Topology

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Assignment №2

Section 13

7. Consider the following topologies on $\mathbb{R}$:

$$\begin{aligned}
\mathcal{T}_1 &= \text{the standard topology,} \\
\mathcal{T}_2 &= \text{the topology of } \mathbb{R}_K, \\
\mathcal{T}_3 &= \text{the finite complement topology,} \\
\mathcal{T}_4 &= \text{the upper limit topology, having all sets } (a, b] \text{ as basis,} \\
\mathcal{T}_5 &= \text{the topology having all sets } (-\infty, a] = \{x \mid x < a\} \text{ as basis.}
\end{aligned}$$

Determine, for each of these topologies, which of the other it contains.

From **Lemma 13.4**, we know that $\mathcal{T}_2$ is strictly finer than $\mathcal{T}_1$.

The finite complement topology will look like $(-\infty, x_0) \cup (x_0, x_1) \cup ....(x_{n-1}, +\infty)$. Then it is easy to notice that $\mathcal{T}_1$ is strictly finer than $\mathcal{T}_3$ (since $\mathcal{T}_3$ is an open set in $\mathcal{T}_1$; also $(2, 3)$ is open in standard topology but not in finite complement topology).

Now, let $B = (a, b)$ be the element in the basis of $\mathcal{T}_1$. Let $x \in B$. Then, we can find element $(a, x]$ in the upper limit topology that clearly contains element x. Hence, the upper limit topology is finer standard topology. In fact $\mathcal{T}_4$ is strictly finer than $\mathcal{T}_1$ since $(2, 3]$ is not open in $\mathcal{T}_1$.

Hence, as of now, we have the relationship $\mathcal{T}_3 \subsetneq \mathcal{T}_1 \subsetneq \mathcal{T}_2, \mathcal{T}_4$.

Let's now find out the relationship between $\mathcal{T}_2$ and $\mathcal{T}_4$.

The upper limit topology is finer than the topology of $\mathbb{R}_K$. To show this let $B = (a, b) - K$ be the element in the basis of $\mathcal{T}_2$. Then let $x \in B$. If $x < 0$, then we can find element $(a, x]$ which is in the basis of the $\mathcal{T}_4$. In this case, it is easy to notice that $x \in (a, x] \in B$. On the other hand, if $x \geq 0$, then let k be the smallest integer such that $\frac{1}{k} < x$. We have $B \cap (\frac{1}{n}, x] = (s, x]$ where $s = a$ if $a > \frac{1}{k}$ and $s = \frac{1}{k}$ if $\frac{1}{k} > a$. It follows that $x \in (s, x] \subset B$, and we get that $(x, s]$ is a basis element for $\mathcal{T}_4$. Hence, we got that $\mathcal{T}_2 \subset \mathcal{T}_4$.

Now our relationship looks a bit better: $\mathcal{T}_3 \subsetneq \mathcal{T}_1 \subsetneq \mathcal{T}_2 \subset \mathcal{T}_4$.

Let's now find the relationship between $\mathcal{T}_5$ and $\mathcal{T}_1$. Let $B = (-\infty, a]$ the element in the basis for $\mathcal{T}_5$. Then let $x \in B$. Notice that $B = \bigcup_{i=1}^{+\infty}(-i, a)$. Knowing that any topology is closed under union, we then know that $B \in \mathcal{T}_1$. Thus, we got that $\mathcal{T}_5 \subset \mathcal{T}_1$. Now, notice that there is no element e such that $e \in (-\infty, 2) \subset (2, 3)$ hence, $\mathcal{T}_1$ is strictly finer than $\mathcal{T}_5$ and $\mathcal{T}_5 \subsetneq \mathcal{T}_1$.

As of now, our relationship is $\mathcal{T}_3, \mathcal{T}_5 \subsetneq \mathcal{T}_1 \subsetneq \mathcal{T}_2 \subset \mathcal{T}_4$.

Finally, let's find out the relationship between $\mathcal{T}_3$ and $\mathcal{T}_5$. They are not comparable! (I looked at the previous exercise which asks for similar but not exactly the same question/proof – **Exercise 6**). Consider two open sets $\mathbb{R} - \{2\}$ and $(-\infty, 4)$ from $\mathcal{T}_3$ and $\mathcal{T}_5$ correspondingly. Then both open sets contain the point 3, however, neither of these open sets contain the open set from the other topology that contains 3. Hence, there is no way to find out whether $\mathcal{T}_3$ is finer than $\mathcal{T}_5$, vice versa.

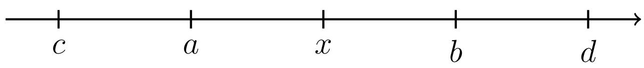
Finally, we have established the relationship $\mathcal{T}_3, \mathcal{T}_5 \subsetneq \mathcal{T}_1 \subsetneq \mathcal{T}_2 \subset \mathcal{T}_4$ with $\mathcal{T}_3$ and $\mathcal{T}_5$ being impossible to compare.

8. (a) Apply Lemma 13.2 to show that the countable collection

$$\mathcal{B} = \{(a, b) \mid a < b, a \text{ and } b \text{ rational}\}$$

is a basis that generates the standard topology on $\mathbb{R}$.

Suppose that (c, d) is the element in the basis of the standard topology. Let $x \in (c, d)$, $c < x < d$. Then $\exists a, b \in \mathbb{Q}$ such that $c < a < x < b < d$. Below is the graphical representation of how the sets would look like (it is obviously not necessary and probably redundant, just learning how to to use the `tikz` package ☺).



We got $x \in (a, b) \subset (c, d)$ with $(c, d) \in \mathcal{B}$. In other words, $\forall x \in B_{\mathbb{R}}$ (where $B_{\mathbb{R}}$ is the open set of $\mathbb{R}$), $\exists C$ such that $x \in C \subset B_{\mathbb{R}}$. We can now apply **Lemma 13.2** and claim that $\mathcal{B}$ is the basis that generates the standard topology on $\mathbb{R}$. $\square$

(b) Show that the collection

$$\mathcal{C} = \{[a, b) \mid a < b, a \text{ and } b \text{ are rational}\}$$

is a basis that generates a topology diffferent from the lower limit topology on $\mathbb{R}$.

Consider the point $\sqrt{5}$ in the open set $[\sqrt{5}, 5)$ of the lower limit topology. Then, since $\mathcal{C}$ can only generate open sets of the topology of the type (a, b) where $a < b$ and $a, b \in \mathbb{Q}$ or unions of such sets, it is clear that there is no basis element in $\mathcal{C}$ which would contain $\sqrt{5}$ and be a subset of $[\sqrt{5}, 5)$. In fact, the topology generated by $\mathcal{C}$ is strictly coarser than the lower limit topology on $\mathbb{R}$. $\square$

Section 16

3. Consider the set $Y = [-1, 1]$ as a subspace of $\mathbb{R}$. Which of the following sets are open in Y? Which are open in $\mathbb{R}$?

$$\begin{aligned} A &= \{x \mid \frac{1}{2} < |x| < 1\}, \\ B &= \{x \mid \frac{1}{2} < |x| \leq 1\}, \\ C &= \{x \mid \frac{1}{2} \leq |x| < 1\}, \\ D &= \{x \mid \frac{1}{2} \leq |x| \leq 1\}, \\ E &= \{x \mid 0 < |x| < 1 \text{ and } 1/x \notin \mathbb{Z}^+\}. \end{aligned}$$

Let's check the openness in Y and $\mathbb{R}$ one by one.

First consider the set $A = \{x \mid \frac{1}{2} < |x| < 1\}$. Notice that A is the union of open intervals since is open in $\mathbb{R}$. Because $Y \subset \mathbb{R}$, A is also open in Y.

Consider the set $B = \{x \mid \frac{1}{2} < |x| \leq 1\}$. We have $1 \in B$, but for any $\delta > 0$, $(1, \delta) \not\subset B$. Hence, B is not open in $\mathbb{R}$. Now notice that $B = Y \cap ((-2, -\frac{1}{2}) \cup (\frac{1}{2}, 2)) = [-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1]$. Now, since, $[-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1]$ is open, B is open in Y.

Now, let's take a look at the set $C = \{x \mid \frac{1}{2} \leq |x| < 1\}$. Suppose, for the sake of contradiction, that C is open in Y. Then $\exists U \in Y$ such that $C = U \cap Y$. In other words, $\exists \delta > 0$ such that $(\frac{1}{2}, \delta) \in U$. Then there would also exist $\delta\prime$ such that $(\frac{1}{2}, \delta\prime) \in C$ which is false. Thus, we have reached the contradiction and the set C is not open in Y. It follows that C is also not open in $\mathbb{R}$.

Consider the set $D = \{x \mid \frac{1}{2} \leq |x| \leq 1\}$. Suppose, for the sake of contradiction, that D is open in Y. Then $\exists U \in Y$ such that $D = U \cap Y$. In other words, $\exists \delta > 0$ such that $(\frac{1}{2}, \delta) \in U$. Then there would also exist $\delta\prime$ such that $(\frac{1}{2}, \delta\prime) \in D$ which is false. Thus, we have reached the contradiction and the set D is not open in Y. It follows that D is also not open in $\mathbb{R}$.

E is open in both Y and $\mathbb{R}$ since it is the union of open intervals.

Finally, we got that A and E are open in both Y and $\mathbb{R}$. The set B is open only in Y. And sets C and D are not open in Y or $\mathbb{R}$.

4. A map $f : X \to Y$ is said to be an **open map** if for every open set U of X, the set $f(U)$ is open in Y. Show that $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are open maps.

At first, let us show that $\pi_1 : X \times Y \to X$ is an open map. Let $U \subset X \times Y$ be an open set and let $x \in \pi_1(U)$. Then $\exists y$ such that $x \times y \in U$. Now, because U is open, there is a basis set $A \times B \in U$ such that $x \times y \in U$. Now, since $A \times B$ is a basis set, A is open in X. Besides, $x \in A = \pi_1(A \times B) \subset \pi_1(U)$. Hence, $\pi_1(U)$ is open. Therefore, $\pi_1 : X \times Y \to X$ is an open map. $\square$

Now, let's show that $\pi_2 : X \times Y \to Y$ is an open map. Let $V \subset X \times Y$ be an open set and let $y \in \pi_1(V)$. Then $\exists x$ such that $x \times y \in V$. Now, because V is open, there is a basis set $A \times B \in V$ such that $x \times y \in V$. Now, since $A \times B$ is a basis set, B is open in Y. Besides, $y \in B = \pi_1(A \times B) \subset \pi_1(V)$. Hence, $\pi_2(V)$ is open. Therefore, $\pi_2 : X \times Y \to Y$ is an open map. $\square$

6. Show that the countable collection

$$\{(a, b) \times (c, d) \mid a < b \text{ and } c < d, \text{and } a, b, c, d \text{ are rational}\}$$

is a basis for $\mathbb{R}^2$.

For simplicity, let's call this set S. Thus, $S = \{(a, b) \times (c, d) \mid a < b \text{ and } c < d, \text{and } a, b, c, d \text{ are rational}\}$.

Suppose that $(a, b) \times (c, d)$ is the element in the basis of the topology for $\mathbb{R}^2$. Let $x \in (a, b) \times (c, d)$. Then $\exists e, f, g, h \in \mathbb{Q}$ with $e < f$ and $g < h$ such that $(e, f) \times (g, h) \subset (a, b) \times (c, d)$. In other words, $\forall x \in B_{\mathbb{R}^2}$ (where $B_{\mathbb{R}^2}$ is the open set in $\mathbb{R}^2$), $\exists C$ such that $x \in C \subset B^2_{\mathbb{R}}$. We can now apply **Lemma 13.2** and claim that S is the basis that generates the topology on $\mathbb{R}^2$. $\square$

10. Let $I = [0, 1]$. Compare the product topology on $I \times I$, the dictionary order topology on $I \times I$, and the topology $I \times I$ inherits the subspace of $\mathbb{R} \times \mathbb{R}$ in the dictionary order topology.

<u>**NOTE**</u> For simplicity let's call the product topology on $I \times I$ – $\mathcal{T}_1$, call the the dictionary order topology on $I \times I$ – $\mathcal{T}_2$, and call he topology $I \times I$ inherits the subspace of $\mathbb{R} \times \mathbb{R}$ in the dictionary order topology – $\mathcal{T}_3$.

Let's first compare $\mathcal{T}_1$ with $\mathcal{T}_2$. Notice that $\mathcal{T}_1$ and $\mathcal{T}_2$ are strictly contained in the subspace topology from the dictionary order on the plane, and hence are incomparable with each other.

$\mathcal{T}_3$ is strictly finer than the $\mathcal{T}_1$ and $\mathcal{T}_2$. $\mathcal{T}_3$ is generated by the basis consisting of the sets $\{x\} \times ((a,b) \cap [0,1])$. Therefore, each basis element $((a,b) \cap [0,1]) \times ((c,d) \cap [0,1])$ of the $\mathcal{T}_1$ is the union of some basis sets of $\mathcal{T}_3$. Similarly, each basis set $(a,b) < (x,y) < (c,d)$ of $\mathcal{T}_2$ topology is the union of some basis sets of $\mathcal{T}_3$. Now, since $\mathcal{T}_1$ and $\mathcal{T}_2$ are not comparable, it follows that $\mathcal{T}_3$ is strictly finer than $\mathcal{T}_1$ and $\mathcal{T}_2$.

At last, we got that $\mathcal{T}_1, \mathcal{T}_2 \subsetneq \mathcal{T}_3$ with $\mathcal{T}_1$ and $\mathcal{T}_2$ being incomparable.

Section 17

3. Show that if A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$.

   Notice that $A \times B = X \times Y - (((X - A) \times Y) \cup (X \times (Y - B)))$. Since A is closed in X, $X - A$ is open in X and thus $(X - A) \times Y$ is open in $X \times Y$. Similarly, since B is closed in Y, $Y - B$ is open in Y and therefore $X \times (Y - B)$ is open in $X \times Y$. Then we have that $((X - A) \times Y) \cup (X \times (Y - B))$ is open and hence $X \times Y - ((X - A) \times Y) \cup (X \times (Y - B))$ is closed. Finally, $A \times B$ is closed in $X \times Y$. $\square$

6. Let A, B, and A_α denote subsets of a space X. Prove the following:

   (a) If $A \subset B$, then $\bar{A} \subset \bar{B}$.

   Notice that $\bar{B}$ is closed and contains B. Now since $A \subset B$, $\bar{B}$ also contains A. Therefore, since $\bar{B}$ is a closed set that contains A, it contains the closure of A and $\bar{A} \subset \bar{B}$. $\square$

   (b) $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

   We have to show that $\overline{A \cup B} \subset \bar{A} \cup \bar{B}$ and $\overline{A \cup B} \supset \bar{A} \cup \bar{B}$.

   Let's first show that $\overline{A \cup B} \subset \bar{A} \cup \bar{B}$. Because $\bar{A}$ and $\bar{B}$ are closed in X, according to **Theorem 17.1**, finite unions of closed sets are closed, and $\bar{A} \cup \bar{B}$ is also closed. Then we know that $A \subset \bar{A}$ and $B \subset \bar{B}$, therefore $A \cup B \subset \bar{A} \cup \bar{B}$. Hence, $\bar{A} \cup \bar{B}$ contains the closure of the set $A \cup B$ and we get $\overline{A \cup B} \subset \bar{A} \cup \bar{B}$. $\square$

   Let's now show that $\overline{A \cup B} \supset \bar{A} \cup \bar{B}$. Let's use the fact that we proved in the part (a) of the exercise. We know that $A \subset A \cup B$ and $B \subset A \cup B$. Therefore, according to the part (a) of the exercise, it follows that $\bar{A} \subset \overline{A \cup B}$ and $\bar{B} \subset \overline{A \cup B}$. Thus, we got that $\overline{A \cup B} \supset \bar{A} \cup \bar{B}$. $\square$

Finally, since we proved both $\overline{A \cup B} \subset \bar{A} \cup \bar{B}$ and $\overline{A \cup B} \supset \bar{A} \cup \bar{B}$, we have $\overline{A \cup B} = \bar{A} \cup \bar{B}$. $\square$

(c) $\overline{\bigcup A_\alpha} \supset \bigcup \bar{A}_\alpha$; give an example where equality fails.

Let $x \notin \overline{\bigcup A_\alpha}$. Then there exists an open neighborhood U of x such that $U \cap \bigcup A_\alpha = \bigcup (U \cap A_\alpha) = \emptyset$ and therefore, $U \cap A_\alpha = \emptyset$. Hence, $x \notin \bar{U}_\alpha$ and since α is arbitrary we have got the results for all A_α. Finally, we got that $x \notin \bigcup \bar{A}_\alpha$. $\square$

Here is the example for when the equality fails:

$$\bigcup_{x \in \mathbb{Q}} \overline{\{q\}} = \bigcup_{x \in \mathbb{Q}} \{q\} = \mathbb{Q}$$
$$\overline{\bigcup_{q \in \mathbb{Q}}} = \overline{\mathbb{Q}} = \mathbb{R}$$

11. Show that the product of two Hausdorff spaces are Hausdorff.

To prove that the product of two Hausdorff spaces are Hausdorff, it is sufficient to show that $\forall x_1, x_2$ such that $x_1 \neq x_2$, $\exists U_1, U_2$ of x_1 and x_2 respectively with $U_1 \cap U_2 = \emptyset$ (U_1, U_2 are neighborhoods). Suppose we have two Hausdorff spaces X and Y. Let $x_1, x_2 \in X$. Then $\exists U_1, U_2 \subset X$ such that $U_1 \cap U_2 = \emptyset$. Now consider the sets $V_1 = U_1 \times Y$ and $V_2 = U_2 \times Y$. Notice that $V_1 \cap V_2 = (U_1 \cap U_2) \times Y = \emptyset \times Y = \emptyset$. Thus, we got that $X \times Y$ is Hausdorff. $\square$

16. Consider the five topologies on $\mathbb{R}$ given in Exercise 7 of §13.

(a) Determine the closure of the set $K = \{1/n \mid n \in \mathbb{Z}^+\}$ under each of these topologies.

Let's consider each of these topologies one by one.

1. $\mathcal{T}_1 =$ the standard topology.

Let $x < 0$. Then notice that $(\frac{5x}{4}, \frac{x}{2})$ is an open neighborhood of x that has no intersection with K. Now, since $K \subset \bar{K}$, it follows that $x \notin \bar{K}$. Now consider the case when $x > 1$. Notice that open set $(x + \frac{1-x}{2}, x - \frac{1-x}{2})$ is a neighborhood of x that does not intersect K. Therefore, since $K \subset \bar{K}$, it follows that $x \notin \bar{K}$. If $x \in (0, 1)$. Then notice that $\exists l \in \mathbb{Z}^+$ such that $\frac{1}{l+1} < x < \frac{1}{l}$. Then this neighborhood has no intersection with K and therefore, $x \notin \bar{K}$. We are now left with point 0. Suppose that U is an open neighborhood of 0. Then there exists the basis element $0 \in (a, b) \subset U$. In fact, there must exist $m \in \mathbb{Z}^+$ such that $\frac{1}{m} < b$ with $\frac{1}{m} \in U \cap K$. The notice that $U \cap K$ is non-empty for any neighborhood U of 0. Therefore, $0 \in \bar{K}$. Finally, $\bar{K} = \{0\}$

2. $\mathcal{T}_2 =$ the topology of $\mathbb{R}_K$.

   In the topology of $\mathbb{R}_K$, $\mathbb{R} - K$ is considered open. Thus, K is closed and, according to **Theorem 17.6**, $\bar{K} = K + \emptyset = K$. Therefore, the closure of the set K is K.

3. $\mathcal{T}_3 =$ the finite complement topology.

   Notice that in the finite complement topology, every closed set is either finite or all of $\mathbb{R}$. Then obviously infinite set K cannot be a subset of any finite set and it follows that $\mathbb{R}$ must be the only closed set that contains K. Therefore, $\bar{K} = \mathbb{R}$.

4. $\mathcal{T}_4 =$ the upper limit topology, having all sets $(a, b]$ as basis.

   Consider the set $\mathbb{R} - K$. Notice that $\mathbb{R} - K = (-\infty, 0] \cup \bigcup_{i \in \mathbb{Z}^+} (\frac{1}{i+1}, \frac{1}{i}) \cup (1, +\infty)$. Then $\forall U \in \mathbb{R} - K$, U is open in the upper limit topology and hence, $\mathbb{R} - K$ is open. Therefore, K is closed and $\bar{K} = K$.

5. $\mathcal{T}_5 =$ the topology having all sets $(-\infty, a] = \{x \mid x < a\}$ as basis.

   Let $x < 0$, then $(-\infty, x]$ has no intersections with K, hence $x \notin \bar{K}$. Now let $x \geq 0$. Then all neighborhoods of x contain a basis set $(-\infty, y)$ where $y > 0$. Let U be a neighborhood of x. Then $\exists z$ such that $\frac{1}{z} < y$. Therefore, $\frac{1}{z} \in U \cup K$. Finally, $x \in \bar{K}$ and $\bar{K} = [0, +\infty)$.

   To summarize, we got that the closure of K under $\mathcal{T}_1$ is $\{0\}$, under $\mathcal{T}_2$ is K, under $\mathcal{T}_3$ is $\mathbb{R}$, under $\mathcal{T}_4$ is K, and under $\mathcal{T}_5$ is $[0, +\infty)$.

(b) Which of these topologies satisfy the Hausdorff axiom? the T_1 axiom?

Once again, let's go through all of the topologies one at a time.

1. $\mathcal{T}_1 =$ the standard topology.

   Notice that $\forall x, y$ such that $x \neq y$, $x \in (x - \frac{|x-y|}{2}, x + \frac{|x-y|}{2})$ and $y \in (y - \frac{|x-y|}{2}, y + \frac{|x-y|}{2})$. Besides, $(x - \frac{|x-y|}{2}, x + \frac{|x-y|}{2}) \cap (y - \frac{|x-y|}{2}, y + \frac{|x-y|}{2}) = \emptyset$. Hence, the standard topology is both Hausdorff and T_1.

2. $\mathcal{T}_2 =$ the topology of $\mathbb{R}_K$.

   We know that the topology on $\mathbb{R}_K$ is finer than the standard topology. This is sufficient to say that it is both Hausdorff and T_1.

3. $\mathcal{T}_3 =$ the finite complement topology.

   Notice that in the finite complement topology, there are no two non-empty open sets U_1 and U_2 such that $U_1 \cap U_2 = \emptyset$. Therefore, finite complement topology is not Hausdorff. However, all the singleton sets are finite and therefore closed. This means that the finite complement topology is T_1.

4. $\mathcal{T}_4 =$ the upper limit topology, having all sets $(a, b]$ as basis.

We know that the upper limit topology is finer than the standard topology. Therefore, it is both Hausdorff and T_1.

5. $\mathcal{T}_5 =$ the topology having all sets $(-\infty, a] = \{x \mid x < a\}$ as basis.

   Suppose that U is an open set and let $x \in U$. Then there exists a basis element $x \in (-\infty, a) \subset U$ where $a > x$. Hence, every real number less than x is in U. Consider $\mathbb{R} - \{0\}$. 1 is in the set, but $0 < 1$ and 0 is not in this set, and hence cannot be open. Thus, $\{0\}$ is not close which means that the topology is not T_1. As being Hausdorff implies being T_1, it is also not Hausdorff.

   To summarize, we got that the closure of $\mathcal{T}_1$ is both Hausdorff and T_1. $\mathcal{T}_2$ is both Hausdorff and T_1, $\mathcal{T}_3$ is T_1 but not Hausdorff, $\mathcal{T}_4$ is both Hausdorff and T_1, and $\mathcal{T}_5$ is neither Hausdorff nor T_1.

19. If $A \subset X$, we define the **boundary** of A by the equation

$$\text{Bd } A = \bar{A} \cap \overline{(X - A)}.$$

(a) Show that Int A and Bd A are disjoint, and $\bar{A} =$ Int $A \cup$ Bd A.

Let us first show that A and Bd are disjoint.

Suppose, for the sake of contradiction, that A and Bd are not disjoint. Then let $x \in$ Int $A \cap$ Bd A. Now, recall that Int A is the union of all open sets **contained** in A. Then $\exists U$ such that $x \in U \subset A$ and U is open. Now, since $x \in$ Bd A, we have $x \in \bar{A} \cap \overline{(X - A)}$ and thus, $x \in \overline{(X - A)}$. But now, since U is an open neighborhood of x (because $x \in U$ and U is open), according to **Theorem 17.5**, U must have an intersection with $X - A$ and we have reached the contradiction as we assumed that $U \subset A$. Therefore, Int A and Bd A are indeed disjoint. $\square$

Now, let's show that $\bar{A} =$ Int $A \cup$ Bd A.

This is the case when we have to show the inclusion both ways. In other words, we have to firsts show that $\bar{A} \subset$ Int $A \cup$Bd A and then $\bar{A} \supset$ Int $A \cup$Bd A.

First, let's show that $\bar{A} \subset$ Int $A \cup$ Bd A. Let $x \in \bar{A}$. Then if there is an open set U containing x and $U \subset A$, obviously $x \in$ Int A as Int A is the union of all open sets contained in A. Therefore $x \in$ Int $A \cup$ Bd A. On the other hand, if there is no open set U such that $x \in U$ and $U \subset A$, then $x \in \overline{X - A}$ (as each open neighborhood of x would have an intersection with $X - A$). Thus, $x \in$ Bd A which means that $x \in$ Int $A \cup$ Bd A. Finally, since $\forall x \in \bar{A}$, $x \in$ Int $A \cup$ Bd A, it must be the case that $\bar{A} \subset$ Int $A \cup$ Bd A. $\square$

Now, let's show that $\bar{A} \supset$ Int $A \cup$ Bd A. By definition, we have that Int $A \subset A \subset \bar{A}$. Now, since Bd $A = \bar{A} \cap \overline{X - A}$, Bd $A \subset \bar{A}$ and therefore, Int $A \cup$ Bd $A \subset \bar{A}$. $\square$

Finally, since we have proven both $\bar{A} \subset \text{Int } A \cup \text{Bd } A$ and $\bar{A} \supset \text{Int } A \cup \text{Bd } A$, it means that $\bar{A} = \text{Int } A \cup \text{Bd } A$. $\square$

(b) Show that $\text{Bd } A = \emptyset \Leftrightarrow A$ is both open and closed.

We know that $\text{Int } A \subset A \subset \bar{A}$. Furthermore, we know that A is open if and only if $A = \text{Int } A$ and A is closed if and only if $A = \bar{A}$. Therefore, A is both open and closed if and only if $\text{Int } A = \bar{A}$. Now, recall that in the part (a) of the exercise, we proved that $\bar{A} = \text{Int } A \cup \text{Bd } A$. Therefore, if $\text{Int } A = \bar{A}$, it follows that $\text{Bd } A = \emptyset$. $\square$

(c) Show that U is open $\Leftrightarrow \text{Bd } U = \bar{U} - U$.

U is open if and only if $\text{Bd } U = \bar{U} - \text{Int } U = \bar{U} - U$. $\square$

(d) If U is open, is it true that $U = \text{Int } \bar{U}$? Justify your answer.

No, it is not.
For instance, consider the open set $\mathbb{R} - \{0\}$. Then $\text{Int } \overline{\mathbb{R} - \{0\}} = \text{Int } \mathbb{R} = \mathbb{R} \neq \mathbb{R} - \{0\}$.

20. Find the boundary and the interior of each of the following subsets of $\mathbb{R}^2$.

NOTE: These are just the boundaries and the interiors of the sets. There are no formal proofs accompanying the observations. All of the observations are made by visualizing the problem.

(a) $A = \{x \times y \mid y = 0\}$

Notice that the set of limit points of A is equal to A as it is the X axis. Therefore, $\bar{A} = A \cup A' = A \cup A = A$. Now, $\text{Bd } A = \bar{A} \cap \overline{(\mathbb{R}^2 - A)}$. Now, also notice that the closure of the complement of X axis is all of $\mathbb{R}^2$. Therefore $\overline{(\mathbb{R}^2 - A)} = \mathbb{R}^2$. Finally, we have $\text{Bd } A = \bar{A} \cap \overline{(\mathbb{R}^2 - A)} = A \cap \mathbb{R}^2 = A$. As for interior, according the formula $\bar{A} = \text{Int } A \cup \text{Bd } A$ and the fact that Int A ad Bd B the correctness of which we proved in **Exercise 19 (a)**, it follows that $\text{Int } A = \emptyset$. Finally, $\text{Bd } A = A$ and $\text{Int } A = \emptyset$.

(b) $B = \{x \times y \mid x > 0 \text{ and } y \neq 0\}$

Notice that the set of limit points of B is $B' = \{(x, y) \mid x \geq 0\}$. Then $\bar{B} = B \cap B' = B' = \{(x, y) \mid x \geq 0\}$. Let's now calculate the closure of the complement of B to then apply our formula for calculating the boundary. Notice that $\overline{\mathbb{R}^2 - B} = \{(x, y) \mid x \leq 0 \text{ or } x > 0 \text{ and } y = 0\}$. Now, we can apply our formula for computing the boundary. We have $\text{Bd } B = \bar{B} \cap \overline{\mathbb{R}^2 - B} =$

$\{(x, y) \mid x = 0 \text{ or } x > 0 \text{ and } y = 0\} = \{0\} \times \mathbb{R} \cup \mathbb{R}^+ \times \{0\}$. Once again, using the results we've got in **Exercise 19 (a)**, we have Int $C = \mathbb{R}^+ \times \mathbb{R}$. Finally, we got that Bd B $= \{0\} \times \mathbb{R} \cup \mathbb{R}^+ \times \{0\}$ and Int B $= B$.

(c) $C = A \cup B$

The set C is the set which consists of all points where $x > 0$ or $y = 0$. The closure of this set is the set of all points such that $x \geq 0$ or $y = 0$. The complement set of the set C is the set where $x \geq 0$ and $y \neq 0$ hold for all points and the closure of it is the set of all points where $x \leq 0$. Then the intersection of the closure of C and the closure of the complement of C is the set of all points where $x = 0$ or both $x < 0$ and $y = 0$. Therefore, Bd $C = \mathbb{R}^- \times \{0\} \cup \{0\} \times \mathbb{R}$ and Int $C = \mathbb{R}^+ \times \mathbb{R}$.

(d) $D = \{x \times y \mid x \text{ is rational}\}$

Every open interval (which is not empty) of $\mathbb{R}$ has infinitely many rational and irrational numbers. Hence, each product of intervals contains infinitely many elements of D and $\mathbb{R}^2 - D$. Therefore, $\bar{D} = \mathbb{R}^2$ (and Bd D is $\mathbb{R}^2$ as well since it is defined as union of the closure of D and the closure of its complement and it cannot be bigger than $\mathbb{R}^2$). Once again, using the formula $\bar{D} = \text{Int } D \cup \text{Bd } D$ and fact that Bd D and Int D are disjoint, we have that Int $D = \emptyset$. At last, we have Bd $D = \mathbb{R}^2$ and Int $D = \emptyset$.

(e) $E = \{x \times y \mid 0 < x^2 - y^2 \leq 1\}$

Once the visualization is done, it is easy to see that the set of all limit points of E is the butterfly-shaped plot. Therefore, the closure of E will be that butterfly-shaped plot (it includes E but also has the wings of the butterfly included). In other words, $\bar{E} = \{(x, y) \mid 0 \leq x^2 - y^2 \leq 1\}$. Now, the closure of $\mathbb{R}^2 - E$ will visually be the diagonal lines plus the area outside of the wings of the butterfly. Therefore, since Bd $E = \bar{E} \cap \overline{\mathbb{R}^2 - E}$, the boundary of E appears to be the skeleton of the butterfly. In other words, Bd $E = \{(x, y) \mid x^2 - y^2 = 1 \text{ or } |x| = |y|\}$. Also, by just looking at the butterfly, it is almost trivial to see (but certainly not trivial to show) that its interior is the space inside and Int $E = \{(x, y) \mid 0 < x^2 + y^2 < 1\}$. Finally, we have that Bd $E = \{(x, y) \mid x^2 - y^2 = 1 \text{ or } |x| = |y|\}$ and Int $E = \{(x, y) \mid 0 < x^2 + y^2 < 1\}$.

(f) $F = \{x \times y \mid x \neq 0 \text{ and } y \leq 1/x\}$

The closure of F is the union of F with Y axis. The reason is that geometrically, this is the region underneath either positive or negative branches of hyperbola $y = \dfrac{1}{x}$ with X axis removed and with branches of hyperbola included in the set (in other words, the set of limit points would contain the Y axis as we can put a square around it and it would include an open neighborhood).

Hence, Bd $F = \{(x, y) \mid x \neq 0 \text{ and } (y = 1/x \text{ or } x = 0)\}$. Once again, the visualization of this shows us that Int $F = \{(x, y) \mid x \neq 0 \text{ and } y < 1/x\}$. Finally, we have Bd $F = \{(x, y) \mid x \neq 0 \text{ and } (y = 1/x \text{ or } x = 0)\}$ and Int $F = \{(x, y) \mid x \neq 0 \text{ and } y < 1/x\}$.