Homework №14

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5.3/5.4 - Cardinalities of Infinite Sets

Section 5.3

1. The bijection is $f : N \to A_k : x \mapsto x + k + 1$

   It is injection since if $f(x_1) = f(x_2)$, it means that $x_1 + k + 1 = x_2 + k + 1$ and thus, $x_1 = x_2$. It is surjection since if we have some $y \in A_k$, then we let $x = y - k - 1$ and we get that $f(x) = y - k - 1 + k + 1 = y$. Hence, the function is both injective and surjective and thus, it is a bijection.

2. (a) If A is finite, then $A - \{a_0\}$ is automatically denumerable (since A is finite). If A is infinite, we can show that $f : A \to A - \{a_0\} : x \mapsto x$ is a bijection and thus, since A is countable, $A - \{a_0\}$ is countable as well.

   (b) If A is finite, then $A - \{a\}$ is automatically denumerable (since A is finite). If A is infinite, we can show that $f : A \to A - \{a\} : x \mapsto x$ is a bijection and thus, since A is countable, $A - \{a\}$ is countable as well.

8. Suppose, for the sake of contradiction, $B - A$ is countable, A is also countable, and B is uncountable,. Notice that $B = (B \cap A) \cup (B - A)$ is countable since it is the union of two countable sets. Hence, we have reached the contradiction since we assumed that B is uncountable and thus, $B - A$ is countable. $\square$

15. Since $A \times A$ is countable, then its subset $B = \{(x, x) \mid x \in A\}$ is also countable. Then, $f : A \to B : x \mapsto (x, x)$ is a bijection and thus, $|A| = |B|$ which means that A is countable as well (since B is countable).

    Proof that $f : A \to B : x \mapsto (x, x)$ is a bijection.
    If $f(x_1) = f(x_2)$, it means that $(x_1, x_1) = (x_2, x_2)$ and thus, $x_1 = x_2$. Hence, f is injective.
    For every $(x_i, x_i) \in B$, we can let $x = x_i$ and we get $f(x_i) = (x_i, x_i)$ and thus f is surjective.

Since we have proven that f is both injective and surjective, it means that f is a bijection.

16. NOT DONE YET.

Section 5.4

6. Suppose, for the sake of contradiction, that the infinite set $I' = \{x \in (0,1) \mid x = .a_1a_2...a_n...$ where each $a_i = 3$ or $8\}$ is countable. Being countable means having a bijection with $\mathbb{Z}^+$. Hence, we have essentially assumed that there exists a function f such that $f : \mathbb{Z}^+ \to I'$ is a bijection. Let's visualize it.

$$\begin{aligned}
1 &\mapsto .a_{1,1}a_{1,2}a_{1,3}... \\
2 &\mapsto .a_{2,1}a_{2,2}a_{2,3}... \\
3 &\mapsto .a_{3,1}a_{3,2}a_{3,3}... \\
&.......................
\end{aligned}$$

Now, let's define a function $transform$ in the following way:

$$transform(x) = \begin{cases} 8 & \text{if } n \text{ is } 3 \\ 3 & \text{if } n \text{ is } 8 \end{cases}$$

Simply put, this function is defined for only two inputs - 3 and 8 - and if the input is 3, it returns 8 while if the input is 8, it returns 3.

Now, consider the number $.transform(a_{1,1})transform(a_{2,2})transform(a_{3,3})...$. We know that it is not equal to the first number $(.a_{1,1}a_{1,2}a_{1,3}...)$ since $transform(a_{1,1}) \neq a_{1,1}$. We also know that it is not equal to the second number in the mapping as $transform(a_{2,2}) \neq a_{2,2}$. It does not equal to the third either because $transform(a_{3,3}) \neq a_{3,3}$. As a result, we've got that:

$$\begin{aligned}
.transform(a_{1,1})transform(a_{2,2})transform(a_{3,3})... &\neq .a_{1,1}a_{1,2}a_{1,3}... \\
.transform(a_{1,1})transform(a_{2,2})transform(a_{3,3})... &\neq .a_{2,1}a_{2,2}a_{2,3}... \\
.transform(a_{1,1})transform(a_{2,2})transform(a_{3,3})... &\neq .a_{3,1}a_{3,2}a_{3,3}... \\
&..................................
\end{aligned}$$

Hence, we have constructed an element of the set I' such that there is no corresponding element in $\mathbb{Z}$ that maps to it and the function $f : \mathbb{Z}^+ \to I'$ is not onto hence, is not a bijection. Finally, we have reached the contradiction and the set I' is not countable.