# ***Real Analysis***

## Assignment №5

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3.2.2 (a) The limit points of $A$ are 1 and $-1$. The limit points of $B$ are all numbers in the closed interval $[0, 1]$.

Limit points of $A$ are $1, -1$ as $\lim_{n\to\infty} A = 1, -1$.
Limit points of $B$ are all numbers in the closed interval $[0, 1]$ since between any two rationals there are infinitely many irrationals, vice versa. Hence, the limit points of $B$ form the closed interval $[0, 1]$.

(b) $A$ **is neither closed nor open**. $B$ **is neither closed nor open**. This is due to the fact that between any two rational numbers, there exists infinitely many irrational numbers.

(c) All points in $A$ except for 1 and $-1$ are isolated. Due to the fact that between any two rational numbers, there exists infinitely many rational numbers, **B has no isolated points**.

(d) $\overline{A} = A \cup \{1, -1\} = \left\{(-1)^n + \dfrac{2}{n} \mid n = 1, 2, 3, \dots\right\} \cup \{-1\}$.
$\overline{B} = A \cup [0, 1] = \{x \in \mathbb{Q} \mid 0 < x < 1\} \cup [0, 1] = [0, 1]$.

3.2.6 (a) This is false.

Let $S = (-\infty, \sqrt{3}) \cup (\sqrt{3}, +\infty)$. Then the union of two open sets $(-\infty, \sqrt{3})$ and $(\sqrt{3}, +\infty)$ is also open. Notice that $S$ contains all numbers in $\mathbb{R}$ except for $\sqrt{3}$ which is irrational. Hence, $S$ contains all rational numbers. However, it does not contain $\sqrt{3}$ which means that it does not contain all real numbers.

$\square$

(b) This is false.

Let us define $S_n = [n, \infty)$. Then we get $\bigcap_n S_n = \emptyset$.

(c) This is true.

Let $S$ be a nonempty open set. Then, as $S$ is not empty, $\exists x$ s.t. $x \in S$. Now, since $S$ is also open, $\exists \epsilon > 0$ s.t. $V_\epsilon(x) \subseteq S$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, $\exists r \in \mathbb{Q}$ s.t. $x - \epsilon < r < x + \epsilon$. Finally, we get $r \in V_\epsilon(x) \subseteq S$.

$\square$

(d) This is false.

Consider $S = \{\frac{1}{n} + \sqrt{3} \mid n \in \mathbb{N}\} \cup \{\sqrt{3}\}$. Then, $S$ is a bounded infinite closed set, however, every number in $S$ is irrational.

$\square$

(e) This is true.

The Cantor set is the intersection of closed sets. Now, since the arbitrary intersectionof closed sets is closed, the Cantor set must be closed.

3.2.8 (a) $\overline{A \cup B}$ **is sometimes open**. If we set $A = B = \mathbb{R}$, then $A \cup B$ is open. On the other hand, if $A = B = [0, 1]$, then $\overline{A \cup B} = [0, 1]$ which means that $\overline{A \cup B}$ is not open.

$\overline{A \cup B}$ **is definitely closed** since for any set $S$, $\overline{S}$ is definitely closed.

$\overline{A \cup B}$ **is sometimes both open and closed (aka *clopen*)**. If $A = B = \mathbb{R}$, then $\overline{A \cup B} = \mathbb{R}$ which is both open and closed. However, if we set $A = B = [0, 1]$, then $\overline{A \cup B} = [0, 1]$ which is not open and thus, $\overline{A \cup B}$ is not both open and closed.

$\overline{A \cup B}$ **can never be neither open nor closed** as $\overline{A \cup B}$ is always closed.

(b) $\overline{A \setminus B}$ **is definitely open** since $\overline{A \setminus B} = \overline{A \cap B^c}$. Then, since $B$ is closed, its complement is open. Finally, as both $A$ and $B^c$ are open, $\overline{A \setminus B}$ is open too.

$\overline{A \setminus B}$ **is sometimes closed**. If we let $A = \mathbb{R}$ and $B = \emptyset$, then $A \setminus B = \mathbb{R}$ is closed. However, if we let $A = (0, 5)$ and $B = [1, 6]$, then $A \setminus B = (0, 1)$ which is not closed.

$\overline{A \setminus B}$ **is sometimes both open and closed (aka *clopen*)**. If we let $A = \mathbb{R}$ and $B = \emptyset$, then $A \setminus B = \mathbb{R}$ is closed. Hence, in this case, $\overline{A \setminus B}$ is both open and closed (open since we showed in $(a)$ that it is always open). However, if we let $A = (0, 5)$ and $B = [1, 6]$, then $A \setminus B = (0, 1)$ is not closed.

$\overline{A \setminus B}$ **can never be neither open nor closed** as $\overline{A \setminus B}$ is always open.

(c) $(A^c \cup B)^c$ **is definitely open**. This is the case since if $A$ is open, $A^c$ is closed. Now, as $B$ is closed $A^c \cup B$ is also closed. Hence, $(A^c \cup B)^c$ is open.

$(A^c \cup B)^c$ **is sometimes closed**. If we let $A = B = \mathbb{R}$, then $(A^c \cup B)^c = \emptyset$ which is closed. On the other hand, if we let $A = (0, 1)$ and $B = \emptyset$, then $(A^c \cup B) = (0, 1)$ which is not closed.

$(A^c \cup B)^c$ **is sometimes both open and closed (aka *clopen*)**. If we let $A = \mathbb{R}$ and $B = \emptyset$, then $A \setminus B = \mathbb{R}$ is closed. Hence, in this case, $\overline{A \setminus B}$ is both open and closed (open since we showed in $(a)$ that it is always open). However, if we let $A = (0, 5)$ and

$B = [1, 6]$, then $A \setminus B = (0, 1)$ is not closed.

$(A^c \cup B)^c$ **can never be neither open nor closed** as $(A^c \cup B)^c$ always open.

(d) Notice that $(A \cap B) \cup (A^c \cap B) = B$.

$(A \cap B) \cup (A^c \cap B) = B$ **is sometimes open**. If $B = \mathbb{R}$ then it is open. However, if $B = [0, 1]$, it is not open.

$(A \cap B) \cup (A^c \cap B) = B$**, by definition, is definitely closed**.

$(A \cap B) \cup (A^c \cap B) = B$ **is sometimes both open and closed (aka *clopen*)**. If $B = \mathbb{R}$, then it is both open and closed. However, if $B = [0, 1]$, then it is not open (but it is still open as it is always open).

$(A \cap B) \cup (A^c \cap B) = B$ **can never be neither open nor closed** as $(A^c \cup B)^c$ always open.

(e) Notice that since $A$ is open, $A^c$ is closed and thus, $\overline{A^c} = A^c$. Hence, we get $\overline{A}^c \cap A^c = \overline{A}^c$.

$\overline{A}^c \cap A^c = \overline{A}^c$ **is definitely open**. This is by definition. As $A$ is open, $\overline{A}$ is closed and its complement $\overline{A}^c$ must be open.

$\overline{A}^c \cap A^c = \overline{A}^c$ **is sometimes closed**. If we let $A = \emptyset$, then $\overline{A}^c = \mathbb{R}$ is closed (and open as well). However, if $A = (0, 1)$, then $\overline{A}^c = (-\infty, 0) \cup (1, +\infty)$ which is not closed.

$\overline{A}^c \cap A^c = \overline{A}^c$ **is sometimes both open and closed (aka *clopen*)**. If we let $A = \emptyset$, then $\overline{A}^c = \mathbb{R}$ which is both open and closed. However, if $A = (0, 1)$, then $\overline{A}^c = (-\infty, 0) \cup (1, +\infty)$ which is not closed.

$\overline{A}^c \cap A^c = \overline{A}^c$ **can never be neither open nor closed** as $(A^c \cup B)^c$ always open.

3.2.14 (a) Let us first show that $E$ is closed if and only if $\overline{E} = E$. We will first prove this directly and then prove its converse.

Suppose that $E$ is closed. Then $E$ must contain all of its limit points. Let us denote the set of all limit points of $E$ as $L$. Then it follows that $L \subseteq E$ but $\overline{E} = E \cup L$. Thus, $\overline{E} = E$.

$\square$

Conversely, suppose that $\overline{E} = E$. Then it follows that $E$ contains all of its limit points since $\overline{E}$ contains all of the limit points of $E$. Hence, $E$ must be closed.

$\square$

Finally, we have shown that $E$ is closed if and only if $\overline{E} = E$.

$\square$

Let us now show that $E$ is open if and only if $E^\circ = E$. Similarly, we will first prove this statement directly and then prove its convers.

Suppose $E$ is open. Then $\forall x \in E, \exists V_\epsilon(x) \subseteq E$. It follows that $x \in E^\circ$ and thus, $E \subseteq E^\circ$. On the other hand, by definition, $E^\circ \subseteq E$. Hence, $E^\circ = E$

$\square$

Conversely suppose $E^\circ = E$. Then, by definition, since $E^\circ$ is open, $E$ must be too.

$\square$

Finally, we have shown that $E$ is open if and only if $E^\circ = E$.

$\square$

(b) Let us first show that $\overline{E}^c = (E^c)^\circ$. In order to prove this, we first show that $\overline{E}^c \subseteq (E^c)^\circ$ and then show that $(E^c)^\circ \subseteq \overline{E}^c$.

Let $x \in \overline{E}^c$. Then, as $\overline{E}^c$ is open, $\exists V_\epsilon(x) \subseteq \overline{E}^c$. Now, since $E \subseteq \overline{E}$, it follows that $\overline{E}^c \subseteq \overline{E^c}$.

Now, let $x \in (E^c)^\circ$. Then $\exists V_\epsilon(x) \subseteq \overline{E}^c \subseteq E^c$. It follows that $V_\epsilon(x) \cap E = \emptyset$.

Now, notice that we $V_\epsilon(x) \cap \overline{E} = \emptyset$. To prove this, suppose, for the sake of contradiction, that $V_\epsilon(x) \cap \overline{E} \neq \emptyset$. Then $\exists y \in \overline{E}$ s.t. $y \in V_\epsilon(x)$ (with $V_\epsilon(x)$ being open). Then there must exists some $\epsilon$-neighborhood of $y$ that is contained in $V_\epsilon(x)$. However, $\epsilon$-neighborhood of $y$ contains points of $E$ which contradicts $V_\epsilon(x) \cap E = \emptyset$ (which we have already shown). Therefore, $V_\epsilon(x) \cap \overline{E} = \emptyset$. Hence, $V_\epsilon(x) \subseteq \overline{E}^c$ and it follows that $(E^c)^\circ \subseteq \overline{E}^c$.

Finally, we have shown both $\overline{E}^c \subseteq (E^c)^\circ$ and then show that $(E^c)^\circ \subseteq \overline{E}^c$. Hence, $\overline{E}^c = (E^c)^\circ$.

$\square$

Let us now show that $\overline{E}^c = (E^c)^\circ$. Recall that we have already shown $\overline{E}^c = (E^c)^\circ$. If we simply substitute $E$ with $E^c$ in this equality, we get $\overline{E^c}^c = ((E^c)^c)^\circ = E^\circ$. Taking the complement of both sides gives us $\overline{E^c} = \overline{E^\circ}^c$.

$\square$