
Real Analysis

Assignment №8

Instructor: Dr. Eric Westlund

David Oniani

Luther College

oniada01@luther.edu

December 18, 2020

4.4.1 (a) **Per Theorem 4.3.4**, we know that products of continuous functions are continuous. Hence, it suffices to show that the function $g(x) = x$ is continuous on all of $\mathbb{R}$. Now, let $\epsilon < 0$ be given and choose $\delta = \epsilon$. Then, we have that $\forall x, y \in \mathbb{R}, |x - y| < \delta$. It follows that $|g(x) - g(y)| = |x - y| < \epsilon$.

$\square$

(b) Let $x_n = n$ and $y_n = n - \dfrac{1}{n}$. Then $|x_n - y_n| \to 0$. However, we have:

$$|f(x_n) - f(y_n)| = \left|n^3 - \left(n - \frac{1}{n}\right)^3\right| = \left|3n + \frac{1}{n^3} - \frac{3}{n}\right| \geq n \to \infty$$

Hence, **by Theorem 4.4.5**, the function f is not uniformly continuous on $\mathbb{R}$.

$\square$

(c) Let $A \subseteq \mathbb{R}$ be any bounded subset. Then $\overline{A}$, the closure of A in $\mathbb{R}$, is compact. **By Theorem 4.4.7 (Uniform Continuity on Compact Sets)**, f is uniformly continuous on $\overline{A}$, and hence on any subset of $\overline{A}$. Thus, f is uniformly continuous on A. Finally, since A was chosen arbitrarily, we get that f is uniformly continuous on any bounded subset of $\mathbb{R}$.

$\square$

4.4.6 (a) Such request is possible.

Let $f(x) = \dfrac{1}{x}$ for $x \in (0,1)$. Then f is continuous on $(0,1)$. Now, let $x_n = \dfrac{1}{n}$. Then x_n is Cauchy. However, $f(x_n) = n$ and thus, $(f(x_n))$ is not Cauchy.

(b) Such request is impossible.

Suppose, for the sake of contradiction, that f is a uniformly continuous function on $(0,1)$, x_n is the Cauchy sequence, and $f(x_n)$ is not a Cauchy sequence. Then $\forall \epsilon > 0, \exists \delta > 0$ s.t. given $|x-y| < \delta$, $|f(x)-f(y)| < \epsilon$. Now, since (x_n) is Cauchy, $\exists N > 0$ s.t. $|x_n - x_m| < \delta$ with $n, m > N$. Then, it follows that $|f(x_n) - f(x_m)| < \epsilon$ which implies that $f(x_n)$ is a Cauchy sequence and we face a contradiction as we have assumed that $f(x)$ is not a Cauchy sequence.

$\square$

(c) Such request is impossible.

Suppose, for the sake of contradiction, that $f : [0,\infty) \to \mathbb{R}$ is a continuous function, (x_n) is a Cauchy sequence, and $f(x_n)$ is not a Cauchy sequence. If $(x_n) \to x$, then $[0,\infty)$ is closed and containts x. Finally, since f is continuous on x, by definition, we get $(x_n) \to x$ and it follows that $f(x_n) \to f(x)$. Thus, such request is impossible.

$\square$

4.4.11 We will first prove the statement directly and then prove its converse.

Suppose that g is continuous function on $O \subseteq \mathbb{R}$. Let $x \in g^{-1}(O)$ which implies that $g(x) \in O$. Since O is open, $\exists \epsilon > 0$ s.t. $V_\epsilon(g(x)) \subseteq O$. Furthermore, as g is continuous, it follows that $\exists \delta > 0$ s.t. if $y \in V_\delta(\epsilon)$, we have $g(y) \in V_\epsilon(g(x)) \subseteq O$. We get that $\forall y \in V_\delta(x), g(y) \in O$ and thus, $y \in g^{-1}(O)$. Now, since y is chosen arbitrarily, $V_\delta(x) \subseteq g^{-1}(O)$. Furthermore, since the choice of x is also arbitrary, every element of $g^{-1}(O)$ has some δ-neighborhood contained in $g^{-1}(O)$. Hence, $g^{-1}(O)$ is open.

$\square$

Conversely, suppose, $g^{-1}(O)$ is open whenever $O \subseteq \mathbb{R}$ with O being open. Let $O = V_\epsilon(g(c))$. Then it follows that $g^{-1}(O)$ is open. Now, since $c \in g^{-1}(O)$ and c is an interior point, $\exists \delta > 0$ s.t $V_\delta(c) \subseteq g^{-1}(O)$. Then, by definition, if $x \in V_\delta(c)$, it follows that $g(x) \in O = V_\epsilon(g(c))$ and we finally got that g is a continuous function.

$\square$

Finally, we have proven both the direct statement and its converse and hence, g is continuous if and only if $g^{-1}(O)$ is open whenever $O \subseteq \mathbb{R}$ is an open set.

$\square$

4.5.7 Notice that $\forall 0 \leq x \leq 1$, we have $0 \leq f(x) \leq 1$. Now, consider the following function:

$$g(x) : [0,1] \to \mathbb{R} : x \mapsto f(x) - x$$

Then g is continuous on $[0,1]$. Furthermore, we have $g(0) = f(0) \geq 0$. However, $g(1) = f(1) - 1 \leq 0$. Hence, **by the Intermediate Value Theorem (Theorem 4.5.1)**, $\exists x_0 \in [0,1]$ s.t. $g(x_0) = 0$. Hence, we have that $f(x_0) = x_0$.

$\square$