Numerical Analysis

Solution of Some Exercises : Chapter 6
Numerical Integration

1. Given
$$I = \int_0^2 x^2 \, e^{-x^2} \, dx.$$
Approximate the value of I using trapezoidal and Simpson's one-third method.

Sol. Given $f(x) = x^2 \, e^{-x^2}$, $a = 0, b = 2$, we have the Trapezoidal rule of the form ($h = 2$)
$$I_T = \frac{2}{2}[f(0 + f(2)] = 0.073263.$$
We have the Simpson's rule ($h = 1$)
$$I_S = \frac{1}{3}[f(0) + 4f(1) + f(2)] = 0.51493.$$

2. Approximate the following integrals using the trapezoidal and Simpson's formulas and compare with exact values.

(a) $I = \int\limits_{-0.25}^{0.25} (\cos x)^2 \, dx.$

(b) $\int\limits_{e}^{e+1} \frac{1}{x \ln x} \, dx.$

Sol.

(a)
$$\begin{aligned}
I_T &= \int\limits_{-0.25}^{0.25} (\cos x)^2 \, dx = \frac{0.5}{2}[f(-0.25) + f(0.25)] = 0.46940. \\
|I_{\text{exact}} - I_T| &= 0.02031. \\
I_S &= \frac{0.25}{3}[f(-0.25) + 4f(0) + f(0.25)] = 0.4898. \\
I_{\text{exact}} &= 2 \int\limits_{0}^{0.25} (\cos x)^2 \, dx = 2 \int\limits_{0}^{0.25} \left[\frac{1 + \cos 2x}{2}\right] \\
&= \left[x + \frac{\sin 2x}{2}\right]_0^{0.25} = 0.48971. \\
|I_{\text{exact}} - I_S| &= 0.00009.
\end{aligned}$$

(b) Similarly
$$\begin{aligned}
I_T &= 0.276086. \\
I_S &= 0.27267045. \\
I_{\text{exact}} &= \int\limits_{1}^{\ln(e+1)} \frac{dt}{t} \quad (\text{by} \ \ \text{taking} \ln x = t) \\
&= \ln(\ln(e + 1)) \\
&= 0.27251388.
\end{aligned}$$

3. Approximate the integral $\int\limits_{1}^{1.5} x^2 \ln x dx$ using the (non-composite) trapezoidal rule. Give a rigorous error bound on this approximation.

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Sol. Here $f(x) = x^2 \ln x$ and $h = (1.5 - 1) = 0.5$, the Trapezoid rule gives

$$\int_1^{1.5} x^2 \ln x dx = \frac{0.5}{2}[f(1) + f(1.5)] = \frac{0.5}{2}[1^2 \ln 1 + 1.5^2 \ln 1.5] = 0.228.$$

The absolute error is

$$|E| = \frac{h^3}{12}|f''(\xi)|$$

for some $\xi \in [1, 1.5]$. Note that

$$|f''(x)| = 3 + 2\ln x$$

is increasing $\forall x > 1$ so

$$|f''(\xi)| \leq |f''(1.5)| = 3 + 2\ln 1.5.$$

Thus

$$|E| = \frac{0.5^3}{12}(3 + 2\ln 1.5) = 0.040.$$

We can conclude:

$$\int_1^{1.5} x^2 \ln x dx = 0.228 \pm 0.040.$$

4. Approximate the integral $\int_0^{0.5} \frac{2}{x-4} dx$ using the (non-composite) Simpson's rule. Give a rigorous error bound on this approximation.

Sol. With $f(x) = 2/(x-4)$ and $h = 0.5/2 = 0.25$ Simpson's Rule gives

$$\int_0^{0.5} \frac{2}{x-4} dx = \frac{h}{3}[f(0) + 4f(0.25) + f(0.5)] = \frac{0.25}{3}\left[\frac{2}{0-4} + 4\frac{2}{0.25-4} + \frac{2}{0.5-4}\right] = -0.26706349.$$

The absolute error is

$$|E| = \frac{h^5}{90} f^{(4)}(\xi)$$

for some $\xi \in [0, 0.5]$. Here

$$|f^{(4)}(x)| = \frac{48}{|x-4|^5}$$

is increasing on $[0, 0.5]$ so

$$|f^{(4)}(\xi)| \leq \frac{48}{|0.5-4|^5} = \frac{48}{3.5^5}.$$

Thus

$$|E| = \frac{0.25^5}{90}|f^{(4)}(\xi)| \leq \frac{0.25^3}{90} \cdot \frac{48}{3.5^3} \approx 10^{-6}.$$

Hence

$$\int_0^{0.5} \frac{2}{x-4} dx = -0.26706349 \pm 0.000001.$$

5. The Trapezoidal rule applied to $\int_0^2 f(x)dx$ gives the value 4, and Simpson's rule gives the value 2. What is $f(1)$?

Sol. By Trapezoidal rule with $h = b - a = 2$, we have

$$\begin{aligned} \frac{2}{2}[f(0) + f(2)] &= 4 \\ \Longrightarrow f(0) + f(2) &= 4. \end{aligned}$$

By Simpson's rule with $h = (b-a)/2 = 1$, we have

$$\begin{aligned}\frac{1}{3}[f(0)+4f(1)+f(2)] &= 2\\ \Longrightarrow f(0)+f(2)+4f(1) &= 6\\ \Longrightarrow 4+4f(1) &= 6\\ \Longrightarrow f(1) &= 1/2.\end{aligned}$$

6. Suppose that $f(0) = 1$, $f(0.5) = 2.5$, $f(1) = 2$, and $f(0.25) = f(0.75) = \alpha$. Find α if the composite Trapezoidal rule with $n = 4$ gives the value 1.75 for $\int\limits_0^1 f(x)dx$.

Sol. By composite Trapezoidal rule with $n = 4$, $h = \dfrac{1-0}{4} = \dfrac{1}{4}$, we have

$$\begin{aligned}\int\limits_{x_0=0}^{x_4=1} f(x)dx &= \frac{1}{4\cdot 2}[f(0)+f(1)+2(f(0.25)+f(0.5)+f(0.75))]\\ &= \frac{1}{8}[1+2+2(\alpha+2.5+\alpha)] = 1.75\\ \Longrightarrow \alpha &= 1.5.\end{aligned}$$

7. Evaluate

$$I = \int_{-1}^{1} \frac{dx}{1+x^2}$$

using trapezoidal and Simpson's rule with 8 subintervals. Compare with the exact value of the integral.

Sol. Given $f(x) = \dfrac{dx}{1+x^2}$, then for $n = 8$ and $h = 0.25$, we have the Simpson's rule of the form

$$\begin{aligned}\int\limits_{x_0=-1}^{x_8=1} &= \frac{1}{3}[f(x_0)+4\{f(x_1)+f(x_3)+f(x_5)+f(x_7)\}+2\{f(x_2)+f(x_4)+f(x_6))+f(x_8)\}]\\ &= \frac{0.25}{3}[f(-1)+4\{f(-0.75)+f(-0.25)+f(0.25)+f(0.75)\}+2\{f(-0.5)+f(0.0)+f(0.5))+f(1)\}]\\ &= \frac{0.25}{3}[0.5+4\{0.64+0.9412+0.9412+0.64)\}+2\{0.8+1.0+0.8)+0.5\}] = 1.570800.\end{aligned}$$

$$I_{\text{exact}} = 2\tan^{-1}(1) = 1.570796.$$

Thus

$$|I_{\text{exact}} - I_S| = 0.000004.$$

8. The quadrature formula $\int\limits_0^2 f(x)dx = c_0 f(0) + c_1 f(1) + c_2 f(2)$ is exact for all polynomials of degree less than or equal to 2. Determine c_0, c_1, and c_2.

Sol. We make the method exact for polynomials up to degree 2.

$$f(x) = 1 \;:\quad 2 = c_0 + c_1 + c_2$$

$$f(x) = x \;:\quad 2 = c_1 + 2c_2$$

$$f(x) = x^2 \;:\quad \frac{4}{3} = c_1 + 4c_2.$$

Solving the above system, we get $c_0 = -\dfrac{1}{3}$, $c_1 = \dfrac{8}{3}$, $c_2 = -\dfrac{1}{3}$.

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9. (a) Find the degree of precision of the quadrature formula

$$\int_{-1}^{1} f(x)dx = f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right).$$

(b) Generalize the integration rule in part (a) to approximate the integral $\int_a^b f(x)dx$.

Sol.

(a) We want the formula

$$\int_{-1}^{1} f(x)dx = f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right)$$

to hold for polynomials $1, x, x^2, \cdots$. Plugging these into the formula, we obtain:

$$f(x) = x^0 = 1; \quad \int_{-1}^{1} 1dx = 2, \quad f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right) = 2,$$

$$f(x) = x; \quad \int_{-1}^{1} xdx = 0, \quad f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right) = 0,$$

$$f(x) = x^2; \quad \int_{-1}^{1} x^2dx = 2/3, \quad f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right) = 2/3,$$

$$f(x) = x^3; \quad \int_{-1}^{1} x^3dx = 0, \quad f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right) = 0,$$

$$f(x) = x^4; \quad \int_{-1}^{1} x^4dx = 2/5, \quad f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right) = 2/9.$$

We can see that the formula provides exact result up to degree 3, therefore degree of precision is 3.

(b) We substitute

$$x = \frac{(b-a)}{2}t + \frac{b+a}{2}, \quad dx = \frac{(b-a)}{2}dt.$$

$$\int_a^b f(x)dx = \int_{-1}^{1} f\left(\frac{(b-a)t+b+a}{2}\right)\frac{(b-a)}{2}dt.$$

Hence

$$\int_a^b f(x)dx = \left[f\left(\frac{(b-a)}{2}\frac{(-\sqrt{3})}{3} + \frac{(b+a)}{2}\right) + f\left(\frac{(b-a)}{2}\frac{\sqrt{3}}{3} + \frac{(b+a)}{2}\right)\right]\frac{b-a}{2}.$$

10. Find the constants c_0, c_1, and x_1 so that the quadrature formula

$$\int_0^1 f(x)dx = c_0 f(0) + c_1 f(x_1)$$

has the highest possible degree of precision.

Sol. We want the given formula to hold for polynomials $1, x, x^2, \cdots$ Plugging these into the formula, we obtain

$$f(x) = 1: \qquad \int_0^1 1dx = 1 = c_0 + c_1$$

$$f(x) = x: \qquad \int_0^1 xdx = \frac{1}{2} = c_1 x_1$$

$$f(x) = x^2: \qquad \int_0^1 x^2 dx = \frac{1}{3} = c_1 x_1^2.$$

We have 3 equations in 3 unknowns and solving we get

$$c_0 = \frac{1}{4},\ c_1 = \frac{3}{4},\ x_1 = \frac{2}{3}.$$

Thus, the quadrature formula is

$$\int_0^1 f(x)dx = \frac{1}{4}f(0) + \frac{3}{4}f\left(\frac{2}{3}\right).$$

The degree of precision of this formula is 2.

11. The length of the curve represented by a function $y = f(x)$ on an interval $[a, b]$ is given by the integral

$$I = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

Use the trapezoidal rule with $n = 20$, compute the length of the graph of the ellipse given with equation $4x^2 + 9y^2 = 36$.

Sol. Write the given equation as

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

and use the parametric equations for the ellipse

$$x = 3\cos t,\ y = 2\sin t,\ t \in [-\pi, \pi].$$

Then by the arc length formula

$$\begin{aligned} L &= \int_{-\pi}^{\pi} \sqrt{x'(t)^2 + y'(t)^2}\, dt \\ &= \int_{-\pi}^{\pi} \sqrt{9\sin^2 t + 4\cos^2 t}\, dt \\ &= \int_{-\pi}^{\pi} \sqrt{4 + 5\sin^2 t}\, dt. \end{aligned}$$

Now can use composite Trapezoidal rule to approximate the integral, where $a = -\pi,\ b = \pi, h = (b - a)/n,\ f(t) = \sqrt{4 + 5\sin^2 t}$. By taking $n = 20$ subintervals, the length is 15.865.

12. A car laps a race track in 84 seconds. The speed of the car at each 6-second interval is determined by using a radar gun and is given from the beginning of the lap, in feet/second, by the entries in the following table.

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Time	0	6	12	18	24	30	36	42	48	54	60	66	72	78	84
Speed	124	134	148	156	147	133	121	109	99	85	78	89	104	116	123

How long is the track?
Sol. Using the composite Simpson's rule, the length of the track is

$$L = \int_{t_0}^{t_f} v(t)\, dt,$$

where $t_0 = 0 =$initial time, $t_f = 84 =$ final time, $h = 6$ and $v(t) =$ speed.

$$L = \frac{6}{3}[124+123+4(134+156+133+109+85+89+116)+2(148+147+121+99+78+104)] = 9858 \text{ feet}.$$

Note: we can also use composite trapezium rule.

13. Evaluate the integral

$$\int_{-1}^{1} e^{-x^2} \cos x\, dx$$

by using the Gauss-Legendre two and three point formula.
Sol. Given interval is $[-1, 1]$ and $f(x) = e^{-x^2} \cos x$.
By 1-point formula

$$I = 2f(0) = 2.$$

By 2-point formula

$$I = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = f(-0.57735) + f(0.57735) = 1.20078.$$

14. Evaluate

$$I = \int_0^1 \frac{\sin x\, dx}{2+x}$$

by subdividing the interval $[0, 1]$ into two equal parts and then by using Gauss-Legendre two point formula.
Sol.

$$I = \int_0^{0.5} \frac{\sin x\, dx}{2+x} + \int_{0.5}^{1} \frac{\sin x\, dx}{2+x} = I_1 + I_2.$$

Now we calculate both the integrals I_1 and I_2, separately. To do so, firstly we change limits.
For I_1, we substitute $x = \dfrac{t+1}{4}$, $dx = \dfrac{dt}{4}$. Thus

$$\begin{aligned} I_1 &= \int_0^{0.5} \frac{\sin x}{2+x} dt \\ &= \int_{-1}^{1} \frac{\sin\left(\frac{t+1}{4}\right)}{2+\frac{t+1}{4}} \frac{dt}{4} \\ &= \int_{-1}^{1} \frac{\sin\left(\frac{t+1}{4}\right)}{t+9} dt \approx f(-0.57735) + f(0.57735) = 0.052637. \end{aligned}$$

Similarly for I_2, we substitute $x = \dfrac{t+3}{4}$, $dx = \dfrac{dt}{4}$. Thus

$$I_2 = \int_{-1}^{1} \frac{\sin\left(\frac{t+3}{4}\right)}{t+11} dt \approx f(-0.57735) + f(0.57735) = 0.12198.$$

Finally $I = I_1 + I_2 = 0.17461.$

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15. A particle of mass m moving through a fluid is subjected to a viscous resistance R, which is a function of the velocity v. The relationship between the resistance R, velocity v, and time t is given by the equation

$$t = \int_{v(t_0)}^{v(t)} \frac{m}{R(u)}\, du$$

Suppose that $R(v) = -v\sqrt{v}$ for a particular fluid, where R is in newtons and v is in meters/second. If $m = 10$ kg and $v(0) = 10$ m/s, approximate the time required for the particle to slow to $v = 5$ m/s.

Sol. Given

$$\begin{aligned} t &= \int_{v(t_0)}^{v(t)} \frac{m}{R(u)}\, du \\ &= \int_{10}^{5} \frac{10}{-u\sqrt{u}}\, du, \text{ here } (m = 10, v(t_0) = 10, v(t) = 5) \\ &= \int_{5}^{10} 10u^{-3/2}\, du. \end{aligned}$$

Using Simpson's rule (for example) with 6 subintervals with $f(u) = 10u^{-3/2}$, $h = (10-5)/6$, we get the required time 2.61993912767802.

16. In statistics it is shown that

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\dfrac{x^2}{2\sigma^2}}\, dx = 1,$$

for any positive σ. The function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\dfrac{x^2}{2\sigma^2}}$$

is the normal density function with mean $\mu = 0$ and standard deviation σ. The probability that a randomly chosen value described by this distribution lies in $[a, b]$ is given by $\int_a^b f(x)dx$. Approximate to within 10^{-3} the probability that a randomly chosen value described by this distribution will lie in

(a) $[-\sigma, \sigma]$

(b) $[-2\sigma, 2\sigma]$

(c) $[-3\sigma, 3\sigma]$.

Sol. To simplify the integral, firstly we substitute $x/\sigma = t$, $dx = \sigma dt$.

(a) In this case,

$$I = \int_{-1}^{1} \frac{1}{\sqrt{2\pi}}\, e^{-\dfrac{t^2}{2}}\, dt.$$

Apply Simpson's rule, with 10 subintervals, $f(t) = \dfrac{1}{\sqrt{2\pi}}\, e^{-\dfrac{t^2}{2}}$, the desired approximation is 0.682698220175433.

(b)

$$I = \int_{-2}^{2} \frac{1}{\sqrt{2\pi}}\, e^{-\dfrac{t^2}{2}}\, dt.$$

Apply Simpson's rule, with 10 subintervals, the desired approximation is 0.954463324707434.

(c)

$$I = \int_{-3}^{3} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{t^2}{2}}\, dt.$$

Apply Simpson's rule, with 10 subintervals, the desired approximation is 0.997195309084966.