Numerical Analysis

Solution of Exercises : Chapter 4[1]
Iterative Techniques in Matrix Algebra

1. Find l_∞ and l_2 norms of the vectors.

(a) $x = (3, -4, 0, \frac{3}{2})^t$.

(b) $x = (\sin k, \cos k, 2^k)^t$ for a fixed positive integer k.

Sol. The l_∞ and l_2 norms of the vector $x = (x_1.x_2, \cdots, x_n)^t$ are defined by

$$\|x\|_\infty = \max_{1\le i\le n} |x_i|$$

$$\|x\|_2 = \left\{\sum_{i=1}^{n} x_i^2\right\}^{\frac{1}{2}}.$$

(a) For $x = (3, -4, 0, \frac{3}{2})^t$, we have

$$\begin{aligned}\|x\|_\infty &= \max\left\{|3|, |-4|, |0|, \left|\frac{3}{2}\right|\right\} = 4\\ \|x\|_2 &= \sqrt{3^2 + (-4)^2 + 0^2 + (3/2)^2} = 5.2202.\end{aligned}$$

(b) For $x = (\sin k, \cos k, 2^k)^t$, we have

$$\begin{aligned}\|x\|_\infty &= \max\left\{|\sin k|, |\cos k|, |2^k|\right\} = 2^k\\ \|x\|_2 &= \sqrt{\sin^2 k + \cos^2 k + (2^k)^2} = \sqrt{1+4^k}.\end{aligned}$$

2. Find the l_∞ norm of the matrix: $\begin{bmatrix} 4 & -1 & 7\\ -1 & 4 & 0\\ -7 & 0 & 4\end{bmatrix}$.

Sol. We have

$$\begin{aligned}\sum_{j=1}^{n}|a_{1j}| &= |a_{11}| + |a_{12}| + |a_{13}| = |4| + |-1| + |7| = 12\\ \sum_{j=1}^{n}|a_{2j}| &= |a_{21}| + |a_{22}| + |a_{23}| = |-1| + |4| + |0| = 5\\ \sum_{j=1}^{n}|a_{3j}| &= |a_{31}| + |a_{32}| + |a_{33}| = |-7| + |0| + |4| = 11.\end{aligned}$$

So we have $\|A\|_\infty = \max\{12, 5, 11\} = 12$.

3. The following linear system $Ax = b$ have x as the actual solution and $\tilde{x}$ as an approximate solution. Compute $\|x - \tilde{x}\|_\infty$ and $\|A\tilde{x} - b\|_\infty$. Also compute $\|A\|_\infty$.

$$\begin{aligned}x_1 + 2x_2 + 3x_3 &= 1\\ 2x_1 + 3x_2 + 4x_3 &= -1\\ 3x_1 + 4x_2 + 6x_3 &= 2,\\ x &= (0, -7, 5)^t\\ \tilde{x} &= (-0.2, -7.5, 5.4)^t.\end{aligned}$$

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Sol. We have

$$
\begin{aligned}
x - \tilde{x} &= \begin{bmatrix} 0 \\ -7 \\ 5 \end{bmatrix} - \begin{bmatrix} -0.2 \\ -7.5 \\ 5.4 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.5 \\ -0.4 \end{bmatrix} \\
\|x - \tilde{x}\|_\infty &= \max\{|0.2|, |0.5|, |-0.4|\} = 0.5 \\
A\tilde{x} &= \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{bmatrix} \times \begin{bmatrix} -0.2 \\ -7.5 \\ 5.4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1.3 \\ 1.8 \end{bmatrix} \\
A\tilde{x} - b &= \begin{bmatrix} 1 \\ -1.3 \\ 1.8 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.3 \\ -0.2 \end{bmatrix} \\
\|A\tilde{x} - b\|_\infty &= \max\{|0|, |-0.3|, |-0.2|\} = 0.3.
\end{aligned}
$$

Also we have (for coefficient matrix A)

$$
\begin{aligned}
\sum_{j=1}^{n} |a_{1j}| &= |a_{11}| + |a_{12}| + |a_{13}| = |1| + |2| + |3| = 6 \\
\sum_{j=1}^{n} |a_{2j}| &= |a_{21}| + |a_{22}| + |a_{23}| = |2| + |3| + |4| = 9 \\
\sum_{j=1}^{n} |a_{3j}| &= |a_{31}| + |a_{32}| + |a_{33}| = |3| + |4| + |6| = 13.
\end{aligned}
$$

Thus we have $\|A\|_\infty = \max\{6, 9, 13\} = 13$.

4. Find the first two iterations of Jacobi and Gauss-Seidel using $x^{(0)} = 0$:

$$
\begin{aligned}
4.63x_1 - 1.21x_2 + 3.22x_3 &= 2.22 \\
-3.07x_1 + 5.48x_2 + 2.11x_3 &= -3.17 \\
1.26x_1 + 3.11x_2 + 4.57x_3 &= 5.11.
\end{aligned}
$$

Sol. Jacobi Iterations:
Iterations are given by

$$
\begin{aligned}
x_1^{(k+1)} &= \frac{1}{4.63}(2.22 + 1.21x_2^{(k)} - 3.22x_3^{(k)}) \\
x_2^{(k+1)} &= \frac{1}{5.48}(-3.17 + 3.07x_1^{(k)} - 2.11x_3^{(k)}) \\
x_3^{(k+1)} &= \frac{1}{4.57}(5.11 - 1.26x_1^{(k)} - 3.11x_2^{(k)}), \quad k = 0, 1, 2, \cdots
\end{aligned}
$$

Start with $x^{(0)} = [0, 0, 0]^t$, the first iteration is

$$x_1^{(1)} = 0.47948$$

$$x_2^{(1)} = -0.57847$$

$$x_3^{(1)} = 1.11816.$$

The second iteration is

$$x_1^{(2)} = -0.44934$$

$$x_2^{(2)} = -0.74039$$

$$x_3^{(2)} = 1.37962.$$

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Gauss-Seidel Iterations:
Iterations are given by

$$\begin{aligned} x_1^{(k+1)} &= \frac{1}{4.63}(2.22 + 1.21x_2^{(k)} - 3.22x_3^{(k)}) \\ x_2^{(k+1)} &= \frac{1}{5.48}(-3.17 + 3.07x_1^{(k+1)} - 2.11x_3^{(k)}) \\ x_3^{(k+1)} &= \frac{1}{4.57}(5.11 - 1.26x_1^{(k+1)} - 3.11x_2^{(k+1)}), \quad k = 0, 1, 2, \cdots \end{aligned}$$

Start with $x^{(0)} = [0, 0, 0]^t$, the first iteration is

$$x_1^{(1)} = 0.47948$$

$$x_2^{(1)} = -0.30985$$

$$x_3^{(1)} = 1.19683.$$

The second iteration is

$$x_1^{(2)} = -0.43384$$

$$x_2^{(2)} = -1.28234$$

$$x_3^{(2)} = 2.11044.$$

5. The linear system

$$\begin{aligned} x_1 - x_3 &= 0.2 \\ -\frac{1}{2}x_1 + x_2 - \frac{1}{4}x_3 &= -1.425 \\ x_1 - \frac{1}{2}x_2 + x_3 &= 2 \end{aligned}$$

has the solution $(0.9, -0.8, 0.7)^T$.

(a) Is the coefficient matrix strictly diagonally dominant?

(b) Compute the spectral radius of the Gauss-Seidel iteration matrix.

(c) Perform four iterations of the Gauss-Seidel iterative method to approximate the solution.

(d) What happens in part (c) when the first equation in the system is changed to $x_1 - 2x_3 = 0.2$?

Sol. Given coefficient matrix is

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -1/2 & 1 & -1/4 \\ 1 & -1.2 & 1 \end{bmatrix}.$$

(a) The given matrix A not strictly diagonally dominant as

$$\begin{aligned} |a_{11}| = 1 &\not> |a_{12}| + |a_{13}| = |0| + |-1| = 1 \\ |a_{22}| = 1 &> |a_{21}| + |a_{23}| = 1/2 + 1/4 = 3/4 \\ |a_{33}| = 1 &\not> |a_{31}| + |a_{32}| = 1 + 1/2 = 3/2. \end{aligned}$$

Hence we are not sure whether the iterative methods converge or not.

(b) Further any iterative method (Gauss-Seidel) converges to unique solution for the system $Ax = b$ if and only if spectral radius $\rho(T_g) < 1$, where T is the iteration matrix.
Firstly we write $A = L + D + U$ where

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 & 0 \\ -1/2 & 0 & 0 \\ 1 & -1/2 & 0 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -1/4 \\ 0 & 0 & 0 \end{bmatrix}.$$

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Iteration matrix of Gauss-Seidel method is given by

$$\begin{aligned} T_g &= -(D+L)^{-1}U \\ &= -\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1 & -1/2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -1/4 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0.75 \\ 0 & 0 & -0.625 \end{bmatrix}. \end{aligned}$$

The eigenvalues of T_g are $0, 0, -0.625$.
Hence $\rho(T_g) = 0.625 < 1$.
As spectral radius less than one, so method will work.

(c) Iterations are given by

$$\begin{aligned} x_1^{(k+1)} &= (0.2 + x_3^{(k)}) \\ x_2^{(k+1)} &= (-1.425 + 0.5x_1^{(k+1)} + 0.25x_3^{(k)}) \\ x_3^{(k+1)} &= (2 - x_1^{(k+1)} + 0.5x_2^{(k+1)}), \quad k = 0, 1, 2, \cdots \end{aligned}$$

Start with $x^{(0)} = [0, 0, 0]^t$, the first iteration is

$$x_1^{(1)} = 0.2$$
$$x_2^{(1)} = -1.325$$
$$x_3^{(1)} = 1.1375.$$

The second iteration is

$$x_1^{(2)} = 1.3375$$
$$x_2^{(2)} = -0.47188$$
$$x_3^{(2)} = 0.42656.$$

The third iteration is

$$x_1^{(3)} = 0.62656$$
$$x_2^{(3)} = -1.00508$$
$$x_3^{(3)} = 0.87090.$$

The fourth iteration is

$$x_1^{(4)} = 1.07090$$
$$x_2^{(4)} = -0.67183$$
$$x_3^{(4)} = 0.59319.$$

(d) By changing the first equation in the system with $x_1 - 2x_3 = 0.2$ and starting with the same initial guess, the four iterations are given by

$$\begin{aligned} x^{(1)} &= [0.2, -1.325, 1.1375]^T \\ x^{(2)} &= [2.475, 0.096875, -0.426563] \\ x^{(3)} &= [-0.65313, -1.85820, 1.72402] \\ x^{(4)} &= [3.64805, 0.83003, -1.23303]. \end{aligned}$$

If we compare with part (c) and exact solution, iterations seems to be diverging.
Let us compute the spectral radius again (just change the coefficient of x_3 in first equation).
In this case, the iterations matrix becomes

$$T_g = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 1.25 \\ 0 & 0 & -1.375 \end{bmatrix}.$$

Clearly eigenvalues of T_g are $0, 0, -1.375$ and hence $\rho(T_g) = 1.375 > 1$. So Gauss-Seidel method will not converge!

6. Comment whether you can apply Gauss-Seidel method for the following system of equations. If so, then perform two iterations by taking initial guess as $x^{(0)} = [0, 0, 0]^t$.

$$\begin{aligned} 12x_1 + 3x_2 - 5x_3 &= 1 \\ x_1 + 5x_2 + 3x_3 &= 28 \\ 3x_1 + 7x_2 + 13x_3 &= 76. \end{aligned}$$

Sol. Firstly we write $A = L + D + U$ where

$$D = \begin{bmatrix} 12 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 13 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 3 & 7 & 0 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 0 & 3 & -5 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

Iteration matrix of Gauss-Seidel method is given by

$$\begin{aligned} T_g &= -(D+L)^{-1}U \\ &= -\begin{bmatrix} 12 & 0 & 0 \\ 1 & 5 & 0 \\ 3 & 7 & 13 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 3 & -5 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \\ &= -\begin{bmatrix} 0.08333 & 0 & 0 \\ -0.01667 & 0.2 & 0 \\ -0.01026 & -0.10769 & 0.07692 \end{bmatrix} \begin{bmatrix} 0 & 3 & -5 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -0.25 & 0.41667 \\ 0 & 0.05 & -0.68333 \\ 0 & 0.03077 & 0.27179 \end{bmatrix}. \end{aligned}$$

The eigenvalues of T_g are $0, 0.16090 + 0.09342i, 0.16090 - 0.09342i$ (eigenvalues are complex number and absolute value of a complex number $a + bi$ is $\sqrt{a^2 + b^2}$).
Hence $\rho(T_g) = 0.18605 < 1$.
As spectral radius less than one, so method will work.
Two iterations are given by

$$\begin{aligned} x_1^{(k+1)} &= \frac{1}{12}(1 - 3x_2^{(k)} + 5x_3^{(k)}) \\ x_2^{(k+1)} &= \frac{1}{5}(28 - 12x_1^{(k+1)} - 3x_3^{(k)}) \\ x_3^{(k+1)} &= \frac{1}{13}(76 - 3x_1^{(k+1)} - 7x_2^{(k+1)}), \quad k = 0, 1, 2, \cdots \end{aligned}$$

Start with $x^{(0)} = [0, 0, 0]^t$, the first iteration is

$$x_1^{(1)} = 0.083333$$

$$x_2^{(1)} = 5.583333$$

$$x_3^{(1)} = 2.820513.$$

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The second iteration is

$$x_1^{(2)} = -0.13729$$
$$x_2^{(2)} = 3.93515$$
$$x_3^{(2)} = 3.75891.$$

7. Find the first two iterations of the SOR method with $\omega = 1.1$ for the following linear systems, using $x^{(0)} = [0, 0, 0[^t$

$$\begin{aligned} 4x_1 + x_2 - x_3 &= 5 \\ -x_1 + 3x_2 + x_3 &= -4 \\ 2x_1 + 2x_2 + 5x_3 &= 1. \end{aligned}$$

Sol. The SOR iterations are given by

$$\begin{aligned} x_1^{(k+1)} &= (1-\omega)x_1^{(k)} + \frac{\omega}{4}(5 - x_2^{(k)} + x_3^{(k)}) \\ x_2^{(k+1)} &= (1-\omega)x_2^{(k)} + \frac{\omega}{3}(-4 + x_1^{(k+1)} - x_3^{(k)}) \\ x_3^{(k+1)} &= (1-\omega)x_3^{(k)} + \frac{\omega}{5}(1 - 2x_1^{(k+1)} - 2x_2^{(k+1)}). \end{aligned}$$

With $\omega = 1.1$ and initial guess $x^{(0)} = [0, 0, 0[^t$, we have

$$x_1^{(1)} = 1.375$$
$$x_2^{(1)} = -0.9625$$
$$x_3^{(1)} = 0.0385.$$

The second iteration is

$$x_1^{(2)} = 1.512775$$
$$x_2^{(2)} = -0.829849$$
$$x_3^{(2)} = -0.084337.$$

8. Compute the condition numbers of the following matrices relative to $\|.\|_\infty$.

(a) $\begin{bmatrix} 0.03 & 58.9 \\ 5.31 & -6.10 \end{bmatrix}$

(b) $\begin{bmatrix} 0.04 & 0.01 & -0.01 \\ 0.2 & 0.5 & -0.2 \\ 1 & 2 & 4 \end{bmatrix}$.

Sol. The condition number of the nonsingular matrix A relative to a norm $\|\cdot\|$ is

$$K(A) = \|A\| \, \|A^{-1}\|.$$

In particular, the condition number of A relative to $\|\cdot\|_\infty$ is

$$K(A) = \|A\|_\infty \, \|A^{-1}\|_\infty.$$

(a) For matrix

$$A = \begin{bmatrix} 0.03 & 58.9 \\ 5.31 & -6.10 \end{bmatrix}, \qquad A^{-1} = \begin{bmatrix} 0.0195 & 0.1882 \\ 0.0170 & 0.0001 \end{bmatrix}.$$

Thus

$$\|A\|_\infty = \max\{|0.03| + |58.9|, |5.31| + |-6.10|\} = 58.93$$
$$\|A^{-1}\|_\infty = \max\{|0.0195| + |0.1882|, |0.0170| + |-0.0001|\} = 0.2077.$$

Thus $K(A) = 58.93 \times 0.2077 = 12.24$.

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(b) We have

$$A = \begin{bmatrix} 0.04 & 0.01 & -0.01 \\ 0.2 & 0.5 & -0.2 \\ 1 & 2 & 4 \end{bmatrix}, \qquad A^{-1} = \begin{bmatrix} 27.586207 & -0.689655 & 0.034483 \\ -11.494253 & 1.954023 & 0.068966 \\ -1.149425 & -0.804598 & 0.206897 \end{bmatrix}.$$

Thus

$$\|A\|_\infty = 7$$
$$\|A^{-1}\|_\infty = 28.31.$$

Thus $K(A) = 198.17$.

9. The linear system $Ax = b$ given by

$$\begin{bmatrix} 1 & 2 \\ 1.0001 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3.0001 \end{bmatrix}$$

has solution $(1, 1)^t$. Use four-digit rounding arithmetic to find the solution of the perturbed system

$$\begin{bmatrix} 1 & 2 \\ 1.000011 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3.00001 \\ 3.00003 \end{bmatrix}$$

Is matrix A ill-conditioned?

Sol. Same as Example 10 from Notes. Matrix is ill-conditioned.

10. Determine the dominant eigenvalue and the corresponding eigenvector of the following matrix using the power method with $x^{(0)} = [1, -1, 2]^t$.

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}.$$

Use stopping criteria with absolute error less than 0.001 for eigenvalue.

Sol. Firstly we need to scale the given $x^{(0)} = [1, -1, 2]^T$ as $x^{(0)} = [1/2, -1/2, 1]^T = [0.5, -0.5, 1]^T$. Let us take accuracy 0.001 (for eigenvalue).
First iteration:

$$y^{(1)} = Ax^{(0)} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0.5 \\ -0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.5 \\ -2 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 0.75 \\ -1 \\ 0.5 \end{bmatrix} = \lambda^{(1)} x^{(1)}.$$

Second iteration:

$$y^{(2)} = Ax^{(1)} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0.75 \\ -1 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 1.5 \\ -2.25 \\ 1.75 \end{bmatrix} = 2.25 \begin{bmatrix} 0.66667 \\ -1 \\ 0.77778 \end{bmatrix} = \lambda^{(2)} x^{(2)}.$$

Here $|\lambda^{(2)} - \lambda^{(1)}| = |2.25 - 2| = 0.25 > 0.001$.
Similarly third iteration:

$$y^{(3)} = Ax^{(2)} = 2.4444 \begin{bmatrix} 0.72727 \\ -1 \\ 0.68182 \end{bmatrix} = \lambda^{(3)} x^{(3)}.$$

$$|\lambda^{(3)} - \lambda^{(2)}| > 0.001.$$

Forth iteration:

$$y^{(4)} = Ax^{(3)} = 2.4091 \begin{bmatrix} 0.69811 \\ -1 \\ 0.71698 \end{bmatrix} = \lambda^{(4)} x^{(4)}.$$

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$$|\lambda^{(4)} - \lambda^{(3)}| > 0.001.$$

Fifth iteration:

$$y^{(5)} = Ax^{(4)} = 2.4151 \begin{bmatrix} 0.71094 \\ -1 \\ 0.70312 \end{bmatrix} = \lambda^{(5)} x^{(5)}.$$

$$|\lambda^{(5)} - \lambda^{(4)}| > 0.001.$$

Sixth iteration: Fifth iteration:

$$y^{(6)} = Ax^{(5)} = 2.4141 \begin{bmatrix} 0.70550 \\ -1 \\ 0.70874 \end{bmatrix} = \lambda^{(5)} x^{(5)}.$$

$$|\lambda^{(6)} - \lambda^{(5)}| < 0.001.$$

Thus dominant eigenvalue is 2.4141 and corresponding eigenvector is $\begin{bmatrix} 0.70550 \\ -1 \\ 0.70874 \end{bmatrix}$.

11. Use the inverse power method to approximate the smallest eigenvalue of the matrix until a tolerance of 10^{-2} is achieved with $x^{(0)} = [1, 0, 0]^t$.

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Sol. In order to get the smallest eigen-value of matrix A, we apply power method on matrix A^{-1}. We have

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

The initial guess is $\mathbf{x^{(0)}} = [\mathbf{1}, \mathbf{0}, \mathbf{0}]^{\mathbf{t}}$.
First iteration:

$$\begin{aligned} A^{-1}\mathbf{x^{(0)}} &= \mathbf{y^{(1)}} \\ A\mathbf{y^{(1)}} &= \mathbf{x^{(0)}} \\ LU\mathbf{y^{(1)}} &= \mathbf{x^{(0)}}. \end{aligned}$$

Firstly we write LU decomposition of matrix A to solve the above system. We apply Gauss elimination on A and thus obtain

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0.5 & 0.33333 & 1 \end{bmatrix}$$

and

$$U = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1.5 & 0.5 \\ 0 & 0 & 1.33333 \end{bmatrix}.$$

Assuming $U\mathbf{y^{(1)}} = \mathbf{z^{(1)}}$, we obtain $L\mathbf{z^{(1)}} = \mathbf{x^{(0)}}$.
Solving $L\mathbf{z^{(1)}} = \mathbf{x^{(0)}}$ using forward substitution, we obtain $\mathbf{z^{(1)}} = [\mathbf{1}, -\mathbf{0.5}, -\mathbf{0.33333}]^{\mathbf{t}}$,
and solving $U\mathbf{y^{(1)}} = \mathbf{z^{(1)}}$ using backward substitution, we get $\mathbf{y^{(1)}} = [\mathbf{0.75}, -\mathbf{0.25}, -\mathbf{0.25}]^{\mathbf{t}}$.
Here we need not to do any scaling and first approximation to the dominant eigen-value of A^{-1} is 1, which is the largest eigenvalue (and thus smallest of A is reciprocal, i.e. 1). Note that eigenvector remain same and which is $[0.75, -0.25, -0.25]$.

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12. Find the eigenvalue of matrix nearest to 3

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

using inverse power method.

Sol. Given matrix is $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$.
Let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of A.
Then $\lambda_1 - 3, \lambda_2 - 3, \lambda_3 - 3$ will be the eigenvalues of $A - 3I$.
The eigenvalue which will be near 3, will provide the difference $\lambda_i - 3$ minimum. Hence $1/\lambda_i - 3$ will be maximum in that case. So we apply power method on $(A - 3I)^{-1}$ which will pick the largest eigenvalue of this matrix and closest to 3.
(I am solving this example by calculating the inverse of matrix and often it is not advisable to do so. You should apply Gauss elimination to solve the system as we did in the previous exercise to get rid from inverse. But in local exams, you can do it by direct inverse.)
Now we have

$$A - 3I = \begin{bmatrix} -1 & -1 & 0 \\ -1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix}.$$

Thus

$$(A - 3I)^{-1} = \begin{bmatrix} 0 & -0.5 & 0.5 \\ -1 & 0.5 & -0.5 \\ 1 & -1 & 0 \end{bmatrix} = B \text{ (say)}.$$

Start with $x^{(0)} = [1, 1, 1]^T$, here are few iterations.
First iteration:

$$y^{(1)} = Bx^{(0)} = \begin{bmatrix} 0 & -0.5 & 0.5 \\ -1 & 0.5 & -0.5 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \lambda^{(1)} x^{(1)}.$$

Second iteration:

$$y^{(2)} = Bx^{(1)} = \begin{bmatrix} 0 & -0.5 & 0.5 \\ -1 & 0.5 & -0.5 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -0.5 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 0.5 \\ -0.5 \\ 1 \end{bmatrix} = \lambda^{(2)} x^{(2)}.$$

Third iteration:

$$y^{(3)} = Bx^{(2)} = \begin{bmatrix} 0 & -0.5 & 0.5 \\ -1 & 0.5 & -0.5 \\ 0.75 & -1.25 & 1 \end{bmatrix} \begin{bmatrix} 0.5 \\ -0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.75 \\ -1.25 \\ 1 \end{bmatrix} = 1.25 \begin{bmatrix} 0.6 \\ -1 \\ 0.8 \end{bmatrix} = \lambda^{(3)} x^{(3)}.$$

Fourth iteration:

$$y^{(4)} = Bx^{(3)} = \begin{bmatrix} 0 & -0.5 & 0.5 \\ -1 & 0.5 & -0.5 \\ 0.75 & -1.25 & 1 \end{bmatrix} \begin{bmatrix} 0.6 \\ -1 \\ 0.8 \end{bmatrix} = \begin{bmatrix} 0.9 \\ -1.5 \\ 1.6 \end{bmatrix} = 1.6 \begin{bmatrix} 0.5625 \\ -0.9375 \\ 1 \end{bmatrix} = \lambda^{(4)} x^{(4)}.$$

Fifth iteration:

$$y^{(5)} = Bx^{(4)} = \begin{bmatrix} 0 & -0.5 & 0.5 \\ -1 & 0.5 & -0.5 \\ 0.75 & -1.25 & 1 \end{bmatrix} \begin{bmatrix} 0.5625 \\ -0.9375 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.96875 \\ -1.53125 \\ 1.5 \end{bmatrix} = 1.5312 \begin{bmatrix} 0.63265 \\ -1 \\ 97959 \end{bmatrix} = \lambda^{(5)} x^{(5)}.$$

Thus if we stop here, then the dominant eigenvalue of $B = (A - 3I)^{-1}$ is 1.5312. So eigenvalue of $A - 3I$ is $1/1.5312 = 0.65308$ and the eigenvalue of A (near 3) is $0.65308 + 3 = 3.65308$.

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Note that the corresponding eigenvector remain same and is $\begin{bmatrix} 0.5625 \\ -0.9375 \\ 1 \end{bmatrix}$. See the reason below.

Remarks: If λ be an eigenvalue of matrix A and corresponding eigenvector is x then

$$\begin{aligned} Ax &= \lambda x \\ Ax - 3Ix &= \lambda x - 3x \\ (A-3I)x &= (\lambda - 3)x \\ (A-3I)^{-1}x &= \frac{1}{\lambda - 3}x. \end{aligned}$$

Note that x remains same.