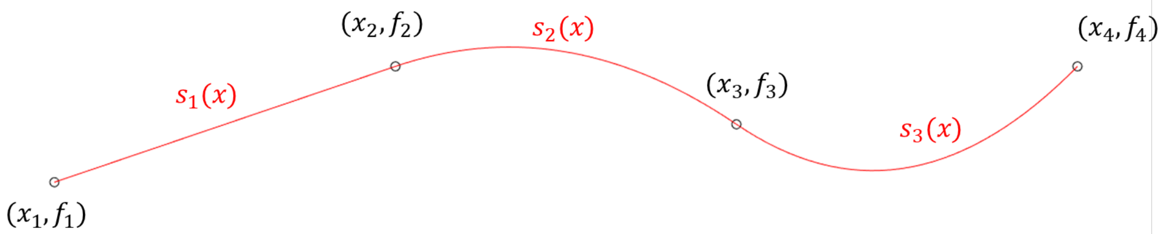
Quadratic splines



$$s_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2$$

For $n = 4$ points, there are 3 intervals $s_1(x), s_2(x), s_3(x)$.

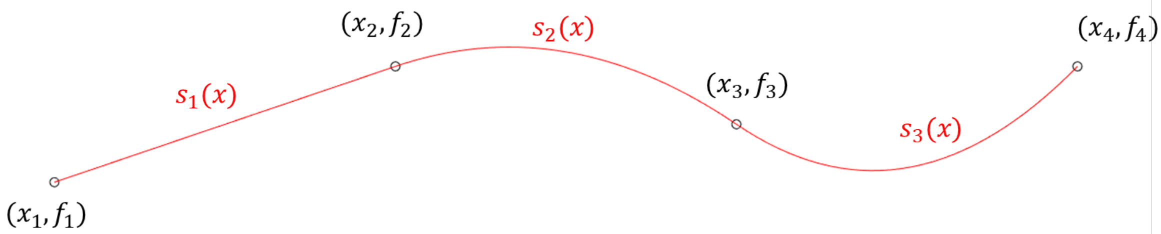
Each segment, there are 3 variables, so total = 3x3 = 9

$$s_1(x) = a_1 + b_1(x - x_1) + c_1(x - x_1)^2$$

$$s_2(x) = a_2 + b_2(x - x_2) + c_2(x - x_2)^2$$

$$s_3(x) = a_3 + b_3(x - x_3) + c_3(x - x_3)^2$$

Condition 1 : Functions must pass through all points.



$s_1(x)$ must pass (x_1, f_1) point.

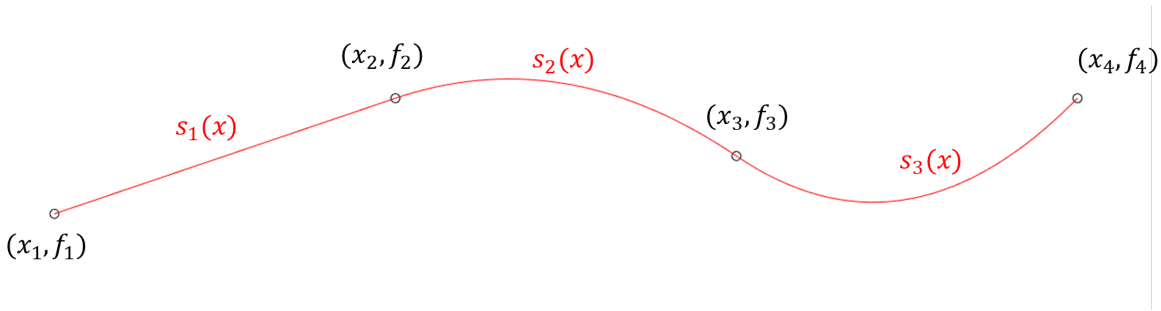
$$s_1(x_1) = f_1$$
$$a_1 + b_1(x_1 - x_1) + c_1(x_1 - x_1)^2 = f_1 \longrightarrow a_1 = f_1$$

Similarly,

$s_2(x)$ must pass (x_2, f_2) point. $a_2 = f_2$

$s_3(x)$ must pass (x_3, f_3) point. $a_3 = f_3$

Condition 2 : Function values of adjacent segment must be equals at the knots



Apply the continuity condition.

$$s_i(x) = f_i + b_i(x - x_i) + c_i(x - x_i)^2 = f_i + b_i h_i + c_i h_i^2$$

$$s_1(x) = f_1 + b_1(x - x_i) + c_1(x - x_i)^2 = f_1 + b_1 h_1 + c_1 h_1^2$$

$$s_2(x) = f_2 + b_2(x - x_i) + c_2(x - x_i)^2 = f_2 + b_2 h_2 + c_2 h_2^2$$

$$s_3(x) = f_3 + b_3(x - x_i) + c_3(x - x_i)^2 = f_3 + b_3 h_3 + c_3 h_3^2$$

The function values of adjacent polynomials must be equals at the knots.

$$f_i + b_i h_i + c_i h_i^2 = f_{i+1} \quad \text{----- (1)}$$

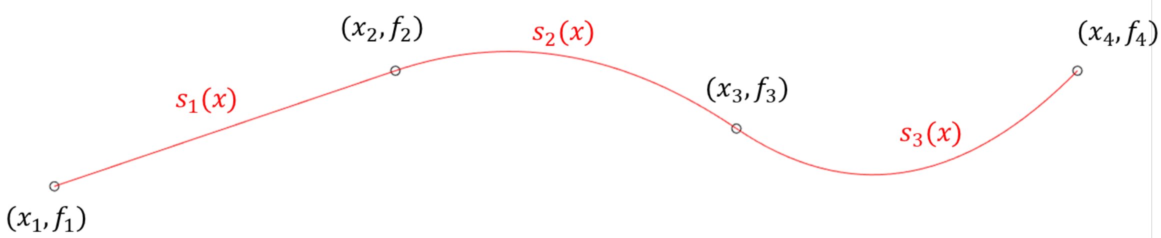
For knot 2,3,4

$$f_1 + b_1 h_1 + c_1 h_1^2 = f_2$$

$$f_2 + b_2 h_2 + c_2 h_2^2 = f_3$$

$$f_3 + b_3 h_3 + c_3 h_3^2 = f_4$$

Condition 3 : The first derivatives at the interior nodes must be equal to ensure smoothness.



$$s_i(x) = f_i + b_i(x - x_i) + c_i(x - x_i)^2$$

$$s'_i(x) = b_i + 2c_i(x - x_i)$$

$$s''_i(x) = 2c_i$$

$$b_i + 2c_i h_i = b_{i+1} \quad \text{----- (2)}$$

At interior knot 2

$$s'_1(x_2) = s'_2(x_2)$$

$$b_1 + 2c_1(x_2 - x_1) = b_2 + 2c_2(x_2 - x_2)$$

$$b_1 + 2c_1 h_1 = b_2$$

For knot 3

$$b_2 + 2c_2 h_2 = b_3$$

Condition 4 : Assume the second derivative is zero at the first point

$$s_1''(x_1) = 2c_1 = 0 \;\rightarrow\; c_1 = 0 \qquad \text{------- (3)}$$

Now we will derive the formula how to solve for all coefficients.

From (2)

$$b_i + 2c_i h_i = b_{i+1}$$
$$c_i = \frac{b_{i+1} - b_i}{2h_i}$$

Substitute in (1)

$$f_i + b_i h_i + c_i h_i^2 = f_{i+1}$$
$$f_i + b_i h_i + \left(\frac{b_{i+1} - b_i}{2h_i}\right) h_i^2 = f_{i+1}$$
$$f_i + \frac{h_i}{2} b_i + \frac{h_i}{2} b_{i+1} = f_{i+1}$$

So we have

$$f_1 + \frac{h_1}{2} b_1 + \frac{h_1}{2} b_2 = f_2 \qquad \text{---------- (4)}$$

$$f_2 + \frac{h_2}{2} b_2 + \frac{h_2}{2} b_3 = f_3 \qquad \text{---------- (5)}$$

From (3)

$$s_1''(x_1) = 2c_1 = 0 \ \rightarrow \ c_1 = 0$$

So we can conclude that

$$b_1 + 2c_1 h_1 = b_2$$
$$b_1 = b_2$$

$$b_1 - b_2 = 0 \qquad \text{---------- (6)}$$

From (4), (5), (6)

$$\begin{bmatrix} \frac{h_1}{2} & \frac{h_1}{2} & 0 \\ 0 & \frac{h_2}{2} & \frac{h_2}{2} \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} f_2 - f_1 \\ f_3 - f_2 \\ 0 \end{bmatrix}$$

Once we solve b_1, b_2, b_3, then we can solve for c_2, c_3

$$b_2 + 2c_2 h_2 = b_3 \quad \longrightarrow \quad c_2 = \frac{(b_3 - b_2)}{2h_2}$$

or

$$f_2 + b_2 h_2 + c_2 h_2^2 = f_3 \quad \longrightarrow \quad c_2 = \frac{(f_3 - f_2 - b_2 h_2)}{h_2^2}$$

$$f_3 + b_3 h_3 + c_3 h_3^2 = f_4 \quad \longrightarrow \quad c_3 = \frac{(f_4 - f_3 - b_3 h_3)}{h_3^2}$$