
Topology

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Assignment №1

Section 2

2. Let $f : A \rightarrow B$ and let $A_i \subset A$ and $B_i \subset B$ for $i = 0$ and $i = 1$. Show that if f^{-1} preserves inclusions, unions, intersections, and differences of sets:

(c) $f^{-1}(B_0 \cap B_1) = f^{-1}(B_0) \cap f^{-1}(B_1)$.

To prove that the set $f^{-1}(B_0 \cap B_1)$ is equal to the set $f^{-1}(B_0) \cap f^{-1}(B_1)$, we have to show that $f^{-1}(B_0 \cap B_1) \subset f^{-1}(B_0) \cap f^{-1}(B_1)$ and $f^{-1}(B_0) \cap f^{-1}(B_1) \subset f^{-1}(B_0 \cap B_1)$.

Case I: $f^{-1}(B_0 \cap B_1) \subset f^{-1}(B_0) \cap f^{-1}(B_1)$

Let $x \in f^{-1}(B_0 \cap B_1)$. Then $f(x) \in B_0 \cap B_1$. Thus, $f(x) \in B_0$ and $f(x) \in B_1$. From this, we get that $x \in f^{-1}(B_0)$ and $x \in f^{-1}(B_1)$. Therefore, $x \in f^{-1}(B_0) \cap f^{-1}(B_1)$. Hence, if $x \in f^{-1}(B_0 \cap B_1)$, then $x \in f^{-1}(B_0) \cap f^{-1}(B_1)$ which means that $f^{-1}(B_0 \cap B_1) \subset f^{-1}(B_0) \cap f^{-1}(B_1)$. \square

Case II: $f^{-1}(B_0) \cap f^{-1}(B_1) \subset f^{-1}(B_0 \cap B_1)$

Let $x \in f^{-1}(B_0) \cap f^{-1}(B_1)$. Then $x \in f^{-1}(B_0)$ and $x \in f^{-1}(B_1)$. Thus, $f(x) \in B_0$ and $f(x) \in B_1$. Finally, we have that $f(x) \in B_0 \cap B_1$ which is equivalent to saying $x \in f^{-1}(B_0 \cap B_1)$. Hence, if $x \in f^{-1}(B_0) \cap f^{-1}(B_1)$, then $x \in f^{-1}(B_0 \cap B_1)$ which means that $f^{-1}(B_0) \cap f^{-1}(B_1) \subset f^{-1}(B_0 \cap B_1)$. \square

We have now proven that $f^{-1}(B_0 \cap B_1) \subset f^{-1}(B_0) \cap f^{-1}(B_1)$ and $f^{-1}(B_0) \cap f^{-1}(B_1) \subset f^{-1}(B_0 \cap B_1)$ and thus, $f^{-1}(B_0 \cap B_1) = f^{-1}(B_0) \cap f^{-1}(B_1)$. \square

- (g) $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$; show that inequality holds if f is injective.

Let's first show that $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$ even if f is not injective.

Let $x \in f(A_0 \cap A_1)$. Then $\exists x' \in A_0 \cap A_1$ such that $f(x') = x$. Now, since $x' \in A_0$ and $x' \in A_1$, we get that $x \in f(A_0)$ and $x \in f(A_1)$ thus, $x \in f(A_0) \cap f(A_1)$. \square

Now let's prove that $f(A_0 \cap A_1) = f(A_0) \cap f(A_1)$ if f is injective. We have already shown that independent of whether f is injective or not, $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$. Thus, we just have to show that $f(A_0) \cap f(A_1) \subset f(A_0 \cap A_1)$ if f is injective.

Let $x \in f(A_0) \cap f(A_1)$. Then $x \in f(A_0)$ and $x \in f(A_1)$. Besides, since f is injective, there exists **unique** x' such that $f(x') = x$. Therefore, $x' \in A_0$ and $x' \in A_1$. Finally, we get that $x \in f(A_0 \cap A_1)$. \square

5. In general, let us denote the **identity function** for a set C by i_C . That is, define $i_C : C \rightarrow C$ to be the function given by the rule $i_C(x) = x$ for all $x \in C$. Given $f : A \rightarrow B$, we say that a function $g : B \rightarrow A$ is a **left inverse** for f if $g \circ f = i_A$; and we say that $h : B \rightarrow A$ is a **right inverse** for f if $f \circ h = i_B$.

- (a) Show that if f has a left inverse, f is injective; and if f has a right inverse, f is surjective.

Let's first show that if f has a left inverse, then f is injective.

Suppose, for the sake of contradiction, that $f : A \rightarrow B$ is function such that it has a left inverse and that f is not injective. Since f is not injective, there exists $x_0, x_1 \in A$ such that $f(x_0) = f(x_1)$ and $x_0 \neq x_1$. Since f has the left inverse, there exists $g : B \rightarrow A$ such that $g \circ f = i_A$. Consider functions $(g \circ f)(x_0)$ and $(g \circ f)(x_1)$. These functions could be rewritten as $g(f(x_0))$ and $g(f(x_1))$. Since $f(x_0) = f(x_1)$, we have that $g(f(x_0)) = g(f(x_1))$. Therefore, we got that $i_A(x_0) = i_A(x_1)$ and thus, $x_0 = x_1$. At last, we have reached the contradiction since initially we assumed that $x_0 \neq x_1$. Hence, if f has a left inverse, then f is injective. \square

Now let's show that if f has a right inverse, then f is surjective.

Suppose $f : A \rightarrow B$ is a function such that it has a right inverse. Then there exists $h : B \rightarrow A$ such that $f \circ h = i_B$. Note that $\forall x \in B$, we have $f(h(x)) = x$ and since $h(x) \in A$, we effectively got that every $x \in B$ has something that maps to it. In other words, f is surjective. \square

- (b) Give an example of a function that has a left inverse but no right inverse.

$f : \mathbb{R} \rightarrow \mathbb{R} - \{0\} : x \mapsto 2^x$. It has the left inverse $g(x) = \log_2|x|$ but no right inverse. From **exercise (a)**, we know that it has no right inverse as the function is not surjective.

- (c) Give an example of a function that has a right inverse but no left inverse.

$f : \mathbb{R} \rightarrow \mathbb{R} \cup \{0\} : x \mapsto x^2$. It has the right inverses $h(x) = \sqrt{x}$ and $h'(x) = -\sqrt{x}$ but no left inverse. From **exercise (a)**, we know that it has no left inverse since as the function is not injective.

- (d) Can a function have more than one left inverse? More than one right inverse?

Yes, it can have more than one left inverse.

Let $A = \{1, 2\}$ and $B = \{1, 2, 3\}$. Then consider the following function:

$$f : A \rightarrow B : x \mapsto x$$

Notice that f has two left inverses. One is $g(x) = x$ if $x \in \{1, 2\}$ and $g(3) = 1$. And the other one is $g'(x) = x$ if $x \in \{1, 2\}$ and $g'(3) = 2$.

Yes, it can also have more than one right inverse.

The example is in **exercise (c)**, but here is another one for diversity:

Let $A = \{1, 2\}$ and $B = \{1\}$. Then consider the following function:

$$f : A \rightarrow B : x \mapsto x$$

Notice that it has two right inverses. One is $h(1) = 1$. And the other one is $h'(1) = 2$.

- (e) Show that if f has both a left inverse g and a right inverse h , then f is bijective and $g = h = f^{-1}$.

From **exercise (a)** we know that if f has both left and right inverses, then f is bijective (it is injective if it has the left inverse and surjective if it has the right inverse hence if it has both left and right inverses, then it is bijective). Now, $\forall x \in B$, there is a unique $y \in A$ such that $f(x) = y$. Then $f(x) = f(h(y)) = y$ and $g(y) = g(f(x)) = x$. Also, as f is injective, $h(y) = x$. Finally, since $h(y) = g(y) = h(f(x)) = g(f(x)) = x$, we get that $g = h = f^{-1}$. \square

Section 3

10. (a) Show that the map $f : (-1, 1) \rightarrow \mathbb{R}$ of Example 9 is order preserving.

We have to show that $f : (-1, 1) \rightarrow \mathbb{R} : x \mapsto \frac{x}{1-x^2}$ is an order-preserving map. In other words, we have to prove that for arbitrary $x_0, x_1 \in (-1, 1)$, if $x_1 > x_0$, then $f(x_1) > f(x_0)$. Suppose $x_0, x_1 \in (-1, 1)$ and $x_1 > x_0$. Then we have:

$$f(x_1) - f(x_0) = \frac{x_1}{1-x_1^2} - \frac{x_0}{1-x_0^2} = \frac{x_1 - x_1x_0^2 - x_0 + x_0x_1^2}{(1-x_1^2)(1-x_0^2)} = \frac{(x_1 - x_0)(x_0x_1 + 1)}{(1-x_1^2)(1-x_0^2)}$$

$$\text{Finally, we got that } f(x_1) - f(x_0) = \frac{x_1 - x_1x_0^2 - x_0 + x_0x_1^2}{(1-x_1^2)(1-x_0^2)} = \frac{(x_1 - x_0)(x_0x_1 + 1)}{(1-x_1^2)(1-x_0^2)}$$

where $x_1 > x_0$. Notice that all the members of the fraction are positive.

$x_1 - x_0 > 0$ since $x_1 > x_0$, $x_0x_1 + 1 > 0$ as $x_0, x_1 \in (-1, 1)$, and $(1-x_1^2)(1-x_0^2)$ is positive since, once again, $x_0, x_1 \in (-1, 1)$. Thus, we have effectively shown that if $x_0, x_1 \in (-1, 1)$ such that $x_1 > x_0$, then $f(x_1) - f(x_0) > 0$. In others, if $x_0, x_1 \in (-1, 1)$ and $x_1 > x_0$, then $f(x_1) > f(x_0)$ and f is indeed an order-preserving map. \square

- (b) Show that the equation $g(y) = 2y/[1 + (1 + 4y^2)^{1/2}]$ defines a function $g : \mathbb{R} \rightarrow (-1, 1)$ that is both a left and a right inverse for f .

Let's first show that it is indeed the left inverse for the function f .

Consider the function composition $g \circ f$. This composition is possible since the codomain of f is the same as the domain of g . Now, let's do some high school algebra:

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) = \frac{2f(x)}{1 + (1 + 4f(x)^2)^{\frac{1}{2}}} = \frac{\frac{2x}{1 - x^2}}{1 + (1 + 4(\frac{x}{1 - x^2})^2)^{\frac{1}{2}}} \\ &= \frac{\frac{2x}{1 - x^2}}{1 + (\frac{(x^2 + 1)^2}{(1 - x^2)^2})^{\frac{1}{2}}} = \frac{\frac{2x}{1 - x^2}}{\frac{1 - x^2 + x^2 + 1}{1 - x^2}} = \frac{\frac{2x}{1 - x^2}}{\frac{2}{1 - x^2}} = x\end{aligned}$$

Hence, we've got that $(g \circ f)(x) = x = i_{(-1,1)}$ thus, is the identity function for the open interval $(-1, 1)$. This is sufficient show for the function g to be the left inverse for the function f . \square

Now, let's show that g is also the right inverse for f . Consider, the function composition $f \circ g$. Once again, we will need to do some high school math:

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) = \frac{g(x)}{1 - g(x)^2} = \frac{\frac{2x}{1 + (1 + 4x^2)^{\frac{1}{2}}}}{1 - (\frac{2x}{1 + (1 + 4x^2)^{\frac{1}{2}}})^2} \\ &= \frac{2x(1 + (1 + 4x^2)^{\frac{1}{2}})}{(1 + (1 + 4x^2)^{\frac{1}{2}})^2 - 4x^2} = \frac{2x(1 + (1 + 4x^2)^{\frac{1}{2}})}{1 + 1 + 4x^2 + 2(1 + 4x^2)^{\frac{1}{2}} - 4x^2} \\ &= \frac{2x(1 + (1 + 4x^2)^{\frac{1}{2}})}{2(1 + (1 + 4x^2)^{\frac{1}{2}})} = x\end{aligned}$$

Thus, we got that $(f \circ g)(x) = x = i_{\mathbb{R}}$. This means that the function composition $f \circ g$ yields the identity function and this means that function g is indeed the right inverse for the function f . \square

Section 6

2. Show that if B is not finite and $B \subset A$, then A is not finite.

Suppose, for the sake of contradiction, that B is not finite, $B \subset A$, and A is finite. Then since $B \subset A$ and A is finite, it must be the case that B is finite too as it is the portion of the set with finite elements. This, however, contradicts our initial assumption that B is not finite thus, we have reached the contradiction and if B is not finite and $B \subset A$, then A is not finite. \square

3. Let X be the two-element set $\{0, 1\}$. Find a bijective correspondence between X^ω and a proper subset of itself.

Let $S = \{A \in X^\omega \mid A = (0, x_1, x_2, \dots)\}$ and define $f : X^\omega \rightarrow S$ such that $f(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3, \dots)$. This is a function that shifts each element of the sequence one place to the right.

7. If A and B are finite, show that the set of all functions $f : A \rightarrow B$ is finite.

Notice that there are $|B|^{|A|}$ different functions of the form $f : A \rightarrow B$. Since A and B are finite, so is $|B|^{|A|}$. Hence, we got that the cardinality of the set of all functions $f : A \rightarrow B$ is finite which implies that the set of all functions $f : A \rightarrow B$ is also finite. \square

Section 7

1. Show that \mathbb{Q} is countably infinite.

To show that the set of rational numbers is countable, it is sufficient to show that there is a way to enumerate (count/list down) all of them. This is the classic visual proof of enumerability/countability of the set of rational numbers.

In order to prove the countability of the set of rational numbers, let's first construct a two-dimensional, infinite matrix. We will construct it in the following way:

1. The first column of the matrix is comprised of non-negative integers.
2. The first row of the matrix is comprised of non-negative integers.
3. The rest of the cells are pairs which are composed by dividing the cell value of the first row by the corresponding column (pairing the division product with its opposite).

The grid will look like this:

0	1	2	3	4	5	etc
1	(1/1, -1/1)	(2/1, -2/1)	(3/1, -3/1)	(4/1, -4/1)	(5/1, -5/1)	etc
2	(1/2, -1/2)	(2/2, -2/2)	(3/2, -3/2)	(4/2, -4/2)	(5/2, -5/2)	etc
3	(1/3, -1/3)	(2/3, -2/3)	(3/3, -3/3)	(4/3, -4/3)	(5/3, -5/3)	etc
4	(1/4, -1/4)	(2/4, -2/4)	(3/4, -3/4)	(4/4, -4/4)	(5/4, -5/4)	etc
5	(1/5, -1/5)	(2/5, -2/5)	(3/5, -3/5)	(4/5, -4/5)	(5/5, -5/5)	etc
etc	etc	etc	etc	etc	etc	etc

Now, our enumeration could proceed as follows: 0, 1/1, -1/1, 1/2, -1/2, 2/1, -2/1, 1/3, -1/3, 2/3, -2/3, 3/3, -3/3 etc. Notice that the enumeration starts at the first entry of the first row and proceeds diagonally. This will guarantee that all of the rational numbers will be listed (notice that we would not be able to list them if we proceeded horizontally or vertically). Thus, the set of rational numbers is countable. \square

5. Determine, for each of the following sets, whether or not it is countable. Justify your answers.

(a) The set A of all functions $f : \{0, 1\} \rightarrow \mathbb{Z}^+$.

It is countable.

Notice that the set of all functions $f : \{0, 1\} \rightarrow \mathbb{Z}^+$ is in bijective correspondence with $\mathbb{Z}^+ \times \mathbb{Z}^+$. Since $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable (we can prove it using the same grid technique as in the previous exercise), $f : \{0, 1\} \rightarrow \mathbb{Z}^+$ is countable.

NOTE: The fact that $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable is **Corollary 7.4**.

(b) The set B_n of all functions $f : \{1, \dots, n\} \rightarrow \mathbb{Z}^+$.

It is countable.

Notice that the set of all functions $f : \{1, \dots, n\} \rightarrow \mathbb{Z}^+$ is in a bijective correspondence with $\underbrace{\mathbb{Z}^+ \times \mathbb{Z}^+ \times \dots \times \mathbb{Z}^+}_{n \text{ times}}$. Now, according to **Theorem 7.6**,

the product of countable sets is countable and hence, the set B_n of all functions $f : \{1, \dots, n\} \rightarrow \mathbb{Z}^+$ is also countable.

(c) The set $C = \bigcup_{n \in \mathbb{Z}^+} B_n$.

It is countable.

$C = \bigcup_{n \in \mathbb{Z}^+} B_n = B_1 \cup B_2 \cup B_3 \cup \dots B_{n-1} \cup B_n$. Now, from the previous question, we know that B_n is countable. This also leads to $B_1, B_2, \dots B_{n-1}$ being countable. Then, according to the **Theorem 7.5**, the union of countable sets is countable and the set C is countable.

(d) The set D of all functions $f : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$.

It is not countable.

Notice that set D is the set of all sequences of positive integers. Thus, using the book's notation, $D = (\mathbb{Z}^+)^\omega$. Then according to **Theorem 7.7**, D is not countable.

Notice that the theorem implies that $\{0, 1\}^\omega$ is uncountable, however, it is obviously sufficient to show in order to say that $(\mathbb{Z}^+)^\omega$ is also not countable. This can also be shown using *Cantor's diagonalization argument*.

(e) The set E of all functions $f : \mathbb{Z}^+ \rightarrow \{0, 1\}$.

It is not countable.

This is similar to the previous question. Once again, according to **Theorem 7.7**, the set E is not countable.

(f) The set F of all functions $f : \mathbb{Z}^+ \rightarrow \{0, 1\}$ that are "eventually zero."

[We say that f is **eventually zero** if there is a positive integer N such that $f(n) = 0$ for all $n \geq N$].

It is countable.

For some N , we can define F_N as the set of all functions where $f(n) = 0 \forall n \geq N$. Then the function $f : F_N \rightarrow \{0, 1\}^{N-1}$ such that

$f(\{x_1, x_2, \dots, x_{N-1}, 0, 0, 0, \dots\}) = (x_1, x_2, x_3, \dots, x_{N-1})$ is a bijection. According to the **Theorem 7.6**, $\{0, 1\}^{N-1}$ is countable and hence, we got that F_N is countable. Now, since F_N is countable for any arbitrary N , according to the **Theorem 7.5**, F is countable too as $F = \bigcup_{N=1}^{\infty} F_N$.

- (g) The set G of functions $f : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ that are eventually 1.

It is countable.

The proof is almost the same as in the previous exercise. For some N , we can define G_N as the set of all functions where $g(n) = 1 \ \forall n \geq N$. Then the function $g : G_N \rightarrow (\mathbb{Z}^+)^{N-1}$ such that

$g(\{x_1, x_2, \dots, x_{N-1}, 1, 1, 1, \dots\}) = (x_1, x_2, x_3, \dots, x_{N-1})$ is a bijection. According to the **Theorem 7.6**, $(\mathbb{Z}^+)^{N-1}$ is countable and hence, we got that G_N is countable. Now, since F_N is countable for any arbitrary N , according to the **Theorem 7.5**, G is countable too as $G = \bigcup_{N=1}^{\infty} G_N$.

- (h) The set H of all functions $f : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ that are eventually constant.

It is countable.

$\forall C \in \mathbb{Z}^+$, we can define H_C to be the set of all $f : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ functions which are eventually C . From the **exercise (f)**, we know that H_C is countable. Then, according to the **Theorem 7.5**, H is countable as well since $H = \bigcup_{C=1}^{\infty} H_C$.

- (i) The set I of all two-element subsets of \mathbb{Z}^+ .

It is countable.

We can define the function $f : I \rightarrow \mathbb{Z}^+$ such that $f(\{x_0, x_1\}) = 3^{x_0} \times 5^{x_1}$. Then f is injective since if $f(x_0, x_1) = f(x_0', x_1')$, then $3^{x_0} \times 5^{x_1} = 3^{x_0'} \times 5^{x_1'}$ and since $x_0, x_1, x_0', x_1' \in \mathbb{Z}^+$ and $\gcd(3, 5) = 1$, it must be the case that $x_0 = x_0'$ and $x_1 = x_1'$. Hence, according to **Theorem 7.1**, the set I is countable.

NOTE: The gcd function returns the greatest common divisor of two numbers.

- (j) The set J of all finite subsets of \mathbb{Z}^+ .

It is countable.

Consider the function $f : J \rightarrow \mathbb{Z}^+$ such that $f(\{x_0, x_1, \dots, x_{n-1}\}) = x_0, x_1, \dots, x_{n-1}$ if $x_1 < x_2 < x_3 < \dots < x_n$. For instance, $f(\{1, 2, 3\}) = 123$ and $f(\{4, 5, 1, 3\}) = 1345$. Then, the map is injective since if $f(S_1) = f(S_2)$ it must be the case that $S_1 = S_2$ (the order matters!). Thus, according to **Theorem 7.1**, the set J is countable.