Homework №4

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Section 3.2

23. Prove that if x and y are integers and $xy - 1$ is even then x and y are odd.

    Suppose, for the sake of contradiction, that x is even or y is even. Then we know that the multiplication xy is even (since we assumed the statement x is even or y is even is right). By the definition of an even number, $xy = 2k$ where $k \in \mathbb{Z}$. Finally, $xy - 1 = 2k - 1$ where $k \in \mathbb{Z}$ which now appears to be the definition of the odd number which contradicts the fact that $xy - 1$ is even.

    Q.E.D.

24. Prove that if x and y are real numbers whose mean is m then at least one of x and y is $\geq m$.

    Suppose, for the sake of contradiction, that x and y are both $< m$. Then by adding the inequalities, we get:

    $$x + y < 2m$$

    And finally,

    $$\frac{x + y}{2} < m$$

    which contradicts the initial statement that the mean of x and y is m.

    Q.E.D.

25. Suppose S is a set of 250 distinct real numbers whose mean is 4. Must there exists $x \in S$ such that $x > 4$? Be sure to prove your answer.

    Yes. Let's prove it!
    Suppose, for the sake of contradiction, that all elements of S are ≤ 4. Then the sum of all the elements will be less ≤ 1000 with equality happening only when all the members of the set are equal to 4 which contradict the initial statement that S is a set of 250 distinct elements. Thus, all elements of S must be strictly < 4. This now means that their sum is less than 1000 which, once again, contradicts the initial statement that the mean of all the elements of S is $250 \times 4 = 1000$.

Q.E.D.

26. Suppose $a, b, c \in \mathbb{Z}$ and $a^2 + b^2 = c^2$. Prove that at least one of a and b is even.

    Suppose, for the sake of contradiction, that both a and b are odd. Then, we can write $a = 2k - 1$ and $b = 2l - 1$ where $k, l \in \mathbb{Z}$. Then, we have:

    $$a^2 + b^2 = 4k^2 - 4k + 1 + 4l^2 - 4l + 1 = 4k^2 + 4l^2 - 4l - 4k + 2 =$$

    $$= 2 \times (2k^2 + 2l^2 - 2l - 2k + 1)$$

    Now, it's easy to see that $a^2 + b^2$ is the multiplication of an even and odd integers (2 is even and $(2k^2 + 2l^2 - 2l - 2k + 1)$ is odd). Finally, we conclude that 2 is only once in the number that is supposed to be a perfect square which means that $a^2 + b^2$ is not a perfect square.

    Q.E.D.

27. Prove that if $x, y \in \mathbb{R}^+$,then $x + y \geq 2\sqrt{xy}$.

    Suppose, for the sake of contradiction, that $x + y < 2\sqrt{xy}$. Then, since $x, y \in \mathbb{R}^+$, we have:

    $$\begin{aligned} x + y &< 2\sqrt{xy} && (1) \\ x^2 + y^2 + 2xy &< 4xy && (2) \\ x^2 + y^2 + 2xy - 4xy &< 0 && (3) \\ x^2 + y^2 - 2xy &< 0 && (4) \\ (x - y)^2 &< 0 && (5) \end{aligned}$$

    Thus, we got that $(x - y)^2 < 0$ which is false since square of a number is always ≥ 0.

    Q.E.D.

28. Prove that if n is an integer, there exist three consecutive integers that sum to n if and only if n is a multiple of 3.

    Let's first prove that if n is not a multiple of 3, one cannot find three consecutive integers with the property that they sum to n.

    (a) Suppose, for the sake of contradiction, that n is not a multiple of 3. Then let's define three consecutive integers, $m, m + 1$ and $m + 2$, where $m \in \mathbb{Z}$. Then we have:

    $$m + m + 1 + m + 2 = 3m + 3 = 3 \times (m + 1)$$

    Thus, we got that the sum of three consecutive integers is a multiple of three which contradicts the statement that n is not a multiple of 3.

    Now, lets prove the second half of the problem. Let's show that if three consecutive integers sum to n, then n is a multiple of 3.

(b) Let $m, m+1, m+2$ where $m \in \mathbb{Z}$ be three consecutive integers. We have:

$$n = m + m + 1 + m + 2 = 3m + 3 = 3 \times (m + 1)$$

Thus, we got that n is a multiple of 3 which proves the iff.

Q.E.D.

29. A subset S of $\mathbb{R}$ has the property that for all $x \in \mathbb{R}$ there exists $y \in S$ such that $|x - y| < 1$. Prove that S is infinite.

Suppose, for the sake of contradiction, that S is finite. Inequality, $|x-y| < 1$ can be transformed into the following system:

$$\begin{cases} x - y < 1 \\ x - y > -1 \end{cases}$$

And from the system above, we get the following system:

$$\begin{cases} y > x - 1 \\ y < x + 1 \end{cases}$$

Hence, we know that y is in the open interval $(x-1, x+1)$. Now, since we also know that $x \in \mathbb{R}$, interval $(x-1, x+1)$ has infinitely many elements in it which contradicts our assumption that S is finite.

Q.E.D.