Topology

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Assignment №4

Section 23

6. Let $A \subset X$. Show that if C is a connected subspace of X that intersects both A and $X - A$, then C intersects Bd A.

   At first, recall that Bd $A = \overline{A} \cap \overline{X - A}$. Now, suppose, for the sake of contradiction, that C is connected and $C \cap \text{Bd } A = \varnothing$. Consider two sets $U_1 = C \cap \overline{A}$ and $U_2 = C \cap \overline{X - A}$. Now, since $C \cap A \subset U_1$ and $C \cap \overline{X - A} \subset U_2$, it follows from our assumptions that U_1 and U_2 are two nonempty subsets of C. Notice that $C = U_1 \cup U_2$ with U_1, U_2 being both open and closed subsets of C. However, $U_1 \cap U_2 = C \cap \overline{A} \cap \overline{X - A} = C \cap \text{Bd } A = \varnothing$ which means that C is disconnected and contradicts the fact that C is connected. Finally, we have reached the contradiction and C intersects Bd A. $\square$

Section 24

1. (c) Show that $\mathbb{R}^n$ and $\mathbb{R}$ are not homeomorphic if $n > 1$.

   Suppose, for the sake of contradiction, that for $n > 1$, $\mathbb{R}^n$ and $\mathbb{R}$ are homeomorphic. Then, by the definition of homeomorphism, there exists a function $f : \mathbb{R} \to \mathbb{R}^n$. Consider $f|_{\mathbb{R}^n - \{0\}} : \mathbb{R}^n - \{0\} \to \mathbb{R} - \{f(0)\}$, $f|_{\mathbb{R}^n - \{0\}}$ is a restriction of f and hence is a homeomorphism. Now, notice that $\mathbb{R}^n - \{0\}$ is a connected space, however, $\mathbb{R} - \{f(0)\}$ is not a connected space and we have reached the contradiction since $f|_{\mathbb{R}^n - \{0\}}$ is a homeomorphism. Finally, we have that for $n > 1$, $\mathbb{R}^n$ and $\mathbb{R}$ are not homeomorphic. In short, by taking away 0, we make $\mathbb{R}$ disconnected, but taking away any point from $\mathbb{R}^n$ leaves it connected. $\square$

3. Let $f : X \to X$ be continuous. Show that if $X = [0, 1]$, there is a point x such that $f(x) = x$. The point x is called a ***fixed point*** of f. What happens if X equals $[0, 1)$ or $(0, 1)$?

In the order topology, X is an ordered set and connected space. Let $a, b \in X$. Let's now pick a midpoint x_1 between $f(a)$ and $f(b)$. Then, according to **Theorem 24.3**, $\exists c_1 \in [a, b]$ such that $f(c_1) = x_1$. Now, if $c_1 = x_1$, we found the fixed point of f and if $c_1 \neq x_1$, we pick the midpoint x_2 between c_1 and x_1. Then $\exists c_2 \in [c_1, x_1]$ or $c_2 \in [x_1, c_1]$ (depending on whether $x_1 > c_1$ or $c_1 > x_1$) such that $f(c_2) = x_2$. Now, if $c_2 = x_2$ then we have found the point and if not we countinue this way. Thinking of computer science, this is a recursive approach to the problem (though, I think recursion comes from math anyway, right?). The simply outline of the algorithm would look something like this:

if $x_n = c_n$ **then**
  hooray! we have found a point! we are done, return the point!
else
  Consider the midpoint between x_n and c_n
end if

After repeating this process, we will end up with two convergent series: $c_1, c_2, c_3, ...$ and $x_1, x_2, x_3, ...$ with the property that $|c_n - x_n| \to 0$ as $n \to \infty$. In others words, we have $\lim_{n\to\infty} |c_n - x_n| \to 0$. This is due to the continuity of f on X. Therefore, we have $|x_n - f(x_n)| = |f(c_n) - f(x_n)| \to 0$ from which we get that $f(x_n) = (x_n)$. $\square$

This fact/theorem does not hold for intervals $[0, 1)$ and $(0, 1)$. This is due to f not being uniformly continuous on these intervals. For instance, a function $f(x) = \dfrac{x+2}{3}$ has a fixed point $x = 1$, but it has no fixed points on intervals $[0, 1)$ or $(0, 1)$.

8. (a) Is a product of path-connected spaces necessarily path-connected?

    Yes.

   (b) If $A \subset X$ and A is path-connected, is $\bar{A}$ necessarily path-connected?

    No. For instance, ***topologist's sine curve***.

   (c) If $f : X \to Y$ is continuous and X is path-connected, is $f(X)$ necessarily path-connected?

    Yes, this is due to the fact that the composition of continuous functions is always continuous.

   (d) If $\{A_\alpha\}$ is a collection of path-connected subspaces of X and if $\bigcap A_\alpha \neq \varnothing$, is $\bigcup A_\alpha$ necessarily path-connected?

    Yes

Section 26

5. Let A and B be disjoint compact subspaces of the Hausdorff space X. Show that there exist disjoint open sets U and V containing A and B, respectively.

   Suppose that A and B are disjoint compact subspaces of the Hausdorff space X. Now, let p be some arbitrary points in A. Then the points of p are not in Y

Section 26

2. (a)