Homework №8

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Additional Proof Practice

53. A subset S of $\mathbb{Z}^+$ is called a P_3-**set** if there exists (not necessarily distinct) elements $x, y, z \in S$ such that $x + y + z$ is prime.

(a) Give some examples of P_3-sets.

$\{1\}$ because $1 + 1 + 1 = 3$ is a prime.
$\{2, 3\}$ because $2 + 3 = 5$ is a prime.
$\{12, 25, 30\}$ because $12 + 25 + 30 = 67$ is a prime.

(b) Prove or Disprove: If A is a P_3-set, and $A \subseteq B \subseteq \mathbb{Z}^+$, then B is a P_3-set.

It's right so let's prove it. Since we know that A is a P_3-set, we know there exist elements (not necessarily distinct) x, y and z such that $x + y + z$ is prime. Since $A \subseteq B$, we know that all elements of A are also in B meaning that x, y and z are in B as well and there exist elements x, y, z (which are also in A) such that $x + y + z$ is a prime.

Q.E.D.

(c) Prove or Disprove: If S is a P_3-set, then so is $S_{+3} := \{x + 3 \mid x \in S\}$.

It's false. Counterexample: Let $S = \{1\}$, then we know that S is a P_3-set since $1 + 1 + 1 = 3$ is a prime. However, $S_{+3} = \{4\}$ and $4 + 4 + 4 = 12$ which is certainly composite.

(d) Prove or Disprove: Every P_3 set contains a prime.

False. Let $S = \{1\}$, then 1 is not a prime but $1+1+1 = 3$ is a prime.

(e) Prove or Disprove: The intersection of two P_3-sets is a P_3-set.

It's false. Let $A = \{1\}$, then A is P_3-set since $1 + 1 + 1 = 3$ is a prime. Let $B = \{2, 3\}$, then $2+3 = 5$ is a prime thus B is also P_3-set. $A \cap B = \emptyset$ which means that there are no elements x, y, z such that $x + y + z$ is a prime and thus, the intersection of two P_3-sets is not necessarily a P_3-set.

(f) Prove or Disprove: Every P_3-set contains an odd integer.

It's true. Suppose, for the sake of contradiction, that S is a P_3-set and it does not contain any odd integers. Thus, it means that all the elements of S are even. Now, since all the elements are even, it means that no matter what 3 elements x, y and z we take, their sum will always be even. On the other hand, the only even prime we have is 2. But unfortunately, there are no three numbers $x, y, z \in \mathbb{Z}^+$ which sum up to 2. The best we can do is $1 + 1 + 1$ which is 3 and is one more than 2. Thus, there is no way to get 2 and otherwise, we won't have 3 elements which sum up to the prime. Hence, we have reached the contradiction and S is a not a P_3-set.

Q.E.D.

(g) Prove or Disprove: Every infinite subset S of $\mathbb{Z}^+$ is a P_3-set.

It's false. Since we already proved that every P_3-set contains an odd integer, we can take a set of all positive even integers which is a subset of $\mathbb{Z}^+$. Let's call this subset E. Then, we know that every element of the subset E is even and sum of any 3 elements (not necessarily distinct) will also be even. However, once again, the only even integer which is a prime is 2 and we cannot get 2 by summing 3 integers which are greater than or equal to 2 (greater than equal because $E = \{2, 4, 6, 8, 10...\}$).

(h) Prove or Disprove: If S is a finite subset of $\mathbb{Z}^+$, then $\mathbb{Z}^+ - S$ is a P_3-set.

It's true. Let's prove it. Since S is a finite set, we know that it cannot contain all the elements of $\mathbb{Z}^+$ because $\mathbb{Z}^+$ is infinite. We already proved that there are infinitely many primes. Then we can find a prime p such that $p - 2 \notin S$. then, we can have a set $L = \{1, p - 2\}$ which is a P_3-set because $1 + 1 + p - 2 = p$ is a prime.

Q.E.D.

54. If a subset S of $\mathbb{Z}^+$ is a P_3-set then the **core** of S is the set

$$\text{core}(S) := \{s \in S \mid S - \{s\} \text{ is not a } P_3\text{-set}\}.$$

(a) What is the core of $S = \{2, 3, 6\}$?

The core of $S = \{2, 3, 6\}$ is $\text{core}(S) = \{2, 3\}$. The reason is that if we take out 2, we are left with 3 and 6 which are both multiples of 3 and any variations of their sums will never be a prime $(3 + 3 + 3$ is a not a prime, $3 + 6 + 3$ is not a prime etc.). If we take out 3, we are left with two even numbers, namely 2 and 6, and still we know that every P_3-set contains an odd integer thus, taking out 3 will leave us with non-P_3-set (any variations of the sums of the even integers will not be even; the only case is 2 but 2 + 2 + 2 = 6 is the best we can do). On the other hand, if we take out 6, S will still be a P_3-set since 2 + 2 + 3 = 7 is a prime. Thus, core$(S) = \{2, 3\}$.

(b) Give an example of a P_3-set whose core is the empty set, or prove none exists.

Here is an example: let $S = \{1\}$, then S is a P_3 set since 1 + 1 + 1 = 3 is a prime. However, if we take out 1, we have $S = \emptyset$ and there are no x, y, z such that $x + y + z$ is a prime.

(c) Give an example of a P_3-set whose core is infinite, or prove none exists.

There is no such P_3-set. Let's prove it. Suppose, for the sake of contradiction, there exists a P_3 set S such that its core is infinite. Let the core be the set $C = \{c_0, c_1, c_2, c_3\ ...\}$. Then, we know that if we took out c_0, the set S would not be P_3. Then, since c_0 affected the outcome of whether S is a P_3-set or not, it means that c_0 plays a role in $x + y + z$. Same goes if we took out c_1. The same for c_2, and the same for c_4, c_5... etc. However, since c_0, c_1 and c_2 play a role in the sum, we know that there are at most 3 different elements of the set in the sum as by the definition $x + y + z$ must be a prime. But here we see that infinitely many elements are in this sum and we reached a contradiction. Thus, there are no P_3-set whose core is finite.

(d) Prove or Disprove: If S is a P_3-set then core(S) is a P_3-set.

It's false. Counterexample: let $S = \{2, 3, 5\}$. Then core$(S) = \{3\}$ since if we take out 2, 3 + 3 + 5 = 11 is still a prime and if we take out 5, 2 + 2 + 3 = 7 is still a prime. However, if we take out 3, all the possible sums are:
2 + 2 + 2 = 6 is not a prime
2 + 2 + 5 = 9 is not a prime
2 + 5 + 5 = 12 is not a prime
5 + 5 + 5 = 15 is not a prime

Thus, we end up with a set core$(S) = \{3\}$ which is not a P_3-set since 3 + 3 + 3 = 9 is the only sum we can get and it is not a prime.

(e) Prove or Disprove: If S and T are P_3-sets then core$(S \cup T) \subseteq$ core$(S) \cap$ core(T).

It's true, let's prove it. Suppose, for the sake of contradiction, that for some two P_3-sets S_1 and S_2, core$(S_1) \cap$ core$(S_2) \subset$ core$(S_1 \cup S_2)$. Then we know that there exists $x \in$ core$(S_1 \cup S_2)$ such that $x \notin$ core$(S_1) \cap$ core(S_2). Now, since $x \in$ core$(S_1) \cap$ core(S_2), it means that $x \in$ core(S_1) and $x \in$ core(S_2).

(f) Prove or Disprove: If S and T are P_3-sets with $S \subseteq T$ then we have $\text{core}(T) \subseteq \text{core}(S)$.

Suppose S_1 and S_2 are two P_3-sets and $S_1 \subseteq S_2$. Then for all $x \in S_1$, $x \in S_2$. If there are k elements in $\text{core}(S_1)$, it means that $\text{core}(S_2)$ will

55. A subset S of $\mathbb{Z}$ is called threequaline if for every $x, y \in S$ one has $3 \mid (x - y)$.

(a) Prove or Disprove: Every subset of a threequaline set is threequaline.

It's false. Let, for the sake of contradiction, that S is a threequaline set and also suppose that all the subsets of S are threequaline. Then we know that an empty set is a subset of every set and S is also a set thus, the emptyset is also a subset of S. However, an empty set has no elements and we cannot find x, y such that $x - y$ is a multiple of 3 and we reached the contradiction.

Q.E.D.

(b) Prove that if S is threequaline than either every element of S is divisible by 3 or none are.

Suppose, for the sake of contradiction, that S is a threequaline set and there exists two elements x, y such that x is a multiple of 3 and y is not a multiple of 3. Then $x - y$ will not be a multiple of 3 and we have reached the contradiction.

Q.E.D.

(c) Prove that if S is threequaline and r and t are integers, then the set $\{rx + t \mid x \in S\}$ is also threequaline.

Since we know that S is a threequaline set, for every $x, y \in S$, $x - y$ is a multiple of 3. Suppose, z_1, z_2 are some elements of the set S. Then, we know that $z_1 - z_2$ is a multiple of 3. The new set will "transform" these elements into $rz_1 + t$ and $rz_2 + t$. On the other hand, $z_1 - z_2 = rz_1 + t - (rz_2 + t) = r(z_1 - z_2)$ which is a multiple of 3 since z_1, z_2 are the members of S and $z_1 - z_2$ is a multiple of 3.

Q.E.D.

(d) Prove that if S and T are threequaline and $S \cap T \neq \emptyset$ then $S \cup T$ is threequaline.

Suppose, for the sake of contradiction, that S_1 and S_2 are threequaline sets and $S_1 \cap S_2 \neq \emptyset$ and let's prove that $S_1 \cup S_2$ is not a threequaline. Since $S_1 \cap S_2 \neq 0$, there exists element x, such that

$x \in S_1$ and $x \in S_2$. Let's consider the following cases:

Case I : x is divisible by 3, thus $x = 3k$ where $k \in \mathbb{Z}$
Case II: x gives remainder of 1 when divided by 3, thus $x = 3k + 1$ where $k \in \mathbb{Z}$
Case III: x gives remainder of 2 when divided by 3, thus $x = 3k + 2$ where $k \in \mathbb{Z}$

In Case I, if $x = 3k$, then all the other elements of S_1 as well as S_2 must be of the type $3l$ where $l \in \mathbb{Z}$ and all the elements in the union of S_1 and S_2 will be the multiples of 3 which means that for all $i, j \in S_1 \cup S_2$, $i - j$ is a multiple of 3. And we reached the contradiction.

In Case II, if $x = 3k + 1$, then all the other elements of S_1 as well as S_2 must be of the type $3l + 1$ where $l \in \mathbb{Z}$ and all the elements in the union of S_1 and S_2 will be the multiples of 3 plus 1 which means that for all $i, j \in S_1 \cup S_2$, $i - j$ is a multiple of 3. And we reached the contradiction.

In Case II, if $x = 3k + 2$, then all the other elements of S_1 as well as S_2 must be of the type $3l + 2$ where $l \in \mathbb{Z}$ and all the elements in the union of S_1 and S_2 will be the multiples of 3 plus 2 which means that for all $i, j \in S_1 \cup S_2$, $i - j$ is a multiple of 3. And we reached the contradiction.

56. A subset S of $\mathbb{R}$ is called **crunched** if there exist integers $m, n \in \mathbb{Z}$ such that for all $x \in S$ we have $m < x < n$.

NOTE: When I mention lower bound or upper bound, I really mean the smallest element or the biggest element of the set.

(a) Give some examples of sets that are and are not crunched.

$S_1 = \{1\}$ is crunched as for all $x \in S$, $0 < x < 2$ $(m = 0, n = 2)$.
$S_2 = \{1, 2, 3\}$ is crunched as for all $x \in S$, $0 < x < 4$ $(m = 0, n = 4)$.
$S_3 = \mathbb{Z}^+$ is not crunched as it has no bounds and we cannot find m, n such that for all $x \in \mathbb{Z}^+, m < x < n$.
$S_4 = \{2, 4, 6...\}$ (a set of positive even numbers) is not crunched as it has no upper bound and we cannot find n such that for all $x \in \mathbb{Z}^+, m < x < n$ (note: we can find m. m can be any integer that is less than or equal to 1 but n cannot be fixed).

(b) Prove or Disprove: All crunched sets are finite.

That's false. Counterexample: let $S = \{1, 1/2, 1/4, ...\}$ (infinite geometric series), then we know that S has an upper bound 1 and the lower bound which is 0. Then, we can say with the great certainty, that for all $x \in S$, $-10 < x < 10$ ($m = -10$, $n = 10$). Thus, crunched sets are not necessarily finite and the initial claim is false.

(c) Prove or Disprove: All finite sets are crunched.

It's true. Let S be a finite set. Then it must have a lower bound (the smallest element), let it be k_1 and the upper bound (the biggest element), let it be k_2. Then let $m = k_1 - 1$ and let $n = k_2 + 1$ and we have that for all $x \in S$, $m < x < n$.

Q.E.D.

(d) Prove or Disprove: Every subset of a crunched set is crunched

It's true. Suppose S is a crunched set. Then we know that for all $x \in S$, there exist m, n such that $m < x < n$. Let S_0 be a subset of S. Then, since all the elements of S_0 are also in S_1, we know that all elements of S_0 are between m and n which makes S_0 crunched. Thus, every subset of a crunched set is crunched.

Q.E.D.

(e) Prove or Disprove: The union of two crunched sets is crunched.

Suppose S_1 and S_2 are two crunched sets. Then we know that for all $x_1 \in S_1$, $m_1 < x_1 < n_1$ and for all $x_2 \in S_2$, $m_2 < x_1 < n_2$. Then, for all z in $S_1 \cup S_2, max(m_1, m_2) < z < max(n_1, n_2)$ which means that $S_1 \cup S_2$ is crunched.

Q.E.D.

57. We call a finite subset S of $\mathbb{Z}$ **balanced** if $|\mathbb{Z}^+ \cap S| = |\mathbb{Z}^- \cap S|$. (Recall that $\mathbb{Z}^- = \{-1, -2, -3, ...\}$).

NOTE: This statement really means that if A is balanced, the number of positive elements in it equals the number of negative elements in it.

(a) Prove or Disprove: If A is a balanced set then so is $A \cup \{0\}$.

It's true. Let's prove it.
Suppose A is a balanced set, then we know that $|\mathbb{Z}^+ \cap A| = |\mathbb{Z}^- \cap A|$. Now, since $0 \notin \mathbb{Z}^+$ and $0 \notin \mathbb{Z}^-$, it means that $\mathbb{Z}^+ \cap A = \mathbb{Z}^+ \cap (A \cup \{0\})$ and $\mathbb{Z}^- \cap A = \mathbb{Z}^- \cap (A \cup \{0\})$ and finally, $|\mathbb{Z}^+ \cap A \cup \{0\}| = |\mathbb{Z}^- \cap (A \cup \{0\})|$
Thus, if A is a balanced set then so is $A \cup \{0\}$.

Q.E.D.

(b) Prove or Disprove: The union of two balanced sets is balanced.

It's false. Counterexample: let $S_1 = \{-1, 1\}$ and $S_2 = \{-1, 5\}$. Then S_1 is balanced since $\mathbb{Z}^+ \cap S_1 = \{1\}$ and $\mathbb{Z}^- \cap S_1 = \{-1\}$ which means that $|\mathbb{Z}^+ \cap S_1| = |\mathbb{Z}^- \cap S_1| = 1$. S_2 is balanced too since $\mathbb{Z}^+ \cap S_2 = \{5\}$ and $\mathbb{Z}^- \cap S_2 = \{-1\}$ which means that $|\mathbb{Z}^+ \cap S_2| = |\mathbb{Z}^- \cap S_2| = 1$. On the other hand, set $S_1 \cup S_2$ is not balanced since $S_1 \cup S_2 = \{-1, 1, 5\}$ and $|\mathbb{Z}^+ \cap (S_1 \cup S_2)| = 2$ while $|\mathbb{Z}^- \cap (S_1 \cup S_2)| = 1$.

(c) Prove or Disprove: The intersection of two balanced sets is balanced.

It's false. Counterexample: let $S_1 = \{-1, 1\}$ and $S_2 = \{-1, 5\}$. Then S_1 is balanced since $\mathbb{Z}^+ \cap S_1 = \{1\}$ and $\mathbb{Z}^- \cap S_1 = \{-1\}$ which means that $|\mathbb{Z}^+ \cap S_1| = |\mathbb{Z}^- \cap S_1| = 1$. S_2 is balanced too since $\mathbb{Z}^+ \cap S_2 = \{5\}$ and $\mathbb{Z}^- \cap S_2 = \{-1\}$ which means that $|\mathbb{Z}^+ \cap S_2| = |\mathbb{Z}^- \cap S_2| = 1$. On the other hand, $S_1 \cap S_2$ is not balanced since $S_1 \cap S_2 = \{-1\}$ and $|\mathbb{Z}^+ \cap (S_1 \cap S_2)| = \emptyset$ while $|\mathbb{Z}^- \cap (S_1 \cap S_2)| = 1$.

(d) Prove or Disprove: For every $n \in \mathbb{Z}^+$ there exists a balanced set S with exactly n elements.

It's true. Let's prove it by construction. Let S be a set and for all $n \in \mathbb{Z}^+$, dump in some elements to make it balanced. If n is even, then we take $n/2$ elements that are positive and $n/2$ elements that are negative which will give us a balanced set since $|\mathbb{Z}^+ \cap S| = |\mathbb{Z}^- \cap S|$. If n is odd, then we can throw in 0 and then take $(n-1)/2$ positive elements and $(n-1)/2$ negative elements. This will guarantee that $|\mathbb{Z}^+ \cap S| = |\mathbb{Z}^- \cap S|$ since there will be exactly same number of positive and negative elements while 0 is neither in $\mathbb{Z}^+$, nor in $\mathbb{Z}^-$.

Q.E.D.

(e) If A is a subset of $\mathbb{Z}$ we denote by $\overline{A}$ the set $\{-a \mid a \in A\}$. Prove or Disprove: For every finite subset of A of $\mathbb{Z}$, the set $A \cup \overline{A}$ is balanced.

It's true. Let's prove it by cases.

Case I: suppose that A is a subset of $\mathbb{Z}$ such that it does not contain 0.
Case II: suppose that A is a subset of $\mathbb{Z}$ such that it does contain 0.

Proof of Case I: If A is a subset of $\mathbb{Z}$ such that it does not contain 0, we know that for all $x \in A$, we have $-x \in \overline{A}$. This means that the number of negative elements in $A \cup \overline{A}$ will be equal to the number of positive elements in $A \cup \overline{A}$ and because of this, $|\mathbb{Z}^+ \cap (A \cup \overline{A})| = |\mathbb{Z}^- \cap (A \cup \overline{A})|$.

Proof of Case II: If A is a subset of $\mathbb{Z}$ such that it does contain 0, for all $x \in A$, we have $-x \in \overline{A}$. This means that if we took out 0 out of $A \cup \overline{A}$, the number of positive and negative elements in $A \cup \overline{A}$ would be equal. On the other hand, 0 plays no role in determining whether $|\mathbb{Z}^+ \cap (A \cup \overline{A})| = |\mathbb{Z}^- \cap (A \cup \overline{A})|$ or not because $0 \notin \mathbb{Z}^+$ and $0 \notin \mathbb{Z}^-$. Thus, $|\mathbb{Z}^+ \cap (A \cup \overline{A})| = |\mathbb{Z}^- \cap (A \cup \overline{A})|$.

Q.E.D.

(f) Prove or Disprove: If A is balanced and $|A|$ is odd, then $0 \in A$.

It's true so let's prove it. Suppose, for the sake of contradiction, that A is balanced and $|A|$ is odd, but $0 \notin A$. Now, since $|A|$ is odd, it means that there is no way to have the number of positive elements be equal to the number of negative elements and we've reached a contradiction.

Q.E.D.

58. We call a subset S of $\mathbb{R}$ **positively scattered** if for every $x \in S$ there exists $y \in \mathbb{R}$ such that $y > x$ and $S \cap (x, y] = \emptyset$.

(a) Is $\mathbb{Z}^+$ positively scattered?

It is. Let's prove it by construction. If we take some element $x \in \mathbb{Z}^+$, then we know that it is positive. If $y > x$, we know that y is also positive and let $y = x + 0.1$. Then, we know that interval $(x, x + 0.1]$ contains no positive integers and thus, $\mathbb{Z}^+ \cap (x, x+01] = \emptyset$. Hence, for all $e \in \mathbb{Z}$, we can construct $(e, e + 0.1]$ (there are no positive integers in this interval) and finally, $\mathbb{Z}^+$ is indeed positively scattered.

Q.E.D.

(b) Is $[2, 3]$ positively scattered?

It is not. Suppose, for the sake of contradiction, that for all $x \in [2, 3]$, there exists y such that $y > x$ and $[2, 3] \cap (x, y] = \emptyset$. Now, since the condition holds for all $x \in [2, 3]$ and $[2, 3]$ is a closed interval, it should hold for 2 as well. Then, let $x = 2$. If the condition holds for $x = 2$, it means that we can find $y > x$ such that $[2, 3] \cap (2, y] = \emptyset$. Since $y > x$, $y = 2 + t$ where $t > 0$. Thus, we have $[2, 3] \cap (2, 2 + t] = \emptyset$, but this is impossible and to see why, let's consider two cases:

Case I: $t \geq 1$
Case I: $0 < t < 1$

If $t \geq 1$, it means that $[2, 3] \cap (2, 2 + t] = (2, 3]$ and we have reach the contradiction.

If $0 < t < 1$, it means that $[2, 3] \cap (2, 2 + t] = (2, 2 + t]$ and interval $(2, 2+t)$ is obviously infinite. Thus, we have, once again, reached the contradiction.

Thus, in all cases we've reached the contradiction and $[2, 3]$ is not positively scattered.

Q.E.D.

(c) Is $\{\frac{1}{n} \mid n \in \mathbb{Z}^+\}$ positively scattered?

It is. Let's prove it by construction. Let $x \in \{\frac{1}{n} \mid n \in \mathbb{Z}^+\}$, then we know that $x = \frac{1}{k}$ where $k \in \mathbb{Z}^+$. Now, let's take $y = \frac{1}{k + 0.5}$. Then, we know that $\{\frac{1}{n} \mid n \in \mathbb{Z}^+\} \cap (\frac{1}{k}, \frac{1}{k + 0.5}] = \emptyset$ because n is always an integer and there are no integer denominators in the interval $(k, k + 0.5]$.

Q.E.D.

(d) Is $\{\frac{1}{n} \mid n \in \mathbb{Z}^+\} \cup \{0\}$ positively scattered?

It is not. Suppose, for the sake of contradiction, that $\{\frac{1}{n} \mid n \in \mathbb{Z}^+\} \cup \{0\}$ is scattered. Then for all $x \in \{\frac{1}{n} \mid n \in \mathbb{Z}^+\} \cup \{0\}$, there exist $y > x$, such that $(\{\frac{1}{n} \mid n \in \mathbb{Z}^+\} \cup \{0\}) \cap (x, y] = \emptyset$. If the condition holds for all $x \in \{\frac{1}{n} \mid n \in \mathbb{Z}^+\} \cup \{0\}$, it should hold for $x = 0$ too ($x = 0$ is the member of the set as well as the set contains element 0). If $x = 0$, there must exist $y > 0$ such that $(\{\frac{1}{n} \mid n \in \mathbb{Z}^+\} \cup \{0\}) \cap (0, y] = \emptyset$. This, however, is impossible since the interval $(0, y]$ is infinite and we can always find n for which $\frac{1}{n} \in (0, y]$. Thus, we've reached the contradiction.

Q.E.D.

(e) Prove or Disprove: A subset of a positively scattered set is positively scattered.