Homework №11

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87. To prove that a relation is an equivalence relation, we must show that the relation is reflexive, symmetric, and transitive.

I. Showing that $\sim$ is reflexive.

$x \sim x = x + 2x = 3x$ Thus, $\sim$ is reflexive.

II. Showing that $\sim$ is symmetric.

Suppose that $x \sim y$, then $x + 2y = 3k$ where $k \in \mathbb{Z}$. Then, if we solve the equation for x, we get $x = 3k - 2y$. Now, consider $y + 2x$. Let's substitute x with $3k - 2y$. We get, $y + 6k - 4y = 6k - 3y = 3 \times (2k - y)$. Hence, if $x + 2y$ is divisible by 3, then $y + 2x$ is also divisible by 3 and the relation is symmetric.

III. Showing that $\sim$ is transitive.

Suppose that $x \sim y$ and $y \sim z$. Then $x + 2y = 3k$ where $k \in \mathbb{Z}$ and $y + 2z = 3l$ where $l \in \mathbb{Z}$. Consider the relation on the variables x and z. The relation is $x \sim z = x + 2z$. Now, from the first equation, let's substitute x and from the second one, substitute z. We get that $x = 3k - 2y$ and $z = \dfrac{3l - y}{2}$. Finally, we get: $x + 2z = 3k - 2y + 2 \times \dfrac{3l - y}{2} = 3k - 2y + 3l - y = 3k + 3l - 3y = 3 \times (k + l - 1)$ and $3 \times (k + l - 1)$ is clearly a multiple of 3. Hence, we got that $\sim$ is transitive.

Now, we proved that the relation $\sim$ is reflexive, symmetric, and transitive and thus, the relation $\sim$ is the equivalence relation. $\square$

88. (a) $\Xi(S)$ is a relation on $\mathcal{P}(S)$ such that it takes the powerset of S

(b) It is not symmetric. Let $A =$

(c)

(d)

Bookwork

4.2

2. $\in$, $\notin$, $\subset$, $=$, $\mathcal{P}$

4. (a) It is not. Consider tuples (a, b) and (b, a). For the relation R to be transitive, since $(a, b) \in R$ and $(b, a) \in R$, it must be the case that $(a, a) \in R$. However, $(a, a) \notin R$. Thus, the relation is not transitive. $\square$

   (b) i. It is. Suppose that $x - y = q_1$ and $y - z = q_2$ where $q_1, q_2 \in \mathbb{Q}$. Let's then sum those two equations up, and we get $x - y + y - z = q_1 + q_2$ and finally $x - z = q_1 + q_2$. Hence, we got that $x - z$ is a sum of two rational numbers and thus is rational itself. Therefore, the relation R is transitive. $\square$

   ii. It is not. Consider $x = \sqrt{2}$, $y = 1$, and $z = \sqrt{2} + 1$. Then $x - y = \sqrt{2} - 1$ thus is irrational and $y - z = 1 - (\sqrt{2} + 1) = -\sqrt{2}$ hence, is also irrational. However, $x - z = \sqrt{2} - (\sqrt{2} + 1) = \sqrt{2} - \sqrt{2} - 1 = -1$ which is rational. Therefore, the relation R is not transitive. $\square$

   iii. It is not. Consider $x = 1$, $y = 2$, and $z = 4$. Then $|x - y| = 1$ and $|y - z| = 2$. However, $|x - z| = 3 > 2$. Hence, the relation R is not transitive. $\square$

12. (a) It is reflexive. Suppose that $x \in R \cap S$. Then $x \in R$ and $x \in S$. Since R and S are reflexive, $(x, x) \in R$ and $(x, x) \in S$. Therefore, $(x, x) \in R \cap S$. $\square$

   (b) It is reflexive. Suppose that $x \in R \cup S$. Then, without a loss of generality, let $x \in S$. Now, since S is reflexive, $(x, x) \in S$ and thus, $(x, x) \in R \cup S$. $\square$

   (e) It is transitive. Suppose that $(x, y), (y, z) \in R \cap S$. Then $(x, y), (y, z) \in R$ and $(x, y), (y, z) \in S$. Since R, S are transitive, $(x, z) \in R$ and $(x, z) \in S$. Finally, $(x, z) \in R \cap S$ and $R \cap S$ is transitive. $\square$

   (f) It is transitive. Suppose that $(x, y), (y, z) \in R \cup S$. Then, without a loss of generality, $(x, y), (y, z) \in R$. Since R is transitive, $(x, z) \in R$ and $(x, z) \in R \cup S$. Therefore, $R \cup S$ is transitive. $\square$

4.4

1. (a) Yes, it is an equivalence relation since it satisfies all the criteria: reflexive, transitive, symmetric.

   1. It is reflexive, since $(a, a),\ (b, b),\ (c, c) \in R$.

   2. It is transitive. $(a, a), (a, c) \in R$ and $(a, c) \in R$; $(c, a), (a, a) \in R$ and $(c, a) \in R$; $(c, c), (c, a) \in R$ and $(c, a) \in R$; $(a, c), (c, c) \in R$ and $(a, c) \in R$.

   3. It is symmetric. $(a, a) \in R$ and $(a, a) \in R$; $(b, b) \in R$ and $(b, b) \in R$; $(c, c) \in R$ and $(c, c) \in R$; $(a, c) \in R$ and $(c, a) \in R$; $(c, a) \in R$ and $(a, c) \in R$.

   (b) No, it is not since $(b, a), (a, c) \in R$ but $(b, c) \notin R$ (it is not transitive).

3. It is not. For a relation to be the equivalence relation, it must be reflexive, transitive, symmetric. It is not transitive since if we have three lines x, y, z in the euclidean space and if $x \perp y$ and $y \perp z$, then $y \not\perp z$ (because $y \parallel z$). As a side note, it is not reflexive either since the line cannot be perpendicular to itself.

9 (a) Equivalence relation R such that xRy if $x, y \leq 0$ or $x, y > 0$.

   (b) Equivalence relation R such that $xRy = 0$ if $x, y < 0$, $x, y = 0$ or $x, y > 0$.

13. To show that $\equiv_2$ is an equivalence relation, we must show that it is reflexive, symmetric, and transitive.

   1. It is reflexive because if $(a, b) \equiv_2 (a, b)$, then $a - a = 0$ is even and $b - b = 0$ is also even.
   2. It is symmetric because if $(a, b) \equiv_2 (c, d)$, then $a - c = 2k$ is even and $b - d = 2l$ where $k, l \in \mathbb{Z}$. Hence, $(c, d) \equiv_2 (a, b)$ because $c - a = 2 \times (-k)$ and $d - b = 2 \times (-l)$
   3. It is transitive because if $(a, b) \equiv_2 (c, d)$ and $(c, d) \equiv_2 (e, f)$, it means that $a - c = 2k,\ b - d = 2l,\ c - e = 2m$ and $d - f = 2n$ where $k, l, m, n \in \mathbb{Z}$. Then, we

have that $a - e = (a - c) + (c - e) = 2k + 2m = 2 \times (k + m)$. On the other hand, $b - f = (b - d) + (d - f) = 2l + 2n = 2 \times (l + n)$. Hence, we got that if $(a, b) \equiv_2 (c, d)$ and $(c, d) \equiv_2 (e, f)$, then $(a, b) \equiv_2 (e, f)$ and therefore, the relation is transitive.

Thus, we have now proven that the relation $\equiv_2$ is reflexive, symmetric, and transitive thus, $\equiv_2$ is an equivalence relation.

A partition of $\equiv_2$ is $\{(a, b), (c, d) \in \mathbb{Z} \times \mathbb{Z} \mid (a, c \bmod 2 = 0 \text{ or } a, c \bmod 2 = 1) \text{ and } (b, d \bmod 2 = 0 \text{ or } b, d \bmod 2 = 1)\}$.

23. $\{(1, 2), (2, 1), (3, 3), (4, 5), (5, 4)\}$

30. To prove that the relation is the equivalence relation, we must prove that it is reflexive, symmetric, and transitive.

    It is reflexive because if $f \sim f$, then $f' = f'$ which is true since given the particular value, the function and then its derivative is the same (the function cannot have more than one output for a single input).

    It is symmetric because if $f \sim g$, it means that $f' = g'$ and since we are dealing with values, we know that if $x = y$, then $y = x$ and the same applied for the derivatives here. Hence, if $f' = g'$, then $g' = f'$. Finally, we got that if $f \sim g$, then $g \sim f$ and the function is symmetric.

    It is transitive because if $f' \sim g'$ and $g' \sim h'$, then $f' = g'$ and $g' = h'$ and thus $f' = g' = h'$ and $f' = h'$. Hence, we got that if $f' \sim g'$ and $g' \sim h'$, then $f' \sim h'$ and the relation is transitive.

    Thus, we have proven that the relation is reflexive, symmetric, and transitive and hence, is the equivalence relation.

31. (a) To show that the relation is the equivalence relation, we must prove that it is reflexive, symmetric, and transitive.

        It is reflexive because if $A_i \sim A_i$, then $|A_i| = |A_i|$ and obviously, $|A_i| = |A_i|$ because the size of the set is constant. Hence, we have shown that the relation is reflexive.

        It is symmetric because if $A_1 \sim A_2$, it means that $|A_1| = |A_2|$ and thus

$|A_2| = |A_1|$. Hence, it is symmetric.

It is transitive because if $A_1 \sim A_2$ and $A_2 \sim A_3$, it means that $|A_1| = |A_2|$ and $|A_2| = |A_3|$. Therefore, $|A_1| = |A_2| = |A_3|$ and we get that $|A_1| = |A_3|$. Thus, we have proven that if $A_1 \sim A_2$ and $A_2 \sim A_3$, then $A_1 \sim A_3$ and the relation is transitive.

At this point, we have shown that the relation is all three: reflexive, symmetric, and transitive and thus, is the equivalence relation.

(b)