Real Analysis

Assignment №9

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5.2.3 (a)

$$\boxed{h'(x) = \lim_{x \to c} \frac{h(x) - h(c)}{x - c} = \lim_{x \to c} \frac{\frac{1}{x} - \frac{1}{c}}{x - c} = \lim_{x \to c} -\frac{1}{cx} = -\frac{1}{x^2}} \quad \square$$

(b) Assuming $g(c) \neq 0$, we have:

$$\boxed{\left(\frac{f}{g}\right)'(c) = f'(c)\frac{1}{g(c)} + \left(-\frac{1}{(g(c))^2}g'(c)f(c)\right) = \frac{f'(c)g(c) - g'(c)f(c)}{(g(c))^2}} \quad \square$$

(c) Assuming $g(c) \neq 0$, we have:

$$
\begin{aligned}
\left(\frac{f}{g}\right)'(c) &= \lim_{x\to c} \frac{\left(\frac{f}{g}\right)(x) - \left(\frac{f}{g}\right)(c)}{x-c} \\
&= \lim_{x\to c} \frac{\frac{f(x)}{g(x)} - \frac{f(c)}{g(x)} + \frac{f(c)}{g(x)} - \frac{f(c)}{g(c)}}{x-c} \\
&= \lim_{x\to c} \frac{\frac{f(x)}{g(x)} - \frac{f(c)}{g(c)}}{x-c} \\
&= \lim_{x\to c} \frac{f(x)g(c) - f(c)g(x)}{g(x)g(c)(x-c)} \\
&= \lim_{x\to c} \frac{g(c)\Big(f(x)-f(c)\Big) - f(c)\Big(g(x)-g(c)\Big)}{g(x)g(c)(x-c)} \\
&= \lim_{x\to c} \frac{g(c)}{g(x)g(c)} \times \lim_{x\to c} \frac{f(x)-f(c)}{x-c} - \lim_{x\to c} \frac{f(c)}{g(x)g(c)} \times \lim_{x\to c} \frac{g(x)-g(c)}{x-c} \\
&= \frac{g(c)}{\Big(g(c)\Big)^2} \times f'(c) - \frac{f(c)}{\Big(g(c)\Big)^2} \times g'(c) \\
&= \boxed{\frac{g(c)f'(c) - f(c)g'(c)}{\Big(g(c)\Big)^2}} \quad \square
\end{aligned}
$$

5.2.7 (a) Let $a = \frac{5}{4}$. For $x = 0$ we have:

$$\lim x \to 0 \frac{x^{\frac{5}{4}} \sin\frac{1}{x} - 0}{x - 0} = \lim_{x\to 0} \sqrt[4]{x} \sin\frac{1}{x}$$

Notice that $\sqrt[4]{x} \leq \sqrt[4]{x} \sin\frac{1}{x} \leq \sqrt[4]{x}$ and $\lim_{x\to 0} \sqrt[4]{x} = 0$. Then it follows by the **Squeeze Theorem** that $\lim_{x\to 0} \sqrt[4]{x}\frac{1}{x} = 0$ and hence, $g_{\frac{5}{4}}(x)$ is differentiable at 0.

Now, for $x \neq 0$ we get:

$$g'_{\frac{5}{4}}(x) = \Big(x^{\frac{5}{4}} \sin\frac{1}{x}\Big)' = \frac{5}{4}\sqrt[4]{x}\sin\frac{1}{x} - \frac{1}{\sqrt[4]{x^3}}\cos\frac{1}{x}$$

Set $x_n = \frac{1}{2n\pi}$ and we have $g'_{\frac{5}{4}}(x) = -\frac{1}{\sqrt[4]{\left(\frac{1}{2n\pi}\right)^3}} = -\sqrt[4]{(2n\pi)^3}$ which is unbounded on $[0,1]$.

Hence, for $a = \frac{5}{4}$, function g_a is differentiable on $\mathbb{R}$ with g'_a unbounded on $[0,1]$.

Finally, we got that $g_{\frac{5}{4}}$ is an example of a function that is differentiable on $\mathbb{R}$ with $g'_{\frac{5}{4}}$ being unbounded on $[0,1]$.

(b) Let $a = \frac{5}{2}$. For $x = 0$ we have:

$$\lim x \to 0 \frac{x^{\frac{5}{2}} \sin \frac{1}{x} - 0}{x - 0} = \lim_{x \to 0} \sqrt{x^3} \sin \frac{1}{x}$$

Then, once again, per the **Squeeze Theorem**, the limit is 0.

Now, for $x \neq 0$ we get:

$$g'_{\frac{5}{2}}(x) = \left(x^{\frac{5}{2}} \sin \frac{1}{x}\right)' = \frac{5}{2}\sqrt{x^3} \sin \frac{1}{x} - \sqrt{x} \cos \frac{1}{x}$$

Functions sin and cos are both bounded and it follows that $\lim_{x \to 0} g'_{\frac{5}{2}}(x) = 0 = g'_{\frac{5}{2}}(0)$. Thus, we have that $g'_{\frac{5}{2}}$ is continuous. Similar to part (a), let $x_n = \frac{1}{2n\pi}$. Then we get $g''_{\frac{5}{2}} = 3\sqrt{2n\pi}$ which is unbounded. Hence, g' is not differentiable at 0.

Finally, we got that $g_{\frac{5}{2}}$ is an example of a function that is differentiable on $\mathbb{R}$ with $g'_{\frac{5}{2}}$ being continuous but not differentiable at 0.

(c) Let $a = 4$. For $x = 0$ we have:

$$g'_4(x) = 4x^3 \sin \frac{1}{x} - x^2 \cos \frac{1}{x}$$
$$g''_4(x) = 12x^2 \sin \frac{1}{x} - 6x \cos \frac{1}{x} + \sin \frac{1}{x}$$

Then, notice that

$$g''_4 = \lim_{x \to 0} \frac{4x^3 \sin \frac{1}{x} - x^2 \cos \frac{1}{x} - 0}{x - 0} = \lim_{x \to 0} 4x^2 \sin \frac{1}{x} - x \cos \frac{1}{x} = 0$$

On the other hand, $\lim_{x \to 0} 12x^2 \sin \frac{1}{x} - 6x \cos \frac{1}{x} + \sin \frac{1}{x}$ does not exist, as the the third term fluctuates between 1 and -1 (the first two do go to 0, but the third one does not).

Finally, we got that g_4 is an example of a function that is differentiable on $\mathbb{R}$ with g'_4 being differentiable on $\mathbb{R}$, but g''_4 not continuous at 0.

5.3.1 (a) Suppose that f is differentiable on a closed interval $[a, b]$ and that f' is continuous on a closed interval $[a, b]$. It follows that $|f'|$ is also continuous on $[a, b]$. Now, per **Theorem 4.4.2 (Extreme Value Theorem)**, $\exists x_0 \in [a, b]$ s.t. $\forall x \in [a, b], |f'(x)| \leq f'(x_0)$. Then, if some $m, n \in [a, b]$ with $m \neq n$, by the **Mean Value Theorem**, $\exists x \in [a, b]$ s.t. $\left|\dfrac{f(m) - f(n)}{m - n}\right| = |f'(x)| \leq f'(x_0)$. Hence, we got that f is Lipschitz on $[a, b]$ with $M = |f'(x_0)|$.

$\square$

5.3.3 (a) As h is differentiable on $[0, 3]$, it follows that h is also continuous on $[0, 3]$. Hence, the function $g(x) = h(x) - x$ is also continuous on $[0, 3]$. Now, notice that $g(0) = h(0) = 1$ and $g(3) = h(3) - 3 = -1$. Then, per **Theorem 4.5.1 (Intermediate Value Theorem)**, there exists $d \in [0, 3]$ s.t. $g(d) = 0$ which means that $h(d) = d$.

$\square$

(b) Once again, since h is differentiable on $[0, 3]$, it follows that h is also continuous on $[0, 3]$. Now, by the **Mean Value Theorem**, $\exists c \in (0, 3)$ s.t. $h'(c) = \dfrac{h(3) - h(0)}{3 - 0} = \frac{1}{3}$.

$\square$

(c) As $h(1) = h(3)$, per **Theorem 5.3.1 (Rolle's Theorem)**, $\exists b \in (1, 3)$ s.t. $h'(b) = 0$. Now, since $0 < \frac{1}{4} < \frac{1}{3}$, by **Theorem 5.2.7 (Darboux's Theorem)**, $\exists x \in A = [b, c]$ (could be $[c, b]$ if $b > c$, but this does not change the logic) s.t. $h'(x) = \frac{1}{4}$ and since $A \subset [0, 3]$, we get $x \in [0, 3]$.

$\square$

5.3.7 Suppose, for the sake of contradiction, that f is differentiable on an interval with $f'(x) \neq 1$ and has two fixed points x and y. Then, we have $f(x) = x$ and $f(y) = y$. Now, by the **Mean Value Theorem**, $\exists c \in (x, y)$ s.t. $\dfrac{f(y) - f(x)}{y - x} = f'(c)$. Substituting $f(x)$ with x and $f(y)$ with y gives us $f'(c) = \dfrac{y - x}{y - x} = 1$. Now, by assumption, we know that $\forall x, f'(x) \neq 1$, however, if we set $x = c$, we get $f'(c) = 1$ and we face a contradiction. Finally, we got that if f is differentiable on an interval with $f'(x) \neq 1$, f can only have at most one fixed point.

$\square$

5.3.11 (a) Let f and g be continuous on an interval containing a, and assume f and g are differentiable on this interval with the possible exception of the point a. Let us consider the following two cases:

(i) $x < 0$

If $x < 0$, for $c \in (x, 0)$, it follows by the **Theorem 5.3.5 (Generalized Mean Value Theorem)** that

$$\frac{f'(c)}{g'(c)} = \frac{f(x) - f(0)}{g(x) - g(0)} = \frac{f(x)}{g(x)}$$

Now, since $x \to 0^-$, it follows that $c \to 0^-$ and thus, we have

$$\lim_{c \to 0^-} \frac{f'(c)}{g'(c)} = \lim_{x \to 0^-} \frac{f(x)}{g(x)}$$

(ii) $x > 0$

Similarly, if $x > 0$, for $c \in (0, x)$, it follows by the **Theorem 5.3.5 (Generalized Mean Value Theorem)** that

$$\frac{f'(c)}{g'(c)} = \frac{f(x) - f(0)}{g(x) - g(0)} = \frac{f(x)}{g(x)}$$

Now, since $x \to 0^+$, it follows that $c \to 0^+$ and thus, we have

$$\lim_{c \to 0^+} \frac{f'(c)}{g'(c)} = \lim_{x \to 0^+} \frac{f(x)}{g(x)}$$

Finally, we got that $\lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \frac{f(x)}{g(x)}$.

$\square$