Topology

Author: David Oniani
Instructor: Dr. Eric Westlund

January 20, 2019

Assignment №4

Section 23

6. Let $A \subset X$. Show that if C is a connected subspace of X that intersects both A and $X - A$, then C intersects Bd A.

   At first, recall that Bd $A = \overline{A} \cap \overline{X - A}$. Now, suppose, for the sake of contradiction, that C is connected and $C \cap \text{Bd } A = \varnothing$. Consider two sets $U_1 = C \cap \overline{A}$ and $U_2 = C \cap \overline{X - A}$. Now, since $C \cap A \subset U_1$ and $C \cap \overline{X - A} \subset U_2$, it follows from our assumptions that U_1 and U_2 are two nonempty subsets of C. Notice that $C = U_1 \cup U_2$ with U_1, U_2 being both open and closed subsets of C. However, $U_1 \cap U_2 = C \cap \overline{A} \cap \overline{X - A} = C \cap \text{Bd } A = \varnothing$ which means that C is disconnected and contradicts the fact that C is connected. Finally, we have reached the contradiction and C intersects Bd A. $\square$

Section 24

1. (c) Show that $\mathbb{R}^n$ and $\mathbb{R}$ are not homeomorphic if $n > 1$.

   Suppose, for the sake of contradiction, that for $n > 1$, $\mathbb{R}^n$ and $\mathbb{R}$ are homeomorphic. Then, by the definition of homeomorphism, there exists a function $f : \mathbb{R} \to \mathbb{R}^n$. Consider $f|_{\mathbb{R}^n - \{0\}} : \mathbb{R}^n - \{0\} \to \mathbb{R} - \{f(0)\}$, $f|_{\mathbb{R}^n - \{0\}}$ is a restriction of f and hence is a homeomorphism. Now, notice that $\mathbb{R}^n - \{0\}$ is a connected space, however, $\mathbb{R} - \{f(0)\}$ is not a connected space and we have reached the contradiction since $f|_{\mathbb{R}^n - \{0\}}$ is a homeomorphism. Finally, we have that for $n > 1$, $\mathbb{R}^n$ and $\mathbb{R}$ are not homeomorphic. In short, by taking away 0, we make $\mathbb{R}$ disconnected, but taking away any point from $\mathbb{R}^n$ leaves it connected. $\square$

3. Let $f : X \to X$ be continuous. Show that if $X = [0, 1]$, there is a point x such that $f(x) = x$. The point x is called a ***fixed point*** of f. What happens if X equals $[0, 1)$ or $(0, 1)$?

In the order topology, X is an ordered set and connected space. Let $a, b \in X$. Let's now pick a midpoint x_1 between $f(a)$ and $f(b)$. Then, according to **Theorem 24.3**, $\exists c_1 \in [a, b]$ such that $f(c_1) = x_1$. Now, if $c_1 = x_1$, we found the fixed point of f and if $c_1 \neq x_1$, we pick the midpoint x_2 between c_1 and x_1. Then $\exists c_2 \in [c_1, x_1]$ or $c_2 \in [x_1, c_1]$ (depending on whether $x_1 > c_1$ or $c_1 > x_1$) such that $f(c_2) = x_2$. Now, if $c_2 = x_2$ then we have found the point and if not we countinue this way. Thinking of computer science, this is a recursive approach to the problem (though, I think recursion comes from math anyway, right?). The simply outline of the algorithm would look something like this:

if $x_n = c_n$ **then**
  hooray! we have found a point! we are done, return the point!
else
  Consider the midpoint between x_n and c_n
end if

After repeating this process, we will end up with two convergent series: $c_1, c_2, c_3, ...$ and $x_1, x_2, x_3, ...$ with the property that $|c_n - x_n| \to 0$ as $n \to \infty$. In others words, we have $\lim_{n\to\infty} |c_n - x_n| \to 0$. This is due to the continuity of f on X. Therefore, we have $|x_n - f(x_n)| = |f(c_n) - f(x_n)| \to 0$ from which we get that $f(x_n) = (x_n)$. $\square$

This fact/theorem does not hold for intervals $[0, 1)$ and $(0, 1)$. This is due to f not being uniformly continuous on these intervals. For instance, a function $f(x) = \dfrac{x+2}{3}$ has a fixed point $x = 1$, but it has no fixed points on intervals $[0, 1)$ or $(0, 1)$.

8. (a) Is a product of path-connected spaces necessarily path-connected?

Yes. Suppose that X and Y are path-connected. Let $x_1 \times x_2, y_1 \times y_2 \in X \times Y$. Notice that $X \times y_1$ is homeomorphic to X and thus, is path-connected. Therefore, there exists a continuous function $f : [0, 1] \to X \times y_1$ with $f(0) = x_1 \times y_1$ and $f(1) = x_2 \times y_2$. Besides, $x_2 \times Y$ is homeomorphic to Y and thus, is path-connected. Therefore, there exists a continuous function g such that $g : [0, 1] \to x_2 \times Y$ with $g(0) = x_2 \times y_1$ and $g(1) = x_2 \times y_2$. Let's now define a function h in the following way:

$$h(x) = \begin{cases} f(\frac{x}{2}) \text{ if } 0 \leq x \leq \frac{1}{2} \\ f(\frac{x}{2} + \frac{1}{2}) \text{ if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

The according to the **Theorem 18.3 (The pasting lemma)**, h is continuous. Besides, $h(0) = f(\frac{0}{2}) = f(0) = x_1 \times y_1$ and $h(1) = g(\frac{1}{2} + \frac{1}{2}) = g(1) = x_2 \times y_2$. Therefore, h is a path from $x_1 \times y_1$ to $x_2 \times y_2$ and because $x_1 \times y_1$ and $x_2 \times y_2$ are arbitrary, we have that $X \times Y$ is path-connected. $\square$

(b) If $A \subset X$ and A is path-connected, is $\bar{A}$ necessarily path-connected?

No. For instance, ***topologist's sine curve***.

(c) If $f : X \to Y$ is continuous and X is path-connected, is $f(X)$ necessarily path-connected?

Yes, this is due to the fact that the composition of continuous functions is always continuous.

(d) If $\{A_\alpha\}$ is a collection of path-connected subspaces of X and if $\bigcap A_\alpha \neq \varnothing$, is $\bigcup A_\alpha$ necessarily path-connected?

Yes. Let $x, y \in \bigcup X_\alpha$ and let $z \in \bigcap A_\alpha$. Then for some b and c, $x \in A_b$ and $y \in A_c$. Besides, $z \in A_b$ and $z \in A_c$. Now, because A_b is path-connected, there is a path f from x to z. On the other hand, because A_c is path-connected, there is a path g from z to y. Now, according to the **Theorem 18.3 (The pasting lemma)**, we can glue these two paths together and make a path h from x to y. $\square$

Section 26

5. Let A and B be disjoint compact subspaces of the Hausdorff space X. Show that there exist disjoint open sets U and V containing A and B, respectively.

Note that since A and B are compact subspaces of a Hausdorff space, they are closed. Then $X - A$ and $X - B$ are open. Since A and B are disjoint, $U = X - B$ contains A and $V = X - A$ contains B. $\square$

Section 27

2. Let X be a metric space with metric d; let $A \subset X$ be nonempty.

(a) Show that $d(x, A) = 0$ if and only if $x \in \bar{A}$.

The function of x described in the problem is continuous, so its set of zeros is a closed set. This closed set contains A therefore, it also contains $\bar{A}$. On the other hand, if $x \notin \bar{A}$ then $\exists \epsilon > 0$ with $U_\epsilon(x) \subset X - \bar{A}$, and in this case it follows that $d(x, A) \geq \epsilon > 0$. $\square$

(b) Show that if A is compact, $d(x, A) = d(x, a)$ for some $a \in A$.

The function $f(a) = d(x, a)$ is continuous and $d(x, A)$ is the greatest lower bound for its set of values. Now, because of the compactness of A, this greatest lower bound is a minimum value that is realized at some point of A (**Theorem 27.4 (Extreme value theorem)**). $\square$

(c) Define the ϵ-neighborhood of A in X to be the set

$$U(A, \epsilon) = \{x \mid d(x, A) < \epsilon\}.$$

Show that $U(A, \epsilon)$ equals to the union of the open balls $B_d(a, \epsilon)$ for $a \in A$.

Note that the ϵ-neighborhood of A in X corresponds to all points in X that are within a distance ϵ of some point in A. It includes all of A. Then $x \in U(A, k)$ if and only if $d(x, a) < k$ for some $a \in A$. It follows that $x \in \bigcup_{a \in A} B(a, k)$. $\square$

6. Let A_0 be the closed interval $[0, 1]$ in $\mathbb{R}$. Let A_1 be the set obtained from A_0 by deleting its "middle third" $(\frac{1}{3}, \frac{2}{3}$. Let A_2 be the set obtained from A_1 by deleting "middle thirds" $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$. In general, define A_n by the equation

$$A_n = A_{n-1} - \bigcup_{k=0}^{\infty} \left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n} \right).$$

The intersection

$$C = \bigcap_{n \in \mathbb{Z}^+} A_n$$

is called the ***Cantor set***; it is a subspace of $[0, 1]$.

(a) Show that C is totally disconnected.

Suppose, for the sake of contradiction, that C is not totally disconnected. Then $\exists [x, y] \subset C$. Let $K \in \mathbb{Z}^+$ with

$$K > \log_3 \left(\frac{1}{y - x} \right).$$

Then for $k > K$, $\frac{1}{3^k} < y - x$. But now since $C = \bigcap A_j$, C must contain intervals (if it contains any intervals at all) with length less than $\dfrac{1}{3^k}$. Therefore, we have reached the contradiction and C contains no intervals. Hence, C it is totally disconnected. $\square$

(b) Show that C is compact.

Notice that C is closed and bounded. Therefore, according to the **Theorem 27.3**, C is compact. $\square$

(c) Show that each set A_n is a union of finitely many disjoint closed intervals of length $1/3^n$; and show that the end points of these intervals lie in C.

Notice that $A_n = \bigcup_{k=0}^{\frac{3^n - 1}{2}} \left(\dfrac{2k}{3^n}, \dfrac{2k+1}{3^n} \right)$. Then $\forall k$, $\dfrac{2k+1}{3^n} - \dfrac{2k}{3^n} = \dfrac{1}{3^n}$. Therefore, each interval in the union has the length $1/3^n$. $\square$

(d) Show that C has no isolated points.

Observe that every point of the Cantor set is a limit point of itself. Therefore, it has no isolated points. $\square$

(e) Conclude that C is uncountable.

C is nonempty since it is the intersection of a nested sequence of closed intervals. Besides, it is Hausdorff, has no isolated points as well as is compact. Now, according to the **Theorem 27.7**, C is uncountable. $\square$