recap

Euler's method approximates ODE's

What is a numerical method?

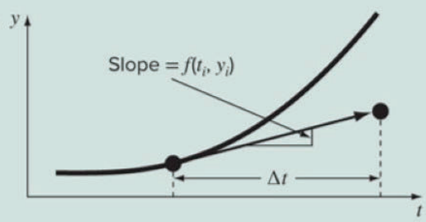
A math tool that calculates an approximation of a solution to a problem

(Ordinary) differential equations

Given $\frac{dy}{dt} \approx \frac{\Delta y}{\Delta t} = f(t, y)$

solve for y as a function of t

$y_{i+1} = y_i + f(t_i, y_i)\,\Delta t$

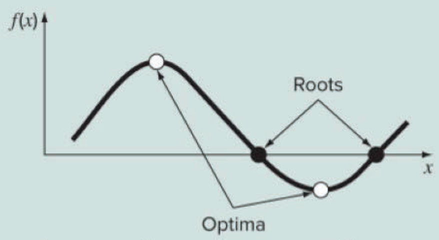


Roots and optimization

Roots: Solve for *x* so that *f*(*x*) = 0

Optimization:

Solve for *x* so that *f′*(*x*) = 0

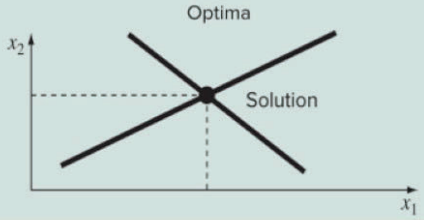


Linear algebraic equations

Given *a*'s and *b*'s, solve for *x*'s

$a_{11}x_1 + a_{12}x_2 = b_1$

$a_{21}x_1 + a_{22}x_2 = b_2$



https://matterofmath.com/tools/math-terms-definitions/#numerical-analysis; https://matterofmath.com/calculus/eulers-method/

Euler's method approximates ODE's

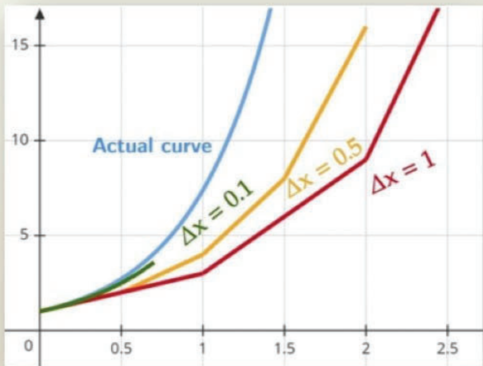
- Given ODE $f'(x) = y'(x) = \frac{dy}{dx}$, initial condition $f(x_0) = y(x_0)$, step size Δx
- Find an approximation of the function f by following: $y_i = y_{i-1} + y'(x_{i-1})\Delta x$

Iteration, i	x_i	y_i	y'_i
	$x_i = x_{i-1} + \Delta x$	$y_i = y_{i-1} + y'(x_{i-1})\Delta x$	$y'_i = \frac{dy}{dx}$
0	x_0	$y_0 = y(x_0)$	$y'(x_0)$
1	$x_1 = x_0 + \Delta x$	$y_1 = y_0 + y'(x_0)\Delta x$	
2			

Euler's method approximates ODE's

- Given ODE $y'(x) = 2y$, initial condition $y(0) = 1$, and step size $\Delta x = 1$
- Question: $y(4) \cong ?$
- Note: to plot the actual curve, an analytical solution is required: $y(x) = e^{2x}$

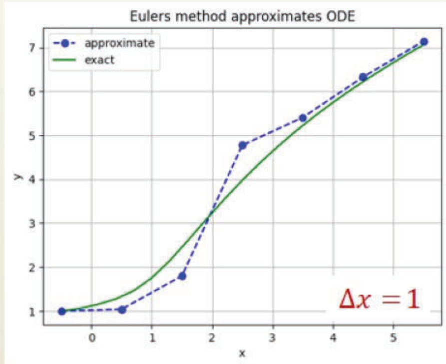
Iteration, i	x_i	y_i	y'_i
	$x_i = x_{i-1} + \Delta x$	$y_i = y_{i-1} + y'(x_{i-1})\Delta x$	$y'_i = \frac{dy}{dx}$
0	0	1	2
1	1	1 + (2)(1) = 3	6
2	2	3 + (6)(1) = 9	18
3	3	9 + (18)(1) = 27	54
4	4	27 + (54)(1) = 81	81



Euler's method approximates ODE's

- Given ODE $y'(x) = e^{x-y} + 4x^3e^{-y},\ y(-0.5) = 1$, using step size $\Delta x = 1$
- Question: $y(5.5) \cong ?$ and what are values of x_i, y_i, y_i' at iterations 1,3,5
- Note: to plot the actual curve, an analytical solution: $y(x) = \ln(e^x + x^4 + 2.049)$

Iteration, i	x_i	y_i	y_i'
	$x_i = x_{i-1} + \Delta x$	$y_i = y_{i-1} + y'(x_{i-1})\Delta x$	$y_i' = \frac{dy}{dx}$
0	-0.5	1	0.03919
1	0.5	1.03919	0.76009
3	2.5	4.77377	0.63095
5	4.5	6.32453	0.81441
6	5.5	7.13894	



Know ...

- what the question is asking, some information may be implicit
  - Given $\frac{dy}{dx} = x - y^2$ and $y(0) = 1$, use Euler's method with two equal steps to estimate $y(0.2)$ → apply $y_i = y_{i-1} + y'(x_{i-1})\Delta x$ two times using $\Delta x = 0.1$
- how to answer the questions, including rounding of numbers
- hand calculations vs. programming
  - some question may be easier to do by hand calculations
  - read/understand the code, know how to extract the information from it
    - example: printing out what's happening at each iterations

Recall from calculus ...

DEFINITION OF DERIVATIVE:

OUTPUT CHANGE

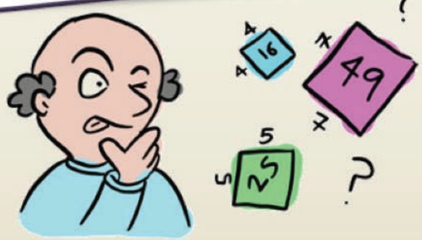
$$f'(x) = \lim_{c \to \emptyset} \frac{f(x+c) - f(x)}{c}$$

INPUT CHANGE

GIVEN AN INPUT CHANGE, THE DERIVATIVE TELLS YOU HOW THE OUTPUT WILL CHANGE.

Derivative Rules

1. Constant Rule: $\frac{d}{dx}(c) = 0$, where c is a constant
2. Power Rule: $\frac{d}{dx}(x^n) = nx^{n-1}$
3. Product Rule: $(fg)' = f'g + fg'$
4. Quotient Rule: $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$
5. Chain Rule: $(f(g(x)))' = f'(g(x))g'(x)$



Trigonometric Functions

$\frac{d}{dx}(\sin x) = \cos x$ $\quad \frac{d}{dx}(\cos x) = -\sin x$ $\quad \frac{d}{dx}(\tan x) = \sec^2 x$

$\frac{d}{dx}(\sec x) = \sec x \tan x$ $\quad \frac{d}{dx}(\csc x) = -\csc x \cot x$ $\quad \frac{d}{dx}(\cot x) = -\csc^2 x$

Exponential & Logarithmic Functions

$\frac{d}{dx}(a^x) = a^x \ln(a)$ $\quad \frac{d}{dx}(e^x) = e^x$

$\frac{d}{dx}(\log_a(x)) = \frac{1}{x\ln(a)}$ $\quad \frac{d}{dx}(\ln(x)) = \frac{1}{x}$

https://robert-dolan.grad.uconn.edu/wp-content/uploads/sites/1419/2016/06/Derivatives-Cheat-Sheet.pdf; https://www.adit.io/posts/2018-02-18-Introduction-To-Calculus-With-Derivatives.html

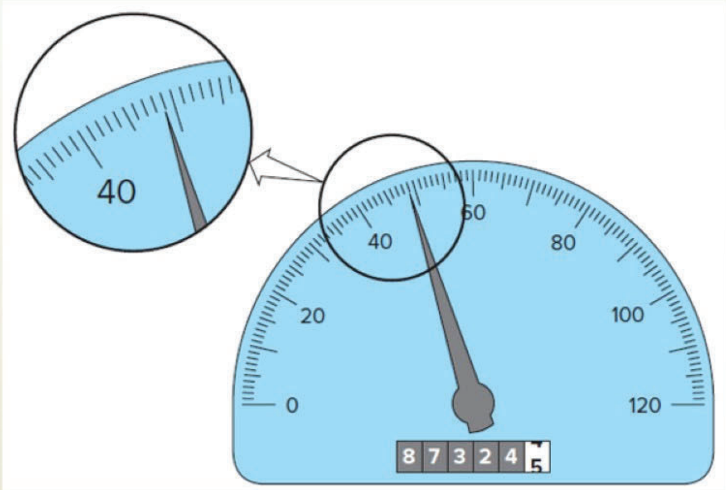
ROUNDOFF AND TRUNCATION ERRORS

Textbook (Python) Part I Chapter 4

Number of significant figures indicates precision

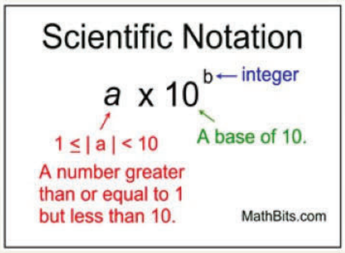
- **Sig figs** → significant digits of a number are those that can be used with confidence → the number of certain digits plus one estimated digit

- How many sig figs in the reading?
- The speedometer –
- The odometer –



How many significant figures?

- 53,800 may have 3, 4, or 5 significant digits
- Use scientific notation!
  - 5.38×10^4
  - 5.380×10^4
  - 5.3800×10^4

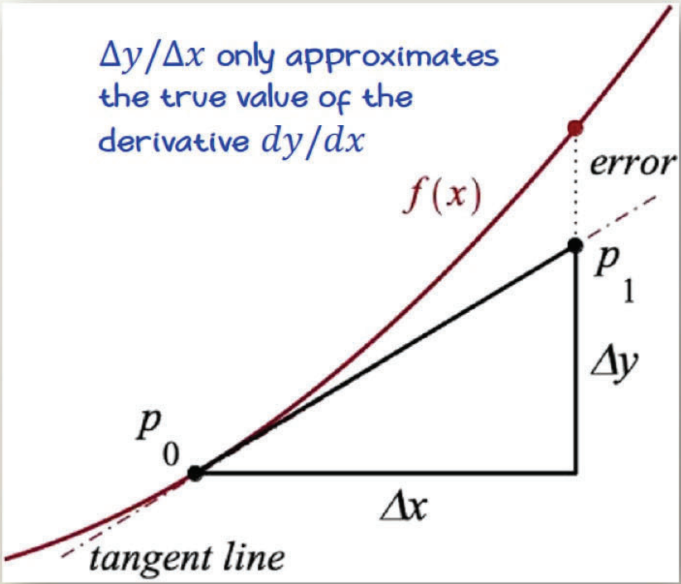


Sig figs will be used in ex, hw, quizzes, and exams. Be sure to do it right.

- Zeros are sometimes used to locate the decimal point not significant figures
  - 0.00001753
  - 0.0001753
  - 0.001753

Two major forms of numerical errors

- **Roundoff error**
  - Due to computer approximation
  - Digital computers cannot represent some numerical quantities exactly
- **Truncation error**
  - Due to math approximation
  - Result from using an approximation in place of an exact math procedure



An example of truncation error

- Estimating $e^{0.5}$ using Maclaurin series expansion

$$e^x \cong \sum_{i=0}^{n} \frac{x^i}{i!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

n	Result
0	
1	
2	
3	
4	
5	

Error definition

- True error, E_t = true value − approximation (+/-)
- True fractional relative error $= \dfrac{\text{true value − approximation}}{\text{true value}}$
- True percent relative error, $\varepsilon_t = \dfrac{\text{true value − approximation}}{\text{true value}} \times 100\%$

- ❑ A 1kg (=1000g) is measured as 990 grams → absolute error = 10g
- ❑ A 100 grams is measured as 90 grams → absolute error = 10g
- ❑ Are the two 10g in the errors the same? Which is more erroneous?

Following Ed8 (pg.61-63), errors et and ea are defined without absolute

An example of truncation error

- Estimating $e^{0.5}$ using Maclaurin series expansion

$$e^x \cong \sum_{i=0}^{n} \frac{x^i}{i!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

n	Result	ε_t (%)
0		
1		
2		
3		
4		
5		

- Calculate the true percent relative errors, answer using 3 sig figs.
- Note that the true value of $e^{0.5}$ is approximated to 15 sig figs as $e^{0.5} \cong 1.64872127070013$

- In real world applications, we usually do not know the true value *(it is known only when we deal with functions that can be solved analytically, i.e. simple systems).*
- $\varepsilon_a = \dfrac{\textit{approximation error}}{\textit{approximation}} \times 100\%$

- Iterative approach, **iterate until a convergence**
- $\varepsilon_a = \dfrac{\textit{current approximation} - \textit{previous approximation}}{\textit{current approximation}} \times 100\%$
- Computations are repeated until stopping criterion is satisfied: $|\varepsilon_a| < \varepsilon_s$
- For results to be correct to at least n sig figs: $\varepsilon_s = (0.5 \times 10^{2-n})\%$

An example of truncation error

- Estimating $e^{0.5}$ using Maclaurin series expansion

$$e^x \cong \sum_{i=0}^{n} \frac{x^i}{i!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

n	Result	ε_t (%)	ε_a (%)
0			
1			
2			
3			
4			
5			

- Set ε_s so that the results are correct to at least 3 sig figs

Recap: when to stop the approximation?

- Estimating $e^{0.5}$ using Maclaurin series expansion

$$e^x \cong \sum_{i=0}^{n} \frac{x^i}{i!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

- When we approximate, the question may specify:

1. **the value of n**, that is, the order until which we should evaluate
   - *Use zero- through fourth-order to approximate ...*
2. **the stopping criterion** in terms of approximated percent relative errors
   - *Add terms one at a time until the error falls below an error criterion corresponding to k sig figs*

The Taylor series

Provides a means to predict a function value at one point in terms of the function value and its derivative at another point

- **Taylor series centered at $x = a$.** Let f be a function and its derivative of all orders are continuous on an interval containing $x = a$. Then, the n^{th} order approximation of the value of the function at x, centered at $x = a$, is given by

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots$$

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(a) \frac{(x-a)^n}{n!}$$

WORKED EXAMPLES

- Approximate the function $f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$ at $x = 1$ using zero- through fourth-order Taylor series centered at $x = 0$

$f(x)$	
$f'(x)$	
$f''(x)$	
$f^{(3)}(x)$	
$f^{(4)}(x)$	

WORKED EXAMPLES

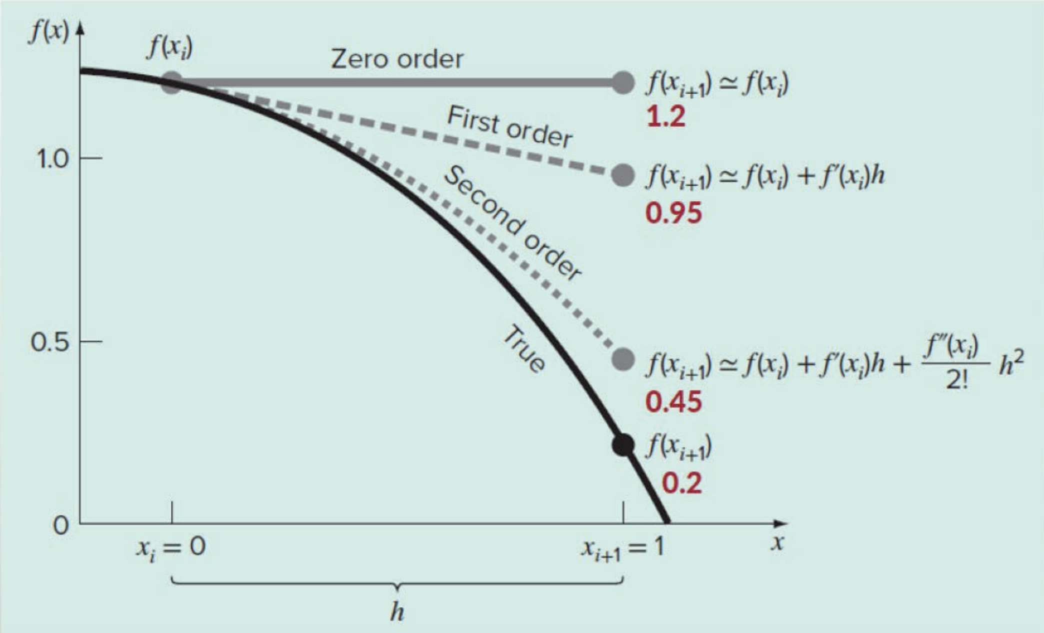
- Approximate the function $f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$ at $x = 1$ using zero- through fourth-order Taylor series centered at $x = 0$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots$$

Order n	Result, $f(1)$
0	
1	
2	
3	
4	

WORKED EXAMPLES

- What happen when terms are added in Taylor series expansion?
- As terms increases, the approximation gets closer to the true func.



PRACTICE PROBLEMS

- Calculate the true errors of the Taylor series approximation of $f(1)$.
  $f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$ about $x = 0$

Order n	Result, $f(1)$	True error, E_t
0		
1		
2		
3		
4		
5		

WORKED EXAMPLES

- Find the Taylor series expansion for e^x centered at $x = a$.

$$\frac{d(e^x)}{dx} = e^x$$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots$$

$f(x) =$

$f'(x) =$

$f''(x) =$

$f^{(3)}(x) =$

Maclaurin series expansion

A special case of the Taylor series expansion, centered at x=0

- To find the Maclaurin series expansion
  - Start with Taylor: $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots$
  - Centered at $a = 0$: $f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f^{(3)}(0)\frac{x^3}{3!} + \cdots$

Maclaurin series expansion

A special case of the Taylor series expansion, centered at x=0

- Find the Maclaurin series expansion for e^x

Taylor series $f(x) = e^a + e^a(x-a) + \frac{e^a}{2!}(x-a)^2 + \frac{e^a}{3!}(x-a)^3 + \cdots$

Setting $a = 0$

Note We used this eqn earlier in the lecture

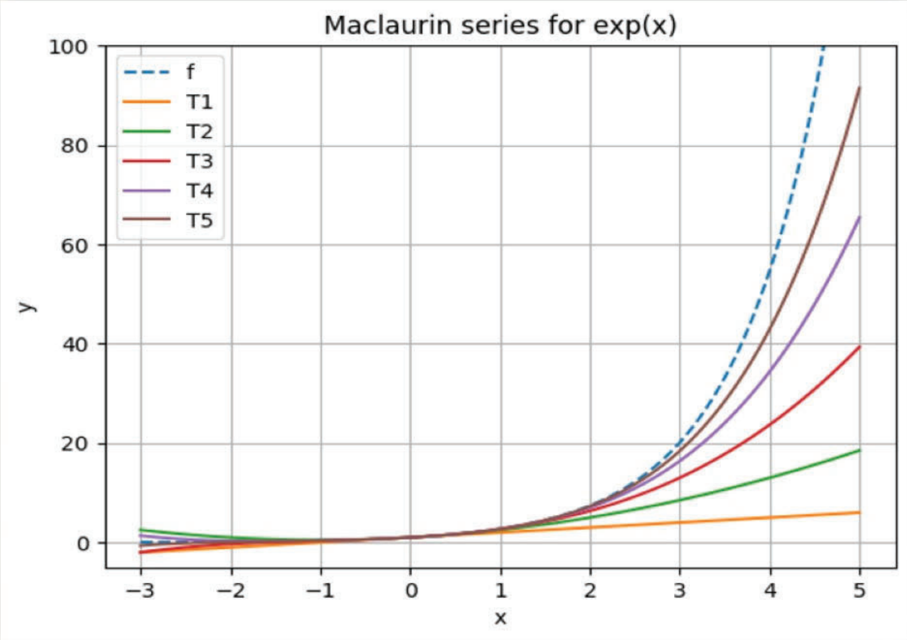
Taylor-Maclaurin

- The function loops until either the stopping criterion is satisfied or max number of iterations is reached
- Each iteration of the loop, add the next term of the series
- *The source code is from the textbook, figure4.2*

```
# function call
itermeth(0.5,1e-6,100)
```

```
import math
def itermeth(x,es=1e-4,maxit=50):
    """
    Maclaurin series expansion of the exponential function
    requires math module
    input:
        x = value at which the series is evaluated
        es = stopping criterion (default = 1e-4)
        maxit = maximum number of iterations (default=50)
    output:
        fx = estimated function value
        ea = approximate relative error (%)
        iter = number of iterations
    """
    # initialization
    iter = 1 ; sol = 1 ; ea = 100
    # iterative calculation
    while True:
        solold = sol
        sol = sol + x**iter / math.factorial(iter)
        iter = iter + 1
        if sol != 0: ea = abs((sol-solold)/sol)*100
        if ea < es or iter == maxit: break
    fx = sol
    return fx,ea,iter
```

Plotting the Maclaurin series for e^x



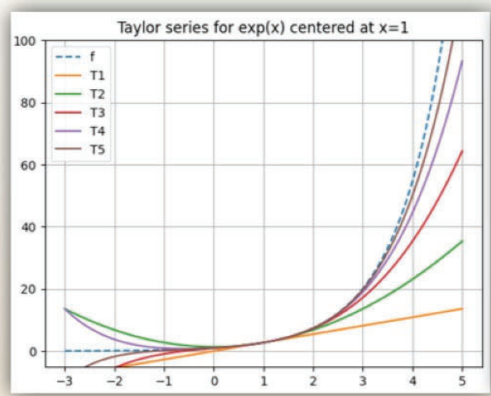
$$f(x) = e^x$$
$$T1(x) = 1 + x$$
$$T2(x) = 1 + x + \frac{x^2}{2}$$
$$T3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

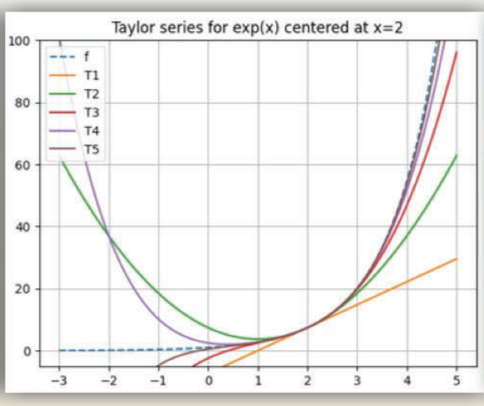
try IT!

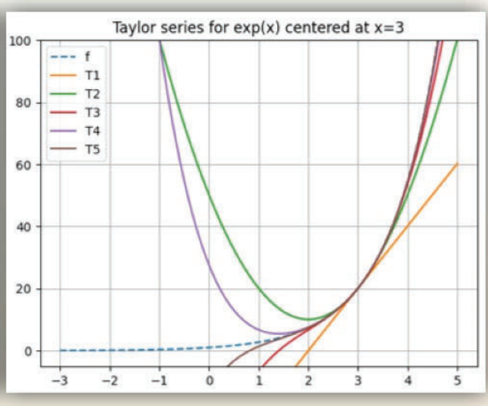
Code Python to plot these curves

Plotting the Taylor series for e^x

Taylor series $f(x) = e^a + e^a(x-a) + \frac{e^a}{2!}(x-a)^2 + \frac{e^a}{3!}(x-a)^3 + \cdots$



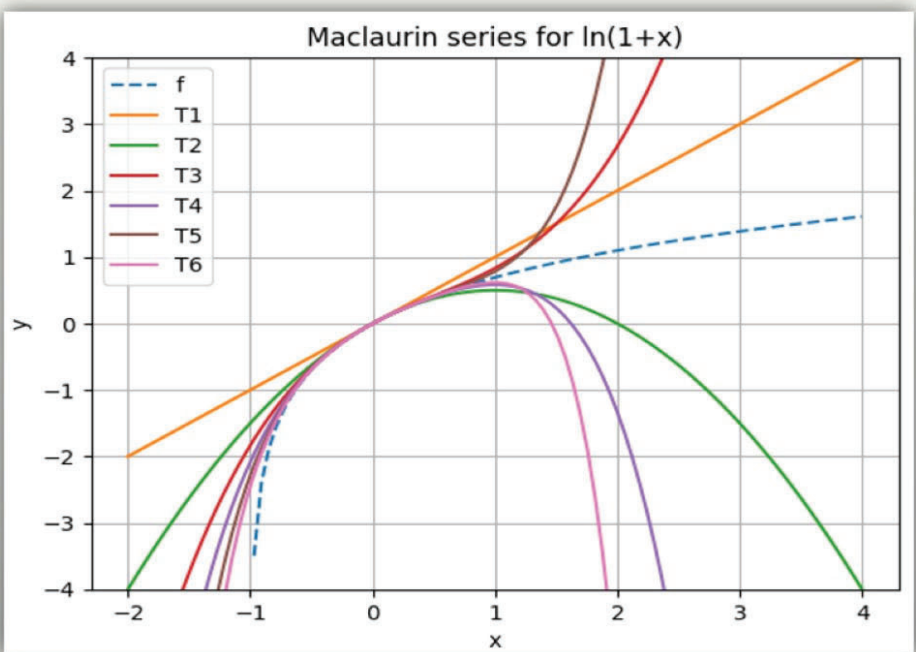




PRACTICE PROBLEMS

- Find the Maclaurin series expansion for $\ln(1+x)$.

WORKED EXAMPLES



- Observe that the Taylor-Maclaurin series converges only when $|x| < 1$
- $|x| < 1$ is the **interval of convergence**
- Finding the interval mathematically involves limits from calculus *(not cover here)*

WORKED EXAMPLES

- Use Maclaurin for $\ln(1+x)$ to estimate $\ln 7$. Add terms one at a time until the error falls below an error criterion corresponding to 3 sig figs.
  - Maclaurin series: $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \dots$, where $|x| < 1$

PRACTICE PROBLEMS

- To evaluate $\ln 25$ using Maclaurin series expansion for $\ln(1+x)$, at what value of x we should we apply the Maclurin series?

- Use Maclaurin for $\ln(1+x)$ to estimate $\ln 25$. Add terms one at a time until the error falls below an error criterion corresponding to 3 sig figs.

PRACTICE PROBLEMS

- Find the coefficient of x^3 in the Taylor series of $\ln(1-x)$ about $x = 0$.

Recap: Taylor-Maclaurin series

- What we have derived with their respective interval of convergence …

- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ $\quad x = (-\infty, \infty)$

- $\ln(1+x) \sum_{n=1}^{\infty} (-1)^{n+1} \left[\frac{x^n}{n}\right] = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ $\quad |x| < 1$

Why Taylor? *If I can tell my computer to compute $e^{0.1}$ and it will give me an exact value, then why approximate it?*

- *Most processors do not have hardware implementations of complex operations like exponential, logarithm, etc. In such cases the programming language may provide a library function for computing those - in other words, someone used a Taylor series or other approximation for you.*
- *Secondly, you may have a function that not even the language supports.*

stackoverflow

https://stackoverflow.com/questions/12678277/in-what-situation-would-a-taylor-series-for-a-polynomial-be-necessary