Real Analysis Exams

Exam №1

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1. Let $a_n = \dfrac{6n+1}{3n+2}$.

   Now, let $N \in \mathbb{N} = \left\lceil \dfrac{1+\epsilon}{\epsilon} \right\rceil \left(\left\lceil \dfrac{a}{b} \right\rceil \text{ is the } \textbf{ceiling} \text{ of } \dfrac{a}{b} \right)$.

   Then $\forall \epsilon > 0$ and $\forall n \geq N$ and we have:

$$
\begin{aligned}
|a_n - 2| &= \left| \frac{6n+1}{3n+2} - 2 \right| \\
&= \left| \frac{-3}{3n+2} \right| \\
&\leq \frac{3}{3N+2} \\
&\leq \frac{3}{3\frac{1+\epsilon}{\epsilon} + 2} = \frac{3\epsilon}{3 + 3\epsilon + 2\epsilon} = \frac{3\epsilon}{5\epsilon + 3} \\
&< \frac{3}{5}\epsilon \\
&< \epsilon
\end{aligned}
$$

   We have now shown that $\forall \epsilon > 0, n \geq N, |a_n - 2| < \epsilon$ and thus, $\lim\limits_{n \to \infty} a_n = 2$.

   □

2. (a) This is true. We can prove this visually. In order to prove that $A \times B$ is countable, it suffices to show that there exists an enumeration of this set.

First, recall that $A \times B = \{(a, b) : a \in A, b \in B\}$.

Now, let $P = A \times B, A = \{a_1, a_2, a_3, \dots\}$, and $B = \{b_1, b_2, b_3, \dots\}$.

We have:

$$P_1 = (a_1, b_1), (a_1, b_2), (a_1, b_3), (a_1, b_4), \dots$$
$$P_2 = (a_2, b_1), (a_2, b_2), (a_2, b_3), (a_2, b_4), \dots$$
$$P_3 = (a_3, b_1), (a_3, b_2), (a_3, b_3), (a_3, b_4), \dots$$
$$P_4 = (a_4, b_1), (a_4, b_2), (a_4, b_3), (a_4, b_4), \dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

We enumerate the sequence diagonal-by-diagonal (of small squares) as follows:

$$(a_1, b_1), (a_2, b_1), (a_1, b_2), (a_3, b_1), (a_2, b_2), (a_1, b_3), \dots$$

This way, all of the elements in P will eventually be listed and hence, we have successfully enumerated P. Thus, P is countable, which means that $A \times B$ is countable.

$\square$

(b) This is true. Let the finite product of countable sets be denoted as

$$\begin{aligned} FP &= A_1 \times A_2 \times A_3 \times \cdots \times A_{n-1} \times A_n \\ &= \{(a_1, a_2, a_3, \dots, a_{n-1}, a_n) \mid a_1 \in A_1, a_2 \in A_2, a_3 \in A_3, \dots, a_{n-1} \in A_{n-1}, a_n \in A_n\} \end{aligned}$$

Let us use induction to show that FP is countable.

Base case: for $n = 1$, we get $FP = A_1$. A_1 is countable by definition and thus, FP is clearly countable.

Inductive step: suppose that $A_1 \times A_2 \times A_3 \times \cdots \times A_{n-1}$ is countable and prove that $A_1 \times A_2 \times A_3 \times \cdots \times A_{n-1} \times A_n$ is countable as well (with $A_1, A_2, A_3, \dots, A_{n-1}, A_n$ all being countable.

Let $A' = A_1 \times A_2 \times A_3 \times \cdots \times A_{n-1}$. Then we have to show that $A' \times A_n$ is countable where A' and A_n are both countable. Now, recall that we have already shown in (a) that the product of two countable sets is countable and thus, $A' \times A_n$ is countable. Hence, we have shown that the finite product of countable sets is countable.

$\square$

(c) This is false. Suppose, for the sake of contradiction, that a countable product of countable sets is countable and let this product be $P = A_1 \times A_2 \times A_3 \times \dots$. Then there must exist an enumeration of P:

$$P_1 = p_{11}, p_{12}, p_{13}, p_{14}, \dots$$
$$P_2 = p_{21}, p_{22}, p_{23}, p_{24}, \dots$$
$$P_3 = p_{31}, p_{32}, p_{33}, p_{34}, \dots$$
$$P_4 = p_{41}, p_{42}, p_{43}, p_{44}, \dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

where p_{ij} is the j^{th} element of P_i.

Then, if P is countable, this enumeration should contain all elements of P.
Now, let us define a sequence p' s.t. the following holds:

$$p' = p'_1, p'_2, p'_3, p'_4, \dots$$

where

$$p'_1 \neq p_{11}$$
$$p'_2 \neq p_{22}$$
$$p'_3 \neq p_{33}$$
$$\dots\dots\dots\dots\dots\dots\dots$$
$$p'_{n-1} \neq p_{n-1}$$
$$p'_n \neq p_{nn}$$
$$\dots\dots\dots\dots\dots\dots\dots$$

Then $p' \neq P_1$ since $p'_1 \neq p_{11}$, $p' \neq P_2$ since $p'_2 \neq p_{22}$, $p' \neq P_3$ since $p'_3 \neq p_{33}$, etc. Hence, we have effectively constructed a sequence that is not in P and we face a contradiction since P has enumerated all sequences. Thus, a countable product of countable sets is not countable.

□

3. Suppose $\sum_{a_n}$ converges conditionally. Then notice that the following stands (thanks for the hint!):

$$p_n = \frac{a_n + |a_n|}{2}$$
$$q_n = \frac{a_n - |a_n|}{2}$$

Now, let us split our proof in two parts: first prove that p_n diverges and then show that q_n diverges as well.

Let us first prove that p_n diverges. Now, suppose for the sake of contradiction, that p_n converges. Then $p_n = \frac{a_n + |a_n|}{2}$ and it follows that $|a_n| = 2p_n - a_n$. We have:

$$\sum |a_n| = \sum 2p_n - \sum a_n$$

Since a_n and p_n converges, we know that the addition/subtraction/multiplication by scalar of convergent sequence will yield a convergent sequence (proved in the homework). Finally, we get that $|a_n|$ converges absolutely and we face a contradiction since it converges conditionally. $\square$

Similarly, we can prove that q_n diverges as well. Suppose for the sake of contradiction, that q_n converges. Then $q_n = \frac{a_n - |a_n|}{2}$ and it follows that $|a_n| = a_n - 2q_n$. We get:

$$\sum |a_n| = \sum a_n - \sum 2q_n$$

Since a_n and q_n converges, we know that the addition/subtraction/multiplication by scalar of convergent sequence will yield a convergent sequence (proved in the homework). Finally, we get that $|a_n|$ converges absolutely and we face a contradiction since it converges conditionally. $\square$

Hence, we have proven that if a_n converges conditionally, then both p_n and q_n must diverge. $\square$

4. Let $(a_n) = \left(\sqrt{5}, \sqrt{5\sqrt{5}}, \sqrt{5\sqrt{5\sqrt{5}}}, \dots\right)$.

   Now, it is easy to notice the pattern:

$$
\begin{aligned}
a_1 &= \sqrt{5} & &= a_1 & &= 5^{\frac{1}{2}} = 5^{1-\frac{1}{2^1}} \\
a_2 &= \sqrt{5\sqrt{5}} & &= \sqrt{5a_1} & &= 5^{\frac{3}{4}} = 5^{1-\frac{1}{2^2}} \\
a_3 &= \sqrt{5\sqrt{5\sqrt{5}}} & &= \sqrt{5a_2} & &= 5^{\frac{7}{8}} = 5^{1-\frac{1}{2^3}} \\
a_4 &= \sqrt{5\sqrt{5\sqrt{5\sqrt{5}}}} & &= \sqrt{5a_3} & &= 5^{\frac{15}{16}} = 5^{1-\frac{1}{2^4}} \\
&\dots\dots\dots\dots & &\dots\dots\dots\dots & &\dots\dots\dots\dots
\end{aligned}
$$

   Then the recursive definition of the sequence can be written as follows:

$$a_{n+1} = \sqrt{5a_n}$$

   Additionally, the formula for the n^{th} element of the sequence is $a_n = 5^{1-\frac{1}{2^n}}$. Then we have:

$$\lim_{x\to\infty} a_n = \lim_{x\to\infty} 5^{1-\frac{1}{2^n}} = 5^{1-0} = 5^1 = 5$$

   We now need to prove that $\lim_{n\to\infty} a_n = 5$ (the sequences converges to 5).
   Let $l = \frac{\epsilon}{2} + 5$ and $k = \frac{1}{1-\log_5^l}$. We then define $N \in \mathbb{N} = \left\lceil \log_2^k \right\rceil$ $\left(\left\lceil \frac{a}{b} \right\rceil \text{ is the } \textbf{ceiling} \text{ of } \frac{a}{b}\right)$.
   Then $\forall \epsilon > 0$ and $\forall n \geq N$ and we have:

$$
\begin{aligned}
|a_n - 5| &= \left|5^{1-\frac{1}{2^n}} - 5\right| \\
&\leq \left|5^{1-\frac{1}{2^N}} - 5\right| = \left|5^{1-\frac{1}{k}} - 5\right| \\
&= \left|5^{1-(1-\log_5^l)} - 5\right| = \left|5^{\log_5^l} - 5\right| = \left|5^{\log_5^l} - 5\right| \\
&= \left|l - 5\right| = \left|\frac{\epsilon}{2} + 5 - 5\right| = \left|\frac{\epsilon}{2}\right| \\
&\leq \frac{\epsilon}{2} \\
&< \epsilon
\end{aligned}
$$

   We have now shown that $\forall \epsilon > 0, n \geq N, |a_n - 5| < \epsilon$ and thus, $\lim_{n\to\infty} a_n = 5$.
   $\square$

   Finally, we have that the recursive definition of the sequence is $a_{n+1} = \sqrt{5a_n}$, the limit of the sequence is 5, and we have also proved this fact.

5. Suppose, for the sake of contradiction, that every convergent subsequence of the sequence (x_n) converges to the same value L, but the sequence does not converge to L. Since $x(n)$ does not converge to L, it follows that $\exists n \geq N$ s.t. $\forall \epsilon > 0, |x_n - L| \geq 0$. Thus, we have found a subsequence x_{n_t} s.t. $\forall n \geq N, |x_{nt} - L| \geq \epsilon$. Let us now find a subsequence that is not in the ϵ-neighborhood of L. We proceed by constructing a subsequence in the following manner:

$$\begin{cases} \text{If } N = 1, n_1 = 1 \text{ with } a_{n_1} \text{ not in the } \epsilon\text{-neighborhood of } L \\ \text{If } N = 2, n_2 = \max(n_1 + 1, 2) \text{ with } a_{n_2} \text{ not in the } \epsilon\text{-neighborhood of } L \\ \text{If } N = 3, n_3 = \max(n_2 + 1, 3) \text{ with } a_{n_3} \text{ not in the } \epsilon\text{-neighborhood of } L \\ \text{If } N = 4, n_3 = \max(n_3 + 1, 4) \text{ with } a_{n_4} \text{ not in the } \epsilon\text{-neighborhood of } L \\ \text{...} \end{cases}$$

This way, we end up with a subsequence x_{nk} s.t. $\forall k \in \mathbb{N}, x_{n_k}$ is not in the ϵ-neighborhood of L. Furthermore, since (x_n) is bounded, it follows that (x_{nk}) is also bounded. Now, per **Bolzano-Weierstrass Theorem**, we get that (x_{nk}) must contain some convergent subsequence. Thus, we have found a convergent subsequence (x_{nkj}) that converges to L (by definition) and we face a contradiction since the subsequence was constructed in a way that none of its terms are in the ϵ-neighborhood of L. Hence, if (x_n) is a bounded sequence of real numbers such that every convergent subsequence of (x_n) converges to the same value L, (x_n) also converges to L.

□

6. (a) Let $f : A \to B : x \mapsto (x, 0.25)$. Then this function is one-to-one. However, f is not onto.

Let us first show that it is one-to-one. Suppose, for the sake of contradiction, that f is not one-to-one. Then $\exists x_1 \neq x_2$ s.t. $f(x_1) = f(x_2)$. We have:

$$f(x_1) = f(x_2)$$
$$(x_1, 0.25) = (x_2, 0.25)$$
$$x_1 = x_2$$

Hence, we got that $x_1 = x_2$ and we face a contradiction since we assumed $x_1 \neq x_2$. Thus, $f : A \to B : x \mapsto (x, 0.25)$ is one-to-one.

□

On the other hand, f is not onto since there is no value of x such that $f(x) = (x, 0.5)$.

Finally, we found a function $f : A \to B : x \mapsto (x, 0.25)$ that is one-to-one, but not onto.

(b) Suppose (a, b) is an input to the function and we want to make the output unique as well. Let us consider the decimal expansions of a and b. We get $a = 0.a_1a_2a_3\dots$ and $b = 0.b_1b_2b_3\dots$ with $a_1, b_1, a_2, b_2, \dots \in \mathbb{N}$. We can then construct number $c = 0.a_1b_1a_2b_2a_3b_3\dots$. Now, since the decimal expansions of a and b are unique, the number built by alternating the digits in the decimal expansions is also unique. Thus, $g : B \to A : (a, b) \mapsto c = g : B \to A : (0.a_1a_2a_3\dots, b_1b_2b_3\dots) \mapsto 0.a_1b_1a_2b_2a_3b_3\dots$ is one-to-one.

However, this function is not onto. Suppose, for the sake of contradiction that g is onto. Consider the following output in the codomain A: $g(x) = 0.9b_19b_29b_3\dots$. The only way for this to happen is if the function has the following form: $g(0.999\dots, 0.b_1b_2b_3\dots)$. Now, recall that $0.999\dots = \dfrac{9}{9} = 1$, but $1 \notin A$ and hence, we face a contradiction. Thus, g is not onto.

□

Finally, we have found a function $g : B \to A : (0.a_1a_2a_3\dots, b_1b_2b_3\dots) \mapsto 0.a_1b_1a_2b_2a_3b_3\dots$ that is one-to-one, but not onto.

(c) Let $u(x) : A \to \mathbb{R} : x \mapsto x + 1$ and let $v(x) : \mathbb{R} \to A : x \mapsto \dfrac{1}{2^x + 1}$. If we now show that both $u : A \to \mathbb{R}$ and $v : \mathbb{R} \to A$ are one-to-one, we have effectively shown that $A \sim \mathbb{R}$.

Let us first show that $u(x) : A \to \mathbb{R} : x \mapsto y$ is one-to-one. Suppose, for the sake of contradiction, that u is not one-to-one. The $\exists x_1 \neq x_2$ s.t. $u(x_1) = u(x_2)$. We have:

$$u(x_1) = u(x_2)$$
$$x_1 + 1 = x_2 + 1$$
$$x_1 = x_2$$

Hence, we got that $x_1 = x_2$ and we face a contradiction since we assumed $x_1 \neq x_2$. Thus, $u : A \to \mathbb{R} : x \mapsto x + 1$ is one-to-one.

$\square$

Let us now prove that $v(x) : \mathbb{R} \to A : x \mapsto \dfrac{1}{2^x + 1}$ is one-to-one. Suppose, for the sake of contradiction, that v is not one-to-one. The $\exists x_1 \neq x_2$ s.t. $v(x_1) = u(v_2)$. We have:

$$v(x_1) = v(x_2)$$
$$\frac{1}{2^{x_1} + 1} = \frac{1}{2^{x_2} + 1}$$
$$2^{x_1} + 1 = 2^{x_2} + 1$$
$$2^{x_1} = 2^{x_2}$$
$$x_1 = x_2$$

Hence, we got that $x_1 = x_2$ and we face a contradiction since we assumed $x_1 \neq x_2$. Thus, $u : \mathbb{R} \to A : x \mapsto \dfrac{1}{2^x + 1}$ is one-to-one.

$\square$

Thus, we found functions $u : A \to \mathbb{R}$ and $v : \mathbb{R} \to A$ such that both u and v are one-to-one. Finally, by **Schroeder-Bernstein Theorem**, we get that $A \sim \mathbb{R}$.

$\square$

(d) Using the results obtained in (a) and (b), we get that $A \sim (0,1) \times (0,1) \sim A \times A$. Thus, $A \sim A \times A$. In (c), we have proved that $A \sim \mathbb{R}$. It follows that $\mathbb{R} \sim \mathbb{R} \times \mathbb{R} \sim \mathbb{R}^2$. Hence, $\mathbb{R} \sim \mathbb{R}^2$.

$\square$