The Topology of Robotic Configuration and Motion Planning

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David Oniani
Luther College
oniada01@luther.edu

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Abstract

This paper explores the topological approach to the problem of robot motion planning. Particularly, we will discuss the safe way to coordinate automated guided vehicles or AGVs. AGVs are mobile robots which are used extensively in manufacturing facilities. Since these robots are costly and cannot tolerate collisions, one of the biggest challenges in designing such facility is setting up mobile robot routes to achieve the safe and efficient coordination of robots. The tools and concepts of topology are naturally employed in this planning process. This paper follows the bottom-up approach by first introducing concepts and then building up on these ideas. It does not assume any background in topology nor in robotics. It is therefore accessible to most undergraduate mathematics students with the knowledge of set theory.

Configuration Spaces

We shall start by introducing the notion of configuration spaces. The idea of configuration spaces come from physics. In classical mechanics, the configuration space is the vector space defined by the generalized coordinates (coordinates that describe the configuration of the physical system). Put it simply, the configuration space is the set of all possible states that could exist in the physical system. For instance, the configuration space of some particle in the room is the set of all points/states of the type (x, y, z) where x, y and z are the coordinates bounded by the room. If the room is a $3 \times 3 \times 3$ cube then we define the configuration space of the particle by

$$C^3(\text{room}) = \{(x, y, z) \mid 0 \leq x, y, z \leq 3\}.$$

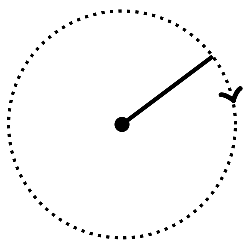
In other words, the configuration space of the particle, is all of $3 \times 3 \times 3$ cube (this example obviously assumes that the particle is allowed to move freely in the room). The configuration space of the same particle in a spherical room, however, would be the set of all points that are in a sphere.

More formally, we call the the states of the particle configurations and the space of all configurations - a configuration space.

Definition. REMOVE THIS!!! A ***configuration*** of an object is a specification of the position of all points of an object. The ***configuration space*** of an object is the space of all configurations.

It appears that the physical notion of configuration spaces is very much connected to that of mathematics. In fact, the idea is the same but rather generalized. To better understand configuration spaces, let us first go through several *classic* examples.

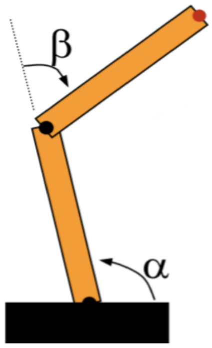
[1] Consider a planar system where we have a rod with a fixed end that can rotate freely. Then it is easy to see that the space of all possible configurations of the rotating rod is a circle.



Circle obtained by the rotational motion of the rod.

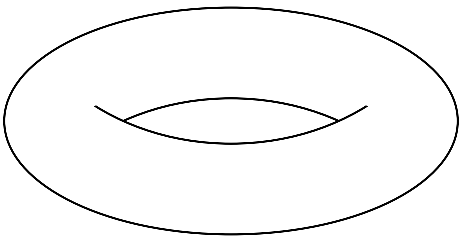
In other words, as the rod rotates, it creates the circle around itself with the radius equal to the length of the rod. This configuration space is also referred to as S^1.

[2] On the other hand, the configuration space of the two-rod system in 3D space where one rod is fixed and the other one is attached to it (also known as a 2R robot) is a torus. This space is also known as $S^1 \times S^1$ configuration space.



A two-rod system.

We already know that a rod with fixed end generates a circle. In this case, we have two rods: one attached to the ground and the other one attached to the end of the first one. Then obviously both of the rods can go through a full circle of states and therefore, create a configuration space which geometrically represents a torus.



A torus obtained by the motion of two-rod system.

As of now, this is all we need to know about the configuration spaces. This idea will be very useful once we learn more about other topological concepts.

Topological Spaces

The fundamental idea in topology is that of a topological space. We will use this idea to then introduce and define other important concepts.

Definition. [3] A ***topology*** on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

(1) $\varnothing$ and X are in $\mathcal{T}$.

(2) The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

(3) The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

A set X for which a topology $\mathcal{T}$ has been specified is called a ***topological space***.

Let us first look at some examples. Consider a set $X = \{a, b, c\}$. Then we can define a topology on X by $\mathcal{T} = \{\varnothing, \{a, b, c\}, \{a, b\}, \{c\}\}$. Observe that $\varnothing, X \in \mathcal{T}$ therefore the first criterion is satisfied. It is easy to see that arbitrary unions will be in $\mathcal{T}$ since the only "interesting" case is when we consider $\{a, b\}$ and $\{c\}$, but in this case $\{a, b\} \cup \{c\} = \{a, b, c\} \in \mathcal{T}$. This satisfies the second requirement. Finally, any arbitrary intersection of the finite subcollections of $\mathcal{T}$ is also in $\mathcal{T}$ and therefore, we conclude that $\mathcal{T}$ is indeed a topology on X. Hence, X is a topological space (note that, properly speaking, a topological space is an ordered pair $(X, \mathcal{T})$, but we often omit mentioning $\mathcal{T}$ and say that X is a topological space).

At this point, you might have already noticed that one could always define more than one topology for a given set. In the previous example, sets $\mathcal{P}(x)$ (powerset of x) and $\{\varnothing, X\}$ are also topologies on X called discrete and indiscrete topologies. In fact, for any set X, $\mathcal{P}(x)$ and $\{\varnothing, X\}$ will always be two distinct topologies on X.

We will now get acquainted with the notion of the *open set*.

Definition. [4] If X is a topological space with topology $\mathcal{T}$, we say that a subset U of X is an **open set** of X if U belongs to the collection $\mathcal{T}$.

Consider our topology $\mathcal{T} = \{\varnothing, \{a, b, c\}, \{a, b\}, \{c\}\}$ on the topological space $X = \{a, b, c\}$. Then notice that $\varnothing, \{a, b, c\}, \{a, b\}, \{c\} \in \mathcal{T}$ and therefore, all are open sets.

Continuous Functions and Homeomorphisms

The notion of continuous functions is familiar to most high-school students. Most people associate them with a nice-looking monotonically increasing or decreasing functions with no leaps or jumps. There are several definitions for a function continuity. Here is the calculus definition.

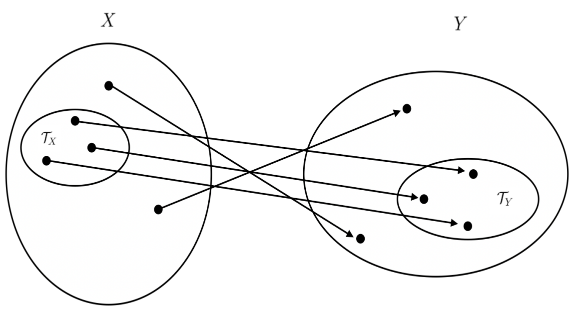
Definition. [5] A function $f(x)$ is continuous at a point $x = c$ if and only if it meets the following three conditions:
1. $f(c)$ exists (c lies in the domain of f).
2. $\lim_{x \to c} f(x)$ exists (f has a limit as $x \to c$).
3. $\lim_{x \to c} f(x) = f(c)$ (the limit equals to the function value).

In topology we cannot really use this definition since the definition assumes that one could take a limit of the function. This, however, is sometimes very difficult or nearly impossible when considering functions defined over more abstract sets such as the configuration space of AGV. Now, we shall introduce more general notion of continuity.

Definition. [6] Let X and Y be topological spaces. A function $f : X \to Y$ is said to be **continuous** if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.

It is important to note that in this definition $f^{-1}(V)$ does not refer to the inverse of the function. Therefore, we are not assuming that $f : X \to Y$ is a bijection. $f^{-1}(V)$ refers to the preimage of the function. In other words, the function $f : X \to Y$ over two topological spaces X and Y is continuous if and only if every open set in the image is mapped by an open set from a preimage.

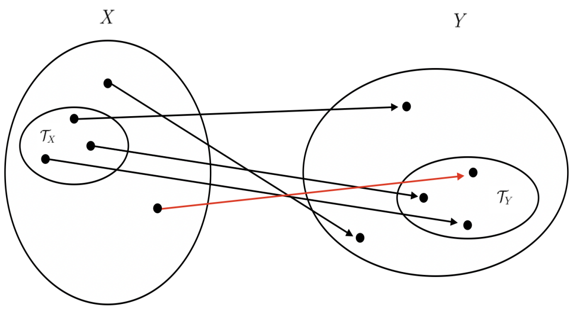
This is the case where the visualization might be useful to see how the definition of continuous functions really works. Consider a continuous function $f : X \to Y$ where X and Y are topological spaces with topologies $\mathcal{T}_X$ and $\mathcal{T}_Y$.



A continuous function $f : X \to Y$.

In the figure above, the big blobs X and Y are topological spaces and smaller blobs $\mathcal{T}_X$ and $\mathcal{T}_Y$ are the topologies on X and Y correspondigly. By the definition, the elements in $\mathcal{T}_X$ or $\mathcal{T}_Y$ are open sets. Then notice that the function $f : X \to Y$ shown above is continuous. All the points which are in $\mathcal{T}_Y$ have a preimage in $\mathcal{T}_X$. Note that there are points in Y that do not have a preimage in $\mathcal{T}_X$, but it has no importance to continuity of f since it must be open sets in Y (not just any set) that must have an open preimage.

The function below, however, is not continuous as there is one point that is not in $\mathcal{T}_X$ but maps to a point in $\mathcal{T}_Y$ (it is highlighted with the red arrow).



A discontinuous function $f : X \to Y$.

Now that we are familiar with the notion of continuous functions, we shall introduce a new idea, that of homeomorphisms.

Definition. [7] Let X and Y be topological spaces; let $f : X \to Y$ be a bijection. If both the function f and the inverse function

$$f^{-1} : Y \to X$$

are continuous, then f is called a homeomorphism.

As one delves deeper in topology, this definition is then taken further and two objects are said to be homeomorphic if one could be obtained by the continuous deformation of the other. We may not use this notion further in the paper yet, it is a helpful way to think about homeomorphisms.

Connectedness and Path Connectedness

One of the important ideas in topology is that of connectedness. This idea is used extensively in various other fields of mathematics such as graph theory and knot theory. Let us first define connectedness.

Definition. [8] Let X be a topological space. A **separation** of X is a pair U, V of disjoint nonempty open subsets of X whose union is X. The space X is said to be **connected** if there does not exist a separation of X.

Consider a topological space $X = \{a, b, c\}$ with a topology $\mathcal{T} = \{\varnothing, \{a, b, c\}, \{a, b\}, \{c\}\}$. Then it is easy to see that X is a disconnected space. Sets $\{a, b\}$ and $\{c\}$ are open since $\{a, b\}, \{c\} \in \mathcal{T}$. Besides, $\{a, b\} \cap \{c\} = \varnothing$ and $\{a, b\} \cup \{c\} = \{a, b, c\} = X$. Hence, $U = \{a, b\}$ and $V = \{c\}$ is a pair of disjoint open subsets of X whose union is X and therefore U, V is a separation of X. This means that X is a disconnected space.

On the other hand, a topological space $Y = \{a, b\}$ with a topology $\mathcal{T} = \{\varnothing, \{a, b\}, \{a\}\}$ is connected as there is no pair of disjoint nonempty open subsets of Y such that their union is Y. In other words Y has no separation. Note that $\varnothing \cup \{a, b\} = \{a, b\} = Y$, but it must be nonempty sets whose union is Y. Therefore, $\varnothing, \{a, b\}$ is not a separation of Y.

Knowing what it means for a topological space to be connected, we can now introduce the notion of path connectedness.

Definition. [9] Given points x and y of the space X, a **path** in X from x to y is a continuous map $f : [a, b] \to X$ of some closed interval in the real line into X, such that $f(a) = x$ and $f(b) = y$. A space X is said to be ***path connected*** if every pair of points of X can be joined by a path in X.

Path connected spaces are certainly related to the connected spaces. In fact, one could easily verify that path connectedness implies connectedness. Therefore, every path-connected space is connected.

Configuration Spaces Revisited - Robotic Configurations

In previous sections, we went through what is called a configuration space. Yet, we did not have a precise definition of it since it varies from field to field. Let us now define what a configuration space means in the robotics context.

Definition. [10] A ***configuration*** of a robot is a specification of the position of all points of a robot. The ***configuration space*** of a robot is the space of all configurations of a robot.

Suppose that we have a robot R_1 that can move freely on a line L. Then we can model the robot as a point with a coordinate x_R. The configuration space for this robot is

$$C^1(L) = \{R_1 \mid R_1 \in L\} = L = L^1.$$

with every point in the room being a unique configuration of the robot. The dimension of C (in this case 1), is also called ***degree(s) of freedom***. In this case, the configuration space C has 1 degree of freedom.

What if we had 2 robots (say R_1 and R_2) that can move freely? Then the configuration space would be C^2 which can be represented as a set

$$C^2(L) = \{(x_{R_1}, x_{R_2}) \mid x_{R_1}, x_{R_2} \in L\} = L \times L = L^2.$$

In general, the configuration space for N robots $R_1, R_2, \ldots, R_N$ that can move freely on a line L is

$$C^N(L) = \{(R_1, R_2, \ldots, R_N) \mid R_1, R_2, \ldots, R_N \in L\}$$
$$= \underbrace{L \times L \times \ldots\ldots L}_{N \text{ times}} = L^N.$$

In fact, we could generalize this notion even further. Instead of considering N robots moving on a line L, suppose that we have N robots moving on some K-dimensional space L^K. Then the configuration for each of the robots will be a K-tuple and the configuration space of all N robots will be a set consisting of N number of K-tuples. Therefore, we have

$$\begin{aligned} C(L^K) = \{&(x_{R_{1,1}}, x_{R_{1,2}}, \ldots, x_{R_{1,K}}),\\ &(x_{R_{2,1}}, x_{R_{2,2}}, \ldots, x_{R_{2,K}}),\\ &(x_{R_{3,1}}, x_{R_{3,2}}, \ldots, x_{R_{3,K}}),\\ &\ldots,\\ &(x_{R_{N,1}}, x_{R_{N,2}}, \ldots, x_{R_{N,K}}),\\ &\mid R_1, R_2, \ldots, R_N \in L^K\} \end{aligned}$$

where $(x_{R_{1,1}}, x_{R_{1,2}}, \ldots, x_{R_{1,K}})$ is the configuration of the robot R_1, $(x_{R_{2,1}}, x_{R_{2,2}}, \ldots, x_{R_{2,K}})$ is the configuration of the robot R_2 etc.

Safe Robotic Configurations

At this point, we have enough machinery to understand safe robotic configurations. Let us model each robot with a point that moves through a topological space represeting the robot routes in the factory.

Consider two robots R_1 and R_2 moving through the line represented by L. Then the configuration space for these robots is

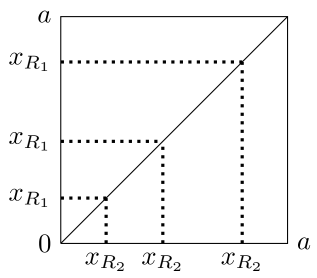
$$C = \{(x_{R_1}, x_{R_2}\} \mid x_{R_1}, x_{R_2} \in L\} = L \times L = L^2.$$

The result here is somewhat natural. If we have one robot moving on a line, we have a configuration space L. If there are two robots, the configuration changes from a single point to a 2-tuple ((x_A, x_B) in the case) which yields a configuration space equal to that of L^2.

Now, since x_{R_1} and x_{R_2} represent the coordinates of the robots, we cannot allow them to be the same. In other words, if $x_{R_1} = x_{R_2}$, we have a collision. We therefore modify our configuration space C to eliminate all the configurations where $x_{R_1} = x_{R_2}$. Let us call this new configuration space SC (safe configuration) with

$$SC = \{(x_{R_1}, x_{R_2}) \mid x_{R_1}, x_{R_2} \in \mathbb{R}, x_{R_1} \neq x_{R_2}\}.$$

It is really helpful to think about this configuration space geometrically. Particularly, let's think about this space as a small chunk of the XOY coordinate system. To do this, we will need to do some transformations. Since L is a line we might as well represent it a closed interval $L = [0, a]$ where $a \in \mathbb{R}$. Now we can say that both robots R_1 and R_2 are moving through the interval $[0, a]$. Let us consider the motion of R_1 and R_2 independently. To do this, we make a copy of the interval $[0, a]$, put one interval as an X axis and the other one as the Y axis.



From the picture above, it is easy to see that the points where $x_{R_1} = x_{R_2}$ are the points on the **diagonal** of the square bounded by interval $[0, a]$ on X axis and its copy on the Y axis. This diagonal is denoted by Δ. Then we can simply say that

$$SC = C - \Delta.$$

In the general case with N robots, it is easy to see that

$$SC^N = \underbrace{L \times L \times \cdots \times L}_{N \text{ times}} - \Delta$$

where

$\Delta = \{x_1, x_2, \ldots, x_N \mid x_i = x_j \text{ for some } i \neq j\}$

Freely Transportable Robots

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