LINEAR ALGEBRAIC EQUATIONS AND MATRICES

Textbook (Python) Part III Chapter 8

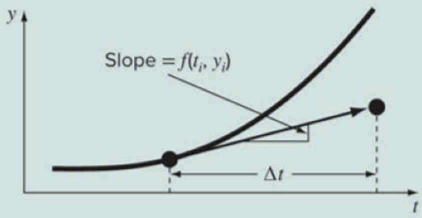
Topics

(Ordinary) differential equations

Given $\frac{dy}{dt} \approx \frac{\Delta y}{\Delta t} = f(t, y)$

solve for y as a function of t

$y_{i+1} = y_i + f(t_i, y_i)\,\Delta t$

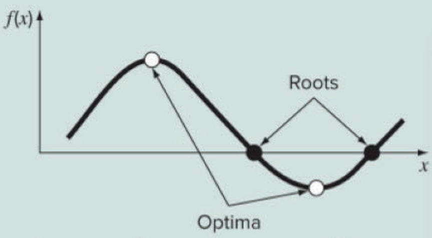


Roots and optimization

Roots: Solve for *x* so that *f*(*x*) = 0

Optimization:

Solve for *x* so that *f′*(*x*) = 0

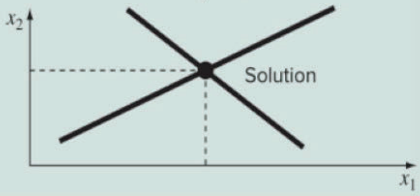


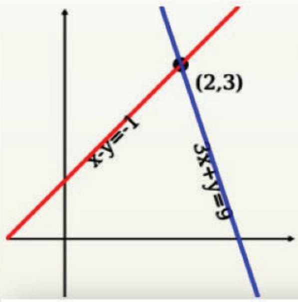
Linear algebraic equations

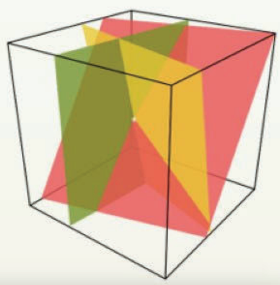
Given *a*'s and *b*'s, solve for *x*'s

$a_{11}x_1 + a_{12}x_2 = b_1$

$a_{21}x_1 + a_{22}x_2 = b_2$







System of Linear Equation

2.0x + 4.0y +6.0z = 18

4.0x + 5.0y +6.0z = 24

3.0x + 1y - 2.0z = 4

Matrix representation

$$A = \begin{bmatrix} 2.0 & 4.0 & 6.0 \\ 4.0 & 5.0 & 6.0 \\ 3.0 & 1.0 & -2.0 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad b = \begin{bmatrix} 18.0 \\ 24.0 \\ 4.0 \end{bmatrix}$$

Matrix notation

- An array of numbers, rectangular in shape, described by its dimension (row × column)

$$
\begin{array}{ccc}
 & \text{Column 3} & \\
 & \downarrow & \\
[A] = & \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & \boxed{a_{23}} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} & \begin{array}{l} \\ \longleftarrow \text{Row 2} \\ \\ \\ \\ \\ \end{array}
\end{array}
$$

Row and column vectors

- A matrix with row dimension $m = 1$ is called **a row vector**
- A matrix with column dimension $n = 1$ is called **a column vector**

$$\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$$

row vector

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

column vector

Square matrices

- Matrices where $m = n$ are called **square matrices**
- A **symmetric matrix** is a square matrix where $a_{ij} = a_{ji}$ for all i's and j's
- A **diagonal matrix** is a square matrix where every element off the main diagonal is zero. A diagonal matrix whose elements on the main diagonal are all equal to 1 is called an **identity matrix**. [A][I] = [I][A] = [A]

$$A = \begin{bmatrix} 1 & 2 & 7 \\ 3 & 6 & 4 \\ 8 & 9 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 1 & 2 \\ 1 & 3 & 7 \\ 2 & 7 & 8 \end{bmatrix} \quad D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 8 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

square **symmetric** **diagonal** **identity**

Matrix equality

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \qquad B = [b_{ij}] = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ b_{21} & \cdots & b_{2n} \\ \vdots & & \vdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix}$$

- Two m**-by-**n matrices are **equal** if and only if $\boldsymbol{a_{ij} = b_{ij}}$ for all i's and j's

$$\begin{bmatrix} 2x + y & 4x \\ 5x - 7 & 4x \end{bmatrix} = \begin{bmatrix} 7 & 7y - 13 \\ y & x + 6 \end{bmatrix} \Rightarrow x + y =$$

Matrix addition/subtraction & scalar multiplication

- Two matrices can be added or subtracted if and only if they are the **same size** m-by-n

$$A + B = [a_{ij} + b_{ij}] = \begin{bmatrix} a_{11} + b_{11} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & \dots & a_{2n} + b_{2n} \\ \vdots & & \vdots \\ a_{m1} + b_{m1} & \dots & a_{mn} + b_{mn} \end{bmatrix}$$

- Scalar multiplication: **multiplying every element** of A by c

$$c[A] = \begin{bmatrix} ca_{11} & \dots & ca_{1n} \\ ca_{21} & \dots & ca_{2n} \\ \vdots & & \vdots \\ ca_{m1} & \dots & ca_{mn} \end{bmatrix}$$

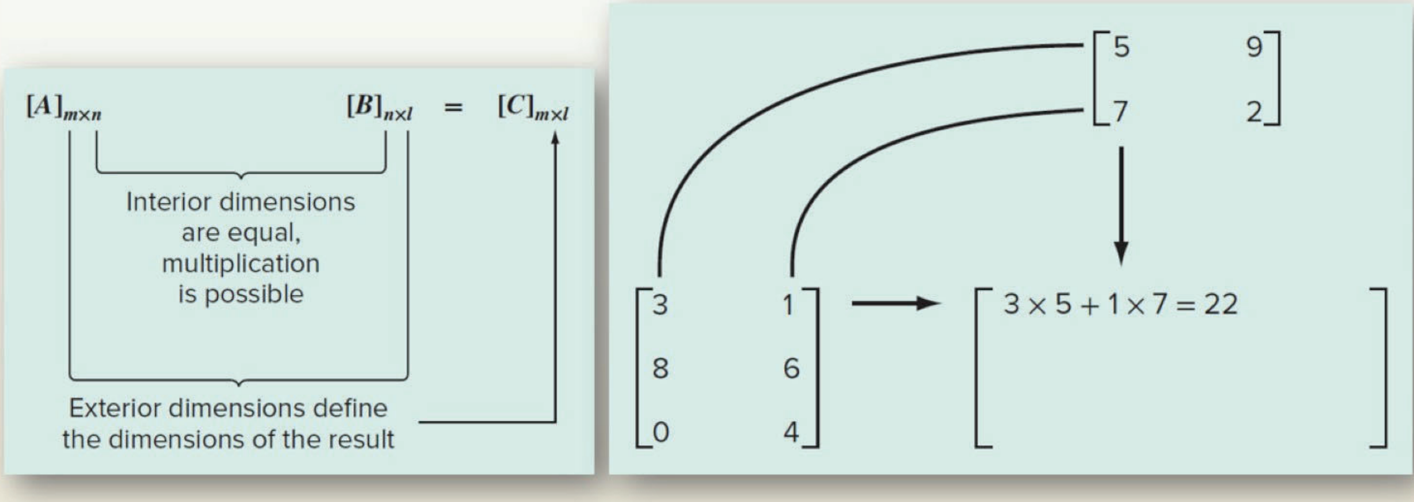
Matrix transpose

- Transpose of a matrix involves **transforming the rows into columns** and **its columns into its rows**. If A is m-by-n, then A^T is n-by-m.

$$[A]^T = \begin{bmatrix} a_{11} & \dots & a_{m1} \\ a_{12} & \dots & a_{m2} \\ \vdots & & \vdots \\ a_{1n} & \dots & a_{mn} \end{bmatrix}$$

$$-2\begin{bmatrix} -4 & 3 \\ 1 & -5 \end{bmatrix} + 4\begin{bmatrix} 3 & -2 \\ 5 & 9 \end{bmatrix}^T =$$

Matrix multiplication



Properties of matrix operations

- [A] + [B] = [B] + [A]
- ([A] + [B]) + [C] = [A] + ([B] + [C])
- $(AB)^T = B^T A^T$
- ([A] [B]) [C] = [A] ([B] [C])
- ([A] + [B]) [C] = [A][C] + [B] [C]
- In general, AB ≠ BA

$$\left(\begin{bmatrix} 8 & 1 & 2 \\ -5 & 6 & 7 \end{bmatrix}\begin{bmatrix} -5 & 1 \\ 0 & 2 \\ -11 & 7 \end{bmatrix}\right)^T \left(\begin{bmatrix} -0.5 \\ 0.25 \end{bmatrix}\right) =$$

Matrix inverse

$$[A][A]^{-1} = [A]^{-1}[A] = [I]$$

- The inverse of a 2-by-2 matrix is

$$[A]^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

$$\begin{bmatrix} 8 & 5 \\ 6 & 4 \end{bmatrix}^{-1} =$$

Determinant of a 2×2 matrix

- **Determinant** of a square matrix A is a scalar representation of a matrix

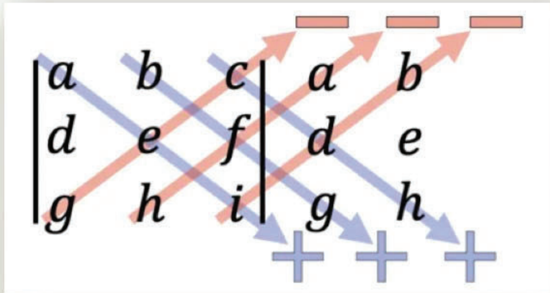
$$det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{21} \cdot a_{12}$$

$$det(A) = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} =$$

Determinant of a 3×3 matrix

- Compute $det(A) = \begin{vmatrix} 5 & 0 & 2 \\ 1 & 3 & 4 \\ -1 & 1 & 0 \end{vmatrix}$



Determinant of a matrix

- **Determinant** of a square matrix A is a scalar representation of a matrix

$$det(A) = \begin{vmatrix} \boxed{a \quad b \quad c} \\ d \quad e \quad f \\ g \quad h \quad i \end{vmatrix} = a\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} - b\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + c\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$= (-1)^{(1+1)}a\begin{vmatrix} e & f \\ h & i \end{vmatrix} + (-1)^{(1+2)}b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + (-1)^{(1+3)}\mathrm{c}\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ei - hf) - b(di - gf) + c(dh - ge)$$

WORKED EXAMPLES

$$det(A) = \begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 1 \end{vmatrix} =$$

Determinant of a matrix

- In general, the determinant of a matrix A is:

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix}$$

$$= (-1)^{1+1} a_{11} \det(A - A[1,:] - A[:,1]) + (-1)^{1+2} a_{12} \det(A - A[1,:] - A[:,2]) + \cdots + (-1)^{1+n} a_{1n} \det(A - A[1,:] - A[:,n])$$

$$= \sum_{j=1}^{n} (-1)^{i+j} \det(\underbrace{A - A[i,:] - A[:,j]}_{\text{A without row } i \text{ and column } j})$$

PRACTICE PROBLEMS

$$det(A) = \begin{vmatrix} 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 4 \\ 5 & 3 & 4 & 1 \\ 4 & 2 & 3 & 5 \end{vmatrix} =$$

Determinant of a matrix will be zero if ...

- it has a row or a column with all elements equal to 0

$$\begin{vmatrix} a & b & c \\ 0 & 0 & 0 \\ g & h & i \end{vmatrix} = 0 \qquad \begin{vmatrix} a & b & 0 \\ d & e & 0 \\ g & h & 0 \end{vmatrix} = 0$$

- it has two equal rows/columns or two proportional rows/columns

$$\begin{vmatrix} a & b & c \\ ka & kb & kc \\ g & h & i \end{vmatrix} = 0 \qquad \begin{vmatrix} a & b & ka \\ d & e & kd \\ g & h & kg \end{vmatrix} = 0$$

- its row/column is the sum or the difference of other row/column

$$\begin{vmatrix} a & b & c \\ d & e & f \\ a+d & b+e & e+f \end{vmatrix} = 0 \qquad \begin{vmatrix} a & b & a+b \\ d & e & d+e \\ f & g & f+g \end{vmatrix} = 0$$

WORKED EXAMPLES

- Use properties of determinant to find $det(A) = \begin{vmatrix} 5^2 & 5^3 & 5^4 \\ 5^3 & 5^4 & 5^5 \\ 5^4 & 5^5 & 5^6 \end{vmatrix}$

Determinant of a matrix remains the same ...

- we can individually factor integers out of a row or a column

$$\begin{vmatrix} ka & kb & kc \\ d & e & f \\ g & h & i \end{vmatrix} = k \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \qquad \begin{vmatrix} a & kb & c \\ d & ke & f \\ g & kh & i \end{vmatrix} = k \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

- we can add or subtract multiples of rows or columns to other rows

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} ma+nd & mb+ne & mc+nf \\ d & e & f \\ g & h & i \end{vmatrix} \qquad \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} ma+nb & b & c \\ md+ne & e & f \\ mg+nh & h & i \end{vmatrix}$$

PRACTICE PROBLEMS

- Use properties of determinant to find $det(A) = \begin{vmatrix} 115 & 106 & 97 \\ 10 & 1 & -8 \\ 106 & 97 & 88 \end{vmatrix}$
- Hint: the middle column is a linear combination of the other columns.

Properties of two determinants

- If A and B are square matrices of the same size, det(AB) = det(A) det(B)
- The determinant of transpose: det(A) = det(A^T)
- If A is n invertible matrix, then det(A^{-1}) = 1/det(A)
- The square matrix A is invertible if and only if det(A) $\neq$ 0

WORKED EXAMPLES

- Let A and B be two invertible matrices of size 3-by-3. If $\det(ABA^T) = 8$ and $\det(AB^{-1}) = 8$, then what is $\det(BA^{-1}B^T)$?

Minor, cofactor, and adjoint

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad M_{m\times n} = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{m1} & m_{m2} & \cdots & m_{mn} \end{bmatrix}$$

- Minor of a matrix: $m_{ij} = \det(A \text{ without row } i \text{ column } j)$
- Cofactor of a matrix: $c_{ij} = (-1)^{i+j} m_{ij}$
- Adjoint of a matrix $= C^T$
- Inverse of a matrix: $A^{-1} = \frac{\text{Adjoint}(A)}{\det(A)}$

WORKED EXAMPLES

- Find the inverse of

$$A = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$


```
A = np.array([[3.6,1.2],[12.9,-9.8]]); A
```
```
array([[ 3.6,  1.2],
       [12.9, -9.8]])
```
```
B = np.array([[3,5],[-8,4]])
```
```
print(A*B); print(A.dot(B)) # watch out!!
(A.transpose())*2
```
```
[[  10.8    6. ]
 [-103.2  -39.2]]
[[   1.2  22.8]
 [117.1  25.3]]
array([[ 7.2,  25.8],
       [ 2.4, -19.6]])
```
```
np.hstack((A,B)) # combine side by side
```
```
matrix([[ 3.6,  1.2,  3. ,  5. ],
        [12.9, -9.8, -8. ,  4. ]])
```
```
np.vstack((A,B[1,:])) #combine vertically
```
```
matrix([[ 3.6,  1.2],
        [12.9, -9.8],
        [-8. ,  4. ]])
```
```
A = np.matrix([[3.6,1.2],[12.9,-9.8]])
A
```
```
matrix([[ 3.6,  1.2],
        [12.9, -9.8]])
```
```
B = np.matrix('3 5 ; -8 4')
```
```
print(A+B) # addition or subtraction
print(A*B) # matrix multiplication
np.transpose(2*A) # transpose
```
```
[[ 6.6  6.2]
 [ 4.9 -5.8]]
[[   1.2  22.8]
 [117.1  25.3]]
matrix([[ 7.2,  25.8],
        [ 2.4, -19.6]])
```
```
np.linalg.det(B) # determinant
```
```
51.999999999999986
```
```
np.linalg.inv(B) # inverse
```
```
matrix([[ 0.07692308, -0.09615385],
        [ 0.15384615,  0.05769231]])
```