Real Analysis

Assignment №1

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1.2.5 (a) Suppose $A, B \subseteq \mathbb{R}$ and $x \in (A \cap B)^c$. Then $x \notin A \cap B$. Hence, $x \in A^c$ or $x \in B^c$. Therefore, $x \in A^c \cup B^c$. Finally, we get that $(A \cap B)^c \subseteq A^c \cup B^c$.

$\square$

(b) Suppose $A, B \subseteq \mathbb{R}$ and $x \in A^c \cup B^c$. Then $x \in A^c$ or $x \in B^c$, which is equivalent to $x \notin A$ or $x \notin B$. Hence, $x \notin A \cap B$. Therefore, $x \in (A \cap B)^c$ and it follows that $(A \cap B)^c \supseteq A^c \cup B^c$. Finally, since we have already proven that $(A \cap B)^c \subseteq A^c \cup B^c$, we get that $(A \cap B)^c = A^c \cup B^c$.

$\square$

(c) Let us first prove that $A^c \cap B^c \subseteq (A \cup B)^c$.

Suppose $A, B \subseteq \mathbb{R}$ and $x \in A^c \cap B^c$. Then $x \in A^c$ and $x \in B^c$. Thus, $x \notin A$ and $x \notin B$. It follows that $x \notin (A \cup B)$. Hence, $x \in (A \cup B)^c$ and $A^c \cap B^c \subseteq (A \cup B)^c$.

$\square$

We have to now prove the converse. Let us now prove that $(A \cup B)^c \subseteq A^c \cap B^c$.

Suppose $A, B \subseteq \mathbb{R}$ and $x \in (A \cup B)^c$. Then $x \notin A \cup B$. Therefore, $x \notin A$ and $x \notin B$. It follows that $x \in A^c$ and $x \in B^c$. Thus, $x \in A^c \cap B^c$ and $(A \cup B)^c \subseteq A^c \cap B^c$.

$\square$

Finally, since $A^c \cap B^c \subseteq (A \cup B)^c$ and $(A \cup B)^c \subseteq A^c \cap B^c$, we get that $(A \cup B)^c = A^c \cap B^c$.

$\square$

1.2.8 (a) Let us define $f : \mathbb{N} \to \mathbb{N} : x \mapsto 2x + 1$. We can now prove that f is $1-1$. Suppose, for the sake of contradiction, that $x_1, x_2 \in \mathbb{N}$ with $x_1 \neq x_2$ and $f(x_1) = f(x_2)$. Then $f(x_1) = f(x_2) \implies 2x_1 + 1 = 2x_2 + 1 \implies x_1 = x_2$. Hence, we face the contradiction and f is $1-1$. However, f is not onto since for $f(x) = 2$, the solution is 0.5 which is not in $\mathbb{N}$. Thus, f is a function that is $1-1$, but not onto.

$\square$

(b) Let us define $f : \mathbb{N} \to \mathbb{N} : x \mapsto \lfloor \frac{x}{4} \rfloor$. Then f is onto since since $\forall x \in \mathbb{N}$, $\exists k = 4x$ with $f(k) = x$. However, f is not $1-1$ since $f(2) = f(3)$. Thus, f if a function that is onto, but not $1-1$.

$\square$

(c) Let us define

$$f : \mathbb{N} \to \mathbb{Z} : x \mapsto \begin{cases} \dfrac{x-1}{2}, & \text{if } n \text{ is odd} \\ -\dfrac{x}{2}, & \text{if } n \text{ is even} \end{cases}$$

Then f is both $1-1$ and onto. Let us first prove that f is $1-1$.

Suppose, for the sake of contradiction, that $x_1, x_2 \in \mathbb{N}$ with $x_1 \neq x_2$ and $f(x_1) = f(x_2)$. Then $f(x_1)$ and $f(x_2)$ must be of the same sign or both be zero. Thus, we have three cases:

1. $f(x_1) = \dfrac{x_1 - 1}{2}$ and $f(x_2) = \dfrac{x_2 - 1}{2}$
   $f(x_1) = f(x_2) \implies \dfrac{x_1 - 1}{2} = \dfrac{x_2 - 1}{2} \implies x_1 = x_2.$
   $\square$

2. $f(x_1) = -\dfrac{x_1}{2}$ and $f(x_2) = -\dfrac{x_2}{2}$
   $f(x_1) = f(x_2) \implies -\dfrac{x_1}{2} = -\dfrac{x_2}{2} \implies x_1 = x_2$
   $\square$

3. $f(x_1) = 0$ and $f(x_2) = 0$
   The only way for this to happen is if $f(x_1) = -\dfrac{x_1}{2}$ and $f(x_2) = -\dfrac{x_2}{2}$ and we get $f(x_1) = f(x_2) \implies -\dfrac{x_1}{2} = -\dfrac{x_2}{2} \implies x_1 = x_2$
   $\square$

$\square$

Now let us prove that f is onto. For $y \in \mathbb{Z}$, we have three cases:

1. y is positive.

   If $y > 0$, we can find $k = 2y + 1$ with $f(k) = y$.

   $\square$

2. y is negative.

   If $y < 0$, we can find $k = -2y$ with $f(k) = y$.

   $\square$

3. y is zero.

   If $y = 0$, we can find $k = 1$ with $f(k) = y = 0$.

   $\square$

$\square$

Finally, we have proven that f is both $1 - 1$ and onto.

$\square$

1.2.9 (a) Given, $f(x) = x^2$, $A = [0, 4]$, and $B = [-1, 1]$, we get:

$$f^{-1}(A) = \{x \mid f(x) \in A\} = [-2, 2]$$
$$f^{-1}(B) = \{x \mid f(x) \in B\} = [-1, 1]$$

We get $f^{-1}(A) = [-2, 2]$ as the negative values plugged into x^2 get positive, with 0 mapping to 0. Since the biggest value of A is 4, x cannot exceed 2, so we have an interval $[0, 2]$. And now, due to properties of $f(x) = x^2$, we add the negative part $[-2, 0)$. Finally, $[0, 2] \cup [-2, 0) = [-2, 2]$

Similarly, $f^{-1}(B) = [-1, 1]$ as $[0, 1] \cup [-1, 0) = [-1, 1]$.

$A \cap B = [0, 4] \cap [-1, 1] = [0, 1]$.

$$f^{-1}(A \cap B) = \{x \mid f(x) \in A \cap B\} = [-1, 1]$$
$$f^{-1}(A) \cap f{-1}(B) = [-2, 2] \cap [-1, 1] = [-1, 1]$$

Hence, $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

Additionally, we have:

$$f^{-1}(A \cup B) = \{x \mid f(x) \in A \cup B\} = [-2, 2]$$
$$f^{-1}(A) \cup f{-1}(B) = [-2, 2] \cup [-1, 1] = [-2, 2]$$

Hence, $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

(b) Let us now prove that the properties shown in (a) are completely general.

1. Suppose $g : \mathbb{R} \to \mathbb{R}$ is a function from reals to reals and $A, B \subseteq \mathbb{R}$. Prove that $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$.

   We have to show both $g^{-1}(A \cap B) \subseteq g^{-1}(A) \cap g^{-1}(B)$ and $g^{-1}(A) \cap g^{-1}(B) \subseteq g^{-1}(A \cap B)$.

   Suppose $x \in g^{-1}(A \cap B)$. Then $g(x) \in A \cap B$. It follows that $g(x) \in A$ and $g(x) \in B$. As a result, $x \in g^{-1}(A)$ and $x \in g^{-1}(B)$. Hence, $x \in g^{-1}(A) \cap g^{-1}(B)$ and $g^{-1}(A \cap B) \subseteq g^{-1}(A) \cap g^{-1}(B)$.

   □

   We now have to prove the converse. Suppose $x \in g^{-1}(A) \cap g^{-1}(B)$. Then $g(x) \in A$ and $g(x) \in B$. Hence, $g(x) \in A \cap B$. Therefore, $x \in g^{-1}(A \cap B)$ and $g^{-1}(A) \cap g^{-1}(B) \subseteq g^{-1}(A \cap B)$.

   □

   We have now shown that $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$

   □

2. Suppose $g : \mathbb{R} \to \mathbb{R}$ is a function from reals to reals and $A, B \subseteq \mathbb{R}$. Prove that $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$.

   We have to show both $g^{-1}(A \cup B) \subseteq g^{-1}(A) \cup g^{-1}(B)$ and $g^{-1}(A) \cup g^{-1}(B) \subseteq g^{-1}(A \cup B)$.

   Suppose $x \in g^{-1}(A \cup B)$. Then $g(x) \in A \cup B$. It follows that $g(x) \in A$ or $g(x) \in B$. As a result, $x \in g^{-1}(A)$ or $x \in g^{-1}(B)$. Hence, $x \in g^{-1}(A) \cup g^{-1}(B)$ and $g^{-1}(A \cup B) \subseteq g^{-1}(A) \cup g^{-1}(B)$.

   □

We now have to prove the converse. Suppose $x \in g^{-1}(A) \cup g^{-1}(B)$. Then $g(x) \in A$ or $g(x) \in B$. Hence, $g(x) \in A \cup B$. Therefore, $x \in g^{-1}(A \cup B)$ and $g^{-1}(A) \cup g^{-1}(B) \subseteq g^{-1}(A \cup B)$.

□

We have now shown that $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$

□

1.3.3 (a) Because $B = \{b \in \mathbb{R} \mid b \text{ is a lower bound for } A\}$, we have that $\inf A \in B$ as $\inf A$ is also a lower bound for A. Thus, B cannot be empty. Furthermore, $\forall a \in A$ and $\forall b \in B$, $b \leq a$. Hence, per the Axiom of Completeness, $\sup(B)$ must exist. Now, since $\inf A$ is the greatest lowest bound for A, it follows that $\forall b \in B, b \leq \inf A$. Therefore, $\inf A$ is the maximum of B and $\inf A = \max B$. Finally, according to the book, *when a maximum exists, then it is also the supremum* and thus $\inf A = \sup B$.

□

(b) We have already shown in (a) that if A is bounded below, then there exists a greatest lower bound. Hence, the existence of the least upper bound implies the existence of the greatest lower bound and there is no need for the axiom to state it explicitly. Sometimes, implicit is better than explicit. The axiom is very minimalistic in design, I like it.

1.3.8 (a)

$$\inf A = \inf \{\frac{1}{n} \mid n \in \mathbb{N}\} = 0$$
$$\sup A = \sup \{\frac{n}{n+1} \mid n \in \mathbb{N}\} = 1$$

(b)

$$\inf A = \inf \{\frac{-1}{n} \mid n \in \mathbb{N}\} = -1$$
$$\sup A = \sup \{\frac{1}{n} \mid n \in \mathbb{N}\} = 1$$

(c) Notice that $\dfrac{n}{3n+1} = \dfrac{1}{3+\dfrac{1}{n}}$. We can divide by n since, by convention, $\mathbb{N}$ does not include 0 (in pure math that is). Then we have:

$$\inf A = \frac{1}{4}$$
$$\sup A = \frac{1}{3}$$

(d) Just like in (c), we, once again, divide both the numerator and the denominator by m and get $\dfrac{1}{1+\dfrac{n}{m}}$. We have:

$$\inf A = 0$$

$$\sup A = 1$$

1.4.1 (a) Let us first show that $ab \in \mathbb{Q}$. Suppose $a = \dfrac{m}{n}, b = \dfrac{p}{q} \in \mathbb{Q}$ s.t. $n, q \neq 0$ and $m, n, p, q \in \mathbb{Z}$. Then $ab = \dfrac{mp}{nq}$ and since $\mathbb{Z}$ is closed under multiplication (with $nq \neq 0$), we get that $ab \in \mathbb{Q}$.

$\square$

Let us now show that $a + b \in \mathbb{Q}$. Suppose $a = \dfrac{m}{n}, b = \dfrac{p}{q} \in \mathbb{Q}$ s.t. $n, q \neq 0$ and $m, n, p, q \in \mathbb{Z}$. Then $a + b = \dfrac{mq + pn}{nq}$ and since $\mathbb{Z}$ is closed under addition (with $nq \neq 0$), we get that $a + b \in \mathbb{Q}$.

$\square$

(b) Let us first show that $a + t \in \mathbb{I}$. Suppose, for the sake of contradiction, that $a = \dfrac{m}{n} \in \mathbb{Q}$ s.t. $m, n \neq 0$, $m, n \in \mathbb{Z}$, and $a + t = \dfrac{j}{k}$ with $k \neq 0$. Then $t = \dfrac{j}{k} - \dfrac{m}{n}$ and thus, $t \in \mathbb{Q}$. Hence, we face a contradiction and $a + t$ is indeed irrational, $a + t \in \mathbb{I}$.

$\square$

Let us now show that $at \in \mathbb{I}$. Suppose, for the sake of contradiction, that $a = \dfrac{m}{n} \in \mathbb{Q}$ s.t. $m, n \neq 0$, $m, n \in \mathbb{Z}$, and $at = \dfrac{j}{k}$ with $k \neq 0$. Then $t = \dfrac{jn}{km}$ and thus, $t \in \mathbb{Q}$. Hence, we face a contradiction and at is indeed irrational, $at \in \mathbb{I}$.

$\square$

(c) The set of irrational numbers is not closed under addition. For example, take $s = \sqrt{3}$ and $t = -\sqrt{3}$, then $s + t = 0$ is not irrational. However, in some cases, it is closed under addition. If $s = \sqrt{2}$ and $t = \sqrt{3}$, then $s + t$ is irrational ($\sqrt{2} + \sqrt{3}$ is irrational).

The set of irrational numbers is not closed under multiplication either. For instance, take $s = \sqrt{3}$ and $t = -\sqrt{3}$ again. Then $st = -3$ which is not irrational. In some cases, much alike addition, it is also closed under multiplication. If $s = \sqrt{2}$ and $t = \sqrt{3}$, then st is irrational ($\sqrt{6}$ is irrational).

1.4.3 Let us denote $\bigcap_{n=1}^{\infty}(0, 1/n)$ as S. Suppose, for the sake of contradiction, that $x \neq \emptyset \in S$. Then, by the definition of $S, x \neq 0$. Hence, $x > 0$. Then, per the **Archimedean Property**, $\exists y \in \mathbb{N}$ s.t. $\dfrac{1}{y} < x$. It follows that $\forall n \geq y, x \notin (0, 1/n)$. Thus, we face a contradiction and $S = \emptyset$.

$\square$

1.5.9 (a) Let us first show that $\sqrt{2}$ is algebraic.

Notice that $\sqrt{2}$ is one of the roots of the equation $x^2 - 2 = 0$. Hence, $\sqrt{2}$ is algebraic.

$\square$

Let us now show that $\sqrt[3]{2}$ is algebraic.

Notice that $\sqrt[3]{2}$ is the root of the equation $x^3 - 2 = 0$. Hence, $\sqrt[3]{2}$ is algebraic.

$\square$

Finally, let us prove that $\sqrt{3} + \sqrt{2}$ is algebraic

Using **Vieta's formulas**, it is easy to see that $\sqrt{3}+\sqrt{2}$ is one of the roots to the equation $x^2 - 2\sqrt{3}x + 1 = 0$. Hence, $\sqrt{3} + \sqrt{2}$ is algebraic.

$\square$

(b) Suppose A_n with $n \in \mathbb{N}$ is the set of algebraic numbers obtained as roots of polynomials with integer coefficients that have degree n. Then let us denote A_{ny} as the set of algebraic numbers s.t. $\forall x \in A_{ny}, x \leq y$ with $y \in \mathbb{N}$. It follows that $\forall y \in \mathbb{N}, A_{ny}$ is countable. Furthermore, $\forall y \in \mathbb{N}, A_{ny}$ is finite. A_n can then be represented as union of many A_{ny} sets:

$$A_n = \bigcup_{y=1}^{\infty} A_{ny}$$

Now, notice that every set in the union must be countable. Additionally, A_n is countable as well. Finally, since the countable union of countable sets is also countable, we conclude that A_n is countable.

$\square$

NOTE: The fact that the coountable union of countable sets is countable can be proven visually. One will need to arrange the sets in the form of the 2D matrix and perform enumeration diagonal-by-diagonal (of the small squares).

(c) Suppose S is the set of all algebraic numbers. In (b), we showed that $\forall n \in \mathbb{N}, A_n$ is countable. Now, recall that the countable union of countable sets is countable. Then, since $S = \bigcap_{n=1}^{\infty} A_n$, S is countable too. Therefore, the set of all algebraic numbers is also countable.

$\square$

The set of transcendental numbers, on the other hand, cannot be countable. The set of real numbers $\mathbb{R}$ is a union of algebraic and transcendental numbers. We already know that the set of algebraic numbers is countable and the set of real numbers $\mathbb{R}$ is uncountable. Since $\mathbb{R}$ is uncountable and the set of algebraic numbers is countable, it must be transcendental numbers that contribute to its uncountability. Thus, the set of transcendental numbers is uncountable.

1.6.4 Suppose, for the sake of contradiction, that $S = \{(a_1, a_2, a_3 \cdots \mid a_n = 0 \text{ or } 1\}$ is countable. Then there must exist an enumeration of these numbers:

$$s_1 = (a_{11}, a_{12}, a_{13}, a_{14}, \dots)$$
$$s_2 = (a_{21}, a_{22}, a_{23}, a_{24}, \dots)$$
$$s_3 = (a_{31}, a_{32}, a_{33}, a_{34}, \dots)$$
$$s_4 = (a_{41}, a_{42}, a_{43}, a_{44}, \dots)$$
$$\dots\dots\dots\dots\dots\dots$$

Now, let us define a sequence of 0s and 1s t s.t. $\forall n \in t$, the following holds:

$$t_n = \begin{cases} 0 & \text{if } s_{nn} \text{ is } 1 \\ 1 & \text{if } s_{nn} \text{ is } 0 \end{cases}$$

with s_{ni} denoting the i^{th} element of S.

Then, since t is a sequence of 0s and 1s, it must be in S. Thus, $t \in S$. However, we have effectively constructed a sequence of numbers that cannot be in S as $\forall s \in S, s \neq t$ (first item in t is different from s_1, the second one is different from s_2, and so on). Thus, we face a contradiction and S cannot be enumerated. Hence, S is uncountable.

$\square$