Homework №11

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77. Suppose A and B are sets and that $f : A \to B$ is a function. Define the function $\mathcal{F} : A \times A \to B \times B : (a_1, a_2) \mapsto (f(a_1), f(a_2))$.

   (a) Prove or Disprove: If f is onto, so is $\mathcal{F}$.

   It's true.

   Proof: Suppose that $f : A \to B$ is an onto function. Then, we know that every element of B has at least one element of A mapped to it. Now, consider $\mathcal{F} : A \times A \to B \times B : (a_1, a_2) \mapsto (f(a_1), f(a_2))$. To prove that $\mathcal{F}$ is onto, we have to show that for every $x \in B \times B$, we can find $y \in A \times A$ that maps to it. Every $x \in B \times B$ will have a type $(f(a_1), f(a_2))$. Let's take a look at $(f(a_1), f(a_2))$. We have $f(a_1) \in B$ and $f(a_2) \in B$ because $f : A \to B$ is onto. Finally, since $B \times B$ is a set of all possible pairs of B, we get that for arbitrary $(a_1, a_2) \in A \times A$, $(f(a_1), f(a_2)) \in B \times B$. And we showed that $\mathcal{F}$ is onto. □

   (b) Prove or Disprove: If f is one-to-one, so is $\mathcal{F}$.

   It is true.

   Suppose $f : A \to B$ is a one-to-one function. Then we know that every $a \in A$, maps to a different $b \in B$. Consider $\mathcal{F} : A \times A \to B \times B : (a_1, a_2) \mapsto (f(a_1), f(a_2))$. Since we know that all the ordered pairs in the set $A \times A$ are different (fortunately, sets do not allow duplicates), we can say that every (a_1, a_2) will map to a different (b_1, b_2).

   Extended proof:

   Suppose, for the sake of contradiction, that $(a_1, a_2), (a_3, a_4) \in A \times A$, $(b_1, b_2) \in B \times B$ where $(a_1, a_2) \neq (a_3, a_4)$, and that $\mathcal{F}((a_1, a_2)) = (b_1, b_2)$ and $\mathcal{F}((a_3, a_4)) = (b_1, b_2)$. Our assumption is equivalent to assuming that $(f(a_1), f(a_2)) = (b_1, b_2)$ and $(f(a_3), f(a_4)) = (b_1, b_2)$ where f is one-to-one (according to the problem description). Since, by our assumption, $(a_1, a_2) \neq (a_3, a_4)$, it means that either $a_1 \neq a_3$ or $a_2 \neq a_4$ or both). In any case, since f is one-to-one, tuples

$(f(a_1), f(a_2))$ and $(f(a_2), f(a_3))$ will be different and thus $(f(a_1), f(a_2)) \neq (f(a_2), f(a_3))$. Hence, we've reached the contradiction and $\mathcal{F}$ is one-to-one.

78. Suppose S is a set.

(a) If there is a function $f : S \to \emptyset$, what does that tell you about S?

Every function g is a subset of the cartesian product of its domain and codomain. Thus, f is a subset of $S \times \emptyset = \emptyset$. Then the only to have a function being a subset of $\emptyset$ is to make its domain an empty set thus $S = \emptyset$ (it is vacuously true). Thus, if $f : S \to \emptyset$ is a function, it tells me that $S = \emptyset$.

(b) If there is an onto function $g : \emptyset \to S$, what does that tell you about S?

It tells me that $S = \emptyset$. It is vacuously onto because everything in the codomain ($\emptyset$) has an element in domain ($\emptyset$) that maps to it. Hence, it is a nothing-to-nothing onto function. Thus, if there is an onto function $g : \emptyset \to S$, it tells me that $S = \emptyset$.

(b) If there is a one-to-one function $h : \emptyset \to S$, what does that tell you about S?

It tells me that $S = \emptyset$. It is vacuously one-to-one because everything in the domain ($\emptyset$) maps to a different element in the codomain ($\emptyset$). In all other cases, where S is non-empty, we have that nothing maps to some element which is vacuously not true. Thus, if there is an onto function $g : \emptyset \to S$, it tells me that $S = \emptyset$.

79. Suppose that S is an arbitrary set, and that $f : S \to \mathcal{P}(S)$ is a function. Prove that f is not onto. [Big Hint: Consider the set $A = \{x \in S \mid x \notin f(x)\}$. Is A in the image of f?]

Consider the set $A = \{x \in S \mid x \notin f(x)\}$. In other words, set A is a set of all elements which are in S but not in $f(x)$. Then we know that $A \in \mathcal{P}(S)$ because A is the set of all elements of S except for the empty set if S contains one. Now, all we have to do is to show that $f(x) \neq A$ where $x \in S$.

To prove that $f(x) \neq A$ for all $x \in S$, consider the following two cases:

1. Case I: $x \in A$.
2. Case II: $x \notin A$.

Case I Proof: If $x \in A$, then we know that $x \notin f(x)$. Thus, $f(x) \neq A$.
Case II Proof: If $x \notin A$, then we know that $x \in f(x)$. Thus, $f(x) \neq A$.

Hence, we got that there is no $x \in S$ such that $f(x) = A$ where $A = \{x \in S \mid x \notin f(x)\}$. Finally, we found an element in $\mathcal{P}(S)$ to which no element in S maps to and thus, f is not onto. $\square$

80. (a) If $f : A \to B$ is a function and $\hat{A}$ is a subset of A, we may define a function $\hat{f} : \hat{A} \to B : \hat{a} \mapsto f(\hat{a})$ called a **restriction** of f to $\hat{A}$. Prove that if $f : A \to B$ is one-to-one, the so is the restriction of f to every subset $\hat{A}$ of A.

    Suppose, for the sake of contradiction, that $f : A \to B$ is a one-to-one function and that $\hat{f} : \hat{A} \to B : \hat{a} \mapsto f(\hat{a})$ is not one-to-one (where $\hat{A}$ is arbitrary subset of A). Since f is one-to-one, know that for all $a_1, a_2 \in A$, if $f(a_1) = f(a_2)$, it means that $a_1 = a_2$. Now, consider $\hat{f} : \hat{A} \to B : \hat{a} \mapsto f(\hat{a})$. Since it is not one-to-one, we have that for some $\hat{a_1} \neq \hat{a_2}$, $f(\hat{a_1}) = f(\hat{a_2})$. Then, since $\hat{A}$ is a subset of A, it means that $a_1, a_2 \in A$ and that $f(a_1) = f(a_2)$ while $a_1 \neq a_2$ and we have reached the contradiction. Hence, if $f : A \to B$ is one-to-one, then $\hat{f} : \hat{A} \to B : \hat{a} \mapsto f(\hat{a})$ is also one-to-one. $\square$

    (b) Find a way analogous to the previous question to replace a function $f : A \mapsto B$ with a function $f : A \mapsto \hat{B}$ if $B \subseteq \hat{B}$. This is called **corestriction**. Prove that every corestriction of a one-to-one function is one-to-one. Is every corestriction of an onto function onto?

    The corestriction of f to $\hat{B}$ is $f : A \to \hat{B} : a \mapsto f(\hat{b})$.

    Proving that every corestriction of a one-to-one function is one-to-one.
    Suppose, for the sake of contradiction, that $f : A \to B$ is a one-to-one function and that $f : A \to \hat{B} : a \mapsto f(\hat{b})$ is not (where $B \subset \hat{B}$). Thus, we have that for any $a_1, a_2 \in A$, if $f(a_1) = f(a_2)$, then $a_1 = a_2$. And we also effectively assumed that there exists $a_3, a_4 \in A$ such that $a_3 \neq a_4$ and $f(\hat{a_3}) = f(\hat{a_4})$. Now consider the function $\hat{f} : A \to \hat{B} : a \mapsto f(\hat{b})$.

81. If A is a set, we call function $f : A \to \mathbb{R}$ **bounded above** if there exists $B \in \mathbb{R}$ such that for every $a \in A$, we have $f(a) \leq B$.

    (a) Prove that function $f : \mathbb{R} \to \mathbb{R} : x \mapsto sin(x)$ is bounded above. Be sure you explicitly use the definition.

Since the function is $sin(x)$ its codomain is the set of all values from -1 to 1. In other words, the codomain of f is a closed interval $[-1, 1]$. Then, we can let $B = 25$ and we have that for all x in $\mathbb{R}$, $-1 \leq f(x) \leq -1$ and thus for all $x \in reals$, $f(x) \leq B$. $\square$

(b) Prove that the function $g : \mathbb{R}^+ \to \mathbb{R} : x \mapsto \ln(x)$ is NOT bounded above. Be sure to explicitly use the definition.

Suppose, for the sake of contradiction, that there exists $B \in \mathbb{R}$, such that for all $a \in A$, we have that $f(a) \leq B$. Then it must be the case that $\ln(x)$ has some fixed upper bound. This is, however, false because $\lim_{x \to +\infty} \ln(x) = +\infty$. Hence, the function $\ln(x)$ has no upper bound and we have reached the contradiction.

(c) Prove or Disprove: No one-to-one function $f : \mathbb{R} \to \mathbb{R}$ is bounded above.

It is true so let's prove it. Suppose, for the sake of contradiction, that there exists $B \in \mathbb{R}$ such that for all x in the f's domain, $f(x) \leq B$. Now, since f is one-to-one, it means that if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$. Thus, f is the function that always increases and there is no upper bound. $\square$

(d)

(e)

(f)

(g)

(h)

(i)