MATH 454: REAL ANALYSIS EXAM 2 12/21/20

This exam is due **Wednesday, January 6, at 11:00 p.m.** Write your solutions in LaTeX or scan your neatly handwritten solutions and upload a pdf to this folder. You may refer to your notes, homework, and textbook. Ask me for clarification on the wording of problems. **Do not discuss the exam with anyone else or use any outside sources.**

1. (7 pts) A set is called a G_δ set if it is the countable intersection of open sets. A set is called an F_σ set if it is the countable union of closed sets.

   (a) Show that any closed interval $[a, b]$ is a G_δ set.

   (b) Show that any half-open interval $(a, b]$ is both a G_δ set and an F_σ set.

   (c) Show that $\mathbb{Q}$ is an F_σ set.

   (d) Show that $\mathbb{R} - \mathbb{Q}$ is a G_δ set.

   (e) Prove that a set is a G_δ set iff its complement is an F_σ set.

2. (6 pts) Let C be the Cantor set. Define $C + C = \{x + y \mid x, y \in C\}$. Prove that $C + C$ equals the closed interval $[0, 2]$.
   Hint: Show $\forall\, r \in [0, 2],\ \exists\, x_n,\ y_n \in C_n$ *such that* $x_n + y_n = r$.

3. (8 pts) Construct a modified Cantor Set F by removing open middle fifths:
   $F_0 = [0, 1]$, $F_1 = [0, 2/5] \cup [3/5, 1]$, etc.

   (a) What is F_2? Find its endpoints and sketch it.

   (b) Prove that F is compact.

   (c) Calculate the length of F.

   (d) Prove that F is uncountable.

   (e) What is the fractal dimension of F?
   See the discussion at the end of Section 3.1.

4. (6 pts) Use Definition 4.2.1 and the definition from Exercise 4.2.9 to prove:

   (a) $\lim_{x \to 3} (x^2 - 5x + 4) = -2$

   (b) $\lim_{x \to \infty} \dfrac{2x}{x + 4} = 2$

5. (5 pts) Use Definition 4.3.1 to prove that $f(x) = \sqrt[4]{x}$ is continuous on $[0, \infty)$.
Hint: See Example 4.3.8.

6. (6 pts) Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ that takes the tenths digit of the decimal expansion of x and replaces it with a 1. For example: $f(2.35) = 2.15$, $f(3) = 3.1$, and $f(\pi) = \pi$. Where is f continuous? Not continuous? Justify your answers.

7. (5 pts) Show that $f(x) = 1/x^2$ is uniformly continuous on $[1, \infty)$, but not on $(0, 1]$.

8. (7 pts) A function $f\colon A \to \mathbb{R}$ is **Lipschitz** if $\exists\, M$ such that

$$\forall\, x, y \in A, \quad \left| \frac{f(x) - f(y)}{x - y} \right| \leq M.$$

   (a) Sketch a function g that is Lipschitz on $[0, 10]$.
   Sketch a function h that is continuous but not Lipschitz on $[0, 10]$.
   In general, describe the graph of a Lipschitz function.

   (b) Show that if f is Lipschitz on A, then f is uniformly continuous on A.

   (c) If f is uniformly continuous on A, is f necessarily Lipschitz on A?