MODULAR ARITHMETIC AND MICROTONAL MUSIC THEORY

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Abstract: In this article we discuss relationships between the cyclic group $\mathbb{Z}_{12}$ and Western tonal music which is embedded in a 12-note division of the octave. We then offer several questions inviting students to explore extensions of these relationships to other n-note octave divisions. The answers to most questions require only basic properties of modular arithmetic and greatest common divisors. However, some questions ask students to explore additional topics such as Cayley graphs of groups. Thus the questions can be adapted to fit into a course that appears as early in the curriculum as introduction to proof or as late as abstract algebra and number theory.

Keywords: group theory, microtonal system, modular arithmetic, music theory

1 INTRODUCTION

This paper highlights a path connecting the mathematics of modular arithmetic and cyclic group theory to the musical concepts of scale and chord. We use our observations to explore alternative n-note divisions of the octave whose scales and chords would be consistent with our Western tradition. Through a series of problems we ask students to tie well-known mathematical models of Western 12-note music (see for example [2], [6], [11], [12], [13]) to explorations of more general n-note systems, many of which were first introduced by music theorists Balzano [1], Zweifel [15], and Gould [9]. The explorations require only basic number and group theory and are thus well-fitted for the undergraduate classroom. Our next section briefly introduces a few key music-theoretic terms. Please consult [4] for more detailed yet still easily accessible descriptions of music terminology.

2 BACKGROUND

When we hear a musician play a note whose frequency is twice that of another, we think the notes sound similar. The two notes sound so similar that musicians give the same name to both. We call the interval separating such notes an *octave* and give all notes that appear one or more octaves apart the same name. Our Western music tradition further divides an octave into twelve pitches. In our work we assume each pitch sounds at a frequency that is $2^{1/12}$ times the previous one. We call this smallest uniform logarithmic distance between pitches a *semitone*. That is, multiplying a note's frequency by $2^{1/12}$ increases the note by one semitone. Musically, our arrangement of pitches reflects *equal-tempered* tuning, the most common tuning for modern keyboard instruments. Figure 1 shows the notes of an octave with their musical and mathematical names. A note's mathematical name represents the number of semitones that notes lies above the C that begins its octave. For example, within a given octave, $D = 2$ lies two semitones above $C = 0$ and thus D's frequency is $2^{2/12}$ times that of C.

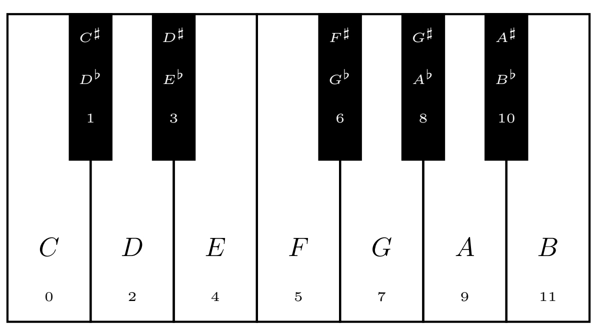


Figure 1. The Octave

A quick look at Figure 1 reveals that some notes have two common musical names, resulting from the standard use of accidentals—sharps

($\sharp$) and flats ($\flat$)—in music. A sharp ($\sharp$) indicates a move up by one semitone while a flat ($\flat$) indicates a move down by one semitone. Since D occurs two semitones above C, $C^\sharp$ and $D^\flat$ represent the same key on the keyboard and thus the same pitch; musically they are *enharmonically equivalent*. We could have included other enharmonic equivalents as labels on our keyboard. For example, F is one semitone above E and thus $E^\sharp$ and F are enharmonic equivalents. In our work we treat enharmonic equivalents interchangeably. By tradition we label our notes without using accidentals whenever possible and thus the white keys in Figure 1 have only one label while the black keys have two.

The *pitch class* A contains the note sounding at a frequency of 440 hz as well as any note separated from this note by one or more octaves; the pitch class $A^\sharp$ contains the note sounding at a frequency of $2^{1/12} \cdot 440$ hz as well as any note separated from this note by one or more octaves; etc. As students of mathematics we frequently encounter such partitions into equivalence classes and can see that the cyclic group $\mathbb{Z}_{12}$ makes sense as our musical model. Figure 2 gives a visualization of this model. Musicians refer to the figure (perhaps without the numbers) as the musical clock; mathematicians might see it (without the names of the notes) as a visual representation of the group $\mathbb{Z}_{12}$.

3 MAJOR DIATONIC SCALES

While in a given key, a composer selects notes (or at least most notes) not from all twelve pitch classes, but from a seven-note *diatonic scale*. The initial note in the scale, known as the *tonic*, gives the scale its name and indicates the scale's tonal center. We play the remaining notes of the scale in clockwise order around the musical clock with pitch class representatives chosen so that the notes played remain within the octave above the tonic. The C-major diatonic scale consists of the white notes on a traditional keyboard, $\{C, D, E, F, G, A, B\} = \{0, 2, 4, 5, 7, 9, 11\}$. The k-major diatonic scale appears as the image of the C-major diatonic scale under a rotation of k semitones in the clockwise direction on the musical clock. For example, we find G seven semitones above C and

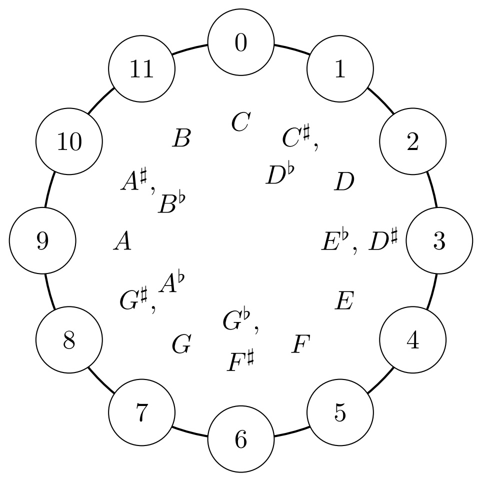


Figure 2. The Musical Clock

thus the G-major diatonic scale contains the notes in the coset $7 + \{0, 2, 4, 5, 7, 9, 11\} = \{7, 9, 11, 0, 2, 4, 6\} = \left\{G, A, B, C, D, E, F^\sharp\right\}$. In this manner we can create 12 distinct major diatonic scales.

Figure 3 shows the C-major diatonic scale as it appears in the musical clock.

While we can certainly observe a pattern within the figure, the notes in the scale seem more closely related if we change the way we picture $\mathbb{Z}_{12}$. The integers 1, 5, 7, and 11 each generate the cyclic group $\mathbb{Z}_{12}$. The musical clock of Figure 2 shows 1 generating $\mathbb{Z}_{12}$ in the clockwise direction and 11 generating the same group in the counter-clockwise direction. The clock in Figure 4 uses 7 to generate $\mathbb{Z}_{12}$ in the clockwise direction and 5 in the counter-clockwise direction. Musicians call Figure 4 the *circle of fifths* because they refer to the seven semitone interval used to generate the circle as a *perfect fifth.*

The pitches in the C-major diatonic scale—marked in gray—form a connected region of seven notes in the circle of fifths. Each connected region of seven notes in the circle of fifths represents the k-major diatonic scale where k is the second note in the region when viewed in the clockwise direction. Musicians refer to the set of notes represented in a

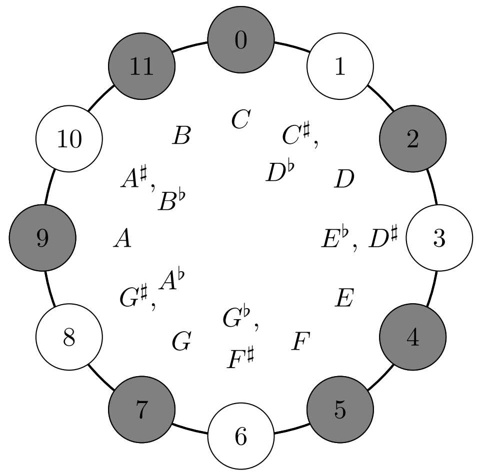


Figure 3. C-major Scale

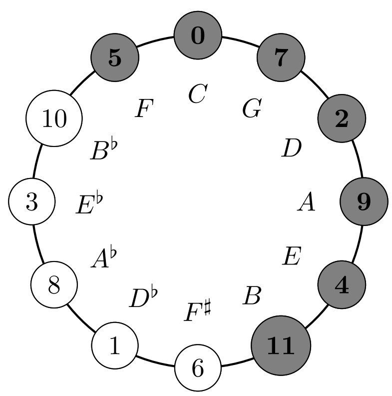


Figure 4. The Circle of Fifths

major diatonic scale as the notes in the major *key* of the same name. Adjacent nodes on the circle of fifths thus represent major keys sharing six of their seven notes. Closeness on the circle of fifths reflects closeness in key. Balzano [1] has more carefully defined this "closeness of key" property of the circle of fifths and refers to it as the $F \to F^\sharp$ property of the related scale. We give his definition next.

Definition 1 ($F \rightarrow F^\sharp$ property, [1]**)** *When transposing a major scale up by a perfect fifth we obtain a scale with all but one element the same and with the note gained only one semitone above the note lost.*

When we rotate the connected region of seven notes defining the C-major diatonic scale by one perfect fifth in the circle of fifths we obtain the connected region of seven notes defining the G-major diatonic scale. Thus in moving from the C-major diatonic scale to the G-major diatonic scale we lose the note $F = 5$ and gain the note $F^\sharp = 6$.

We have presented a few basic properties of diatonic scales. In the next section, we generalize these properties to microtonal systems. The interested reader can learn of deeper mathematical connections within diatonic scale theory by consulting [10] and [5]. Considering generalizations of these more advanced aspects of diatonic scale theory may lead to intriguing new investigations.

4 GENERALIZING MAJOR DIATONIC SCALES

A *microtonal musical system* divides the octave in a way that includes intervals smaller than a semitone. In this paper we propose a series of questions exploring n-note microtonal systems with equal-tempered tuning and use $\mathbb{Z}_n$ to model such systems. Thus, for us a *microtone* represents the smallest interval in our equal-tempered n-note system and plays the same role for the n-note system that the semitone plays in 12-note equal temperament.

As a first step in creating generalized n-note microtonal systems we must choose an interval to mirror the important role the perfect fifth plays in the Western 12-note system. The fifth note of the C-major diatonic scale lies seven semitones, a perfect fifth, above $C = 0$. When generalizing this interval for the n-note system Balzano [1], a pioneer in group-theoretic techniques for microtonal music, chose to name the interval mirroring the role of the perfect fifth based on its location in the generalized C-major scale as well. However, in our work we simply call this interval a *generalized perfect fifth* in order to highlight the role the interval plays in the microtonal system. Our first definition of a generalized n-note microtonal system (Definition 2) captures the important role the generalized perfect fifth and the $F \to F^\sharp$ property play in the structure of major diatonic scales. Of course we generalize the $F \to F^\sharp$ property to reflect the fact that when transposing a generalized major diatonic scale up by a generalized perfect fifth we obtain a scale with all but one element the same and with the note gained one microtone above the note lost.

Definition 2 *A generalized n-note microtonal system with generalized perfect fifth $\mathrm{p}5$ is a division of the octave into n microtones such that*

1. *each note has a frequency $2^{1/n}$ times the previous one; that is we require our microtonal system to reflect equal-tempered tuning,*
2. *our generalized perfect fifth leads to a generalized circle of fifths; mathematically this means $\mathrm{p}5$ generates $\mathbb{Z}_n$, and*
3. *the generalized major diatonic scales appear as connected regions in the generalized circle of fifths and these regions enjoy the $F \to F^\sharp$ property.*

In our Western 12-note keyboard the seven-semitone perfect fifth contains one more than half the number of semitones in the octave. Our first question explores generalized n-note microtonal systems with a generalized perfect fifth of $\frac{n}{2} + 1$ microtones.

Question 1 *Which even values of n permit generalized n-note microtonal systems with $\mathrm{p}5 = \frac{n}{2} + 1$? What is the length of the major diatonic scales in such a system?*

Typically, students begin their work for Question 1 by looking at several values of $p5 = \frac{n}{2} + 1$. They quickly observe that $p5$ and n are relatively prime only when n is a multiple of 4. Using the Euclidean algorithm students then observe $\gcd(2k+1, 4k) = 1$, showing that when $n = 4k$, $p5 = 2k + 1$ generates a generalized circle of fifths. They can then use the extended Euclidean algorithm to find the length, ℓ, of the generalized diatonic scales since the $F \to F^\sharp$ property requires

$(p5)(\ell) \bmod n = 1$. They should find $\ell = \frac{n}{2} + 1 = p5$. We offer a full solution to this question in the online appendix.

Our work in Question 1 allows us to determine the sets of notes in each diatonic scale. However, we have not yet learned how to determine the tonic for each scale. Building on the work of Balzano [1] and Zweifel [15], Gould [9] suggests a method for determining the tonic in the 20-note diatonic system. He bases his method on harmonic considerations which we address in Section 5. Below we generalize his method to all n-note systems with $p5 = \frac{n}{2} + 1$.

Definition 3 (Generalized Tonic, [9]**)** *Consider each generalized diatonic scale as a connected region of* $\frac{n}{2} + 1 = p5$ *notes in the generalized circle of fifths. When viewing this region in the clockwise direction, the generalized tonic lies one microtone above the last note in the region.*

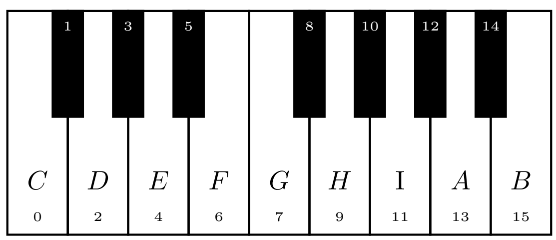
For example, the connected region $(9, 0, 11, 2, 13, 4, 15, 6, 17, 8, 19)$ in the 20-note microtonal system with $p5 = 11$ serves as our generalized C-major diatonic scale because $19 + 1 \bmod 20 = 0$. When playing this scale we must re-order the notes so that they begin at $C = 0$ and appear in clockwise order around the musical clock.

In Section 3 we observed that the tonic appears as the second clockwise note in a traditional diatonic scale when viewed as a connected region of seven notes in the 12-note circle of fifths. Our next question asks students to determine if Gould's method of identifying the tonic corresponds with this alternative method for other microtonal systems.

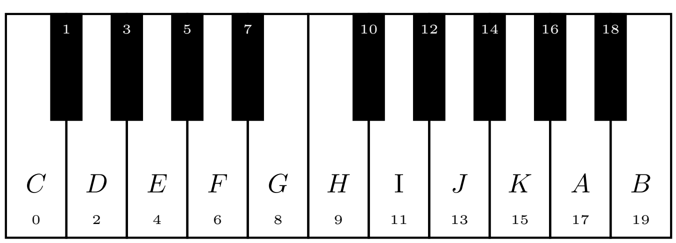
Question 2 *In our Western 12-note system a diatonic scale's tonic appears as the second clockwise note in the scale when viewed as a connected region of seven notes in the circle of fifths. Consider an n-note microtonal systems (n even) with* $p5 = \frac{n}{2} + 1$. *For which values of n, if any, is Gould's method of identifying the tonic (see Definition 3) consistent with this alternative method?*

Examples suggest that the conditions are equivalent for all n-note microtonal systems with $p5 = \frac{n}{2} + 1$. The key observation needed to prove this conjecture stems from the $F \to F^\sharp$ property which allows us to define the last note in the connected region defining the diatonic scale using only the first note of the region and the generalized perfect fifth. We provide the details of the argument in the online appendix.

With these results in mind, for any n-note microtonal system with $p5 = \frac{n}{2} + 1$ we take the name of the generalized diatonic scale from the second note of the connected region defining the scale in the generalized circle of fifths, knowing that this choice coincides with Gould's suggestion. Using this method we can propose keyboards for n-note microtonal systems with $p5 = \frac{n}{2}+1$, taking the white keys for our keyboard from the C-major diatonic scale. Figure 5 gives the keyboards for two such microtonal systems. Gould [9] first suggested the 20-note keyboard shown in the figure.



Generalized 16-note microtonal keyboard with $p5 = 9$



Generalized 20-note microtonal keyboard with $p5 = 11$

Figure 5. Microtonal Keyboards

5 CONSONANT TRIADS

In this section and the next we discuss the harmonic structure of our n-note microtonal systems, focusing on an important family of chords, the major and minor triads. Whereas we play the notes of a scale in succession, we play the notes of a chord in unison. As their name suggests the major and minor triads contain three notes. The chord's root note lends the chord its name. In addition to their root notes, both major and minor triads contain a note a perfect fifth above the root. Thus, the major and minor triads differ only in how we determine their remaining note.

We partition the seven-semitone perfect fifth into a four-semitone interval called a major third (M3) and a three-semitone interval called a minor third (m3). A major triad contains a root note, a note a major third above the root, and a note a perfect fifth above the root while a minor triad contains a root note, a note a minor third above the root, and a note a perfect fifth above the root. For example the C-major triad contains the notes $\{C, E, G\} = \{0, 4, 7\}$ played in unison and the C-minor triad contains the notes $\left\{C, E^{\flat}, G\right\} = \{0, 3, 7\}$ again played in unison. When playing a scale we always choose pitch class representatives within the octave above the tonic. We do not place any such restrictions on chords and thus a chord's root note might not be the lowest sounding note in the chord. Next, we update our definition of the generalized major diatonic scale with generalized perfect fifth $p5$ and generalized major third $M3$ to guarantee the notes of the generalized major triads occur within the corresponding generalized major diatonic scales.

Definition 4 *A generalized n-note microtonal system with perfect fifth $p5$ and major third $M3$ is a division of the octave into n microtones such that*

1. *each note has a frequency $2^{1/n}$ times the previous one; that is we require our microtonal system to reflect equal-tempered tuning,*
2. *our generalized perfect fifth leads to a generalized circle of fifths;*

mathematically this means $p5$ *generates* $\mathbb{Z}_n$*,*

3. *the generalized major diatonic scales appear as connected regions in the generalized circle of fifths and these regions enjoy the* $F \to F^\sharp$ *property, and*
4. *the generalized major third* $M3$ *and smaller generalized minor third* $m3$ *partition the generalized perfect fifth* $p5$ *(i.e.* $M3 + m3 = p5$*) with* $M3$ *appearing as an element of the* C*-major diatonic scale.*

In our traditional 12-note system the major third consists of $\frac{n}{4} + 1$ semitones. Our next question asks if making similar choices in a generalized n-note microtonal system places any limitations on n.

Question 3 *What additional requirements, if any, on* n *are needed to ensure that the generalized* n*-note microtonal system with* $n = 4k$*,* $p5 = 2k + 1$*, and* $M3 = k + 1$ *meets the requirements of Definition 4?*

Our solution to Question 1 shows that as long as $n = 4k$ we satisfy requirements (1)–(3). Again, we should encourage students to explore and they will quickly see that requirement (4) fails whenever $n = 4k$ is divisible by 8 and holds otherwise. Students may find it difficult to begin the proof, but encouraging them to think about the circle of fifths helps. Using their work from Question 2, the students can characterize the notes in the generalized C-major n-note diatonic scales as those in the set

$$\Big\{(x)(p5) \bmod n\Big\}_{x=-1}^{(p5)-2}.$$

With this characterization in mind they need only solve the equation $k + 1 = (x)(p5) \bmod n$ for x to determine whether $k + 1$ lies within the generalized C-major diatonic scale. Of course they should expect different answers for even and odd values of k, encouraging students to investigate the solutions separately and thus employ a technique that appears often in elementary number theory problems.

6 CONSONANT TRIADS AND CAYLEY GRAPHS

In [15], Zweifel extends Balzano's original requirements of a good microtonal system with perfect fifth $p5$ and major third $M3$ to include all equal-tempered systems with $n \geq 12$ for which

1. the generalized perfect fifth generates a generalized circle of fifths,
2. major diatonic scales have the $F \to F^\sharp$ property, and
3. the notes in the major diatonic scale form a simple cycle with alternating $M3$ and $m3$ arcs in what has come to be known as the Balzano diagram for the microtonal system but mathematically can be realized as the Cayley graph for $\mathbb{Z}_n$ with generators $\{p5, M3, m3 = p5 - M3\}$.

Definition 5 (Cayley Graph of $\mathbb{Z}_n$, adapted from [8]**)** *The Cayley graph of the group $\mathbb{Z}_n$ with generating set S, denoted Cay$(\mathbb{Z}_n, S)$, has one vertex for each element of $\mathbb{Z}_n$. For each $a \in S$ and $x, y \in \mathbb{Z}_n$ we connect x to y via an a-colored arc if and only if $y = x + a$ mod n.*

If we add clockwise arrows to the arcs of our musical clock (Figure 2) and circle of fifths (Figure 4) we obtain the Cayley graphs for $\mathbb{Z}_{12}$ with generating sets $\{1\}$ and $\{7\}$ respectively. Figure 6 shows the Cayley graph of $\mathbb{Z}_{12}$ with generating set $\{p5 = 7, M3 = 4, m3 = 3\}$. In the graph we identify any vertices with the same label. For example, the ② we draw in each corner of the graph represents a single vertex. These identifications allow us to depict the graph on the two-dimensional page.

By simply ignoring the gray (p5) arrows, we can easily see that the shaded notes in Figure 6—the notes in the C-major diatonic scale—form a simple cycle involving alternating $M3$ and $m3$ arcs. This cycle points to a construction of diatonic scales from alternating major and minor triads, revealing a rich chordal structure. Crans, Fiore, and Satyendra investigate many topics related to this structure in their article [6]. There the authors refer to the graph of Figure 6 as the Oettingen/Riemann Tonnetz, a name coming from neo-Riemannian music theory. In our next question we determine if our generalized n-note diatonic scales meet the requirements set out by Zweifel leading to the same rich chordal structure.

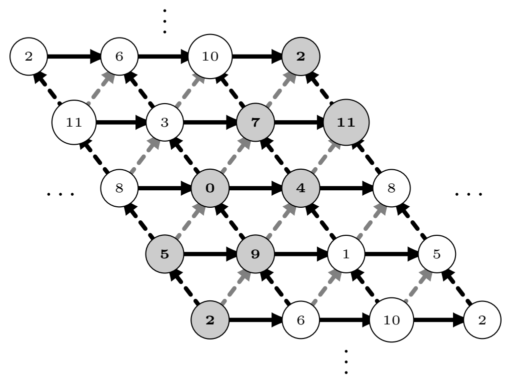


Figure 6. Cay($\mathbb{Z}_{12}$ $\{3, 4, 7\}$)

Question 4 *In our generalized n-note microtonal system with $n = 4k$, $p5 = 2k + 1$ and $M3 = k + 1$ we require the generalized major third to be an element of the C-major generalized diatonic scale. For $n \geq 12$ is this requirement equivalent to Zweifel's stipulation that the generalized diatonic scales appear as simple cycles involving alternating $M3$ and $m3$ arcs in Cay $(\mathbb{Z}_n, \{p5, M3, m3\})$?*

After drawing several Cayley graphs, students should suspect that the two requirements are equivalent. Our proof (which is in the online appendix) makes use of the following observation: if $M3$ lies within the C-major diatonic scale, we can include the arc connecting 0 to $M3$ in our simple cycle, whereas if $M3$ is not within the C-major diatonic scale our path cannot contain this arc.

7 AN APPROACH INVOLVING RATIOS

Previously we defined a perfect fifth as an interval of seven semitones and this definition reflects the way we realize the perfect fifth in our modern equal-tempered tuning of the Western 12-note system. However the *just perfect fifth* represents the interval between two notes whose frequencies occur at a 3 : 2 ratio. Notes whose frequencies are separated by small whole-number ratios tend to be more pleasing to our ears, that is more consonant, than other ratios. The octave with its $2:1$ ratio is the most consonant and the just perfect fifth with a 3:2 ratio is the second most consonant. Yet, with equal-tempered tuning we cannot obtain both exact ratios. Equal-tempered tuning sacrifices the just perfect fifth for a system in which music arranged in any key sounds equally good. In Western 12-note music we approximate the perfect fifth as seven semitones so that notes separated by a perfect fifth occur in a $2^{7/12} : 1 \approx 2.9966 : 2$ ratio. This seems to be a very close approximation. Our next question asks students to determine if this approximation could be improved by using a different microtonal system with $p5 = \frac{n}{2} + 1$.

Question 5 *Suppose we form a generalized n-note microtonal system (see Definition 4) with $n = 4k$, $p5 = 2k + 1$, and $M3 = k + 1$. Which value of n leads to the system with a generalized perfect fifth that best approximates the just perfect fifth? How does the ratio of the generalized perfect fifth change as $n = 4k$ grows past this optimal value? Repeat the same investigations for the just major third with its $5 : 4$ ratio.*

Basic algebraic manipulations show that our Western 12-note system leads to the best approximation of the just perfect fifth as well as the just major third. Although, the approximation of the just perfect fifth is much closer.

These results indicate choosing a generalized perfect fifth other than $\frac{n}{2}+1$ may improve the sound of our n-note microtonal systems. Suppose we no longer require that $p5 = \frac{n}{2} + 1$ but rather choose $p5$ so that it mirrors the just perfect fifth's $3 : 2$ ratio as closely as possible while still generating a generalized circle of fifths with the $F \to F^\sharp$ property. Similarly, consider choosing $M3$ to be the best approximation of the $5 : 4$ ratio existing within our C-major diatonic scale. Such choices allow us to combine the just-interval approach of many microtonal theorists (see for example [7]) with the algebraic approach first introduced by Balzano [1] and lead to many open questions for student exploration.

8 CONCLUSIONS

The questions posed in this paper allow students to explore a nontraditional application using basic mathematic tools. In answering the questions, students use the standard tools they are learning in class—the extended Euclidean algorithm, modular inverses, etc.—to solve nonstandard problems designed to given them a sense of exploration. Individually, the questions could simply be added to homework assignments or even used as in-class examples. The third author typically discusses the circle of fifths when introducing the idea of isomorphic groups in her abstract algebra courses. Question 1 makes a nice homework question following this in-class presentation.

Taken together the questions could serve as an end-of-semester project for a student with some musical background. In fact, the first two authors drafted many of the solutions found in the online appendix as part of their undergraduate thesis work at Appalachian State University. Questions 1, 2, 3, and 5 would work well for a project in a techniques of proof course following a unit discussing modular arithmetic. Alternatively, students could read the solutions to these questions and use what they have learned to explore four and eight-note divisions of the octave, generating generalized circles of fifths, keyboards, etc. for these wider octave divisions. As a course project, an abstract algebra student might choose to study Cayley graphs and use $Cay\left(\mathbb{Z}_{12}, \{p5, M3, m3\}\right)$ to illustrate properties of chord progressions (see [6] for details). Using our Question 4 as a starting point, this student could then extend his investigations to microtonal systems.

Easley Blackwood's *Twelve Microtonal Etudes for Electronic Music Media*, funded in part by the National Endowment for the Humanities and recorded in [3], "explore tonal and modal behavior of all the equal tunings of 13 through 24 notes (to the octave)" [3, liner notes]. A music student with sufficient mathematical background may wish to compare and contrast Blackwood's use of scale, chord and key with the ideas presented in this paper. Alternatively, such a student might choose to use the definitions given within this paper to guide her own microtonal composition.

Please view this paper online to listen to generalized major diatonic scales as well as generalized major and minor triads for several generalized n-note microtonal systems as discussed in Sections 4 and 5. The online version also provided audio files related to n-note microtonal systems with generalized perfect fifths and generalized major thirds chosen to best approximate just intervals as described in Section 7. In addition, the website includes the simple MATLAB code used to generate the sound files, which was adapted from code provided in [14].

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