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# *Real Analysis Exams*

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## Exam №1

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1. Let  $a_n = \frac{6n+1}{3n+2}$ .

Now, let  $N \in \mathbb{N} = \left\lceil \frac{1+\epsilon}{\epsilon} \right\rceil$  ( $\left\lceil \frac{a}{b} \right\rceil$  is the **ceiling** of  $\frac{a}{b}$ ).

Then  $\forall \epsilon > 0$  and  $\forall n \geq N$  and we have:

$$\begin{aligned} |a_n - 2| &= \left| \frac{6n+1}{3n+2} - 2 \right| \\ &= \left| \frac{-3}{3n+2} \right| \\ &\leq \frac{3}{3N+2} \\ &\leq \frac{3}{3\frac{1+\epsilon}{\epsilon} + 2} = \frac{3\epsilon}{3 + 3\epsilon + 2\epsilon} = \frac{3\epsilon}{5\epsilon + 3} \\ &< \frac{3}{5}\epsilon \\ &< \epsilon \end{aligned}$$

We have now shown that  $\forall \epsilon > 0, n \geq N, |a_n - 2| < \epsilon$  and thus,  $\lim_{n \rightarrow \infty} a_n = 2$ .

□

2. (a) This is true. We can prove this visually. In order to prove that  $A \times B$  is countable, it suffices to show that there exists an enumeration of this set.

First, recall that  $A \times B = \{(a, b) : a \in A, b \in B\}$ .

Now, let  $P = A \times B$ ,  $A = \{a_1, a_2, a_3, \dots\}$ , and  $B = \{b_1, b_2, b_3, \dots\}$ .

We have:

$$P_1 = (a_1, b_1), (a_1, b_2), (a_1, b_3), (a_1, b_4), \dots$$

$$P_2 = (a_2, b_1), (a_2, b_2), (a_2, b_3), (a_2, b_4), \dots$$

$$P_3 = (a_3, b_1), (a_3, b_2), (a_3, b_3), (a_3, b_4), \dots$$

$$P_4 = (a_4, b_1), (a_4, b_2), (a_4, b_3), (a_4, b_4), \dots$$

.....

We enumerate the sequence diagonal-by-diagonal (of small squares) as follows:

$$(a_1, b_1), (a_2, b_1), (a_1, b_2), (a_3, b_1), (a_2, b_2), (a_1, b_3), \dots$$

This way, all of the elements in  $P$  will eventually be listed and hence, we have successfully enumerated  $P$ . Thus,  $P$  is countable, which means that  $A \times B$  is countable.

□

- (b) This is true. Let the finite product of countable sets be denoted as

$$\begin{aligned} FP &= A_1 \times A_2 \times A_3 \times \dots \times A_{n-1} \times A_n \\ &= \{(a_1, a_2, a_3, \dots, a_{n-1}, a_n) \mid a_1 \in A_1, a_2 \in A_2, a_3 \in A_3, \dots, a_{n-1} \in A_{n-1}, a_n \in A_n\} \end{aligned}$$

Let us use induction to show that  $FP$  is countable.

**Base case:** for  $n = 1$ , we get  $FP = A_1$ .  $A_1$  is countable by definition and thus,  $FP$  is clearly countable.

**Inductive step:** suppose that  $A_1 \times A_2 \times A_3 \times \dots \times A_{n-1}$  is countable and prove that  $A_1 \times A_2 \times A_3 \times \dots \times A_{n-1} \times A_n$  is countable as well (with  $A_1, A_2, A_3, \dots, A_{n-1}, A_n$  all being countable).

Let  $A' = A_1 \times A_2 \times A_3 \times \dots \times A_{n-1}$ . Then we have to show that  $A' \times A_n$  is countable where  $A'$  and  $A_n$  are both countable. Now, recall that we have already shown in (a) that the product of two countable sets is countable and thus,  $A' \times A_n$  is countable. Hence, we have shown that the finite product of countable sets is countable.

□

- (c) This is false. Suppose, for the sake of contradiction, that a countable product of countable sets is countable and let this product be  $P = A_1 \times A_2 \times A_3 \times \dots$ . Then there must exist an enumeration of  $P$ :

$$P_1 = p_{11}, p_{12}, p_{13}, p_{14}, \dots$$

$$P_2 = p_{21}, p_{22}, p_{23}, p_{24}, \dots$$

$$P_3 = p_{31}, p_{32}, p_{33}, p_{34}, \dots$$

$$P_4 = p_{41}, p_{42}, p_{43}, p_{44}, \dots$$

$$\dots\dots\dots$$

where  $p_{ij}$  is the  $j^{th}$  element of  $P_i$ .

Then, if  $P$  is countable, this enumeration should contain all elements of  $P$ .

Now, let us define a sequence  $P'$  s.t. the following holds:

$$P' = p'_1, p'_2, p'_3, p'_4, \dots$$

where

$$p'_1 \neq p_{11}$$

$$p'_2 \neq p_{22}$$

$$p'_3 \neq p_{33}$$

$$\dots\dots\dots$$

$$p'_{n-1} \neq p_{n-1}$$

$$p'_n \neq p_{nn}$$

$$\dots\dots\dots$$

Then  $P' \neq P_1$  since  $p'_1 \neq p_{11}$ ,  $P' \neq P_2$  since  $p'_2 \neq p_{22}$ ,  $P' \neq P_3$  since  $p'_3 \neq p_{33}$ , etc. Hence, we have effectively constructed a sequence  $P'$  that is not in  $P$  and we face a contradiction since we assumed that  $P$  enumerates all sequences. Thus, a countable product of countable sets is not countable.

□

3. Suppose  $\sum a_n$  converges conditionally. Then notice that the following stands (thanks for the hint!):

$$p_n = \frac{a_n + |a_n|}{2}$$

$$q_n = \frac{a_n - |a_n|}{2}$$

Now, let us split our proof in two parts: first prove that  $p_n$  diverges and then show that  $q_n$  diverges as well.

Let us first prove that  $p_n$  diverges. Now, suppose for the sake of contradiction, that  $p_n$  converges. Then  $p_n = \frac{a_n + |a_n|}{2}$  and it follows that  $|a_n| = 2p_n - a_n$ . We have:

$$\sum |a_n| = \sum 2p_n - \sum a_n$$

Since  $a_n$  and  $p_n$  converges, we know that the addition/subtraction/multiplication by scalar of convergent sequence will yield a convergent sequence (proved in the homework). Finally, we get that  $|a_n|$  converges absolutely and we face a contradiction since it converges conditionally.  $\square$

Similarly, we can prove that  $q_n$  diverges as well. Suppose for the sake of contradiction, that  $q_n$  converges. Then  $q_n = \frac{a_n - |a_n|}{2}$  and it follows that  $|a_n| = a_n - 2q_n$ . We get:

$$\sum |a_n| = \sum a_n - \sum 2q_n$$

Since  $a_n$  and  $q_n$  converges, we know that the addition/subtraction/multiplication by scalar of convergent sequence will yield a convergent sequence (proved in the homework). Finally, we get that  $|a_n|$  converges absolutely and we face a contradiction since it converges conditionally.  $\square$

Hence, we have proven that if  $a_n$  converges conditionally, then both  $p_n$  and  $q_n$  must diverge.  $\square$

4. Let  $(a_n) = (\sqrt{5}, \sqrt{5\sqrt{5}}, \sqrt{5\sqrt{5\sqrt{5}}}, \dots)$ .

Now, it is easy to notice the pattern:

$$\begin{array}{lll}
a_1 = \sqrt{5} & = a_1 & = 5^{\frac{1}{2}} = 5^{1-\frac{1}{2^1}} \\
a_2 = \sqrt{5\sqrt{5}} & = \sqrt{5a_1} & = 5^{\frac{3}{4}} = 5^{1-\frac{1}{2^2}} \\
a_3 = \sqrt{5\sqrt{5\sqrt{5}}} & = \sqrt{5a_2} & = 5^{\frac{7}{8}} = 5^{1-\frac{1}{2^3}} \\
a_4 = \sqrt{5\sqrt{5\sqrt{5\sqrt{5}}}} & = \sqrt{5a_3} & = 5^{\frac{15}{16}} = 5^{1-\frac{1}{2^4}} \\
\dots\dots\dots & \dots\dots\dots & \dots\dots\dots
\end{array}$$

Then the recursive definition of the sequence can be written as follows:

$$a_{n+1} = \sqrt{5a_n}$$

Additionally, the formula for the  $n^{th}$  element of the sequence is  $a_n = 5^{1-\frac{1}{2^n}}$ . Then we have:

$$\lim_{x \rightarrow \infty} a_n = \lim_{x \rightarrow \infty} 5^{1-\frac{1}{2^n}} = 5^{1-0} = 5^1 = 5$$

We now need to prove that  $\lim_{n \rightarrow \infty} a_n = 5$  (the sequences converges to 5).

Let  $l = \frac{\epsilon}{2} + 5$  and  $k = \frac{1}{1 - \log_5^l}$ . We then define  $N \in \mathbb{N} = \left\lceil \log_2^k \right\rceil$  ( $\left\lceil \frac{a}{b} \right\rceil$  is the **ceiling** of  $\frac{a}{b}$ ).

Then  $\forall \epsilon > 0$  and  $\forall n \geq N$  and we have:

$$\begin{aligned}
|a_n - 5| &= \left| 5^{1-\frac{1}{2^n}} - 5 \right| \\
&\leq \left| 5^{1-\frac{1}{2^N}} - 5 \right| = \left| 5^{1-\frac{1}{k}} - 5 \right| \\
&= \left| 5^{1-(1-\log_5^l)} - 5 \right| = \left| 5^{\log_5^l} - 5 \right| = \left| 5^{\log_5^l} - 5 \right| \\
&= \left| l - 5 \right| = \left| \frac{\epsilon}{2} + 5 - 5 \right| = \left| \frac{\epsilon}{2} \right| \\
&\leq \frac{\epsilon}{2} \\
&< \epsilon
\end{aligned}$$

We have now shown that  $\forall \epsilon > 0, n \geq N, |a_n - 5| < \epsilon$  and thus,  $\lim_{n \rightarrow \infty} a_n = 5$ .

□

Finally, we have that the recursive definition of the sequence is  $a_{n+1} = \sqrt{5a_n}$ , the limit of the sequence is 5, and we have also proved this fact.

5. Suppose, for the sake of contradiction, that every convergent subsequence of the sequence  $(x_n)$  converges to the same value  $L$ , but the sequence does not converge to  $L$ . Since  $(x_n)$  does not converge to  $L$ , it follows that  $\exists n \geq N$  s.t.  $\forall \epsilon > 0, |x_n - L| \geq \epsilon$ . Thus, we have found a subsequence  $(x_{n_t})$  s.t.  $\forall n \geq N, |x_{n_t} - L| \geq \epsilon$ . Let us now find a subsequence that is not in the  $\epsilon$ -neighborhood of  $L$ . We proceed by constructing a subsequence in the following manner:

$$\left\{ \begin{array}{l} \text{If } N = 1, n_1 = 1 \text{ with } a_{n_1} \text{ not in the } \epsilon\text{-neighborhood of } L \\ \text{If } N = 2, n_2 = \max(n_1 + 1, 2) \text{ with } a_{n_2} \text{ not in the } \epsilon\text{-neighborhood of } L \\ \text{If } N = 3, n_3 = \max(n_2 + 1, 3) \text{ with } a_{n_3} \text{ not in the } \epsilon\text{-neighborhood of } L \\ \text{If } N = 4, n_4 = \max(n_3 + 1, 4) \text{ with } a_{n_4} \text{ not in the } \epsilon\text{-neighborhood of } L \\ \dots\dots\dots \end{array} \right.$$

This way, we end up with a subsequence  $x_{n_k}$  s.t.  $\forall k \in \mathbb{N}, x_{n_k}$  is not in the  $\epsilon$ -neighborhood of  $L$ . Furthermore, since  $(x_n)$  is bounded, it follows that  $(x_{n_k})$  is also bounded. Now, per **Bolzano-Weierstrass Theorem**, we get that  $(x_{n_k})$  must contain some convergent subsequence. Thus, we have found a convergent subsequence  $(x_{n_{k_j}})$  that converges to  $L$  (by definition) and we face a contradiction since the subsequence was constructed in a way that none of its terms are in the  $\epsilon$ -neighborhood of  $L$ . Hence, if  $(x_n)$  is a bounded sequence of real numbers such that every convergent subsequence of  $(x_n)$  converges to the same value  $L$ ,  $(x_n)$  also converges to  $L$ .

□

6. (a) Let  $f : A \rightarrow B : x \mapsto (x, 0.25)$ . Then this function is one-to-one. However,  $f$  is not onto.

Let us first show that it is one-to-one. Suppose, for the sake of contradiction, that  $f$  is not one-to-one. Then  $\exists x_1 \neq x_2$  s.t.  $f(x_1) = f(x_2)$ . We have:

$$\begin{aligned} f(x_1) &= f(x_2) \\ (x_1, 0.25) &= (x_2, 0.25) \\ x_1 &= x_2 \end{aligned}$$

Hence, we got that  $x_1 = x_2$  and we face a contradiction since we assumed  $x_1 \neq x_2$ . Thus,  $f : A \rightarrow B : x \mapsto (x, 0.25)$  is one-to-one.

□

On the other hand,  $f$  is not onto since there is no value of  $x$  such that  $f(x) = (x, 0.5)$ .

Finally, we found a function  $f : A \rightarrow B : x \mapsto (x, 0.25)$  that is one-to-one, but not onto.

- (b) Suppose  $(a, b)$  is an input to the function and we want to make the output unique as well. Let us consider the decimal expansions of  $a$  and  $b$ . We get  $a = 0.a_1a_2a_3\dots$  and  $b = 0.b_1b_2b_3\dots$  with  $a_1, b_1, a_2, b_2, \dots \in \mathbb{N}$ . We can then construct number  $c = 0.a_1b_1a_2b_2a_3b_3\dots$ . Now, since the decimal expansions of  $a$  and  $b$  are unique, the number built by alternating the digits in the decimal expansions is also unique. Thus,  $g : B \rightarrow A : (a, b) \mapsto c = g : B \rightarrow A : (0.a_1a_2a_3\dots, b_1b_2b_3\dots) \mapsto 0.a_1b_1a_2b_2a_3b_3\dots$  is one-to-one.

However, this function is not onto. Suppose, for the sake of contradiction that  $g$  is onto. Consider the following output in the codomain  $A$ :  $g(x) = 0.9b_19b_29b_3\dots$ . The only way for this to happen is if the function has the following form:  $g(0.999\dots, 0.b_1b_2b_3\dots)$ . Now, recall that  $0.999\dots = \frac{9}{9} = 1$ , but  $1 \notin A$  and hence, we face a contradiction. Thus,  $g$  is not onto.

□

Finally, we have found a function  $g : B \rightarrow A : (0.a_1a_2a_3\dots, b_1b_2b_3\dots) \mapsto 0.a_1b_1a_2b_2a_3b_3\dots$  that is one-to-one, but not onto.

- (c) Let  $u(x) : A \rightarrow \mathbb{R} : x \mapsto x + 1$  and let  $v(x) : \mathbb{R} \rightarrow A : x \mapsto \frac{1}{2^x + 1}$ . If we now show that both  $u : A \rightarrow \mathbb{R}$  and  $v : \mathbb{R} \rightarrow A$  are one-to-one, we have effectively shown that  $A \sim \mathbb{R}$ .

Let us first show that  $u(x) : A \rightarrow \mathbb{R} : x \mapsto y$  is one-to-one. Suppose, for the sake of

contradiction, that  $u$  is not one-to-one. The  $\exists x_1 \neq x_2$  s.t.  $u(x_1) = u(x_2)$ . We have:

$$u(x_1) = u(x_2)$$

$$x_1 + 1 = x_2 + 1$$

$$x_1 = x_2$$

Hence, we got that  $x_1 = x_2$  and we face a contradiction since we assumed  $x_1 \neq x_2$ .

Thus,  $u : A \rightarrow \mathbb{R} : x \mapsto x + 1$  is one-to-one.

□

Let us now prove that  $v(x) : \mathbb{R} \rightarrow A : x \mapsto \frac{1}{2^x + 1}$  is one-to-one. Suppose, for the sake of contradiction, that  $v$  is not one-to-one. The  $\exists x_1 \neq x_2$  s.t.  $v(x_1) = v(x_2)$ . We have:

$$v(x_1) = v(x_2)$$

$$\frac{1}{2^{x_1} + 1} = \frac{1}{2^{x_2} + 1}$$

$$2^{x_1} + 1 = 2^{x_2} + 1$$

$$2^{x_1} = 2^{x_2}$$

$$x_1 = x_2$$

Hence, we got that  $x_1 = x_2$  and we face a contradiction since we assumed  $x_1 \neq x_2$ .

Thus,  $u : \mathbb{R} \rightarrow A : x \mapsto \frac{1}{2^x + 1}$  is one-to-one.

□

Thus, we found functions  $u : A \rightarrow \mathbb{R}$  and  $v : \mathbb{R} \rightarrow A$  such that both  $u$  and  $v$  are one-to-one. Finally, by **Schroeder-Bernstein Theorem**, we get that  $A \sim \mathbb{R}$ .

□

- (d) Using the results obtained in (a) and (b), we get that  $A \sim (0, 1) \times (0, 1) \sim A \times A$ . Thus,  $A \sim A \times A$ . In (c), we have proved that  $A \sim \mathbb{R}$ . It follows that  $\mathbb{R} \sim \mathbb{R} \times \mathbb{R} \sim \mathbb{R}^2$ . Hence,  $\mathbb{R} \sim \mathbb{R}^2$ .

□