Real Analysis Exams

Exam №4

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1. (a) Notice that we have:

$$\limsup a_n = \lim_{n\to\infty} \left(4 + \frac{1}{n}\right) \cos\left(\frac{n\pi}{4}\right) = 4 \times 1 = 4$$

   Similarly, $\liminf a_n = 4 \times (-1) = -4$. Hence, $\limsup a_n = 4$ and $\liminf a_n = -4$.

   (b) Notice that the subsequence $a_k = \left(4 + \frac{1}{8k}\right) \cos(2\pi k)$ ($n = 8k$ with $k \in \mathbb{N}$) converges to 0 since $\cos(2\pi k) = 0$ and thus, every $a_k = \left(4 + \frac{1}{8k}\right) \times 0 = 0$.

   (c) This set will be $A = (-4, 4)$.

2. Let $\epsilon > 0$ be given, x_0 be fixed, and let $\delta = \min\left(1, \frac{\epsilon}{5(1+2|x_0|)}\right)$. Then for $|x - x_0| < \delta$ we have:

$$\begin{aligned}
|f(x) - f(x_0)| &= |5x^2 + 3 - 5x_0^2 - 3| \\
&= |5(x^2 - x_0^2)| \\
&= 5|x - x_0||x - x_0 + 2x_0| \\
&< 5\delta(|x - x_0| + |2x_0|) \\
&< 5\delta(\delta + 2|x_0|) \\
&\leq 5\frac{\epsilon}{5(1+2|x_0|)}(\delta + 2|x_0|) \\
&\leq \frac{\epsilon}{1+2|x_0|}(1 + 2|x_0|) \\
&= \epsilon \qquad\qquad (\text{Thus, } |f(x) - f(x_0)| < \epsilon)
\end{aligned}$$

Hence, $f(x) = 5x^2 + 3$ is continuous at each point $x_0 \in \mathbb{R}$.

□

3. (a) Counterexample: let us define

$$f_n : [0,1] \to \mathbb{R} : x \mapsto \begin{cases} n & \text{if } 0 < x < \frac{1}{n} \\ 0 & \text{if } x = 0 \text{ or } \frac{1}{n} \leq x \leq 1 \end{cases}$$

Then notice that $\int_0^1 f_n = 1$ and the pointwise limit of f_n is $f(x) = 0$ ($x \in [0,1]$) for which we have $\int_0^1 f = 0$. Hence, we have $\int_0^1 f_n \to 1 \neq \int_0^1 f = 0$. Finally, we get that every f_n is Riemann-integrable, but f is not.

(b) As $f_n \to f$ uniformly, pick n_1 s.t. the following holds:

$$|f_{n_1}(x) - f(x)| < \frac{\epsilon}{3 \cdot (b-a)}$$

Now, since every f_n is integrable, take n_2 such that

$$|U(f_{n_1}, P_{n_2}) - L(f_{n_1}, P_{n_2})| < \frac{\epsilon}{3}.$$

Now, choose $n = \max(n_1, n_2)$,

Notice that

$$\begin{aligned}
|U(f,P_n)-U(f_n,P_n)| &\leq \sum_{x_k} |f(x_k)-f_n(x_k)|\Delta x_k \\
&< \sum_{x_k} \frac{\epsilon}{3(b-a)}\Delta x_k \\
&= \frac{\epsilon}{3(b-a)} \sum_{x_k} \Delta x_k \\
&= \frac{\epsilon}{3(b-a)}(b-a) \\
&= \frac{\epsilon}{3}
\end{aligned}$$

Now, notice that over $[x_k, x_{k+1}]$, $|\sup f(x) - \sup f_n(x)| \leq |f_n(x) - f(x)|$ (since every point of f_n is close to f).

A similar results holds for

$$|L(f,P_{n_2}) - L(f_n, P_{n_2})| < \frac{\epsilon}{3}$$

Hence, we have:

$$\begin{aligned}
|U(f,P_n)-L(f,P_n)| &\leq \Big|U(f,P_n)-U(f_n,P_n)+U(f_n,P_n)-L(f_n,P_n)-\Big(L(f,P_n)-L(f_n,P_n)\Big)\Big| \\
&\leq |U(f,P_n)-U(f_n,P_n)| + |U(f_n,P_n)-L(f_n,P_n)| + |L(f,P_n)-L(f_n,P_n)| \\
&< \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3} = \epsilon
\end{aligned}$$

Finally, we got that if $f_n \to f$ uniformly on $[a,b]$, then f is integrable on $[a,b]$.

□

(c) We have already shown in (b) part of the exercise that f is integrable on $[a,b]$. We now need to show that the following holds:

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$$

It follows by the uniform convergence of f_n that

$$\begin{aligned}
|f_n - f| &< \epsilon & &\Longrightarrow f-\epsilon < f_n < f+\epsilon & (1)\\
\int_a^b \Big(f-\epsilon\Big) < \int_a^b f_n &< \int_a^b \Big(f+\epsilon\Big) & &\Longrightarrow \Big|\int_a^b f_n - \int_a^b f\Big| < \epsilon(b-a) & (2)
\end{aligned}$$

Hence, we can make the difference $|\int_a^b f_n - \int_a^b f|$ arbitrarily small and we get $\lim_{n\to\infty}\int_a^b f_n = \int_a^b f$.

□

4. Yes, G is differentiable at 1. In order to show this, we must show that the following limit exists:

$$\lim_{x \to 1} \frac{G(x) - G(1)}{x - 1}$$

Notice that the antiderivative A of function g can be calcuated as:

$$A = \begin{cases} x & \text{if } x \neq 1 \\ 2x & \text{if } x = 1 \end{cases}$$

We have:

$$G(1) = \int_0^1 g = A'(1) - A'(0) = 2 - 1 = 1$$

Plugging in the values into the limit formula above, we get:

$$\lim_{x \to 1} \frac{\int_0^x g - 1}{x - 1}$$

Now, clearly $\lim_{x \to 1^-} \frac{1-1}{x-1} = 0$ and $\lim_{x \to 1^+} \frac{1-1}{x-1} = 0$. Hence, $\lim_{x \to 1} \frac{\int_0^x g - 1}{x-1} = 0$ and always exists. Finally, we got that G is differentiable at 1.

□

5. No, H is not differentiable at 1. In order to show this, we must show that the following limit does not exist:

$$\lim_{x \to 1} \frac{H(x) - H(1)}{x - 1}$$

Notice that the antiderivative A of function h can be calcuated as:

$$A = \begin{cases} x & \text{if } x < 1 \\ 2x & \text{if } x \geq 1 \end{cases}$$

We have:

$$G(1) = \int_0^1 g = A'(1) - A'(0) = 2 - 1 = 1$$

Plugging in the values into the limit formula above, we get:

$$\lim_{x \to 1} \frac{\int_0^x h - 1}{x - 1}$$

Now, clearly $\lim_{x \to 1^-} \frac{1-1}{x-1} = 0$ and $\lim_{x \to 1^+} \frac{1-2}{x-1} \neq 0$. Hence, $\lim_{x \to 1^-} \neq \lim_{x \to 1^+}$ and the two-sided limits are not the same which implies that the limit does not exist. Finally, we got that H is not differentiable at 1.

□

6. Let $A(t)$ be the antiderivative of $t^2 \sin(t^2)$. We have:

$$\begin{aligned}
\frac{d}{dx}\int_0^{x^2} t^2 \sin(t^2)dt &= \frac{d}{dx}A(x^2) - A(0) \\
&= 2xA'(x^2) \\
&= 2x \times (x^2)^2 \sin((x^2)^2 \\
&= 2x^5 \sin(x^4)
\end{aligned}$$

7. Since f is continuous on $[a, b]$, it follows that f achieves both the absolute maximum M and the absolute minimum m in $[a, b]$. Suppose, without a loss of generality, that these points are c_1 and c_2 with $c_1 < c_2$. We then have:

$$m(b-a) \leq \int_a^b f \leq M(b-a) \tag{3}$$

$$m \leq \frac{1}{b-a}\int_a^b f \leq M \tag{4}$$

Now, it follows by **Intermediate Value Theorem** that $\exists c \in (c_1, c_2) \subset [a, b]$ s.t. $f(c) = \frac{1}{b-a}\int_a^b f$.

$\square$

8. Suppose, for the sake of contradiction, that f has Generalized Riemann integrals Q_1 and Q_2 with $Q_1 \neq Q_2$ and let $\epsilon > 0$ be given. Then,.it follows that $\exists \delta_1(x)$ s.t. $\forall \delta_1(x)$-fine tagged partitions, $|R(f, P) - Q_1| < \frac{\epsilon}{2}$. Similarly, $\exists \delta_2(x)$ s.t. $\forall \delta_2(x)$-fine tagged partitions, $|R(f, P) - Q_2| < \frac{\epsilon}{2}$. Now, let $\delta(x) = \min(\delta_1(x), \delta_2(x))$. It follows by **Theorem 8.1.5** that there exists a tagged partition $(P, \{c_k\})$ s.t. it is both $\delta_1(x)$-fine and $\delta_2(x)$-fine. We have:

$$|Q_1 - Q_2| \leq |Q_1 - R(f, P)| + |R(f, P) - Q_2| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Hence, we got that $Q_1 = Q_2$ and we face a contradiction since we have assumed that $Q_1 \neq Q_2$. Finally, we conclude that if f has a generalized Riemann integral on $[a, b]$, then the value of the integral $\int_a^b f$ is unique.

$\square$

9. (a) Below find the plots.

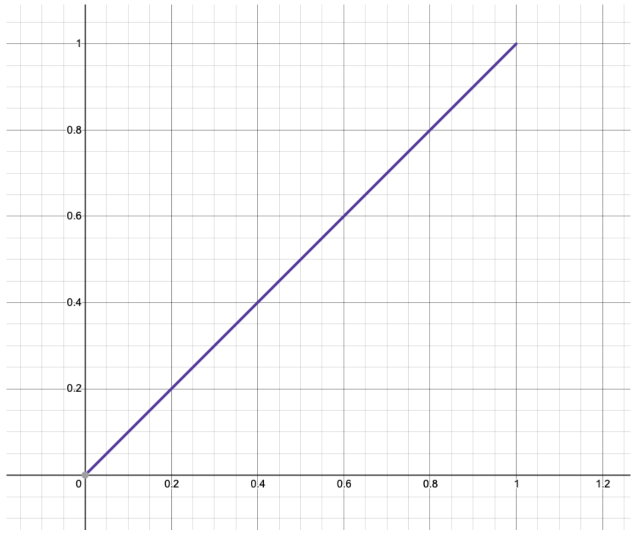

Figure 1: Plot of f_0

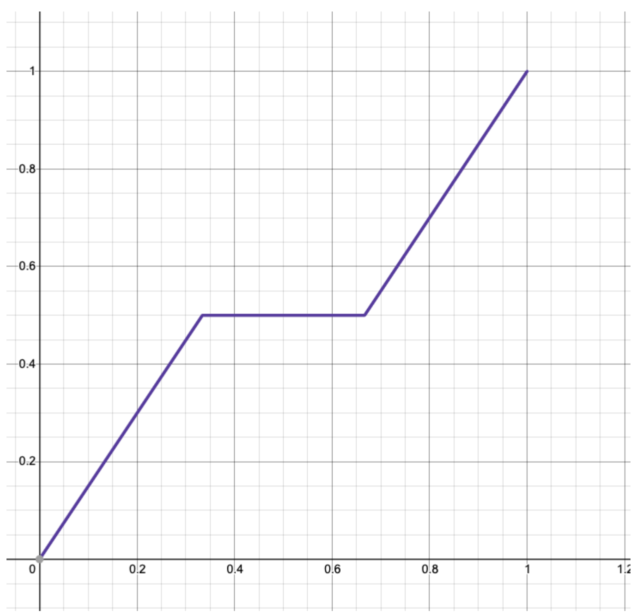

Figure 2: Plot of f_1

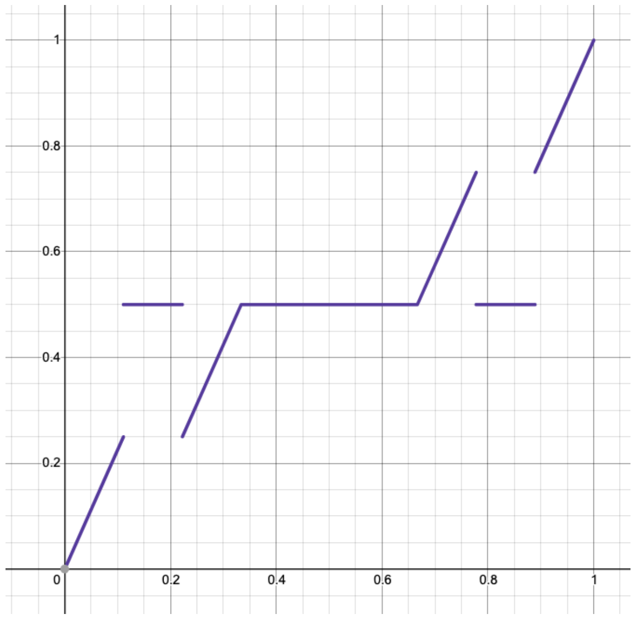

Figure 3: Plot of f_2

(b)

$$f_3(x) = \begin{cases} \dfrac{f_2(3x)}{2} & \text{if } 0 \le x \le \dfrac{1}{3} \\ \dfrac{1}{2} & \text{if } \dfrac{1}{3} < x < \dfrac{2}{3} \\ \dfrac{(f_2(3x-3)+2)}{2} & \text{if } \dfrac{2}{3} \le x \le 1 \end{cases}$$

Below find the graph of f_3:

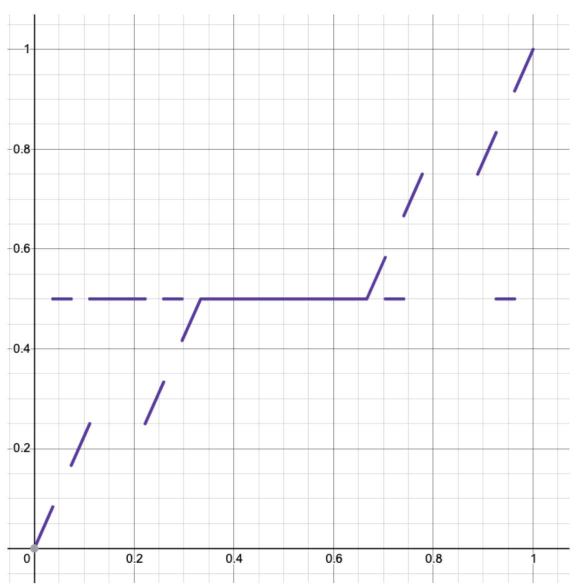

Figure 4: Plot of f_3

(c) It is similar to the cantor set in a way that the function iteratively turns the open middle third of its values to $\frac{1}{2}$.

The difference is that in Cantor's set, the open middle third was completely removed, but in this case it is just transformed into $\frac{1}{2}$.

(d) Notice that at each step, the function just turns some portion(s) of the closed interval $[0,1]$ into $\frac{1}{2}$. Hence, there are no discontinuities and the function has to be continuous.

$\square$

(e) The function is not differentiable at non-endpoints of $[0,1]$. Hence, we have:

$$f'(x) = \begin{cases} 0 & \text{if } x \in \{0,1\} \\ \\ none & \text{otherwise} \end{cases}$$

Now, let us show that the function is not differentiable at non-endpoints. Suppose, for the sake of contradiction, $(x_n) \to x$ and $(y_n) \to x$ s.t. $\forall n, x_n < y_n$ and asume that the derivative exists. We have:

$$\frac{f(y_n) - f(x_n)}{y_n - x_n} \to f'(x)$$

Now, we can see that $y_n - x_n = \frac{1}{3^n}$, but $f(y_n) - f(x_n) = \frac{1}{2^n}$, hence $f'(x) \to \infty$ and we face the contradiction. Thus, the derivative cannot exist.

$\square$

(f) As mentioned in (d), as n approaches infinity, all of the interval $[0,1]$ will be $\frac{1}{2}$. Notice that $\forall n \in \mathbb{N}$ (including $n \to \infty$), $f(0) = 0$ and $f(1) = 1$. To see this, one could think about the graph construction or about the fact that in the end, for $x = 0$, the function will have a multiplication by 0 (yields 0), similar reasoning could be applied for obtaining $f(1) = 1$. Hence, we have:

$$\int_0^1 f = f(1) - f(0) = 1 - 0 = 1$$

Even easier way to see that $\int_0^1 f = 1$ is to notice that the function itself does not remove any points from the interval $[0,1]$, it just maps to $\frac{1}{2}$ and hence, the total number of points always stays the same.