Lecture-13

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Lecture 13: Numerical Analysis (UMA011)

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Fixed point iteration

Example of FPI:

Find the root of an equation $x^3 - 7x + 2 = 0$ by using fixed point iteration method with the accuracy of 10^{-2} = tol

Solution: find an interval by IVT

Step 1

$f(0) = 2 = +ve$ ✓

$f(1) = 1 - 7 + 2 = -ve$

$\Rightarrow$ Root lies in $[0,1]$ ✓

Step 2 To find $g(x)$

1) $x = x + x^3 - 7x + 2$

✗ $x = x^3 - 6x + 2 = g_1(x)$

$g_1(x) \in C[0,1]$

$g_1(0) = 2 \notin [0,1]$

2) $x^3 = 7x - 2$

✗ $x = (7x-2)^{1/3} = g_2(x)$

$g_2(x) \notin C[0,1]$ not continuous on $[0,1]$

3) $7x = x^3 + 2$

$x = \frac{x^3+2}{7} = g_3(x)$

\# $g_3(x) \in C[0,1]$

\# $g_3(0) = \frac{2}{7} \in [0,1]$, $g_3(1) = \frac{3}{7} \in [0,1]$

$g_3'(x) = \frac{3x^2}{7} > 0 \quad \forall x \in [0,1]$

$\Rightarrow g_3(x)$ is increasing on $[0,1]$

$\Rightarrow$ Minimum value of $g_3(x)$ occurs at $x = 0$ i.e. $g_3(0) = \frac{2}{7} \in [0,1]$

& Maximum value " " at $x = 1$ i.e. $g_3(1) = \frac{3}{7} \in [0,1]$

$\Rightarrow g_3(x) \in [0,1] \quad \forall x \in [0,1]$.

\# $g_3'(x) = \frac{3x^2}{7}$ & $|g_3'(x)| = \left|\frac{3x^2}{7}\right| < 1 \quad \forall x \in [0,1]$ ✓

$g_3''(x) = \frac{6x}{7} > 0 \quad \forall x \in [0,1]$

$\Rightarrow g_3'(x)$ is increasing on $[0,1]$

$|g_3'(0)| = |\text{min value}| = |0| < 1$

$g_3'(1) = |\text{max value}| = \left|\frac{3}{7}\right| < 1$

$\Rightarrow |g_3'(x)| < 1 \quad \forall x \in [0,1]$ ✓

$g(x) = \frac{x^3+2}{7}$ $\qquad g(x) = x - \frac{x^3 - 7x + 2}{3x^2 - 7}$

Step 3 Take $p_0 = 0.5$

$p_1 = g(0.5) = \frac{(0.5)^3 + 2}{7} = 0.30357$

$p_2 = g(0.30357) = \frac{(0.30357)^3 + 2}{7} = 0.28971$

$|p_1 - p_2| \not< 10^{-2}$

$p_3 = g(0.28971) = \frac{(0.28971)^3 + 2}{7} = 0.28919$

$|p_2 - p_3| < 10^{-2}$

$p_3 = 0.28919$ is the fixed pt for $g(x)$ & root for $f(x) = 0$.

Fixed point iteration

$x_{n+1} = g(x_n) \to 1$

Example:

The iterates $x_{n+1} = 2 - (1+c)x_n + cx_n^3$ converge to $p = 1$ for some constant c. Find the value or bound for c for which convergence occurs.

Solution: $x_{n+1} = 2 - (1+c)x_n + cx_n^3 = g(x_n)$

$\Rightarrow g(x) = 2 - (1+c)x + cx^3$ at $(1-\delta, 1+\delta)$, $\delta > 0$

$g'(x) = 0 - (1+c) + 3cx^2$

using 3rd cond. (given) $\because x_{n+1}$ is convergent

$|g'(x)| < 1 \quad \because (x_{n+1} \to 1) \quad \forall x \in (1-\delta, 1+\delta)$

$|-(1+c) + 3cx^2| < 1 \quad \forall x \in (1-\delta, 1+\delta)$

at $x = 1$

$|-(1+c) + 3c(1)| < 1$

$|-1 - c + 3c| < 1 \Rightarrow |-1 + 2c| < 1$

$-1 < -1 + 2c < 1$

$0 < 2c < 2$

$\boxed{0 < c < 1}$ ✓

Fixed point iteration

Exercise:

1. Find the root of an equation $x^3 - 2x^2 - 5 = 0$ by using fixed point iteration method with the accuracy of 10^{-2}.
2. Let A be a given positive constant and $g(x) = 2x - Ax^2$:
   (a) Show that $1/A$ is a fixed point for $g(x)$.
   (b) Find an interval about $1/A$ for which fixed-point iteration converges, provided p_0 is in that interval.

Hint:- $|g'(x)| < 1$

Order of convergence:

$p_n \to p$ with order α

Definition:

Suppose $\{p_n\}_{n=0}^{\infty}$ is a sequence that converges to p, with $p_n \neq p$ for all n. If positive constants λ and α exist with

$$\boxed{\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} = \lambda,}$$

then $\{p_n\}_{n=0}^{\infty}$ converges to p of order α, with asymptotic error constant λ.

$\frac{1}{n}$, $\frac{1}{n^2}$, $\frac{1}{n^3}$

(i) If $\alpha = 1$ (and $\lambda < 1$), the sequence is linearly convergent.
(ii) If $\alpha = 2$, $\lambda \neq 0$ the sequence is quadratically convergent.
In general, a sequence with a high order of convergence converges more rapidly than a sequence with a lower order.

for example $\frac{1}{n}$: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \ldots \to 0$

$\frac{1}{n^2}$: $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16} \ldots \to 0$

$\frac{1}{n^4}$: $1, \frac{1}{16}, \frac{1}{81} \ldots \to 0$

$p_1, p_2, p_3 \ldots p_n$

$p_{n+1} - p = \epsilon_{n+1}$, $p_n - p = \epsilon_n$, $\lim_{n\to\infty} \frac{\epsilon_{n+1}}{(\epsilon_n)^{\alpha}} = \lambda_n$ $\downarrow$ decreasing sequence.

$\frac{|p_2 - p|}{|p_1 - p|^2} = \lambda_1$ $\qquad \lambda_n \to$ decreasing

$\frac{p_3 - p}{|p_2 - p|^2} = \lambda_2$

$\frac{p_4 - p}{|p_3 - p|^2} = \lambda_3$

Order of convergence:

Order of convergence of bisection method:

Let $\langle p_n \rangle$ be any sequence which converges to p by using bisection method in $[a,b]$

Now, $|p_n - p| \le \frac{|b-a|}{2^n} \quad \forall n$

& $|p_{n+1} - p| \le \frac{|b-a|}{2^{n+1}}$

for $\alpha = 1$

Take $\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^1} \le \frac{|b-a|}{2^{n+1}} * \frac{2^n}{|b-a|} = \frac{1}{2} < 1$ $(= \lambda,\ \lambda < 1)$

for $\alpha = 2$

$\frac{|p_{n+1} - p|}{|p_n - p|^2} \le \frac{|b-a|}{2^{n+1}} * \frac{(2^n)^2}{|b-a|^2}$

$= \frac{1}{|b-a|} 2^{2n-n-1} = \frac{2^{n-1}}{|b-a|}$

$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^2} \le \lim_{n\to\infty} \frac{2^{n-1}}{|b-a|} \to \infty$

Bisection method generates atleast linear order convergence but not quadratic.