Homework №5

Author: David Oniani
Instructor: Tommy Occhipinti

September 23, 2018

Section 3.2

23. Prove that if x and y are integers and $xy - 1$ is even then x and y are odd.

    Let's prove it by contrapositive. Contrapositive of the initial statement (which is equivalent to the initial statement) is:

    If x is even or y is even, then $xy - 1$ is odd.

    If x is even or y is even, xy is even. Then we can write that $xy = 2k$ where $k \in \mathbb{Z}$. Then, $xy - 1 = 2k - 1 = 2(k - 1) + 1$ where $k \in \mathbb{Z}$. Now, let $t = l - 1$ where $l \in \mathbb{Z}$ and we get $xy - 1 = 2t + 1$. Thus, $xy - 1$ is odd.

    Q.E.D.

24. Prove that if x and y are real numbers whose mean is m then at least one of x and y is $\geq m$.

    Suppose, for the sake of contradiction, that x and y are both $< m$. Then,

    $$x < m$$

    $$y < m$$

    By adding the inequalities, we get:

    $$x + y < 2m$$

    And finally,

    $$\frac{x + y}{2} < m$$

    which contradicts the initial statement that the mean of x and y is m.

    Q.E.D.

25. Suppose S is a set of 250 distinct real numbers whose mean is 4. Must there exists $x \in S$ such that $x > 4$? Be sure to prove your answer.

Yes. Let's prove it!
Suppose, for the sake of contradiction, that all elements of S are ≤ 4. Then the sum of all the elements will be less ≤ 1000 with equality happening only when all the members of the set are equal to 4 which contradict the initial statement that S is a set of 250 distinct elements. Thus, only one of the elements of S is allowed to be equal to 4. Finally, we get two cases:

1. All 250 elements of S are less than 4.
2. 249 elements of S are less than 4 and one is equal to 4.

If all 250 elements of S are less than 4, then their sum is less than $4 \times 250 = 1000$ and their mean is less than $1000/250 = 4$ which contradicts the initial statement that the mean of all elements of S is 4.
If 249 elements of S are less than 4 and one is equal to 4, then the sum of 249 elements which are less than 4 is less than $249 \times 4 = 996$. Then let this sum of 249 numbers be equal to $996 - k$ where $k > 0$. Then the sum of all the elements including the one which equals 4 is:

$$996 - k + 4 = 1000 - k \text{ where } k > 0$$

Using the fact above, we get that the mean of all the elements of S is $(1000 - k)/250$ where $k > 0$. And finally, we get:

$$\frac{1000 - k}{250} = 4 - \frac{k}{250} \text{ where } k > 0$$

And $4 - \frac{k}{250}$ where $k > 0$ is clearly less than 4 which contradicts the initial claim that the mean of all elements of S is 4.

Q.E.D.

26. Suppose $a, b, c \in \mathbb{Z}$ and $a^2 + b^2 = c^2$. Prove that at least one of a and b is even.

Suppose, for the sake of contradiction, that both a and b are odd. Then, we can write $a = 2k - 1$ and $b = 2l - 1$ where $k, l \in \mathbb{Z}$. Then, we have:

$$a^2 + b^2 = 4k^2 - 4k + 1 + 4l^2 - 4l + 1 = 4k^2 + 4l^2 - 4l - 4k + 2 =$$

$$= 2 \times (2k^2 + 2l^2 - 2l - 2k + 1)$$

Now, it's easy to see that $a^2 + b^2$ is the multiplication of an even and odd integers (2 is even and $2k^2 + 2l^2 - 2l - 2k + 1$ is odd). $2k^2 + 2l^2 - 2l - 2k + 1$ is odd since $2k^2 + 2l^2 - 2l - 2k + 1 = 2 \times (k^2 + l^2 - l - k) + 1$ and if we let $t = k^2 + l^2 - l - k$ where $t \in \mathbb{Z}$ (since $k^2 + l^2 - l - k \in \mathbb{Z}$), then we have that $2k^2 + 2l^2 - 2l - 2k + 1 = 2t + 1$ which is an even number plus one which is always odd. Finally, we conclude that 2 is only once in the number that is supposed to be a perfect square as $2k^2 + 2l^2 - 2l - 2k + 1$ is odd and is not a multiple of 2 which means that $a^2 + b^2$ is not a perfect square which contradicts the initial claim that the sum $a^2 + b^2$ is the perfect square.

Q.E.D.

27. Prove that if $x, y \in \mathbb{R}^+$, then $x + y \geq 2\sqrt{xy}$.

Suppose, for the sake of contradiction, that $x + y < 2\sqrt{xy}$. Then, since $x, y \in \mathbb{R}^+$, we have:

$$x + y < 2\sqrt{xy} \tag{1}$$
$$x^2 + y^2 + 2xy < 4xy \tag{2}$$
$$x^2 + y^2 + 2xy - 4xy < 0 \tag{3}$$
$$x^2 + y^2 - 2xy < 0 \tag{4}$$
$$(x - y)^2 < 0 \tag{5}$$

Thus, we got that $(x-y)^2 < 0$ which is false since the square of a number is always ≥ 0. Finally, since by assuming that $x+y < 2\sqrt{xy}$ where $x, y \in \mathbb{R}^+$, we basically got the nonsensical inequality $(x - y)^2 < 0$, something has to be wrong with this assumption and we got that if $x, y \in \mathbb{R}^+$, then $x + y \geq 2\sqrt{xy}$

Q.E.D.

28. Prove that if n is an integer, there exist three consecutive integers that sum to n if and only if n is a multiple of 3.

Let's first prove that if n is not a multiple of 3, one cannot find three consecutive integers with the property that they sum to n.

(a) Suppose, for the sake of contradiction, that n is not a multiple of 3. Then let's define three consecutive integers, $m, m + 1$ and $m + 2$, where $m \in \mathbb{Z}$. Then we have:

$$m + m + 1 + m + 2 = 3m + 3 = 3 \times (m + 1)$$

Thus, we got that the sum of three consecutive integers is a multiple of 3 which contradicts the statement that n is not a multiple of 3.

Now, lets prove the second half of the problem. Let's show that if three consecutive integers sum to n, then n is a multiple of 3.

(b) Let $m, m + 1, m + 2$ where $m \in \mathbb{Z}$ be three consecutive integers. We have:

$$n = m + m + 1 + m + 2 = 3m + 3 = 3 \times (m + 1)$$

Thus, we got that n is a multiple of 3 which proves the iff.

Q.E.D.

29. A subset S of $\mathbb{R}$ has the property that for all $x \in \mathbb{R}$ there exists $y \in S$ such that $|x - y| < 1$. Prove that S is infinite.

Suppose, for the sake of contradiction, that S is finite. Inequality, $|x-y| < 1$ can be transformed into the following system:

$$\begin{cases} x - y < 1 \\ x - y > -1 \end{cases}$$

And from the system above, we get the following system:

$$\begin{cases} y > x - 1 \\ y < x + 1 \end{cases}$$

Hence, we know that y is in the open interval $(x-1, x+1)$. Now, since we also know that $x \in \mathbb{R}$, interval $(x-1, x+1)$ has infinitely many elements in it which contradicts our assumption that S is finite.

Q.E.D.

30. A subset S of $\mathbb{Z}$ is called **non-differential** if for every $x, y \in S$ we have $x - y \notin S$. Here are some statements about non-differential sets. Decide which statements are true and which are false, and provide a proof or counterexample for each as appropriate.

Before going right into the proof, note that in any set we can do self-subtration (e.g, if the set is $\{1, 3\}$), we can write $1 - 1 = 0$. HOWEVER, we are not going to consider those trivial cases. We are going to consider it if and only if element 0 is in the set (because all the self-subtractions are zero and it is only the case when the element 0 is in the set when such subtractions make sense in proving or disproving something).

(a) Every non-differential set is finite.

This is false. Counterexample:
Let $S = \{1, 3, 5, 7, 9, 11...\}$ thus, S is a set of all positive odd integers. Then we know that for every x, y, $x - y$ is even. But all the members of S are odd. Thus, For every $x, y \in S$, $x - y \notin S$ and S is an infinite set which also turns out to be non-differential and the initial statement is false.

(b) The intersection of two non-differential sets is non-differential.

This is true. Let's prove it.
Let $x, y \in A \cap B$. Then, since $x, y \in A$, $x - y \notin A$ as well as $x - y \notin A \cap B$.

Q.E.D.

(c) The union of two non-differential sets is non-differential.

This is false. Counterexample:
Let $A = \{1, 3\}$ then A is non-differential since $1 - 3 \notin A$ and $3 - 1 \notin A$. Now, let $B = \{1, 4\}$, then B is non-differential too as $1 - 4 \notin B$ and $4 - 1 \notin B$. Finally, we get $A \cup B = \{1, 3, 4\}$ which is <u>NOT</u> non-differential because $4 - 3 = 1 \in A \cup B$.

(d) No non-differential set contains the element 0.

It's true.
For a set to be non-differential there should be no x, y such that $x - y \in S$. For the sake of contradiction, suppose that we have a non-differential set A such that $0 \in A$. If A has more than one elements, let the other element (any element which is not 0) be k. Then we get $k - 0 = k \in A$ which contradicts

the initial statement that A is non-differential as we found two elements $x = 0$ and $y = k$ such that $x - y \in A$. If A has only one element which is 0, then it is <u>NOT</u> non-differential anyway, because $0 - 0 = 0 \in A$. Hence, no non-differential set contains element 0.

(e) Every subset of a non-differential set is non-differential.

It's true.

Suppose we have two sets, A and B such that $B \subseteq A$ and A is non-differential. Let $x \in B$, then we know that there exists no y in A such that $x - y \in A$. Since such y does not exist in A, it does not exist in B is as well since it is the subset of A.

Q.E.D.

(f) There is no non-differential set with exactly 5 elements.

It's false. Counterexample:

Let $A = \{1, 3, 8, 19, 50\}$, then we have:

$$1 - 3 = -2 \notin A$$
$$3 - 1 = 2 \notin A$$
$$3 - 8 = -5 \notin A$$
$$8 - 3 = 5 \notin A$$
$$8 - 19 = -11 \notin A$$
$$19 - 8 = 11 \notin A$$
$$19 - 50 = -31 \notin A$$
$$50 - 19 = 31 \notin A$$

(g) If S is non-differential, so is $\mathbb{Z} - S$.

It's false. Counterexample:

Let $A = \{1, 3\}$. A is non-differential since $1 - 3 = -2 \notin A$ and $3 - 1 = 2 \notin A$. Then we know that $Z - A$ would include numbers $7, 8, 15$. But $15 - 8 = 7 \in Z - A$ which is not non-differential.

(h) If S is a non-differential set, then so is the $S_{+3} = \{x + 3 \mid x \in S\}$.

It's false. Counterexample:

Let $A = \{1, 3, 8, 19, 50\}$, then A is non-differential since:

$$1 - 3 = -2 \notin A$$
$$3 - 1 = 2 \notin A$$
$$3 - 8 = -5 \notin A$$
$$8 - 3 = 5 \notin A$$
$$8 - 19 = -11 \notin A$$
$$19 - 8 = 11 \notin A$$
$$19 - 50 = -31 \notin A$$
$$50 - 19 = 31 \notin A$$

$A_{+3} = \{4, 7, 11, 22, 53\}$. Now, notice that $22 - 11 = 11 \in A_{+3}$ thus, we found two elements $x = 22$ and $y = 11$ such that $x - y \in A$ and so A is <u>NOT</u> non-differential. Thus, the initial statement is false.

31. A subset A of $\mathbb{R}$ is called **cofinite** if $\mathbb{R} - A$ is finite. Here are some statements about cofinite sets. Decide which statements are true and which are false, and provide a proof or counterexample for each as appropriate

Before jumping in the proofs, let's make a little note. If A is cofinite, it is some subset of $\mathbb{R}$. Then, we can represent it as $A = \mathbb{R} - F$ where F is some finite set.
Why finite?
If F be infinite, then $\mathbb{R} - (\mathbb{R} - F) = F$ is also infinite and this contradicts the fact that A is cofinite. Thus, we know (and will use) the fact that any cofinite set A can be represented as $\mathbb{R} - F$ where F is a finite set.

(a) If $A \subseteq B$ and B is cofinite then A is cofinite.

It's false. Counterexample:
Let $B = \mathbb{R} - \{0, 1\}$. Then B is cofinite as $\mathbb{R} - B = \{0, 1\}$. Now let $A = \{-1, -2\}$, then $A \subseteq B$, however, $\mathbb{R} - A$ is not finite as R has an infinite number of elements and subtracting only a finite number of elements (2 elements) still leaves it will infinitely many.

(b) There exist two cofinite sets A and B with the property that $A \cap B = \emptyset$.

It's false. Suppose, for the sake of contradiction, that A and B are cofinite sets. Then we know that both A and B are of the form $\mathbb{R} - F$ where F is some finite set (if F is infinite, $\mathbb{R} - (\mathbb{R} - F) = F$ and the set is NOT cofinite). Let $A = \mathbb{R} - C$ and $B = \mathbb{R} - D$. Then we know that both A and B contain sets $\mathbb{R} - C - D$. Thus $\mathbb{R} - C - D \subseteq A \cap B$ which is never an $\emptyset$ since sets C and D are finite and $\mathbb{R} - C - D$ is infinite.

(c) If A is cofinite, then A contains a positive integer.

It's true.
We know that $\mathbb{R}$ contains all the positive integers. For A to be cofinite it $\mathbb{R} - A$ should be finite thus, it should have a finite number of elements. If A has no positive integers, it means that $\mathbb{R} - A$ is infinite since it contain at least all the positive integers. Thus, A is not cofinite and we've encountered a contradiction. And finally, the statement if A is cofinite, then A contains a positive integer is true.

(d) The intersection of two cofinite sets is cofinite.

It's true.
Suppose $A = \mathbb{R} - F$ and $B = \mathbb{R} - G$ are cofinite sets (F and G are finite). Then, their intersection will be:

$$A \cap B = \mathbb{R} - F - G = \mathbb{R} - (F \cup G)$$

And we get:

$$\mathbb{R} - (\mathbb{R} - (F \cup G)) = F \cup G$$

Now, since F and G are finite, $F \cup G$ is also finite and we proved that the intersection of the two cofinite sets is cofinite.

Q.E.D.

(e) The union of two cofinite sets is cofinite.

It's true. Suppose $A = \mathbb{R} - F$ and $B = \mathbb{R} - G$ are cofinite sets (F and G are finite). Then, their union will be:

$$A \cup B = (\mathbb{R} - F) \cup (\mathbb{R} - G) = \mathbb{R} - (F \cap G)$$

And then we get:

$$\mathbb{R} - (\mathbb{R} - (F \cap G)) = F \cap G$$

Now, since F and G are finite, so is $F \cap G$ and the union of two cofinite sets is cofinite.

Q.E.D.

(f) If A and B are cofinite then $A - B$ is finite.

It's true.
Suppose $A = \mathbb{R} - F$ and $B = \mathbb{R} - G$ are cofinite sets (F and G are finite). Then, $A - B = (\mathbb{R} - F) - (\mathbb{R} - G) = G - F$. Now, since F and G are finite, $G - F$ is finite (even if $F = G$, empty set is considered finite with the cardinality zero).

Q.E.D.

(e) Every cofinite set is infinite.

It's true. Let A be a cofinite set. Then, we know that it is some subset of $\mathbb{R}$ and we can write it as $A = \mathbb{R} - F$ where F is some set. Then, we have:

$$\mathbb{R} - (\mathbb{R} - F) = F$$

Now, since A is cofinite, then F has to be finite by the definition of the cofinite set. Then we get that $\mathbb{R} - F$ is infinite since F is finite and $\mathbb{R}$ minus any finite set is always infinite.

Q.E.D.

32. We say that a subset S of $\mathbb{Z}$ is angled if for every $x, y, z \in S$ we have $x + y > z$.

(a)

$$S = \{3\} \text{ since } 3 + 3 > 3$$
$$S = \{3, 4\} \text{ since } 3 + 4 > 3,\ 3 + 4 > 4,\ 3 + 3 > 3, \text{ and } 4 + 4 > 4$$

$$S = \{3, 4, 5\}$$
$$\text{since } 4 + 5 > 3,\ 3 + 5 > 4,\ 3 + 4 > 5,\ 3 + 3 > 3,\ 4 + 4 > 4, \text{ and } 5 + 5 > 5$$

(b)

$$S = \{3, 2, 7\} \text{ as } 3 + 2 < 7$$
$$S = \{12, -13, 29, 47\} \text{ as 12 - 13} < 29$$
$$S = \{1, 2, 3, 4, 5\} \text{ as } 1 + 2 < 4$$

(c) Can 0 be an element of an angled set?

No.

If the set contains element 0, then $0 + 0 = 0$ thus, $0 + 0 \not> 0$ and the set is not angled.

(d) Prove or disprove: If S is angled and $x \in S$ then $x > 0$.

It's true so let's prove it.

Suppose, for the sake of contradiction, that $x \leq 0 \in S$ where S is angled. Then, we must have that $x + x > 2x$. But if $x \leq 0$, $x + x$ is always less than or equal to $2x$. Thus, we've encountered a contradiction and if S is angled and $x \in S$ then $x > 0$.

Q.E.D.

(e) Prove or disprove: If S is angled then there exists $c \in \mathbb{Z}$ such that for every $x \in S$ we have $x < c$.

It's true.

Suppose, for the sake of contradiction, that we cannot find $c \in \mathbb{Z}$ such that for every $x \in S$ we have $x < c$. Then, it is clearly the case that S contains the biggest element of $\mathbb{Z}$. BUT, unfortunately, there is no "BIGGEST" element in $\mathbb{Z}$ as if we pick some element x to be the biggest, we can always take $x + 1$ which will be bigger than x. Thus, we've encountered a contradiction and if S is angled then there exists $c \in \mathbb{Z}$ such that for every $x \in S$ we have $x < c$.

Q.E.D.

(f) Prove or disprove: There exists $c \in \mathbb{Z}^+$ such that if S is angled and $x \in S$ then we have $x < c$.

It's true.

In (d), we proved that if S in angled and $x \in S$, then $x > 0$. Let x be an element of S, then we can set $c = x + 1$ (c is positive since $x > 0$) which will be greater than x.

(g) Prove or disprove: Every angled set is finite.

It's true.

Suppose, for the sake of contradiction, that S is an infinite angled set. Then, for every $x, y, z, x + y > z$. We also know, from the previous proofs, that all elements of S are positive. Now, since all elements of S have to be positive and it's infinite, we know that there is no biggest element in the set. Then, if we take two fixed elements x and y, we can certainly find z (we can change z until we get the value bigger than $x+y$). such that $z \geq x+y$, since the set is infinite and thus, we've encountered a contradiction. Hence, every angled set is finite.

(h) Prove or disprove: For every $n \in \mathbb{Z}^+$ there exists an angled set S such that $|S| = n$.

It's true.

Let's construct a set in the following way:

$$A = \{a_1, a_2, a_3, ...a_n\} \text{ where } a_1 \leq a_2 \leq a_3 \; ... \leq a_n \text{ and}$$
$$a_1, a_2, a_3... \text{ are consecutive positive integers.}$$

Then, our primary concern is that $2a_1 > a_n$. If $a_1, a_2, a_3...$ are conse positive integers, then $a_n = a_1 + n - 1$ and we have:

$$2a_1 > a_1 + n - 1$$

and finally, we get:

$$a_1 > n - 1$$

Thus, the set $A = \{n, n+1, n+2, n+3, n+4, ... \; 2n\}$ is now angled because for every $x, y, z \in \mathbb{Z}, x + y > z$.

Q.E.D.

33. Use quantifiers to precisely write down, in mathematical language, the definition given for $\sum_{n=0}^{\infty} a_n = X$ outlined in the video at the 4:00 minute mark.

    This sum means that when we generate a list of numbers by cutting off the sums at finite points:

$$\begin{aligned} s_0 &= a_0 \\ s_1 &= a_0 + a_1 \\ s_2 &= a_0 + a_1 + a_2 \\ s_3 &= a_0 + a_1 + a_2 + a_3 \\ s_4 &= a_0 + a_1 + a_2 + a_3 + a_4 \\ s_5 &= a_0 + a_1 + a_2 + a_3 + a_4 + a_5 \\ s_6 &= a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 \\ s_7 &= a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 \\ s_8 &= a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 \\ s_9 &= a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 \\ &\cdots \end{aligned}$$

    Then, these sums approach X in the sense that no matter how small is the distance, at some point down list, all the numbers start falling within this distance of X. Thus, the further we proceed with the list $s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_10, s_11...$ the more these numbers approach X and the smaller the distance between X and the sum.

34. Explain why it makes sense, in a way, for $1 - 1 + 1 - 1 + 1 - 1 + ...$ to equal $1/2$, as is suggested in the video at about 6:45. Is this what you learned in Calculus II?

    We can cut the line of the length 1 in two pieces with proportions $(1 - p)$ and p. Then, we can cut p in two with same proportions

$((1-p)/p)$ and continue doing it infinite. Finally, we can sum up the pieces to get the equation:

$$(1-p) + p(1-p) + p^2(1-p) + p^3(1-p) + ... = 1$$

Now, we can divide both sides by $1-p$ and we get:

$$1 + p + p^2 + p^3 + ... = \frac{1}{1-p}$$

If we plug in, $p = -1$, we get:

$$1 - 1 + 1 - 1 + 1 - 1 + 1... = \frac{1}{2}$$

which seems to be true. Unfortunately, it is not true. One can calculate the sum this way if and only if $-1 < p < 1$ meaning that one cannot simply plug $p = -1$ or $p = 12$ and get the sum for the infinite geometric series. This sum is sometimes 1 and sometimes -1. That's what Calc II says.

35. In the sense of distance discussed at 12:45, how far apart are 5 and 13? How about -1 and -15?

    5 and 13 are 1/8 apart from each other.
    -1 and -15 are 1/16 apart from each other.

Bookwork

1. Let a be an integer. Prove: If a^2 is even, then a is even.

   Let's prove the contrapositive. The contrapositive of the initial statement is: If a is not even, a^2 is not even (where a is an integer). Then suppose an integer a is not even, thus is odd. Since a is odd, we can write $a = 2k + 1$ where $k \in \mathbb{Z}$ and we have:

   $$a^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$

   Now, let $l = 2k^2 + 2k$ where $l \in \mathbb{Z}$ (since $2k^2 + 2k \in \mathbb{Z}$). And finally, we have:

   $$a^2 = (2k+1)^2 = 4k^2+4k+1 = 2(2k^2+2k)+1 = 2l+1 \text{ where } l \in \mathbb{Z}$$

   Thus, we got that a^2 is of the form $2l + 1$ which means that a^2 is odd thus, not even.

   Q.E.D.

5(a) Prove that $\sqrt{3} \notin Q$.

   Suppose, for the sake of contradiction, that $\sqrt{3} \in Q$. Then, we can represent $\sqrt{3}$ as $\frac{a}{b}$ where $a, b \in \mathbb{Z}$. Let's assume that a and b do not have any common factors and if they do, let's cancel them out and write already cancelled-out form. Thus, assume that the fraction $\frac{a}{b}$ is already cancelled out and a and b do not have common factors. Then we have:

   $$\frac{a}{b} = \sqrt{3}$$
   $$\frac{a^2}{b^2} = 3$$
   $$a^2 = 3b^2$$

   Hence, we got that a^2 is divisible by 3 which means that a is also divisible by 3. Now, let $a = 3k$ where $k \in \mathbb{Z}$. After substituting a, we get:

   $$3b^2 = (3k)^2 = 9k^2$$
   $$b^2 = 3k^2$$

   Thus, we got that b^2 is divisble by 3 which means that b is also divisble by 3. However, we assumed that a and b had no common factors and now, we encounter the contradiction.

   Q.E.D.

6(d) Prove that $r + \sqrt{2} \notin \mathbb{Q}$ where $r \in \mathbb{Q}$.

   Suppose, for the sake of contradiction, that $r + \sqrt{2} \in \mathbb{Q}$. Now, since $r, r + \sqrt{2} \in \mathbb{Q}$, we can write $r = \frac{a}{b}$ and $r + \sqrt{2} = \frac{c}{d}$ where $a, b, c, d \in \mathbb{Z}$. Then we have:

   $$\frac{a}{b} + \sqrt{2} = \frac{c}{d}$$

$$\frac{bc - ad}{bd} = \sqrt{2}$$

Now, let $x = bc - ad$ and $y = bd$ (where $x, y \in \mathbb{Z}$ as $bc - ad, bd \in \mathbb{Z}$). And finally, we got that $\frac{x}{y} = \sqrt{2}$ which is false since $\sqrt{2}$ is irrational and cannot be represented as a fraction of two even integers.

Q.E.D.

Proof that $\sqrt{2}$ is irrational:

Suppose, for the sake of contradiction, that $\sqrt{2} \in Q$. Then, we can represent $\sqrt{2}$ as $\frac{a}{b}$ where $a, b \in \mathbb{Z}$. Let's assume that a and b do not have any common factors and if they do, let's cancel them out and write already cancelled-out form. Thus, assume that the fraction $\frac{a}{b}$ is already cancelled out and a and b do not have common factors. Then we have:

$$\frac{a}{b} = \sqrt{2}$$
$$\frac{a^2}{b^2} = 2$$
$$a^2 = 2b^2$$

Hence, we got that a^2 is divisible by 3 which means that a is also divisible by 2. Now, let $a = 2k$ where $k \in \mathbb{Z}$. After substituting a, we get:

$$2b^2 = (2k)^2 = 4k^2$$
$$b^2 = 2k^2$$

Thus, we got that b^2 is divisble by 2 which means that b is also divisble by 2. However, we assumed that a and b had no common factors and now, we encounter the contradiction.

Q.E.D.

9. Prove: If x is irrational, then $\sqrt{x}$ is irrational.

   Suppose, for the sake of contradiction, that x is irrational and $\sqrt{x}$ is rational. Then we can represent $\sqrt{x}$ as $\frac{a}{b}$ where $a, b \in \mathbb{Z}$ and have no common factors. Then, by squaring both sides, we get:

   $$\frac{a^2}{b^2} = x$$

   This, now means that x can be represented as the fraction of two integers as if a, b are integers, so are a^2, b^2 and we've encountered a contradiction. Thus, if x is irrational, then $\sqrt{x}$ is irrational.

   Q.E.D.

11. Prove: $\sqrt[4]{2} \notin \mathbb{Q}$.

Suppose, for the sake of contradiction, that $\sqrt[4]{2} \in \mathbb{Q}$. Then we can represent $\sqrt[4]{2}$ as $\frac{a}{b}$ where $a, b \in \mathbb{Z}$ and have no common factors. Then, by squaring both sides, we get:

$$\frac{a^2}{b^2} = \sqrt{2}$$

This, however, means that $\sqrt{2}$ can be represented as a fraction of two integers (because a, b are integers, so are a^2, b^2) which is clearly impossible since we've already proven that $\sqrt{2}$ is irrational, thus cannot be represented as a fraction of two integers. Hence, we've encountered a contradiction and $\sqrt[4]{2} \notin \mathbb{Q}$.

Q.E.D.