Real Analysis Exams

Exam №4

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1. (a) Placeholder.

   (b) Placeholder.

   (c) Placeholder.

2. Let $\epsilon > 0$ be given, x_0 be fixed, and let $\delta = \min\left(1, \dfrac{\epsilon}{5(1+2|x_0|)}\right)$. Then for $|x - x_0| < \delta$ we have:

$$\begin{aligned}
|f(x) - f(x_0)| &= |5x^2 + 3 - 5x_0^2 - 3| \\
&= |5(x^2 - x_0^2)| \\
&= 5|x - x_0||x - x_0 + 2x_0| \\
&< 5\delta(|x - x_0| + |2x_0|) \\
&< 5\delta(\delta + 2|x_0|) \\
&\leq 5\frac{\epsilon}{5(1+2|x_0|)}(\delta + 2|x_0|) \\
&\leq \frac{\epsilon}{1+2|x_0|}(1 + 2|x_0|) \\
&= \epsilon && \text{(Thus, } |f(x) - f(x_0)| < \epsilon\text{)}
\end{aligned}$$

Hence, $f(x) = 5x^2 + 3$ is continuous at each point $x_0 \in \mathbb{R}$.

$\square$

3. (a) Placeholder.

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   (c) Placeholder.

4. Placeholder.

5. Placeholder.

6. Placeholder.

7. Since f is continuous on $[a, b]$, it follows that f achieves both the absolute maximum M and the absolute minimum m in $[a, b]$. Suppose, without a loss of generality, that these points are c_1 and c_2 with $c_1 < c_2$. We then have:

$$m(b-a) \le \int_a^b f \le M(b-a) \tag{1}$$

$$m \le \frac{1}{b-a} \int_a^b f \le M \tag{2}$$

Now, it follows by **Intermediate Value Theorem** that $\exists c \in (c_1, c_2) \subset [a, b]$ s.t. $f(c) = \frac{1}{b-a} \int_a^b f$.

$\square$

8. Suppose, for the sake of contradiction, that f has Generalized Riemann integrals Q_1 and Q_2 with $Q_1 \neq Q_2$ and let $\epsilon > 0$ be given. Then,.it follows that $\exists \delta_1(x)$ s.t. $\forall \delta_1(x)$-fine tagged partitions, $|R(f, P) - Q_1| < \frac{\epsilon}{2}$. Similarly, $\exists \delta_2(x)$ s.t. $\forall \delta_2(x)$-fine tagged partitions, $|R(f, P) - Q_2| < \frac{\epsilon}{2}$. Now, let $\delta(x) = \min(\delta_1(x), \delta_2(x))$. It follows by **Theorem 8.1.5** that there exists a tagged partition $(P, \{c_k\})$ s.t. it is both $\delta_1(x)$-fine and $\delta_2(x)$-fine. We have:

$$|Q_1 - Q_2| \le |Q_1 - R(f, P)| + |R(f, P) - Q_2| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Hence, we got that $Q_1 = Q_2$ and we face a contradiction since we have assumed that $Q_1 \neq Q_2$. Finally, we conclude that if f has a generalized Riemann integral on $[a, b]$, then the value of the integral $\int_a^b f$ is unique.

$\square$

9. (a) Placeholder.

   (b) Placeholder.

   (c) Placeholder.

   (d) Placeholder.

   (e) Placeholder.

   (f) Placeholder.