Real Analysis

Assignment №11

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6.5.1 (a) Notice that $g(x) = \frac{(-1)^{n-1}x^n}{n} \leq x^n$. Then x^n is a geometric series $\sum_{n=1}^{\infty} x^n$ which converges absolutely on $(-1, 1)$. Therefore, $g(x)$ is defined on set $(-1, 1)$. Now, since every term in $g(x)$ is continuous, by uniform convergence $g(x)$ is also continuous. Furthermore, at point $x = 1$, we have:

$$\begin{aligned} g(1) &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \\ &= \frac{1}{1 \times 2} + \frac{1}{3 \times 4} + \frac{1}{5 \times 6} \dots \\ &\leq \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \\ &\leq \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \end{aligned}$$

Now, notice that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$ is a p-series with $p = 2$. Thus, $g(x)$ is also defined on $(-1, 1]$. If $x = -1$, we have $g(-1) = -(1 + \frac{1}{2} + \frac{1}{3} + \dots)$ which is a harmonic series that diverges. Hence, $g(x)$ diverges on $[-1, 1]$. Finally, if $|x| > 1, g(x)$ diverges since $\lim_{n\to\infty} \frac{(-x)^{n-1}}{n} \neq 0$.

(b) Notice that $g'(x) = 1 + (-x) + (-x)^2 + (-x)^3 + \dots$ which is a geometric series that converges if $|(-x)| < 1$. Hence, $g'(x)$ is defined on $(-1, 1)$. The sum of the series is $\frac{1}{1+x}$.

6.5.2 (a) Let $a_n = 0 \ \forall n \geq 0$.

(b) This is impossible since all power series converge at 0.

(c) Let $a_0 = 0$ and let $a_n = \frac{1}{n^2} \ \forall n \geq 1$. Then $\sum a_n x^n$ converges since $\frac{1}{n^2}$ converges absolutely if $x = \pm 1$ (we showed that $\frac{1}{n^2}$ converges multiple times over the course of the class). However, if $|x| > 1$, we have $\lim_{n\to\infty} \frac{x^n}{n^2} = \infty$ and the power series diverges.

(d) This is impossible since $\sum |a_n(-1)^n| = \sum |a_n(1)^n|$.

(e) Let a_n be the following:

$$a_n = \begin{cases} \frac{(-1)^{\frac{n}{2}}}{n} \text{ if } n \text{ is even} \\ 0 \text{ otherwise} \end{cases}$$

Then we have $\sum_{n=1}^{\infty} = \frac{(-1)^n x^{2n}}{2n}$ which converges conditionally at both $x \pm 1$.

6.5.7 (a) Let $\sum a_n x^n$ be a power series s.t. $a_n \neq 0$ and let $L = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$. Then, if $|x| < \frac{1}{L}$ and $L \neq 0$, we have:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \times |x| < L \times \frac{1}{L} = 1$$

Finally, by **Exercise 2.7.9 (Ratio Test)** $\sum a_n x^n$ converges if $x \in (-\frac{1}{L}, \frac{1}{L})$.

(b) Let $L = 0$. Then we have:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = L \times |x| = 0 < 1$$

It follows that $\sum a_n x^n$ converges $\forall x \in \mathbb{R}$.

(c) Let $L' = \lim_{n\to\infty} s_n$ where $s_n = \sup\left\{ \left| \frac{a_{k+1}}{a_k} \right| : k \geq n \right\}$. Now, $\forall \epsilon > 0, L' + \epsilon > 1, \exists N \in \mathbb{N}$ s.t. $|\frac{a_n}{a_{n+1}}| < L' + \epsilon \ \forall n \geq N$. For $x \in (-\frac{1}{L}, \frac{1}{L})$, let $0 < \delta < \frac{1}{L'} - |x|$. We have:

$$\left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| < (L' + \epsilon)(\frac{1}{L'} - \delta) = 1 + \epsilon \times \left(\frac{1}{L'} - \delta \right) - \delta L'$$

Now, choose ϵ s.t. the following holds:

$$\epsilon < \frac{\delta L'}{\frac{1}{L'} - \delta} \text{ with } L' + \epsilon < 1$$

Then $|\frac{a_{n+1}x^{n+1}}{a_n x_n}| < 1 \ \forall n \geq N$ and thus, $\sum_1^{\infty} a_n x^n$ converges.

6.5.8 (a) Suppose that $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$ converges in an interval $(-R, R)$. Let $x = 0$, then $a_0 = b_0$. It follows by **Theorem 6.5.6** that the differentiated series also converges and hence for $x = 0$, the equality of the differentiated series gives us $a_1 + \sum_{n=2}^{\infty} na_n 0^{n-1} = b_1 + \sum_{n=2}^{\infty} nb_n 0^{n-1}$. Thus, we have $a_1 = b_1$. Then, once again, employing **Theorem 6.5.6**, we get that $a_2 = b_2$. If we continue in this fashion, we get $a_n = b_n \; \forall n = 0, 1, 2, \ldots$.

$\square$

6.6.5 (a) Let $f(x) = e^x$. Then we have $f^n(x) = e^x$ and the Taylor coefficients are of the form $a_n = \frac{e^0}{n!} = \frac{1}{n!}$. Thus, $f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$. Now, applying the **ratio test**, we get:

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{n!}{(n+1)!} \right| = 0$$

Hence, the Taylor series converges absolutely in all of $\mathbb{R}$. Furthermore, if we let $x = \pm 1$, it follows by the **Theorem 6.5.4 (Abel's Theorem)** that the Taylor series converges on $[-R, R]$.

(b) Using the fact that the series converges unifotmly on $[-R, R]$ (shown in (a)), $f'(x)$ can be given by differentiating every term in the sequence. We have:

$$f'(x) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=1}^{\infty} \frac{x^n}{n!} = f(x)$$

(c) Notice that e^{-c} is given by $f(x) = 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \cdots = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots$. Then we have:

$$\begin{aligned} e^x \times e^{-x} &= (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots) \times (1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots) \\ &= 1 + x(1-1) + x^2(\frac{1}{2!} + \frac{1}{2!} - 1) + x^3(\frac{1}{3!} - \frac{1}{3!} + \frac{1}{2!} - \frac{1}{2!}) \\ &= 1 + 0 + 0 + 0 + \ldots \\ &= 1 \end{aligned}$$

Hence, $e^x \times e^{-x} = 1$.