Real Analysis Exams

Exam №2

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1. (a) Let $S_n = (a - \frac{1}{n}, b + \frac{1}{n})$. Then S_n is open. Now, it is easy to see that $S = \cap_{n=1}^{\infty} S_n$ is G_δ set. Furthermore, $[a, b] \subseteq S$ as $\forall n, [a, b] \subseteq S_n$. Now, suppose, for the sake of contradiction, that $x \in S$ and $x \notin [a, b]$. We have two cases:

   (i) $x < a \implies \exists n$ s.t. $a - x > \frac{1}{n}$ and thus, $x < a - \frac{1}{n}$. It follows that $x \notin S_n$ and $x \notin S$. Hence, we face a contradiction and $x \geq a$.

   (ii) $x > b \implies \exists n$ s.t. $x - b > \frac{1}{n}$ and thus, $x > b + \frac{1}{n}$. It follows that $x \notin S_n$ and $x \notin S$. Hence, we face a contradiction and $x \leq b$.

   Finally, from these two cases, we got that $x \geq a$ and $x \leq b$ and thus $x \in [a, b]$. Therefore, $[a, b]$ is G_δ set.

   (b) Let $S_n = (a, b + \frac{1}{n})$. Then, by the argument presented in (a) part of the exercise, $S = \cap_{n=1}^{\infty} S_n = (a, b]$ and thus $(a, b]$ is G_δ.

   Now, suppose, for the sake of contradiction, that $U_n = [a + \frac{1}{n}, b], x \in U_n$, and $x \notin (a, b]$. Let us now consider these two cases:

   (i) $x \leq a \implies x < a + \frac{1}{n} \implies x \notin U_n$ and we face a contradiction.

(ii) $x > b \implies x \notin U_n$ and we face a contradiction.

Thus $x > a$ and $x \leq b$ which implies that $x \in (a, b]$ and therefore, $(a, b]$ is F_σ. Hence, we have shown that any arbitrary half-open interval $(a, b]$ is both G_δ and F_σ.

(c) To prove that $\mathbb{Q}$ is F_σ, we need to find a countable collection of closed subsets of $\mathbb{Q}$ whose union is $\mathbb{Q}$. Now, since $\mathbb{Q}$, there exists a bijective function $f : \mathbb{N} \to \mathbb{Q}$. Then, $\forall n \in \mathbb{N}$, set $S_n = \{f(n)\}$ is closed. We have $\mathbb{Q} = \bigcup_{n=1}^{\infty} \{S_n\}$ is a union of closed sets. Thus, by definition, $\mathbb{Q}$ is F_σ set.

(d) Notice that $\mathbb{R} - \mathbb{Q}$ is the set of irrational numbers which is the complement of the rational numbers in $\mathbb{R}$. Hence, $\mathbb{I} = \mathbb{R} - \mathbb{Q} = \mathbb{Q}^c$. From (c) we know that $\mathbb{Q}$ can be represented as a countable union of closed sets. Then, per **De Morgan's Law**, we get that $\mathbb{I}$ is the countable intersection of open sets (complement of a closed set is an open set). Hence, by definition, we get that $\mathbb{R} - \mathbb{Q}$ is G_δ.

(e) Since this is a if and only if question, let us first prove the statement directly and then prove its converse.

(i) Let us first show that a set is a G_δ set if its complement is an F_σ set.

Suppose that we have a set S which is a G_δ set. Then, by definition, $S = \bigcap_{n=1}^{\infty} S_n$ where every S_n is an open set. Then, by **De Morgan's Law**, it follows that $S^c = \bigcup_{n=1}^{\infty} S_n^c$ (with S_n^c being closed as the complement of an open set is a closed set) and by definition, S^c is a F_σ set.

$\square$

(ii) Let us now prove the converse, that if a set is a complement of a F_σ set, then it is a G_δ set.

Suppose that we have a set S which is a F_σ set. Then, by definition, $S = \bigcup_{n=1}^{\infty} S_n$ where every S_n is a closed set. Then, by **De Morgan's Law**, it follows that $S^c = \bigcap_{n=1}^{\infty} S_n^c$ (with S_n^c being open as the complement of a closed set is an open set) and by definition, S^c is a F_σ set.

$\square$

Finally, we have proven that a set is a G_δ set if and only if its complement is an F_σ set.

$\square$

2. Let us first prove that $\frac{1}{2}\mathbb{C}+\frac{1}{2}\mathbb{C}=[0,1]$. Recall that the Cantor set $\mathbb{C}$ is the set of all numbers in $[0,1]$ that in the **ternary system** can be represented as the sequence of 0s and 2s only. Then $\frac{1}{2}\mathbb{C}$ must only contain 0s and 1s. Now, let $r \in [0,1]$. If we show that $\exists x, y \in \frac{1}{2}\mathbb{C}$ s.t $x+y \in [0,1]$, then we have effectively shown that $\frac{1}{2}\mathbb{C}+\frac{1}{2}\mathbb{C}=[0,1]$. Let us construct x and y in the following manner:

   * Let x have 0s in the same places where it is in r and let x have 1s when the corresponding digit in r is either 1 or 2.
   * Let y have 0s in the same places where r has $0s$ or 1s. Let y have 1s when the corresponding digit in r is 2.

   Hence, we split all 2s in r in a way that half goes to x and half goes to y, and all 1s of r were given to x. Thus, $x+y=r$. For instance, if $r=0.120120\ldots$, then $x=0.110110\ldots$ and $y=0.010010\ldots$. It follows that $x+y=0.120120\cdots=r$. Now, since we have already shown that $\frac{1}{2}\mathbb{C}+\frac{1}{2}\mathbb{C}=[0,1]$, we can just multiply both sides of the equation by 2 and we get $\mathbb{C}+\mathbb{C}=[0,2]$.

   $\square$

3. (a) $F_2=\left[0,\frac{4}{25}\right]\cup\left[\frac{6}{25},\frac{2}{5}\right]\cup\left[\frac{3}{5},\frac{19}{25}\right]\cup\left[\frac{21}{25},1\right]$.

   Below find the sketch (blue segments are included and red segments are not included).

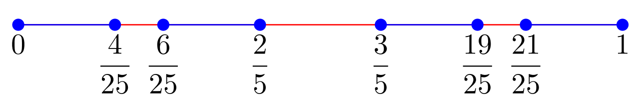

Figure 1: Sketch of F_2.

   (b) Notice that by definition, F is bounded by $[0,1]$. Recall that arbitrary intersection of closed sets is closed (we have proved this in the past as a part of an exercise). Now, every F_n is closed and since F is an intersection of such sets, it follows that F closed. Finally, F is both closed and bounded and by **Theorem 3.3.8 (Heine–Borel Theorem)**, F is compact.

   $\square$

(c) Notice that in order to construct F, we first remove 1 interval of length $\frac{1}{5}$. Then we remove 4 intervals of length $\frac{1}{25}$, then 16 intervals of the length $\frac{1}{125}$, etc. Thus, the removed intervals form the infinite geometric series of the following form:

$$\frac{1}{5} \times \left(\frac{4}{5}\right)^0, \frac{4}{5} \times \left(\frac{4}{5}\right)^1, \frac{1}{5} \times \left(\frac{4}{5}\right)^2 \dots$$

Recall that the sum of such series is calculated by the formula $S = \frac{s_1}{1-r}$ where s_1 is the first element of the sequence and r is the ratio/quotient (next element over the previous one). Then, we have $S = \frac{\frac{1}{5}}{1 - \frac{4}{5}} = \frac{\frac{1}{5}}{\frac{1}{5}} = 1$. Now, notice that we had the length of 1 initially as the length of $[0,1]$ is 1. We subtracted $S = 1$ from 1 and get $1 - 1 = 0$. Thus, the length of F is 0.

(d) Suppose, for the sake of contradiction, that $S = \{s_1, s_2, \dots\}$ is countable. We now need to find some point $x \in F$ s.t. $x \notin S$. Notice that $F \subseteq [0, \frac{2}{5}], [\frac{3}{5}, 1]$ with $s_1 \notin [0, \frac{2}{5}] \cup [0, \frac{2}{5}]$ (i.e., s_1 is not in either of the two intervals). Let us denote $[0, \frac{2}{5}]$ as I_1. Then $s_1 \notin I_1$. Similarly, after removing the middle fifth of I_1, s_1 will not be in neither of the resulting two intervals and $s_2 \notin I_2$ (where I_2 is one of the intervals [does not matter which one] obtained after removing the middle half of I_1, so $I_2 \subset I_1$). If we continue in this fashion, after m steps, we get $I_m \subset I_{m-1} \cdots \subset I_2 \subset I_1$ with $s_m \notin I_m$. Finally, we have found a point $x \in \bigcap_{n=1}^{\infty} I_n$ s.t. $x \in F$ but $\forall n, x \neq s_n$. Hence, we face a contradiction and F is uncountable.

$\square$

(e) Notice that F_1 has 2 intervals of the length $\frac{2}{5}$. F_2, on the other hand, has 2^2 intervals of the size $\frac{2^2}{5^2}$. F_3 consists of 2^3 intervals of the size $\frac{2^3}{5^3}$ and so forth. Hence, in general, F_n consists of 2^n intervals of the size $\frac{2^n}{5^n} = \left(\frac{2}{5}\right)^n$. Hence, magnifying F by the factor of $\frac{5}{2}$ will give us 2 additional copies of F. The equation will be $2 = \left(\frac{5}{2}\right)^n$ (for instance, $[0,1] \times 2.5 = [0, 2.5]$ and after taking the middle-fifth, we get two copies $[0,1]$ and $[1.5, 2.5]$) and thus, the dimension of F is

$$\boxed{\dim F = \frac{\log 2}{\log \frac{5}{2}} = \frac{\log 2}{\log 5 - \log 2}}$$

4. (a) According to **Definition 4.2.1 (Functional Limit)**, we have to show that $\forall\epsilon > 0, \exists\delta > 0$ s.t. $0 < |x-3| < \delta \implies |x^2-5x+4-(-2)| < \epsilon$. Let $\epsilon > 0$ be given. Let $\delta = -0.5 + \sqrt{0.25+\frac{\epsilon}{2}}$ ($\delta > 0$ since $\sqrt{0.25+\frac{\epsilon}{2}} > 0.5$). Then suppose that $|x-3| = |x-3| < -0.5 + \sqrt{0.25+\frac{\epsilon}{2}}$ Notice that:

$$\begin{aligned} |x^2-5x+4-(-2)| &= \left|x^2-5x+6\right| \\ &= \left|(x-2)(x-3)\right| \\ &= \left|-0.5+\sqrt{0.25+\frac{\epsilon}{2}}+1\right| \times \left|-0.5+\sqrt{0.25+\frac{\epsilon}{2}}\right| \\ &< \left|\sqrt{0.25+\frac{\epsilon}{2}}+0.5\right| \times \left|\sqrt{0.25+\frac{\epsilon}{2}}-0.5\right| \\ &= \left|0.25+\frac{\epsilon}{2}-0.25\right| \\ &= \left|\frac{\epsilon}{2}\right| = \frac{\epsilon}{2} < \epsilon \end{aligned}$$

Hence, we showed that $\forall\epsilon > 0, \exists\delta = -0.5 + \sqrt{0.25+\frac{\epsilon}{2}}$ s.t. $0 < |x-3| < \delta \implies |x^2-5x+4-(-2)| < \epsilon$.

$\square$

(b) Per **Exercise 4.2.9 (b)** that I have completed as a part of the assignment, we can say $\lim_{x\to\infty} f(x) = L$ if $\forall\epsilon > 0, \exists M > 0$ s.t. if $x > M$ we have $|f(x)-L| < \epsilon$. Let us now show that $\lim_{x\to\infty} \frac{2x}{x+4} = 2$. Let $\epsilon > 0$ be given and let $M = \frac{8}{\epsilon}$. Then if $x > M$, we have $x > \frac{8}{\epsilon}$. We have $\frac{2x}{x+4} = |\frac{2\frac{8}{\epsilon}}{\frac{8}{\epsilon}+4} - 2| = \frac{8}{\frac{8}{\epsilon}+4} = \frac{2\epsilon}{\epsilon+2} = \epsilon - \frac{4}{\epsilon+2} < \epsilon$.

Hence, $\lim_{x\to\infty} \frac{2x}{x+4} = 2$.

$\square$

5. We need to prove that $\forall c \in [0, \infty)$ and $\forall \epsilon > 0, \exists \delta > 0$ s.t. whenever $|x - c| < \delta$ (with $x \in [0, \infty)$), it follows that $|\sqrt[4]{x} - \sqrt[4]{c}| < \epsilon$. Let $\epsilon > 0$ be given. Now, let us consider the following two cases:

(1) $c = 0$

If $c = 0$, let $\delta = \epsilon^4$. Then $|x - c| = |x - 0| = |x| < \epsilon^4$. Now, $|\sqrt[4]{x} - \sqrt[4]{0}| = |\sqrt[4]{x}| < \epsilon$ is true as if we raise both sides of the inequality to the power of four, we get $|x| < \epsilon^4$ which is true. Hence, we have that $|x - c| < \delta$ implies $|\sqrt[4]{x} - \sqrt[4]{c}| < \epsilon$.

□

(2) $c > 0$

If $c > 0$, let $\delta = \epsilon\sqrt[4]{c}$. Then $|x - c| < \epsilon\sqrt[4]{c}$. Consider $|\sqrt[4]{x} - \sqrt[4]{c}|$. Now, notice that:

$$
\begin{aligned}
|\sqrt[4]{x} - \sqrt[4]{c}| &= |\sqrt{x} - \sqrt{c} \times \frac{1}{\sqrt[4]{x} + \sqrt[4]{c}}| \\
&= |\sqrt{x} - \sqrt{c}| \times \frac{1}{\sqrt[4]{x} + \sqrt[4]{c}} \\
&= |x - c| \times \frac{1}{(\sqrt[4]{x} + \sqrt[4]{c})(\sqrt{x} + \sqrt{c})} \\
&< \frac{|x - c|}{\sqrt[4]{c^3}} \\
&\leq \frac{|x - c|}{\sqrt[4]{c}} \\
&< \frac{\epsilon\sqrt[4]{c}}{\sqrt[4]{c}} = \epsilon
\end{aligned}
$$

Hence, we have that $|x - c| < \delta$ implies $|\sqrt[4]{x} - \sqrt[4]{c}| < \epsilon$.

□

Thus, we have now shown that $\forall c \in [0, \infty)$ and $\forall \epsilon > 0, \exists \delta > 0$ s.t. whenever $|x - c| < \delta$ (with $x \in [0, \infty)$), it follows that $|\sqrt[4]{x} - \sqrt[4]{c}| < \epsilon$.

□

6. Note that function $f : \mathbb{R} \to \mathbb{R}$ would not be well-defined if repeating 9s were allowed. If repeating 9s are allowed, then the decimal expansion of the number is not unique since $1 = 0.9999...$ and the function $f : \mathbb{R} \to \mathbb{R}$ is not well-defined. Hence, we do not allow for repeating 9s.

   $f : \mathbb{R} \to \mathbb{R}$ is not continuous at points in $\left\{ \frac{n}{10} \mid n \in \mathbb{Z}, n \neq 0 \right\}$. Hence, $f : \mathbb{R} \to \mathbb{R}$ is not continuous at points $\cdots - 0.9, -0.8, \ldots, -0.1, 0.1, 0.2, \ldots 0.8, 0.9 \ldots$.

   Consider an real number $\left\{ \frac{n}{10} \mid n \in \mathbb{Z}, n \neq 0 \right\}$. Notice that $r = a.b$ s.t. $a \in \mathbb{Z}$ and $b \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Then, due to the density property, $\exists (x_n) \subseteq \mathbb{R}$ s.t. $(x_n) \to r$. In fact, we can build (x_n) ourselves. For $r = a.b$ (with $a \in \mathbb{Z}$ and $b \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, by considering the following two cases:

   (i) $b = 0$

   If $b = 0$, $r = a.0$.

   Now, if $a > 0$, pick $x_n = (a - 1).9999 \cdots \to r$. Then $f(r) = a.1$ and $f(x_n) = (a - 1).1999 \cdots = (a - 1).12$. Thus, we have $\lim f(x_n) \neq f(r)$ and the function is not continuous at r.

   If $a < 0$, pick $x_n = (a + 1).9999 \cdots \to r$. Then $f(r) = a.1$ and $f(x_n) = (a + 1).1999 \cdots = (a - 1).12$. Thus, we have $\lim f(x_n) \neq f(r)$ and the function is not continuous at r.

   (ii) $b \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

   If $b \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $r = a.b$ with $b \neq 0$.

   Now, pick $x_n = a.(b - 1)9999 \cdots \to r$. Then $f(r) = a.1$ and $f(x_n) = a.1999 \cdots = a.12$. Thus, we have $\lim f(x_n) \neq f(r)$ and the function is not continuous at r.

   Finally, we have shown that $f : \mathbb{R} \to \mathbb{R}$ is not continuous at points in $\left\{ \frac{n}{10} \mid n \in \mathbb{Z}, n \neq 0 \right\}$.

   $\square$

   It is easy to see that $f : \mathbb{R} \to \mathbb{R}$ is continuous at all points that are not in $\left\{ \frac{n}{10} \mid n \in \mathbb{Z}, n \neq 0 \right\}$.

   Recall that for a function $f : \mathbb{R} \to \mathbb{R}$ to be continuous, it must be the case that $\forall (x_n) \to c$, (with $x_n \in \mathbb{R}$), it follows that $f(x_n) \to f(c)$ (**Theorem 4.3.2 (Characterizations of Continuity) (iii)**. Consider an arbitrary real number $r = a.b_1b_2b_3b_4\cdots \in \mathbb{R}$. Then, due to the density property, $\exists(x_n) \subseteq \mathbb{R}$ s.t. $(x_n) \to r$. Notice that $f(r) = a.1b_2b_3b_4\ldots$ and $f(x_n) = a.1b_2b_3b_4\ldots$ (This is due to $r \notin \left\{\frac{n}{10} \mid n \in \mathbb{Z}, n \neq 0\right\}$. In other words, there is no way to change anything in the first position that will affect the rest of the expansion and thus, $\lim f(x_n) = f(r)$). Hence, we got that $f(x_n) \to f(r)$ and $f : \mathbb{R} \to \mathbb{R}$ is continuous at all points not in $\left\{\frac{n}{10} \mid n \in \mathbb{Z}, n \neq 0\right\}$.

□

7. Let us first prove that $f(x) = 1/x^2$ is uniformly continuous on $[0, \infty)$. Let $x, y \in [1, \infty)$ and let $\epsilon > 0$ be set. Then we have:

$$\begin{aligned}\left|\frac{1}{x^2} - \frac{1}{y^2}\right| &= \left|\frac{y^2 - x^2}{x^2y^2}\right| \\ &= \left|\frac{(x+y)(x-y)}{x^2y^2}\right| \\ &= \frac{x+y}{x^2y^2}|x-y|\end{aligned}$$

Since $x, y \in [1, \infty)$, it follows that $\frac{x+y}{x^2y^2} \leq 2$ and for $x, y \in [1, \infty)$, we have:

$$\left|\frac{1}{x^2} - \frac{1}{y^2}\right| \leq 2|x-y|$$

Now, let $\delta = \frac{\epsilon}{2}$. Then we have $|x-y| < \delta$ and it follows that $\left|\frac{1}{x^2} - \frac{1}{y^2}\right| < \epsilon$. Hence, by **Definition 4.4.4 (Uniform Continuity)**, $f(x) = 1/x^2$ is uniformly continuous on $[0, \infty)$.

□

Let us now prove that $f(x) = 1/x^2$ is not uniformly continuous on the interval $(0, 1]$. Suppose, for the sake of contradiction, that $f(x)$ is uniformly continuous on $(0, 1]$. Then for $\epsilon > 0$ there must exist $\delta > 0$ s.t. $\forall x, y \in (0, 1]$ with $|x-y| < \delta$, it follows that $|f(x) - f(y)| < \epsilon$. Now, let $x = \frac{2}{n}$ and $y = \frac{1}{n}$ with $n \geq 2$. We have that $|x-y|$ implies

$$\left|\frac{1}{x^2} - \frac{1}{y^2}\right| = \left|\frac{\frac{3}{n^2}}{\frac{4}{n^2}}\right| = \frac{3}{4} < \epsilon$$

Now, we got that $\epsilon > \frac{3}{4}$, however, $|f(x) - f(y)| < \epsilon$ must be true $\forall \epsilon > 0$. Hence, we face a contradiction and $f(x) = 1/x^2$ is not uniformly continuous on $(0, 1]$.

□

Finally, we have shown that $f(x) = 1/x^2$ is uniformly continuous on $[0, \infty)$, but not on $(0, 1]$.

$\square$

8. (a) $g = \sin x$ is Lipschitz on $[0, 10]$. Below find the sketch:

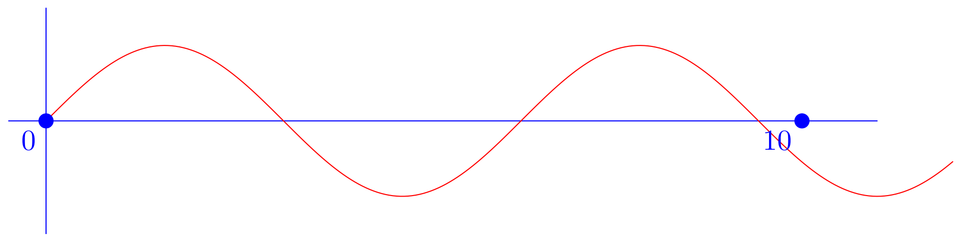

Figure 2: Plot of $g = \sin x$ which is Lipschitz on $[0, 10]$.

$h = \sqrt{x}$, on the other hand, is continuous, but not Lipschitz on $[0, 10]$. Below find the sketch:

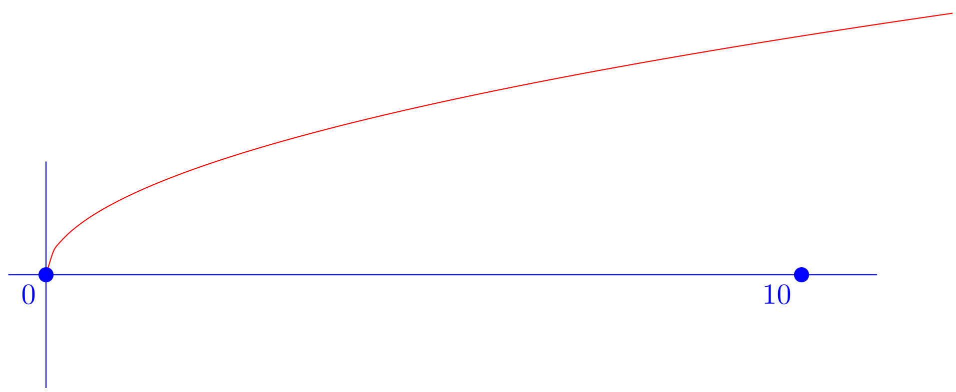

Figure 3: Plot of $h = \sqrt{x}$ which is continuous, but not Lipschitz on $[0, 10]$.

In a more general sense, a function being Lipschitz means that it does not become infinitely steep at some point. In other words, the slope of the line joining $(x, f(x))$ and $(y, f(y))$ is always bounded by some M. Therefore, the graph will be more or less uniform in terms of the slope (as was the case with $\sin x$).

(b) Let $\epsilon > 0$ be given and let $\delta = \frac{\epsilon}{M}$. Now, if $|x - y| < \delta$ with $x, y \in A$. Thus, we have:

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq M \implies f(x) - f(y) \leq M|x - y| < M\delta = \epsilon$$

Hence, $f : A \to \mathbb{R}$ is uniformly continuous on A.

$\square$

(c) No, if f is uniformly continuous on A, f is not necessarily Lipschitz on A.

Counterexample: Consider $f : [0,1] \to \mathbb{R} : x \mapsto \sqrt{x}$. Then it is easy to see that f is uniformly continuous on $[0,1]$. However, f is not Lipschitz as $\forall M > 0$ if we take $x \in (0, \frac{1}{M^2})$ and set $y = 0$, we get:

$$\left| \frac{f(x) - f(y)}{x - y} \right| = \frac{1}{\sqrt{x}} > M$$

Hence, uniform continuoity A does not imply the Lipschitz property on A.

$\square$