Real Analysis

Assignment №8

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4.4.1 (a) **Per Theorem 4.3.4**, we know that products of continuous functions are continuous. Hence, it suffices to show that the function $g(x) = x$ is continuous on all of $\mathbb{R}$. Now, let $\epsilon < 0$ be given and choose $\delta = \epsilon$. Then, we have that $\forall x, y \in \mathbb{R}, |x - y| < \delta$. It follows that $|g(x) - g(y)| = |x - y| < \epsilon$.

$\square$

(b) Let $x_n = n$ and $y_n = n - \dfrac{1}{n}$. Then $|x_n - y_n| \to 0$, but $|f(x_n) - f(y_n)| = 3n - \dfrac{1}{n(3 - \frac{1}{n^2})} \to \infty$. Hence, **by Theorem 4.4.5**, the function f is not uniformly continuous on $\mathbb{R}$.

$\square$

(c) Let $A \subset \mathbb{R}$ be any bounded subset. Then $\overline{A}$, the closure of A in $\mathbb{R}$, is compact. **By Theorem 4.4.8**, f is uniformly continuous on $\overline{A}$, and hence on any subset of $\overline{A}$. In particular, f is uniformly continuous on A.

$\square$

4.4.6 (a) Such request is possible.

Let $f(x) = \dfrac{1}{x}$ for $x \in (0,1)$. Then it follows that f is continuous on $(0,1)$. Now, let $x_n = \dfrac{1}{n}$. Then x_n is Cauchy. However, $f(x_n) = n$ and thus, $(f(x_n))$ is not Cauchy.

(b) Such request is impossible.

Suppose, for the sake of contradiction, that f is a uniformly continuous function on $(0,1)$ and x_n is the Cauchy sequence. Now, if $f(x_n)$ is not Cauchy, then $\exists \epsilon_0$ s.t. $\forall N \in \mathbb{N}, f|(x_n) - f(y_n)| \geq \epsilon_0$ for all $m, n \geq N$. Finally, since $|x_n - y_n| \to 0$ we face a contradiction as we assumed that f is uniformly continuous. Thus, such request is impossible.

□

(c) Such request is impossible.

If $(x_n) \to x$, then $[0, \infty)$ is closed and containts x. Finally, since f is continuous on x, by the definition of continuity, we get $(x_n) \to x$ and it follows that $f(x_n) \to f(x)$. Thus, such request is impossible.

□

4.4.11 We will first prove the statement directly and then prove its converse.

Let us suppose that g is continuous on O. Let $c \in g^{-1}(O)$. Then $\forall \epsilon > 0, \exists \delta > 0$ s.t. if $x \in V_\delta(c)$, it follows that $g(x) \in V_\delta(g(c))$. Choose ϵ s.t. $V_\epsilon(g(c)) \subseteq O$. Then $x \in V_\delta(c)$ implies that $g(x) \in V_\epsilon(g(c)) \subseteq O$ and thus, $x \in g^{-1}(O)$. Finally, we get $V_\delta(c) \subseteq g^{-1}(O)$ and $g^{-1}(O)$ is open.

□

Conversely, suppose that for every open $O \subseteq \mathbb{R}$, $g^{-1}(O)$ is an open set. Let $O = V_\epsilon(g(c))$. Then $g^{-1}(O)$ is open. Now, since $c \in g^{-1}(O)$ and c is an interior point, $\exists \delta > 0$ s.t $V_\delta(c) \subseteq g^{-1}(O)$. Then, by definition, if $x \in V_\delta(c)$, it follows that $g(x) \in O = V_\epsilon(g(c))$ which shows that g is a continuous function.

□

Finally, we have proven both the direct statement and its converse and hence, g is continuous if and only if $g^{-1}(O)$ is open whenever $O \subseteq \mathbb{R}$ is an open set.

□

4.5.7 We have $\forall 0 \leq x \leq 1$, we have $0 \leq f(x) \leq 1$. Consider a function $g : [0,1] \to \mathbb{R}$, given by $g(x) = f(x) - x$. Then g is continuous on $[0,1]$. Now $g(0) = f(0) \geq 0$, but $g(1) = f(1) - 1 \leq 0$. Hence, **by the Intermediate Value Theorem (Theorem 4.5.1)**, $\exists x_0 \in [0,1]$ s.t. $g(x_0) = 0$. Hence, we have that $f(x_0) = x_0$.

$\square$