# Numerical Integration

Rawesak Tanawongsuwan, Ph.D.
rawesak.tan@mahidol.ac.th

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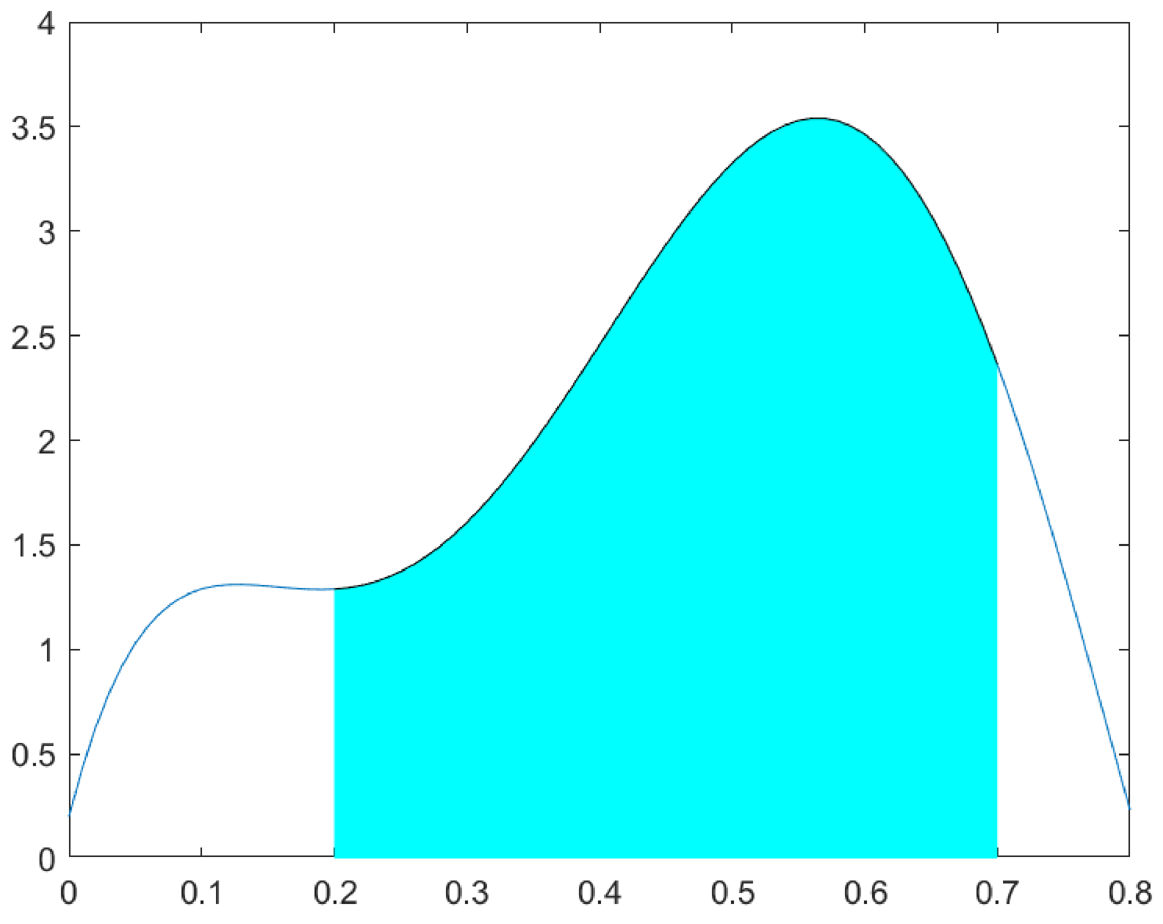
# What is Integration?

To integrate = to bring together, as parts, into a whole; to unite; to indicate the total amount

$$I = \int_a^b f(x)dx$$

The integral of the function $f(x)$ with respect to the independent variable $x$, evaluated between the limits $x = a$ to $x = b$ (Area under the curve)



# Numerical integration

Simple functions → Analytical evaluation

Complicated functions or underlying function is unknown and defined only by measurement at discrete points → Numerical integration

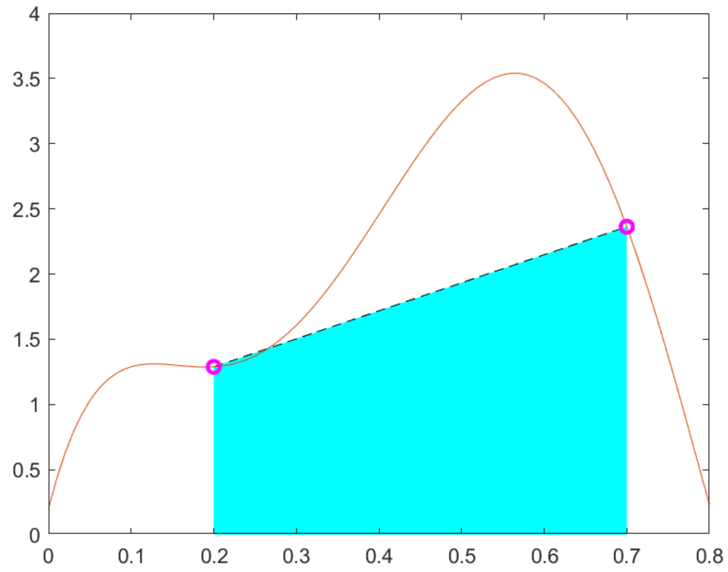
# Newton-Cotes Formulas

- Replacing a complicated function with a polynomial that is easy to integrate

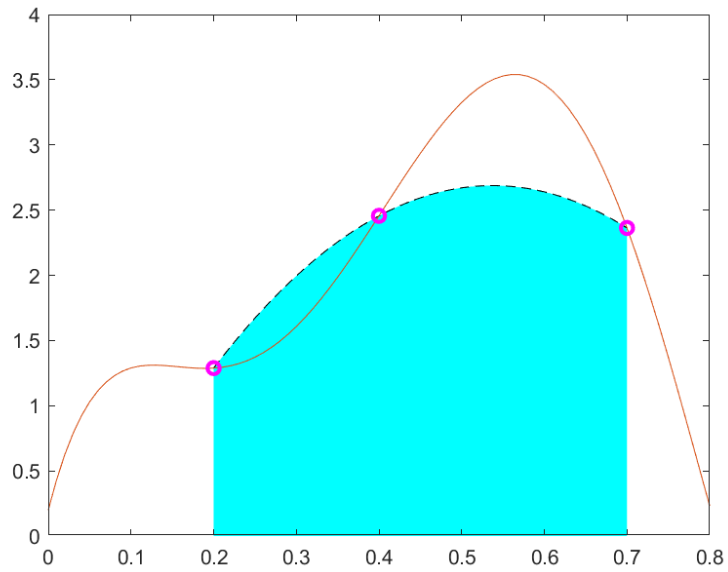
$$I = \int_a^b f(x)dx \cong \int_a^b f_n(x)dx$$

where

$$f_n(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + a_n x^n$$




Linear




Quadratic

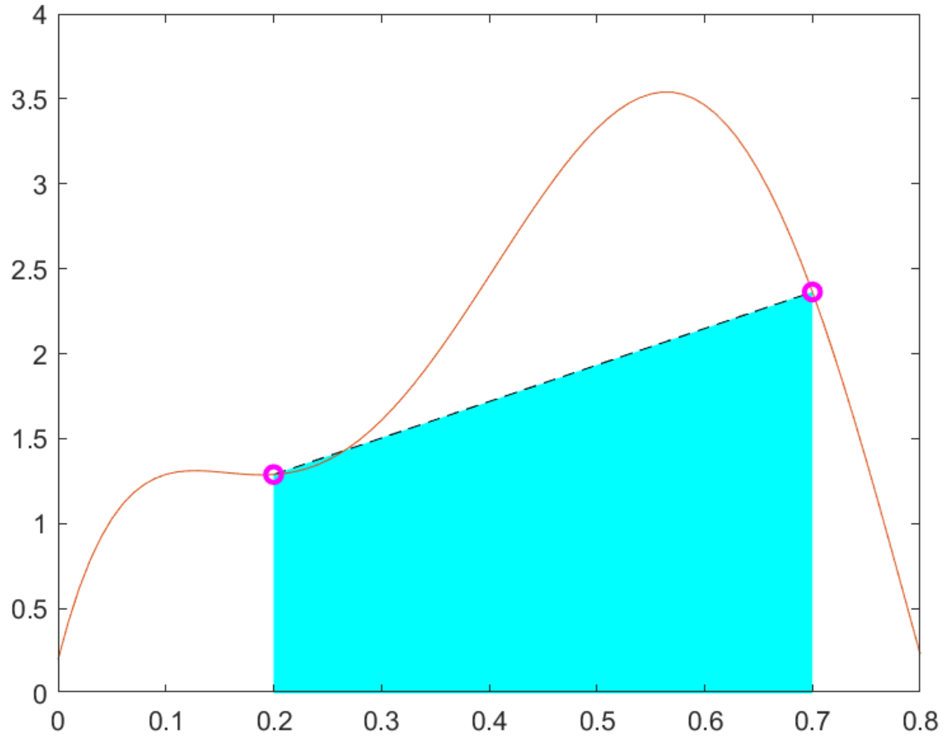
# The trapezoidal rule

The first of the Newton-Cotes closed integration formulas

$$I = \int_a^b \left[ f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \right] dx$$

$$I = (b - a)\frac{f(a) + f(b)}{2}$$

$$I = \text{width} \times \text{average height}$$



# Error of the trapezoidal rule

$$E_t = -\frac{1}{12} f''(\xi)(b-a)^3$$

where $\xi$ lies somewhere in the interval from $a$ to $b$

If $f(x)$ is linear, then $f''(x)$ is zero. That means $E_t = 0$ or the trapezoidal rule will be exact.

If $f(x)$ is nonlinear, $f''(x)$ is non-zero. That mean $E_t \neq 0$ or some error can occur.

# Example : the trapezoidal rule

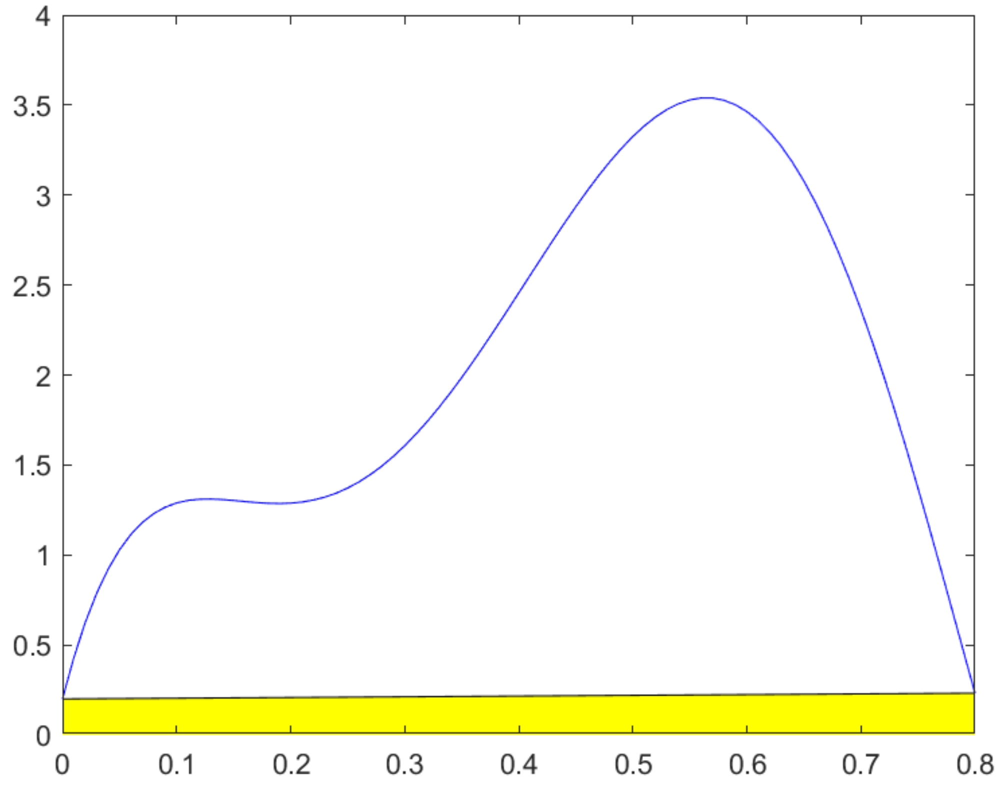
Let $f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$ → Compute $\int_0^{0.8} f(x)dx$

Using analytical evaluation = $1.640533$

Using the trapezoidal rule

$f(0) = 0.2, \qquad f(0.8) = 0.232$

$$I = (0.8 - 0)\frac{0.2 + 0.232}{2} = 0.1728$$



# Example → Error analysis

Since in this example, we can evaluate the integral analytically, so we can compute error from using the trapezoidal rule

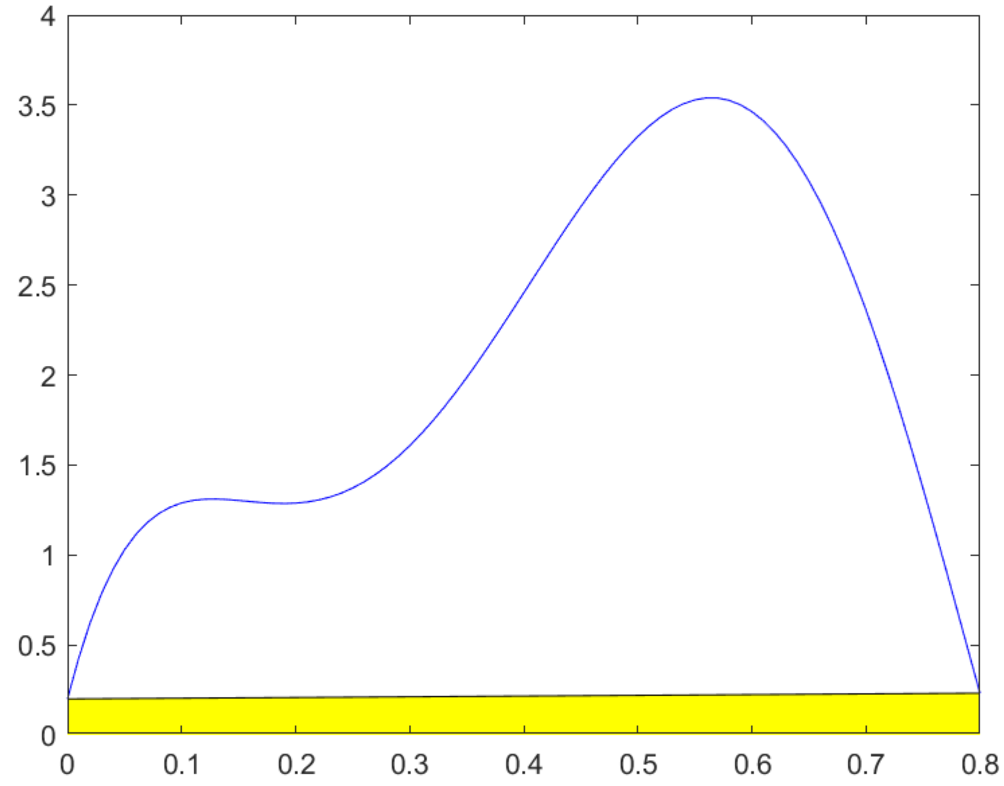
$$E_t = true\ value\ - approximation$$
$$= 1.640533 - 0.1728$$
$$= 1.467733$$

A percent relative error

$$\varepsilon_t = \frac{true\ value\ - approximation}{true\ value} 100\%$$

$$= \frac{1.467733}{1.640533} 100\%$$

$$= 89.5\%$$ → Large error



# Example → Error analysis (2)

In actual situations, we usually have no foreknowledge of the true value. Therefore, an approximate error estimate is required. $E_t = -\frac{1}{12} f''(\xi)(b-a)^3$

So we need to compute $f''(x) = -400 + 4{,}050x - 10{,}800x^2 + 8{,}000x^3$

The average value of the second derivative can be computed as

$$\bar{f}''(x) = \frac{\int_0^{0.8} (-400 + 4{,}050x - 10{,}800x^2 + 8{,}000x^3)dx}{0.8 - 0} = -60$$

$$E_a = -\frac{1}{12}(-60)(0.8-0)^3 = 2.56$$

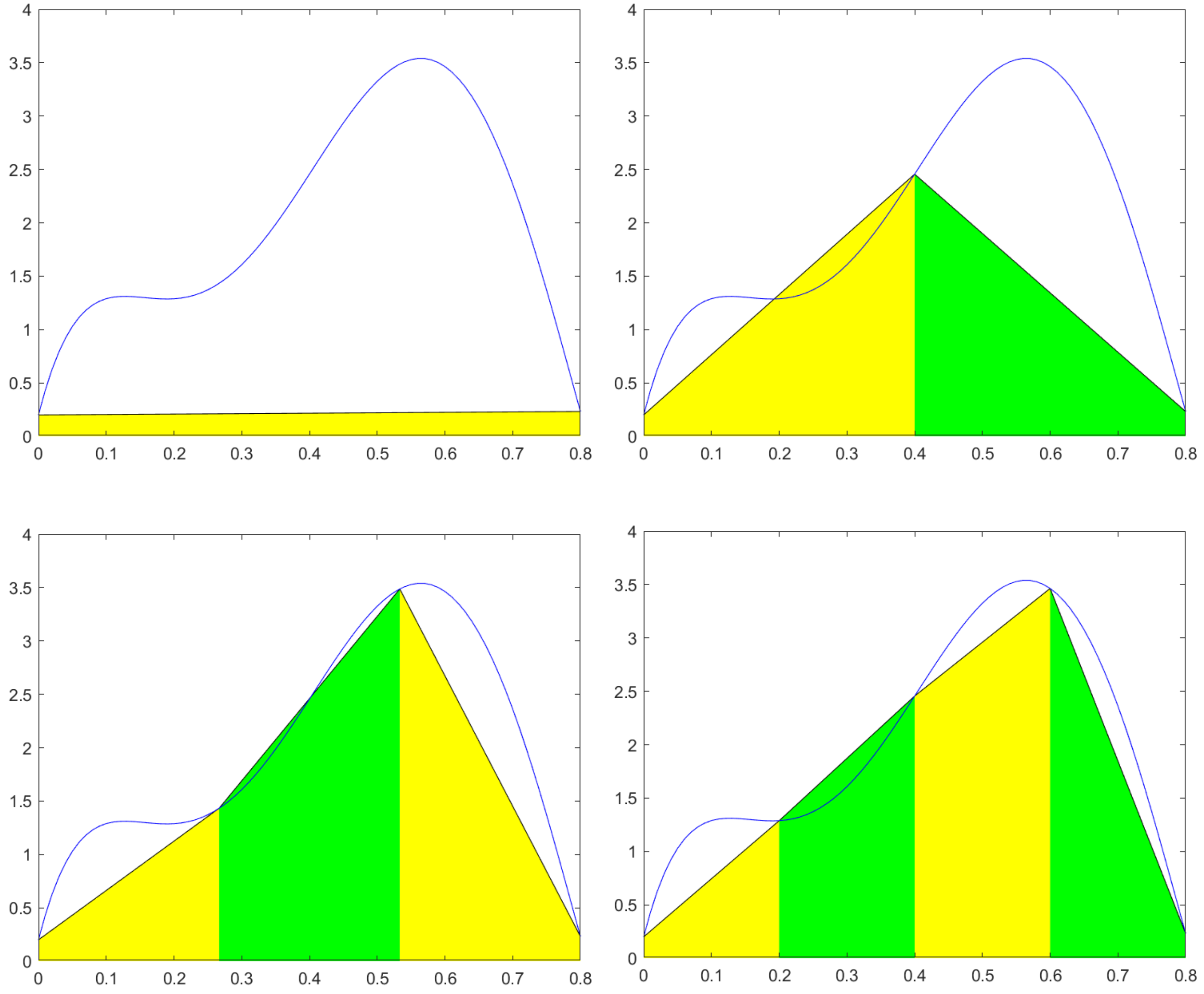
Note that we use $E_a$ instead of $E_t$ to represent the approximate error because computing $f''(x)$ in this case is not exact due to the large interval

# The composite trapezoidal rule (1)

- Divide the integration interval from $a$ to $b$ into a number of segments and apply the method to each segment.

- Let's define $ns$ = number of segments

- Figures in this slide show $ns = 1, 2, 3, 4$ segments

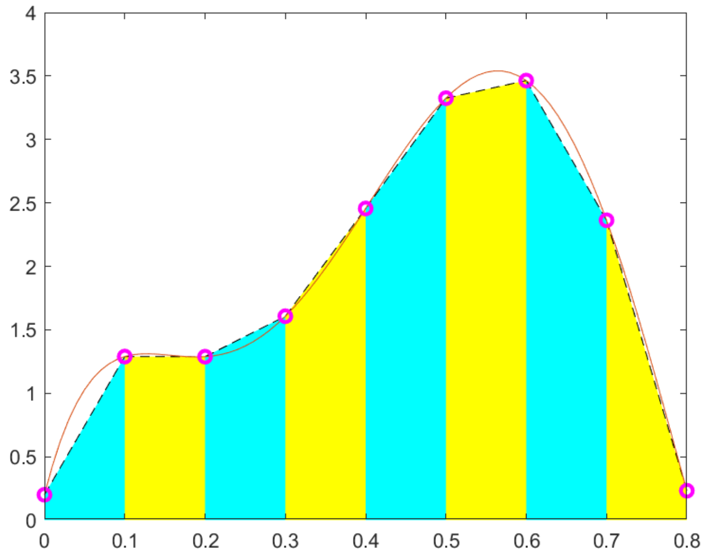


# The composite trapezoidal rule (2)

- In the composite trapezoidal rule, there are $ns$ segments.
- There are $n + 1$ equally spaced base points $(x_0, x_1, x_2, \ldots, x_n)$.
- In this case, $ns$ segments = $n$ segments.

- In the trapezoidal rule, $h = \text{width of segment} = \frac{b-a}{ns} = \frac{b-a}{n}$

- In the figure shown, there are 8 trapezoidal segments, so there are 8+1 points.




- $h = \text{width of segment} = \frac{0.8-0}{8} = 0.1$

# The composite trapezoidal rule (3)

Let's assume that there are $n+1$ equally spaced base points $(x_0, x_1, x_2, \ldots, x_n)$.

$$I = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)dx$$

$$= h\frac{f(x_0)+f(x_1)}{2} + h\frac{f(x_1)+f(x_2)}{2} + \cdots + h\frac{f(x_{n-1})+f(x_n)}{2}$$

$$= \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right]$$

$$= (b-a)\frac{\left[f(x_0)+2\sum_{i=1}^{n-1} f(x_i)+f(x_n)\right]}{2n}$$

$$= width\ \times average\ height$$

# Error of the composite trapezoidal rule

$$E_t = -\frac{1}{12}(x_1 - x_0)^3 f''(\xi_1) - \frac{1}{12}(x_2 - x_1)^3 f''(\xi_2) - \cdots - \frac{1}{12}(x_n - x_{n-1})^3 f''(\xi_n) = -\frac{1}{12}\left(\frac{b-a}{n}\right)^3 \sum_{i=1}^{n} f''(\xi_i)$$

since $x_1 - x_0 = x_2 - x_1 = \cdots = x_n - x_{n-1} = \frac{b-a}{n}$

and $f''(\xi_i)$ is the second derivative at a point $\xi_i$ located in segment $i$.

$f''(\xi_i)$ can be estimated by average value of the second derivative for the entire interval

$$\bar{f}''(\xi_i) \cong \frac{\sum_{i=1}^{n} f''(\xi_i)}{n} \quad \longrightarrow \quad \sum_{i=1}^{n} f''(\xi_i) \cong n\bar{f}''$$

$$E_a = -\frac{(b-a)^3}{12n^2}\bar{f}''$$

If the number of segments ($n$) is doubled, the truncation error will be quartered.

# Example : the composite trapezoidal rule

Let $f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$

Compute $\int_0^{0.8} f(x)dx$ using $ns = 2$ segments

The exact value of the integration = $1.640533$

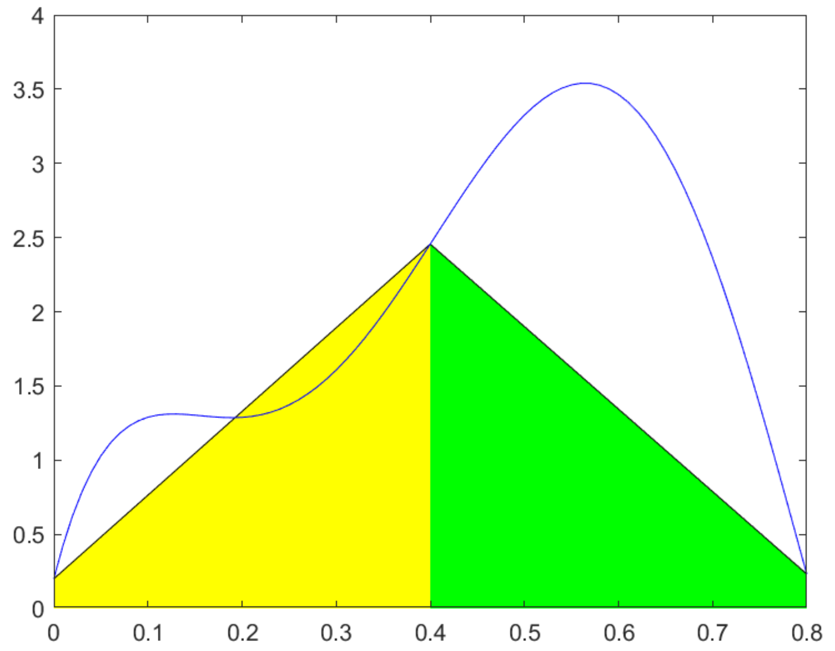
For $ns = n = 2 \left(h = \frac{0.8-0}{2} = 0.4\right) \longrightarrow f(0) = 0.2, \quad f(0.4) = 2.456, \quad f(0.8) = 0.232$

$$I = 0.8\frac{0.2 + 2(2.456) + 0.232}{4} = 1.0688$$

$$E_t = 1.640533 - 1.0688 = 0.57173$$

$$\varepsilon_t = 34.9\%$$

$$E_a = -\frac{0.8^3}{12(2)^2}(-60) = 0.64$$



# Example : the composite trapezoidal rule

| $ns$ | $h$ | $I$ | $\varepsilon_t\ (\%)$ |
|---|---|---|---|
| 2 | 0.4 | 1.0688 | 34.9 |
| 3 | 0.2667 | 1.3696 | 16.5 |
| 4 | 0.2 | 1.4848 | 9.5 |
| 5 | 0.16 | 1.5399 | 6.1 |
| 6 | 0.1333 | 1.5703 | 4.3 |
| 7 | 0.1143 | 1.5887 | 3.2 |
| 8 | 0.1 | 1.6008 | 2.4 |
| 9 | 0.0889 | 1.6091 | 1.9 |
| 10 | 0.08 | 1.6150 | 1.6 |

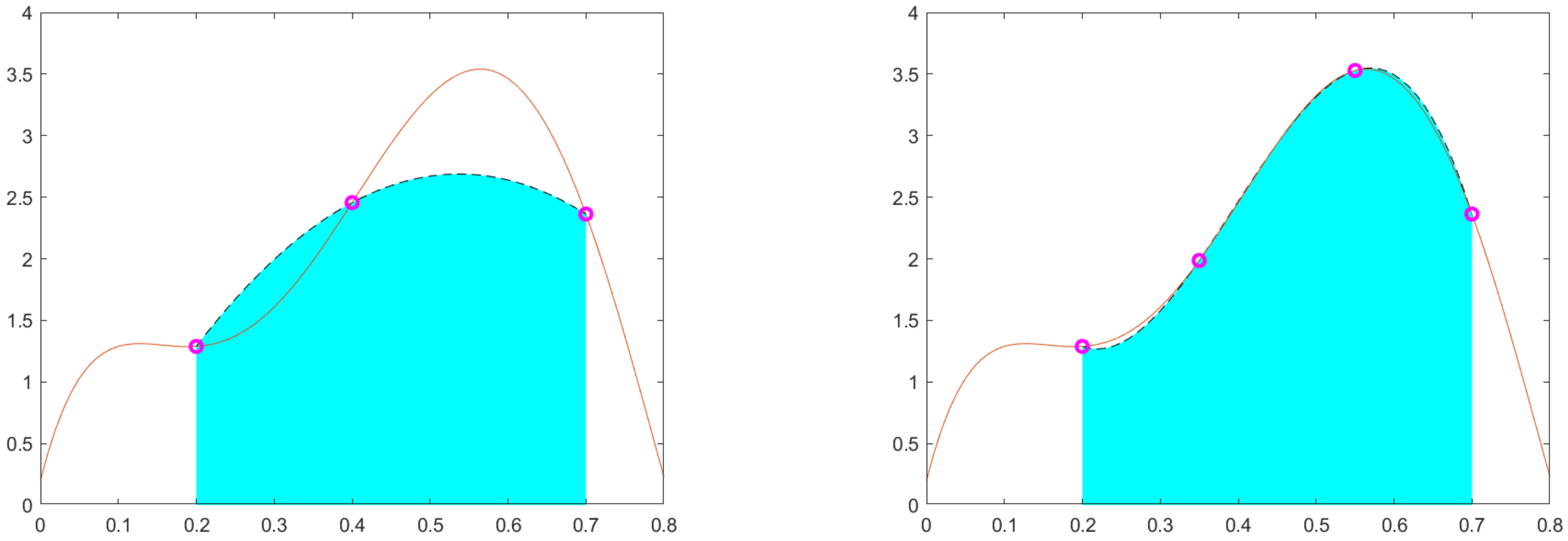
# MATLAB Program for Trapezoidal Rule

```
function I = trap(func,a,b,n,varargin)
% trap: composite trapezoidal rule quadrature
%  I = trap(func,a,b,n,p1,p2,...):
%             composite trapezoidal rule
% input:
%  func = function handle to function to be integrated
%  a, b = integration limits
%  n = number of segments (default = 100)
%  p1,p2,... = additional parameters used by func
% output:
%  I = integral estimate

if nargin<3,error('at least 3 input arguments required'),end
if ~(b>a),error('upper bound must be greater than lower'),end
if nargin<4||isempty(n),n=100;end
x = a; h = (b - a)/n;
s=func(a,varargin{:});
for i = 1 : n-1
 x = x + h;
 s = s + 2*func(x,varargin{:});
end
s = s + func(b,varargin{:});
I = (b - a) * s/(2*n);
```

# Simpson’s rules

- Another way to obtain a more accurate estimate of an integral is to use higher-order polynomials to connect the points.
- The formulas that result from taking the integrals under these polynomials are called ***Simpson’s rules***.



# Simpson's 1/3 rule (second-order polynomial)

$$I = \int_{x_0}^{x_2} \left[ \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2) \right] dx$$

$I = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)]$ → $h = \frac{b-a}{2}$ → $h$ is half of segment

$$I = (b - a) \frac{f(x_0) + 4f(x_1) + f(x_2)}{6}$$

$$x_0 = a, \qquad x_1 = \frac{a + b}{2}, \qquad x_2 = b$$

# Error of the Simpson’s 1/3 rule (single segment)

$$E_t = -\frac{1}{90} h^5 f^{(4)}(\xi)$$

where $h = \frac{b-a}{2}$

$$E_t = -\frac{(b-a)^5}{2880} f^{(4)}(\xi)$$

where $\xi$ lies somewhere in the interval from $a$ to $b$.

Simpson’s 1/3 rule is more accurate than the trapezoidal rule $-\frac{1}{12} f''(\xi)(b-a)^3$

Since the Simplson’s 1/3 rule is based on 3 points, the error should be proportional to the third derivative. However, it is proportional to the fourth derivative!

In other words, it yields exact results for cubic polynomials even though it is derived from a parabola.

# Example : the Simpson's 1/3 rule

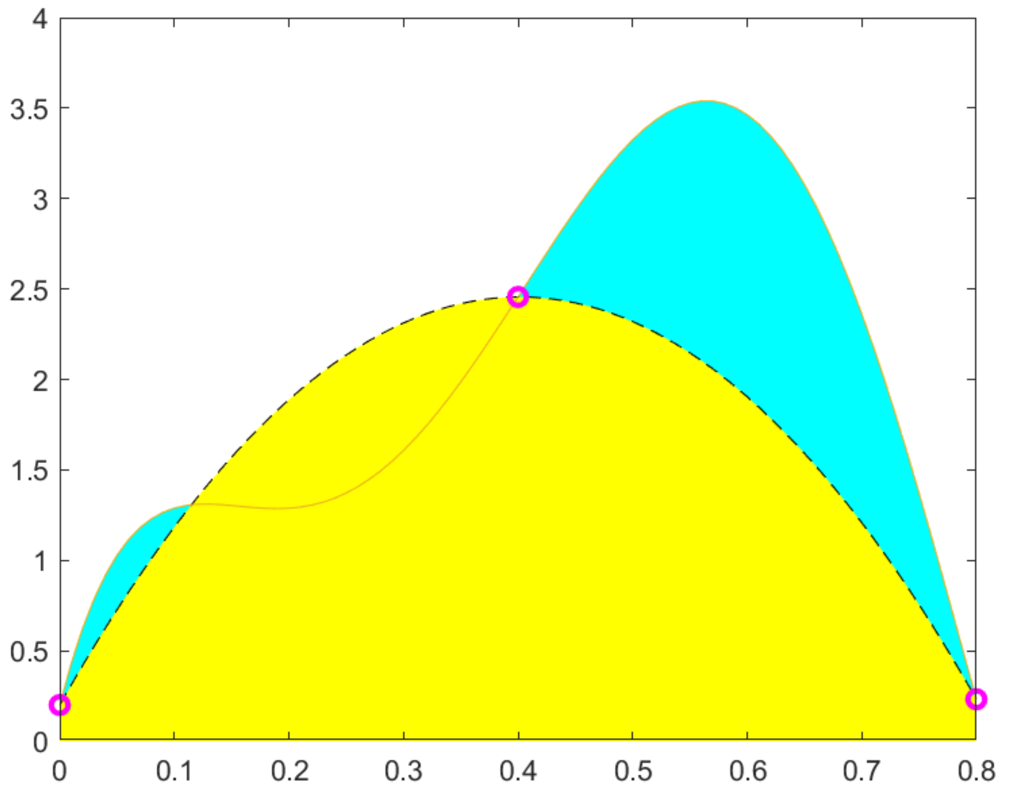
Let $f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$ → Compute $\int_0^{0.8} f(x)dx$

Using analytical evaluation = $1.640533$

Using the Simpson's 1/3 rule

$f(0) = 0.2,\qquad f(0.4) = 2.456,\qquad f(0.8) = 0.232$

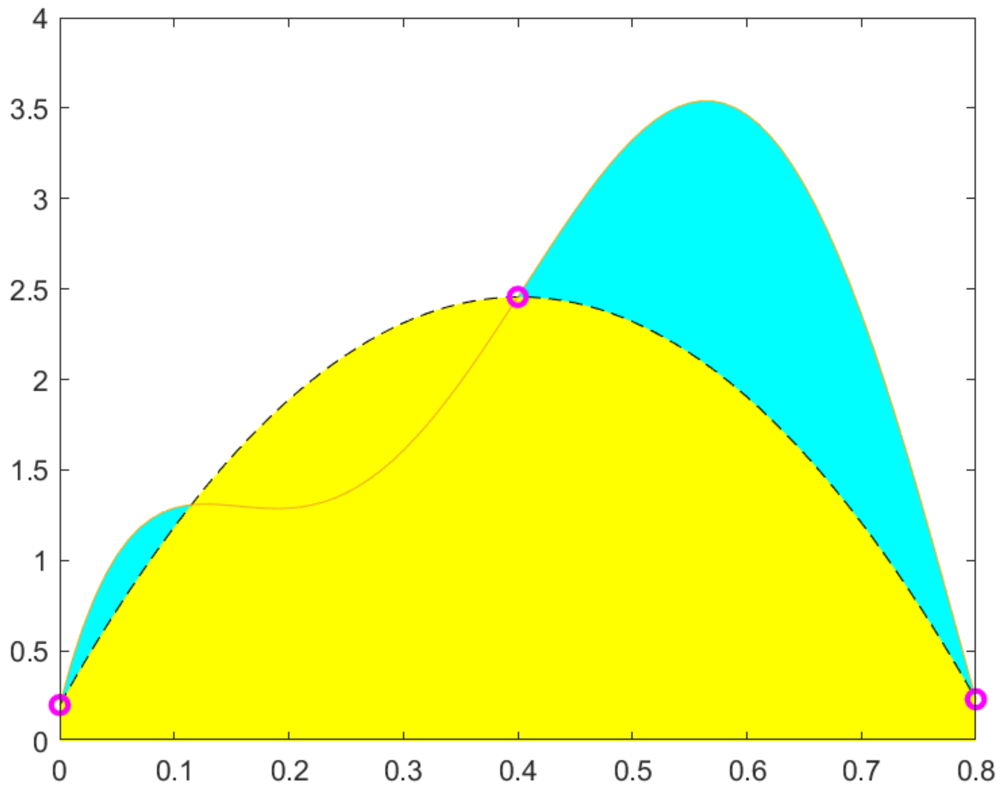
$$I = (0.8 - 0)\frac{0.2 + 4(2.456) + 0.232}{6} = 1.367467$$



# Example → the Simpson's 1/3 rule (Error analysis)

Since in this example, we can evaluate the integral analytically, so we can compute error from using the Simpson's 1/3 rule

$$E_t = true\ value\ - approximation$$
$$= 1.640533 - 1.367467$$
$$= 0.2730667$$




A percent relative error

$$\varepsilon_t = \frac{true\ value\ - approximation}{true\ value} 100\%$$

$$= \frac{0.2730667}{1.640533} 100\%$$

$$= 16.6\%$$ → Smaller error compared to the trapezoidal rule

# Example → the Simpson's 1/3 rule (Error analysis)(2)

In actual situations, we usually have no foreknowledge of the true value. Therefore, an approximate error estimate is required. $E_t = -\frac{(b-a)^5}{2880} f^{(4)}(\xi)$

So we need to compute $f^{(4)}(x) = -21{,}600 + 48{,}000x$

The average value of the fourth derivative can be computed as

$$\bar{f}^{(4)}(x) = \frac{\int_0^{0.8}(-21{,}600 + 48{,}000x)dx}{0.8 - 0} = -2{,}400$$

$$E_a = -\frac{(0.8-0)^5}{2880}(-2400) = 0.2730667$$

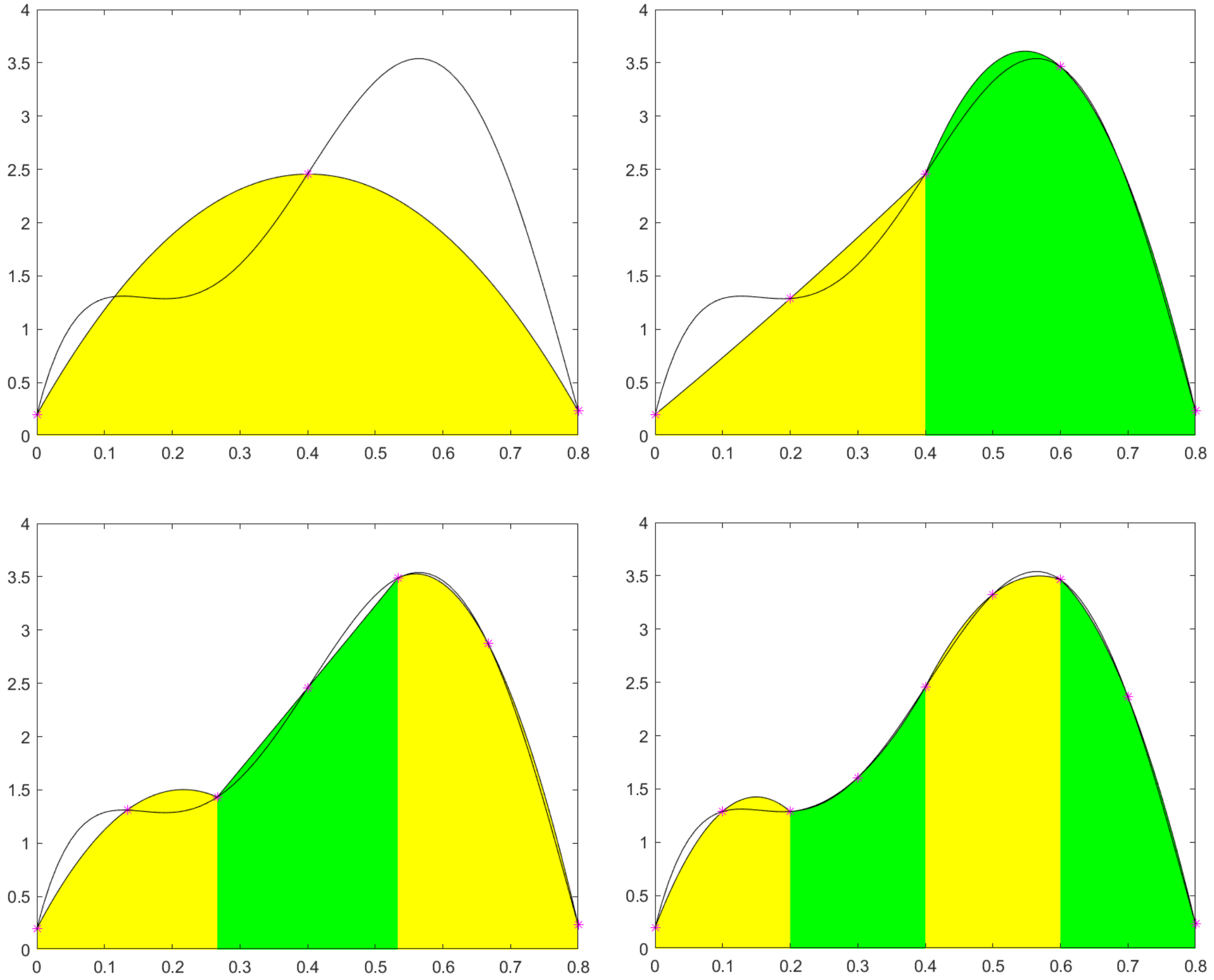
Note that we use $E_a$ instead of $E_t$ to represent the approximate error because computing $f^{(4)}(x)$ in this case is not exact due to the large interval

# The composite Simpson's 1/3 rule (1)

- Divide the integration interval from $a$ to $b$ into a number of segments and apply the method to each segment.

- Let's define $ns$ = number of segments

- Figures in this slide show $ns = 1, 2, 3, 4$ segments

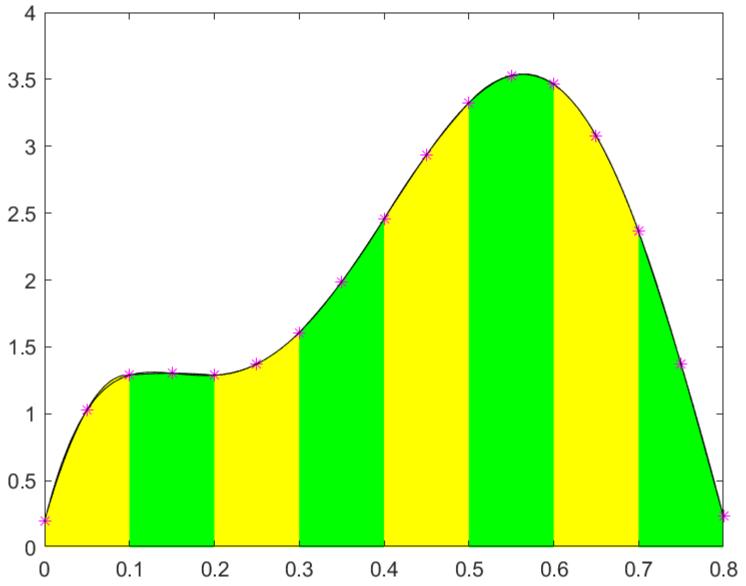


# The composite Simpson's 1/3 rule (2)

- In the composite Simpson's 1/3 rule, there are $ns$ segments.
- There are $n + 1$ equally spaced base points $(x_0, x_1, x_2, \dots, x_n)$.
- In this case, $ns$ segments = $\frac{n}{2}$ segments.

- In the Simpson's 1/3 rule,

$$h = \text{half width of segment} = \frac{b-a}{2*ns} = \frac{b-a}{n}$$




- In the figure shown, there are $ns = 8$ segments $= \frac{n}{2} = \frac{16}{2}$, there are $2 * 8 + 1 \; = \; 16 + 1 = 17$ points.

- $h = \text{half width of segment} = \frac{0.8-0}{16} = 0.05$

# The composite Simpson's 1/3 rule (3)

Let's assume that there are $n+1$ equally spaced base points $(x_0, x_1, x_2, \dots, x_n)$.

$$I = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \cdots + \int_{x_{n-2}}^{x_n} f(x)dx$$

$$= \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] + \frac{h}{3}[f(x_2) + 4f(x_3) + f(x_4)] + \cdots + \frac{h}{3}[f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

$$= \frac{h}{3}\left[f(x_0) + 4\sum_{i=1,3,5,\dots}^{n-1} f(x_i) + 2\sum_{j=2,4,6,\dots}^{n-2} f(x_j) + f(x_n)\right]$$

$$= (b-a)\frac{\left[f(x_0) + 4\sum_{i=1,3,5,\dots}^{n-1} f(x_i) + 2\sum_{j=2,4,6,\dots}^{n-2} f(x_j) + f(x_n)\right]}{3n}$$

# Error of the composite Simpson's 1/3 rule

$$E_t = -\frac{1}{90}h_1^5 f^{(4)}(\xi_1) - \frac{1}{90}h_2^5 f^{(4)}(\xi_2) - \cdots - \frac{1}{90}h_n^5 f^{(4)}(\xi_n) = -\frac{1}{90}h^5 \sum_{i=1}^{n} f(4)(\xi_i)$$

where $h_1 = h_2 = \cdots = h_n = h = \frac{b-a}{2n}$

$$E_t = -\frac{1}{90}\left(\frac{b-a}{2n}\right)^5 \sum_{i=1}^{n} f^{(4)}(\xi_i)$$

where $f^{(4)}(\xi_i)$ is the fourth derivative at a point $\xi_i$ located in segment $i$.

$f^{(4)}(\xi_i)$ can be estimated by average value of the fourth derivative for the entire interval

$$\bar{f}^{(4)}(\xi_i) \cong \frac{\sum_{i=1}^{n} f^{(4)}(\xi_i)}{n} \quad \longrightarrow \quad \sum_{i=1}^{n} f^{(4)}(\xi_i) \cong n\bar{f}^{(4)}$$

$$E_a = -\frac{(b-a)^5}{2880n^4}\bar{f}^{(4)}(\xi_i)$$

If the number of segments $(n)$ is doubled, the truncation error will be reduced by a factor of 16.

# Example : the composite Simpson's 1/3 rule

Let $f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$

Compute $\int_0^{0.8} f(x)dx$ using 2 segments ($ns = 2,\ n = 4,$ and $4 + 1 = 5$ points)

The exact value of the integration = $1.640533$

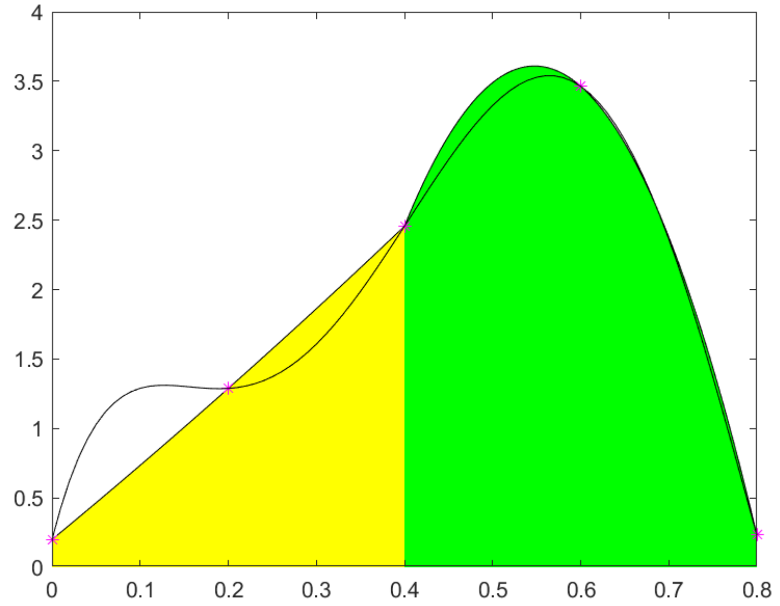
For $ns = 2, n = 4$ $\left(h = \frac{0.8-0}{4} = 0.2\right)$ $f(0) = 0.2, f(0.2) = 1.288, f(0.4) = 2.456, f(0.6) = 3.464, f(0.8) = 0.232$

$$I = 0.8\frac{0.2 + 4(1.288 + 3.464) + 2(2.456) + 0.232}{12} = 1.623467$$

$$E_t = 1.640533 - 1.623467 = 0.017067$$

$$\varepsilon_t = 1.04\%$$

$$E_a = -\frac{0.8^5}{2880(4)^4}(-2400) = 0.0011$$



# Simpson's 3/8 rule (third-order polynomial)

A third-order Lagrange polynomial can be fit to four points and integrated to yield

$$I = \int_{x_0}^{x_3} \left[ \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1) + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3) \right] dx$$

$I = \frac{3h}{8}[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)]$ → $h = \frac{b-a}{3}$ → $h$ is a third of segment

$$I = (b-a)\frac{f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)}{8}$$

$$x_0 = a, \qquad x_1 = \frac{2a+b}{3}, \qquad x_2 = \frac{a+2b}{3}, \qquad x_3 = b$$

# Error of the Simpson’s 3/8 rule (single segment)

$$E_t = -\frac{3}{80} h^5 f^{(4)}(\xi)$$

where $h = \frac{b-a}{3}$

$$E_t = -\frac{(b-a)^5}{6480} f^{(4)}(\xi)$$

where $\xi$ lies somewhere in the interval from $a$ to $b$.

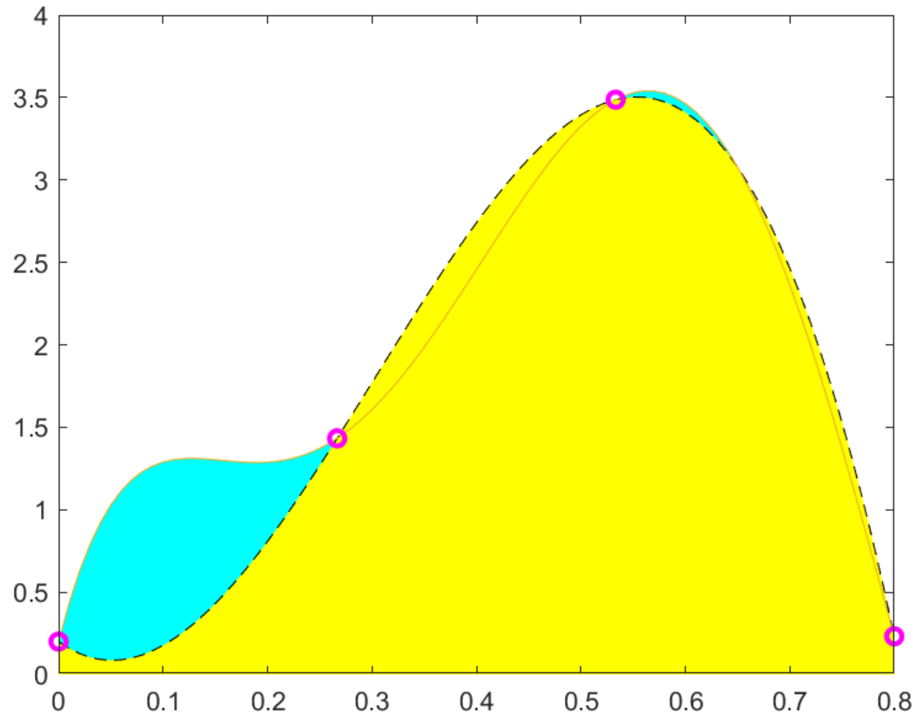
Simpson’s 3/8 rule is more accurate than the trapezoidal rule $-\frac{1}{12} f''(\xi)(b-a)^3$

Since the Simplson’s 3/8 rule is based on 4 points, the error is proportional to the fourth derivative.

# Example : the Simpson's 3/8 rule

Let $f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$

Compute $\int_0^{0.8} f(x)dx$ using one segment (4 points)




Using analytical evaluation = $1.640533$

Using the Simpson's 3/8 rule

$f(0) = 0.2, \qquad f(0.2667) = 1.432724$

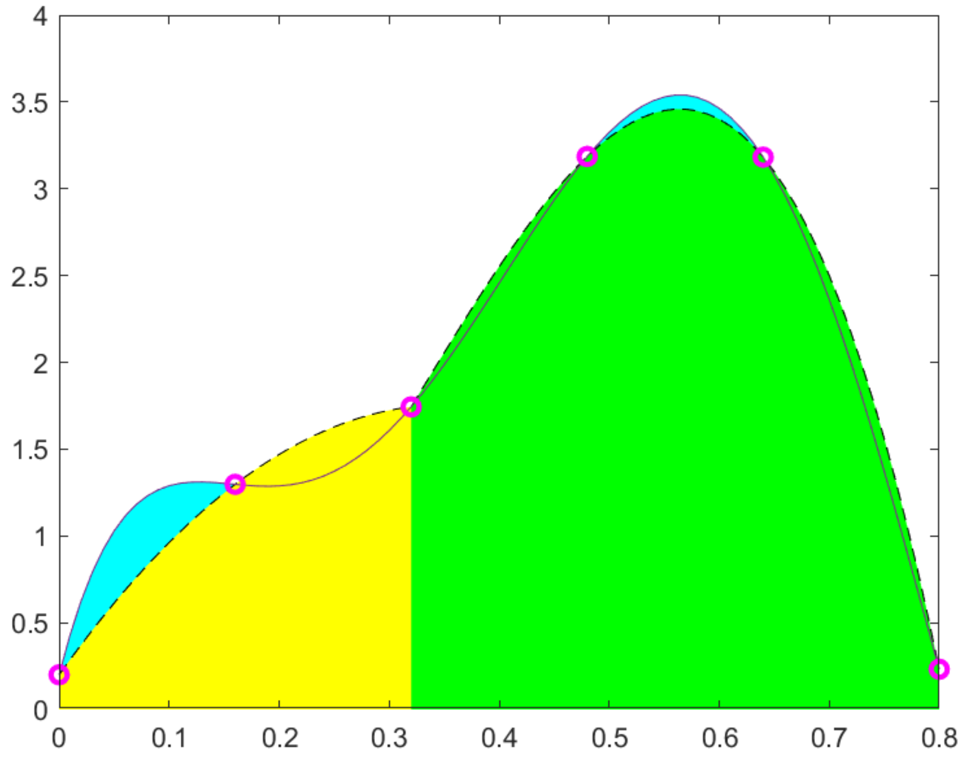
$f(0.5333) = 3.487177, \qquad f(0.8) = 0.232$

$$I = (0.8 - 0)\frac{0.2 + 3(1.432724 + 3.487177) + 0.232}{8} = 1.51917$$

# Example : the Simpson's 3/8 rule

Let $f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$

Compute $\int_0^{0.8} f(x)dx$ using 2 segments (6 points, $h = 0.16$)




We need to combine 1/3 and 3/8 rules

$f(0) = 0.2, f(0.16) = 1.296919, f(0.32) = 1.743393$

$f(0.48) = 3.186015, f(0.64) = 3.181929, f(0.8) = 0.232$

The first segment (3 points) uses Simpson's 1/3 rule.

$$I = (0.32 - 0)\frac{0.2 + 4(1.296919\ ) + 1.743393}{6} = 0.3803237$$

The second segment (4 points) uses Simpson's 3/8 rule.

$$I = (0.8 - 0.32)\frac{1.743393 + 3(3.186015 + 3.181929) + 0.232}{8} = 1.264754$$

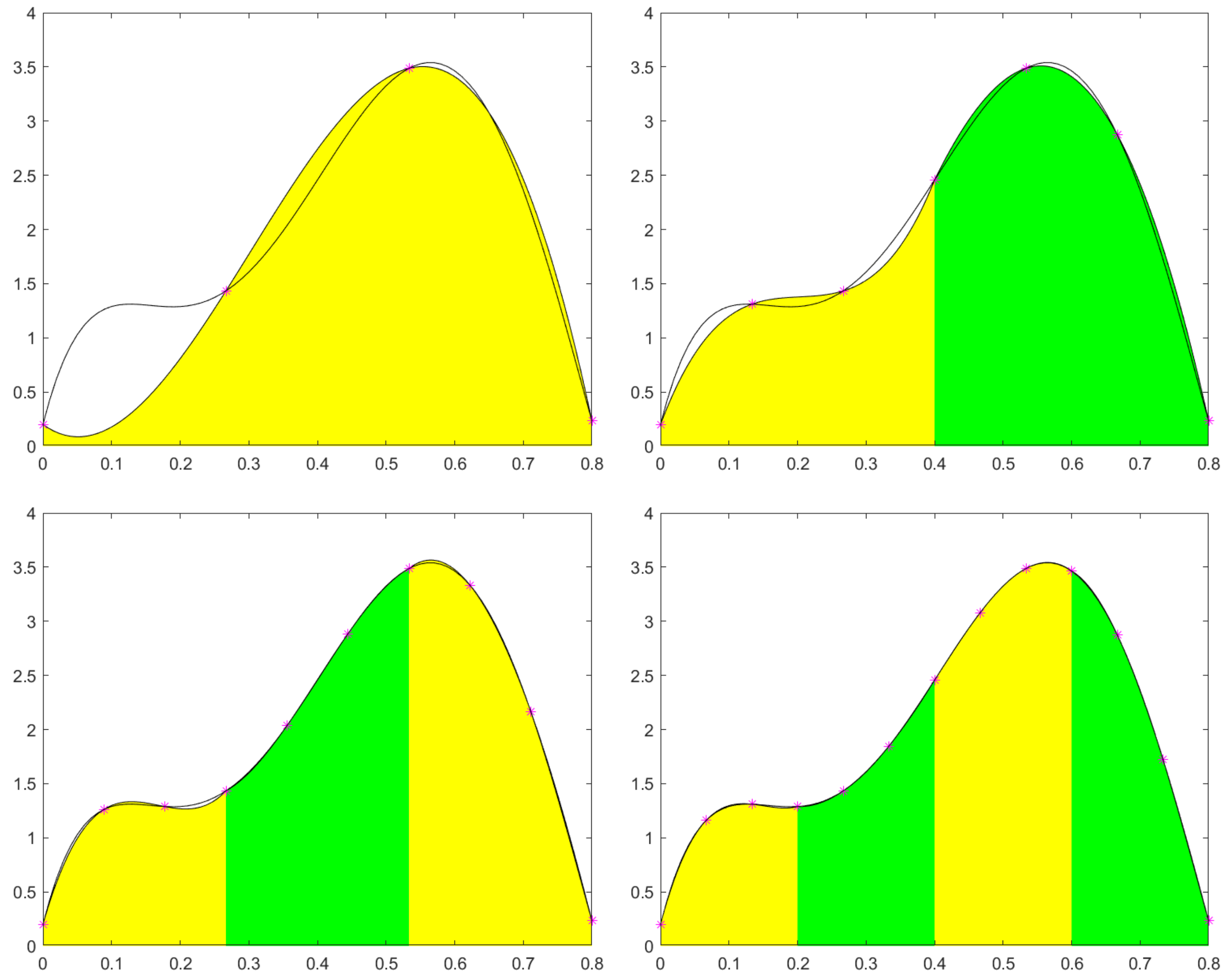
The total integral $I = 0.3803237 + 1.264754 = 1.645077$

# The composite Simpson's 3/8 rule (1)

- Divide the integration interval from $a$ to $b$ into a number of segments and apply the method to each segment.

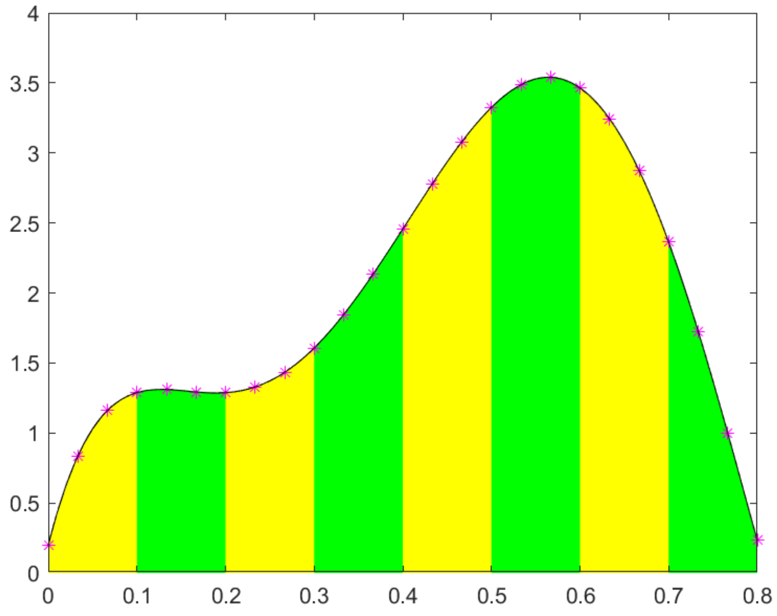
- Let's define $ns$ = number of segments

- Figures in this slide show $ns = 1, 2, 3, 4$ segments



# The composite Simpson's 3/8 rule (2)

- In the composite Simpson's 3/8 rule, there are $ns$ segments.
- There are $n + 1$ equally spaced base points $(x_0, x_1, x_2, \ldots, x_n)$.
- In this case, $ns$ segments = $\frac{n}{3}$ segments.




- In the Simpson's 3/8 rule,

$$h = \text{a third width of segment} = \frac{b-a}{3*ns} = \frac{b-a}{n}$$

- In the figure shown, there are $ns = 8$ segments $= \frac{n}{3} = \frac{24}{3}$, so there are $3 * 8 + 1 \ = 24 + 1 = 25$ points.

- $h = \text{a third width of segment} = \frac{0.8-0}{24} = 0.0333$

# The composite Simpson's 3/8 rule (3)

Let's assume that there are $n+1$ equally spaced base points $(x_0, x_1, x_2, \dots, x_n)$.

$$I = \int_{x_0}^{x_3} f(x)dx + \int_{x_3}^{x_6} f(x)dx + \dots + \int_{x_{n-3}}^{x_n} f(x)dx$$

$$= \frac{3h}{8}[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)] + \frac{3h}{8}[f(x_3) + 3f(x_4) + 3f(x_5) + f(x_6)] + \cdots$$
$$+ \frac{3h}{8}[f(x_{n-3}) + 3f(x_{n-2}) + 3f(x_{n-1}) + f(x_n)]$$

$$= \frac{3h}{8}\left[f(x_0) + 3\sum_{i=1,4,7,\dots}^{n-2} f(x_i) + 3\sum_{j=2,5,8,\dots}^{n-1} f(x_j) + 2\sum_{j=3,6,9,\dots}^{n-3} f(x_j) + f(x_n)\right]$$

$$= 3(b-a)\frac{\left[f(x_0) + 3\sum_{i=1,4,7,\dots}^{n-2} f(x_i) + 3\sum_{j=2,5,8,\dots}^{n-1} f(x_j) + 2\sum_{j=3,6,9,\dots}^{n-3} f(x_j) + f(x_n)\right]}{8n}$$

# Example : the composite Simpson's 3/8 rule

Let $f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$

Compute $\int_0^{0.8} f(x)dx$ using 2 segments ($ns = 2,\ n = 6,$ and $6 + 1 = 7$ points)

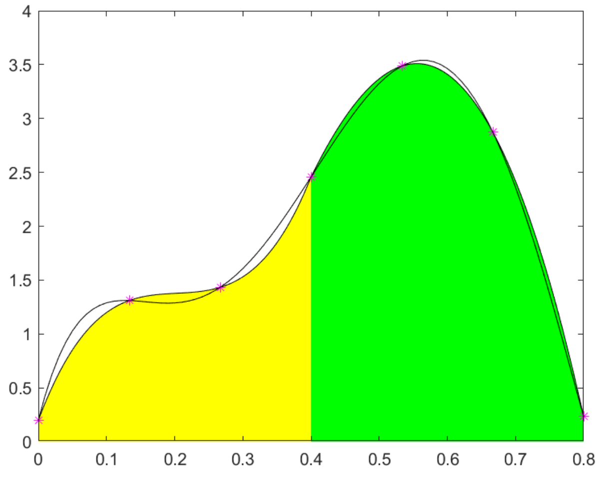
The exact value of the integration $=$ 1.640533

For $ns = 2, n = 6$ $\left(h = \frac{0.8-0}{6} = 0.13333\right)$

$f(0) = 0.2, f(0.13333) = 1.31019, f(0.26667) = 1.43272, f(0.4) = 2.4560,$

$f(0.53333) = 3.48718, f(6.66667) = 2.87490, f(0.8) = 0.2320$

$$\frac{3h}{8}\left[f(x_0) + 3\sum_{i=1,4,7,\dots}^{n-2} f(x_i) + 3\sum_{j=2,5,8,\dots}^{n-1} f(x_j) + 2\sum_{j=3,6,9,\dots}^{n-3} f(x_j) + f(x_n)\right]$$




$$I = \frac{3*0.13333}{8}[0.2 + 3(1.31019 + 3.48718) + 3(1.43272 + 2.87490) + 2*2.4560 + 0.2320] = 1.632948$$

$$E_t = 1.640533 - 1.623948 = 0.007585$$

$$\varepsilon_t = 0.46\%$$

# Integration with Unequal Segments

Previous formulas were simplified based on equispaced data points—though this is not always the case.

The trapezoidal rule may be used with data containing unequal segments:

$$I = h_1 \frac{f(x_0)+f(x_1)}{2} + h_2 \frac{f(x_1)+f(x_2)}{2} + \cdots + h_n \frac{f(x_{n-1})+f(x_n)}{2}$$

where $h_i =$ the width of segment $i$

# Example : Trapezoidal rule with unequal segments

Let $f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$

Compute $\int_0^{0.8} f(x)dx$ from the data for $f(x)$ with unequally spaced values of $x$.

| $x$ | $f(x)$ |
|---|---|
| 0.00 | 0.200000 |
| 0.12 | 1.309729 |
| 0.22 | 1.305241 |
| 0.32 | 1.743393 |
| 0.36 | 2.074903 |
| 0.40 | 2.456000 |
| 0.44 | 2.842985 |
| 0.54 | 3.507297 |
| 0.64 | 3.181929 |
| 0.70 | 2.363000 |
| 0.80 | 0.232000 |

Using analytical evaluation $= \; 1.640533$

Using the trapezoidal rule with unequal segments

$$I = 0.12\,\frac{0.2+1.309729}{2} + 0.10\,\frac{1.309729+1.305241}{2} + \cdots$$

$$+0.10\,\frac{2.363000+0.232000}{2} = 1.594801$$

$$E_t = 1.640533 - 1.594801 = 0.045732$$

$$\varepsilon_t = \frac{0.045732}{1.640533}100\% = 2.8\%$$

# Integration Code for Unequal Segments

```
function I = trapuneq(x,y)
% trapuneq: unequal spaced trapezoidal rule quadrature
%   I = trapuneq(x,y):
%   Applies the trapezoidal rule to determine the integral
%   for n data points (x, y) where x and y must be of the
%   same length and x must be monotonically ascending
% input:
%   x = vector of independent variables
%   y = vector of dependent variables
% output:
%   I = integral estimate

if nargin<2,error('at least 2 input arguments required'),end
if any(diff(x)<0),error('x not monotonically ascending'),end
n = length(x);
if length(y)~=n,error('x and y must be same length'); end
s = 0;
for k = 1:n-1
  s = s + (x(k+1)-x(k))*(y(k)+ y(k +1))/2;
end
I = s;
```

# MATLAB Functions

- MATLAB has built-in functions to evaluate integrals based on the trapezoidal rule

- **`z = trapz(y)`**
  **`z = trapz(x, y)`**

- produces the integral of `y` with respect to `x`. If `x` is omitted, the program assumes $h$ = 1.

- **`z = cumtrapz(y)`**
  **`z = cumtrapz(x, y)`**

- produces the cumulative integral of `y` with respect to `x`. If `x` is omitted, the program assumes $h$ = 1.