Lecture-14

Monday, August 29, 2022 6:19 PM

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Lecture 14: Numerical Analysis (UMA011)

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Order of convergence:

Order of convergence of fixed point iteration method:

Let $g \in C[a,b]$ be such that $g(x) \in [a,b], \forall x \in [a,b]$. Suppose, in addition, that g' is continuous on $[a,b]$ and a positive constant $k < 1$ exists with $|g'(x)| \le k < 1$, for all $x \in (a,b)$

$\Rightarrow \{p_n\} \to p$, $g(p) = p$

(i) If $g'(p) \neq 0$, then for any number $p_0 \neq p$ in $[a,b]$, then the sequence $p_n = g(p_{n-1})$, $n \ge 1$ converges only linearly to the unique fixed point p in $[a,b]$.

Taylor's series (poly.) on $g(x)$ about the pt. p.

$g(p_n) \approx g(p) + g'(p)(p_n - p)$

$g(p_n) = g(p) + (p_n - p)\, g'(c_n)$, $p_n < c_n < p$

$p_{n+1} = p + (p_n - p)\, g'(c_n)$

or by L.M.V.T. $\left\{ \left| \dfrac{g(p_n) - g(p)}{p_n - p} \right| = g'(c_n) \right.$

$\lim_{n\to\infty} \left| \dfrac{p_{n+1} - p}{(p_n - p)^1} \right| = \lim_{n\to\infty} |g'(c_n)| = \lambda$

$= |g'(p)| < 1$

$= (\lambda < 1)$

$\because$ $p_n < c_n < p$; $p_n \to p$, $c_n \to p$, $p \to p$ (by Sandwitch thm)

If g is cont. $a_n \to a$ then $g(a_n) \to g(a)$

$\Rightarrow p_n \to p$ linearly if $g'(p) \neq 0$

Order of convergence:

Order of convergence of fixed point iteration method:

(ii) If $g'(p) = 0$ and $g''(x)$ is continuous function with $|g''(x)| < M$ on an open neighbourhood of p, the there exists a $\delta > 0$ such that, for $p_0 \in [p-\delta, p+\delta]$ the sequence defined by $p_n = g(p_{n-1})$, when $n \ge 1$, converges at least quadratically to p. Moreover, for sufficiently large values of n,

$$|p_{n+1} - p| < \frac{M}{2}|p_n - p|^2.$$

Taylor series (poly) for $g(x)$ about p is given by

$g(p_n) = g(p) + (p_n - p)\underset{0}{g'(p)} + \dfrac{(p_n-p)^2}{2!} g''(c_n)$, $p_n < c_n < p$

$p_{n+1} = p + 0 + \dfrac{(p_n - p)^2}{2!} g''(c_n)$

$p_{n+1} - p = (p_n - p)^2 \dfrac{g''(c_n)}{2!}$ — (*)

$p_n \to p$, $c_n \to p$, $p \to p$ & g'' is cont then $g''(c_n) \to g''(p)$

$\lim_{n\to\infty} \left| \dfrac{p_{n+1} - p}{(p_n - p)^2} \right| = \lim_{n\to\infty} \left| \dfrac{g''(c_n)}{2!} \right|$

$= \left| \dfrac{g''(p)}{2!} \right| < \dfrac{M}{2} \neq 0$, $= \lambda$

$\Rightarrow \{p_n\} \to p$ with order 2 or atleast quadratically.

Moreover $\left| \dfrac{p_{n+1} - p}{(p_n - p)^2} \right| = \left| \dfrac{g''(c_n)}{2!} \right| < \dfrac{M}{2}$

$\Rightarrow |p_{n+1} - p| < \dfrac{M}{2}(p_n - p)^2$

Order of convergence:

Order of convergence of fixed point iteration method:

In general, if $g'(p) = 0$, $g''(p) = 0, \cdots, g^{m-1}(p) = 0$, then the sequence defined by $p_n = g(p_{n-1})$, when $n \ge 1$, converges at least of order m to p.

but $g^m(p) \neq 0$ then $p_n \to p$ with order m

If $g'(p) \neq 0$ $p_n \to p$ with order 1

If $g'(p) = 0$, $g''(p) \neq 0$, $p_n \to p$ — " — 2

If $g'(p) = 0$, $g''(p) = 0$, $g'''(p) \neq 0$, $p_n \to p$ — " — 3

Order of convergence:

Order of convergence of Newton's method:

The sequence generated by N.M is given by

$p_n = p_{n-1} - \dfrac{f(p_{n-1})}{f'(p_{n-1})}$, $n \ge 1$

$p_n = g(p_{n-1}) \to p$ (which converges to p)

Here $g(x) = x - \dfrac{f(x)}{f'(x)}$, $g(p) = p - \dfrac{f(p)}{f'(p)}$, $f'(p) \neq 0$

$= p - \dfrac{0}{f'(p)} = p$

$g'(x) = 1 - \dfrac{f'(x)f'(x) - f(x)f''(x)}{(f'(x))^2}$

$= 1 - \dfrac{(f'(x))^2 - f(x)f''(x)}{(f'(x))^2}$

$= \cancel{1} - \cancel{1} + \dfrac{f(x)f''(x)}{(f'(x))^2} = \dfrac{f(x)f''(x)}{(f'(x))^2}$

Put $x = p$

$g'(p) = \dfrac{f(p) f''(p)}{(f'(p))^2} = 0$, $f(p) = 0$ $\Rightarrow p_n \to p$ with order atleast 2 or atleast quadratically

Order of convergence:

Example:

Given that the iterates $x_{n+1} = \frac{2}{3}x_n + \frac{a}{3x_n^2}$, $a \in \mathbb{R}$ converges to $p = a^{1/3}$. Find the order of convergence of the iteration scheme.

$x_{n+1} = \dfrac{2}{3}x_n + \dfrac{a}{3x_n^2} = g(x_n)$, $a \in \mathbb{R}$

$g(x) = \dfrac{2}{3}x + \dfrac{a}{3x^2}$ Here $p = a^{1/3}$ is the fixed pt. for $g(x)$

$g(a^{1/3}) = \dfrac{2}{3}a^{1/3} + \dfrac{a}{3a^{2/3}}$

$= \dfrac{2}{3}a^{1/3} + \dfrac{1}{3}a^{1/3} = \dfrac{3}{3}a^{1/3} = a^{1/3}$

g has a fixed pt at $x = a^{1/3}$

$g'(x) = \dfrac{2}{3} - \dfrac{2}{3}\dfrac{a}{x^3}$

Put $p = a^{1/3}$

$g'(p) = \dfrac{2}{3} - \dfrac{2}{3}\dfrac{\cancel{a}}{\cancel{a^{3/3}}} = 0$ $\Rightarrow x_n \to p$ atleast quadratically

$g''(x) = 0 - \dfrac{2}{3}a\dfrac{(-3)}{x^4} = \dfrac{2a}{x^4}$

Put $p = a^{1/3}$

$g''(a^{1/3}) = \dfrac{2a}{(a^{1/3})^4} \neq 0$

$\Rightarrow x_n \to p$ quadratically or order is 2

Order of convergence:

Exercise:

1. What is the order of convergence of the iteration

$$x_{n+1} = \frac{x_n(x_n^2 + 3a)}{3x_n^2 + a},\ a \in \mathbb{R}$$

as it converges to the fixed point $p = \sqrt{a}$?

2. The iterates $x_{n+1} = 2 - (1+c)x_n + cx_n^3$ converges to $p = 1$ for some values of constant c (provided that x_0 is sufficiently close to p). For what values of c, if any, convergence is quadratic.