Session : Basic Logic

1: Propositional Logic

Logic

T 1) All professors drink tea
T 2) Anyone who drinks tea is a scientist
✓ 3) Therefore all professor are scientist

- Study of valid reasoning.
- Allows us to represent knowledge in precise, mathematical way.
- Allows us to make valid inferences using a set of precise rules.
- **Many applications in CS:**

  AI, programming languages, databases, computer architecture, automated testing and program analysis etc.

Propositional Logic

Proposition - Statement / Sentence

Atomic / Primitive

Compound

T

F

operations

T or F

- Simplest logic is propositional logic.
- Building blocks of propositional logic are propositions.
- A proposition is a statement that is either true or false.
- Examples:

1. UCS405 is a course in discrete mathematics in TIET :- True
2. Patiala is located in Haryana :- False
3. Pay attention, Attend all lectures :- Not a proposition
4. $\boxed{x + y>4}$:- Not a proposition
5. Earth is the only planet that contains life. –T/F

$2+2=4 \Rightarrow T$

$2+x=4$ — Not a proposition

Propositional Variables

- Each proposition will be represented by a propositional variable.
- Propositional variables are usually represented as lower-case letters, such as p, p_1, p_2, q, etc.
- Each variable can take one of two values: true or false.
- What is truth value of “India hosted 2019 ICC Cricket World Cup”?

  False
- A truth table is a table showing the truth value of a propositional logic formula as a function of its inputs.

Logical Connectives

Unary - 1 prop.

Binary p q

p, q, r

- Three basic logical connectives:

Negation

1. **Logical NOT**: ¬p (not p)

   Also called logical negation.

$\neg p$

$\sim p$

p: It is raining

q: It is cold

True T 1

False F 0

2. **Logical AND**: p ∧ q (p and q)

   Also called logical conjunction.

$p \wedge q$

$p \vee q$

w/l

$\boxed{p \vee q}$ $\boxed{p \oplus q}$

$p \underline{\vee} q$

3. **Logical OR**: p ∨ q (p or q)

   Also called logical disjunction

Inclusive -

Exclusive - Either this or that

$\oplus$

p q	$p \underline{\vee} q$
0 0	0
0 1	1
1 0	1
1 1	0

2^2

- Propositions formed using logical connectives are called **compound propositions**

Truth Table

p	q	¬p	p ∧ q	p ∨ q
T	T	F	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	F

Example: Construct a truth table for $(p \vee q) \vee (\neg r)$

① $\neg$ ② $\wedge$ ③ $\vee$

⇒ Conditional Proposition

Implication $p \to q$

Bicondition $p \leftrightarrow q$

p : Premise, Hypothesis, Antecedent

q - Conclusion, Consequence

If $\boxed{p \to q}$ T, F

$\boxed{q \to p}$: converse

$p \to q$ i) If p then q

Greek Logic

p, q

$p \to q \equiv \neg q \to \neg p$: Contra +ve

p	q	$p \to q$	$q \to p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

T → T

F → T

F → F T

If moon is made of cheese

Then earth is round

F → F

True

ii) Q if P

iii) P only if Q

iv) P is sufficient condition for Q

v) Q is necessary condⁿ for P

vi) Q whenever P

vii)

Summary

- Propositional Logic
- Propositional Variables
- Basic Logical Connectives
- Truth Table

Session : Basic Logic

2: Conditional and Biconditional Statements

Conditional Statements

- The conditional statement $p \rightarrow q$ is the proposition "if p, then q."
- The conditional statement $p \rightarrow q$ is false when p is true and q is false, and true otherwise.
- The statement $p \rightarrow q$ is called a conditional statement because it asserts that q is true on the condition that p holds.
- A conditional statement has two parts, a **hypothesis and a conclusion.**
- The part after the "if" is the hypothesis, and the part after the "then" is the conclusion
- A conditional statement is also called an **implication**.

Expressing Conditional Statements

"if *p*, then *q*"

"if *p*, *q*"

"*p* is sufficient for *q*"

"*q* if *p*"

"*q* when *p*"

"a necessary condition for *p* is *q*"

"*q* unless $\neg p$"

"*p* implies *q*"

"*p* only if *q*"

"a sufficient condition for *q* is *p*"

"*q* whenever *p*"

"*q* is necessary for *p*"

"*q* follows from *p*"

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

→ vacuously True

Default case

Converse, Inverse and Contrapositive

- Converse
  - The converse of a conditional is formed by switching the hypothesis and the conclusion.
  - The converse of $p \rightarrow q$ is $q \rightarrow p$
- Contrapositive
  - Negate the hypothesis and the conclusion of the converse
  - The contrapositive of $\neg p \rightarrow \neg q$, is $\neg q \rightarrow \neg p$.
- Inverse
  - Negate the hypothesis and the conclusion
  - The inverse of $p \rightarrow q$, is $\neg p \rightarrow \neg q$

Example

$(t \to (\neg u \to s))$ $\quad \neg q \to \neg p$

$\neg(s \to (u \vee t))$ $\quad p \to q \quad q \to p$

- Conditional statement is "If it rained last night, then the sidewalk is wet."

It is not the case $\quad (t \wedge \neg u) \to s$

$\neg p \to \neg q$

- The converse of the conditional statement is

  "If the sidewalk is wet, then it rained last night."

- The contrapositive of the conditional statement is

  "If the sidewalk is not wet, then it did not rain last night."

- The inverse of the conditional statement is

  "If it did not rain last night, then the sidewalk is not wet."

s : Anil goes out for a walk

t : The moon is out

u : It is snowing

Biconditional Statement

- The biconditional statement $p \leftrightarrow q$ is the proposition "p if and only if q."
- The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise.
- Biconditional statements are also called bi-implications.

Truth Table

p	q	p → q	p ↔ q
T	T	T	T
T	F	F	F
F	T	T	F
F	F	T	T

Example: Construct a truth table for the formula ¬p∧(p → q)

$[\, p \rightarrow q \equiv \neg p \vee q$

p	q	$p \rightarrow q$	$\neg p$	$\neg p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Logically Equivalent

Always F

$\boxed{p \wedge \neg p} \Rightarrow$ False

$p \vee \neg p \Rightarrow$ True

Relations b/w propositions

Propositional Equivalence

1) Tautology - Always True
2) Contradiction - Always False
3) Neither Tautology nor Contradiction or Contingency

Summary

- Conditional Statement
- Converse, Inverse and Contrapositive
- Biconditional Statement

p, q

$\underline{P}, \underline{Q}$

$p_1, p_2, \ldots, p_n$

$\underline{P} \equiv \underline{Q}$

P & Q – T
P & Q – F

De Morgan's Law – P & Q, $\wedge, \vee, \neg$

Thm – $p \to q \equiv \neg q \to \neg p$ ✓ $\neg p$

$P \equiv \neg(p \vee q)$

$\neg p \wedge \neg q$

If you fix my computer then I will give you Rs 500

p : Fix my com

q : I will pay you Rs 500

Session : Basic Logic

3: Practice on Propositional Logic

Dual $p \to p^d$

$\wedge$ $\vee$

$p \vee (p \wedge q) = p$
$p \wedge (p \vee q) = p$

$\neg\neg p \equiv p$

$\neg(p \wedge q) \equiv \neg p \vee \neg q$

$p \vee T \equiv T$
$p \wedge F \equiv F$

$p \vee F = p$
$p \wedge T = p$

Operator Precedence

$\vee, \wedge, \neg, \rightarrow, \leftrightarrow$

- How do we parse this statement?

$$\neg x \rightarrow y \vee z \rightarrow x \vee y \wedge z$$

- Operator precedence for propositional logic:

$$(\neg x) \rightarrow ((y \vee z) \rightarrow (x \vee (y \wedge z)))$$

$\neg$

$\wedge$

$\vee$

$\rightarrow$

$\leftrightarrow$

Above expression is same as:

$(\neg x) \rightarrow ((y \vee z) \rightarrow (x \vee (y \wedge z)))$

Example 1

- **Construct Truth Table of** $(p \lor \neg q) \rightarrow (p \land q)$.

Translating into Propositional Logic

and , but $\wedge$

or $\oplus$

General rule for translation:

- Step 1 find logical connectives
- Step 2 break the sentence into elementary propositions
- Step 3 rewrite the sentence in propositional logic

Example 2

$p \to q$

compound

$(a \vee b) \to c$

- If (you are older than 18 or you are with your parents) then you can drive a two-wheeler.
- Atomic (elementary) propositions:

  a= you are older than 18

  b= you are with your parents

  c=you can drive a two-wheeler
- Translation: $\boxed{a \vee b \to c}$

Example 3

$p \to q$ if p then q

q if p

- You can have free coffee if you are senior citizen and it is a Tuesday
- Atomic (elementary) propositions:

  a= You can have free coffee

  b= you are senior citizen

  c= it is a Tuesday

- Translation: $b \wedge c \to a$

Example 4

- Consider following propositional variables:

  a: I will get up early this morning

  b: There is a lunar eclipse this morning

  c: There are no clouds in the sky this morning

  d: I will see the lunar eclipse

$p \to q$ If p then q

q if p

- Convert the following sentence in to propositional form:

  $\neg d$ $\neg a$

  "I won't see the lunar eclipse if I don't get up early this morning"

  Translation: ¬a → ¬d

Example 5

- Consider following propositional variables:

  a: I will get up early this morning

  b: There is a lunar eclipse this morning

  c: There are no clouds in the sky this morning

  d: I will see the lunar eclipse

$p \to q$ If P, Q

$\wedge$

- Convert the following sentence in to propositional form:

$a \wedge \neg c \to \neg d$

"If I get up early this morning, but it's cloudy outside, I won't see the lunar eclipse.

Translation: a ∧ ¬c → ¬d

Example 6

Let us use the following propositions:

a: It is raining.

b: I will go to college.

c: The computers are broken.

d: I have a headache.

Expression	**English meaning**
$a \wedge d$	It is raining and I have a headache.
$c \rightarrow \neg b$	If the computers are broken then I will not go to college.
$a \vee d \rightarrow \neg b$	If it is raining or I have a headache then I will not go to college.

Session : Basic Logic

4:Propositional Equivalences

True False

Tautology and Contradiction

- Some propositions are interesting since their values in the truth table are always the same
- A compound proposition that is always true for all possible truth values of the propositions is called a **tautology**.
- A compound proposition that is always false is called a **contradiction**.
- A proposition that is neither a tautology nor contradiction is called a **contingency.**
- Example: $p \vee \neg p$ is a tautology

Example

Show that the statement $(p \vee q) \wedge [(\neg p) \wedge (\neg q)]$ is contradiction

F

F

F

F

Logical equivalence

$p \to q \equiv \neg q \to \neg p$

Converse

$q \to p$

- Two propositions p and q are logically equivalent if their truth tables are the same.
- Also, p and q are logically equivalent if p ↔ q is a tautology.
- If p and q are logically equivalent, we write p ≡ q.

$p \leftrightarrow q$ T

0 0 T

1 1 T

Example

Look at the these two compound propositions: $p \rightarrow q$ and $q \lor \neg p$

$p \quad q \quad p \rightarrow q \quad q \lor \neg p$

$(p \rightarrow q) \leftrightarrow (q \lor \neg p)$

Equivalence	Name
$p \wedge T \equiv p, \quad p \vee F \equiv p$	Identity laws
$p \vee T \equiv T, \quad p \wedge F \equiv F$	Domination laws
$p \vee p \equiv p, \quad p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \vee q) \equiv \neg p \wedge \neg q$ $\neg(p \wedge q) \equiv \neg p \vee \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv T, \quad p \wedge \neg p \equiv F$	Negation laws

Example

- Prove $\neg(p \lor (\neg p \land q)) \equiv \neg p \land \neg q$

$\neg(p \lor (\neg p \land q)) \equiv \neg p \land \neg(\neg p \land q)$	DeMorgan
$\equiv \neg p \land (p \lor \neg q)$	DeMorgan
$\equiv (\neg p \land p) \lor (\neg p \land \neg q)$	Distributivity
$\equiv F \lor (\neg p \land \neg q)$ $F \lor p$	Because $\neg p \land p \equiv F$
$\equiv \neg p \land \neg q$	Because $F \lor r \equiv r$ for any r (Identity Law)

Example

$p \to q \equiv \neg q \to \neg p$

Use the logical equivalences to show that ¬(p ∨ ¬(p ∧ q)) is a contradiction.

¬(p ∨ ¬(p ∧ q))

$\neg p \wedge (p \wedge q)$

⇔ ¬p ∧ ¬(¬(p ∧ q))	De Morgan's Law
⇔ ¬p ∧ (p ∧ q)	Double Negation Law
⇔ (¬p ∧ p) ∧ q	Associative Law
⇔ F ∧ q	Contradiction
⇔ q ∧ F	Commutative Law
⇔ F	Domination Law

$(\neg p \wedge p) \wedge q$

$F \wedge q$

F

Session : Basic Logic

5:Normal Forms

Normal Forms

- We can convert any proposition in two normal forms –

1. Disjunctive normal form
2. Conjunctive normal form

DNF

Disjunctive normal form(DNF)/SOP

abc + $\bar{a}$bc

$(a \wedge b \wedge c) \vee (\bar{a} \wedge b \wedge c)$

$\neg a$

$\sim a$

- Every compound proposition in the propositional variables p, q, r, ..., is uniquely equivalent to a proposition that is formed by taking the disjunction of conjunctions of some combination of that variables or their negations.
- This is called the disjunctive normal form of a proposition.
- Example

  (A∧B)∨(A∧C)∨(B∧C∧D)
- The individual conjunctions that make up the disjunctive normal form are called minterms.

Method to construct DNF

- Construct a truth table for the proposition.
- Use the rows of the truth table where the proposition is True to construct minterms
  - If the variable is true, use the propositional variable in the minterm
  - If a variable is false, use the negation of the variable in the minterm
- Connect the minterms with V's(OR's).

Example

DNF/SOP : 1, 1

Find the disjunctive normal form for the proposition p → q

p	**q**	**p → q**
T	T	T
T	F	F
F	T	T
F	F	T

$(p \wedge q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q)$

p → q is true when either
1. p is true and q is true,
2. or p is false and q is true,
3. or p is false and q is false.

CNF/POS

$(\neg p \vee q) \wedge (\quad)$

The disjunctive normal form is then (p ∧ q) ∨ (¬p ∧ q) ∨ (¬p ∧ ¬q)

Conjunctive normal form(CNF)/POS

- Conjunctive normal form of a proposition is the equivalent form that consists of a “conjunction of disjunctions.”
- Example

  $(A \vee B) \wedge (A \vee C) \wedge (B \vee C \vee D)$

- If we want to get the conjunctive normal form of a proposition, construct the disjunctive normal form of its negation and then negate again and apply De Morgan’s Laws.

Example: Find the conjunctive normal form of the proposition (p∧¬q)∨r.

Solution:

T DNF - F

(1) Negate: ¬[(p ∧ ¬q) ∨ r] ⇔ (¬p ∨ q) ∧ ¬r.

(2) Find the disjunctive normal form of (¬p ∨ q) ∧ ¬r:

(¬p ∨ ¬q ∨ ¬r)

p	q	r	¬p	¬r	(¬p ∨ q)	(¬p ∨ q) ∧ ¬r
T	T	T	F	F	T	F
T	T	F	F	T	T	T
T	F	T	F	F	F	F
T	F	F	F	T	F	F
F	T	T	T	F	T	F
F	T	F	T	T	T	T
F	F	T	T	F	T	F
F	F	F	T	T	T	T

The disjunctive normal form for (¬p ∨ q) ∧ ¬r is (p ∧ q ∧ ¬r) ∨ (¬p ∧ q ∧ ¬r) ∨ (¬p ∧ ¬q ∧ ¬r)

The conjunctive normal form for (p∧ ¬q)∨r is then the negation of this last expression, which, by De Morgan's Laws, is
(¬p ∨ ¬q ∨ r) ∧ (p ∨ ¬q ∨ r) ∧ (p ∨ q ∨ r).

Well Formed Formula

- Any expression that obeys the syntactic rules of propositional logic is called a *well-formed formula*, or *WFF*
- The well-formed formulas of propositional logic are obtained by using the construction rules below:
  - An atomic proposition p is a well-formed formula.
  - If p is a well-formed formula, then so is ¬p.
  - If p and q are well-formed formulas, then so are p ∨ q , p ∧ q , and p → q.

Session : Basic Logic

6:Propositional Inference Rules

Argument

- In mathematics, an argument is a sequence of propositions (called premises) followed by a proposition (called conclusion)
- A valid argument is one that, if all its premises are true, then the conclusion is true
- Ex: "If it rains, I drive to school."

  "It rains."

  ∴ "I drive to school."
- Rules of inference are templates for building valid arguments

$F \rightarrow T$

$T \rightarrow T$ T

$p \rightarrow q$

$\Rightarrow [(P_1 \wedge P_2 \wedge \dots \wedge P_n) \rightarrow q]$

Tautology

$p \rightarrow q$

p

$\therefore q$

Premises P_1, P_2, P_3

∴ Conclusion

Argument

$p \rightarrow q$

If p then q

Valid Argument Form

- In the previous example, the argument belongs to the following form:

  $p \to q$
  p
  $\therefore q$

- Indeed, the above form is valid no matter what propositions are substituted to the variables

- This is called a valid argument form

- By definition, if a valid argument form consists

– premises: $p1$, $p2$, … , pk

– conclusion: q

then $(p1 \wedge p2 \wedge \ldots \wedge pk) \to q$ is a tautology

Rules of Inference

$p \to q$

1. Addition

premise: p

conclusion: p ∨ q

Corresponding Tautology: $p \to (p \vee q)$

T T/F

F T T

2. Simplification

premise: p ∧ q

conclusion: p

Corresponding Tautology: $[(p \wedge q) \to p]$ T

Rules of Inference

3. Modus Ponens (method of affirming) or law of detachment

premises: p, p → q

① p
② p → q
∴ q

conclusion: q

Corresponding Tautology: $(p \wedge (p \rightarrow q)) \rightarrow q$

4. Modus Tollens (method of denying)

premises: ¬q, p → q

① ¬q
② p → q
∴ ¬p

conclusion: ¬p

Corresponding Tautology: $(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$

Rules of Inference

5. Hypothetical Syllogism

premises: $p \rightarrow q$, $q \rightarrow r$

conclusion: $p \rightarrow r$

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

Corresponding Tautology: $((p \rightarrow q) \land (q \rightarrow r)) \rightarrow (p \rightarrow r)$

6. Disjunctive Syllogism

premises: $\neg p$, $p \lor q$

conclusion: q

$$\begin{array}{l} \neg p \\ p \lor q \\ \hline \therefore q \end{array}$$

Corresponding Tautology: $((p \lor q) \land \neg p) \rightarrow q$

Rules of Inference

7. Conjunction

premises: p, q

conclusion: p ∧ q

$$\begin{array}{l} p \\ q \\ \hline \therefore p \wedge q \end{array}$$

Corresponding Tautology: $((p) \wedge (q)) \rightarrow (p \wedge q)$

8. Resolution

premises: p ∨ q, ¬p ∨ r

conclusion: q ∨ r

$$\begin{array}{l} p \vee q \\ \neg p \vee r \\ \hline \therefore q \vee r \end{array}$$

Corresponding Tautology: $((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$

Rules of Inference

9. Constructive Dilemma

premises: $(p \rightarrow q) \land (r \rightarrow s)$

$(p \lor r)$

conclusion: $(q \lor s)$

Corresponding Tautology: $(((p \rightarrow q) \land (r \rightarrow s)) \land (p \lor r)) \rightarrow (q \lor s)$

10. Destructive Dilemma

premises: $(p \rightarrow q) \land (r \rightarrow s)$

$(\neg q \lor \neg s)$

conclusion: $(\neg p \lor \neg r)$

Corresponding Tautology: $(((p \rightarrow q) \land (r \rightarrow s)) \land (\neg q \lor \neg s)) \rightarrow (\neg p \lor \neg r)$

Example 1

- State which rule of inference is the basis of the following argument.

"It is below freezing now Therefore, it is either below freezing or raining now."

Solution:

Let P: It is below freezing now.

Q: It is raining now.

premise: p

conclusion: p ∨ q

$$\frac{p}{\therefore p \vee q}$$

This is an argument that uses the addition rule.

Example 2

- State which rule of inference is the basis of the following argument.

“**It is below freezing and raining now. Therefore, it is raining now**.”

Solution:

Let P: It is below freezing now.

Q: It is raining now.

premise: $p \land q$
conclusion: p

This is an argument that uses the Simplification rule.

Example 3

- State which rule of inference is the basis of the following argument.

  "if it rains today, then we will not have a barbecue today.

  If we do not have a barbecue today, then we will have a barbecue tomorrow.

  Therefore if it rains today, then we will have a barbecue tomorrow."

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$

Solution:

Let p: It is raining today.

q: we will not have a barbecue today.

r: we will have a barbecue tomorrow.

p

premises: p → q, q → r

conclusion: $p \to r$

This is an argument that uses the Hypothetical Syllogism rule.

Example 4

- It is known that
  1. It is not sunny this afternoon, and it is colder than yesterday.
  2. We will go swimming only if it is sunny.
  3. If we do not go swimming, we will play basketball.
  4. If we play basketball, we will go home early.
- Can you conclude "we will go home early"?

t

Solution

Premises:

A. It is not sunny this afternoon, and it is colder than yesterday.

B. We will go swimming only if it is sunny.

C. If we do not go swimming, we will play basketball.

D. If we play basketball, we will go home early.

q if p

- To simplify the discussion, let

  ① $\neg p \wedge q$

  ② $r \to p$

  ③

  p := It is sunny this afternoon

  q := It is colder than yesterday

  r := We will go swimming

  s := We will play basketball

  t := We will go home early

- We will give a valid argument with premises: $\neg p \wedge q$, $r \to p$, $\neg r \to s$, , $s \to t$ conclusion: t

Step	Reason
1. $\neg p \wedge q$	Premise A
2. $\neg p$	Simplification using (1)
3. $r \to p$	Premise B
4. $\neg r$	Modus Tollens using (2) and (3)
5. $\neg r \to s$	Premise C
6. s	Modus Ponens using (4) and (5)
7. $s \to t$	Premise D
8. t	Modus Ponens using (6) and (7)

$p \wedge q$

p

Example 5

- Show that the premises

  1. If you send me an email message, then I will finish writing the program.
  2. If you do not send me an email message, then I will go to sleep early.
  3. If I go to sleep early, then I will wake up feeling refreshed.

  Leads to the conclusion

  I do not finish writing the program, then I will wake up feeling refreshed.

Step	Reason
1. $p \rightarrow q$	Premise
2. $\neg q \rightarrow \neg p$	Contrapositive
3. $\neg p \rightarrow r$	Premise
4. $\neg q \rightarrow r$	Hypothetical Syllogism
5. $r \rightarrow s$	Premise
6. $\neg q \rightarrow s$	Hypothetical Syllogism

Session : Basic Logic

7:Predicate Logic

Limitation of Propositional Logic

If $x>3$, then $y=5$
else $y=7$

$P(x)$ - Unary
$P(x,y)$ - Binary

Quantifier

- Every COE student must study discrete mathematics
- Deepak is a COE student
  - – So Deepak must study discrete mathematics

  This idea can't be expressed with propositional logic

P: x is an odd integer
2, 3

$\rightarrow$
$\leftrightarrow$

$\Rightarrow \left[\begin{array}{l} x > 3 \\ x+y=7 \end{array} \right]$

- What propositional logic allows to express:
  - If Deepak is a COE student he must study discrete mathematics
  - Deepak is a COE student
- So Deepak must study discrete mathematics

She is tall & fair
x lives in city y in year z

Predicates

$P(c_1, c_2, \ldots, c_n)$ Proposition

$P(x_1, x_2, \ldots, x_n)$ – n-ary

$P(x, y, z)$ $S(x, y)$

Predicate

$\boxed{x+y=2} \rightarrow Sum(x, y, z)$ 2 3 5 $M(x, y)$ $\boxed{x>3} \rightarrow T$ $\forall$ U

- A predicate is a statement that contains variables (predicate variables) and that may be true or false depending on the values of these variables.

Universe of Discourse

Example: P(x) = “x^2 is greater than x” is a predicate

- The domain of a predicate variable is the collection of all possible values that the variable may take.

e.g. the domain of x in P(x): integer

$\boxed{\text{Binding}}$

→ Valid
→ Satisfiable in U
→ Unsatisfiable in U

- Predicate logic is an extension of Propositional logic.

- It adds the concept of predicates and quantifiers to better capture the meaning of statements that cannot be adequately expressed by propositional logic.

Example

- Let $P(x, y)$ = "$x > y$".

  Domain: integers, i.e. both x and y are integers.
- $P(4, 3)$ means "$4 > 3$", so $P(4, 3)$ is TRUE;
- $P(1, 2)$ means "$1 > 2$" , so $P(1, 2)$ is FALSE;
- $P(3,4)$ is false (in general, $P(x, y)$ and $P(y, x)$ not equal).

Binding using

Quantifiers

- The variable of predicates is quantified by quantifiers.
- There are two types of quantifier in predicate logic –

1. Universal Quantifier
2. Existential Quantifier.

Universal Quantifiers

∀ ∃ Asympt

- Universal quantifier states that the statement within its scope is true for every value of the specific variable.
- It is denoted by the symbol ∀.
- ∀x P(x) is read as for every value of x, P(x) is true.
- Example: Let P(x) denote x > x - 1.

  What is the truth value of ∀x P(x) ?

  Assume the universe of discourse of x is all real numbers.

  Answer: Since every number x is greater than itself minus 1.

  Therefore, ∀x P(x) is true

$\boxed{\forall x P(x)}$

for all x
for any x
every x
each x
arb - x

Ef- U: Z

T → $\forall x [x < x+1]$

F → $\forall x [x = 3]$

$\forall x [x = x+1] \rightarrow$ F

Existential quantifier

- Existential quantifier states that the statement within its scope is true for some values of the specific variable.
- It is denoted by the symbol ∃.
- ∃x P(x) is read as for some values of x, P(x) is true.
- Example Let T(x) denote x > 5 and x is from Real numbers.
- What is the truth value of ∃ x T(x)?
  - Answer: Since 10 > 5 is true.
  - Therefore, it is true that ∃ x T(x).

$\exists x\, P(x)$

$\exists x\, [x < x+1]$ — T

$\exists x\, [x = 3]$ — T

$\exists x\, [x = x+1]$ — F

$\exists! x\, P(x)$

$\exists! x\, [x < x+1]$ — F

$\exists! x\, [x = 3]$ — T

Nested Quantification

$P(x,y,z)$

$\underline{\forall x}\, P(x,y,z)$

$\forall x\, P(x,2,2)$

$\exists z\, \forall x\, \underbrace{P(x,z,z)}_{T \text{ or } F}$

- A proposition can have multiple quantifier
  - "All rabbits are faster than all tortoises."
  - Domains: R={rabbits}, T={tortoises}
  - Predicate C(x, y): Rabbit x is faster than tortoise y
- In symbols
  - ∀x ∈ R, ∀y ∈ T, C(x,y)
- In words
  - For any rabbit x, and for any tortoise y, x is faster than y.

$\forall x\, \forall y\, P(x,y)$

$\forall y\, \forall x$

$\forall x\, \forall y\, P(x,y)$

$\exists x\, \forall y$

$\left.\begin{array}{l}\exists x\, \forall y\, P(x,y) \\ \forall y\, \exists x\, P(x,y)\end{array}\right]$? $\underline{NO}$

a) $\forall x \exists y$ [x is married to y] — T

$\exists y \forall x$ [ ] — F

$U = \mathbb{Z}$

$\forall x \exists y [x + y = 0]$ — T

$\boxed{y = -x}$

$\exists y \forall x [x + y = 0]$ —

$\forall x \forall y \exists! z [x + y = z]$

$\forall x \exists! z \forall y [x + y = z]$ — False

1 + 3 5

Translation

$U = Z$

$N(x)$ = x is non -ve Z Z^+

$E(x)$ = x is even

$O(x)$ = x is odd

$P(x)$ = x is prime

$\forall x \, [P(x) \rightarrow N(x)]$

Even prime is 2

$\forall x \, [E(x) \wedge P(x) \Rightarrow x = 2]$

a) There exists an even integer - $\exists x \, E(x)$

Every int. is even or odd — $\forall x \, [E(x) \vee O(x)]$

There is one & only one even prime $\exists! x \, [E[x] \wedge P[x]]$

Not all integers are odd $\Rightarrow$ $\neg \forall x \, O(x) \Rightarrow \exists x \, \neg O(x)$

$\neg \forall x \, [P[x] \Rightarrow O(x)]$

Truth Value of Quantified Statements

Statement	When True	When False
$\forall x \in D, P(x)$	P(x) is true for every x.	There is one x for which P(x) is false.
$\exists x \in D, P(x)$	There is at least one x for which P(x) is true.	P(x) is false for every x.

Negation of Quantification

- Negation of a universal quantification becomes an existential quantification.

$$\neg (\forall x \in D, P(x)) \equiv \exists x \in D, \neg P(x)$$

- Example:

Not all students study hard ≡ There is at least one student who do not study hard

- Negation of an existential quantification becomes an universal quantification.

$$\neg (\exists x \in D, P(x)) \equiv \forall x \in D, \neg P(x)$$

- Example:

It is not the case that some students in this class are from Jaipur≡ All students in this class are not from Jaipur

Contd..

- $\neg (\forall x \in D, P(x) \wedge Q(x))$

  $\equiv \exists x \in D, \neg(P(x) \wedge Q(x))$ (Negation of Quantification)

  $\equiv \exists x \in D, (\neg P(x) \vee \neg Q(x))$ (DeMorgan Law)

- Example: Not all students in this class are using Facebook and (also) Google+

  There is some (at least one) student in this class who is not using Facebook or not using Google+ (or may be using neither)

Summary of Session

- Propositional Logic
- Logical Connectives
- Truth tables
- Logical Equivalences
- Normal Forms
- Propositional inference rules
- Predicate Logic