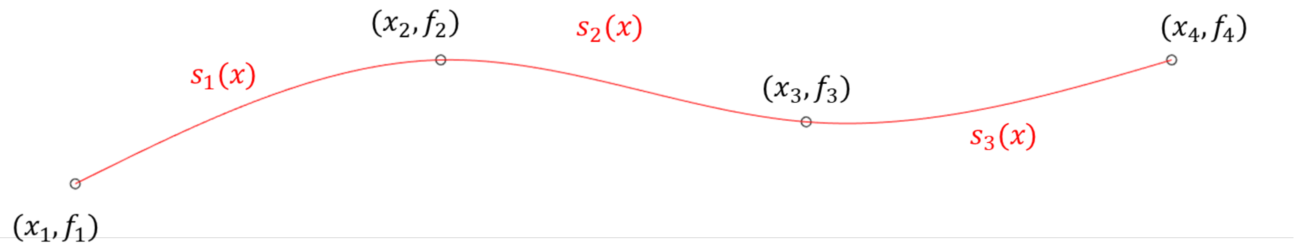
Cubic splines



$$s_i(x) = f_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$$

For $n = 4$ points, there are 3 intervals $s_1(x), s_2(x), s_3(x)$.

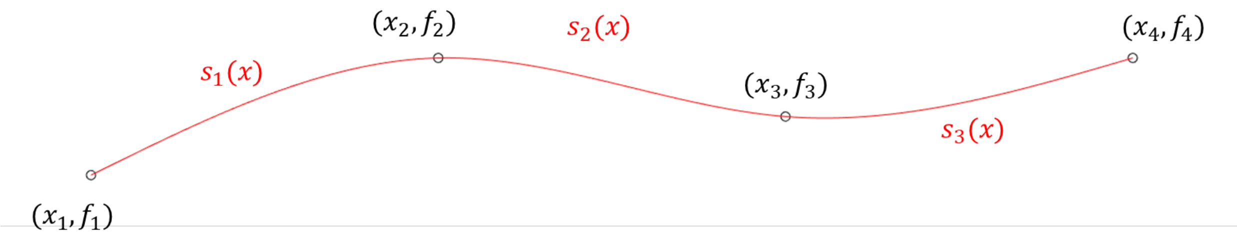
Each segment, there are 4 variables, so total = 4x3 = 12

$$s_1(x) = f_1 + b_1(x - x_i) + c_1(x - x_i)^2 + d_1(x - x_i)^3$$

$$s_2(x) = f_2 + b_2(x - x_i) + c_2(x - x_i)^2 + d_2(x - x_i)^3$$

$$s_3(x) = f_3 + b_3(x - x_i) + c_3(x - x_i)^2 + d_3(x - x_i)^3$$

Condition 1 : Functions must pass through all points.



$s_1(x)$ must pass (x_1, f_1) point.

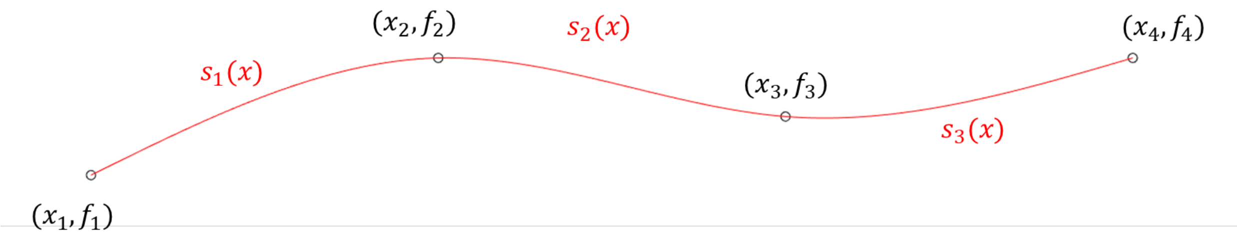
$$s_1(x_1) = f_1$$
$$a_1 + b_1(x_1 - x_1) + c_1(x_1 - x_1)^2 + d_1(x_1 - x_1)^3 = f_1 \longrightarrow a_1 = f_1$$

Similarly,

$s_2(x)$ must pass (x_2, f_2) point. $\quad a_2 = f_2$

$s_3(x)$ must pass (x_3, f_3) point. $\quad a_3 = f_3$

Condition 2 : Function values of adjacent segment must be equals at the knots



Apply the continuity condition.

$$s_i(x) = f_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3 = f_i + b_i h_i + c_i h_i^2 + d_i h_i^3$$

$$s_1(x) = f_1 + b_1(x - x_i) + c_1(x - x_i)^2 + d_1(x - x_i)^3 = f_1 + b_1 h_1 + c_1 h_1^2 + d_1 h_1^3$$

$$s_2(x) = f_2 + b_2(x - x_i) + c_2(x - x_i)^2 + d_2(x - x_i)^3 = f_2 + b_2 h_2 + c_2 h_2^2 + d_2 h_2^3$$

$$s_3(x) = f_3 + b_3(x - x_i) + c_3(x - x_i)^2 + d_3(x - x_i)^3 = f_3 + b_3 h_3 + c_3 h_3^2 + d_3 h_3^3$$

The function values of adjacent polynomials must be equals at the knots.

$$f_i + b_i h_i + c_i h_i^2 + d_i h_i^3 = f_{i+1} \qquad \text{------------- (1)}$$

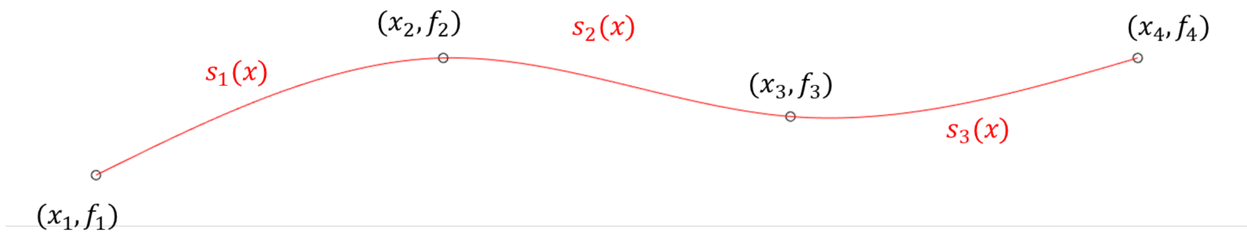
For knot 2,3,4

$$f_1 + b_1 h_1 + c_1 h_1^2 + d_1 h_1^3 = f_2$$

$$f_2 + b_2 h_2 + c_2 h_2^2 + d_2 h_2^3 = f_3$$

$$f_3 + b_3 h_3 + c_3 h_3^2 + d_3 h_3^3 = f_4$$

Condition 3 : The first derivatives at the interior nodes must be equal to ensure smoothness.



$$s_i(x) = f_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$$

$$s'_i(x) = b_i + 2c_i(x - x_i) + 3d_i(x - x_i)^2$$

$$s''_i(x) = 2c_i + 6d_i(x - x_i)$$

$$b_i + 2c_i h_i + 3d_i h_i^2 = b_{i+1} \qquad \text{------------- (2)}$$

For knot 2,3

$$b_1 + 2c_1 h_1 + 3d_1 h_1^2 = b_2$$

$$b_2 + 2c_2 h_2 + 3d_2 h_2^2 = b_3$$

Condition 4 : The second derivatives at the interior nodes must be equal to ensure smoothness.

$$c_i + 3d_i h_i = c_{i+1} \qquad \text{------------- (3)}$$

For knot 2,3

$$c_1 + 3d_1 h_1 = c_2$$

$$c_2 + 3d_2 h_2 = c_3$$

Condition 5 : Assume the second derivative is zero at the first point and the last point

At $x = x_1$, $s_1''(x_1) = 2c_1 + 6d_1(x_1 - x_1) = 0 \rightarrow c_1 = 0$ ------------- (4)

At $x = x_4$, $s_3''(x_4) = 2c_3 + 6d_3(x_4 - x_3) = 2c_3 + 6d_3 h_3 = 0$ ------------- (5)

Now we will derive the formula how to solve for all coefficients.

From (3)

$$c_i + 3d_i h_i = c_{i+1}$$

$$d_i = \frac{c_{i+1} - c_i}{3h_i} \qquad \text{------------- (6)}$$

Substitute (5) in (1)

$$f_i + b_i h_i + c_i h_i^2 + d_i h_i^3 = f_{i+1}$$

$$f_i + b_i h_i + c_i h_i^2 + (\frac{c_{i+1} - c_i}{3h_i})h_i^3 = f_{i+1}$$

$$f_i + b_i h_i + \frac{h_i^2}{3}(2c_i + c_{i+1}) = f_{i+1}$$

$$b_i = \frac{f_{i+1} - f_i}{h_i} - \frac{h_i}{3}(2c_i + c_{i+1}) \qquad \text{------------------- (7)}$$

Or we can reduce the index by 1 to

$$b_{i-1} = \frac{f_i - f_{i-1}}{h_{i-1}} - \frac{h_{i-1}}{3}(2c_{i-1} + c_i) \qquad \text{------------------- (8)}$$

Substitute (6) in (2)

$$b_i + 2c_i h_i + 3d_i h_i^2 = b_{i+1}$$

$$b_i + 2c_i h_i + 3(\frac{c_{i+1} - c_i}{3h_i})h_i^2 = b_{i+1}$$

$$b_{i+1} = b_i + h_i(c_i + c_{i+1})$$

Or we can reduce the index by 1 to

$$b_i = b_{i-1} + h_{i-1}(c_{i-1} + c_i) \qquad \text{------------------- (9)}$$

Substitute (7), (8) in (9)

$$b_i = b_{i-1} + h_{i-1}(c_{i-1} + c_i)$$

$$\frac{f_{i+1} - f_i}{h_i} - \frac{h_i}{3}(2c_i + c_{i+1}) = \frac{f_i - f_{i-1}}{h_{i-1}} - \frac{h_{i-1}}{3}(2c_{i-1} + c_i) + h_{i-1}(c_{i-1} + c_i)$$

$$h_{i-1}c_{i-1} + 2(h_{i-1} + h_i)c_i + h_i c_{i+1} = 3\frac{f_{i+1} - f_i}{h_i} - 3\frac{f_i - f_{i-1}}{h_{i-1}}$$

Let's define

$$f[x_i, x_j] = \frac{f_i - f_j}{x_i - x_j}$$

$$h_{i-1}c_{i-1} + 2(h_{i-1} + h_i)c_i + h_i c_{i+1} = 3(f[x_{i+1}, x_i] - f[x_i, x_{i-1}])$$

Now this last equation depends on c's coefficients.

For interior knots 2,3

$$h_1c_1 + 2(h_1 + h_2)c_2 + h_2c_3 = 3(f[x_3, x_2] - f[x_2, x_1]) \qquad \text{-------------- (10)}$$

$$h_2c_2 + 2(h_2 + h_3)c_3 + h_3c_4 = 3(f[x_4, x_3] - f[x_3, x_2]) \qquad \text{-------------- (11)}$$

From (4)

At $x = x_1$, $s_1''(x_1) = 2c_1 + 6d_1(x_1 - x_1) = 0 \ \rightarrow \ c_1 = 0$ ------------- (12)

At $x = x_4$, $s_3''(x_4) = 2c_3 + 6d_3(x_4 - x_3) = 2c_3 + 6d_3h_3 = 0$

Let's define $c_n = c_{n-1} + 3d_{n-1}h_{n-1} = 0$

$$c_4 = c_3 + 3d_3h_3 = 0 \qquad \text{------------- (13)}$$

Writing (10)-(13)

$$\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{h_1} & \mathbf{2}(\boldsymbol{h_1} + \boldsymbol{h_2}) & \boldsymbol{h_2} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{h_2} & \mathbf{2}(\boldsymbol{h_2} + \boldsymbol{h_3}) & \boldsymbol{h_3} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \boldsymbol{c_1} \\ \boldsymbol{c_2} \\ \boldsymbol{c_3} \\ \boldsymbol{c_4} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{3}(\boldsymbol{f}[\boldsymbol{x_3}, \boldsymbol{x_2}] - \boldsymbol{f}[\boldsymbol{x_2}, \boldsymbol{x_1}]) \\ \mathbf{3}(\boldsymbol{f}[\boldsymbol{x_4}, \boldsymbol{x_3}] - \boldsymbol{f}[\boldsymbol{x_3}, \boldsymbol{x_2}]) \\ \mathbf{0} \end{bmatrix}$$

Once we solve c_1, c_2, c_3, c_4, then we can solve for b's and d's using (6) and (7)

$$b_i = \frac{f_{i+1} - f_i}{h_i} - \frac{h_i}{3}(2c_i + c_{i+1})$$

$$f_i + b_ih_i + c_ih_i^2 + d_ih_i^3 = f_{i+1}$$