Lecture-17

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Lecture 17: Numerical Analysis (UMA011)

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System of linear equations

$1x+2y=7$
$3x-5y=9$

$AX=b$

$A=\begin{bmatrix}1 & 2\\ 3 & -5\end{bmatrix}_{2\times 2}, \quad b=\begin{bmatrix}7\\9\end{bmatrix}, \quad X=\begin{bmatrix}x\\y\end{bmatrix}=?$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ...a_{1n}x_n = b_1\\ a_{21}x_1 + a_{22}x_2 + ...a_{2n}x_n = b_2\\ ....\\ a_{n1}x_1 + a_{n2}x_2 + ...a_{nn}x_n = b_n. \end{cases} \tag{1.1}$$

This is a linear system of n equation in n unknowns $x_1, x_2......x_n$. This system can simply be written in the matrix equation form

Ax=b

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \times \underbrace{\begin{bmatrix} x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix}}_{x=} = \underbrace{\begin{bmatrix} b_1\\ b_2\\ \vdots\\ b_n\end{bmatrix}}_{b=} \tag{1.2}$$

System of linear equations

Direct Methods

1. Gauss Elimination
2. LU Decomposition(Factorization)

Gauss- Elimination Method:-

To solve larger system of linear equation, we consider a following $n \times n$ system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \quad (E_1)\\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \quad (E_2)\\ &\ldots\ldots\\ a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + \cdots + a_{in}x_n &= b_i \quad (E_i)\\ &\ldots\ldots\\ a_{n1}x_1 + a_{n2}x_2 + a_{n3} + \cdots + a_{nn}x_n &= b_n \quad (E_n). \end{aligned}$$

Here E_i denote the i-th row of the coefficients matrix A, $i = 1, 2, \cdots, n$.

① write Augmented Matrix. $[A:b]$ or $[A|b]$

$$[A:b] = \begin{array}{c} E_1\\ E_2\\ \vdots\\ E_n \end{array}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & : b_1\\ a_{21} & a_{22} & \cdots & a_{2n} & : b_2\\ \vdots & & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} & : b_n \end{bmatrix}$$

If $a_{11}=0$ find p s.t $a_{p1}\neq 0$ $\Rightarrow E_p \leftrightarrow E_1$

② To Reduce A into upper triangular Matrix.

$E_2 \to E_2 - \underbrace{\frac{a_{21}}{a_{11}}}_{=m_{21}} E_1$

$E_3 \to E_3 - \underbrace{\frac{a_{31}}{a_{11}}}_{=m_{31}} E_1 \quad \cdots \quad E_n \to E_n - \underbrace{\frac{a_{n1}}{a_{11}}}_{=m_{n1}} E_1$

Let $a_{11} \neq 0$ and eliminate x_1 from $E_2, E_3, \cdots, E_n$.

Define multipliers $m_{j1} = \dfrac{a_{j1}}{a_{11}}$, for each $j = 2, 3, \cdots, n+1$

We write each entry in E_j as $E_j - m_{j1}E_1$ [redacted].

$$\boxed{E_j \to E_j - \underbrace{\frac{a_{j1}}{a_{11}}}_{=m_{j1}} E_1} \quad j = 2, 3, 4, \cdots n+1$$

$$\sim \begin{array}{c} E_1\\ E_2\\ \vdots\\ E_n \end{array}\begin{bmatrix} a_{11} & a_{12}^* & \cdots & a_{1n}^* & : b_1^*\\ 0 & a_{22}^* & \cdots & a_{2n}^* & : b_2^*\\ 0 & a_{32}^* & & & \\ \vdots & & & & \\ 0 & a_{n2}^* & \cdots & a_{nn}^* & : b_n^* \end{bmatrix}$$

If $a_{22}^* = 0$ then find q s.t $a_{q2}^* \neq 0$ $E_2 \leftrightarrow E_q$

$E_3 \to E_3 - \dfrac{a_{32}^*}{a_{22}^*} E_2, \quad E_4 \to E_4 - \dfrac{a_{42}^*}{a_{22}^*} E_2$

$\cdots \quad E_n \to E_n - \dfrac{a_{n2}^*}{a_{22}^*} E_2$

$$\boxed{E_j \to E_j - \frac{a_{j2}^*}{a_{22}^*} E_2} \quad j = 3, 4, \cdots n.$$

$$\sim \begin{bmatrix} a_{11}^* & a_{12}^* & \cdots & a_{1n}^* & : b_1^*\\ 0 & a_{22}^{**} & \cdots & a_{2n}^{**} & : b_2^{**}\\ 0 & 0 & & & \\ \vdots & \vdots & & & \\ 0 & 0 & \cdots & a_{nn}^{**} & : b_n^{**} \end{bmatrix} \qquad E_j \to E_j - \frac{a_{ji}}{a_{ii}} E_i; \; i = 1, \cdots n-1, \; j = i+1, i+2, \cdots n$$

We repeat this procedure and follow a sequential procedure for $j = 2, 3, \cdots, n$ and perform the operations

$$E_j \longrightarrow E_j - (a_{ji}/a_{ii}E_i), \quad \text{for each } j = i+1, i+2, \cdots, n.$$

[redacted]

provided $a_{ii} \neq 0$. This eliminates x_i in each row below the i-th values. [redacted] The resulting matrix is triangular and has [redacted]

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1\\ 0 & a_{22} & \cdots & a_{2n} & b_2\\ \vdots & & & a_{n-1,n} & b_{n-1}\\ 0 & 0 & \cdots & a_{nn} & b_n \end{bmatrix} \to \text{upper triangular Matrix}$$

Step 3 Use backward substitution.

Solving the n-th equation for x_n gives

$a_{nn}x_n = b_n$

$$x_n = \frac{b_n}{a_{nn}},$$

$a_{n-1,n-1}x_{n-1} + a_{n-1,n}x_n = b_{n-1}$

$\Rightarrow x_{n-1} = \dfrac{b_{n-1} - a_{n-1,n}x_n}{a_{n-1,n-1}}$

Solving the $(n-1)$st equation for x_{n-1} and using the known value for x_n yields (back substitution)

$$x_{n-1} = \frac{b_{n-1} - a_{n-1,n}x_n}{a_{n-1,n-1}}$$

$x_1 = \dfrac{b_1 - (a_{12}x_2 - \cdots - a_{1n}x_n)}{a_{11}}$

Continuing this process, we obtain

$$x_i = \frac{b_i - (a_{i,i+1}x_{i+1} + a_{i,i+2}x_{i+2} + \cdots + a_{in}x_n)}{a_{ii}} = \frac{b_i - \sum_{j=i+1}^{n} a_{ij}x_j}{a_{ii}},$$

$i = n-1$

for each $i = n-1, n-2, \cdots, 2, 1$.

System of linear equations

Example

Use Gaussian elimination with backward substitution and two-digit rounding arithmetic to solve the following linear system. (The exact solution to each system is $x_1 = -1,\ x_2 = 1,\ x_3 = 3$.)

$$\begin{aligned} -x_1 + 4x_2 + x_3 &= 8\\ \frac{5}{3}x_1 + \frac{2}{3}x_2 + \frac{2}{3}x_3 &= 1\\ 2x_1 + x_2 + 4x_3 &= 11. \end{aligned}$$

Solution:

$$[A:b] = \begin{array}{c} E_1\\ E_2\\ E_3 \end{array}\begin{bmatrix} -1 & 4 & 1 & : 8\\ 1.7 & 0.67 & 0.67 & : 1\\ 2 & 1 & 4 & : 11 \end{bmatrix}$$

$E_2 \to E_2 - \dfrac{1.7}{-1} E_1$

$E_3 \to E_3 - \dfrac{2}{-1} E_1$

$E_2 \to E_2 + 1.7 E_1$

$E_3 \to E_3 + 2 E_1$

$0.67 + 1.7 * 4$
$0.67 + 1.7 * 1$
$1 + 1.7 * 8$
$1 + 2 * 4$
$4 + 2 * 1$
$11 + 2 * 8$

$$\sim \begin{array}{c} E_1\\ E_2\\ E_3 \end{array}\begin{bmatrix} -1 & 4 & 1 & : 8\\ 0 & 7.5 & 2.4 & : 15\\ 0 & 9 & 6 & : 27 \end{bmatrix}$$

$E_3 \to E_3 - \dfrac{9}{7.5} E_2$

$E_3 \to E_3 - 1.2 E_2$ by taking 2-digits rounding

$$\sim \begin{bmatrix} -1 & 4 & 1 & : 8\\ 0 & 7.5 & 2.4 & : 15\\ 0 & 0 & 3.1 & : 9 \end{bmatrix}$$

$6 - 1.2 * 2.4$
$27 - 1.2 * 15$

↳ upper triangular Matrix

Use backward sub.

$3.1x_3 = 9$

$x_3 = \dfrac{9}{3.1} = 2.9 \qquad \Rightarrow x_3 = 2.9$

$7.5x_2 + 2.4x_3 = 15$

$x_2 = \dfrac{15 - 2.4 * 2.9}{7.5} = 1.1 \qquad x_2 = 1.1$

$-x_1 + 4x_2 + x_3 = 8$

$-x_1 = 8 - 4x_2 - x_3$

$\Rightarrow x_1 = -8 + 4x_2 + x_3$

$x_1 = -0.7$

$X = \begin{bmatrix} -0.7\\ 1.1\\ 2.9 \end{bmatrix}$ Ans.

System of linear equations:

Exercise:

1. Using four-digit arithmetic operations, solve the following system of equations by Gaussian elimination

$$\begin{aligned} 0.729x_1 + 0.81x_2 + 0.9x_3 &= 0.6867\\ x_1 + x_2 + x_3 &= 0.8338\\ 1.331x_1 + 1.21x_2 + 1.1x_3 &= 1.000. \end{aligned}$$

This system has exact solution, rounded to four places $x_1 = 0.2245$, $x_2 = 0.2814$, $x_3 = 0.3279$. Compare your answers!