Real Analysis Exams

Exam №3

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1. (a) Yes, it is continuous at 0. Recall that a function is continuous at $x = 0$ if $\lim_{x \to 0} f(x) = f(0) = 0$. Notice that $\forall x \neq 0, |\sin(\frac{1}{x^2})| \leq 1$. We then have that $|f(x)| \leq x^4$ (multiply both sides of the inequality by x^4). Now, since $\lim_{x \to 0} x^4 = 0$ and $\lim_{x \to 0}(-x^4) = 0$, it follows by **Squeeze Theorem** that $\lim_{x \to 0} = 0 = f(0)$. Hence, $f(x)$ is continuous at 0.

   (b) Yes, it is differentiable at 0. Recall that a function is differentiable at $x = 0$ if the limit $\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$ exists. We have:

$$\begin{aligned}
\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} &= \lim_{x \to 0} \frac{f(x) - 0}{x} \\
&= \lim_{x \to 0} \frac{x^4 \sin(\frac{1}{x^2})}{x} \\
&= \lim_{x \to 0} x^3 \sin\left(\frac{1}{x^2}\right)
\end{aligned}$$

   Now, notice that $x^3 \sin(\frac{1}{x^2})$ is bounded by $-|x^3|$ and $|x^3|$ and once again, by **Squeeze Theorem**, it follows that f is differentiable at 0.

   (c) $f'(x)$ is continuous at 0. Away from 0, we have $f'(x) = 4x^3 \sin(\frac{1}{x^2}) - 2x \cos(\frac{1}{x^2})$. Then $\lim_{x \to 0} f'(x) = 0$ and thus, $\lim_{x \to 0} f'(x) = 0 = f'(0)$. Hence, $f'(x)$ is continuous at 0.

(d) No, it is not differentiable at 0. Recall that a function is differentiable at $x = 0$ if the limit $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$ exists. We have:

$$\begin{aligned}
\lim_{x\to 0} \frac{f'(x) - f'(0)}{x - 0} &= \lim_{x\to 0} \frac{f(x) - 0}{x} \\
&= \lim_{x\to 0} \frac{4x^3 \sin\left(\frac{1}{x^2}\right) - 2x \cos\left(\frac{1}{x^2}\right)}{x} \\
&= \lim_{x\to 0} 4x^2 \sin\left(\frac{1}{x^2}\right) - 2\cos\left(\frac{1}{x^2}\right)
\end{aligned}$$

Now, notice that $\lim_{x\to 0} 4x^2 \sin\left(\frac{1}{x^2}\right) - 2\cos\left(\frac{1}{x^2}\right)$ does not exist and thus, $f'(x)$ is not differentiable at 0.

2. (a) Since $f(x)$ is differentiable on $[0, 4]$, it is also continuous on $[0, 4]$. Now, let $g(x) = f(x) - x$. Then g is again both differentiable and continuous on $[0, 4]$. We have $g(0) = f(0) = 2$ and $g(4) = f(4) - 4 = -3$. Now, since g is continuous, it follows by the **Theorem 4.5.1 (Intermediate Value Theorem)** $\exists c \in (0, 4)$ s.t. $g(c) = 0$ and for this c, we will have $f(c) = g(c) + c = 0 + c = c$. Hence, $f(x)$ has a fixed point on $[0, 4]$

$\square$

(b) Notice that $\frac{f(4)-f(0)}{4-0} = \frac{1-2}{4} = -\frac{1}{4}$. Then it follows by **Theorem 5.3.2 (Mean Value Theorem)** that $\exists c \in (0, 4)$ s.t. $f'(c) = -\frac{1}{4}$. On the other hand, we know that $f'(1) = 2$. Finally, it follows by **Theorem 5.2.7 (Darboux's Theorem)** that $\exists c \in (0, 4)$ s.t. $f'(c) = 0$.

$\square$

3. (a) For any fixed x, we have:

$$\lim_{n\to\infty} \frac{x^n e^{-x}}{n!} = \lim_{n\to\infty} \frac{x^n}{n! e^x} = 0$$

The reason the limit is 0 is that as n approaches infinity, $n!e^x$ grows a lot faster than x^n (one could also use **Squeeze Theorem** to show that the limit is 0).

(b) For any fixed x, we have:

$$|g_n(x) - g(x)| = \left|\frac{x^n e^{-x}}{n!} - 0\right| = \left|\frac{x^n}{n! e^x}\right|$$

Now, we need to pick N s.t. $\forall n \geq N$, $\left|\frac{x^n}{n!e^x}\right| < \epsilon$ holds. Now, although it is possible to do for every $x \in [0, \infty)$, there is no way to choose a single value of N that will work for all values of x at the same time. Thus, such N does not exist. Finally, we conclude that the sequence of functions (g_n) does not uniformly converge to g on $[0, \infty)$.

4. (a)

$$f_n'(x) = \left(xe^{-nx^2}\right)' = e^{-nx^2}(1 - 2nx^2)$$

(b) We need to solve the equation $f_n'(x) = 0$. We have:

$$e^{-nx^2}(1 - 2nx^2) = 0 \implies x = \pm\frac{1}{\sqrt{2n}}$$

Hence, the global maximum occurs at $x = \frac{1}{\sqrt{2n}}$ and the global minimum occurs at $x = -\frac{1}{\sqrt{2n}}$.

Let us sketch $f_n(x)$ for $n = 2$. For $n = 2$, we have the function $f_2(x) = xe^{-2x^2}$.

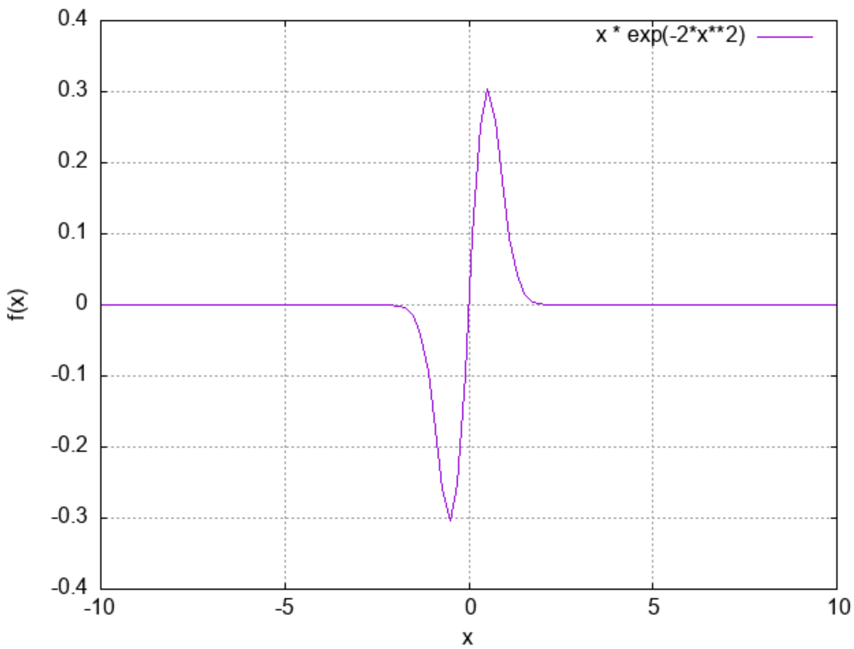

Figure 1: Plot of $g = xe^{-2x^2}$.

(c)

$$\begin{aligned} f(x) &= \lim_{n\to\infty} f_n(x) \\ &= \lim_{n\to\infty} \left(xe^{-nx^2}\right) \\ &= \lim_{n\to\infty} \frac{x}{e^{nx^2}} \\ &= 0 \qquad\qquad \text{(notice that as } n \to \infty, e^{nx^2} \to \infty) \end{aligned}$$

(d) Since the global maximum of $f(x)$ is $\frac{1}{\sqrt{2n}}$, we can let $N = \lceil \frac{1}{\epsilon^2} \rceil$. Then $\forall n \geq N$, we have:

$$|f_n(x) - f(x)| = |f_n(x)| = |xe^{-nx^2}| \leq \frac{1}{\sqrt{2n}} \leq \frac{1}{\sqrt{2N}} \leq \frac{\epsilon}{\sqrt{2}} < \epsilon$$

Now, we could find $N \in \mathbb{N}$ s.t. $\forall n \geq N, |f_n(x) - f(x)| < \epsilon$ holds and hence, f_n converges uniformly to f on $\mathbb{R}$.

(e)

$$\lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} e^{-nx^2}(1 - 2nx^2) = \lim_{n\to\infty} \frac{-2nx^2 + 1}{e^{nx^2}} = 0$$

(notice that as $n \to \infty, e^n$ grows a lot faster than $2n$)

Since $f(x) = 0$, it follows that $f'(x) = 0$ and finally, we have $f'(x) = \lim_{n\to\infty} f_n'(x) = 0$.

5. (a) Notice that the following holds:

$$\left|\frac{\cos(3^n x)}{2^n}\right| \leq \frac{1}{2^n}$$

Now, recall that $\frac{1}{2^n}$ converges (showed many times over the course of the class). Then, it follows by **Corollary 6.4.5 (Weierstrass M-Test)** that $g(x) = \sum_{n=1}^{\infty} \frac{\cos(3^n x)}{2^n}$ converges uniformly on $\mathbb{R}$. And since the uniform convergence implies continuity, it follows that $g(x) = \sum_{n=1}^{\infty} \frac{\cos(3^n x)}{2^n}$ is continuous on $\mathbb{R}$.

(b) Notice that we have:

$$g'(x) = \sum_{n=1}^{\infty} -\left(\frac{3}{2}\right)^n \sin(3^n x)$$

Unfortunately, in this case we cannot apply **Corollary 6.4.5 (Weierstrass M-Test)** as $\left(\frac{3}{2}\right)^n$ is not bounded. Hence, this is the difference between part (a) and part (b) of the exercise (we cannot determine if g is differentiable on $\mathbb{R}$).

As a side note, recall that this is the Weierstrass function of the form $\sum_{n=0}^{\infty} a^n \cos(b^n x)$ which is a nowhere-differentiable function. Hence, $g'(x)$ is not differentiable on $\mathbb{R}$.

6. For $x \notin \mathbb{Q}$, we can show $f_n(x)$ is continuous, since for $x < r_n$, we can choose a small enough δ such that $f_n(y) = 0$ for $y \in V_\delta(x)$. Similar logic can be applied when $x > r_n$. Now, notice that

$$f_n(x) \leq \frac{1}{2^n}$$

Then it follows by **Corollary 6.4.5 (Weierstrass M-Test)** that $f(x)$ converges uniformly.

Now, since f_n are all continuous, and f converges uniformly, we have that f is continuous.

Furthermore, since every $f_n(x)$ is increasing, f is monotonely increasing. Thus, for $x < y$, we get:

$$\begin{aligned}
\forall n \; f_n(x) &\leq f_n(y) \\
\sum_{n=1}^{k} f_n(x) &\leq \sum_{n=1}^{k} f_n(y) \\
\lim_k \sum_{n=1}^{k} f_n(x) &\leq \lim_k \sum_{n=1}^{k} f_n(y) \\
f(x) &\leq f(y)
\end{aligned}$$

Hence, we got that f is increasing on $\mathbb{R}$.

$\square$

7. (a) We have:

$$\begin{aligned}
\ln(1+x) &= \sum_{n\geq 1} \frac{f^{(n)}(0)}{n!} x^n \\
&= \sum_{n\geq 1} \frac{(-1)^{n-1}(n-1)!}{n!} x^n \\
&= \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} x^n \\
&= \sum_{n\geq 1} \frac{(-1)^{n}}{n+1} x^{n+1} \\
&= x - \frac{x^2}{3} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots
\end{aligned}$$

Hence the taylor series representation is $x - \frac{x^2}{3} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$.

(b) When $x = 1$, we get alternating harmonic series that we know converges [shown many times over the course of the class]. It diverges when $x = -1$ as we get the $-\frac{1}{n}$ which is the negative harmonic series that we know diverges [shown many times over the course of the class]). Then it follows that the series converges when $-1 < x \leq 1$. Hence, the interval of convergence of the series is $(-1, 1]$ (the radius of convergence is 1).

(c) Yes, it does. Let us apply the ratio test. We get:

$$\lim_{n\to\infty} \frac{\frac{(-1)^{n+1}}{n+2} x^{n+2}}{\frac{(-1)^n}{n+1} x^{n+1}} = -\frac{n+1}{n+2} x = -x$$

Note that the series converges uniformly if $|-x| = |x| < 1$. Hence, it converges on $(-1, 1)$. We now need to check the endpoints $x = -1$ and $x = 1$. Notice that if $x = -1$, the series does not converge uniformly as $\ln(1 + -1) = \ln(0)$ which is undefined (negative infinity). For $x = 1$, it follows by **Leibnitz' test for alternating series**, that the series converges. Hence, the Taylor series converges uniformly to f on $(-1, 1]$ which is its interval of convergence.