Real Analysis

Assignment №1

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5. (a) Suppose $A, B \subseteq \mathbb{R}$ and $x \in (A \cap B)^c$. Then $x \notin A \cap B$. Hence, $x \in A^c$ or $x \in B^c$. Therefore, $x \in A^c \cup B^c$. Finally, we get that $(A \cap B)^c \subseteq A^c \cup B^c$.

   $\square$

   (b) Suppose $A, B \subseteq \mathbb{R}$ and $x \in A^c \cup B^c$. Then $x \in A^c$ or $x \in B^c$, which is equivalent to $x \notin A$ or $x \notin B$. Hence, $x \notin A \cap B$. Therefore, $x \in (A \cap B)^c$ and it follows that $(A \cap B)^c \supseteq A^c \cup B^c$. Finally, since we have already proven that $(A \cap B)^c \subseteq A^c \cup B^c$, we get that $(A \cap B)^c = A^c \cup B^c$.

   $\square$

   (c) Let us first prove that $A^c \cap B^c \subseteq (A \cup B)^c$.

   Suppose $A, B \subseteq \mathbb{R}$ and $x \in A^c \cap B^c$. Then $x \in A^c$ and $x \in B^c$. Thus, $x \notin A$ and $x \notin B$. It follows that $x \notin (A \cup B)$. Hence, $x \in (A \cup B)^c$ and $A^c \cap B^c \subseteq (A \cup B)^c$.

   $\square$

   Let us now prove that $(A \cup B)^c \subseteq A^c \cap B^c$.

   Suppose $A, B \subseteq \mathbb{R}$ and $x \in (A \cup B)^c$. Then $x \notin A \cup B$. Therefore, $x \notin A$ and $x \notin B$. It follows that $x \in A^c$ and $x \in B^c$. Thus, $x \in A^c \cap B^c$ and $(A \cup B)^c \subseteq A^c \cap B^c$.

$\square$

Finally, since $A^c \cap B^c \subseteq (A \cup B)^c$ and $(A \cup B)^c \subseteq A^c \cap B^c$, we get that $(A \cup B)^c = A^c \cap B^c$.

$\square$

8. (a) Let us define $f : \mathbb{N} \to \mathbb{N} : x \mapsto 2x + 1$. We can now prove that f is $1 - 1$. Suppose, for the sake of contradiction, that $x_1, x_2 \in \mathbb{N}$ with $x_1 \neq x_2$ and $f(x_1) = f(x_2)$. Then $f(x_1) = f(x_2) \implies 2x_1 + 1 = 2x_2 + 1 \implies x_1 = x_2$. Hence, we face the contradiction and f is $1 - 1$. However, f is not onto since for $f(x) = 2$, the solution is 0.5 which is not in $\mathbb{N}$. Thus, f is a function that is $1 - 1$, but not onto.

   $\square$

   (b) Let us define $f : \mathbb{N} \to \mathbb{N} : x \mapsto \lfloor \frac{x}{4} \rfloor$. Then f is onto since since $\forall x \in \mathbb{N}$, $\exists k = 4x$ with $f(k) = x$. However, f is not $1 - 1$ since $f(2) = f(3)$. Thus, f if a function that is onto, but not $1 - 1$.

   $\square$

   (c) Let us define

$$f : \mathbb{N} \to \mathbb{Z} : x \mapsto \begin{cases} \frac{x-1}{2}, & \text{if } n \text{ is odd} \\ -\frac{x}{2}, & \text{if } n \text{ is even} \end{cases} \tag{1}$$

   Then f is both $1 - 1$ and onto. Let us first prove that f is $1 - 1$.

   Suppose, for the sake of contradiction, that $x_1, x_2 \in \mathbb{N}$ with $x_1 \neq x_2$ and $f(x_1) = f(x_2)$. Then $f(x_1)$ and $f(x_2)$ must be of the same sign or both be zero. Thus, we have three cases:

   1. $f(x_1) = \frac{x_1 - 1}{2}$ and $f(x_2) = \frac{x_2 - 1}{2}$
      $f(x_1) = f(x_2) \implies \frac{x_1 - 1}{2} = \frac{x_2 - 1}{2} \implies x_1 = x_2$.
      $\square$
   2. $f(x_1) = -\frac{x_1}{2}$ and $f(x_2) = -\frac{x_2}{2}$
      $f(x_1) = f(x_2) \implies -\frac{x_1}{2} = -\frac{x_2}{2} \implies x_1 = x_2$
      $\square$
   3. $f(x_1) = 0$ and $f(x_2) = 0$
      The only way for this to happen is if $f(x_1) = -\frac{x_1}{2}$ and $f(x_2) = -\frac{x_2}{2}$ and we get $f(x_1) = f(x_2) \implies -\frac{x_1}{2} = -\frac{x_2}{2} \implies x_1 = x_2$
      $\square$

$\square$

Now let us prove that f is onto. For $y \in \mathbb{Z}$, we have three cases:

1. y is positive.

   If $y > 0$, we can find $k = 2y + 1$ with $f(k) = y$.

   $\square$

2. y is negative.

   If $y < 0$, we can find $k = -2y$ with $f(k) = y$.

   $\square$

3. y is zero.

   If $y = 0$, we can find $k = 1$ with $f(k) = y = 0$.

   $\square$

$\square$

Finally, we have proven that f is both $1-1$ and onto.

$\square$

9. (a)

   (b)

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