Real Analysis

Assignment №10

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6.2.1 (a)

$$\boxed{f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{1+nx^2} = \frac{1}{x}}$$

(b) Notice that

$$f(x) = \frac{1}{x} - \frac{1}{\frac{1}{nx} + x} = \frac{1}{nx^3 + x}$$

Then it is easy to see that the convergence is **not uniform** since $\forall \epsilon > 0$ and $\forall n \in \mathbb{N}$, we can always have $x = \frac{1}{2n}$ s.t. $\frac{1}{nx^3+x} = \frac{8n^2}{4n+1} > \epsilon$ which shows that $|f(x) - f_n(x)| > \epsilon$.

(c) Similar to (b), it is **not uniform** on $(0, 1)$ since $\forall \epsilon > 0$ and $\forall n \in \mathbb{N}$, we can always have $x = \frac{1}{2n}$ s.t. $\frac{1}{nx^3+x} = \frac{8n^2}{4n+1} > \epsilon$ which shows that $|f(x) - f_n(x)| > \epsilon$.

(d) We have:

$$\Big|\frac{x}{nx^2 + x} - \frac{1}{x}\Big| = \Big|\frac{1}{nx^3 + x}\Big| < \frac{1}{n}$$

Hence, $\forall \epsilon > 0$ and $\exists N$ s.t. $\forall n \geq N, \frac{1}{n} < \epsilon$ and $|f(x) - f_n(x) < \epsilon|$. Finally, we got that the convergence **is uniform** on $(1, \infty)$.

6.2.3 (a) We have:

$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \frac{x}{1+x^n} = x \text{ if } x < 1$$
$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \frac{x}{1+x^n} = \frac{1}{2} \text{ if } x = 1$$
$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \frac{x}{1+x^n} = 0 \text{ if } x > 1$$
$$\textit{and}$$
$$\lim_{n\to\infty} h_n(x) = 0 \text{ if } x = 0$$
$$\lim_{n\to\infty} h_n(x) = 1 \text{ if } x > 0$$

Hence, (g_n) converges pointwise to

$$g(x) = \begin{cases} x \text{ if } x < 1 \\ \frac{1}{2} \text{ if } x = 1 \\ 0 \text{ if } x > 1 \end{cases}$$

And (h_n) converges pointwise to

$$g(x) = \begin{cases} 0 \text{ if } x - 1 \\ 1 \text{ if } x > 1 \end{cases}$$

(b) It follows by **Theorem 6.2.6 (Continuous Limit Theorem)**, that both (g_n) (pick $x = 1$) and (h_n) (pick $x = 0$) do not converge uniformly on $[0, \infty)$.

(c) For (g_n) consider a half-open interval $[0, 1)$. Then we have:

$$|g_n(x) - g(x)| = \frac{x^n}{1+x^n} < x^n$$

Now, as $x < 1$, it follows that $\forall \epsilon > 0, \exists N$ s.t. $\forall n \geq N, x^n < \epsilon$ and therefore the convergence is uniform on $[0, 1)$.

For (h_n) consider a half-open interval $[1, \infty)$. Pick $N = 2$, then $\forall \epsilon > 0$ and $\forall n \geq N$ we have:

$$|h_n(x) - h(x)| = 1 - 1 = 0 < \epsilon$$

Hence, the convergence is uniform on $[1, \infty)$.

6.2.5 We first prove the the theorem directly and then its converse. Suppose (f_n) converges uniformly on A to some function f. Let $\epsilon > 0$. Then, by definition, $\exists N \in \mathbb{N}$ s.t. $\forall x \in A, n \geq N$ implies $|f_n(x) - f(x)| < \frac{\epsilon}{2}$. Now, if $m, n \geq N$, then, by the triangle inequality, we have:

$$\begin{aligned} |f_n(x) - f_m(x)| &= |f_n(x) - f(x) + f(x) - f_m(x)| \\ &\leq |f_n(x) - f(x)| + |f(x) - f_m(x)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &< \epsilon \end{aligned}$$

Hence, $|f_n(x) - f_m(x) < \epsilon|$.

□

Conversely, suppose that $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall x \in A$ and $\forall m, n \geq N, |f_n(x) - f_m(x)| < \epsilon$. Now, notice that $(f_n(x))$ is a Cauchy sequence and per **Theorem 2.6.4 (Cauchy Criterion)**, it converges. Now, since this is true $\forall x \in A$, we can define $f(x) = \lim_{n\to\infty}(f_n(x))$ and we now have to show that f_n converges to f uniformly. Let $\epsilon > 0$ be given. Then we know that $\exists N \in \mathbb{N}$ s.t. $x \in A$ and $\forall m, n \geq N, |f_n(x) - f_m(x)| < \epsilon$. Then it follows by the **Algebraic Limit Theorem** that $\lim_{m\to\infty} f_n(x) - f_m(x) = f_n(x) - f_m(x)$. Finally, per the **Order Limit Theorem**, we get that for $x \in A$ and $\forall n \geq N, |f_n(x) - f(x)| < \epsilon$. Hence, we got that f_n uniformly converges to f.

□

Finally, we have shown that a sequence of functions (f_n) defined on a set $A \subseteq \mathbb{R}$ converges uniformly on A if and only if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \epsilon$ whenever $m, n \geq N$ and $x \in A$.

□

6.3.1 (a)

(b)

6.3.3 (a)

(b)

6.4.3 (a) Placeholder

6.4.5 (a) Placeholder