Real Analysis

Assignment №3

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2.4.1 (a) Let us first show that (x_n) is monotonically decreasing. We can use induction for this proof.

Base case: $x_1 = 3$ and $x_2 = \frac{1}{4 - x_1} = \frac{1}{1} = 1$. It follows that $x_2 - x_1 = 1 - 3 = -2 < 0$. Hence, the base case is satisfied.

Inductive hypothesis: suppose $x_n - x_{n+1} > 0$. We now have to show that $x_{n+1} - x_{n+2} > 0$.

$$\begin{aligned} x_{n+1} - x_{n+2} &= \frac{1}{4 - x_n} - \frac{1}{4 - x_{n+1}} \\ &= \frac{x_n - x_{n+1}}{(4 - x_n)(4 - x_{n+1})} > 0 \end{aligned}$$

Thus, by assuming that $x_n - x_{n+1} > 0$, we got that $x_{n+1} - x_{n+2} > 0$ as well. Hence, $\forall n \in \mathbb{N}, x_n > x_{n+1}$. Additionally, (x_n) is a bounded sequence since $\forall n \in \mathbb{N}, 0 < x_n < 5$. Finally, by **Monotone Convergence Theorem**, we get that (x_n) converges.

$\square$

(b) As $\lim x_n$ exists, let $\lim x_n = X$. Then $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. if $n \geq N, |x_n - X| < \epsilon$. Now, since $n + 1 > n \geq N$, we get that $|x_{n+1} - X| < \epsilon$ and hence, $\lim x_{n+1} = X$.

$\square$

(c) If we take the limits of both sides, we know from (b) that $\lim x_n = \lim x_{n+1}$. Then, by **Algebraic Limit Theorem**, we get the following:

$$\lim x_{n+1} = \frac{1}{4 - \lim x_n} = \lim x_n \tag{1}$$

$$\lim x_n = \frac{1}{4 - \lim x_{n+1}} \tag{2}$$

$$\lim x_n^2 - 4 \lim x_n + 1 = 0 \tag{3}$$

From (3), we have that $\lim x_n = 2 \pm \sqrt{3}$. Finally, since $x_1 < 2 + \sqrt{3}$ and the sequence is monotonically decreasing, it follows that the $\lim x_n = 2 - \sqrt{3}$.

2.4.7 (a) Since (a_n) is bounded, $\exists B_1, B_2$ s.t. $\forall n \in N, B_1 < a_n < B_2$. It follows that $\{a_m \mid m \geq n + 1\} \subseteq \{a_m \mid m \geq n\}$. We get:

$$A \leq \sup \{a_m \mid m \geq n + 1\} = y_{n+1} \leq \sup \{a_m \mid m \geq n\} = y_n \leq B_2$$

Finally, since (y_n) is both bounded and decreasing, by **Monotone Convergence Theorem**, we conclude that (y_n) converges.

(b) Let $x_n = \inf \{a_m \mid m \geq n\}$. Similar to part (a), x_n converges by **Monotone Convergence Theorem**. To see this, all one needs to do is to reverse all the inequalities in part (a). In this case, it is both bounded and increasing (in lieu of decreasing). We can then define limit inferior as $\liminf a_n = \lim x_n$ and it will always exist.

(c) As (a_n) is bounded, we get:

$$x_n = \inf \{a_m \mid m \geq n\} \leq \sup \{a_m \mid m \geq n\} = y_n$$

Then, by **Order Limit Theorem**, it follows that $\forall n \in \mathbb{N}$:

$$\liminf a_n = \lim_{n \to \infty} x_n \leq \lim_{n \to \infty} y_n = \limsup a_n$$

$\square$

One such example can be a sequence $t_n = (-1)^n$. In this case, we would have $\liminf a_n = -1$ and $\limsup a_n = 1$. Thus, we get that $\liminf a_n \leq \limsup a_n$ since $-1 < 1$ and the inequality is strict.

(d) Let us first show that if $\liminf a_n = \limsup a_n$, then $\lim a_n$ exists. Suppose that $\liminf a_n = \limsup a_n$. Then, $\forall n \in \mathbb{N}$, we get:

$$x_n = \inf \{a_m \mid m \geq n\} \leq a_n \leq \sup \{a_m \mid m \geq n\} = y_n$$

Then, we get that $\lim_{n\to\infty} x_n = \liminf a_n = \limsup a_n = \lim_{n\to\infty} y_n$ and it follows by the **Squeeze Theorem** (proven in the previous homework) that a_n converges to $\liminf a_n = \limsup a_n = \lim a_n$.

$\square$

Conversely, suppose that $\liminf a_n$ exists. Then $\lim a_n = A$ and $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N, A - \epsilon < a_n < A + \epsilon, \forall \epsilon > 0$. Then $A - \epsilon \leq \liminf a_n$ and $\limsup a_n \leq A + \epsilon$. Finally, we get that:

$$A - \epsilon \leq \liminf a_n \leq \limsup a_n \leq A + \epsilon$$

and since ϵ is chosen arbitrarily, we get that $\liminf a_n = \limsup a_n = \lim a_n$.

$\square$

Thus, we have now shown that $\liminf a_n = \limsup a_n$ if and only if $\lim a_n$ exists. And in this case, all three share the same value.

$\square$

2.5.1 (a) Per Bolzano-Weierstrass theorem, such sequence cannot exist.

(b) Such sequence exists. Let $(a_n) = \left(\frac{1}{2}, -1, \frac{1}{4}, \frac{1}{2}, \frac{1}{8}, -\frac{1}{3}, \frac{1}{16}, \frac{1}{4}, \dots\right)$. Then notice that the odd terms make up a sequence $(b_n) = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots\right)$ which converges to $\frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$. On the other hand, the even terms make up a sequence $(c_n) = \left(-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \dots\right)$, which represents a sequence $\frac{(-1)^n}{n}$ and converges 0. Thus, we found a sequence that does not contain 0, but its subsequences converge to 0 and 1.

(c) Such sequence exists. To construct such sequence it suffices to make each number appear infinitely many times. Let $(s_n) = \left(1, 1, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{3}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \dots\right)$. Hence, at a step $k \in \mathbb{N}$, we add a new number $\frac{1}{k}$ to the sequence. This ensures that the sequence contains subsequences converging to every point in the infinite set $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\}$ and thus, such sequence exists.

(d) There is no such sequence. Let $S = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\}$. Then, since $\lim_{n\to\infty} S = 0$, there must exist at least one sequence that converges to 0. On the other hand, 0 is not in the set S. Hence, such sequence cannot exist.

2.5.9 As sequence (a_n) is bounded, $\exists B > 0$ s.t. $\forall n \in \mathbb{N}, -B < a_n < B$. It follows that $\forall x$ s.t. $x < -B, x \in S$. Hence, $S \neq \emptyset$. Additionally, since $x < a_n < B$ for infinitely many n, B is the upper bound for S. Then we get, per the **Axiom of Completeness**, that $s = \sup S$ exists. Now, suppose, for the sake of contradiction, that the sequence does not converge to $s = \sup S$ and $a_{n_m} \geq \sup S$ (a_{n_m} represents the m^{th} element of the sequence a_n). Then $\exists \epsilon > 0$ s.t. $\forall m \geq M$, we get $a_{n_m} - s \geq \epsilon$. Since there are infinitely many of such m, let $M = \max\{m_t\} + 1$ where $\{m_t\}$ is any finite subset of ms. However, now we have effectively found $e = \sup S + \epsilon$ which is an element the set S ($e \in S$) and we face a contradiction since $\forall x \in S, x \leq \sup S$. Thus, the sequence (a_n) converges to $\sup S$.

$\square$