Real Analysis Exams

Exam №3

Instructor: Dr. Eric Westlund

David Oniani
Luther College
oniada01@luther.edu

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1. (a) Placeholder.
   (b) Placeholder.
   (c) Placeholder.
   (d) Placeholder.
2. (a) Placeholder.
   (b) Placeholder.
3. (a) Placeholder.
   (b) Placeholder.
4. (a) Placeholder.
   (b) Placeholder.
   (c) Placeholder.
   (d) Placeholder.
   (e) Placeholder.

5. (a) Notice that the following holds:

$$\left|\frac{\cos(3^n x)}{2^n}\right| \leq \frac{1}{2^n}$$

Now, recall that $\frac{1}{2^n}$ converges (showed many times over the course of the class). Then, it follows by **Corollary 6.4.5 (Weierstrass M-Test)** that $g(x) = \sum_{n=1}^{\infty} \frac{\cos(3^n x)}{2^n}$ converges uniformly on $\mathbb{R}$. And since the uniform convergence implies continuity, it follows that $g(x) = \sum_{n=1}^{\infty} \frac{\cos(3^n x)}{2^n}$ is continuous on $\mathbb{R}$.

(b) Notice that we have:

$$g'(x) = \sum_{n=1}^{\infty} -\left(\frac{3}{2}\right)^n \sin(3^n x)$$

Unfortunately, in this case we cannot apply **Corollary 6.4.5 (Weierstrass M-Test)** as $\left(\frac{3}{2}\right)^n$ is not bounded. However, recall that this is the Weierstrass function of the form $\sum_{n=0}^{\infty} a^n \cos(b^n x)$ which is a nowhere-differentiable function. Hence, $g'(x)$ is not differentiable on $\mathbb{R}$.

6. For $x \notin \mathbb{Q}$, we can show $f_n(x)$ is continuous, since for $x < r_n$, we can choose a small enough δ such that $f_n(y) = 0$ for $y \in V_\delta(x)$. Similar logic can be applied for when $x > r_n$. Now, notice that

$$f_n(x) \leq \frac{1}{2^n}$$

Then it follows by **Corollary 6.4.5 (Weierstrass M-Test)** that $f(x)$ converges uniformly.

Now, since f_n are all continuous, and f converges uniformly, we have that f is continuous.

Furthermore, since every $f_n(x)$ is increasing, f is monotonely increasing. Thus, for $x < y$, we get:

$$\begin{gathered}
\forall n \; f_n(x) \leq f_n(y) \\
\sum_{n=1}^{k} f_n(x) \leq \sum_{n=1}^{k} f_n(y) \\
\lim_k \sum_{n=1}^{k} f_n(x) \leq \lim_k \sum_{n=1}^{k} f_n(y) \\
f(x) \leq f(y)
\end{gathered}$$

Hence, we got that f is increasing on $\mathbb{R}$.

□

7. (a) Placeholder.

   (b) Placeholder.

   (c) Placeholder.