Homework №8

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October 20, 2018

Additional Proof Practice

53. A subset S of $\mathbb{Z}^+$ is called a P_3**-set** if there exists (not necessarily distinct) elements $x, y, z \in S$ such that $x + y + z$ is prime.

(a) Give some examples of P_3-sets.

$\{1\}$ because $1 + 1 + 1 = 3$ is a prime.
$\{2, 3\}$ because $2 + 2 + 3 = 7$ is a prime.
$\{12, 25, 30\}$ because $12 + 25 + 30 = 67$ is a prime.

(b) Prove or Disprove: If A is a P_3-set, and $A \subseteq B \subseteq \mathbb{Z}^+$, then B is a P_3-set.

It's right so let's prove it. Since we know that A is a P_3-set, we know there exist elements (not necessarily distinct) x, y and z such that $x + y + z$ is prime. Since $A \subseteq B$, we know that all elements of A are also in B meaning that x, y and z are in B as well. Then there exist elements $x, y, z \in B$ (which are also in A) such that $x + y + z$ is a prime.

Q.E.D.

(c) Prove or Disprove: If S is a P_3-set, then so is $S_{+3} := \{x + 3 \mid x \in S\}$.

It's false. Counterexample: Let $S = \{1\}$, then we know that S is a P_3-set since $1 + 1 + 1 = 3$ is a prime. However, $S_{+3} = \{4\}$ and $4 + 4 + 4 = 12$ which is certainly not a prime ($12 = 2^2 * 3^1$).

(d) Prove or Disprove: Every P_3 set contains a prime.

False. Counterexample: Let $S = \{1\}$, then 1 is not a prime but $1 + 1 + 1 = 3$ is a prime.

(e) Prove or Disprove: The intersection of two P_3-sets is a P_3-set.

It's false. Let $A = \{1\}$, then A is P_3-set since $1 + 1 + 1 = 3$ is a prime. Let $B = \{2, 3\}$, then $2 + 2 + 3 = 7$ is a also prime thus B is also a P_3-set. On the other hand, $A \cap B = \emptyset$ which means that there are no elements x, y, z such that $x + y + z$ is a prime and thus, the intersection of two P_3-sets is not necessarily a P_3-set.

(f) Prove or Disprove: Every P_3-set contains an odd integer.

It's true. Suppose, for the sake of contradiction, that S is a P_3-set and it does not contain any odd integers. Then it must be the case that all the elements of S are even. Now, since all the elements are even, it means that no matter what 3 elements x, y and z we take, their sum will always be even. On the other hand, the only even prime we have is 2, but unfortunately there are no three numbers $x, y, z \in \mathbb{Z}^+$ which sum up to 2. The best we can do is $1 + 1 + 1$ which is 3 and is one more than 2 (in all other cases the sum of three positive integers will be greater than 2). Thus, there is no way to get 2 and otherwise, we won't have 3 elements which sum up to the prime. Hence, we have reached the contradiction and S is a not a P_3-set.

Q.E.D.

(g) Prove or Disprove: Every infinite subset S of $\mathbb{Z}^+$ is a P_3-set.

It's false. Counterexample: Since we already proved that every P_3-set contains an odd integer, we can take a set of all positive even integers which is a subset of $\mathbb{Z}^+$. Let's call this subset E. Then, we know that every element of the subset E is even and sum of any 3 elements (not necessarily distinct) will also be even. However, once again, the only even integer which is a prime is 2 and we cannot get 2 by summing 3 integers which are greater than or equal to 2 (greater than equal to 2 because $E = \{2, 4, 6, 8, 10...\}$).

(h) Prove or Disprove: If S is a finite subset of $\mathbb{Z}^+$, then $\mathbb{Z}^+ - S$ is a P_3-set.

It's true. Let's prove it by construction. Since S is a finite set, we know that it cannot contain all the elements of $\mathbb{Z}^+$ because $\mathbb{Z}^+$ is infinite. We already proved that there are infinitely many primes (in the previous chapter(s)). Thus, since S is finite, it cannot possibly contain all the primes. Then we can find a prime p and integer k such that $k, p - 2k \notin S$. Finally, let's have a look at the set $\mathbb{Z}^+ - S = \{k, p - 2k...\}$ which is a P_3-set because $p - 2k + k + k = p$ is a prime.

Q.E.D.

54. If a subset S of $\mathbb{Z}^+$ is a P_3-set then the **core** of S is the set

$$\text{core}(S) := \{s \in S \mid S - \{s\} \text{ is not a } P_3\text{-set}\}.$$

(a) What is the core of $S = \{2, 3, 6\}$?

The core of $S = \{2, 3, 6\}$ is core$(S) = \{2, 3\}$. The reason is that if we take out 2, we are left with 3 and 6 which are both multiples of 3 and any variations of their sums will never be a prime ($3 + 3 + 3$ is a not a prime, $3 + 6 + 3$ is not a prime etc.). If we take out 3, we are left with two even numbers, namely 2 and 6, and still we know that every P_3-set contains an odd integer thus, taking out 3 will leave us with non-P_3-set (any variations of the sums of the even integers will not be even; the only case is 2 but $2 + 2 + 2 = 6$ is the best we can do). On the other hand, if we take out 6, S will still be a P_3-set since $2 + 2 + 3 = 7$ is a prime. Thus, core$(S) = \{2, 3\}$.

(b) Give an example of a P_3-set whose core is the empty set, or prove none exists.

Here is an example: Let $S = \{1, 2, 3\}$. If we take out 1, we are left with $A = \{2, 3\}$ which is still a P_3-set because $2 + 2 + 3 = 7$ is a prime. If we take out 2, we are left with $B = \{1, 3\}$ which is still a P_3 set because $1 + 1 + 1 = 3$ is a prime. Finally, if we take out 3, we are left with $C = \{1, 2\}$ which, once again, is a P_3-set. Thus, we've showed exhaustively that no matter what we take out of S, it is still a P_3-set and hence, core$(S) = \emptyset$.

(c) Give an example of a P_3-set whose core is infinite, or prove none exists.

There is no such P_3-set. In fact, there is no way for the core of the set to be bigger than 3. Let's prove it. Suppose, for the sake of contradiction, there exists a P_3-set S such that its core is infinite. Let the core be the set $C = \{c_0, c_1, c_2, c_3 \ldots\}$. Then we know that $S - \{c_0\}$ is not a P_3-set. Thus, c_0 must have contributed to some summation of 3 elements (among which is c_0). Then, we also know that $S - \{c_1\}$ is not a P_3 set as well as $S - \{c_2\}$ is not a P_3-set. We also know that $S - \{c_4\}$ is not a P_3-set and we've reached the contradiction. The reason we've reached the contradiction is that if there are 3 elements such that taking out either of those makes S a non-P_3-set, they, in the worst case, are in the same sum ($x = c_0, y = c_1, z = c_2$) and then taking out c_4 should not make S non-P_3-set (because c_0, c_1, and c_3 are still there). On the other hand, if c_0, c_1, c_2 are not in the same sum, we've reached even "worse" contradiction since leaving c_0 for instance and taking out c_1 would still result in P_3-set (the same logic, if c_0, c_1, c_2 are in the different sums it means that leaving one of them would still give us a P_3 set). Without a loss of generality, if two of the c_1, c_2, c_3 are in the same sum, it means that c_0, c_1 (took them arbitrarily, without a loss of generality) are in some sum in which c_2 plays no role. Then taking out c_0 or c_1 would still leave us with a P_3-set because c_2 is there and if we take out c_2, then c_0, c_1 and some other member of the sum is still in the set and we, once again, reach the contradiction. Thus, we've considered all the cases and have reached the contradiction.

Q.E.D.

(d) Prove or Disprove: If S is a P_3-set then core(S) is a P_3-set.

It's false. Counterexample: Consider the set $S = \{2, 3, 5\}$. Then core(S) $= \{3\}$ since if we take out 2, $3 + 3 + 5 = 11$ is still a prime and if we take out 5, $2 + 2 + 3 = 7$ is still a prime. However, if we take out 3, all the possible sums are:
$2 + 2 + 2 = 6$ is not a prime
$2 + 2 + 5 = 9$ is not a prime
$2 + 5 + 5 = 12$ is not a prime
$5 + 5 + 5 = 15$ is not a prime

Thus, we end up with a set core(S) $= \{3\}$ which is not a P_3-set since $3 + 3 + 3 = 9$ is the only sum we can get and it is not a prime.

(e) Prove or Disprove: If S and T are P_3-sets then core($S \cup T$) $\subseteq$ core(S) $\cap$ core(T).

It's true, let's prove it. Let $x \in$ core($S \cup T$). Then $x \in$ core(S) or $x \in$ core(T) (not the **exclusive** or). Now then, without a loss of generality and for the sake of contradiction, assume that $x \in$ core(S) and $x \notin$ core(T). If $x \in$ core(S) and $x \notin$ core(T), it means that $x \notin$ core($S \cup T$) and we've reached the contradiction.

Q.E.D.

(f) Prove or Disprove: If S and T are P_3-sets with $S \subseteq T$ then we have core(T) $\subseteq$ core(S).

Suppose S and T are two P_3-sets and $S \subseteq T$. Then for all $x \in S$, $x \in T$. Let $y \in$ core(T). Then since T has all the elements of S in addition to some other elements which S does not have (if they have the same elements, they are obviously equal and if $T = S$, core(T) $\subseteq$ core(S)), it is harder to make the set a non-P_3 because of simply having more elements to deal with. Thus, if T has simply more elements which do not contribute to any summation $x + y + z = p$ where p is prime then core(S) $=$ core(T) and if there exist such numbers, then one has to take out the additions (e.g. T has extra number m which in not in S and plays a role in some sum, then the core of T will obviously shrink). Thus, if S and T are P_3-sets and $S \subseteq T$, then core(T) $\subseteq$ core(S).

55. A subset S of $\mathbb{Z}$ is called threequaline if for every $x, y \in S$ one has $3 \mid (x-y)$.

(a) Prove or Disprove: Every subset of a threequaline set is threequaline.

It's false. Suppose, for the sake of contradiction, that S is a threequaline set and also suppose that all the subsets of S are threequaline. Then we know that an empty set is a subset of every set and S is a set thus, the emptyset is also a subset of S. However, the empty set has no elements and we cannot find x, y such that $x - y$ is a multiple of 3 and we've reached the contradiction.

Q.E.D.

(b) Prove that if S is threequaline than either every element of S is divisible by 3 or none are.

Suppose, for the sake of contradiction, that S is a threequaline set and there exists two elements x, y such that x is a multiple of 3 and y is not a multiple of 3. Then, without a loss of generality, consider the following two cases:

Case I: $x = 3k$ and $y = 3l + 1$ where $k, l \in \mathbb{Z}$
Case II: $x = 3k$ and $y = 3l + 2$ where $k, l \in \mathbb{Z}$.

Case I Proof: if $x = 3k$ and $y = 3l+1$, $x+y = 3k+3l+1 = 3(k+l)+1$ which is not a multiple of 3.

Case II Proof: if $x = 3k$ and $y = 3l+2$, $x+y = 3k+3l+2 = 3(k+l)+2$ which is not a multiple of 3.

Q.E.D.

(c) Prove that if S is threequaline and r and t are integers, then the set $\{rx + t \mid x \in S\}$ is also threequaline.

Since we know that S is a threequaline set, for every $x, y \in S$, $x - y$ is a multiple of 3. Suppose, z_1, z_2 are some elements of the set S. Then, we know that $z_1 - z_2$ is a multiple of 3. The new set will "transform" these elements into $z_1^{'} = rz_1 + t$ and $z_2^{'} = rz_2 + t$. On the other hand, $z_1^{'} - z_2^{'} = rz_1 + t - (rz_2 + t) = r(z_1 - z_2)$ which is a multiple of 3 since z_1, z_2 are the members of S and $z_1 - z_2$ is a multiple of 3.

Q.E.D.

(d) Prove that if S and T are threequaline and $S \ \cap \ T \neq \emptyset$ then $S \ \cup \ T$ is threequaline.

Suppose, for the sake of contradiction, that S_1 and S_2 are threequaline sets and $S_1 \ \cap \ S_2 \neq \emptyset$ and let's prove that $S_1 \ \cup \ S_2$ is not a threequaline. Since $S_1 \cap S_2 \neq 0$, there exists element x, such that $x \in S_1$ and $x \in S_2$. Let's consider the following cases:

Case I : x is divisible by 3, thus $x = 3k$ where $k \in \mathbb{Z}$
Case II: x gives remainder of 1 when divided by 3, thus $x = 3k + 1$ where $k \in \mathbb{Z}$
Case III: x gives remainder of 2 when divided by 3, thus $x = 3k + 2$ where $k \in \mathbb{Z}$

In Case I, if $x = 3k$, then all the other elements of S_1 as well as S_2 must be of the type $3l$ where $l \in \mathbb{Z}$ and all the elements in the union of S_1 and S_2 will be the multiples of 3 which means that for all $i, j \in S_1 \cup S_2$, $i - j$ is a multiple of 3. And we reached the contradiction.

In Case II, if $x = 3k + 1$, then all the other elements of S_1 as well as S_2 must be of the type $3l + 1$ where $l \in \mathbb{Z}$ and all the elements in the union of S_1 and S_2 will be the multiples of 3 plus 1 which means that for all $i, j \in S_1 \cup S_2$, $i - j$ is a multiple of 3. And we reached the contradiction.

In Case II, if $x = 3k + 2$, then all the other elements of S_1 as well as S_2 must be of the type $3l + 2$ where $l \in \mathbb{Z}$ and all the elements in the union of S_1 and S_2 will be the multiples of 3 plus 2 which means that for all $i, j \in S_1 \cup S_2$, $i - j$ is a multiple of 3. And we reached the contradiction.

56. A subset S of $\mathbb{R}$ is called **crunched** if there exist integers $m, n \in \mathbb{Z}$ such that for all $x \in S$ we have $m < x < n$.

NOTE: When I mention lower bound or upper bound, I really mean the smallest element or the biggest element of the set.

(a) Give some examples of sets that are and are not crunched.

$S_1 = \{1\}$ is crunched as for all $x \in S$, $0 < x < 2$ $(m = 0, n = 2)$.
$S_2 = \{1, 2, 3\}$ is crunched as for all $x \in S$, $0 < x < 4$ $(m = 0, n = 4)$.
$S_3 = \mathbb{Z}^+$ is not crunched as it has no bounds and we cannot find m, n such that for all $x \in \mathbb{Z}^+, m < x < n$.
$S_4 = \{2, 4, 6...\}$ (a set of positive even numbers) is not crunched as it has no upper bound and we cannot find n such that for all $x \in \mathbb{Z}^+, m < x < n$ (note: we can find m. m can be any integer that is less than or equal to 1 but n cannot be fixed).

(b) Prove or Disprove: All crunched sets are finite.

That's false. Counterexample: let $S = \{1, 1/2, 1/4, ...\}$ (infinite geometric series), then we know that S has an upper bound 1 and the lower bound which is 0. Then, we can say with the great certainty, that for all $x \in S$, $-10 < x < 10$ ($m = -10$, $n = 10$). Thus, crunched sets are not necessarily finite and the initial claim is false.

(c) Prove or Disprove: All finite sets are crunched.

It's true. Let S be a finite set. Then it must have a lower bound (the smallest element), let it be k_1 and the upper bound (the biggest element), let it be k_2. Then let $m = k_1 - 1$ and let $n = k_2 + 1$ and we have that for all $x \in S$, $m < x < n$.

Q.E.D.

(d) Prove or Disprove: Every subset of a crunched set is crunched

It's true. Suppose S is a crunched set. Then we know that for all $x \in S$, there exist m, n such that $m < x < n$. Let S_0 be a subset of S. Then, since all the elements of S_0 are also in S_1, we know that all elements of S_0 are between m and n which makes S_0 crunched. Thus, every subset of a crunched set is crunched.

Q.E.D.

(e) Prove or Disprove: The union of two crunched sets is crunched.

Suppose S_1 and S_2 are two crunched sets. Then we know that for all $x_1 \in S_1$, $m_1 < x_1 < n_1$ and for all $x_2 \in S_2$, $m_2 < x_1 < n_2$. Then, for all z in $S_1 \cup S_2, max(m_1, m_2) < z < max(n_1, n_2)$ which means that $S_1 \cup S_2$ is crunched.

Q.E.D.

57. We call a finite subset S of $\mathbb{Z}$ **balanced** if $|\mathbb{Z}^+ \cap S| = |\mathbb{Z}^- \cap S|$. (Recall that $\mathbb{Z}^- = \{-1, -2, -3, ...\}$).

NOTE: This statement really means that if A is balanced, the number of positive elements in it equals the number of negative elements in it.

(a) Prove or Disprove: If A is a balanced set then so is $A \cup \{0\}$.

It's true. Let's prove it.
Suppose A is a balanced set, then we know that $|\mathbb{Z}^+ \cap A| = |\mathbb{Z}^- \cap A|$. Now, since $0 \notin \mathbb{Z}^+$ and $0 \notin \mathbb{Z}^-$, it means that $\mathbb{Z}^+ \cap A = \mathbb{Z}^+ \cap (A \cup \{0\})$ and $\mathbb{Z}^- \cap A = \mathbb{Z}^- \cap (A \cup \{0\})$ and finally, $|\mathbb{Z}^+ \cap A \cup \{0\}| = |\mathbb{Z}^- \cap (A \cup \{0\})|$
Thus, if A is a balanced set then so is $A \cup \{0\}$.

Q.E.D.

(b) Prove or Disprove: The union of two balanced sets is balanced.

It's false. Counterexample: let $S_1 = \{-1, 1\}$ and $S_2 = \{-1, 5\}$. Then S_1 is balanced since $\mathbb{Z}^+ \cap S_1 = \{1\}$ and $\mathbb{Z}^- \cap S_1 = \{-1\}$ which means that $|\mathbb{Z}^+ \cap S_1| = |\mathbb{Z}^- \cap S_1| = 1$. S_2 is balanced too since $\mathbb{Z}^+ \cap S_2 = \{5\}$ and $\mathbb{Z}^- \cap S_2 = \{-1\}$ which means that $|\mathbb{Z}^+ \cap S_2| = |\mathbb{Z}^- \cap S_2| = 1$. On the other hand, set $S_1 \cup S_2$ is not balanced since $S_1 \cup S_2 = \{-1, 1, 5\}$ and $|\mathbb{Z}^+ \cap (S_1 \cup S_2)| = 2$ while $|\mathbb{Z}^- \cap (S_1 \cup S_2)| = 1$.

(c) Prove or Disprove: The intersection of two balanced sets is balanced.

It's false. Counterexample: let $S_1 = \{-1, 1\}$ and $S_2 = \{-1, 5\}$. Then S_1 is balanced since $\mathbb{Z}^+ \cap S_1 = \{1\}$ and $\mathbb{Z}^- \cap S_1 = \{-1\}$ which means that $|\mathbb{Z}^+ \cap S_1| = |\mathbb{Z}^- \cap S_1| = 1$. S_2 is balanced too since $\mathbb{Z}^+ \cap S_2 = \{5\}$ and $\mathbb{Z}^- \cap S_2 = \{-1\}$ which means that $|\mathbb{Z}^+ \cap S_2| = |\mathbb{Z}^- \cap S_2| = 1$. On the other hand, $S_1 \cap S_2$ is not balanced since $S_1 \cap S_2 = \{-1\}$ and $|\mathbb{Z}^+ \cap (S_1 \cap S_2)| = \emptyset$ while $|\mathbb{Z}^- \cap (S_1 \cap S_2)| = 1$.

(d) Prove or Disprove: For every $n \in \mathbb{Z}^+$ there exists a balanced set S with exactly n elements.

It's true. Let's prove it by construction. Let S be a set and for all $n \in \mathbb{Z}^+$, dump in some elements to make it balanced. If n is even, then we take $n/2$ elements that are positive and $n/2$ elements that are negative which will give us a balanced set since $|\mathbb{Z}^+ \cap S| = |\mathbb{Z}^- \cap S|$. If n is odd, then we can throw in 0 and then take $(n-1)/2$ positive elements and $(n-1)/2$ negative elements. This will guarantee that $|\mathbb{Z}^+ \cap S| = |\mathbb{Z}^- \cap S|$ since there will be exactly same number of positive and negative elements while 0 is neither in $\mathbb{Z}^+$, nor in $\mathbb{Z}^-$.

Q.E.D.

(e) If A is a subset of $\mathbb{Z}$ we denote by $\overline{A}$ the set $\{-a \mid a \in A\}$. Prove or Disprove: For every finite subset of A of $\mathbb{Z}$, the set $A \cup \overline{A}$ is balanced.

It's true. Let's prove it by cases.

Case I: suppose that A is a subset of $\mathbb{Z}$ such that it does not contain 0.
Case II: suppose that A is a subset of $\mathbb{Z}$ such that it does contain 0.

Proof of Case I: If A is a subset of $\mathbb{Z}$ such that it does not contain 0, we know that for all $x \in A$, we have $-x \in \overline{A}$. This means that the number of negative elements in $A \cup \overline{A}$ will be equal to the number of positive elements in $A \cup \overline{A}$ and because of this, $|\mathbb{Z}^+ \cap (A \cup \overline{A})| = |\mathbb{Z}^- \cap (A \cup \overline{A})|$.

Proof of Case II: If A is a subset of $\mathbb{Z}$ such that it does contain 0, for all $x \in A$, we have $-x \in \overline{A}$. This means that if we took out 0 out of $A \cup \overline{A}$, the number of positive and negative elements in $A \cup \overline{A}$ would be equal. On the other hand, 0 plays no role in determining whether

$|\mathbb{Z}^+ \cap (A \cup \overline{A})| = |\mathbb{Z}^- \cap (A \cup \overline{A})|$ or not because $0 \notin \mathbb{Z}^+$ and $0 \notin \mathbb{Z}^-$. Thus, $|\mathbb{Z}^+ \cap (A \cup \overline{A})| = |\mathbb{Z}^- \cap (A \cup \overline{A})|$.

Q.E.D.

(f) Prove or Disprove: If A is balanced and $|A|$ is odd, then $0 \in A$.

It's true so let's prove it. Suppose, for the sake of contradiction, that A is balanced and $|A|$ is odd, but $0 \notin A$. Now, since $|A|$ is odd, it means that there is no way to have the number of positive elements be equal to the number of negative elements and we've reached a contradiction.

Q.E.D.

58. We call a subset S of $\mathbb{R}$ **positively scattered** if for every $x \in S$ there exists $y \in \mathbb{R}$ such that $y > x$ and $S \cap (x, y] = \emptyset$.

NOTE: The statement above really means that for every $x \in S$, there exists some $y > 0 \in \mathbb{R}$ such that $S \cap (x, x + y] = \emptyset$.

(a) Is $\mathbb{Z}^+$ positively scattered?

It is. Let's prove it by construction. If we take some element $x \in \mathbb{Z}^+$, then we know that it is positive. If $y > x$, we know that y is also positive and let $y = x + 0.1$. Then, we know that interval $(x, x + 0.1]$ contains no positive integers and thus, $\mathbb{Z}^+ \cap (x, x+01] = \emptyset$. Hence, for all $e \in \mathbb{Z}$, we can construct $(e, e + 0.1]$ (there are no positive integers in this interval) and finally, $\mathbb{Z}^+$ is indeed positively scattered.

Q.E.D.

(b) Is $[2, 3]$ positively scattered?

It is not. Suppose, for the sake of contradiction, that for all $x \in [2, 3]$, there exists y such that $y > x$ and $[2, 3] \cap (x, y] = \emptyset$. Now, since the condition holds for all $x \in [2, 3]$ and $[2, 3]$ is a closed interval, it should hold for 2 as well. Then, let $x = 2$. If the condition holds for $x = 2$, it means that we can find $y > x$ such that $[2, 3] \cap (2, y] = \emptyset$. Since $y > x$, $y = 2 + t$ where $t > 0$. Thus, we have $[2, 3] \cap (2, 2 + t] = \emptyset$, but this is impossible and to see why, let's consider two cases:

Case I: $t \geq 1$
Case I: $0 < t < 1$

If $t \geq 1$, it means that $[2, 3] \cap (2, 2 + t] = (2, 3]$ and we have reach the contradiction.

If $0 < t < 1$, it means that $[2,3] \cap (2, 2+t] = (2, 2+t]$ and interval $(2, 2+t)$ is obviously infinite. Thus, we have, once again, reached the contradiction.

Thus, in all cases we've reached the contradiction and $[2,3]$ is not positively scattered.

Q.E.D.

(c) Is $\{\frac{1}{n} \mid n \in \mathbb{Z}^+\}$ positively scattered?

It is. Let's prove it by construction. Let $x \in \{\frac{1}{n} \mid n \in \mathbb{Z}^+\}$, then we know that $x = \frac{1}{k}$ where $k \in \mathbb{Z}^+$. Now, let's take $y = \frac{1}{k+0.5}$. Then, we know that $\{\frac{1}{n} \mid n \in \mathbb{Z}^+\} \cap (\frac{1}{k}, \frac{1}{k+0.5}] = \emptyset$ because n is always an integer and there are no integer denominators in the interval $(k, k+0.5]$.

Q.E.D.

(d) Is $\{\frac{1}{n} \mid n \in \mathbb{Z}^+\} \cup \{0\}$ positively scattered?

It is not. Suppose, for the sake of contradiction, that $\{\frac{1}{n} \mid n \in \mathbb{Z}^+\} \cup \{0\}$ is scattered. Then for all $x \in \{\frac{1}{n} \mid n \in \mathbb{Z}^+\} \cup \{0\}$, there exist $y > x$, such that $(\{\frac{1}{n} \mid n \in \mathbb{Z}^+\} \cup \{0\}) \cap (x, y] = \emptyset$. If the condition holds for all $x \in \{\frac{1}{n} \mid n \in \mathbb{Z}^+\} \cup \{0\}$, it should hold for $x = 0$ too ($x = 0$ is the member of the set). If $x = 0$, there must exist $y > 0$ such that $(\{\frac{1}{n} \mid n \in \mathbb{Z}^+\} \cup \{0\}) \cap (0, y] = \emptyset$. This, however, is impossible since the interval $(0, y]$ is infinite and we can always find n for which $\frac{1}{n} \in (0, y]$. Thus, we've reached the contradiction.

Q.E.D.

(e) Prove or Disprove: A subset of a positively scattered set is positively scattered.

It's true.
Suppose, for the sake of contradiction, that the set S_0 is a subset of the positively scattered set S and that S_0 is not positively scattered. Then we know that for some $x \in S_0$, there is no $y \in \mathbb{R}$ such that $y > x$ and $S \cap (x, y] = \emptyset$. S_0, however, is a subset of S which means that if $x \in S_0$, then $x \in S$. Finally, we have that for some $x \in S$ there is no y such that $S \cap (x, x+y] = \emptyset$ which means that S is not positively scattered. Thus, we've reached the contradiction since we assumed that S was positively scattered.

Q.E.D.

(f) Prove or Disprove: The union of two positively scattered sets is positively scattered.

Suppose, for the sake of contradiction, that $S = S_1 \cup S_2$ and let $x \in S$. Then we know that $x \in S_1$ or $x \in S_2$. Without the loss of generality suppose that $x \in S_1$, we can find $y > 0$ such that $(x, x+y]$ not in S_1. Then, the only way we cannot find $k > 0$ such that $(x, x+k) \cap S \neq \emptyset$ is if S_2 contains the part of the interval thus, infinitely many numbers in the range $(x, x+k]$. But then S_2 would not be positively scattered, because since it contains all the elements in the range $(x, x+k]$, it means that for all $l \in \mathbb{R}$, $(x, l] \cap S_2 \neq \emptyset$. Thus, we have reached the contradiction and the union of two scattered sets is indeed scattered.

(g) Prove or Disprove: If S is a positively scattered then $\mathbb{R} - S$ is not positively scattered.

It's true. Suppose, for the sake of contradiction, that S is a positively scattered set. Then, consider $\mathbb{R} - S$. If $\mathbb{R} - S$ is scattered, it must be the case that all the sets in $\mathbb{R}$ are scattered which is not true since we have already shown that set $\{\frac{1}{n} \mid n \in \mathbb{Z}^+\} \cup \{0\}$ is not scattered.

(h) Prove or Disprove: If S is positively scattered then the set $\overline{S} := \{-x \mid x \in S\}$ is positively scattered.

It's true. Suppose, S is a positively scattered set. Let $x \in S$. We know that for some y, $(x, x+y] \cap S = \emptyset$. Then we also know that in $\overline{S}$, x will be mapped to $-x$. Note that $\overline{S}$ will be absolutely "symmetric" which means that if y worked for S, it will also work for $\overline{S}$. In "symmetric" I really mean the difference between two elements next to each other $(x_1 - x_2 = -x_2 - (-x_1))$. Then we can take the same y, and for $-x \in \overline{S}$, $(-x, -x+y] \cap \overline{S} = \emptyset$.

Q.E.D.