Functions of random variable

(X, f_X)

$\downarrow$

p.m.f or p.d.f.

$E(X) = \int_{\mathbb{R}} x f_X \, dx$.

Suppose $g: \mathbb{R} \to \mathbb{R}$

$$\boxed{E(g(X)) = \int_{\mathbb{R}} g(x) f_X \, dx}$$

Question $E(g(X)) = g(E(X))$?

if $g: x \longrightarrow ax$ then $E(g(x)) = g(E(x))$

Example X is r.v. with p.m.f.

x	-3	6	9
$f(x)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$

, $g: x \longrightarrow (2x+1)^2$

$E(g(x)) = ?$

$g(E(x)) = \left(2 \cdot \frac{11}{2} + 1\right)^2$
$= 144$

$E(X) = \cancel{E}\ -3 \cdot \frac{1}{6} + 6 \cdot \frac{1}{2} + 9 \cdot \frac{1}{3} = \frac{11}{2}$

$E(g(x)) = g(-3)\frac{1}{6} + g(6)\frac{1}{2} + g(9) \cdot \frac{1}{3}$

$= 209$

Therefore
$E(g(x))$
$\neq g(E(x))$

Markov Inequality

Thm X be a non-negative r.v. and for $a > 0$

$$P(X \geq a) \leq \frac{E(X)}{a} \longrightarrow 0$$

as $a \to \infty$

pf Suppose p.d.f of X is f_X

then $E(X) = \int_{-\infty}^{\infty} x f(x)\, dx = \int_{0}^{\infty} x f(x)\, dx$

$$\geq \int_{a}^{\infty} x f(x)\, dx$$

$$\geq \int_{a}^{\infty} a f(x)\, dx$$

$$= a \int_{a}^{\infty} f(x)\, dx$$

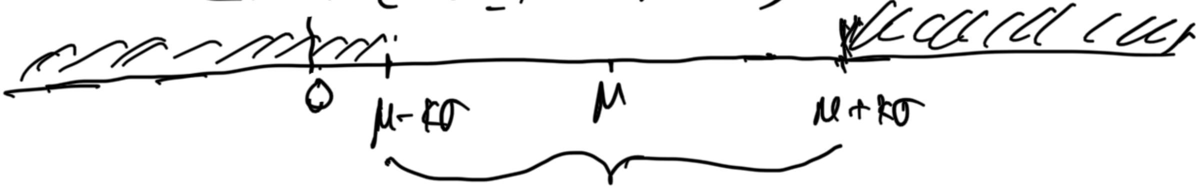
$$= a\, P(X \geq 0)$$

Therefore $P(X \geq a) \leq \frac{E(X)}{a}$

Chebyshev Inequality X is r.v. with $E(X) = \mu$ and variance σ^2, then for any $k > 0$

$$P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$$

$$\Leftrightarrow P(\mu - k\sigma \leq X \leq \mu + k\sigma) \leq \frac{1}{k^2}$$



Example

$$X = \begin{cases} -1 & \text{with probability } \frac{1}{2k^2} \\ 0 & \quad " \qquad 1-\frac{1}{k^2} \\ 1 & \quad " \qquad \frac{1}{2k^2} \end{cases}$$

$$\mu(X) = 0 \qquad \sigma = \frac{1}{k}$$

$$P(|X-\mu| \geq k\sigma) = \frac{1}{k^2}$$

proof of Chebyshev

use Markov inequality for

$$X_1 = (X - E(X))^2$$

Application X is a r.v. with $\mu = 10$, $\sigma^2 = 4$

$$P(5 < X < 15) = P(|X-10| < 5)$$

where $k\sigma = 5$

$\Rightarrow k2 = 5$

$\Rightarrow k = \frac{5}{2}$

$$|X-\mu| \leq k\sigma$$

$$= P(|X-10| < \tfrac{5}{2}\sigma)$$

$$= 1 - P(|X-10| \geq \tfrac{5}{2}\sigma)$$

$$\geq 1 - \frac{1}{(\frac{5}{2})^2} \quad \text{(by Chebyshev inequality)}$$

$$= \frac{21}{25}$$