Lecture-12

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Lecture 12: Numerical Analysis (UMA011)

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Find the root of $f(x)=0$ in $[a,b]$ by F.P.I.

$\Downarrow$

(1) $g(x)=x$

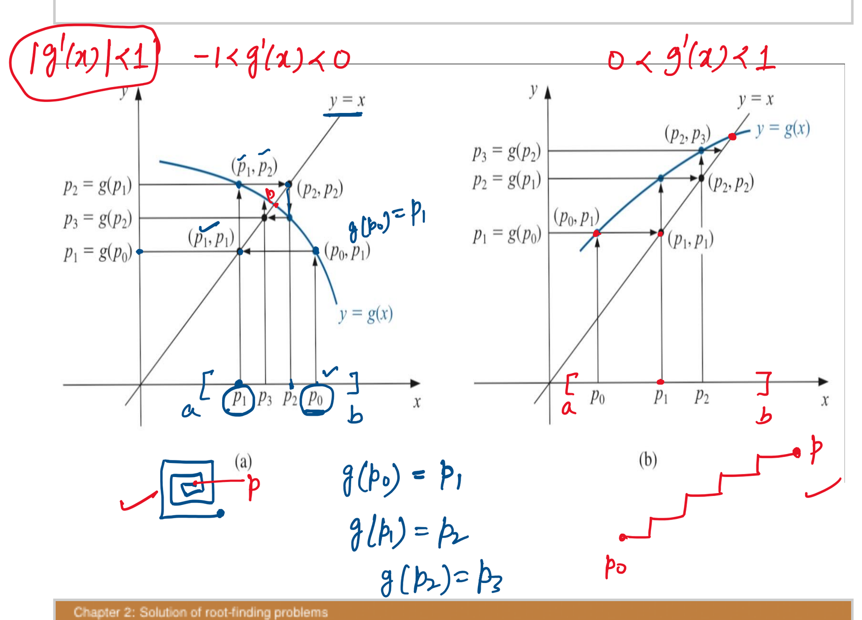
(2) Take $p_0 \in [a,b]$

(3) Make a seq^n $p_{n+1} = g(p_n) \to p$

Find $g(x)$ s.t.
(a) $g(x) \in C[a,b]$
(b) $g(x) \in [a,b]$ $\forall x \in [a,b]$
(c) $|g'(x)| < 1$ $\forall x \in (a,b)$

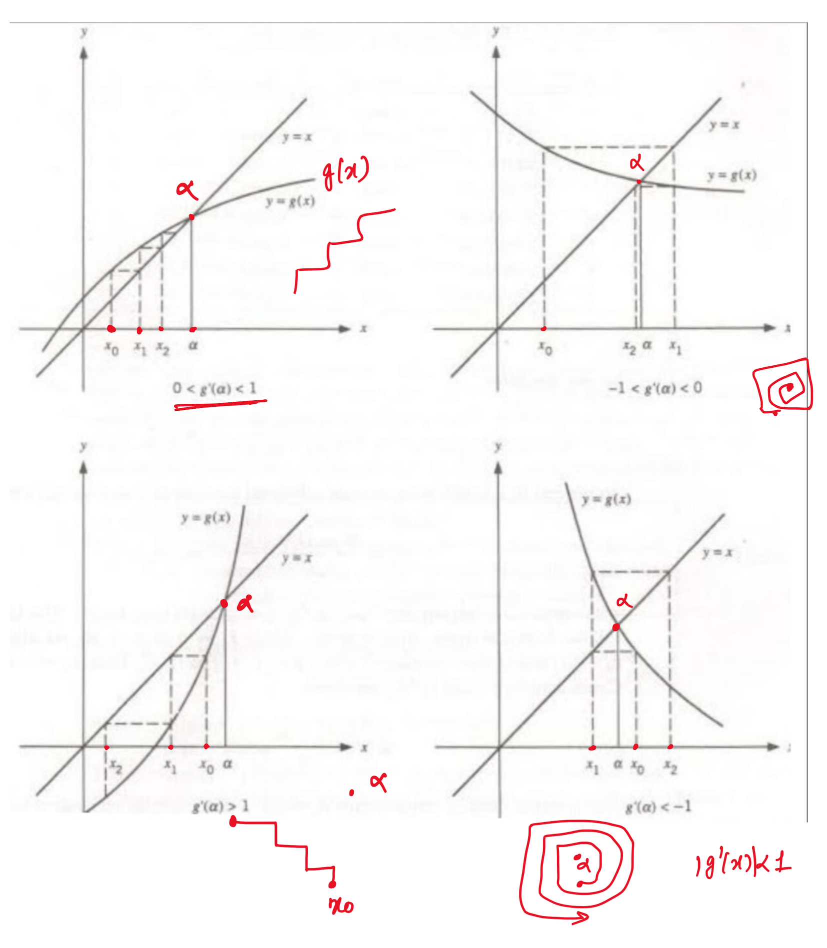
Root finding problem
Convergence through graphics:

$|g'(x)|<1$ $\quad -1<g'(x)<0 \quad\quad 0<g'(x)<1$



$g(p_0) = p_1$
$g(p_1) = p_2$
$g(p_2) = p_3$

Root finding problem
$|g'(x)| < 1$ **is required:**



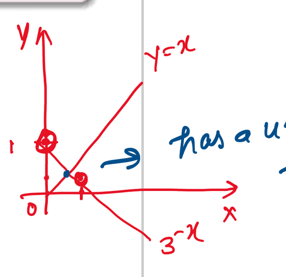
Fixed point iteration

Converse is not true:
If the conditions for the convergence (three conditions on $g(x)$) of a fixed point are satisfied then there is a guarantee for the existence and uniqueness of a fixed point on a given interval $p \in [a,b]$ but if we have one fixed point in a given interval then condition may or may not be satisfied.

Fixed point iteration

Counter example:
Show that the conditions for the convergence of a fixed point do not ensure a unique fixed point of $g(x) = 3^{-x}$ on the interval $[0,1]$, even though a unique fixed point on this interval does exist.

Solution: Let $g(x) = 3^{-x}$



has a unique fixed pt. in $[0,1]$

(i) $g(x)$ is continuous on $[0,1]$

(ii) $g'(x) = 3^{-x}\ln(3)(-1) < 0$ on $[0,1]$

$\Rightarrow$ $g(x)$ is decreasing on $[0,1]$

Max. value of $g(x) = g(0) = 3^0 = 1 \in [0,1]$

Min value of $g(x) = g(1) = 3^{-1} = \frac{1}{3} \in [0,1]$

$\Rightarrow g(x) \in [0,1]$ $\forall x \in [0,1]$.

(iii) $|g'(x)| = |3^{-x}\ln(3)|$

$g'(0) = |3^0 \ln(3)| = |\ln(3)| = 1.09 > 1$

$\Rightarrow$ $g(x)$ does not satisfy $|g'(x)| < 1$ $\forall x \in [0,1]$

Fixed point iteration

Exercise:
1. Show that the conditions for the convergence of a fixed point do not ensure a unique fixed point of $g(x) = \frac{x^2-1}{3}$ on the interval $[3,4]$, even though a unique fixed point on this interval does exist.

Fixed point iteration

Example of FPI:
Find the root of an equation $x^3 + 4x^2 - 10 = 0$ ($f(x)=0$) by using fixed point iteration method with the accuracy of 10^{-2}.

Solution: from IVT.

$f(x) = x^3+4x^2-10$
$f(0) = -10 = -ve$
$f(1) = 1+4-10 = -ve$
$f(2) = 8+16-10 = +ve$

$\Rightarrow$ Root lies in between $[1,2]$.

To find $g(x)=x$

(1) $x = x^3+4x^2+x-10$ ✗
$= g_1(x) \in C[1,2]$
$g_1(1) = 1+4+1-10 = -4 \notin [1,2]$

(2) ✗ $-x = x^3+4x^2-x-10$
$x = -x^3-4x^2+x+10$
$= g_2(x) \in C[1,2]$
$g_2(1) = -1-4+1+10 \notin [1,2]$

(3) ✗ $x^3 = 10-4x^2$
$x = (10-4x^2)^{1/3} = g_3(x)$
$(-6)^{1/3} \notin [1,2]$

(4) ✗ $4x^2 = 10-x^3$
$x = \frac{\sqrt{10-x^3}}{2} = g_4(x)$ $\in C[1,2]$
$g_4(1) = \frac{\sqrt{10-1}}{2} = \frac{3}{2} \in [1,2]$
$g_4(2) = \frac{\sqrt{10-8}}{2} = \frac{1}{\sqrt{2}} \notin [1,2]$

(5) ✗ $x^2+4x = \frac{10}{x}$
$x = \sqrt{\frac{10}{x}-4x} = g_5(x)$
$g_5(2) = \sqrt{\frac{10}{2}-8}$ is not defined in $[1,2]$

(6) ✓ $x^2(x+4) = 10$

$x = \sqrt{\frac{10}{x+4}} = g_6(x)$

$\boxed{g_6(x) \in C[1,2]}$

$g_6(1) = \sqrt{\frac{10}{5}} = \sqrt{2} \in [1,2]$

$g_6(2) = \sqrt{\frac{10}{6}} = \sqrt{\frac{5}{3}} \in [1,2]$

$g_6(x) = \sqrt{\frac{10}{x+4}}$

$g_6'(x) = \frac{\sqrt{10}\left(\frac{-1}{2}\right)}{(x+4)^{3/2}} < 0$

$\Rightarrow g_6(x)$ is decreasing on $[1,2]$

max-value $g_6(1) \in [1,2]$
min. value $g_6(2) \in [1,2]$

$\boxed{g_6(x) \in [1,2] \ \forall x \in [1,2]}$

$g_6'(x) = \frac{-\sqrt{10}}{2(x+4)^{3/2}}$

$g_6''(x) = \frac{-\sqrt{10}}{2}\left(\frac{-3}{2}\right)\frac{1}{(x+4)^{5/2}} > 0$ $\forall x \in [1,2]$

$g_6'(x)$ is increasing on $[1,2]$

Min value of $|g_6'(x)| = \left|\frac{-\sqrt{10}}{2(5)^{3/2}}\right| < 1$ at $x=1$

Max. value of $|g_6'(x)| = \left|\frac{-\sqrt{10}}{2(6)^{3/2}}\right| < 1$ at $x=2$

$\Rightarrow$ $\boxed{|g'(x)| < 1 \quad \forall x \in (1,2)}$

(7) $g_7(x) = x - \frac{x^3+4x^2-10}{3x^2+8x}$ $=0$

$g_6(x) = g(x) = \frac{\sqrt{10}}{\sqrt{x+4}}$

Take $p_0 = 1.5 \in [1,2]$

$p_1 = g(p_0) = \frac{\sqrt{10}}{\sqrt{1.5+4}} = 1.3484$ $\quad |p_1 - p_2| \nless 10^{-2}$

$p_2 = g(p_1) = \frac{\sqrt{10}}{\sqrt{1.3484+4}} = 1.3649$

$p_3 = g(p_2) = \frac{\sqrt{10}}{\sqrt{1.3649+4}} = 1.3652$ $\quad |p_2 - p_3| < 10^{-2}$

$\boxed{p_3 = 1.3652}$ Ans.
$\hookrightarrow$ fixed pt. for $g(x)$
& Root for $f(x)=0$

Fixed point iteration

Example of FPI:
Find the root of an equation $x^3 - 7x + 2 = 0$ by using fixed point iteration method with the accuracy of 10^{-2}.

Solution:

Do it yourself!