Topology

Author: David Oniani
Instructor: Dr. Eric Westlund

January 7, 2019

Exercises

2. Let $f : A \to B$ and let $A_i \subset A$ and $B_i \subset B$ for $i = 0$ and $i = 1$. Show that if f^{-1} preserves inclusions, unions, intersections, and differences of sets:

   (c) $f^{-1}(B_0 \cap B_1) = f^{-1}(B_0) \cap f^{-1}(B_1)$.

   To prove that the set $f^{-1}(B_0 \cap B_1)$ is equal to the set $f^{-1}(B_0) \cap f^{-1}(B_1)$, we have to show that $f^{-1}(B_0 \cap B_1) \subseteq f^{-1}(B_0) \cap f^{-1}(B_1)$ and $f^{-1}(B_0) \cap f^{-1}(B_1) \subseteq f^{-1}(B_0 \cap B_1)$.

   Case I: $f^{-1}(B_0 \cap B_1) \subseteq f^{-1}(B_0) \cap f^{-1}(B_1)$

   Let $x \in B_0 \cap B_1$. Then $x \in B_0$ and $x \in B_1$. Besides, $f(x) = A_p$ where $A_p \subseteq A$ (the author calls it the preimage). Now, since $x \in B_0 \cap B_1$, its preimage is in $f^{-1}(B_0 \cap B_1)$. On the other hand, as $x \in B_0$, its preimage lies in $f^{-1}(B_0)$ and as $x \in B_1$, its preimage also lies in $f^{-1}(B_1)$. In other words, the preimage of x lies in $f^{-1}(B_0) \cap f^{-1}(B_1)$. Therefore, $f^{-1}(B_0 \cap B_1) \subseteq f^{-1}(B_0) \cap f^{-1}(B_1)$.

   Case II: $f^{-1}(B_0) \cap f^{-1}(B_1) \subseteq f^{-1}(B_0 \cap B_1)$

   Let $x_0 \in B_0$ and $x_1 \in B_1$. Then the preimages of x_0 and x_1 are in $f^{-1}(B_0)$ and $f^{-1}(B_1)$ respectively. Thus, x mapping to $f^{-1}(B_0) \cap f^{-1}(B_1)$ has the preimage that maps to both $f^{-1}(B_0)$ and $f^{-1}(B_1)$. In other words, $x \in B_0$ and $x \in B_1$ which means that the preimage of x also lies in $f^{-1}(B_0 \cap B_1)$. Therefore, $f^{-1}(B_0) \cap f^{-1}(B_1) \subseteq f^{-1}(B_0 \cap B_1)$.

   We have now proven that $f^{-1}(B_0 \cap B_1) \subseteq f^{-1}(B_0) \cap f^{-1}(B_1)$ and $f^{-1}(B_0) \cap f^{-1}(B_1) \subseteq f^{-1}(B_0 \cap B_1)$ and thus, $f^{-1}(B_0 \cap B_1) = f^{-1}(B_0) \cap f^{-1}(B_1)$.

   Q.E.D.

   (g) $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$; show that inequality holds if f is injective.

   Let $x \in A_0 \cap A_1$. Then $x \in A_0$ and $x \in A_1$.

   sadasdasd

   Q.E.D.