

Homework №6

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36. Prove that if S and T are shifty sets (in the sense of a previous exercise) then $S \cup T$ is also a shifty set.

Let's first remember what it means to be shifty. A subset S of \mathbb{Z} is called shifty if for every $x \in S$, $x - 1 \in S$ or $x + 1 \in S$.

Let's consider the following two cases:

1. Let $x \in S$ and prove that $x - 1$ or $x + 1$ is in $S \cup T$
2. Let $y \in T$ and prove that $y - 1$ or $y + 1$ is in $S \cup T$

Let's first consider the case when $x \in S$ and prove that $x - 1$ or $x + 1$ is in $S \cup T$. If $x \in S$, since S is shifty, it means that either $x - 1 \in S$ or $x + 1 \in S$. Union $S \cup T$ will have all the members of S thus, it means that either $x - 1 \in S \cup T$ or $x + 1 \in S \cup T$.

Now let's show that if $y \in T$, $y - 1$ or $y + 1$ is in $S \cup T$. If $y \in T$, since T is shifty, it means that either $y - 1 \in T$ or $y + 1 \in T$. Union $S \cup T$ will have all the members of T thus, it means that either $y - 1 \in S \cup T$ or $y + 1 \in S \cup T$.

Thus, we considered all the cases and proved that if S and T are shifty sets (in the sense of a previous exercise) then $S \cup T$ is also a shifty set.

Q.E.D.

37. Prove that $n \in \mathbb{Z}$ then $1 + (-1)^n(2n - 1)$ is a multiple of 4.

Since $n \in \mathbb{Z}$, it is either even or odd. Let's consider two cases:

1. n is even
2. n is odd

If n is even, then $n = 2k$ where $k \in \mathbb{Z}$. We get:

$$1 + (-1)^n(2n - 1) = 1 + (-1)^{2k}(2 \times 2k - 1) = 1 + 4k - 1 = 4k, \text{ where } k \in \mathbb{Z}$$

Hence, we got that if n is even, $1 + (-1)^n(2n - 1) = 4k$ where $k \in \mathbb{Z}$. Thus, if n is even, $1 + (-1)^n(2n - 1)$ is a multiple of 4.

If n is odd, then $n = 2k + 1$ where $k \in \mathbb{Z}$. We have:

$$1 + (-1)^n(2n - 1) = 1 + (-1)^{2k+1}(2 \times (2k + 1) - 1) = 1 - (4k + 2 - 1) = -4k \text{ where } k \in \mathbb{Z}$$

Thus, we got that if n is odd, $1 + (-1)^n(2n - 1) = -4k$ where $k \in \mathbb{Z}$. Hence, if n is odd, $1 + (-1)^n(2n - 1) = -4k$ is a multiple of 4.

Finally, since integers could either be odd or even, we considered all the cases (n is even and n is odd) and in both of the cases, $1 + (-1)^n(2n - 1)$ is a multiple of 4. Thus, if $n \in \mathbb{Z}$, $1 + (-1)^n(2n - 1)$ is a multiple of 4.

Q.E.D.

38 Prove that if $n \in \mathbb{Z}^+$ is odd then $n^2 - 1$ is divisible by 8.

Suppose n is odd. Then $n = 2k + 1$ where $k \in \mathbb{Z}^+ \cup \{0\}$. Then we have:

$$n^2 - 1 = (2k + 1)^2 - 1 = 4k^2 + 4k + 1 - 1 = 4k^2 + 4k = 4k \times (k + 1)$$

Thus, if n is odd, $n^2 - 1 = 4k \times (k + 1)$ where $k \in \mathbb{Z}^+ \cup \{0\}$. Since k is in the set of positive integers or equals 0, we can consider the following three cases:

1. k is 0
2. k is even
3. k is odd

If $k = 0$, $n^2 - 1 = 4k \times (k + 1) = 4 \times 0 \times (0 + 1) = 0$ which is divisible by 8.

If k is even, $k = 2l$ where $l \in \mathbb{Z}^+$ (not including zero since we already considered that case above). Then $n^2 - 1 = 4k \times (k + 1) = 8 \times (l \times (2l + 1))$ which is divisible by 8.

If k is odd, $k = 2l + 1$ where $l \in \mathbb{Z}^+$. Then $n^2 - 1 = 4k \times (k + 1) = 4 \times (2l + 1) \times (2l + 2) = 8 \times ((2l + 1) \times (l + 1))$ which is divisible by 8.

Thus, we considered all the cases and we proved that if $n \in \mathbb{Z}^+$ is odd then $n^2 - 1$ is divisible by 8.

Q.E.D.

39. Prove that every integer can be written as the sum of exactly 3 distinct integers. (For example, $5 = 4 + 2 + (-1)$).

Suppose k is an integer. Then, we should find three distinct integers such that they sum to k . Now, let's consider the following integers:

$x = k + 1$, $y = -k - 1$, and $z = k$. It is clear that $x \neq z$ and $y \neq z$.

If $x = z$, we have $k + 1 = k \iff 1 = 0$ which is nonsensical.

If $y = z$, we have $-k - 1 = k \iff k = \frac{-1}{2}$ which is also not possible because k is an integer.

Thus $x \neq z$ and $y \neq z$, but we do not know if $x \neq y$. Hence, we have to

consider cases.

Now, consider two cases:

1. $x \neq y$
2. $x = y$

If $x \neq y$, we already know that $x \neq z$ and $y \neq z$ and thus we found three distinct integers which sum up to k , namely $x = k + 1$, $y = -k - 1$, and $z = k$.

If $x = y$, we get $k + 1 = -k - 1$ and $k = -1$. However, even if $k = -1$, we can find three distinct integers, namely $x = -2$, $y = 2$, and $z = -1$.

Thus, we considered all the possible cases and we always find three integers which sum up to some random integer k . Hence, every integer can be written as the sum of exactly 3 distinct integers.

Q.E.D.

40. Prove that if x, y , and z are integers then at least one of $x + y$, $x + z$, and $y + z$ is even.

Suppose, for the sake of contradiction, that $x + y$, $x + z$, and $y + z$ are all odd.

Now, without a loss of generality, consider the following four cases:

1. x, y, z are all even
2. x, y, z are all odd
3. x, y are even and z is odd
4. x, y are odd and z is even

If x, y, z are all even, $x = 2j$, $y = 2k$, $z = 2l$ where $j, k, l \in \mathbb{Z}$. Thus, we get: $x + y = 2j + 2k = 2 \times (j + k)$ which is even.

If x, y, z are all odd, $x = 2j + 1$, $y = 2k + 1$, $z = 2l + 1$ where $j, k, l \in \mathbb{Z}$. Thus, we get: $x + y = 2j + 1 + 2k + 1 = 2j + 2k + 2 = 2 \times (j + k + 1)$ which is even.

If x, y are even and z is odd, $x = 2j$, $y = 2k$, $z = 2l + 1$ where $j, k, l \in \mathbb{Z}$. Thus, we get: $x + y = 2j + 2k = 2 \times (j + k)$ which is even.

If x, y are odd and z is even, $x = 2j + 1$, $y = 2k + 1$, $z = 2l$ where $j, k, l \in \mathbb{Z}$. Thus, we get: $x + y = 2j + 1 + 2k + 1 = 2j + 2k + 2 = 2 \times (j + k) + 1$ which is even.

Thus, we've considered all the possible cases and proved that if x, y , and z are integers then at least one of $x + y$, $x + z$, and $y + z$ is even.

Q.E.D.

41. Prove or Disprove: There exist prime number p and q such that $p - q = 97$.

There are no prime numbers p and q such that $p - q = 97$. Let's prove it! First, notice that since p and q are prime, $p, q > 0$. Also, from $p - q = 97$, we get $p = 97 + q$ and since $p, q > 0$, $p > q$. Thus, we have $0 < q < p$. Now, since $p - q$ is odd, it is the case that one of p, q is odd and the other one is even (because difference of even as well as odd numbers is even). However, we know that there is only one even prime number, namely 2. Let's consider following two cases:

1. $p = 2$
2. $q = 2$

Since $0 < q < p$, we know that p cannot be 2 (if $p = 2$, there is no prime number below 2, so q cannot be prime).

If $q = 2$, $p - 2 = 97$ and $p = 99 = 3 \times 3 \times 11$ which is not prime.

Thus, we considered all the cases and there are no prime numbers, p and q , of which the difference is 97.

Q.E.D.

42. For $n \in \mathbb{Z}^+$, we define the n^{th} triangular number to be $T_n := 1 + 2 + \dots + n$. Thus we have $T_1 = 1, T_2 = 3, T_3 = 6, T_4 = 10, T_5 = 15$, and so on. We will prove later that $T_n = \frac{n(n+1)}{2}$, and you should use that formula for this problem. Prove that for $n \in \mathbb{Z}^+$, T_n is odd if and only if n is 1 or 2 more than a multiple of 4.

Since it is the "if and only if" proof, let's split the proof in two cases:

1. Prove that if n is 1 or 2 more than a multiple of 4, T_n is odd.
2. Prove that if n is not 1 and is not 2 more than a multiple of 4, T_n is not odd (thus is even).

Let's first prove that if n is 1 or 2 more than a multiple of 4, T_n is odd.

If n is 1 more than a multiple of 4, $n = 4k + 1$ where $k \in \mathbb{Z}^+$. Then we have:

$$T_n = \frac{n(n+1)}{2} = \frac{(4k+1)((4k+1)+1)}{2} = \frac{(4k+1)(4k+2)}{2} = (4k+1) \times (2k+1).$$

Thus, we have that if n is 1 more than a multiple of 4, $T_n = (4k+1) \times (2k+1)$, where $k \in \mathbb{Z}^+$, which is a multiplication of two odd numbers and thus is odd.

If n is 2 more than a multiple of 4, $n = 4k + 2$ where $k \in \mathbb{Z}^+$. Then we have:

$$T_n = \frac{n(n+1)}{2} = \frac{(4k+2)((4k+2)+1)}{2} = \frac{(4k+2)(4k+3)}{2} = (2k+1) \times (4k+3)$$

Hence, we got that $T_n = (2k + 1) \times (4k + 1) = (2k + 1) \times (4k + 3)$ which is the multiplication of two odd numbers and thus, if n is 2 more than a multiple of 4, T_n is odd.

Now, let's prove that if n is not 1 or 2 more than a multiple of 4, T_n is even.

If n is not 1 and is not two more than a multiple of four, we have the following five cases to consider:

1. $n = 4k$ where $k \in \mathbb{Z}^+$
2. $n = 4k + 3$ where $k \in \mathbb{Z}^+$

If $n = 4k$ where $k \in \mathbb{Z}^+$, $T_n = \frac{4k(4k + 1)}{2} = 2 \times (k(4k + 1))$ which is even.

If $n = 4k + 3$ where $k \in \mathbb{Z}^+$, $T_n = \frac{(4k + 3)(4k + 4)}{2} = 2 \times ((4k + 3)(2k + 2))$ which is even.

NOTE: SINCE WE CONSIDERED INTEGERS STARTING AT 4 (SINCE WE ASSUMED THAT $n = 4k, 4k + 1, 4k + 2, 4k + 3$ where $k \in \mathbb{Z}^+$, the smallest number we get is $n = 4 \times 1 = 4$), NOW WE NEED TO CHECK T_n for 1, 2, and 3.

If $n = 1$, $T_n = \frac{1(1 + 1)}{2} = 1$ which is odd and indeed, $1 = 0 + 1$ and 0 is a multiple of 4 (thus, 3 has $4l + 1$ form where $l \in \mathbb{Z}$).

If $n = 2$, $T_n = \frac{2(2 + 1)}{2} = 3$ which is odd and indeed, $2 = 0 + 2$ and 0 is a multiple of 4 (thus, 3 has $4l + 2$ form where $l \in \mathbb{Z}$).

If $n = 3$, $T_n = \frac{3(3 + 1)}{2} = 6$ which is even and indeed, $3 = 0 + 3$ and 0 is a multiple of 4 (thus, 3 has $4l + 3$ form where $l \in \mathbb{Z}$).

Thus, we considered all the cases and proved that for $n \in \mathbb{Z}^+$, T_n is odd if and only if n is 1 or 2 more than a multiple of 4.

Q.E.D.

Bookwork

2. Solve the equality $|x + 1| < |x^2 - 1|$. Interpret results geometrically.

Let's consider the following two cases:

1. $x + 1 \geq 0$
2. $x - 1 \leq 0$

Let's first consider the case $x + 1 \geq 0$.

If $x + 1 \geq 0$, the inequality will have a form:

$$x + 1 < |x^2 - 1|$$

Now, let's consider another two cases:

1. $x^2 - 1 \geq 0$
2. $x^2 - 1 \leq 0$

If $x^2 - 1 \geq 0$, the equality we get is:

$$x + 1 < x^2 - 1$$

And we get:

$$x^2 - x - 2 > 0$$

And finally, $x \in (-\infty, -1) \cup (2, +\infty)$.

But since $x + 1 \geq 0$, $x \geq -1$ and we get $x \in (2, +\infty)$

If $x^2 - 1 \leq 0$, the equality we get is:

$$x + 1 < -x^2 + 1$$

And we get:

$$x^2 + x < 0$$

And finally, $x \in (-1, 0)$.

It also satisfies $x + 1 \geq 0$.

Now, let's consider the second case, $x + 1 \leq 0$.

If $x + 1 \leq 0$, the inequality will have a form:

$$-x - 1 < |x^2 - 1|$$

Now, as previously, let's consider another two cases:

1. $x^2 - 1 \geq 0$
2. $x^2 - 1 \leq 0$

If $x^2 - 1 \geq 0$, the equality we get is:

$$-x - 1 < x^2 - 1$$

And we get:

$$x^2 + x > 0$$

And finally, $x \in (-\infty, -1) \cup (0, +\infty)$.

But since $x + 1 \leq 0$, $x \leq -1$ and we get $x \in (-\infty, -1)$

If $x^2 - 1 \leq 0$, the equality we get is:

$$-x - 1 < -x^2 + 1$$

And we get:

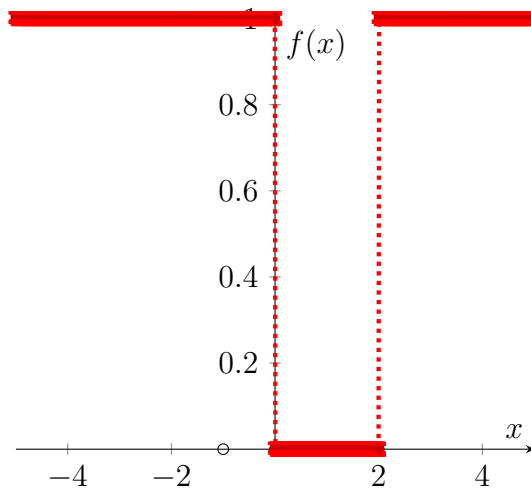
$$x^2 - x - 2 < 0$$

And finally, $x \in (-1, 2)$.

But since $x + 1 \leq 0$, $x \leq -1$ and we do not get any solutions for this inequality.

Thus, we got the following answers: $x \in (2, +\infty)$, $x \in (-1, 0)$, $x \in (-\infty, -1)$.

Finally, by merging these answers, we get: $x \in (-\infty, -1) \cup (-1, 0) \cup (2, +\infty)$.



This line shows that $x \in (-\infty, -1) \cup (-1, 0) \cup (2, +\infty)$. Part where $x \in (0, 2)$ is in between two horizontal dotted lines meaning that it is not the part of the solution. Also, $x = -1$ is also not in the set (you can see a little circle there which basically means that the point is out and $x \neq -1$).

4. (a)

Show that any integer can be written in the form $10q + r$ where $0 \leq r \leq 9$.

I don't really like the way this is phrased, so I will rephrase what we have to prove. We have to prove that:

for every integer k , there exists q and r such that $k = 10q + r$ and $0 \leq r \leq 9$.

Notice that if an integer ends on 0, it is always divisible by 10. Thus, it is just the problem of picking r so that $k - r$ ends on 0 (if $k - r$ ends on zero we divide the result by 10 and get q).

Let's consider the following 10 cases for the possible endings of the number.

- * Number ends on 0
 - * Number ends on 1
 - * Number ends on 2
 - * Number ends on 3
 - * Number ends on 4
 - * Number ends on 5
 - * Number ends on 6
 - * Number ends on 7
 - * Number ends on 8
 - * Number ends on 9
-
- * If number ends on 0, we can pick $r = 0$, and then k is divisible by 10 and we get the value of q .
 - * If number ends on 1, we can pick $r = 1$, and then k is divisible by 10 and we get the value of q .
 - * If number ends on 2, we can pick $r = 2$, and then k is divisible by 10 and we get the value of q .
 - * If number ends on 3, we can pick $r = 3$, and then k is divisible by 10 and we get the value of q .
 - * If number ends on 4, we can pick $r = 4$, and then k is divisible by 10 and we get the value of q .
 - * If number ends on 5, we can pick $r = 5$, and then k is divisible by 10 and we get the value of q .
 - * If number ends on 6, we can pick $r = 6$, and then k is divisible by 10 and we get the value of q .
 - * If number ends on 7, we can pick $r = 7$, and then k is divisible by 10 and we get the value of q .

- * If number ends on 8, we can pick $r = 8$, and then k is divisible by 10 and we get the value of q .
- * If number ends on 9, we can pick $r = 9$, and then k is divisible by 10 and we get the value of q .

Thus, we considered all the possible cases and proved that any integer can be written in the form $10q + r$ where $0 \leq r \leq 9$.

Q.E.D.

(b)

Show that the square of any integer ends in 0, 1, 4, 5, 6, or 9.

There are 9 numbers on which an integer could end.

These numbers are: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

Then let's consider the following 10 cases.

- * Number ends on 0
- * Number ends on 1
- * Number ends on 2
- * Number ends on 3
- * Number ends on 4
- * Number ends on 5
- * Number ends on 6
- * Number ends on 7
- * Number ends on 8
- * Number ends on 9

If number ends on 0, its square ends on the same number as 0^2 ends on, thus on 0.

If number ends on 1, its square ends on the same number as 1^2 ends on, thus on 1.

If number ends on 2, its square ends on the same number as 2^2 ends on, thus on 4.

If number ends on 3, its square ends on the same number as 3^2 ends on, thus on 9.

If number ends on 4, its square ends on the same number as 4^2 ends on, thus on 6.

If number ends on 5, its square ends on the same number as 5^2 ends on, thus on 5.

If number ends on 6, its square ends on the same number as 6^2 ends on, thus on 6.

If number ends on 7, its square ends on the same number as 7^2 ends on, thus on 9.

If number ends on 8, its square ends on the same number as 8^2 ends on, thus on 4.

If number ends on 9, its square ends on the same number as 9^2 ends on, thus on 1.

Thus, the ending numbers we got are: 0, 1, 4, 9, 6, 5, 6, 9, 4, 1.

Now, if we eliminate the duplicates, we get: 0, 1, 4, 5, 6, and 9.

Q.E.D.

8. Can one form a ten-digit integer by putting a digit between 0 and 9 in the empty boxes in the given table as follows: The digit in the box labeled 0 indicates the number of times 0 appears in the number, the digit in box 1 indicated the number of times 1 appears in the number, etc.?

0	1	2	3	4	5	6	7	8	9

(For instance, if 9 is placed in box 0, then the remaining boxes must be filled with 0 in order that the nine zeros actually appear. In this case, however, we reach a contradiction since box 9 cannot contain 0 as at least one 9 appears in the number. Thus the desired ten-digit number cannot have 9 as its leftmost digit.)(Hint: Consider the possible ways of filling the box labeled 0.) How many such numbers exist?

- ** If 9 is placed in box 0, we reach the contradiction since box 9 cannot contain 0 as at least one 0 appears in the number.
- ** If 8 is placed in box 0, we still cannot find such number since the only possibility would be 8000000010 but then 1 is not in a number and if we have 8100000010 we have two 1s. Thus, we reach the contradiction.
- ** If 7 is placed in box 0, we have one 7 (we cannot have two 7s because $2 \times 7 = 14$ and we have 10 boxes) meaning that box under label 7 should be one. Then we must also have 1 under box labeled 1. But now we have a number 7100000100 but we actually have 2 ones, so it should be 7200000100, but now we must have 2 ones but only have 1 and we cannot add more since we also must have 7 zeros. Thus, 7 cannot be placed under the box 0. Thus, we reach the contradiction.
- ** If 6 is placed under the box 0, box 6 should be 1 (we cannot have two 6s because $2 \times 6 = 12 > 10$). Then we should have at least one 1. However, now we have two 1s, thus box labeled 2 must be 1 and box labeled 1 must be two and we are actually left with six 0s. Hooray, we got a number! It's 6210001000. It has ten digits, six 0s, two 1s, one 2, zero 3s, zero 4s, zero 5s, one 6s, zero 7s, zero 8s, zero 9s.
- ** If 5 is placed under the box 0, box 5 should be 1 (we cannot have two 5s, because $2 \times 5 = 10$ and then it means that somewhere else, other than under the box labeled 0, we will have another 5 which will then result in some other box having some number under it which is not possible since we have 10 boxes in total). If we have, one 5, we end up with 5210010000 since we must have at least one 1, but then we will end up with two 1s (since box under label 5 is also 1). But now, we have a problem since we have 6 zeros. Thus, we reach the contradiction.

- ** If 4 is placed under the box 0, with the same reasoning, we end up with 4210100000 but then we have five 0s. Thus, we reach the contradiction.
- ** If 3 is placed under the box 0, with the same reasoning, we end up with 3211000000 but then we have five 0s. Thus, we reach the contradiction.
- ** If 2 is placed under the box 0, we have to write 1 under the box 2, but now we have one 1 as well hence, we have to write 2 under box labeled 2 and with the same reasoning as above, we cannot fill in all of these numbers. Thus, we reach the contradiction.
- ** Case when 1 is placed under the box labeled 0 is also not possible.
- ** If 0 is placed under the label 0, it means that at least one number occurs twice, but then it means that we will end up with 11 digits in the end.

Thus, we can construct a ten-digit number with this approach. The number is 6210001000. And there is only one number like this (6210001000). Hence, the answer is yes and there is only one such number which is 6210001000.