Homework №3

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Section 2.2

7. List three elements of each of the following sets.

   (a) $\mathbb{Q} \cap (2, 3)$

   Since $(2, 3)$ is an open range from 2 to 3 (not including 2 and 3), all of it is in the $\mathbb{Q}$ and the answer of the intersection is $(2, 3)$.

   Here are three elements that are the result set: $2.1, 2.2, 2.3$

   (b) $\{2^n - 1 \mid n \in \mathbb{Z}^+\}$

   If $n = 1$, then $2^n - 1 = 1$
   If $n = 2$, then $2^n - 1 = 3$
   If $n = 3$, then $2^n - 1 = 7$

   Here are three elements that are the result set: $1, 3, 7$

   (c) $\{n \in \mathbb{Z}^+ \mid n^2 + 1 \text{ is prime}\}$

   If $n = 1$, then $n^2 + 1 = 2$ and 2 is a prime
   If $n = 2$, then $n^2 + 1 = 5$ and 3 is a prime
   If $n = 4$, then $n^2 + 1 = 17$ and 17 is a prime

   Here are three elements that are the result set: $2, 3, 17$

8. Let $A = [4, 7]$ and $B = (6, 8)$, both subsets of the universal set $\mathbb{R}$. Write each of the following sets as naturally as possible:

   (a) $A \cup B$

   $A \cup B = [4, 8)$. Thus, the set $A \cup B$ is the set of all real numbers from 4 to 8, including 4 but not including 8.

(b) $A \cap B$

$A \cap B = (6, 7]$. Thus, the set $A \cap B = (6, 7]$ is the set of all real numbers from 6 to 7, not including 6 but including 7.

(c) A^C

$A^C = (-\infty, 4) \cup (7, +\infty)$. Thus, the set A^C is the set of all real numbers but those in the range $[4, 7]$, not including 4 and 7 (also called the *complement* of A).

(d) B^C

$B^C = (-\infty, 6] \cup [8, +\infty)$. Thus, the set B^C is the set of all real numbers but those in the range $(6, 8)$, including 4 and 7 (also called the *complement* of B).

(e) $A - B$

$A - B = [4, 6]$. Thus, the set B^C is the set of all real numbers in the range $[4, 6]$, 4 and 6 inclusive.

9. A **bi-partition** of a set S is a set $\{A, B\}$ of two subsets A and B of S such that $A \cup B = S$ and $A \cap B = \emptyset$.

(a) List all bi-partitions of the set $\{1, 2, 3\}$.

The list of all bi-partitions of the set $\{1, 2, 3\}$ is:

$$\begin{array}{c} \{1\} \text{ and } \{2,3\} \\ \{2\} \text{ and } \{1,3\} \\ \{3\} \text{ and } \{1,2\} \end{array}$$

The list above shows that there are three bi-partitions for the given set.

(b) Explain why, every subset A of a set S is an element of exactly one bi-partition of S. (Hint: First explain why every such A is an element of at least one bi-partition of S, then explain why it cannot be be an element of more than one bi-partition of S.)

A is an element of at least one bi-partition since the list of bi-partitions of B are really the sets of paired subsets of B. If we list all the bi-partition subset pairs, we will get all the subsets of B. Then if we have all the subsets of B, A is just one of them and it has to be one of the bi-partitions.

(c) It is a fact that we will prove later that if $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, then S has exactly 1024 subsets. How many bi-partitions does S have?

Actually, let's prove it now and then count the number of bi-partitions.
So, for each element of the set, we have two options, either put in the subset or leave it off. We have 10 elements in the set S. $2^{10} = 1024$.

Q.E.D

Now, let's go ahead and count the number of bi-partitions for the set.

$$\begin{gathered}
\{1\} \text{ and } \{2,3,4,5,6,7,8,9,10\} \\
\{2\} \text{ and } \{1,3,4,5,6,7,8,9,10\} \\
\{3\} \text{ and } \{1,2,4,5,6,7,8,9,10\} \\
\{4\} \text{ and } \{1,2,3,5,6,7,8,9,10\} \\
\{5\} \text{ and } \{1,2,3,4,6,7,8,9,10\} \\
\{6\} \text{ and } \{1,2,3,4,5,7,8,9,10\} \\
\{7\} \text{ and } \{1,2,3,4,5,6,8,9,10\} \\
\{8\} \text{ and } \{1,2,3,4,5,6,7,9,10\} \\
\{9\} \text{ and } \{1,2,3,4,5,6,7,8,10\} \\
\{10\} \text{ and } \{1,2,3,4,5,6,7,8,9\}
\end{gathered}$$

Thus, we see that the number of the bi-partitions for the set S is 10.

2. Write down two sets each having infinitely many elements. Do parts (a)–(c) for these sets.

Let's look at two infinite sets $A = \{x \in \mathbb{Z}^+ \mid 2x\}$ and $B = \{x \in \mathbb{Z}^+ \mid 2x - 1\}$.

(a) What is their union?

$A \cup B = \mathbb{Z}^+$

(a) What is their intersection?

$A \cap B = \emptyset$

(a) What is their set difference? (both of them)

$A - B = A$
$B - A = B$

6. List the elements in each of the following sets:

(h) $\{x \in \mathbb{R} \mid x^3 - 3x = 0\}$

If $x^3 - 3x = 0$, then the solutions to the equation are $0, \sqrt{3}$ and $-\sqrt{3}$.
Thus the elements of the set are $0, \sqrt{3}, -\sqrt{3}$.

(h) $\{x \in \mathbb{R} \mid x^2 + 4x + 5 = 0\}$

Equation $x^2 + 4x + 5 = 0$ has no solutions.
Thus the set is $\emptyset$.

7. (c). Show that $\{x \in \mathbb{R} \mid (x - 1/2)(x - 1/3) < 0\} \subseteq \{x \in \mathbb{R} \mid 0 < x < 1\}$. Check your answer.

The solution set for the equation $(x - 1/2)(x - 1/3)$ is $\{1/2, 1/3\}$. Thus, $\{x \in \mathbb{R} \mid (x - 1/2)(x - 1/3) < 0\} = \{1/2, 1/3\}$. $\{x \in \mathbb{R} \mid 0 < x < 1\}$ the set of all elements of the open range $(0, 1)$. $1/2 \in (0, 1)$ and $1/3 \in (0, 1)$.

Q.E.D

14. A set A is called *full* if any element of A is also a subset of A. In other words, A is full if $x \in A$ implies $x \subseteq A$.

(a) Show that $\{\emptyset\}$ is full.

Set $\{\emptyset\}$ has only one element, namely $\emptyset$ (empty set), and $\emptyset \subseteq \{\emptyset\}$. Hence, $\{\emptyset\}$ is full.

Q.E.D

(b) Find a full set having exactly two elements.

$\{\emptyset, \{\emptyset\}\}$.

(c) Find a full set having exactly three elements.

$\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$.

19. Russell's Paradox. Let $S = \{A \mid A \text{ is a set and } A \notin A\}$. Suppose that S itself is a set.

(a) Show that if S is a member of itself, then S cannot be a member of itself.

If $S \in S$, then $S \notin S$ as S is a subset of S and for every subset A of S, if $A \in S$, then $A \notin S$.

Q.E.D

(b) Show that if S is not a member of itself, then S must be a member of itself.

Since S is a set, it has to have members. Thus, if S is not a member of itself, then S must be a member of itself.

Q.E.D

Section 2.3

10. An integer n is called doubly even if there exist even integers x and y such that $n = xy$.

    (a) Is 12 doubly even? Prove your answer.

    Yes, it is doubly even since $12 = 2 \times 6$ where 2 and 6 are even.

    Q.E.D.

    (b) Is 98 doubly even? Prove your answer.? Prove your answer.

    No, it is not doubly even since $98 = 2 \times 47$ where 2 and 47 is odd. Thus, since 47 cannot be further divided by 2 (and obtain integer), it is not double even.

    Q.E.D.

    (c) Write the negation of the statement "n is doubly even" without using the word "not."

    n is not divisible by 4.

    Explanation:
    Notice that if n is doubly even, by the definition, it is the multiplication of two even integers that is $n = xy$. Let $x = 2l$ and $y = 2k$ $(l,\ k \in \mathbb{Z}^+)$, then we have $n = 2l * 2k = 4lk$. Thus, if n is doubly even, then it is divisible by 4.

    (d) For what positive integers n is $n!$ doubly even? Prove your answer.

    For $n \geq 4$. Let's prove it!
    For a number n to be doubly-even, it should be divisible by 4 (if $n = xy$ where x and y are even, then let $x = 2i$ and $y = 2j$ where $i, j \in \mathbb{Z}$ and we get $n = 4ij$ which proves that n is indeed divisible by 4). From this fact, we can deduce that $n!$ must be divisible by 4 which happens for $n \geq 4$. Thus, if $n \geq 4$, $n!$ is doubly even.

    Q.E.D.

11. A subset S of $\mathbb{Z}$ is called *shifty* if for every $x \in S,\ x - 1 \in S,$ or $x + 1 \in S$.

    (a) Give an example of a shifty set with 5 elements.

    $S = \{1, 2, 3, 4, 5\}$

    (b) Give an example of a shifty set that contains 10 and -10 but does not contain 0.

    $S = \{-10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

(c) Is $\{n \in \mathbb{Z}^+ \mid n \text{ is not a multiple of 5 or 11}\}$ shifty? Why or why not?

For the set not to be shifty, we should have an element n for which either $n-1$ must be either divisible by 5 or 11 and $n+1$ should either be divisible by 5 or 11. Then, all we have to do is solve the systems of the following Diophantine equations:

$$\begin{cases} n-1 = 5k, \ \ where\ k \in \mathbb{Z}^+ \\ n+1 = 11l, \ \ where\ l \in \mathbb{Z}^+ \end{cases}$$

or

$$\begin{cases} n-1 = 11k, \ \ where\ k \in \mathbb{Z}^+ \\ n+1 = 5l, \ \ where\ l \in \mathbb{Z}^+ \end{cases}$$

Let's only consider the first system. From the first equation, we get:

$$n = 5k + 1$$

Then, if we substitute n in the second equation, we get the following Diophantine equation:

$$11l - 5k = 2$$

It's easy to see that $l = 2$ and $k = 4$. Then we have $n = 5 \times 4 + 1 = 21$. And we finally, 21 is an element of the set for which $21 - 1 = 20$ which is divisible by 5 and $21 + 1 = 22$ is divisible by 11. Thus, $\{n \in \mathbb{Z}^+ \mid n \text{ is not a multiple of 5 or 11}\}$ is not shifty.

(d) $\{n \in \mathbb{Z}^+ \mid n \text{ is a multiple of 5 or } n+1 \text{ is a multiple of 5}\}$. Why or why not?

This set will be shifty. To see, let's group the elements of the set in pairs. All the pairs will have the following type $(5k-1,\ 5k)$, where $k \in \mathbb{Z}^+$. Then, we know that for for $5k-1$, $5k$ is in the set and for $5k$, $5k-1$ is in the set which makes the set shifty.

(e) Write the negation of the statement "S is shifty" without using the word "not."

There exists $x \in S$ such that $x - 1 \notin S$ and $x + 1 \notin S$.

(f) Does every non-empty shifty set contain an even integer? Why or why not?

Yes. In order for the set S to be shifty, for every element (positive integer) n, either $n-1$ or $n+1$ must be in the set. If n is odd, then $n-1$ and $n+1$ are even and if n is even, then $n-1$ and $n+1$ are odd. Thus, for a set to be shifty, it must contain an even integer.

2. If $\sum a_n$ is a convergent infinite series, then $\lim_{n\to\infty} a_n = 0$.

   (a) If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ is not a convergent series.

(b) If $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ is a convergent series.
It's true.

6. If $\sum |a_n|$ is convergent, then $\sum a_n$ converges.

   (a) If $\sum a_n$ is not convergent, then $\sum |a_n|$ does not converge.

   (b) If $\sum |a_n|$ is not convergent, then $\sum a_n$ does not converge.
   It's true

8. If f has a local (relative) maximum at the real number a or a local (relative) minimum at a, then $f^{'}(a) = 0$ or $f^{'}(a)$ does not exist.

   (a) If $f^{'}(a)$ exists and $f^{'}(a) \neq 0$, then f does not have a local (relative) maximum at the real number a or a local (relative) minimum at a.

   (b) If $f^{'}(a) = 0$ or $f^{'}(a)$ does not exist, then f has a local (relative) maximum at the real number a or a local (relative) minimum at a.
   It's true.

12. To the ideal mathematician, what is a proof?

    To the "ideal mathematician", the proof is the process which is comprised of 3 stages:

    1. Writing down the axioms of the theory in a formal language with a given list of symbols or alphabet.
    2. Writing down the hypothesis of the theorem in the same symbolism.
    3. Showing that it is possible to transform the hypothesis step by step, using the rules of logic, till the conclusion is reached.

13. The ideal mathematician regards it as obvious that an extraterrestrial intelligence capable of intergalactic travel would recognize the binary expansion of π. Do you believe this? Explain why.

    Interesting question. Actually, I am not sure. If the question was about something not involving π (or such "universal" constants), it would be easier... It is possible that they simply cannot create something like circle (for some reasons which I am not sure of yet). One possible thing might be some very very weird gravity or atmosphere or the way their universe works (it might be that the universe is "anti-circle"). Secondly, "intelligence" is a very ambiguous thing and the definition depends on a person. For some, it's knowing maths, for some it's philosophy, for some it is everything altogether etc. Hence, due to the limited information about the aliens and their environment/universe as well as the vague "intelligence" definition, my answer is I DON'T KNOW.

14 How does this article make you feel about studying mathematics?

Good.