Real Analysis Exams

Exam №2

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1. (a) Placeholder

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   (e) Placeholder

2. Let us first prove that $\frac{1}{2}\mathbb{C}+\frac{1}{2}\mathbb{C}=[0,1]$. Recall that the Cantor set $\mathbb{C}$ is the set of all numbers in $[0,1]$ that in the **ternary system** can be represented as the sequence of 0s and 2s only. Then $\frac{1}{2}\mathbb{C}$ must only contain 0s and 1s. Now, let $r\in[0,1]$. If we show that $\exists x, y \in \frac{1}{2}\mathbb{C}$ s.t $x+y\in[0,1]$, then we have effectively shown that $\frac{1}{2}\mathbb{C}+\frac{1}{2}\mathbb{C}=[0,1]$. Let us construct x and y in the following manner:

   * Let x have 0s in the same places where it is in r and let x have 1s when the corresponding digit in r is either 1 or 2.

   * Let y have 0s in the same places where r has $0s$ or 1s. Let y have 1s when the corresponding digit in r is 2.

Hence, we split all 2s in r in a way that half goes to x and half goes to y, and all 1s of r were given to x. Thus, $x + y = r$. For instance, if $r = 0.120120\dots$, then $x = 0.110110\dots$ and $y = 0.010010\dots$. It follows that $x + y = 0.120120\dots = r$. Now, since we have already shown that $\frac{1}{2}\mathbb{C} + \frac{1}{2}\mathbb{C} = [0, 1]$, we can just multiply both sides of the equation by 2 and we get $\mathbb{C} + \mathbb{C} = [0, 2]$.

□

3. (a) Placeholder

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4. (a) According to **Definition 4.2.1 (Functional Limit)**, we have to show that $\forall \epsilon > 0, \exists \delta > 0$ s.t. $0 < |x - 3| < \delta \implies |x^2 - 5x + 4 - (-2)| < \epsilon$. Let $\epsilon > 0$ be given. Let $\delta = -0.5 + \sqrt{0.25 + \frac{\epsilon}{2}}$ ($\delta > 0$ since $\sqrt{0.25 + \frac{\epsilon}{2}} > 0.5$). Then suppose that $|x - 3| = |x - 3| < -0.5 + \sqrt{0.25 + \frac{\epsilon}{2}}$ Notice that:

$$\begin{aligned}
|x^2 - 5x + 4 - (-2)| &= \left|x^2 - 5x + 6\right| \\
&= \left|(x-2)(x-3)\right| \\
&= \left|-0.5 + \sqrt{0.25 + \frac{\epsilon}{2}} + 1\right| \times \left|-0.5 + \sqrt{0.25 + \frac{\epsilon}{2}}\right| \\
&< \left|\sqrt{0.25 + \frac{\epsilon}{2}} + 0.5\right| \times \left|\sqrt{0.25 + \frac{\epsilon}{2}} - 0.5\right| \\
&= \left|0.25 + \frac{\epsilon}{2} - 0.25\right| \\
&= \left|\frac{\epsilon}{2}\right| = \frac{\epsilon}{2} < \epsilon
\end{aligned}$$

   Hence, we showed that $\forall \epsilon > 0, \exists \delta = -0.5 + \sqrt{0.25 + \frac{\epsilon}{2}}$ s.t. $0 < |x - 3| < \delta \implies |x^2 - 5x + 4 - (-2)| < \epsilon$.

   □

   (b) Per **Exercise 4.2.9 (b)** that I have completed as a part of the assignment, we can say $\lim_{x\to\infty} f(x) = L$ if $\forall \epsilon > 0, \exists M > 0$ s.t. if $x > M$ we have $|f(x) - L| < \epsilon$. Let us now show that $\lim_{x\to\infty} \dfrac{2x}{x+4} = 2$. Let $\epsilon > 0$ be given and let $M = \dfrac{8}{\epsilon}$. Then if $x > M$, we have $x > \frac{8}{\epsilon}$. We have $\frac{2x}{x+4} = |\frac{2\frac{8}{\epsilon}}{\frac{8}{\epsilon}+4} - 2| = \frac{8}{\frac{8}{\epsilon}+4} = \frac{2\epsilon}{\epsilon+2} = \epsilon - \frac{4}{\epsilon+2} < \epsilon$.

Hence, $\lim_{x\to\infty} \frac{2x}{x+4} = 2$.

$\square$

5. We need to prove that $\forall c \in [0,\infty)$ and $\forall \epsilon > 0, \exists \delta > 0$ s.t. whenever $|x-c| < \delta$ (with $x \in [0,\infty)$), it follows that $|\sqrt[4]{x} - \sqrt[4]{c}| < \epsilon$. Let $\epsilon > 0$ be given. Now, let us consider the following two cases:

   (1) $c = 0$

   If $c = 0$, let $\delta = \epsilon^4$. Then $|x-c| = |x-0| = |x| < \epsilon^4$. Now, $|\sqrt[4]{x} - \sqrt[4]{0}| = |\sqrt[4]{x}| < \epsilon$ is true as if we raise both sides of the inequality to the power of four, we get $|x| < \epsilon^4$ which is true. Hence, we have that $|x-c| < \delta$ implies $|\sqrt[4]{x} - \sqrt[4]{c}| < \epsilon$.

   $\square$

   (2) $c > 0$

   If $c > 0$, let $\delta = \epsilon\sqrt[4]{c}$. Then $|x-c| < \epsilon\sqrt[4]{c}$. Consider $|\sqrt[4]{x} - \sqrt[4]{c}|$. Now, notice that:

$$\begin{aligned}
|\sqrt[4]{x} - \sqrt[4]{c}| &= |\sqrt{x} - \sqrt{c} \times \frac{1}{\sqrt[4]{x} + \sqrt[4]{c}}| \\
&= |\sqrt{x} - \sqrt{c}| \times \frac{1}{\sqrt[4]{x} + \sqrt[4]{c}} \\
&= |x-c| \times \frac{1}{(\sqrt[4]{x} + \sqrt[4]{c})(\sqrt{x} + \sqrt{c})} \\
&< \frac{|x-c|}{\sqrt[4]{c^3}} \\
&\leq \frac{|x-c|}{\sqrt[4]{c}} \\
&< \frac{\epsilon\sqrt[4]{c}}{\sqrt[4]{c}} = \epsilon
\end{aligned}$$

   Hence, we have that $|x-c| < \delta$ implies $|\sqrt[4]{x} - \sqrt[4]{c}| < \epsilon$.

   $\square$

   Thus, we have now shown that $\forall c \in [0,\infty)$ and $\forall \epsilon > 0, \exists \delta > 0$ s.t. whenever $|x-c| < \delta$ (with $x \in [0,\infty)$), it follows that $|\sqrt[4]{x} - \sqrt[4]{c}| < \epsilon$.

   $\square$

6. Note that function $f : \mathbb{R} \to \mathbb{R}$ would not be well-defined if repeating 9s were allowed. If repeating 9s are allowed, then the decimal expansion of the number is not unique since $1 = 0.9999...$ and the function $f : \mathbb{R} \to \mathbb{R}$ is not well-defined. Hence, we do not allow for repeating 9s.

   $f : \mathbb{R} \to \mathbb{R}$ is continuous PLACEHOLDER

   $f : \mathbb{R} \to \mathbb{R}$ is not continuous PLACEHOLDER

7. Let us first prove that $f(x) = 1/x^2$ is uniformly continuous on $[0, \infty)$. Let $x, y \in [1, \infty)$ and let $\epsilon > 0$ be set. Then we have:

$$\begin{aligned} \Big|\frac{1}{x^2} - \frac{1}{y^2}\Big| &= \Big|\frac{y^2 - x^2}{x^2y^2}\Big| \\ &= \Big|\frac{(x+y)(x-y)}{x^2y^2}\Big| \\ &= \frac{x+y}{x^2y^2}|x-y| \end{aligned}$$

   Since $x, y \in [1, \infty)$, it follows that $\dfrac{x+y}{x^2y^2} \leq 2$ and for $x, y \in [1, \infty)$, we have:

$$\Big|\frac{1}{x^2} - \frac{1}{y^2}\Big| \leq 2|x-y|$$

   Now, let $\delta = \dfrac{\epsilon}{2}$. Then we have $|x-y| < \delta$ and it follows that $\Big|\dfrac{1}{x^2} - \dfrac{1}{y^2}\Big| < \epsilon$. Hence, by **Definition 4.4.4 (Uniform Continuity)**, $f(x) = 1/x^2$ is uniformly continuous on $[0, \infty)$.

   $\square$

   Let us now prove that $f(x) = 1/x^2$ is not uniformly continuous on the interval $(0, 1]$. Suppose, for the sake of contradiction, that $f(x)$ is uniformly continuous on $(0, 1]$. Then for $\epsilon > 0$ there must exist $\delta > 0$ s.t. $\forall x, y \in (0, 1]$ with $|x - y| < \delta$, it follows that $|f(x) - f(y)| < \epsilon$. Now, let $x = \dfrac{2}{n}$ and $y = \dfrac{1}{n}$ with $n \geq 2$. We have that $|x - y|$ implies

$$\Big|\frac{1}{x^2} - \frac{1}{y^2}\Big| = \Big|\frac{\frac{3}{n^2}}{\frac{4}{n^2}}\Big| = \frac{3}{4} < \epsilon$$

   Now, we got that $\epsilon > \dfrac{3}{4}$, however, $|f(x) - f(y)| < \epsilon$ must be true $\forall \epsilon > 0$. Hence, we face a contradiction and $f(x) = 1/x^2$ is not uniformly continuous on $(0, 1]$.

   $\square$

Finally, we have shown that $f(x) = 1/x^2$ is uniformly continuous on $[0, \infty)$, but not on $(0, 1]$.

$\square$

8. (a) Placeholder

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