Real Analysis

Assignment №10

Instructor: Dr. Eric Westlund

David Oniani

Luther College

oniada01@luther.edu

January 12, 2021

6.2.1 (a)

$$\boxed{f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{1+nx^2} = \frac{1}{x}}$$

(b) Notice that

$$f(x) = \frac{1}{x} - \frac{1}{\frac{1}{nx} + x} = \frac{1}{nx^3 + x}$$

Then it is easy to see that the convergence is **not uniform** since $\forall \epsilon > 0$ and $\forall n \in \mathbb{N}$, we can always have $x = \frac{1}{2n}$ s.t. $\frac{1}{nx^3+x} = \frac{8n^2}{4n+1} > \epsilon$ which shows that $|f(x) - f_n(x)| > \epsilon$.

(c) Similar to (b), it is **not uniform** on $(0, 1)$ since $\forall \epsilon > 0$ and $\forall n \in \mathbb{N}$, we can always have $x = \frac{1}{2n}$ s.t. $\frac{1}{nx^3+x} = \frac{8n^2}{4n+1} > \epsilon$ which shows that $|f(x) - f_n(x)| > \epsilon$.

(d) We have:

$$\left|\frac{x}{nx^2+x} - \frac{1}{x}\right| = \left|\frac{1}{nx^3+x}\right| < \frac{1}{n}$$

Hence, $\forall \epsilon > 0$ and $\exists N$ s.t. $\forall n \geq N, \frac{1}{n} < \epsilon$ and $|f(x) - f_n(x) < \epsilon|$. Finally, we got that the convergence **is uniform** on $(1, \infty)$.

6.2.3 (a) We have:

$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \frac{x}{1+x^n} = x \text{ if } x < 1$$
$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \frac{x}{1+x^n} = \frac{1}{2} \text{ if } x = 1$$
$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \frac{x}{1+x^n} = 0 \text{ if } x > 1$$
$$\textit{and}$$
$$\lim_{n\to\infty} h_n(x) = 0 \text{ if } x = 0$$
$$\lim_{n\to\infty} h_n(x) = 1 \text{ if } x > 0$$

Hence, (g_n) converges pointwise to

$$g(x) = \begin{cases} x \text{ if } x < 1 \\ \frac{1}{2} \text{ if } x = 1 \\ 0 \text{ if } x > 1 \end{cases}$$

And (h_n) converges pointwise to

$$g(x) = \begin{cases} 0 \text{ if } x - 1 \\ 1 \text{ if } x > 1 \end{cases}$$

(b) It follows by **Theorem 6.2.6 (Continuous Limit Theorem)**, that both (g_n) (pick $x = 1$) and (h_n) (pick $x = 0$) do not converge uniformly on $[0, \infty)$.

(c) For (g_n) consider a half-open interval $[0, 1)$. Then we have:

$$|g_n(x) - g(x)| = \frac{x^n}{1+x^n} < x^n$$

Now, as $x < 1$, it follows that $\forall \epsilon > 0, \exists N$ s.t. $\forall n \geq N, x^n < \epsilon$ and therefore the convergence is uniform on $[0, 1)$.

For (h_n) consider a half-open interval $[1, \infty)$. Pick $N = 2$, then $\forall \epsilon > 0$ and $\forall n \geq N$ we have:

$$|h_n(x) - h(x)| = 1 - 1 = 0 < \epsilon$$

Hence, the convergence is uniform on $[1, \infty)$.

6.2.5 We first prove the the theorem directly and then its converse. Suppose (f_n) converges uniformly on A to some function f. Let $\epsilon > 0$. Then, by definition, $\exists N \in \mathbb{N}$ s.t. $\forall x \in A, n \geq N$ implies $|f_n(x) - f(x)| < \frac{\epsilon}{2}$. Now, if $m, n \geq N$, then, by the triangle inequality, we have:

$$\begin{aligned} |f_n(x) - f_m(x)| &= |f_n(x) - f(x) + f(x) - f_m(x)| \\ &\leq |f_n(x) - f(x)| + |f(x) - f_m(x)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &< \epsilon \end{aligned}$$

Hence, $|f_n(x) - f_m(x) < \epsilon|$.

□

Conversely, suppose that $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall x \in A$ and $\forall m, n \geq N, |f_n(x) - f_m(x)| < \epsilon$. Now, notice that $(f_n(x))$ is a Cauchy sequence and per **Theorem 2.6.4 (Cauchy Criterion)**, it converges. Now, since this is true $\forall x \in A$, we can define $f(x) = \lim_{n\to\infty}(f_n(x))$ and we now have to show that f_n converges to f uniformly. Let $\epsilon > 0$ be given. Then we know that $\exists N \in \mathbb{N}$ s.t. $x \in A$ and $\forall m, n \geq N, |f_n(x) - f_m(x)| < \epsilon$. Then it follows by the **Algebraic Limit Theorem** that $\lim_{m\to\infty} f_n(x) - f_m(x) = f_n(x) - f_m(x)$. Finally, per the **Order Limit Theorem**, we get that for $x \in A$ and $\forall n \geq N, |f_n(x) - f(x)| < \epsilon$. Hence, we got that f_n uniformly converges to f.

□

Finally, we have shown that a sequence of functions (f_n) defined on a set $A \subseteq \mathbb{R}$ converges uniformly on A if and only if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \epsilon$ whenever $m, n \geq N$ and $x \in A$.

□

6.3.1 (a) Let $\epsilon > 0$ be given. Then on the closed interval $[0, 1]$, we have:

$$|g_n(x) - g_m(x)| = \left|\frac{x^n}{n} - \frac{x^m}{m}\right|$$

Now, if we have $m, n > N$ with $\epsilon > \frac{1}{N}$, we then get:

$$|g_n(x) - g_m(x)| < \left|\frac{x^n}{n}\right| \leq \frac{1}{N} < \epsilon$$

It follows that (g_n) converges uniformly on $[0,1]$. We have:

$$0 \leq g(x) = \lim_{n\to\infty} \frac{x^n}{n} \leq \lim_{n\to\infty} \frac{1}{n} = 0$$

Thus, $\forall x \in [0,1], g(x) = 0$. Hence, $g(x)$ is differentiable on $[0,1]$ and $\forall x \in [0,1], g'(x) = 0$.

(b) Notice that $g_n'(x) = n \times \frac{x^{n-1}}{n} = x^{n-1}$. Now, notice that we have:

$$h(x) = \lim_{n\to\infty} g_n(x) = \begin{cases} 0 \text{ if } 0 \leq x \leq 1 \\ 1 \text{ if } x = 1 \end{cases}$$

Then the convergence is not uniform. To show this we set $x_n = \frac{1}{\sqrt[n]{3}}$. We get:

$$|g'_{n+1}(x_n) - h(x_n)| = \frac{1}{3}$$

Now $\forall n \in \mathbb{N}, \exists x \in [0,1]$ s.t. $|g_n(x) - g(x)| \geq \frac{1}{3}$ and therefore, the convergence is not uniform. Also, notice that g' and h are not the same since $g'(1) = 0$ and $h(1) = 1$.

6.3.3 (a) Notice that we have:

$$\Big(f_n(x)\Big)' = \frac{1 - nx^2}{(1+nx^2)^2}$$

Then the maximum and minimum should occur when $1 - nx^2 = 0$ or in other words, $x = \pm\frac{1}{\sqrt{n}}$. Now, we get $f'(\frac{1}{\sqrt{n}}) = \frac{1}{4\sqrt{n}}$ and $f'(-\frac{}{\sqrt{n}}) = -\frac{1}{4\sqrt{n}}$. Now, it follows that $\frac{1}{4\sqrt{n}}$ is the maximum since $f_n(0) = 0$ and if $x > 0, f_n(x) > 0$. Moreover, notice that if $x < \frac{1}{4\sqrt{n}}, f'(x) > 0$ and thus, f is increasing. Similarly, if $x > \frac{1}{4\sqrt{n}}, f'(x) < 0$ and f is decreasing. Therefore, $\frac{1}{4\sqrt{n}}$ is the maximum. By the similar argument, $-\frac{1}{4\sqrt{n}}$ is the minimum. Now, let $\epsilon > 0$ be given and choose $N = \frac{1}{16\epsilon^2}$. Then for $n > N$, we have $n > \frac{1}{16\epsilon^2} \to \epsilon > \frac{1}{4\sqrt{n}}$. Thus, $\forall x \in \mathbb{R}$, we get $|f_n(x) - 0| \leq \frac{1}{4\sqrt{n}} < \epsilon$ and it follows that f_n converges uniformly to 0.

$\square$

The limit function is $f(x) = \lim_{n\to\infty} \frac{x}{1+nx^2} = 0$.

(b) From the part (a), we have:

$$\Big(f_n(x)\Big)' = \frac{1 - nx^2}{(1+nx^2)^2} = \frac{1 - nx^2}{n^2x^4 + 2nx^2 + 1}$$

Now, assuming $x \neq 0$, we can divide both the numerator and the denominator by n^2 (given $n \neq 0$) and get:

$$\Big(f_n(x)\Big)' = \frac{\frac{1}{n^2} - \frac{x^2}{n}}{\frac{1}{n^2} + 2\frac{x^2}{n} + x^4}$$

Since $\forall m \in \mathbb{N}, \lim \frac{1}{n^m} = 0$, it follows by the **Algebraic Limit Theorem** that $\lim f_n'(x) = 0$. Now, suppose that $x = 0$. Then $f_n' = 0$ and if $x \neq 0$, we get $\lim f_n'(x) = f'(x)$.

6.4.3 (a) Let $k \in \mathbb{R}$ be fixed. Then we have:

$$\begin{aligned}
|g(x) - g(k)| &= \Big|\sum_{n=1}^{\infty} \frac{\cos(2^n x)}{2^n} - \sum_{n=1}^{\infty} \frac{\cos(2^n k)}{2^n}\Big| \\
&= \Big|\sum_{n=1}^{\infty} \frac{\cos(2^n x) - \cos(2^n k)}{2^n}\Big| \\
&= \Big|\sum_{n=1}^{\infty} 2\sin\Big(\frac{2^n(x+k)}{2}\Big) \times \sin\Big(\frac{2^n(k-x)}{2}\Big) \times \frac{1}{2^n}\Big| \\
&= \Big|\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} \sin(2^{n-1}(x+k)) \sin(2^{n-1}(k-x))\Big|
\end{aligned}$$

And by applying the triangle inequality, we get:

$$|g(x) - g(k)| \leq \Big|\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} \sin(2^{n-1}(x+k)) \sin(2^{n-1}(k-x))\Big| \quad \leq \sum_{n=1}^{\infty} \frac{1}{2^{n-1}|\sin(2^{n-1}(k-x))|}$$

Hence, $g(x)$ is continuous on all of $\mathbb{R}$.

□

6.4.5 (a) Let $h_n(x) = \frac{x^n}{n^2}$. Then $h_n(x)$ is continuous on the closed interval $[-1, 1]$. Furthermore, $\frac{x^n}{n^2} \leq \frac{1}{n^2}$ and since $\sum_{n=1}^{\infty} n^{-2}$ converges, by **Corollary 6.4.5 (Weierstrass M-Test)**, $\sum_{n=1}^{\infty} h_n(x)$ also converges. Finally, by **Theorem 6.4.2 (Term-by-term Continuity Theorem)**, $h(x)$ is continuous on $[-1, 1]$.