
Real Analysis

Assignment №6

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3.3.1 Per **Heine-Borel Theorem**, K is closed and bounded. Then since K is bounded, the least upper bound and greatest lower bound properties of $\mathbb{R}$ imply that $\sup K$ and $\inf K$ both exist.

Let us first prove that $\sup K \in K$. Suppose, for the sake of contradiction, that $\sup K \notin K$. Then it follows that $\sup K$ is the limit point of K. However, since K is closed, $\sup K \in K$ and we face a contradiction. Thus, $\sup K \in K$

$\square$

Let us now prove that $\inf K \in K$. Similarly, suppose, for the sake of contradiction, that $\inf K \notin K$. Then it follows that $\inf K$ is the limit point of K. However, since K is closed, $\inf K \in K$ and we face a contradiction. Thus, $\inf K \in K$

$\square$

Finally, we have shown that if K is compact and nonempty, then $\sup K$ and $\inf K$ both exist and are elements of K.

$\square$

3.3.2 (a) $\mathbb{N}$ **is not compact**.

This is the case since a sequence $a_n = (n)$ has no subsequence that is convergent and **by Theorem 3.3.1**, $\mathbb{N}$ is not compact.

(b) $\mathbb{Q} \cap [0, 1]$ **is not compact.**

Recall that $\mathbb{Q}$ is dense in $\mathbb{R}$. Then there must exist a sequence $(a_n) \subseteq \mathbb{Q}$ s.t. $(a_n) \to \frac{1}{\sqrt{3}}$. Now, since every ϵ-neighborhood contains rational numbers, it is a limit point. However, $\frac{1}{\sqrt{3}} \notin \mathbb{Q}$ as $\frac{1}{\sqrt{3}}$ is irrational. Thus, **by Theorem 3.3.1** $\mathbb{Q} \cap [0, 1]$ cannot be compact.

(c) **The Cantor set is compact.**

The Cantor set is an intersection of closed sets, and hence, it is closed. Additionally, the Cantor set is a subset of $[0, 1]$ and thus, it is bounded. Now, since the Cantor set is both closed and bounded, it follows **by Theorem 3.3.1** that the Cantor set is compact.

(d) **The set** $\{1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \mid n \in \mathbb{N}\}$ **is not compact.**

Notice that $\lim_{n\to\infty}\{1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \mid n \in \mathbb{N}\} = \lim_{n\to\infty} \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. Hence, the sum is a limit point of the set. However, the sum is not in the set and thus, **by Theorem 3.3.1**, it follows that the set $\{1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \mid n \in \mathbb{N}\}$ is not compact.

(e) **The set** $\{1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots \mid n \in \mathbb{N}\}$ **is compact.**

Let us denote this set by S. Then notice that $S_1 = 1$ and the rest of the terms can be calculated using the formula $S_n = \frac{n-1}{n}$ with $n > 1$. Also notice that $\lim_{n\to\infty} \frac{n}{n-1} = 1$. Hence, the set S contains its own only limit point. Additionally, it is easy to see that every element of the set lies between 0 and 1 (inclusive). Thus, **by Theorem 3.3.1**, we get that The set $\{1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots \mid n \in \mathbb{N}\}$ is compact.

3.3.11 From 3.3.2 we know that there are three sets which are not compact:

1. $\mathbb{N}$
2. $\mathbb{Q} \cap [0, 1]$
3. $\{1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \mid n \in \mathbb{N}\}$

For $\mathbb{N}$, we can take an open cover $C_1 = \{(n - 0.5, n + 0.5) \mid n \in \mathbb{N}\}$. Then C_1 has no finite subcover.

For $\mathbb{Q} \cap [0, 1]$, we can take an open cover $C_2 = \{(-3, \frac{1}{\sqrt{3}} - \frac{1}{n}), (\frac{1}{\sqrt{3}} + \frac{1}{n}, 3) \mid n \in \mathbb{N}\}$. Then C_2 has no finite subcover.

For $\{1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2} \mid n \in \mathbb{N}\}$, we can take an open cover $C_3 = \{(0, \sum_{m=1}^{n} \frac{1}{m^2}) \mid n \in \mathbb{N}\}$. Then C_3 has no finite subcover.

3.4.5 Suppose, for the sake of contradiction and without a loss of generality, that A and B are nonempty subsets of $\mathbb{R}$ s.t. $A \cap B \neq \emptyset$ and there exist disjoint open sets U and V s.t. $A \subseteq U$ and $B \subseteq V$. Let $x \in A \cap \overline{B}$. Then it follows that $x \in U \cap \overline{V}$. Now, since $U \cap V = \emptyset$, we get that $x \in U$ and $x \in \overline{V}$. Now, notice that any ϵ-neighborhood $V_\epsilon(x)$ contains an element of V. Thus, $V_\epsilon(x)$ is not contained in U. We get $A \cap B = \emptyset$ and we face a contradiction since we assumed that $A \cap B \neq \emptyset$. Hence, A and B are separated.

□

3.4.7 (a) Recall that the set of irrational numbers $\mathbb{I}$ is dense in $\mathbb{R}$. This means that $\forall r_1, r_2 \in \mathbb{R}$ with $r_1 < r_2, \exists i \in \mathbb{I}$ s.t. $r_1 < i < r_2$. Let $U = \mathbb{Q} \cap (-\infty, i)$ and let $V = \mathbb{Q} \cap (i, +\infty)$. Then it is easy to see that $Q = U \cap V$ where U and V are separated. Now, it is clear that $\overline{U} \subset (-\infty, i]$ and thus, $\overline{U} \cap V = \emptyset$. Similarly, $\overline{V} \subset [i, -\infty)$ and therefore, $U \cap \overline{V} = \emptyset$. Finally, we get that $\mathbb{Q}$ is totally disconnected since given any two distinct points r_1, r_2 there exist separated sets U and V with $r_1 \in U, r_2 \in V$, and $U \cap V = \mathbb{Q}$.

□

(b) **Yes, the set of irrational numbers is totally disconnected**.

Let us now show this fact.

Notice that $\mathbb{I} = \mathbb{R} - \mathbb{Q}$. Recall that $\mathbb{Q}$ is dense in $\mathbb{R}$. It follows that $\forall i_1, i_2 \in \mathbb{I}$ with

$i_1 < i_2, \exists q \in \mathbb{Q}$ s.t. $i_1 < q < i_2$. Now, let $U = \mathbb{I} \cap (-\infty, q)$ and let $V = \mathbb{I} \cap (q, +\infty)$. Then $U \cup V = \mathbb{I}$. Let $X = (-\infty, q)$ and let $Y = (q, +\infty)$. Then $U \subseteq X$ and $Y \subseteq V$. Notice that X and Y are totally disconnected sets. Then, according to what we showed in **Exercise 3.4.5**, U and V must be separated. Hence, $\mathbb{I}$ is totally disconnected.

□

3.4.9 (a) Notice that the length of O, the following stands:

$$|O| \leq \sum_{n=1}^{\infty} 2(\frac{1}{2^n})$$

It follows that O does not fully cover $\mathbb{R}$ and thus, $F = O^c$ is nonempty. Now, O contains all rational numbers and thus $F = O^c \subseteq \mathbb{I}$. Furthermore, O is an open set as it is constructed by an arbirary countable union of open neighborhoods. Hence, $F = O^c$ must be closed. Hence, we have shown that F is a closed, nonempty set consisting only of irrational numbers.

□

(b) **No, F does not contain any nonempty open intevals and yes, F is totally disconnected.**

Let us now prove these facts.

Let $r_1, r_2 \in \mathbb{R}$ with $r_1 < r_2$. Recall that $\mathbb{Q}$ is dense in $\mathbb{R}$. Then $\exists q \in \mathbb{Q}$ s.t. $r_1 < q < r_2$. Now, since $q \in O$, $(r_1, r_2) \cap O = \emptyset$ and thus $(r_1, r_2) \not\subseteq F$. Hence, F does not contain any nonempty open intevals.

□

F is totally disconnected since it is a subset of irrational numbers $\mathbb{I}$ (shown in **Exercise 3.4.9 (a)**). We have shown in **Exercise 3.4.7 (b)** that the set of irrational numbers $\mathbb{I}$ is totally disconnected. Hence, F is also totally disconnected.

□