Real Analysis Exams

Exam №4

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1. (a) Notice that we have:

$$\limsup a_n = \lim_{n\to\infty} \left(4 + \frac{1}{n}\right) \cos\left(\frac{n\pi}{4}\right) = 4 \times 1 = 4$$

Similarly, $\liminf a_n = 4 \times (-1) = -4$. Hence, $\limsup a_n = 4$ and $\liminf a_n = -4$.

(b) Notice that the subsequence $a_k = \left(4 + \frac{1}{8k}\right) \cos(2\pi k)$ ($n = 8k$ with $k \in \mathbb{N}$) converges to 0 since $\cos(2\pi k) = 0$ and thus, every $a_k = \left(4 + \frac{1}{8k}\right) \times 0 = 0$.

(c) This set will be $A = (-4, 4)$.

2. Let $\epsilon > 0$ be given, x_0 be fixed, and let $\delta = \min\left(1, \dfrac{\epsilon}{5(1+2|x_0|)}\right)$. Then for $|x - x_0| < \delta$ we have:

$$\begin{aligned}
|f(x) - f(x_0)| &= |5x^2 + 3 - 5x_0^2 - 3| \\
&= |5(x^2 - x_0^2)| \\
&= 5|x - x_0||x - x_0 + 2x_0| \\
&< 5\delta(|x - x_0| + |2x_0|) \\
&< 5\delta(\delta + 2|x_0|) \\
&\leq 5\frac{\epsilon}{5(1+2|x_0|)}(\delta + 2|x_0|) \\
&\leq \frac{\epsilon}{1+2|x_0|}(1 + 2|x_0|) \\
&= \epsilon \qquad\qquad (\text{Thus, } |f(x) - f(x_0)| < \epsilon)
\end{aligned}$$

Hence, $f(x) = 5x^2 + 3$ is continuous at each point $x_0 \in \mathbb{R}$.

$\square$

3. (a) Counterexample: let us define

$$f_n : [0,1] \to \mathbb{R} : x \mapsto \begin{cases} n \text{ if } 0 < x < \frac{1}{n} \\ 0 \text{ if } x = 0 \text{ or } \frac{1}{n} \leq x \leq 1 \end{cases}$$

Then notice that $\int_0^1 f_n = 1$ and the pointwise limit of f_n is $f(x) = 0$ ($x \in [0,1]$) for which we have $\int_0^1 f = 0$. Hence, we have $\int_0^1 f_n \to 1 \neq \int_0^1 f = 0$. Finally, we get that every f_n is Riemann-integrable, but f is not.

(b) As $f_n \to f$ uniformly, pick n_1 s.t. the following stands:

$$|f_{n_1}(x) - f(x)| < \frac{\epsilon}{3 \cdot (b - a)}$$

Now, since every f_n is integrable, take n_2 such that

$$|U(f_{n_1}, P_{n_2}) - L(f_{n_1}, P_{n_2})| < \frac{\epsilon}{3}.$$

Now, choose $n = \max(n_1, n_2)$,

Notice that

$$\begin{aligned}
|U(f,P_n) - U(f_n,P_n)| &\leq \sum_{x_k} |f(x_k) - f_n(x_k)| \Delta x_k \\
&< \sum_{x_k} \frac{\epsilon}{3(b-a)} \Delta x_k \\
&= \frac{\epsilon}{3(b-a)} \sum_{x_k} \Delta x_k \\
&= \frac{\epsilon}{3(b-a)} (b-a) \\
&= \frac{\epsilon}{3}
\end{aligned}$$

Now, notice that over $[x_k, x_{k+1}]$, $|\sup f(x) - \sup f_n(x)| \leq |f_n(x) - f(x)|$ (since every point of f_n is close to f).

A similar results holds for

$$|L(f, P_{n_2}) - L(f_n, P_{n_2})| < \frac{\epsilon}{3}$$

Hence, we have:

$$\begin{aligned}
|U(f,P_n) - L(f,P_n)| &\leq \Big| U(f,P_n) - U(f_n,P_n) + U(f_n,P_n) - L(f_n,P_n) - \Big(L(f,P_n) - L(f_n,P_n) \Big) \Big| \\
&\leq |U(f,P_n) - U(f_n,P_n)| + |U(f_n,P_n) - L(f_n,P_n)| + |L(f,P_n) - L(f_n,P_n)| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon
\end{aligned}$$

Finally, we got that if $f_n \to f$ uniformly on $[a,b]$, then f is integrable on $[a,b]$.

□

(c) We have already shown in (b) part of the exercise that f is integrable on $[a,b]$. We now need to show that the following stands:

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$$

It follows by the uniform convergence of f_n that

$$\begin{aligned}
|f_n - f| &< \epsilon & &\implies f - \epsilon < f_n < f + \epsilon & (1) \\
\int_a^b \Big(f - \epsilon\Big) &< \int_a^b f_n < \int_a^b \Big(f + \epsilon\Big) & &\implies \Big|\int_a^b f_n - \int_a^b f\Big| < \epsilon(b-a) & (2)
\end{aligned}$$

Hence, we can make the difference $|\int_a^b f_n - \int_a^b f|$ arbitrarily small and we get $\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$.

□

4. Placeholder.

5. Placeholder.

6. Placeholder.

7. Since f is continuous on $[a, b]$, it follows that f achieves both the absolute maximum M and the absolute minimum m in $[a, b]$. Suppose, without a loss of generality, that these points are c_1 and c_2 with $c_1 < c_2$. We then have:
$$m(b-a) \leq \int_a^b f \leq M(b-a) \tag{3}$$
$$m \leq \frac{1}{b-a}\int_a^b f \leq M \tag{4}$$
Now, it follows by **Intermediate Value Theorem** that $\exists c \in (c_1, c_2) \subset [a, b]$ s.t. $f(c) = \frac{1}{b-a}\int_a^b f$.
$\square$

8. Suppose, for the sake of contradiction, that f has Generalized Riemann integrals Q_1 and Q_2 with $Q_1 \neq Q_2$ and let $\epsilon > 0$ be given. Then,.it follows that $\exists \delta_1(x)$ s.t. $\forall \delta_1(x)$-fine tagged partitions, $|R(f, P) - Q_1| < \frac{\epsilon}{2}$. Similarly, $\exists \delta_2(x)$ s.t. $\forall \delta_2(x)$-fine tagged partitions, $|R(f, P) - Q_2| < \frac{\epsilon}{2}$. Now, let $\delta(x) = \min(\delta_1(x), \delta_2(x))$. It follows by **Theorem 8.1.5** that there exists a tagged partition $(P, \{c_k\})$ s.t. it is both $\delta_1(x)$-fine and $\delta_2(x)$-fine. We have:
$$|Q_1 - Q_2| \leq |Q_1 - R(f, P)| + |R(f, P) - Q_2| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$
Hence, we got that $Q_1 = Q_2$ and we face a contradiction since we have assumed that $Q_1 \neq Q_2$. Finally, we conclude that if f has a generalized Riemann integral on $[a, b]$, then the value of the integral $\int_a^b f$ is unique.
$\square$

9. (a) Placeholder.
   (b) Placeholder.
   (c) Placeholder.
   (d) Placeholder.
   (e) Placeholder.
   (f) Placeholder.