$ has a gamma distribution with p.d.f.

$$f(y) = \frac{1}{(10000)^2 \Gamma 2} y^{2-1} e^{-y/10000} \quad ; \quad y > 0$$

$$\because f(y) = \begin{cases} \frac{\beta^\alpha e^{-\beta x} y^{\alpha - 1}}{\Gamma \alpha} & ; \quad y > 0 \\ 0 & ; \quad o/w \end{cases}$$

Here, $\alpha = 2$, $\beta = \frac{1}{10,000}$

Since, the city has a daily stock of 30,000 gallons, the probability that the ~~city~~ stock is insufficient for a (single) day is

$$P(X > 30{,}000) = P(X - 20{,}000 > 30000 - 20000)$$

$$= P(Y > 10000)$$

$$= \int_{10000}^{\infty} f(y)\, dy$$

$$= \int_{10000}^{\infty} \frac{y e^{-y/10000}}{(10000)^2}\, dy$$

put $\frac{y}{10000} = z$

$\frac{dy}{10000} = dz$

$$= \int_{1}^{\infty} z e^{-z}\, dz$$

$$= \left(-z e^{-z}\right)_1^{\infty} + \int_1^{\infty} e^{-z}\, dz$$

$$= 0 + e^{-1} - \left(e^{-z}\right)_1^{\infty}$$

$$= e^{-1} - \left(0 - e^{-1}\right) = 2/e$$

$$\boxed{P(X > 30{,}000) = 2/e}$$

Relation between Normal distribution and Gamma distribution :

Remark : If $X \sim N(\mu, \sigma^2)$

then $\frac{z^2}{2} \sim \Gamma\left(\frac{1}{2}\right)$

Where $z = \frac{X-\mu}{\sigma}$ is the standard normal variate.

Result : If $X \sim N(\mu, \sigma^2)$, obtain p.d.f of $U = \frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2$, i.e., $\frac{Z^2}{2}$.

Solution- : Since $X \sim N(\mu, \sigma^2)$, which implies that $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$

$\downarrow$

Standard Normal Variate.

Hence the p.d.f. of Z is

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \rightarrow \text{even function}$$

The p.d.f. of $U = \frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2 = \frac{Z^2}{2}$ is found by

$$g(y) = f(z)\left|\frac{dz}{dy}\right|$$

As $U = \frac{z^2}{2} \quad \Rightarrow \quad z = \pm\sqrt{2y}$

$$\Rightarrow \quad \frac{dz}{dy} = \pm\frac{1}{\sqrt{2y}}\, dU$$

Then

$$g(y) = f(z)\left|\frac{dz}{dy}\right| = 2\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\right)\frac{1}{\sqrt{2y}}$$

(as $f(z)$ is an even function)

$$g(y) = 2\left(\frac{1}{\sqrt{2\pi}} e^{-y}\right)\frac{1}{\sqrt{2y}}$$

$$g(y) = \frac{1}{\sqrt{\pi}} e^{-y}\, y^{-1/2} = \frac{1}{\Gamma(1/2)} e^{-y}\, y^{1/2-1}$$

$$g(y) = \frac{1}{\Gamma(1/2)} e^{-y}\, y^{1/2-1} \sim \Gamma\left(\frac{1}{2}\right)$$

which is the p.d.f of gamma distribution with parameter $\alpha = 1/2$.

Hence $U = \frac{z^2}{2}$ is a $\Gamma(1/2)$ variate.

Standard Gamma dist^n.	Two parameter Gamma dist^n.
Mean $= \alpha$	Mean $= \alpha/\beta$
Variance $= \alpha$	Variance $= \alpha/\beta^2$
M.g.f. $M_X(t) = (1-t)^{-\alpha}$; $\lvert t\rvert < 1$	M.g.f. $M_X(t) = \left(1 - t/\beta\right)^{-\alpha}$; $\lvert t\rvert < \beta$

Mean of standard Gamma distribution :

$$X \sim \Gamma(\alpha)$$

$$f(x) = \begin{cases} \dfrac{e^{-x} x^{\alpha-1}}{\Gamma\alpha} & ; x > 0 \\ 0 & ; o/w \end{cases}$$

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx = \int_0^{\infty} x \frac{e^{-x} x^{\alpha-1}}{\Gamma\alpha} dx$$

$$= \frac{1}{\Gamma\alpha} \int_0^{\infty} e^{-x} x^{(\alpha+1)-1} dx = \frac{\Gamma(\alpha+1)}{\Gamma\alpha}$$

$$E(X) = \frac{\alpha \cancel{\Gamma\alpha}}{\cancel{\Gamma\alpha}}$$

$$\therefore \boxed{E(X) = \alpha}$$

Mean of two parameter Gamma distribution :

If $X \sim \Gamma(\alpha, \beta)$ so its p.d.f. is given by

$$f(x) = \begin{cases} \dfrac{\beta^{\alpha} e^{-\beta x} x^{\alpha-1}}{\Gamma\alpha} & ; \quad x > 0 \\ 0 & ; \quad o/w \end{cases}$$

$$E(X) = \int_x x f(x)\, dx = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_0^{\infty} x \frac{\beta^{\alpha} e^{-\beta x} x^{\alpha-1}}{\Gamma\alpha}\, dx$$

$$= \int_0^{\infty} \frac{(\beta x)^{\alpha} e^{-\beta x}}{\Gamma\alpha}\, dx$$

Put $\beta x = y$, $\beta\, dx = dy$

$$= \frac{1}{\beta\Gamma\alpha}\int_0^{\infty} y^{\alpha} e^{-y}\, dy = \frac{1}{\beta\Gamma\alpha}\int_0^{\infty} e^{-y} y^{(\alpha+1)-1}\, dy$$

$$= \frac{\Gamma(\alpha+1)}{\beta\Gamma\alpha} = \frac{\alpha\cancel{\Gamma\alpha}}{\beta\cancel{\Gamma\alpha}}$$

$\Gamma n = \int_0^{\infty} e^{-x} x^{n-1} dx$

$\Gamma(n+1) = n\Gamma n$

$$\therefore \boxed{E(X) = \alpha/\beta}$$

Variance of standard Gamma distribution :

If $X \sim \Gamma(\alpha)$, so its p.d.f. is given by

$$f(x) = \begin{cases} \dfrac{e^{-x} x^{\alpha-1}}{\Gamma\alpha} & ; \; x > 0 \\ 0 & ; \; o/w \end{cases}$$

and $Var(X) = E(X^2) - (E(X))^2$

$$\therefore \quad E(X^2) = \int_x x^2 f(x)\, dx$$

$$= \int_0^\infty \frac{x^2 e^{-x} x^{\alpha-1}}{\Gamma\alpha}\, dx$$

$$= \frac{1}{\Gamma\alpha}\int_0^\infty e^{-x} x^{\alpha+1}\, dx = \frac{\Gamma\alpha+2}{\Gamma\alpha}$$

$$= \frac{(\alpha+1)(\alpha)\cancel{\Gamma\alpha}}{\cancel{\Gamma\alpha}}$$

$\because$ For any $n > 1$

$\Gamma n+1 = n\,\Gamma n$

$\Gamma n = (n-1)\,\Gamma n-1$

$$\therefore \quad E(X^2) = \alpha(\alpha+1)$$

So $Var(X) = E(X^2) - (E(X))^2 = \alpha(\alpha+1) - \alpha^2$

$$\boxed{Var(X) = \alpha}$$

Therefore, Mean = Variance in the standard Gamma distn.

Variance of two-parameter Gamma distribution :

A random variable $X \sim \Gamma(\alpha, \beta)$, so its p.d.f. is given by

$$f(x) = \begin{cases} \dfrac{\beta^{\alpha} e^{-\beta x} x^{\alpha-1}}{\Gamma\alpha} & ; \; x > 0 \\ 0 & ; \; o/w \end{cases}$$

and $\quad Var(X) = E(X^2) - (E(X))^2$

$$E(X^2) = \int_x x^2 f(x)\, dx = \int_0^\infty x^2 \frac{\beta^\alpha e^{-\beta x} x^{\alpha-1}}{\Gamma\alpha}\, dx$$

$$= \frac{\beta^\alpha}{\Gamma\alpha} \int_0^\infty x^{\alpha+1} e^{-\beta x}\, dx$$

put $\beta x = y$

$\beta dx = dy$

$dx = \dfrac{dy}{\beta}$

$$= \frac{1}{\beta \Gamma\alpha} \int_0^\infty (\beta x)^{\alpha+1} e^{-\beta x}\, dx$$

$$= \frac{1}{\beta^2 \Gamma\alpha} \int_0^\infty e^{-y} y^{\alpha+1}\, dy$$

$$= \frac{\Gamma(\alpha+2)}{\beta^2 \Gamma\alpha} = \frac{(\alpha+1)\alpha \cancel{\Gamma\alpha}}{\beta^2 \cancel{\Gamma\alpha}}$$

$$\therefore \boxed{E(X^2) = \frac{\alpha(\alpha+1)}{\beta^2}}$$

$\because$ Mean $= E(x) = \alpha/\beta$

$\therefore$ $Var(x) = E(x^2) - (E(x))^2$

$$Var(x) = \frac{\alpha(\alpha+1)}{\beta^2} - \left(\frac{\alpha}{\beta}\right)^2$$

$$= \cancel{\frac{\alpha^2}{\beta^2}} + \frac{\alpha}{\beta^2} - \cancel{\frac{\alpha^2}{\beta^2}}$$

$$\boxed{Var(x) = \frac{\alpha}{\beta^2}}$$

Hence, $\text{Variance} = \frac{\text{Mean}}{\beta}$

If $\beta < 1$, then Variance > Mean

If $\beta > 1$, then Variance < Mean

If $\beta = 1$ then Mean = Variance.

Note that: In discrete distribution, Poisson distribution has the property that Mean = Variance. While in Continuous distribution, Standard Gamma distribution has this property.

Moment generating function of standard gamma distribution:

Ist Kind (Standard gamma distribution):

$X \sim \Gamma(\alpha)$, so its p.d.f. is given by

$$f(x) = \begin{cases} \dfrac{e^{-x} x^{\alpha-1}}{\Gamma\alpha} & ; \quad x > 0 \\ 0 & ; \quad \text{elsewhere.} \end{cases}$$

The moment generating function is given by

$$M_X(t) = E\left(e^{tX}\right) = \int_x e^{tx} f(x)\, dx$$

$$= \int_0^\infty e^{tx} \frac{e^{-x} x^{\alpha-1}}{\Gamma\alpha} dx$$

$$M_X(t) = \frac{1}{\Gamma\alpha} \int_0^\infty e^{-(1-t)x} x^{\alpha-1} dx \quad \text{provided } |t| < 1$$

put $x(1-t) = y \Rightarrow dx(1-t) = dy$

$$\Rightarrow dx = \frac{dy}{(1-t)}$$

$$M_X(t) = \frac{1}{\Gamma\alpha} \int_0^\infty e^{-y} \left(\frac{y}{1-t}\right)^{\alpha-1} \frac{dy}{(1-t)}$$

$$M_X(t) = \frac{1}{\Gamma\alpha\,(1-t)^{\alpha}} \int_0^{\infty} e^{-y} y^{\alpha-1}\, dy \quad \text{provided } |t|<1$$

$$M_X(t) = \frac{1}{\cancel{\Gamma\alpha}\,(1-t)^{\alpha}} \cdot \cancel{\Gamma\alpha}$$

$$\boxed{M_X(t) = (1-t)^{-\alpha} \; ; \quad \text{provided } |t|<1}$$

M.g.f. for two parameter gamma distribution:

If $X \sim \Gamma(\alpha, \beta)$, so its p.d.f. is given by

$$f(x) = \begin{cases} \dfrac{\beta^{\alpha} e^{-\beta x} x^{\alpha-1}}{\Gamma\alpha} & ; \quad x>0 \\ 0 & ; \quad o/w \end{cases}$$

$$M_X(t) = E(e^{tX}) = \int_x e^{tx} f(x)\, dx$$

$$= \int_0^{\infty} e^{tx} \frac{\beta^{\alpha} e^{-\beta x} x^{\alpha-1}}{\Gamma\alpha}\, dx$$

$$M_X(t) = \frac{\beta^\alpha}{\Gamma\alpha}\int_0^\infty e^{-(\beta-t)x}\, x^{\alpha-1}\, dx$$

provided $|t|<\beta$

put $(\beta-t)x = y \Rightarrow x = \frac{y}{\beta-t}$

$(\beta-t)\,dx = dy \Rightarrow dx = \frac{dy}{\beta-t}$

$$\therefore \quad M_X(t) = \frac{\beta^\alpha}{\Gamma\alpha}\int_0^\infty e^{-y}\left(\frac{y}{\beta-t}\right)^{\alpha-1}\frac{dy}{(\beta-t)}$$

$$M_X(t) = \frac{\beta^\alpha}{\Gamma\alpha\,(\beta-t)^\alpha}\int_0^\infty e^{-y}\, y^{\alpha-1}\, dy$$

$$= \frac{\beta^\alpha}{\cancel{\Gamma\alpha}\,(\beta-t)^\alpha}\,\cancel{\Gamma\alpha} = \frac{\beta^\alpha}{(\beta-t)^\alpha}\ ; \quad |t|<\beta$$

$$M_X(t) = \frac{1}{\left(1-t/\beta\right)^\alpha}\ ; \quad |t|<\beta$$

$$\boxed{M_X(t) = \left(1-\frac{t}{\beta}\right)^{-\alpha} \quad \text{provided } |t|<\beta}$$

Mean and Variance by moment generating function :

If $X \sim \Gamma(\alpha)$ then

$$M_X(t) = (1-t)^{-\alpha} \quad ; \quad |t| < 1$$

For standard Gamma dist^n

We know that

$$E(X^r) = \frac{d^r}{dt^r}\left(M_X(t)\right)\Bigg|_{t=0}$$

$$\text{Mean} = E(X) = \frac{d}{dt}\left(M_X(t)\right)\Bigg|_{t=0}$$

$$= \frac{d}{dt}\left((1-t)^{-\alpha}\right)\Bigg|_{t=0}$$

$$= \alpha(1-t)^{-\alpha-1}\Bigg|_{t=0} = \alpha$$

$$\boxed{\text{Mean} = E(X) = \alpha}$$

$$E(X^2) = \frac{d^2}{dt^2}\left(M_X(t)\right)\Bigg|_{t=0} = \frac{d}{dt}\left(\alpha(1-t)^{-\alpha-1}\right)\Bigg|_{t=0}$$

$$= \alpha(\alpha+1)(1-t)^{-\alpha-2}\Bigg|_{t=0}$$

$$\boxed{E(X^2) = \alpha(\alpha+1)}$$

$$\therefore \quad Var(X) = E(X^2) - (E(X))^2$$

$$Var(X) = \alpha(\alpha+1) - \alpha^2$$

$$\boxed{Var(X) = \alpha}$$

Therefore, $\boxed{\text{Mean} = \text{variance}}$ in the gamma distribution.

<u>For two parameter gamma distribution :</u>

If $X \sim \text{Gamma}(\alpha, \beta)$ and its m.g.f. is given by

$$M_X(t) = \left(1 - \frac{t}{\beta}\right)^{-\alpha} \text{ provided } |t| < \beta.$$

We know that

$$E(X^r) = \frac{d^r}{dt^r}\left(M_X(t)\right)\Big|_{t=0} = \text{Coefficient of}$$

$\frac{t^r}{r!}$ in the expression of $M_X(t)$.

$$\therefore \quad E(X) = \frac{d}{dt}\left(M_X(t)\right)\Big|_{t=0} = \frac{d}{dt}\left(\left(1 - t/\beta\right)^{-\alpha}\right)\Big|_{t=0}$$

$$E(X) = \frac{\alpha}{\beta}\left(1 - t/\beta\right)^{-\alpha-1}\Big|_{t=0} = \frac{\alpha}{\beta}$$

$\because$ Mean $= E(X) = \alpha/\beta$

and $$E(X^2) = \frac{d^2}{dt^2}\left(M_X(t)\right)\Bigg|_{t=0}$$

$$= \frac{d}{dt}\left(\frac{\alpha}{\beta}\left(1-\frac{t}{\beta}\right)^{-\alpha-1}\right)\Bigg|_{t=0}$$

$$= \frac{\alpha(\alpha+1)}{\beta^2}\left(1-t/\beta\right)^{-\alpha-2}\Bigg|_{t=0}$$

$$\boxed{E(X^2) = \frac{\alpha(\alpha+1)}{\beta^2}}$$

$\because$ $$Var(X) = E(X^2) - (E(X))^2$$

$$Var(X) = \frac{\alpha(\alpha+1)}{\beta^2} - \left(\frac{\alpha}{\beta}\right)^2$$

$$\boxed{Var(X) = \frac{\alpha}{\beta^2}}$$

\#\#

Or

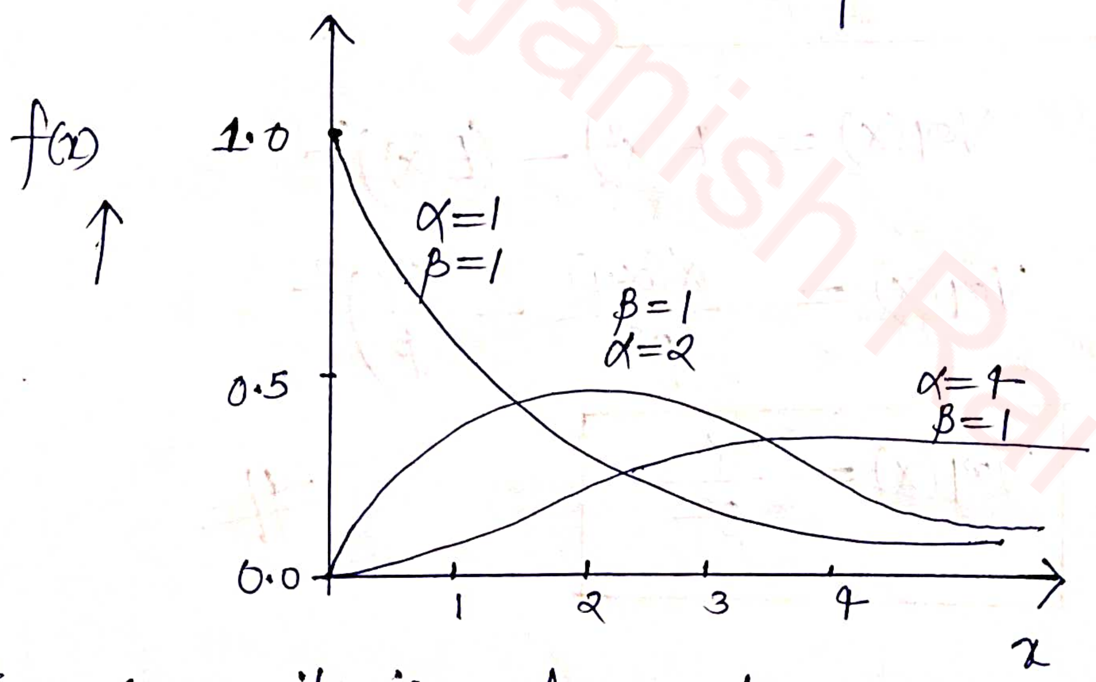
A random variable X has a gamma distribution, with parameters α and β, if its density function is given by

$$f(x;\alpha,\beta)=\begin{cases}\dfrac{1}{\beta^{\alpha}\Gamma\alpha}\,x^{\alpha-1}e^{-x/\beta} & x>0\\[2ex] 0 & o/w\end{cases}$$

where $\alpha>0$, and $\beta>0$

Mean $= E(X) = \alpha\beta$

$Var(X) = \alpha\beta^2$



For $\alpha=1$, it is reduces to exponential distribution

$$f(x)=\begin{cases}\frac{1}{\beta}e^{-x/\beta} & ; \; x>0\\[1ex] 0 & ; \; \text{elsewhere.}\end{cases}$$

Mean $= E(X) = \beta$

$Var(X) = \beta^2$

Question: In a biomedical study with rats, a dose-response investigation is used to determine the effect of the dose of a toxicant on their survival time. The toxicant is one that is frequently discharge into the atmosphere from jet fuel. For a certain dose of the toxicant, the study determines that the survival time, in weeks, has a gamma distribution with $\alpha = 5$ and $\beta = 10$. What is probability that a rat survives no longer than 60 weeks?

Solution:

$$X \sim \text{Gamma}(\alpha, \beta)$$

$$f(x) = \begin{cases} \dfrac{1}{\beta^\alpha \Gamma\alpha} e^{-x/\beta} x^{\alpha-1} & x > 0 \\ 0 & o/w \end{cases}$$

Let the random variable X be the survival time (time to death). The required probability is

$$P(X \leq 60) = \int_0^{60} \frac{1}{(10)^5 \Gamma 5} e^{-x/10} x^4 \, dx$$

put $x/10 = y \qquad dx = 10y$

$$= \frac{1}{(10)^5 \sqrt{5}} \int_0^6 e^{-y} (10y)^4 \, 10 dy$$

$$= \frac{1}{(10)^5 \sqrt{5}} (10)^5 \int_0^6 e^{-y} y^{5-1} dy$$

$$= \frac{1}{\sqrt{5}} \int_0^6 e^{-y} y^{5-1} dy = F(6,5)$$

From table
$$F(6,5) = 0.715$$

$$\therefore \boxed{P(X \leq 60) = 0.715} \quad \text{Ans}$$