Topology

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Assignment №1

Chapter 2

2. Let $f : A \to B$ and let $A_i \subset A$ and $B_i \subset B$ for $i = 0$ and $i = 1$. Show that if f^{-1} preserves inclusions, unions, intersections, and differences of sets:

   (c) $f^{-1}(B_0 \cap B_1) = f^{-1}(B_0) \cap f^{-1}(B_1)$.

   To prove that the set $f^{-1}(B_0 \cap B_1)$ is equal to the set $f^{-1}(B_0) \cap f^{-1}(B_1)$, we have to show that $f^{-1}(B_0 \cap B_1) \subset f^{-1}(B_0) \cap f^{-1}(B_1)$ and $f^{-1}(B_0) \cap f^{-1}(B_1) \subset f^{-1}(B_0 \cap B_1)$.

   Case I: $f^{-1}(B_0 \cap B_1) \subset f^{-1}(B_0) \cap f^{-1}(B_1)$

   Let $x \in f^{-1}(B_0 \cap B_1)$. Then $f(x) \in B_0 \cap B_1$. Thus, $f(x) \in B_0$ and $f(x) \in B_1$. From this, we get that $x \in f^{-1}(B_0)$ and $x \in f^{-1}(B_1)$. Therefore, $x \in f^{-1}(B_0) \cap f^{-1}(B_1)$. Hence, if $x \in f^{-1}(B_0 \cap B_1)$, then $x \in f^{-1}(B_0) \cap f^{-1}(B_1)$ which means that $f^{-1}(B_0 \cap B_1) \subset f^{-1}(B_0) \cap f^{-1}(B_1)$. $\square$

   Case II: $f^{-1}(B_0) \cap f^{-1}(B_1) \subset f^{-1}(B_0 \cap B_1)$

   Let $x \in f^{-1}(B_0) \cap f^{-1}(B_1)$. Then $x \in f^{-1}(B_0)$ and $x \in f^{-1}(B_1)$. Thus, $f(x) \in B_0$ and $f(x) \in B_1$. Finally, we have that $f(x) \in B_0 \cap B_1$ which is equivalent to saying $x \in f^{-1}(B_0 \cap B_1)$. Hence, if $x \in f^{-1}(B_0) \cap f^{-1}(B_1)$, then $x \in f^{-1}(B_0 \cap B_1)$ which means that $f^{-1}(B_0) \cap f^{-1}(B_1) \subset f^{-1}(B_0 \cap B_1)$. $\square$

   We have now proven that $f^{-1}(B_0 \cap B_1) \subset f^{-1}(B_0) \cap f^{-1}(B_1)$ and $f^{-1}(B_0) \cap f^{-1}(B_1) \subset f^{-1}(B_0 \cap B_1)$ and thus, $f^{-1}(B_0 \cap B_1) = f^{-1}(B_0) \cap f^{-1}(B_1)$. $\square$

   (g) $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$; show that inequality holds if f is injective.

Let's first show that $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$ even if f is not injective.

Let $x \in f(A_0 \cap A_1)$. Then $\exists x' \in A_0 \cap A_1$ such that $f(x') = x$. Now, since $x' \in A_0$ and $x' \in A_1$, we get that $x \in f(A_0)$ and $x \in f(A_1)$ thus, $x \in f(A_0) \cap f(A_1)$. $\square$

Now let's prove that $f(A_0 \cap A_1) = f(A_0) \cap f(A_1)$ if f is injective. We have already shown that independent of whether f is injective or not, $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$. Thus, we just have to show that $f(A_0) \cap f(A_1) \subset f(A_0 \cap A_1)$ if f is injective.

Let $x \in f(A_0) \cap f(A_1)$. Then $x \in f(A_0)$ and $x \in f(A_1)$. Besides, since f is injective, there exists <u>**unique**</u> x' such that $f(x') = x$. Therefore, $x' \in A_0$ and $x' \in A_1$. Finally, we get that $x \in f(A_0 \cap A_1)$. $\square$

5. In general, let us denote the ***identity function*** for a set C by i_C. That is, define $i_C : C \to C$ to be the function given by the rule $i_C(x) = x$ for all $x \in C$. Given $f : A \to B$, we say that a function $g : B \to A$ is a ***left inverse*** for f if $g \circ f = i_A$; and we say that $h : B \to A$ is a ***right inverse*** for f if $f \circ h = i_B$.

   (a) Show that if f has a left inverse, f is injective; and if f has a right inverse, f is surjective.

   Let's first show that if f has a left inverse, then f is injective.

   Suppose, for the sake of contradiction, that $f : A \to B$ is function such that it has a left inverse and that f is not injective. Since f is not injective, there exists $x_0, x_1 \in A$ such that $f(x_0) = f(x_1)$ and $x_0 \neq x_1$. Since f has the left inverse, there exists $g : B \to A$ such that $g \circ f = i_A$. Consider functions $(g \circ f)(x_0)$ and $(g \circ f)(x_1)$. These functions could be rewritten as $g(f(x_0))$ and $g(f(x_1))$. Since $f(x_0) = f(x_1)$, we have that $g(f(x_0)) = g(f(x_1))$. Therefore, we got that $i_A(x_0) = i_A(x_1)$ and thus, $x_0 = x_1$. At last, we have reached the contradiction since initially we assumed that $x_0 \neq x_1$. Hence, if f has a left inverse, then f is injective. $\square$

   Now let's show that if f has a right inverse, then f is surjective.

   Suppose $f : A \to B$ is a function such that it has a right inverse. Then there exists $h : B \to A$ such that $f \circ h = i_B$. Note that $\forall x \in B$, we have $f(h(x)) = x$ and since $h(x) \in A$, we effectively got that every $x \in B$ has something that maps to it. In other words, f is surjective. $\square$

   (b) Give an example of a function that has a left inverse but no right inverse.

   $f : \mathbb{Z} \to \mathbb{R} : x \mapsto x$.

   (c) Give an example of a function that has a right inverse but no left inverse.

   $f : \mathbb{Z} \to \mathbb{Z}^+ \cup \{0\} : x \mapsto x^2$.

   (d) Can a function have more than one left inverse? More than one right inverse?

Yes, it can have more than one left inverse.
Consider the following functions:

$$
\begin{aligned}
f &: \{1,2,3\} \to \{1,2,3\} \\
g &: \{1,2,3\} \to \{1,2,3\} \\
g' &: \{1,2,3\} \to \{1,2,3\}
\end{aligned}
$$

(e) Show that if f has both a left inverse g and a right inverse h, then f is bijective and $g = h = f^{-1}$.

Solution to e.

Chapter 3

10. (a) Show that the map $f : (-1, 1) \to \mathbb{R}$ of Example 9 is order preserving.

We have to show that $f : (-1,1) \to \mathbb{R} : x \mapsto \dfrac{x}{1-x^2}$ is an order-preserving map. In other words, we have to prove that for arbitrary $x_0, x_1 \in (-1,1)$, if $x_1 > x_0$, then $f(x_1) > f(x_0)$. Suppose $x_0, x_1 \in (-1, 1)$ and $x_1 > x_0$. Then we have:

$$f(x_1) - f(x_0) = \frac{x_1}{1-x_1^2} - \frac{x_0}{1-x_0^2} = \frac{x_1 - x_1x_0^2 - x_0 + x_0x_1^2}{(1-x_1^2)(1-x_0^2)} = \frac{(x_1-x_0)(x_0x_1+1)}{(1-x_1^2)(1-x_0^2)}$$

Finally, we got that $f(x_1) - f(x_0) = \dfrac{x_1 - x_1x_0^2 - x_0 + x_0x_1^2}{(1-x_1^2)(1-x_0^2)} = \dfrac{(x_1-x_0)(x_0x_1+1)}{(1-x_1^2)(1-x_0^2)}$ where $x_1 > x_0$. Notice that all the members of the fraction are positive. $x_1 - x_0 > 0$ since $x_1 > x_0$, $x_0x_1 + 1 > 0$ as $x_0, x_1 \in (-1, 1)$, and $(1-x_1^2)(1-x_0^2)$ is positive since, once again, $x_0, x_1 \in (0, 1)$. Thus, we have effectively shown that if $x_0, x_1 \in (0,1)$ such that $x_1 > x_0$, then $f(x_1) - f(x_0) > 0$. In others, if $x_0, x_1 \in (0, 1)$ and $x_1 > x_0$, then $f(x_1) > f(x_0)$ and f is indeed an order-preserving map. $\square$

(b) Show that the equation $g(y) = 2y/[1 + (1 + 4y^2)^{1/2}]$ defines a function $g : \mathbb{R} \to (-1, 1)$ that is both a left and a right inverse for f.

Solution.