Getting Started with $\LaTeX$

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15. Prove the following statement is false by providing a counterexample: If $n \in \mathbb{Z}^+$ is odd and $n > 1$ then there exists a non-negative integer i and a prime p such that $n = 2^i + p$.

    This is the type of statement where "for every" part is hidden. In other words, the statement could be translated in the following way: "For every $n > 1 \in \mathbb{Z}^+$ there exists a non-negative integer i and a prime p such that $n = 2^i + p$".

    Thus, all we have to do is find an integer $n > 1$, $i \in \mathbb{Z}^+$, and $p \in \mathbb{P}$ where $\mathbb{P}$ is a set of prime numbers. Now, notice that $n = 2$ is a counterexample. If $n = 2$, then i has to be either 0 or 1 (since i is non-negative and if $i > 1$, $2^i > 2$ and the equality will not hold). Hence, for $n = 2$, we have $i = 0$ or $i = 1$. Let's only consider the first example. If $n = 2$ and $i = 0$, $2^i = 1$ and p has to be 1 which is not prime. Thus, we found n for which the statement is false which proves that the initial statement is indeed false.

    Q.E.D.

16. Prove the following statement is false by providing a counterexample: If S and T are shifty sets (in the sense of a previous exercise), then $S \cap T$ is also a shifty set.

    Definition of the shifty set: A subset S of $\mathbb{Z}$ is called *shifty* if for every $x \in S,\ x - 1 \in S$ or $x + 1 \in S$.

    Suppose S and T and shifty. Let $S = \{1, 2, 4, 5\}$ and $T = \{2, 3, 6, 7\}$. Then $S \cap T = \{2\}$ and $1, 3 \notin S \cap T$ so we found an example for which S and T are shifty, but $S \cap T$ is not.

    Q.E.D.

17. Prove that if x is odd, then x^3 is odd.

    Suppose x is odd. Then, by the definition of an odd number, we have:

    $$x = 2k + 1, \text{ where } k \in \mathbb{Z}$$

Now, we can plug $2k+1$ into x^3. We get:

$$x = (2k+1)^3 = 8k^3 + 12k^2 + 6k + 1 = 2 \times (4k^3 + 6k^2 + 3k) + 1$$

Let's introduce a new variable l and set it equal to $(4k^3 + 6k^2 + 3k)$. Then we can rewrite x as $x = 2l + 1$. Finally, we conclude that since $(4k^3 + 6k^2 + 3k) \in \mathbb{Z}$, $l \in \mathbb{Z}$ and $2l + 1$ is odd which means that x is also odd.

18. Suppose that m and n are doubly even (in the sense of an earlier exercise):

    Definition of the *doubly even* integer: An integer n is called doubly even if there exist even integers x and y such that $n = xy$.

    a. Prove that mn is doubly even.

        According to the definition, an integer k is doubly even if there exist even integers m and n such that $k = mn$. Then we can write that

Q.E.D.