Math 220 Homework – Occhipinti – Fall 2018

Before starting homework, here are some guidelines to receiving full credit on your solutions, and to getting the most out of the homework.

1. This is a writing course. You must write your solutions in sentence form using word and not symbols such as $\Longrightarrow$, $\exists$, or $\forall$. Note that symbols that are part of expressions and equations (such as $=$, $\in$, $\geq$, etc.) are all appropriate.

2. You are encouraged to work on homework with classmates. However, you must write up the solutions that you turn in separately. Turning in solutions that are too similar to a classmate's is a violation of the code of academic integrity. Don't risk it!

3. This packet contains problems as well as lists of problems from the textbook. You must turn in all questions from the packet, when assigned, as well as the underlined questions from the textbook.

4. Make sure you clearly label what question you are answering on all homework you turn in.

5. **The solutions you turn in must be typeset in LaTeX**. The easiest way to do this is to go to www.overleaf.com/latex/templates and choose one of their homework templates that you like. You can find a great deal of help here: wiki.carleton.edu/display/carl/Carleton+LaTeX+Workshop .

6. Leave lots of space on your homework for feedback, as the feedback from the grader will be a large part of how you learn.

7. Turn in solutions you can be proud of. One good technique to accomplish this is to read your solution aloud to yourself. Can you read it without adding a lot of extra words? Does it sound like English? Does it make sense?

8. Homework will be due in lecture. Late homework will be accepted only with prior consent of the instructor.

9. I'm always happy to discuss homework problems with you up until the day before they are due. You are encouraged to stop by my office when I am there, whether it is during my official office hours or not.

Contents

Getting Started with LaTeX

1. Render the following text in LaTeX as exactly as you can:

   The quadratic formula asserts that if $a, b, c \in \mathbb{R}$ and $a \neq 0$ then the roots of the polynomial $ax^2 + bx + c$ are given by

   $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

   The case of the cubic $ax^3 + bx^2 + cx + d$ is more complicated, and one must first compute $\Delta_0 = b^2 - 3ac$, $\Delta_1 = 2b^3 - 9abc + 27ad^2$ and

   $$C = \sqrt[3]{\frac{\Delta_1 \pm \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2}}.$$

   In this case, one finds that one root of the cubic is given by

   $$x = -\frac{1}{3a}\left(b + C + \frac{\Delta_0}{C}\right).$$

   A similar formula exists for the roots of the quartic $ax^4 + bx^3 + cx^2 + dx + e$, however it is even more complicated. Remarkably, Abel and Ruffini proved in 1824 that no similar formula exists for the roots of the quintic $ax^5 + bx^4 + cx^3 + dx^2 + ex + f$, or indeed for any polynomial of degree ≥ 5.

Video: What's the probability you live in an odd numbered house?

Description: This video explains the tools a statistician would use to address the question in the title. If you'd like to learn more about these topics, I'd recommend taking our introductory statistics sequence Math 321/322.

Watch the video here: `http://www.youtube.com/watch?v=wydlZ9lcEiQ`

2. How convinced are you by the evidence and argument presented that there are more odd numbered houses than even numbered houses? Which do you find more convincing, the argument or the numerical evidence? [Your answer here should probably be about 4 sentences.]

3. Suppose you looked at all numbers from 1 to 1000000. What percentage of them do you think <u>start</u> with an odd number?

Paper: The Strong Law of Small Numbers

Description: This paper gives data in evidence of 35 different conjectures, and asks you to guess based on the data whether the conjecture will be true or false. Then, in the second half, it gives you the answers, when they are known. I encourage you to actually guess for each conjecture, as you read it, if it will be true or false. A few parts of the paper will use mathematical language that you are likely unfamiliar with, but this should not discourage you, or harm your understanding of the paper overall.

Read the Paper here (you will have to download the PDF): `http://bit.ly/smallnumbers`

4. Summarize in 2-4 sentences what you learned from reading this paper.

5. Which of the 35 conjectures in the paper is your favorite? Explain both what the conjecture is, and what the result is. Why do you find it compelling?

2.1 - Logical Operators: And, Or, Implies, ...

Exercises: <u>2</u>, 3, <u>4</u>, 6a, <u>6b</u>, <u>7ab</u>

6. Write the negation of the following statements as naturally as possible. Do not simply write "it is not the case that..." For some statements you are given the context, but you should not negate that part.

   (a) About an integer $n \geq 2$:

   n is prime or $n + 1$ is prime.

   (b) About matrices A and B:

   The determinant of A is negative and the determinant of B is positive.

   (c) About functions $f(x)$ and $g(x)$:

   If $f(x)$ and $g(x)$ are differentiable, then so is $f(x) + g(x)$.

   (d) About an integer n in $\mathbb{Z}^+$:

   $n! + 1$ is a perfect square if and only if $n = 4$ or $n = 5$

   (e) The populations of Decorah, IA and Minneapolis, MN are the same.

2.2 - Sets

Exercises: 2, 6fg, 6hi, 7c, 14abc, 19ab

7. List three elements of each of the following sets.

   (a) $\mathbb{Q} \cap (2, 3)$

   (b) $\{2^n - 1 \mid n \in \mathbb{Z}^+\}$

   (c) $\{n \in \mathbb{Z}^+ \mid n^2 + 1 \text{ is prime}\}$

8. Let $A = [4, 7]$ an $B = (6, 8)$, both subsets of the universal set $\mathbb{R}$. Write each of the following sets as naturally as possible:

   (a) $A \cup B$

   (b) $A \cap B$

   (c) A^C

   (d) B^C

   (e) $A - B$

9. A **bi-partition** of a set S is a set $\{A, B\}$ of two subsets A and B of S such that $A \cup B = S$ and $A \cap B = \emptyset$.

   (a) List all bi-partitions of the set $S = \{1, 2, 3\}$.

   (b) Explain why, every subset A of a set S is an element of exactly one bi-partition of S. (Hint: First explain why every such A is an element of at least one bi-partition of S, then explain why it cannot be be an element of more than one bi-partition of S.)

   (c) It is a fact that we will prove later that if $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, then S has exactly 1024 subsets. How many bi-partitions does S have?

2.3 - Logical Quantifiers: For all and There exists

Exercises: 2, 4, 5, 6, 7, 8, 11

10. An integer n is called *doubly even* if there exist even integers x and y such that $n = xy$.

    (a) Is 12 doubly even? Prove your answer.

    (b) Is 98 doubly even? Prove your answer.

    (c) Write the negation of the statement "n is doubly even" without using the word "not."

    (d) For what positive integers n is $n!$ doubly even? Prove your answer.

11. A subset S of $\mathbb{Z}$ is called *shifty* if for every $x \in S$, $x - 1 \in S$ or $x + 1 \in S$.

    (a) Give an example of a shifty set with 5 elements.

    (b) Give an example of a shifty set that contains 10 and -10 but does not contain 0.

    (c) Is $\{n \in \mathbb{Z}^+ \mid n \text{ is not a multiple of 5 or 11}\}$ shifty? Why or why not?

    (d) Is $\{n \in \mathbb{Z}^+ \mid n \text{ is a multiple of 5 or } n+1 \text{ is a multiple of 5}\}$ shifty? Why or why not?

    (e) Write the negation of the statement "S is shifty" without using the word "not."

    (f) Does every non-empty shifty set contain an even integer? Why or why not?

Reading: The Ideal Mathematician

Description: This chapter from The Mathematical Experience by Davis and Hersh gives some insight into the life of an average mathematician. It focuses, in particular, on how mathematicians view proof.

Link: This is posted on KATIE

12. To the ideal mathematician, what is a proof?

13. The ideal mathematician regards it as obvious that an extraterrestrial intelligence capable of intergalactic travel would recognize the binary expansion of π. Do you believe this? Explain why.

14. How does this article make you feel about studying mathematics?

3.1 - Direct Proof

Exercises: <u>1</u>, 2, 4, <u>5</u>, 7, <u>10</u>, 18, <u>19a</u>

15. Prove the following statement is false by providing a counterexample: If $n \in \mathbb{Z}^+$ is odd and $n > 1$ then there exists a non-negative integer i and a prime p such that $n = 2^i + p$.

16. Prove the following statement is false by providing a counterexample: If S and T are shifty sets (in the sense of a previous exercise), then $S \cap T$ is also a shifty set.

17. Prove that if x is odd, then x^3 is odd.

18. Suppose that m and n are doubly even (in the sense of an earlier exercise):

    (a) Prove that mn is doubly even

    (b) Prove that $m + n$ is doubly even

19. Prove that if m is even but not doubly even then $m + 2$ is doubly even.

20. Prove or Disprove: if A and B are sets then there exists a set C such that $A \cup B = A \cup C$.

Video: Fermat's Last Theorem - Numberphile

Description: This video discusses the history Fermat's Last Theorem, the most famous problem in all of mathematics.

Link: `http://www.youtube.com/watch?v=qiNcEguuFSA`

21. What does Fermat's Last Theorem say? How many years passed, roughly, from when Fermat originally wrote down his Last Theorem and when it was proved?

22. Given that he very rarely wrote them down, Fermat clearly did not have the reverence for proofs that ideal mathematician from our recent reading did. Based on what you learned about Fermat from the video, why do you think that is?

3.2 - Proof by Contradiction and Proof by Contrapositive

Exercises: 1, 2, 5a, 6d, 8, 9, 11, 14

23. Prove that if x and y are integers and $xy - 1$ is even then x and y are odd.

24. Prove that if x and y are real numbers whose mean is m then at least one of x and y is $\geq m$.

25. Suppose S is a set of 250 distinct real numbers whose mean is 4. Must there exists $x \in S$ such that $x > 4$? Be sure to prove your answer.

26. Suppose $a, b, c \in \mathbb{Z}$ and $a^2 + b^2 = c^2$. Prove that at least one of a and b is even.

27. Prove that if $x, y \in \mathbb{R}^+$, then $x + y \geq 2\sqrt{xy}$.

28. Prove that if n is an integer, there exist three consecutive integers that sum to n if and only if n is a multiple of 3.

29. A subset S of $\mathbb{R}$ has the property that for all $x \in \mathbb{R}$ there exists $y \in S$ such that $|x - y| < 1$. Prove that S is infinite.

30. A subset S of $\mathbb{Z}$ is called **non-differential** if for every $x, y \in S$ we have $x - y \notin S$. Here are some statements about non-differential sets. Decide which statements are true and which are false, and provide a proof or counterexample for each as appropriate.

    (a) Every non-differential set is finite.

    (b) The intersection of two non-differential sets is non-differential.

    (c) The union of two non-differential sets is non-differential.

    (d) No non-differential set contains the element 0.

    (e) Every subset of a non-differential set is non-differential.

    (f) There is no non-differential set with exactly 5 elements.

    (g) If S is a non-differential set, so is $\mathbb{Z} - S$.

    (h) If S is a non-differential set, then so is the set $S_{+3} = \{x + 3 \mid x \in S\}$.

31. A subset A of $\mathbb{R}$ is called **cofinite** if $\mathbb{R} - A$ is finite. Here are some statements about cofinite sets. Decide which statements are true and which are false, and provide a proof or counterexample for each as appropriate.

   (a) If $A \subseteq B$ and B is cofinite then A is cofinite.
   (b) There exist two confinite sets A and B with the property that $A \cap B = \emptyset$.
   (c) If A is cofinite, then A contains a positive integer.
   (d) The intersection of two cofinite sets is cofinite.
   (e) The union of two cofinite sets is cofinite.
   (f) If A and B are cofinite then $A - B$ is finite.
   (g) If A is cofinite and $A \subseteq B \subseteq \mathbb{R}$ then B is cofinite.
   (h) Every cofinite set is infinite.

32. We say that a subset S of $\mathbb{Z}$ is **angled** if for every $x, y, z \in S$ we have $x + y > z$.

   (a) Give some examples of angled sets
   (b) Give some examples of sets that are not angled.
   (c) Can 0 be an element of an angled set?
   (d) Prove or disprove: If S is angled and $x \in S$ then $x > 0$.
   (e) Prove or disprove: If S is angled then there exists $c \in \mathbb{Z}$ such that for every $x \in S$ we have $x < c$.
   (f) Prove or disprove: There exists $c \in \mathbb{Z}^+$ such that if S is angled and $x \in S$ then we have $x < c$.
   (g) Prove or disprove: Every angled set is finite.
   (h) Prove or disprove: For every $n \in \mathbb{Z}^+$ there exists an angled set S such that $|S| = n$.

Video: What does it feel like to invent math?

Link: `http://www.youtube.com/watch?v=XFDM1ip5HdU`

33. Use quantifiers to precisely write down, in mathematical language, the definition given for $\sum_{n=0}^{\infty} a_n = X$ outlined in the video at the 4:00 minute mark.

34. Explain why it makes sense, in a way, for $1 - 1 + 1 - 1 + 1 - 1 + ...$ to equal $1/2$, as is suggested in the video at about 6:45. Is this what you learned in Calculus II?

35. In the sense of distance discussed at 12:45, how far apart are 5 and 13? How about -1 and 15?

3.4 - Proof by Cases

Exercises: 1, 2, 3, 4ab, 6, 8

36. Prove that if S and T are shifty sets (in the sense of a previous exercise) then $S \cup T$ is also a shifty set.

37. Prove that if $n \in \mathbb{Z}$ then $1 + (-1)^n(2n - 1)$ is a multiple of 4.

38. Prove that if $n \in \mathbb{Z}^+$ is odd then $n^2 - 1$ is divisible by 8.

39. Prove that every integer can be written as the sum of exactly 3 distinct integers. (For example, $5 = 4 + 2 + (-1)$.)

40. Prove that if x, y, and z are integers then at least one of $x+y$, $x+z$, and $y+z$ is even.

41. Prove or Disprove: There exist prime numbers p and q such that $p - q = 97$.

42. For $n \in \mathbb{Z}^+$, we define the n^{th} triangular number to be $T_n := 1 + 2 + ... + n$. Thus we have $T_1 = 1$, $T_2 = 3$, $T_3 = 6$, $T_4 = 10$, $T_5 = 15$, and so on. We will prove later that $T_n = \dfrac{n(n+1)}{2}$, and you should use that formula for this problem. Prove that for $n \in \mathbb{Z}^+$, T_n is odd if and only if n is 1 or 2 more than a multiple of 4.

3.3 - Mathematical Induction

Exercises: 1, 3, 5, 11abcd, 12abcd, 13abc, 16, 17, 18b

43. Prove that for all $n \in \mathbb{Z}^+$, $11^n - 6$ is divisible by 5.

44. Prove that for $n \in \mathbb{Z}^+$, we have $1^2 + 2^2 + 3^2 + ... + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

45. Prove that for all $n \in \mathbb{Z}^+$, $2^{n+2} + 3^{2n+1}$ is divisible by 7.

46. Let F_n denote the n^{th} Fibonacci number. Prove that $\displaystyle\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$.

47. Prove that for all $n \in \mathbb{Z}_{\geq 12}$ we have $n! > 5^n$.

48. Suppose that $x \neq 0$ is a real number and $x + \frac{1}{x} \in \mathbb{Z}$. Prove that $x^n + 1/x^n \in \mathbb{Z}$ for all $n \in \mathbb{Z}^+$. (Hint: $(x + \frac{1}{x})(x^n + \frac{1}{x^n})$ is probably nicer than you think!)

Video: Visualizing irrationality with triangular squares

Description: Earlier we discussed proofs that $\sqrt{2}$ and $\sqrt{3}$ are irrational. This video gives entirely different proofs that rely on some truly beautiful animations and combinatorial geometry.

Link: `http://www.youtube.com/watch?v=yk6wbvNPZW0`

49. At about 2:00 he discusses the fact that $1 + 3 + 5 + 7 + ... + (2n - 1) = n^2$. That is, the sum of the first n odd numbers is n^2. Through triangular squares, he gives a striking visual proof of this fact. Give a proof of this fact using mathematical induction (without any need to reference triangular squares).

50. At about 6:35 he shows a diagram and says "at this point, the dark green overlap will be exactly as large as the white empty area." Why is that?

51. At 7:20, he suggests that there is an even easier way to reach a contradiction. What is it?

52. At the very end of the video, he gives a demonstration that $T_{20} + T_{20} + T_{20} = T_{35}$, and uses that identity to produce the smaller identity $T_5 + T_5 + T_5 = T_9$, much like the key step in the proofs earlier in the video that $\sqrt{3}$ and $\sqrt{2}$ are irrational. Why is it not a contradiction here?

Additional Proof Practice

53. A subset S of $\mathbb{Z}^+$ is called a P_3**-set** if there exist (not necessarily distinct) elements $x, y, z \in S$ such that $x + y + z$ is prime.

    (a) Give some examples of P_3-sets.

    (b) Prove or Disprove: If A is a P_3-set, and $A \subseteq B \subseteq \mathbb{Z}^+$, then B is a P_3-set.

    (c) Prove or Disprove: If S is a P_3-set, then so is $S_{+3} := \{x + 3 \mid x \in S\}$.

    (d) Prove or Disprove: Every P_3-set contains a prime.

    (e) Prove or Disprove: The intersection of two P_3-sets is a P_3-set.

(f) Prove or Disprove: Every P_3-set contains an odd integer.

(g) Prove or Disprove: Every infinite subset S of $\mathbb{Z}^+$ is a P_3-set.

(h) Prove or Disprove: If S is a finite subset of $\mathbb{Z}^+$, then $\mathbb{Z}^+ - S$ is a P_3-set.

54. If a subset S of $\mathbb{Z}^+$ is a P_3-set then the **core** of S is the set

$$\text{core}(S) := \{s \in S \mid S - \{s\} \text{ is not a } P_3 - set\}.$$

(a) What is the core of $S = \{2, 3, 6\}$?

(b) Give an example of a P_3-set whose core is the empty set, or prove none exists.

(c) Give an example of a P_3-set whose core is infinite, or prove none exists.

(d) Prove or Disprove: If S is a P_3-set then $core(S)$ is a P_3-set.

(e) Prove or Disprove: If S and T are P_3-sets then $core(S \cup T) \subseteq core(S) \cap core(T)$.

(f) Prove or Disprove: If S and T are P_3-sets with $S \subseteq T$ then we have $core(T) \subseteq core(S)$.

55. A subset S of $\mathbb{Z}$ is called **threequaline** if for every $x, y \in S$ one has $3 \mid (x - y)$.

(a) Prove or Disprove: Every subset of a threequaline set is threequaline.

(b) Prove that if S is threequaline than either every element of S is divisible by 3 or none are.

(c) Prove that if S is threequaline and r and t are integers, then the set $\{rx+t \mid x \in S\}$ is also threequaline.

(d) Prove that if S and T are threequaline and $S \cap T \neq \emptyset$ then $S \cup T$ is threequaline.

56. A subset S of $\mathbb{R}$ is called **crunched** if there exist integers $m, n \in \mathbb{Z}$ such that for all $x \in S$ we have $m < x < n$.

(a) Give some examples of sets that are and are not crunched.

(b) Prove or Disprove: All crunched sets are finite.

(c) Prove or Disprove: All finite sets are crunched.

(d) Prove or Disprove: Every subset of a crunched set is crunched.

(e) Prove or Disprove: The union of two crunched sets is crunched.

57. We call a <u>finite</u> subset S of $\mathbb{Z}$ **balanced** if $|\mathbb{Z}^+ \cap S| = |\mathbb{Z}^- \cap S|$. (Recall that $\mathbb{Z}^- = \{-1, -2, -3, ...\}$)

   (a) Prove or Disprove: If A is a balanced set then so is $A \cup \{0\}$.
   (b) Prove or Disprove: The union of two balanced sets is balanced.
   (c) Prove or Disprove: The intersection of two balanced sets is balanced.
   (d) Prove or Disprove: For every $n \in \mathbb{Z}^+$ there exists a balanced set S with exactly n elements.
   (e) If A is a subset of $\mathbb{Z}$ we denote by $\overline{A}$ the set $\{-a \mid a \in A\}$. Prove or disprove: For every finite subset A of $\mathbb{Z}$, the set $A \cup \overline{A}$ is balanced.
   (f) Prove or Disprove: If A is balanced and $|A|$ is odd, then $0 \in A$.

58. We call a subset S of $\mathbb{R}$ **positively scattered** if for every $x \in S$ there exists $y \in \mathbb{R}$ such that $y > x$ and $S \cap (x, y] = \emptyset$.

   (a) Is $\mathbb{Z}^+$ positively scattered?
   (b) Is $[2, 3]$ positively scattered?
   (c) Is $\{\frac{1}{n} \mid n \in \mathbb{Z}^+\}$ positively scattered?
   (d) Is $\{\frac{1}{n} \mid n \in \mathbb{Z}^+\} \cup \{0\}$ positively scattered?
   (e) Prove or Disprove: A subset of a positively scattered set is positively scattered.
   (f) Prove or Disprove: The union of two positively scattered sets is positively scattered.
   (g) Prove or Disprove: If S is positively scattered then $\mathbb{R} - S$ is not positively scattered.
   (h) Prove or Disprove: If S is positively scattered then the set $\overline{S} := \{-x \mid x \in S\}$ is positively scattered.

Reading: A Mathematician's Apology by G. H. Hardy

Description: This essay is Hardy's defense of the study of pure mathematics. Published when Hardy was 63, this essay defends his choice to spend his life studying number theory rather than something more "useful." As this essay was published in 1940, at the height of World War II and only a few years before the Manhattan Project and the dawn of nuclear weapons, Hardy shies away from the usual justification that pure mathematics is worth studying because it may be useful to the sciences some day.

Link: `https://bit.ly/2cyag14`

59. What is it, to Hardy, that makes a mathematical idea significant?

60. Some mathematicians believe that mathematics is fundamentally a creation of humans, in the same way that literature is. Others believe that mathematics exists in its own reality, and that as mathematicians we merely observe it. Which does Hardy believe? Which more accurately reflects your understanding of mathematics?

61. Hardy says "I am interested in mathematics only as a creative art." What is it that makes you interested in mathematics? Do you think of mathematics primarily as a creative art? Explain your answer.

62. Hardy writes:

    > But science works for evil as well as for good (an particularly, of course, in time of war); and both Gauss and lesser mathematicians may be justified in rejoicing that there is one science at any rate, and that their own, whose very remoteness from ordinary human activities should keep it gentle and clean.

    Unbeknown to Hardy, likely even as he penned these words, French and British military intelligence was using a great deal of number theory (the supposed queen of mathematics because of its uselessness) to intercept and decode German communications that had been encrypted with the Enigma cipher.

    Today, modern cryptography has its roots in number theory, and is used for both good (online shopping) and evil (organizing terrorist attacks). Based on what he writes in this essay, how do you think Hardy's feelings for number theory would change based on these developments?

63. At every mathematics seminar where the audience includes non-mathematicians, someone in the crowd inevitably raises their hand during the question session and says "Can you explain to me what this could be used for?" In a few sentences, what do you think Hardy would say to this question?

4.1- Set Operations

Exercises: 1, 3a, 3bcef, 4a, 6, 7ab, 8abc, 9ab, 21abc, 24, 25, 26

64. List three elements of each of the following sets:

    (a) $\mathcal{P}(\{2, 7\})$
    (b) $\mathbb{Q} \times (\mathbb{R} - \mathbb{Q})$
    (c) $\bigcup_{i=3}^{5} \{2^i, 2^i + 1\}$
    (d) $\mathcal{P}(\mathcal{P}(\{1, 2\}))$
    (e) $\mathcal{P}([0, 1]) \times \mathcal{P}([0, 1])$
    (f) $\mathcal{P}([0, 1] \times [0, 1])$

65. Prove or Disprove: If A and B are sets, then $\mathcal{P}(A - B) = \mathcal{P}(A) - \mathcal{P}(B)$.

66. If for all $n \in \mathbb{Z}^+$ we have $A_n = \{n, n+1, n+2, ..., 2n\}$, find $\bigcap_{i=10}^{20} A_i$. Can you generalize this result?

67. If S is a set, define $\Xi(S)$ to be the set $\{(x, y) \in \mathcal{P}(S) \times \mathcal{P}(S) \mid x \cap y \neq \emptyset\}$.

    (a) Explain why $(\{1, 3\}, \{2, 3\})$ is an element of $\Xi(\{1, 2, 3\})$.
    (b) If $S = \{1, 2, 3, 4\}$, give 3 examples of elements of $\mathcal{P}(S) \times \mathcal{P}(S)$ that are not in $\Xi(S)$.
    (c) Explain carefully in words what $\Xi(S)$ is.
    (d) Prove that if S is a subset of T then $\Xi(S)$ is a subset of $\Xi(T)$.
    (e) Prove that if S is finite, then $|\Xi(S)| \geq |S|$. (Hint: To do this, explain how to find a different element of $\Xi(S)$ for each $s \in S$. How do you know they are different?)

68. Here are five similar looking definitions that apply to subsets S of $\mathbb{R}$.

    - S is **spread out** if for every $x, y \in S$ if $x \neq y$ then $|x - y| \geq 1$.
    - S is called **broad** if for every $x \in \mathbb{R}$ there exists $y \in S$ such that $|x - y| \leq 1$.
    - S is called **clumped** if for every $x, y \in S$ one has $|x - y| \leq 1$.
    - S is called **bunched** if there exists $x \in S$ such that for every $y \in S$ one has $|x - y| \leq 1$.
    - S is called **clustered** if for every $x \in S$ there exists $y \in S - \{x\}$ such that $|x - y| \leq 1$.

(a) Decide which of the above five definitions apply to each of the following sets

   i. $\{1, 2, 3\}$
   ii. $\mathbb{R}$
   iii. $\mathbb{Z}$
   iv. $[2, 4]$
   v. The set of even integers
   vi. $\cup_{i \in \mathbb{Z}}[2i, 2i + 1]$

(b) Give examples of the following, or explain why none exists.

   i. A broad, spread out set
   ii. A spread out, clumped set
   iii. A subset of $\mathbb{R}$ that is not broad, not clumped, and not spread out.
   iv. An infinite clumped set

(c) For each of the five properties above, prove or disprove the statement: A subset of a set with PROPERTY is also PROPERTY.

(d) For each of the five properties above, prove or disprove the statement: If S and T are sets with PROPERTY than $S \cup T$ is a set with PROPERTY.

(e) For each of the five properties above, prove or disprove the statement: If S and T are sets with PROPERTY than $S \cap T$ is a set with PROPERTY.

Video: Ramanujan, 1729, and Fermat's Last Theorem

Description: Matt Parker discusses Hardy (author of A Mathematician's Apology)'s relationship with Ramanujan, and discusses a famous story of the taxi cab number 1729, as well as surprising connections between this story and an attempted proof of Fermat's Last Theorem.

Link: `http://www.youtube.com/watch?v=_o0cIpLQApk`

69. How did Ramanujan and Hardy meet?

70. In the video Parker discusses the near miss "solution" $65601^3 + 67402^3 = 83802^3$ to Fermat's Last Theorem. Explain how you could know this equation was not correct without doing any calculation at all.

71. What is the connection between 1729 and Fermat's Last Theorem?

4.3 - Functions

Exercises: 1abcdef, 2abcde, 5abcdef, 6abc, 8, 11, 12a, 12b, 19ab, 19c, 20, 25a, 26, 27abc, 29

72. For each A and B below, give an example of a function from A to B that is non-constant and not given piecewise.

    (a) $A = (0, 1)$, $B = [0, 1]$

    (b) $A = [0, 1]$, $B = (0, 1)$

    (c) $A = \mathbb{R}$, $B = 2\mathbb{Z}$ (the set of even integers)

    (d) A is the set of finite subsets of $\mathbb{Z}^+$, $B = \mathbb{Z}^+$

    (e) $A = \mathbb{R}$, $B = [-1, 1]$

    (f) $A = \mathbb{Z}_{\geq 2}$, B is the set of primes

73. Consider the function $f : \mathbb{Z} \to \mathbb{Z} : n \mapsto 2n + 1$. Is it one-to-one? Is it onto? Be sure to prove your answers.

74. (a) Prove the function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R} : (x, y) \mapsto (x + y, x - y)$ is a bijection.

    (b) Prove the function $g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z} : (x, y) \mapsto (x + y, x - y)$ is NOT a bijection.

75. Consider the function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Q}$ given by $f(a, b) = \dfrac{a}{|b| + 1}$. Is f one-to-one? Is f onto? Be sure to prove your answers.

76. Consider the function $g : \mathcal{P}(\{1, 2, 3, 4, 5\}) \to \mathcal{P}(\{1, 2, 3, 4\}) : A \mapsto A - \{5\}$. Is g one-to-one? Is it onto? Be sure to prove your answers.

77. Suppose that A and B are sets and that $f : A \to B$ is a function. Define the function $\mathcal{F} : A \times A \to B \times B : (a_1, a_2) \mapsto (b_1, b_2)$.

    (a) Prove or Disprove: If f is onto, so is $\mathcal{F}$.

    (b) Prove or Disprove: If f is one-to-one, so is $\mathcal{F}$.

78. Suppose S is a set.

    (a) If there is a function $f : S \to \emptyset$, what does that tell you about S?

    (b) If there is an onto function $g : \emptyset \to S$, what does that tell you about S?

    (c) If there is a one-to-one function $h : \emptyset \to S$, what does that tell you about S?

79. Suppose that S is an arbitrary set, and that $f : S \to \mathcal{P}(S)$ is a function. Prove that f is not onto. [Big Hint: Consider the set $A = \{x \in S \mid x \notin f(x)\}$. Is A in the image of f?]

80. (a) If $f : A \to B$ is a function and $\hat{A}$ is a subset of A, we may define a function $\hat{f} : \hat{A} \to B : \hat{a} \mapsto f(\hat{a})$ called the **restriction** of f to $\hat{A}$. Prove that if $f : A \to B$ is one-to-one, then so is the restriction of f to every subset $\hat{A}$ of A.

    (b) Find a way analogous to the previous question to replace a function $f : A \to B$ with a function $f : A \to \hat{B}$ if $B \subseteq \hat{B}$. This is called **corestriction**. Prove that every corestriction of a one-to-one function is one-to-one. Is every corestriction of an onto function onto?

81. If A is a set, we call a function $f : A \to \mathbb{R}$ **bounded above** if there exists $B \in \mathbb{R}$ such that for every $a \in A$ we have $f(a) \leq B$.

    (a) Prove the function $f : \mathbb{R} \to \mathbb{R} : x \mapsto \sin(x)$ is bounded above. Be sure you explicitly use the definition.

    (b) Prove the function $g : \mathbb{R}^+ \to \mathbb{R} : x \mapsto \ln(x)$ is NOT bounded above. Be sure you explicitly use the definition.

    (c) Prove or Disprove: No one-to-one function $f : \mathbb{R} \to \mathbb{R}$ is bounded above.

    (d) Prove or Disprove: If A is a set, $f : A \to \mathbb{R}$ is a function and $g : \mathbb{R} \to \mathbb{R}$ is a bounded above function, then the function $g \circ f : A \to \mathbb{R}$ is bounded above.

    (e) Prove or Disprove: If A is a set and $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ are two functions such that for every $a \in A$ one has $f(a) \leq g(a)$ and g is bounded above, then f is bounded above.

    (f) Prove or Disprove: If $f : A \to \mathbb{R}$ is not onto, then f is bounded above.

    (g) Prove or Disprove: If $f : A \to \mathbb{R}$ is bounded above, then so is the function $\overline{f} : A \to \mathbb{R} : a \mapsto -f(a)$.

    (h) Prove or Disprove: If $f : A \to \mathbb{R}$ is not bounded above, then the set $\mathbb{Z}^+ \cap \mathbf{im}(f)$ is infinite. (Recall $\mathbf{im}(f)$ denotes the image of f.

    (i) Prove or Disprove: If A is a set and $f : A \to \mathbb{R}$ is onto, then f is not bounded above.

82. We call a function $f : \mathbb{R} \to \mathbb{R}$ *weakly onto* if for every $y \in \mathbb{R}$ there exists $x \in \mathbb{R}$ such that $|f(x) - y| < 1$.

   (a) Consider the function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = \lfloor x \rfloor$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x. Is g onto? Is g weakly onto?

   (b) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is onto, then f is weakly onto. (This somewhat justifies calling it weakly onto.)

   (c) Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are weakly onto. Must $f \circ g$ be weakly onto?

   (d) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is weakly onto. Must the function g defined by $g(x) = \frac{1}{2}f(x)$ be weakly onto?

   (e) A function $f : \mathbb{R} \to \mathbb{R}$ is called a *jiggle* if for every $x \in \mathbb{R}$, we have $|f(x) - x| < \frac{1}{1+x^2}$. The function $f(x) = x$ is called the *trivial jiggle*. Give an example of a non-trivial jiggle.

   (f) Prove or disprove: Every jiggle is weakly onto.

Video: What is i^i?

Description: This video discusses the amazing Euler identity $e^{i\pi} = -1$ and related ideas to calculate the surprising value of i^i. Recall for this video that $i = \sqrt{-1}$.

Link: `https://www.youtube.com/watch?v=9tlHQOKMHGA`

83. What is $e^{2\pi i}$? What is $e^{\pi i/4}$?

84. The function $exp : \mathbb{R} \to \mathbb{R}^+ : x \mapsto e^x$ is a bijection, and hence it has an inverse function. What is that inverse function called?

85. Is the function $exp : \mathbb{C} \to \mathbb{C} : x \mapsto e^x$ one-to-one? Explain why or why not. (Here $\mathbb{C}$ denotes the set of complex numbers.)

86. How is your answer to the previous question relevant to the methods used in the video?

4.2/4.4: Relations

Exercises: 4.2: 1, 2, 3abcde, 4ab, 12ab, 12cd, 12ef

4.4: 1ab, 2ab, 3, 4a, 5, 6abcde, 9ab, 11, 12, 13, 23, 30, 31, 36

87. Define a relation $\sim$ on $\mathbb{Z}$ by $x \sim y$ if $x + 2y$ is a multiple of 3. Prove that $\sim$ is an equivalence relation.

88. For any given set S, the set $\Xi(S)$ defined in a previous exercise is a subset of $\mathcal{P}(S) \times \mathcal{P}(S)$, and hence defines a relation on $\mathcal{P}(S)$. Denote this relation by R. (If letting S be an arbitrary set bugs you while working on this exercise, you may instead take $S = \mathbb{Z}$.)

    (a) Describe the relation R in words. (e.g. write a sentence that starts with "$A\,R\,B$ if...")

    (b) Is R symmetric?

    (c) Is R transitive?

    (d) Is R reflexive? (Be careful!)

5.2 - Combinatorics of Finite Sets

Exercises: 1, 3, 10ab, 11ab, 14, 15, 16abc, 17a, 19, 20

89. Suppose that S is a finite set. Give an example of each of the following, or prove that no example exists. [Note: Your example must work for every finite set S.]

    (a) An injective function $S \to \mathcal{P}(S)$

    (b) A surjective function $S \to \mathcal{P}(S)$

    (c) A surjective function $\mathcal{P}(S) \to S$

    (d) An injective function $S \to S \times S$

    (e) A surjective function $S \times S \to S$

90. Prove that in a room containing twelve people whose ages are positive integers not greater than 122 (the largest recorded human age) that it is possible to find two disjoint groups of people the sums of whose ages are the same. (For example, if the group contains people of ages $1, 5, 11, 13, 21, 23, 40, 44, 48, 57, 59$, and 89, then one may take as one group the 23 and 89 year old and the other the 5, 48, and 59 year olds. Both groups ages sum to 112.)

Video: Gaps Between Primes

Description: This video discusses Yitang Zhang's recent (2013) result on prime gaps, one of the most important theorems of the past decade. Notably, Zhang was almost 60 when he published this result, a striking counterexample to Hardy's claim that mathematics is only for the young.

Link: `http://youtu.be/vkMXdShDdtY`

91. State clearly the result that Zhang proved.

92. Explain why the statement "There exist infinitely many pairs of consecutive primes p_1 and p_2 such that $|p_1 - p_2| < 70,000,000$" implies the statement "There is at least one integer N such that there exist infinitely many pairs of consecutive primes p_1 and p_2 with the property that $p_2 - p_1 = N$.

5.1 - Cardinality

Exercises: 1ab, 2a, 2b, 2e, 3ab, 4ab (Recall I=(0,1)),7, 8, 13a, 13b, 15ab

93. Let A be the set of integers that are divisible by 3, B be the set of integers that leave a remainder of 1 when divided by 3, and C be the set of integers that leave a remainder of 2 when divided by 3. Show that $|A| = |B| = |C|$.

94. Are the sets $[0, 1]$ and $[0, 1)$ equinumerous? If so, provide an explicit bijection between them. If not, prove that no such bijection exists.

95. Are $\mathbb{Z}^+$ and $\mathcal{P}(\mathbb{Z}^+)$ equinumerous? If so, provide an explicit bijection between them. If not, prove that no such bijection exists. [Big Hint: You will want to revisit an earlier exercise from section 4.3.]

Video: Pi Is (still) Wrong.

Description: In this video, Vi Hart argues that mathematicians have made our lives more complicated by choosing π (the ratio of a circle's circumference to its diameter) as the main circle constant, and argues for an alternative circle constant.

Link: `http://www.youtube.com/watch?v=jG7vhMMXagQ`

96. What is $\sin(\tau/8)$? Explain why.

97. Rewrite the well known geometry formulas $C = 2\pi r$, $A = \pi r^2$ and $V = \frac{4}{3}\pi r^3$ using τ instead of π. Do you think each of these formulas looks better or worse with τ instead of π?

5.3/5.4 - Cardinalities of Infinite Sets

Exercises: 5.3 : 1, <u>2ab</u>, <u>8</u>, <u>15</u>, <u>16</u>

5.4 : 1, <u>6</u>

98. Suppose that A is a set and that $a \in A$. Prove that A is infinite if and only if $|A| = |A - \{a\}|$. [Do not reference any theorems from the chapter that we have not gone over in class. You will have to produce a bijection between A and $A - \{a\}$ in the case that A is an arbitrary infinite set.]

99. Prove that $\mathcal{P}(\mathbb{Z}^+)$ is uncountable.

Video: Who Cares about Topology?

Description: In this video the presenter outlines the proof of the inscribed rectangle theorem. This is a beautiful illustration of how sometimes, in order to solve a problem, you have to make it more complicated. The second video is optional, but highly recommended, as it gives a beautiful illustration of how disparate areas of mathematics connect.

Link: `http://www.youtube.com/watch?v=AmgkSdhK4K8`

Link (Optional Part II): `http://www.youtube.com/watch?v=FhSFkLhDANA`

100. At 8:00 in the video, the text on screen says "Pair of loop points $\implies$ Point in unit square." What is really meant by this? (Hint: the implies arrow is probably a bit misleading.)

101. At around 11:00 he explains that he wants the coordinates (x, y) and (y, x) to be represented by the same point. Why is that, in terms of the original problem about rectangles?

102. What theorem from topology is needed in order to complete the proof?

Video: Gödel's Incompleteness Theorem

Description: This is our last topic for the course, Gödel's incompleteness theorem, which defines the limitations of mathematical reasoning and proof. If you find this interesting, I highly recommend watching the optional follow-up video. For further reading, I highly recommend the book *Gödel, Escher, Bach* by Douglas Hofstadter, which discusses connections between logic, computer science, music, and the philosophy of consciousness. It might change your view of the universe!

Link: `http://www.youtube.com/watch?v=O4ndIDcDSGc`

Link (Optional Part II): `http://www.youtube.com/watch?v=mccoBBf0VDM`

103. What is the problem with the set of all sets that don't contain themselves?

104. The presenter argues that if the Riemann Hypothesis is proven to be undecidable, then it is necessarily true. Does the same argument apply to the Twin Prime Conjecture?

105. This video makes reference to Hilbert's problems. This is a list a list of 23 problems stated by David Hilbert in 1900 which he believed to be the most important for mathematicians to solve. One hundred years later, the Clay Mathematics Institute (in honor of Hilbert) published the list of 7 problems (the Millennium Prize Problems) that they believed were most important for this century. Using wikipedia, how many of Hilbert's problems remain unsolved? How many of the Millennium Prize Problems remain unsolved? There is only one problem that is on both Hilbert's original list as well as the Clay Math list, what is it?