
Topology

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Assignment №5

Section 31

1. Show that if X is regular, every pair of points of X have neighborhoods whose closures are disjoint.

Since X is regular, by the definition, $\forall x, y \in X \exists U, V$ with $x \in U, y \in V$ and $U \cap V = \emptyset$ (with U and V being open sets). Now, recall that X is regular if and only if given a point x of X and a neighborhood U of x , there is a neighborhood V of x such that $\bar{V} \subset U$ (**Lemma 31.1 (a)**). Then, according to **Lemma 31.1 (a)**, $\exists U', V'$ such that $U' \subset U$ and $V' \subset V$. Now, because $U \cap V = \emptyset$, it follows that $\bar{U}' \cap \bar{V}' = \emptyset$. \square

2. Show that if X is normal, every pair of disjoint closed sets have neighborhoods whose closures are disjoint.

Let A and B be disjoint closed sets. Then by the definition of normality, $\exists U, V$ such that $U \cap V = \emptyset, A \subset U$, and $B \subset V$ (with U and V being open sets). Now, recall that X normal if and only if given a closed set A and an open set U containing A , there is an open set V containing A such that $\bar{V} \subset U$ (**Lemma 31.1 (b)**). Then it follows from **Lemma 31.1 (b)** that $\exists U', V'$ with $A \subset U'$ and $B \subset V'$ such that $\bar{U}' \subset U$ and $\bar{V}' \subset V$. Now, since $U \cap V = \emptyset$, we get $\bar{U}' \cap \bar{V}' = \emptyset$. \square

3. Show that every order topology is regular.

Suppose that X is an ordered set. Let $x \in X$ and let $U = (a, b)$ be the neighborhood of x . Also, let $A = (a, x)$ and $B = (x, b)$. Now, according to the **Lemma 31.1 (a)**, X is regular if and only if given a point x of X and a neighborhood U of x , there is a neighborhood V of x such that $\bar{V} \subset U$. Then it follows that we have the following four cases:

1. If $A = \emptyset$ and $v \in B$, then $x \in (a, v) \subset [x, v) \subset \overline{[x, v)} \subset [x, v] \subset (a, b)$.
2. If $B = \emptyset$ and $u \in A$, then $x \in (u, b) \subset (u, x] \subset \overline{(u, x]} \subset [u, x] \subset (a, b)$.
3. If $u \in A$ and $v \in B$, then $x \in (u, v) \subset \overline{(u, v)} \subset [u, v] \subset (a, b)$.
4. If $A = B = \emptyset$, then $(a, b) = \{x\}$ is both open and closed (since every order topology is Hausdorff)

Finally, we have considered all the cases and have exhaustively shown that every order topology is regular. \square

Section 32

1. Show that a closed subspace of a normal space is normal.

Suppose that Y is a closed subspace of the normal space X . Now, recall that every simply ordered set is a Hausdorff space in the order topology; The product of two Hausdorff spaces is a Hausdorff space; A subspace of a Hausdorff space is a Hausdorff space. (**Theorem 17.11**). Then according to the **Theorem 17.11**, Y is Hausdorff. Now let A and B be disjoint closed subspaces of Y . Since A and B are closed in X , they can be separated in X by open sets U and V (with $U \cap V = \emptyset$). Then $U \cap Y$ and $V \cap Y$ are open sets in Y separating A and B . Therefore, Y is normal. \square