Real Analysis

Assignment №4

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December 14, 2020

2.6.2 (a) Such sequence exists. $a_n = \dfrac{(-1)^n}{n}$ is a Cauchy sequence that is not monotone since it alternates, but converges to 0.

(b) Per **Lemma 2.6.3**, such sequence cannot exist.

(c) Such sequences cannot exist. A divergent monotone sequence implies that the sequence is unbounded. Unbounded and monotone sequence, on the other hand, cannot contain a convergent subsequence. Hence, by **Cauchy Criterion (Theorem 2.6.4)**, it cannot contain a Cauchy subsequence.

(d) Such sequence exists. Let us define

$$a_n = \begin{cases} n \text{ if } n \in \mathbb{N} \text{ is odd} \\ 0 \text{ if } n \in \mathbb{N} \text{ is even} \end{cases}$$

Then it is easy to see that sequence is unbounded since $\forall k \in \mathbb{N}, a_{2k+1} = 2k + 1 > k$. On the other hand the subsequence formed by the even-termed elements is comprised of only zeros and hence, converges to 0. Therefore, the subsequence is Cauchy. Hence, we found an unbounded sequence containing a subsequence that is Cauchy.

2.6.3 (a) Since x_n and y_n are both Cauchy sequences, $\forall \epsilon > 0, \exists N_1, N_2 \in \mathbb{N}$ s.t. $|x_{m_1} - x_{n_1}| < \dfrac{\epsilon}{2}$ and $|y_{m_2} - y_{n_2}| < \dfrac{\epsilon}{2}$ with $m_1, n_1 \geq N_1$ and $m_2, n_2 \geq N_2$. Then let $N = \max\{N_1, N_2\}$. It follows that $\forall m, n \geq N, |x_m - x_n| < \dfrac{\epsilon}{2}$ and $|y_m - y_n| < \dfrac{\epsilon}{2}$. Finally, using the triangle inequality, we get:

$$\begin{aligned} |(x_m + y_m) - (x_n + y_n)| &= |(x_m - x_n) - (y_m - y_n)| \\ &\leq |x_m - x_n| + |y_m - y_n| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Thus, we conclude that $(x_n + y_n)$ is a Cauchy sequence.

□

(b) Since x_n and y_n are both Cauchy sequence, they are also bounded and hence, $\exists X, Y > 0$ s.t. $\forall n \in \mathbb{N}, |x_n| < X$ and $|y_n| < Y$. Additionally, $\forall \epsilon > 0, \exists N_1, N_2 \in \mathbb{N}$ s.t. $|x_{m_1} - x_{n_1}| < \dfrac{\epsilon}{2}$ and $|y_{m_2} - y_{n_2}| < \dfrac{\epsilon}{2}$ with $m_1, n_1 \geq N_1$ and $m_2, n_2 \geq N_2$. Then let $N = \max\{N_1, N_2\}$. We get:

$$\begin{aligned} |x_m y_m - x_n y_n| &= |x_m(y_m - y_n) + y_n(x_m - x_n)| \\ &\leq |x_m||y_m - y_n| + |y_n||x_m - x_n| \\ &< \frac{X\epsilon}{2} + \frac{Y\epsilon}{2} = \frac{\epsilon}{2}(X + Y). \end{aligned}$$

Thus, we found $\epsilon' = \dfrac{\epsilon}{2}(X + Y) > 0$ with N s.t. $m, n \geq N$ and $|x_m y_m - x_n y_n| < \epsilon'$. Hence, we conclude that $(x_n y_n)$ is a Cauchy sequence.

□

2.7.5 We have to prove that the series $\sum_{n=1}^{\infty} 1/n^p$ converges if and only if $p > 1$. By **Cauchy Condensation Test**, the series converges if and only if $\sum_{n=1}^{\infty} \dfrac{2^n}{2^{np}} = \sum_{n=1}^{\infty} 2^{n(1-p)}$ converges. Now, we need to get $2^{n(1-p)}$ less than 1 as otherwise, the sequence will diverge (all terms will be defferent and ≥ 1). $2^{n(1-p)} < 1$ if and only if $n(1-p) < 0$. Since $n \in \mathbb{N}$, we can safely divide both sides of the inequality by n. We get $1 - p < 0$ and thus, $p > 1$. Hence, we have proven that the series $\sum_{n=1}^{\infty} 1/n^p$ converges if and only if $p > 1$.

NOTE: We did not take the classic "prove it directly and prove its converse" approach since every statement used in the proof was if and only if statement. ***Cauchy Condensation Test*** *is if and only if and* $2^{n(1-p)} < 1$ *when* $n(1-p) < 0$ *is if and only if.*

2.7.8 (a) True. By **Theorem 2.7.3**, $\sum a_n$ converges absolutely. It follows that $\lim a_n = 0$. Thus, $|a_n|$ is bounded and $\exists B > 0$ s.t. $\forall n \in \mathbb{N}, |a_n| \leq B$. Now, by **Algebraic Limit Theorem for Series (Theorem 2.7.1)**, $\sum B|a_n|$ and $B|a_n| \geq a_n^2$. Finally, per **Comparison Test (Theorem 2.7.4)**, $\sum a_n^2$ converges.

□

(b) False. Let $a_n = b_n = \dfrac{(-1)^n}{\sqrt{n}}$. Now, $\lim b_n = 0$ and thus, (b_n) converges. Additionally, $\lim \dfrac{1}{\sqrt{n}} = 0$ and $\dfrac{1}{\sqrt{n}}$ is decreasing. Hence, by **Alternating Series Test (Theorem 2.7.7)**, we get that $\sum a_n$ converges. Thus, both $\sum a_n$ and $\lim (b_n)$ converge. However, $a_n b_n = \dfrac{1}{n}$ which is harmonic series and it does not converge.

(c) True. Suppose, for the sake of contradiction, that $\sum a_n$ converges conditionally and $\sum n^2 a_n$ converges. Then, $\lim n^2 a_n = 0$ and $\exists N$ s.t. $\forall n \geq N, |n^2 a_n| < 1$. We then get $|a_n| < \dfrac{1}{n^2}$. Now, per **Comparison Test (Theorem 2.7.4)**, $\sum a_n$ converges absolutely and we face the contradiction since $\sum a_n$ converges conditionally. Thus, $\sum a_n$ converges conditionally, then $\sum n^2 a_n$ diverges.

□

2.7.9 (a) Suppose $r < r' < 1$. Since $\lim_{n\to\infty} |\dfrac{a_{n+1}}{a_n}| = r < 1$, let $\epsilon = r' - r > 0$. Then $\exists N \in \mathbb{N}$ s.t. $\forall n \geq \mathbb{N}$, the following is true:

$$
\begin{aligned}
&||\frac{a_{n+1}}{a_n}| - r| < \epsilon \\
&|\frac{a_{n+1}}{a_n}| < r + \epsilon \\
&|\frac{a_{n+1}}{a_n}| < r + r' - r \\
&|\frac{a_{n+1}}{a_n}| < r' \\
&|a_{n+1}| \leq |a_n| r'
\end{aligned}
$$

□

(b) Since $|r'| < 1$, $\sum (r')^n$ is a convergent geometric series. Then, by **Algebraic Limit Theorem for Series (Theorem 2.7.1)**, $|a_N| \sum (r')^n$.

□

(c) Notice that $\sum |a_n| = \sum_{n=1}^{N} |a_n| + \sum_{n=N+1}^{\infty} |a_n|$. Now, it is easy to see that $N \sum_{n=N+1}^{\infty} a_n$ converges by **Comparison Test (Theorem 2.7.4)** since $N \sum_{n=N+1}^{\infty} \leq |a_N| \sum_{n=N+1}^{\infty} r'^{n-N}$

(the fact that $|a_N| \sum_{n=N+1}^{\infty} r'^{n-N}$ converges was proved in part (b) of this exercise). Hence, $\sum a_n$ converges as well. Finally, as $\sum |a_n|$ converges absolutely, by **Absolute Convergence Test (Theorem 2.7.6)**, $\sum a_n$ conveges absolutely as well.

$\square$

2.8.1 From **Section 2.1**, we know

$$(a_{ij}) = \begin{cases} \frac{1}{2^{j-i}} \text{ if } j > i \\ -1 \text{ if } j = i \\ 0 \text{ if } j < i \end{cases}$$

Now, if we set $j = 1$, it is easy to see that $a_{i1} = (-1, \frac{1}{2}, \frac{1}{4}, \dots)$ and excluding the first term (-1), a_{i1} is a sequence whose sum converges. Recall that the formula for the sum is $\frac{b_n q - b_1}{q-1}$ where b_1 is the first term, b_n is the last term, and q is the quotient/ratio (current element over previous element). Thus, we get:

$$\sum_{i=1}^{k} a_{i1} = -1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^{n-1}} = -1+\frac{\frac{1}{2^{n-1}} \times \frac{1}{2} - \frac{1}{2}}{\frac{1}{2}-1} = -1+\frac{\frac{1}{2^{n-1}} \times \frac{1}{2} - \frac{1}{2}}{-\frac{1}{2}} = -1+1-\frac{1}{2^{n-1}} = -\frac{1}{2^{n-1}}$$

In general, $\forall j < k$, we have:

$$\sum_{i=1}^{k} a_{ij} = -1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^{n-j}} = -1+\frac{\frac{1}{2^{n-j}} \times \frac{1}{2} - \frac{1}{2}}{\frac{1}{2}-1} = -1+\frac{\frac{1}{2^{n-j}} \times \frac{1}{2} - \frac{1}{2}}{-\frac{1}{2}} = -1+1-\frac{1}{2^{n-j}} = -\frac{1}{2^{n-j}}$$

Finally, we get:

$$s_{nn} = \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij} = -1 + \sum_{j=1}^{n-1} -\frac{1}{2^{n-j}} = -1 - (1 - \frac{1}{2^{n-1}}) = -2 + \frac{1}{2^{n-1}}$$

Now, since $\lim_{n\to\infty} \frac{1}{2^{n-1}} = 0$, we get that $\sum_{i,j}^{\infty} a_{ij} = \lim_{n\to\infty} s_{nn} = \lim n \to \infty - 2 + \frac{1}{2^{n-1}} = -2 + \lim_{n\to\infty} \frac{1}{2^{n-1}} = -2 + 0 = -2$. Hence, we get $s_{nn} = -2$.

The iterated sums would give us $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = -1 + \sum_{i=2}^{\infty} -\frac{1}{2^{i-1}}$. Now, recall that the sum can be computed by the infinite geometric series formula $\frac{b_1}{q-1}$. Finally, we get $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = -1 + \frac{-\frac{1}{2}}{-\frac{1}{2}} = -1+1 = 0$. Similarly, $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}(-1+\sum_{i=1}^{\infty} \frac{1}{2^i}) = \sum_{j=1}^{\infty} 0 = 0$.