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# *Real Analysis Exams*

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## Exam №3

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1. (a) Yes, it is continuous at 0. Recall that a function is continuous at  $x = 0$  if  $\lim_{x \rightarrow 0} f(x) = f(0) = 0$ . Notice that  $\forall x \neq 0, |\sin(\frac{1}{x^2})| \leq 1$ . We then have that  $|f(x)| \leq x^4$  (multiply both sides of the inequality by  $x^4$ ). Now, since  $\lim_{x \rightarrow 0} x^4 = 0$  and  $\lim_{x \rightarrow 0} (-x^4) = 0$ , it follows by **Squeeze Theorem** that  $\lim_{x \rightarrow 0} 0 = f(0)$ . Hence,  $f(x)$  is continuous at 0.
- (b) Yes, it is differentiable at 0. Recall that a function is differentiable at  $x = 0$  if the limit  $\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0}$  exists. We have:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} &= \lim_{x \rightarrow 0} \frac{f(x) - 0}{x} \\ &= \lim_{x \rightarrow 0} \frac{x^4 \sin(\frac{1}{x^2})}{x} \\ &= \lim_{x \rightarrow 0} x^3 \sin\left(\frac{1}{x^2}\right) \end{aligned}$$

Now, notice that  $x^3 \sin(\frac{1}{x^2})$  is bounded by  $-|x^3|$  and  $|x^3|$  and once again, by **Squeeze Theorem**, it follows that  $f$  is differentiable at 0.

- (c)  $f'(x)$  is continuous at 0. Away from 0, we have  $f'(x) = 4x^3 \sin(\frac{1}{x^2}) - 2x \cos(\frac{1}{x^2})$ . Then  $\lim_{x \rightarrow 0} f'(x) = 0$  and thus,  $\lim_{x \rightarrow 0} f'(x) = 0 = f'(0)$ . Hence,  $f'(x)$  is continuous at 0.

- (d) No, it is not differentiable at 0. Recall that a function is differentiable at  $x = 0$  if the limit  $\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0}$  exists. We have:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{f'(x) - f'(0)}{x - 0} &= \lim_{x \rightarrow 0} \frac{f(x) - 0}{x} \\ &= \lim_{x \rightarrow 0} \frac{4x^3 \sin\left(\frac{1}{x^2}\right) - 2x \cos\left(\frac{1}{x^2}\right)}{x} \\ &= \lim_{x \rightarrow 0} 4x^2 \sin\left(\frac{1}{x^2}\right) - 2 \cos\left(\frac{1}{x^2}\right) \end{aligned}$$

Now, notice that  $\lim_{x \rightarrow 0} 4x^2 \sin\left(\frac{1}{x^2}\right) - 2 \cos\left(\frac{1}{x^2}\right)$  does not exist and thus,  $f'(x)$  is not differentiable at 0.

2. (a) Since  $f(x)$  is differentiable on  $[0, 4]$ , it is also continuous on  $[0, 4]$ . Now, let  $g(x) = f(x) - x$ . Then  $g$  is again both differentiable and continuous on  $[0, 4]$ . We have  $g(0) = f(0) = 2$  and  $g(4) = f(4) - 4 = -3$ . Now, since  $g$  is continuous, it follows by the **Theorem 4.5.1 (Intermediate Value Theorem)**  $\exists c \in (0, 4)$  s.t.  $g(c) = 0$  and for this  $c$ , we will have  $f(c) = g(c) + c = 0 + c = c$ . Hence,  $f(x)$  has a fixed point on  $[0, 4]$

□

- (b) Notice that  $\frac{f(4) - f(0)}{4 - 0} = \frac{1 - 2}{4} = -\frac{1}{4}$ . Then it follows by **Theorem 5.3.2 (Mean Value Theorem)** that  $\exists c \in (0, 4)$  s.t.  $f'(c) = -\frac{1}{4}$ . On the other hand, we know that  $f'(1) = 2$ . Finally, it follows by **Theorem 5.2.7 (Darboux's Theorem)** that  $\exists c \in (0, 4)$  s.t.  $f'(c) = 0$ .

□

3. (a) For any fixed  $x$ , we have:

$$\lim_{n \rightarrow \infty} \frac{x^n e^{-x}}{n!} = \lim_{n \rightarrow \infty} \frac{x^n}{n! e^x} = 0$$

The reason the limit is 0 is that as  $n$  approaches infinity,  $n!e^x$  grows a lot faster than  $x^n$  (one could also use **Squeeze Theorem** to show that the limit is 0).

- (b) For any fixed  $x$ , we have:

$$|g_n(x) - g(x)| = \left| \frac{x^n e^{-x}}{n!} - 0 \right| = \left| \frac{x^n}{n! e^x} \right|$$

Now, we need to pick  $N$  s.t.  $\forall n \geq N$ ,  $\left| \frac{x^n}{n! e^x} \right| < \epsilon$  holds. Now, although it is possible to do for every  $x \in [0, \infty)$ , there is no way to choose a single value of  $N$  that will work for all values of  $x$  at the same time. Thus, such  $N$  does not exist. Finally, we conclude that the sequence of functions  $(g_n)$  does not uniformly converge to  $g$  on  $[0, \infty)$ .

4. (a)

$$f'_n(x) = \left(xe^{-nx^2}\right)' = e^{-nx^2}(1 - 2nx^2)$$

(b) We need to solve the equation  $f'_n(x) = 0$ . We have:

$$e^{-nx^2}(1 - 2nx^2) = 0 \implies x = \pm \frac{1}{\sqrt{2n}}$$

Hence, the global maximum occurs at  $x = \frac{1}{\sqrt{2n}}$  and the global minimum occurs at  $x = -\frac{1}{\sqrt{2n}}$ .

Let us sketch  $f_n(x)$  for  $n = 2$ . For  $n = 2$ , we have the function  $f_2(x) = xe^{-2x^2}$ .

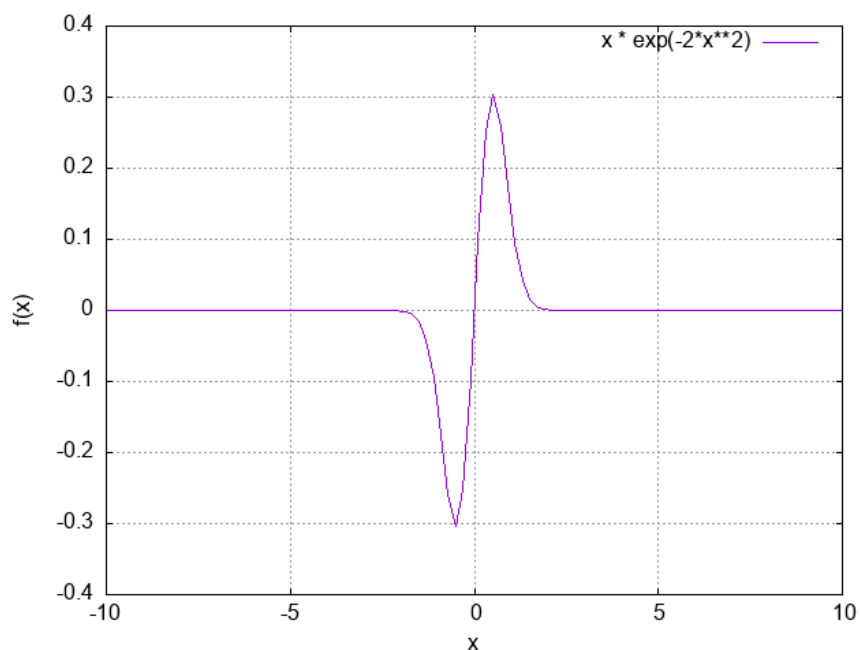


Figure 1: Plot of  $g = xe^{-2x^2}$ .

(c)

$$\begin{aligned} f(x) &= \lim_{n \rightarrow \infty} f_n(x) \\ &= \lim_{n \rightarrow \infty} \left(xe^{-nx^2}\right) \\ &= \lim_{n \rightarrow \infty} \frac{x}{e^{nx^2}} \\ &= 0 \end{aligned} \quad \text{(notice that as } n \rightarrow \infty, e^{nx^2} \rightarrow \infty \text{)}$$

(d) Since the global maximum of  $f(x)$  is  $\frac{1}{\sqrt{2n}}$ , we can let  $N = \lceil \frac{1}{\epsilon^2} \rceil$ . Then  $\forall n \geq N$ , we have:

$$|f_n(x) - f(x)| = |f_n(x)| = |xe^{-nx^2}| \leq \frac{1}{\sqrt{2n}} \leq \frac{1}{\sqrt{2N}} \leq \frac{\epsilon}{\sqrt{2}} < \epsilon$$

Now, we could find  $N \in \mathbb{N}$  s.t.  $\forall n \geq N, |f_n(x) - f(x)| < \epsilon$  holds and hence,  $f_n$  converges uniformly to  $f$  on  $\mathbb{R}$ .

(e)

$$\lim_{n \rightarrow \infty} f'_n(x) = \lim_{n \rightarrow \infty} e^{-nx^2} (1 - 2nx^2) = \lim_{n \rightarrow \infty} \frac{-2nx^2 + 1}{e^{nx^2}} = 0$$

(notice that as  $n \rightarrow \infty, e^n$  grows a lot faster than  $2n$ )

Since  $f(x) = 0$ , it follows that  $f'(x) = 0$  and finally, we have  $f'(x) = \lim_{n \rightarrow \infty} f'_n(x) = 0$ .

5. (a) Notice that the following holds:

$$\left| \frac{\cos(3^n x)}{2^n} \right| \leq \frac{1}{2^n}$$

Now, recall that  $\frac{1}{2^n}$  converges (showed many times over the course of the class). Then, it follows by **Corollary 6.4.5 (Weierstrass M-Test)** that  $g(x) = \sum_{n=1}^{\infty} \frac{\cos(3^n x)}{2^n}$  converges uniformly on  $\mathbb{R}$ . And since the uniform convergence implies continuity, it follows that  $g(x) = \sum_{n=1}^{\infty} \frac{\cos(3^n x)}{2^n}$  is continuous on  $\mathbb{R}$ .

(b) Notice that we have:

$$g'(x) = \sum_{n=1}^{\infty} -\left(\frac{3}{2}\right)^n \sin(3^n x)$$

Unfortunately, in this case we cannot apply **Corollary 6.4.5 (Weierstrass M-Test)** as  $\left(\frac{3}{2}\right)^n$  is not bounded. Hence, this is the difference between part (a) and part (b) of the exercise (we cannot determine if  $g$  is differentiable on  $\mathbb{R}$ ).

As a side note, recall that this is the Weierstrass function of the form  $\sum_{n=0}^{\infty} a^n \cos(b^n x)$  which is a nowhere-differentiable function. Hence,  $g'(x)$  is not differentiable on  $\mathbb{R}$ .

6. For  $x \notin \mathbb{Q}$ , we can show  $f_n(x)$  is continuous, since for  $x < r_n$ , we can choose a small enough  $\delta$  such that  $f_n(y) = 0$  for  $y \in V_\delta(x)$ . Similar logic can be applied when  $x > r_n$ . Now, notice that

$$f_n(x) \leq \frac{1}{2^n}$$

Then it follows by **Corollary 6.4.5 (Weierstrass M-Test)** that  $f(x)$  converges uniformly.

Now, since  $f_n$  are all continuous, and  $f$  converges uniformly, we have that  $f$  is continuous.

Furthermore, since every  $f_n(x)$  is increasing,  $f$  is monotonely increasing. Thus, for  $x < y$ , we get:

$$\begin{aligned}\forall n \quad f_n(x) &\leq f_n(y) \\ \sum_{n=1}^k f_n(x) &\leq \sum_{n=1}^k f_n(y) \\ \lim_k \sum_{n=1}^k f_n(x) &\leq \lim_k \sum_{n=1}^k f_n(y) \\ f(x) &\leq f(y)\end{aligned}$$

Hence, we got that  $f$  is increasing on  $\mathbb{R}$ .

□

7. (a) We have:

$$\begin{aligned}\ln(1+x) &= \sum_{n \geq 1} \frac{f^{(n)}(0)}{n!} x^n \\ &= \sum_{n \geq 1} \frac{(-1)^{n-1}(n-1)!}{n!} x^n \\ &= \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} x^n \\ &= \sum_{n \geq 1} \frac{(-1)^n}{n+1} x^{n+1} \\ &= x - \frac{x^2}{3} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\end{aligned}$$

Hence the Taylor series representation is  $x - \frac{x^2}{3} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$ .

(b) When  $x = 1$ , we get alternating harmonic series that we know converges [shown many times over the course of the class]. It diverges when  $x = -1$  as we get the  $-\frac{1}{n}$  which is the negative harmonic series that we know diverges [shown many times over the course of the class]). Then it follows that the series converges when  $-1 < x \leq 1$ . Hence, the interval of convergence of the series is  $(-1, 1]$  (the radius of convergence is 1).

(c) Yes, it does. Let us apply the ratio test. We get:

$$\lim_{n \rightarrow \infty} \frac{\frac{(-1)^{n+1}}{n+2} x^{n+2}}{\frac{(-1)^n}{n+1} x^{n+1}} = -\frac{n+1}{n+2} x = -x$$

Note that the series converges uniformly if  $|-x| = |x| < 1$ . Hence, it converges on  $(-1, 1)$ .

We now need to check the endpoints  $x = -1$  and  $x = 1$ . Notice that if  $x = -1$ , the

series does not converge uniformly as  $\ln(1 + -1) = \ln(0)$  which is undefined (negative infinity). For  $x = 1$ , it follows by **Leibnitz' test for alternating series**, that the series converges. Hence, the Taylor series converges uniformly to  $f$  on  $(-1, 1]$  which is its interval of convergence.