This exam is due **Wednesday, January 27, at 11:00 p.m.** Write your solutions in LaTeX or scan your neatly handwritten solutions and upload a pdf to this folder. You may refer to your notes, homework, and textbook. Ask me for clarification on the wording of problems. **Do not discuss the exam with anyone else or use any outside sources.**

1. (4 pts) Define a sequence $a_n = \left(4 + \frac{1}{n}\right)\cos\left(\frac{n\pi}{4}\right)$.

   (a) What are lim sup a_n and lim inf a_n?

   (b) Give an example of a convergent subsequence of (a_n).

   (c) Determine the set A of all real numbers a for which there exists a subsequence of (a_n) converging to a.

2. (4 pts) Use the ϵ-δ definition of continuity to prove that $f(x) = 5x^2 + 3$ is continuous at each point $x_0 \in \mathbb{R}$.

3. (9 pts) Suppose f_n is integrable on $[a, b]$, $\forall\, n \in \mathbb{N}$. Prove the following:

   (a) If $f_n \to f$ pointwise on $[a, b]$, then f is not necessarily integrable on $[a, b]$.

   (b) If $f_n \to f$ uniformly on $[a, b]$, then f is integrable on $[a, b]$.

   (c) If $f_n \to f$ uniformly on $[a, b]$, then $\lim\limits_{n\to\infty} \int_a^b f_n = \int_a^b f$.

4. (4 pts) Define $g(x) = \begin{cases} 1 & \text{if } x \neq 1 \\ 2 & \text{if } x = 1. \end{cases}$ Let $G(x) = \int_0^x g$. Is G differentiable at 1?

5. (4 pts) Define $h(x) = \begin{cases} 1 & \text{if } x < 1 \\ 2 & \text{if } x \geq 1. \end{cases}$ Let $H(x) = \int_0^x h$. Is H differentiable at 1?

6. (4 pts) Let $F(x) = \int_0^{x^2} t^2 \sin(t^2)\, dt$. Find $F'(x)$; carefully justify each step.

7. (6 pts) Prove that if f is continuous on $[a, b]$, then $\exists\, c \in (a, b)$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f$$

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8. (6 pts) Prove Theorem 8.1.7: If f has a generalized Riemann integral on $[a, b]$, then the value of the integral $\int_a^b f$ is unique.

9. (9 pts) Consider the sequence of functions $f_n \colon [0,1] \to \mathbb{R}$ defined as follows:

$$f_0(x) = x$$

$$f_1(x) = \begin{cases} 3x/2 & \text{if} \quad 0 \leq x \leq 1/3 \\ 1/2 & \text{if} \quad 1/3 < x < 2/3 \\ (3x-1)/2 & \text{if} \quad 2/3 \leq x \leq 1 \end{cases}$$

$$f_2(x) = \begin{cases} f_1(3x)/2 & \text{if} \quad 0 \leq x \leq 1/3 \\ f_1(x) & \text{if} \quad 1/3 < x < 2/3 \\ (f_1(3x-2)+1)/2 & \text{if} \quad 2/3 \leq x \leq 1 \end{cases}$$

(a) Sketch the graphs of f_0, f_1, and f_2.

(b) Write a piecewise definition for f_3 and graph it.

(c) How are these functions related to the Cantor set?

(d) Consider the sequence of functions f_n obtained by iterating this process. Let $f(x) = \lim_{n\to\infty} f_n(x)$. Show that f is continuous on $[0, 1]$.

(e) What is $f'(x)$? Justify your answer.

(f) What is $\int_0^1 f$? Justify your answer.