Homework №5

Author: David Oniani
Instructor: Tommy Occhipinti

September 23, 2018

Section 3.2

23. Prove that if x and y are integers and $xy - 1$ is even then x and y are odd.

Let's prove it by contrapositive. Contrapositive of the initial statement (which is equivalent to the initial statement) is:

If x is even or y is even, then $xy - 1$ is odd.

If x is even or y is even, xy is even. Then we can write that $xy = 2k$ where $k \in \mathbb{Z}$. Then, $xy - 1 = 2k - 1 = 2(k - 1) + 1$ where $k \in \mathbb{Z}$. Now, let $t = l - 1$ where $l \in \mathbb{Z}$ and we get $xy - 1 = 2t + 1$. Thus, $xy - 1$ is odd.

Q.E.D.

24. Prove that if x and y are real numbers whose mean is m then at least one of x and y is $\geq m$.

Suppose, for the sake of contradiction, that x and y are both $< m$. Then by adding the inequalities, we get:

$$x + y < 2m$$

And finally,

$$\frac{x + y}{2} < m$$

which contradicts the initial statement that the mean of x and y is m.

Q.E.D.

25. Suppose S is a set of 250 distinct real numbers whose mean is 4. Must there exists $x \in S$ such that $x > 4$? Be sure to prove your answer.

Yes. Let's prove it!
Suppose, for the sake of contradiction, that all elements of S are ≤ 4. Then the sum of all the elements will be less ≤ 1000 with equality happening only when all the members of the set are equal to 4 which contradict the initial statement that S is a set of 250 <u>distinct</u> elements. Thus, only one of the elements of S is allowed to be equal to 4. Finally, we get two cases:

1. All 250 elements of S are less than 4.
2. 249 elements of S are less than 4 and one is equal to 4.

If all 250 elements of S are less than 4, then their sum is less than $4 \times 250 = 1000$ and their mean is less than $1000/4 = 250$ which contradicts the initial statement that the mean of all elements of S is 250.

If 249 elements of S are less than 4 and one is equal to 4, then the sum of 249 elements which are less than 4 is less than $249 \times 4 = 996$. Then let this sum of 249 numbers be equal to $996 - k$ where $k > 0$. Then the sum of all the elements including the one which equals 4 is:

$$996 - k + 4 = 1000 - k \text{ where } k > 0$$

Using the fact above, we get that the mean of all the elements of S is $(1000 - k)/250$ where $k > 0$. And finally, we get:

$$\frac{1000 - k}{250} = 4 - \frac{k}{250} \text{ where } k > 0$$

And $4 - \dfrac{k}{250}$ where $k > 0$ is clearly less than 4 which contradicts the initial claim that the mean of all elements of S is 4.

Q.E.D.

26. Suppose $a, b, c \in \mathbb{Z}$ and $a^2 + b^2 = c^2$. Prove that at least one of a and b is even.

Suppose, for the sake of contradiction, that both a and b are odd. Then, we can write $a = 2k - 1$ and $b = 2l - 1$ where $k, l \in \mathbb{Z}$. Then, we have:

$$a^2 + b^2 = 4k^2 - 4k + 1 + 4l^2 - 4l + 1 = 4k^2 + 4l^2 - 4l - 4k + 2 =$$
$$= 2 \times (2k^2 + 2l^2 - 2l - 2k + 1)$$

Now, it's easy to see that $a^2 + b^2$ is the multiplication of an even and odd integers (2 is even and $2k^2 + 2l^2 - 2l - 2k + 1$ is odd). $2k^2 + 2l^2 - 2l - 2k + 1$ is odd since $2k^2 + 2l^2 - 2l - 2k + 1 = 2 \times (k^2 + l^2 - l - k) + 1$ and if we let $t = k^2 + l^2 - l - k$ where $t \in \mathbb{Z}$ (since $k^2 + l^2 - l - k \in \mathbb{Z}$), then we have that $2k^2 + 2l^2 - 2l - 2k + 1 = 2t + 1$ which is an even number plus one which is always odd. Finally, we conclude that 2 is only once in the number that is supposed to be a perfect square as $2k^2 + 2l^2 - 2l - 2k + 1$ is odd and is not a multiple of 2 which means that $a^2 + b^2$ is not a perfect square which contradicts the initial claim that the sum $a^2 + b^2$ is the perfect square.

Q.E.D.

27. Prove that if $x, y \in \mathbb{R}^+$, then $x + y \geq 2\sqrt{xy}$.

Suppose, for the sake of contradiction, that $x + y < 2\sqrt{xy}$. Then, since $x, y \in \mathbb{R}^+$, we have:

$$x + y < 2\sqrt{xy} \tag{1}$$
$$x^2 + y^2 + 2xy < 4xy \tag{2}$$
$$x^2 + y^2 + 2xy - 4xy < 0 \tag{3}$$
$$x^2 + y^2 - 2xy < 0 \tag{4}$$
$$(x - y)^2 < 0 \tag{5}$$

Thus, we got that $(x-y)^2 < 0$ which is false since the square of a number is always ≥ 0. Finally, since by assuming that $x+y < 2\sqrt{xy}$ where $x, y \in \mathbb{R}^+$, we basically got the nonsensical inequality $(x-y)^2 < 0$, something has to be wrong with this assumption and we got that if $x, y \in \mathbb{R}^+$, then $x + y \geq 2\sqrt{xy}$

Q.E.D.

28. Prove that if n is an integer, there exist three consecutive integers that sum to n if and only if n is a multiple of 3.

Let's first prove that if n is not a multiple of 3, one cannot find three consecutive integers with the property that they sum to n.

(a) Suppose, for the sake of contradiction, that n is not a multiple of 3. Then let's define three consecutive integers, $m, m+1$ and $m+2$, where $m \in \mathbb{Z}$. Then we have:

$$m + m + 1 + m + 2 = 3m + 3 = 3 \times (m + 1)$$

Thus, we got that the sum of three consecutive integers is a multiple of 3 which contradicts the statement that n is not a multiple of 3.

Now, lets prove the second half of the problem. Let's show that if three consecutive integers sum to n, then n is a multiple of 3.

(b) Let $m, m+1, m+2$ where $m \in \mathbb{Z}$ be three consecutive integers. We have:

$$n = m + m + 1 + m + 2 = 3m + 3 = 3 \times (m + 1)$$

Thus, we got that n is a multiple of 3 which proves the iff.

Q.E.D.

29. A subset S of $\mathbb{R}$ has the property that for all $x \in \mathbb{R}$ there exists $y \in S$ such that $|x - y| < 1$. Prove that S is infinite.

Suppose, for the sake of contradiction, that S is finite. Inequality, $|x-y| < 1$ can be transformed into the following system:

$$\begin{cases} x - y < 1 \\ x - y > -1 \end{cases}$$

And from the system above, we get the following system:

$$\begin{cases} y > x - 1 \\ y < x + 1 \end{cases}$$

Hence, we know that y is in the open interval $(x-1, x+1)$. Now, since we also know that $x \in \mathbb{R}$, interval $(x-1, x+1)$ has infinitely many elements in it which contradicts our assumption that S is finite.

Q.E.D.

30. A subset S of $\mathbb{Z}$ is called **non-differential** if for every $x, y \in S$ we have $x - y \notin S$. Here are some statements about non-differential sets. Decide which statements are true and which are false, and provide a proof or counterexample for each as appropriate.

Before going right into the proof, note that in any set we can do self-subtration (e.g, if the set is $\{1, 3\}$), we can write $1 - 1 = 0$. HOWEVER, we are not going to consider those trivial cases. We are going to consider it if and only if element 0 is in the set (because all the self-subtractions are zero and it is only the case when the element 0 is in the set when such subtractions make sense in proving or disproving something).

(a) Every non-differential set is finite.

This is false. Counterexample:
Let $S = \{1, 3, 5, 7, 9, 11...\}$ thus, S is a set of all positive odd integers. Then we know that for every x, y, $x - y$ is even. But all the members of S are odd. Thus, For every $x, y \in S$, $x - y \notin S$ and S is an infinite set which also turns out to be non-differential and the initial statement is false.

(b) The intersection of two non-differential sets is non-differential.

This is true. Let's prove it.
Let $x, y \in A \cap B$. Then, since $x, y \in A$, $x - y \notin A$ as well as $x - y \notin A \cap B$.

Q.E.D.

(c) The union of two non-differential sets is non-differential.

This is false. Counterexample:
Let $A = \{1, 3\}$ then A is non-differential since $1 - 3 \notin A$ and $3 - 1 \notin A$. Now, let $B = \{1, 4\}$, then B is non-differential too as $1 - 4 \notin B$ and $4 - 1 \notin B$. Finally, we get $A \cup B = \{1, 3, 4\}$ which is <u>NOT</u> non-differential because $4 - 3 = 1 \in A \cup B$.

(d) No non-differential set contains the element 0.

It's true.
For a set to be non-differential there should be no x, y such that $x - y \in S$. For the sake of contradiction, suppose that we have a non-differential set A such that $0 \in A$. If A has more than one elements, let the other element (any element which is not 0) be k. Then we get $k - 0 = k \in A$ which contradicts the initial statement that A is non-differential as we found two elements $x = 0$ and $y = k$ such that $x - y \in A$. If A has only one element which is 0, then it is <u>NOT</u> non-differential anyway, because $0 - 0 = 0 \in A$. Hence, no non-differential set contains element 0.

(e) Every subset of a non-differential set is non-differential.

It's true.
Suppose we have two sets, A and B such that $B \subseteq A$ and A is non-differential. Let $x \in B$, then we know that there exists no y in A such that $x - y \in A$. Since such y does not exist in A, it does not exist in B is as well since it is the subset of A.

Q.E.D.

(f) There is no non-differential set with exactly 5 elements.

It's false. Counterexample:
Let $A = \{1, 3, 8, 19, 50\}$, then we have:

$$1 - 3 = -2 \notin A$$
$$3 - 1 = 2 \notin A$$
$$3 - 8 = -5 \notin A$$
$$8 - 3 = 5 \notin A$$
$$8 - 19 = -11 \notin A$$
$$19 - 8 = 11 \notin A$$
$$19 - 50 = -31 \notin A$$
$$50 - 19 = 31 \notin A$$

(g) If S is non-differential, so is $\mathbb{Z} - S$.

It's false. Counterexample:
Let $A = \{1, 3\}$. A is non-differential since $1 - 3 = -2 \notin A$ and $3 - 1 = 2 \notin A$. Then we know that $Z - A$ would include numbers $7, 8, 15$. But $15 - 8 = 7 \in Z - A$ which is not non-differential.

(h) If S is a non-differential set, then so is the $S_{+3} = \{x + 3 \mid x \in S\}$.

It's false. Counterexample:
Let $A = \{1, 3, 8, 19, 50\}$, then A is non-differential since:

$$1 - 3 = -2 \notin A$$
$$3 - 1 = 2 \notin A$$
$$3 - 8 = -5 \notin A$$
$$8 - 3 = 5 \notin A$$
$$8 - 19 = -11 \notin A$$
$$19 - 8 = 11 \notin A$$
$$19 - 50 = -31 \notin A$$
$$50 - 19 = 31 \notin A$$

$A_{+3} = \{1, 4, 11, 22, 54\}$. Now, notice that $22 - 11 = 11 \in A_{+3}$ thus, we found two elements $x = 22$ and $y = 11$ such that $x - y \in A$ and so A is <u>NOT</u> non-differential. Thus, the initial statement is false.

31. A subset A of $\mathbb{R}$ is called **cofinite** if $\mathbb{R} - A$ is finite. Here are some statements about cofinite sets. Decide which statements are true and which are false, and provide a proof or counterexample for each as appropriate

Before jumping in the proofs, let's make a little note. If A is cofinite, it is some subset of $\mathbb{R}$. Then, we can represent it as $A = \mathbb{R} - F$ where F is some finite set.
Why finite?
If F be infinite, then $\mathbb{R} - (\mathbb{R} - F) = F$ is also infinite and this contradicts the fact that A is cofinite. Thus, we know (and will use) the fact that any <u>cofinite</u> set A can be represented as $\mathbb{R} - F$ where F is a finite set.

(a) If $A \subseteq B$ and B is cofinite then A is cofinite.

It's false. Counterexample:
Let $B = \mathbb{R} - \{0, 1\}$. Then B is cofinite as $\mathbb{R} - B = \{0, 1\}$. Now let $A = \{-1, -2\}$, then $A \subseteq B$, however, $\mathbb{R} - A$ is not finite as R has an infinite number of elements and subtracting only a finite number of elements (2 elements) still leaves it will infinitely many.

(b) There exist two cofinite sets A and B with the property that $A \cap B = \emptyset$.

It's false. Suppose, for the sake of contradiction, that A and B are cofinite sets. Then we know that both A and B are of the form $\mathbb{R} - F$ where F is some finite set (if F is infinite, $\mathbb{R} - (\mathbb{R} - F) = F$ and the set is <u>NOT</u> cofinite). Let $A = \mathbb{R} - C$ and $B = \mathbb{R} - D$. Then we know that both A and B contain sets $\mathbb{R} - C - D$. Thus $\mathbb{R} - C - D \subseteq A \cap B$ which is never an $\emptyset$ since sets C and D are finite and $\mathbb{R} - C - D$ is infinite.

(c) If A is cofinite, then A contains a positive integer.

It's true.
We know that $\mathbb{R}$ contains all the positive integers. For A to be cofinite it $\mathbb{R} - A$ should be finite thus, it should have a finite number of elements. If A has no positive integers, it means that $\mathbb{R} - A$ is infinite since it contain at least all the positive integers. Thus, A is not cofinite and we've encountered a contradiction. And finally, the statement if A is cofinite, then A contains a positive integer is true.

(d) The intersection of two cofinite sets is cofinite.

It's true.
Suppose $A = \mathbb{R} - F$ and $B = \mathbb{R} - G$ are cofinite sets (F and G are finite). Then, their intersection will be:

$$A \cap B = \mathbb{R} - F - G = \mathbb{R} - (F \cup G)$$

And we get:

$$\mathbb{R} - (\mathbb{R} - (F \cup G)) = F \cup G$$

Now, since F and G are finite, $F \cup G$ is also finite and we proved that the intersection of the two cofinite sets is cofinite.

Q.E.D.

(e) The union of two cofinite sets is cofinite.

It's true. Suppose $A = mathbbR - F$ and $B = \mathbb{R} - G$ are cofinite sets (F and G are finite). Then, their union will be:

$$A \cup B = (\mathbb{R} - F) \cup (\mathbb{R} - G) = \mathbb{R} - (F \cap G)$$

And then we get:

$$\mathbb{R} - (\mathbb{R} - (F \cap G)) = F \cap G$$

Now, since F and G are finite, so is $F \cap G$ and the union of two cofinite sets is cofinite.

Q.E.D.

(f) If A and B are cofinite then $A - B$ is finite.

It's true.
Suppose $A = \mathbb{R} - F$ and $B = \mathbb{R} - G$ are cofinite sets (F and G are finite). Then, $A - B = (\mathbb{R} - F) - (\mathbb{R} - G) = G - F$. Now, since F and G are finite, $G - F$ is finite (even if $F = G$, empty set is considered finite with the cardinality zero).

Q.E.D.

(e) Every cofinite set is infinite.

It's true. Let A be a cofinite set. Then, we know that it is some subset of $\mathbb{R}$ and we can write it as $A = \mathbb{R} - F$ where F is some set. Then, we have:

$$\mathbb{R} - (\mathbb{R} - F) = F$$

Now, since A is cofinite, then F has to be finite by the definition of the cofinite set. Then we get that $\mathbb{R} - F$ is infinite since F is finite and $\mathbb{R}$ minus any finite set is always infinite.

Q.E.D.

32. We say that a subset S of $\mathbb{Z}$ is angled if for every $x, y, z \in S$ we have $x + y > z$.

(a)

$$S = \{3\} \text{ since } 3 + 3 > 3$$
$$S = \{3, 4\} \text{ since } 3 + 4 > 3,\ 3 + 4 > 4,\ 3 + 3 > 3, \text{ and } 4 + 4 > 4$$

$$S = \{3, 4, 5\}$$
$$\text{since } 4 + 5 > 3,\ 3 + 5 > 4,\ 3 + 4 > 5,\ 3 + 3 > 3,\ 4 + 4 > 4, \text{ and } 5 + 5 > 5$$

(b)

$$S = \{3, 2, 7\} \text{ as } 3 + 2 < 7$$
$$S = \{12, -13, 29, 47\} \text{ as 12 - 13} < 29$$
$$S = \{1, 2, 3, 4, 5\} \text{ as } 1 + 2 < 4$$

(c) Can 0 be an element of an angled set?

No.
If the set contains element 0, then $0 + 0 = 0$ thus, $0 + 0 \not> 0$ and the set is not angled.

(d) Prove or disprove: If S is angled and $x \in S$ then $x > 0$.

It's true so let's prove it.
Suppose, for the sake of contradiction, that $x \leq 0 \in S$ where S is angled. Then, we must have that $x + x > 2x$. But if $x \leq 0$, $x + x$ is always less than or equal to $2x$. Thus, we've encountered a contradiction and if S is angled and $x \in S$ then $x > 0$.

Q.E.D.

(e) Prove or disprove: If S is angled then there exists $c \in \mathbb{Z}$ such that for every $x \in S$ we have $x < c$.

It's true.
Suppose, for the sake of contradiction, that we cannot find $c \in \mathbb{Z}$ such that for every $x \in S$ we have $x < c$. Then, it is clearly the case that S contains the biggest element of $\mathbb{Z}$. BUT, unfortunately, there is no "BIGGEST" element in $\mathbb{Z}$ as if we pick some element x to be the biggest, we can always take $x + 1$ which will be bigger than x. Thus, we've encountered a contradiction and if S is angled then there exists $c \in \mathbb{Z}$ such that for every $x \in S$ we have $x < c$.

Q.E.D.

(f) Prove or disprove: There exists $c \in \mathbb{Z}^+$ such that if S is angled and $x \in S$ then we have $x < c$.

CONTINUE HERE!! CONTINUE HERE!!

Q.E.D.