Homework №6

Author: David Oniani
Instructor: Tommy Occhipinti

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36. Prove that if S and T are shifty sets (in the sense of a previous exercise) then $S \cup T$ is also a shifty set.

    Let's first remember what it means to be shifty. A subset S of $\mathbb{Z}$ is called shifty if for every $x \in S$, $x - 1 \in S$ or $x + 1 \in S$.
    Let's consider the following two cases:

    1. Let $x \in S$ and prove that $x - 1$ or $x + 1$ is in $S \cup T$
    2. Let $y \in T$ and prove that $y - 1$ or $y + 1$ is in $S \cup T$

       Let's first consider the case when $x \in S$ and prove that $x - 1$ or $x + 1$ is in $S \cup T$. If $x \in S$, since S is shifty, it means that either $x - 1 \in S$ or $x + 1 \in S$. Union $S \cup T$ will have all the members of S thus, it means that either $x - 1 \in S \cup T$ or $x + 1 \in S \cup T$.

       Now let's show that if $y \in T$, $y - 1$ or $y + 1$ is in $S \cup T$. If $y \in T$, since T is shifty, it means that either $y - 1 \in T$ or $y + 1 \in T$. Union $S \cup T$ will have all the members of T thus, it means that either $y - 1 \in S \cup T$ or $y + 1 \in S \cup T$.

       Thus, we considered all the cases and proved that if S and T are shifty sets (in the sense of a previous exercise) then $S \cup T$ is also a shifty set.

    Q.E.D.

37. Prove that $n \in \mathbb{Z}$ then $1 + (-1)^n(2n - 1)$ is a multiple of 4.

    Since $n \in \mathbb{Z}$, it is either even or odd. Let's consider two cases:

    1. n is even
    2. n is odd

    If n is even, then $n = 2k$ where $k \in \mathbb{Z}$. We get:

    $1 + (-1)^n(2n - 1) = 1 + (-1)^{2k}(2 \times 2k - 1) = 1 + 4k - 1 = 4k$, where $k \in \mathbb{Z}$

    Hence, we got that if n is even, $1 + (-1)^n(2n - 1) = 4k$ where $k \in \mathbb{Z}$. Thus, if n is even, is a multiple of 4.

If n is odd, then $n = 2k$ where $k \in \mathbb{Z}$. We have:

$$1+(-1)^n(2n-1) = 1+(-1)^{2k+1}(2\times(2k+1)-1) = 1-(4k+2-1) = -4k \text{ where } k \in \mathbb{Z}$$

Thus, we got that if n is odd, $1 + (-1)^n(2n-1) = -4k$ where $k \in \mathbb{Z}$. Hence, if n is odd, $1 + (-1)^n(2n-1) = -4k$ is a multiple of 4.
Finally, since integers could either be odd or even, we considered all the cases (n is even and n is odd) and in both of the cases, $1+(-1)^n(2n-1)$ is a multiple of 4.

Q.E.D.

38 Prove that if $n \in \mathbb{Z}^+$ is odd then $n^2 - 1$ is divisible by 8.

Suppose n is odd. Then $n = 2k + 1$ where $k \in \mathbb{Z}^+ \cup \{0\}$. Then we have:

$$n^2 - 1 = (2k+1)^2 - 1 = 4k^2 + 4k + 1 - 1 = 4k^2 + 4k = 4k \times (k+1)$$

Thus, if n is odd, $n^2 - 1 = 4k \times (k+1)$ where $k \in \mathbb{Z}^+ \cup \{0\}$. Since k is in the set of positive integers or equls 0, we can consider the following three cases:

1. k is 0
2. k is even
3. k is odd

If $k = 0$, $n^2-1 = 4k \times (k+1) = 4 \times 0 \times (0+1) = 0$ which is divisible by 8.

If k is even, $k = 2l$ where $l \in \mathbb{Z}^+$ (not including zero since we already considered that case above). Then $n^2-1 = 4k \times (k+1) = 8 \times (l \times (2l+1))$ which is divisible by 8.

If k is even, $k = 2l + 1$ where $l \in \mathbb{Z}^+$. Then $n^2 - 1 = 4k \times (k+1) = 4 \times (2l+1) \times (2l+2) = 8 \times ((2l+1) \times (l+1))$ which is divisible by 8.

Thus, we considered all the cases and we proved that if $n \in \mathbb{Z}^+$ is odd then $n^2 - 1$ is divisible by 8.

Q.E.D.

39. Prove that every integer can be written as the sum of exactly 3 distinct integers. (For example, $5 = 4 + 2 + (-1)$).

Suppose k is an integer. Then, we should find three distinct integers such that they sum to k. Now, let's consider the following integers:
$x = k + 1$, $y = -k - 1$, and $z = k$. It is clear that $x \neq z$ and $y \neq z$.
If $x = z$, we have $k + 1 = k \iff 1 = 0$ which is nonsensical.
If $y = z$, we have $-k - 1 = k \iff k = \dfrac{-1}{2}$ which is also not possible because k is an integer.
Thus $x \neq z$ and $y \neq z$, but we do not know if $x \neq y$. Hence, we have to consider cases.

Now, consider two cases:

1. $x \neq y$
2. $x = y$

If $x \neq y$, we already know that $x \neq z$ and $y \neq z$ and thus we found three distinct integers which sum up to k, namely $x = k + 1$, $y = -k - 1$, and $z = k$.

If $x = y$, we get $k + 1 = -k - 1$ and $k = -1$. However, even if $k = -1$, we can find three distinct integers, namely $x = -2$, $y = 2$, and $z = -1$.

Thus, we considered all the possible cases and we always find three integers which sum up to k.

Q.E.D.

40. Prove that if $x, y,$ and z are integers then at least one of $x + y$, $x + z$, and $y + z$ is even.

Suppose, for the sake of contradiction, that $x + y$, $x + z$, and $y + z$ are all odd.

Now, without a loss of generality, consider the following three cases:

1. x, y, z are all even
2. x, y, z are all odd
3. x, y are even and z is odd
4. x, y are odd and z is even

If x, y, z are all even, $x = 2j$, $y = 2k$, $z = 2l$ where $j, k, l \in \mathbb{Z}$. Thus, we get: $x + y = 2j + 2k = 2 \times (j + k)$ which is even.

If x, y, z are all odd, $x = 2j + 1$, $y = 2k + 1$, $z = 2l + 1$ where $j, k, l \in \mathbb{Z}$. Thus, we get: $x + y = 2j + 1 + 2k + 1 = 2j + 2k + 2 = 2 \times (j + k + 1)$ which is even.

If x, y are even and z is odd, $x = 2j$, $y = 2k$, $z = 2l + 1$ where $j, k, l \in \mathbb{Z}$. Thus, we get: $x + y = 2j + 2k = 2 \times (j + k)$ which is even.

If x, y are odd and z is even, $x = 2j+1$, $y = 2k+1$, $z = 2l$ where $j, k, l \in \mathbb{Z}$. Thus, we get: $x + y = 2j + 1 + 2k + 1 = 2j + 2k + 2 = 2 \times (j + k) + 1$ which is even.
Thus, we've considered all the possible cases and proved that if $x, y,$ and z are integers then at least one of $x + y$, $x + z$, and $y + z$ is even.

Q.E.D.

41. Prove or Disprove: There exist prime number p and q such that $p - q = 97$.

There are no prime numbers p and q such that $p - q = 97$. Let's prove it! First, notice that since p and q are prime, $p, q > 0$. Also, from $p - q = 97$, we get $p = 97 + q$ and since $p, q > 0$, $p > q$. Thus, we have $0 < q < p$. Now, since $p - q$ is odd, it is the case that one of p, q is odd and the other one is even (because difference of even as well as odd numbers is even). However, we know that there is only one even prime number, namely 2. Let's consider following two cases:

1. $p = 2$
2. $q = 2$

Since $0 < q < p$, we know that p cannot be 2 (if $p = 2$, there is no prime number below 2, so q cannot be prime).

If $q = 2$, $p - 2 = 97$ and $p = 95 = 5 \times 19$ which is also not prime.

Thus, we considered all the cases and there are no prime numbers, p and q, which sum up to 95.

Q.E.D.

42. For $n \in \mathbb{Z}^+$, we define the n^{th} triangular number to be $T_n := 1 + 2 + ... + n$. Thus we have $T_1 = 1, T_2 = 3, T_3 = 6, T_4 = 10, T_5 = 15$, and so on. We will prove later that $T_n = \dfrac{n(n+1)}{2}$, and you should use that formula for this problem. Prove that for $n \in \mathbb{Z}^+$, T_n is odd if and only if n is 1 or 2 more than a multiple of 4.

Since it is the "if and only if" proof, let's split the proof in two cases:

1. Prove that if n is 1 or 2 more than a multiple of 4, T_n is odd.
2. Prove that if n is not 1 and is not 2 more than a multiple of 4, T_n is not odd (thus is even).

Let's first prove that if n is 1 or 2 more than a multiple of 4, T_n is odd.

If n is 1 more than a multiple of 4, $n = 4k + 1$ where $k \in \mathbb{Z}^+$. Then we have:

$$T_n = \frac{n(n+1)}{2} = \frac{(4k+1)((4k+1)+1)}{2} = \frac{(4k+1)((4k+2)}{2} = (2k+1)\times(2k+1)$$

. Thus, we have that if n is 1 more than a multiple of 4, $T_n = (2k+1)\times(2k+1)$ which is a multiplication of two odd numbers and thus is odd.

If n is 2 more than a multiple of 4, $n = 4k + 2$ where $k \in \mathbb{Z}^+$. Then we have:

$$T_n = \frac{n(n+1)}{2} = \frac{(4k+2)((4k+2)+1)}{2} = \frac{(4k+2)((4k+3)}{2} = (2k+1)\times(4k+1)$$

Hence, we got that $T_n = (2k+1)\times(4k+1) = (2k+1)\times(2(2k)+1)$ which the multiplication of two odd numbers and thus, if n is 2 more than a multiple of 4, T_n is odd.

Now, let's prove that if n is not 1 or 2 more than a multiple of 4, T_n is even.

If n is not 1 and is not two more than a multiple of four, we have the following five cases to consider:

1. $n = 4k$ where $k \in \mathbb{Z}^+$
2. $n = 4k + 3$ where $k \in \mathbb{Z}^+$

If $n = 4k$ where $k \in \mathbb{Z}^+$, $T_n = \dfrac{4k(4k+1)}{2} = 2 \times (k(4k+1))$ which is even.

If $n = 4k+3$ where $k \in \mathbb{Z}^+$, $T_n = \dfrac{(4k+3)(4k+4)}{2} = 2 \times ((4k+3)(2k+2))$ which is even.

NOTE: SINCE WE CONSIDERED INTEGERS STARTING AT 4 (SINCE WE ASSUMED THAT $n = 4k, 4k+1, 4k+2, 4k+3$ where $k \in \mathbb{Z}^+$), NOW WE NEED TO CHECK T_n for 1,2, and 3.

If $n = 1$, $T_n = \dfrac{1(1+1)}{2} = 1$ which is odd and indeed, $1 = 0 + 1$ and 0 is a multiple of 4 (thus, 3 has $4l + 1$ form where $l \in \mathbb{Z}$)..
If $n = 2$, $T_n = \dfrac{2(2+1)}{2} = 3$ which is odd and indeed, $2 = 0 + 2$ and 0 is a multiple of 4 (thus, 3 has $4l + 2$ form where $l \in \mathbb{Z}$)..
If $n = 3$, $T_n = \dfrac{3(3+1)}{2} = 6$ which is even and indeed, $3 = 0 + 3$ and 0 is a multiple of 4 (thus, 3 has $4l + 3$ form where $l \in \mathbb{Z}$).

Thus, we considered all the cases and proved that for $n \in \mathbb{Z}^+$, T_n is odd if and only if n is 1 or 2 more than a multiple of 4.

Q.E.D.

Bookwork

2. Solve the equality $|x+1| < |x^2-1|$. Interpret results geometrically.

   Let's consider the following two cases:

   1. $x+1 \geq 0$
   2. $x-1 \leq 0$

   Let's first consider the case $x+1 \geq 0$.
   If $x+1 \geq 0$, the inequality will have a form:

   $$x+1 < |x^2-1|$$

   Now, let's consider another two cases:

   1. $x^2-1 \geq 0$
   2. $x^2-1 \leq 0$

   If $x^2-1 \geq 0$, the equality we get is:

   $$x+1 < x^2-1$$

   And we get:

   $$x^2-x-2 > 0$$

   And finally, $x \in (-\infty,-1) \cup (2,+\infty)$.
   But since $x+1 \geq 0$, $x \geq -1$ and we get $x \in (2,+\infty)$

   If $x^2-1 \leq 0$, the equality we get is:

   $$x+1 < -x^2+1$$

   And we get:

   $$x^2+x < 0$$

   And finally, $x \in (-1,0)$.
   It also satisfies $x+1 \geq 0$.

   Now, let's consider the second case, $x+1 \leq 0$.
   If $x+1 \leq 0$, the inequality will have a form:

   $$-x-1 < |x^2-1|$$

   Now, as previously, let's consider another two cases:

   1. $x^2-1 \geq 0$
   2. $x^2-1 \leq 0$

If $x^2 - 1 \geq 0$, the equality we get is:

$$-x - 1 < x^2 - 1$$

And we get:

$$x^2 + x > 0$$

And finally, $x \in (-\infty, -1) \cup (0, +\infty)$.
But since $x + 1 \leq 0$, $x \leq -1$ and we get $x \in (-\infty, -1)$

If $x^2 - 1 \leq 0$, the equality we get is:

$$-x - 1 < -x^2 + 1$$

And we get:

$$x^2 - x - 2 < 0$$

And finally, $x \in (-1, 2)$.
But since $x + 1 \leq 0$, $x \leq -1$ and we do not get any solutions for this inequality.

Thus, we got the following answers: $x \in (2, +\infty)$, $x \in (-1, 0)$, $x \in (-\infty, -1)$.
Finally, by merging these answers, we get: $x \in (-\infty, -1) \cup (-1, 0) \cup (2, +\infty)$.

4. (a)

Show that any integer can be written in the form $10q + r$ where $0 \leq r \leq 9$.

I don't really like the way this is phrased, so I will rephrase what we have to prove. We have to prove that:

for every integer k, there exists q and r such that $k = 10q + r$ and $0 \leq r \leq 9$

Notice that if an integer ends on 0, it is always divisible by 10. Thus, it is just the problem of picking r so that $k - r$ end on 0.
Let's consider the following 10 cases for the possible endings of the number.

- Number ends on 0
- Number ends on 1
- Number ends on 2
- Number ends on 3
- Number ends on 4
- Number ends on 5
- Number ends on 6
- Number ends on 7
- Number ends on 8

* Number ends on 9

* If number ends on 0, we can pick $r = 0$, and then k is divisble by 10 and we get the value of q.

* If number ends on 1, we can pick $r = 1$, and then k is divisble by 10 and we get the value of q.

* If number ends on 2, we can pick $r = 2$, and then k is divisble by 10 and we get the value of q.

* If number ends on 3, we can pick $r = 3$, and then k is divisble by 10 and we get the value of q.

* If number ends on 4, we can pick $r = 4$, and then k is divisble by 10 and we get the value of q.

* If number ends on 5, we can pick $r = 5$, and then k is divisble by 10 and we get the value of q.

* If number ends on 6, we can pick $r = 6$, and then k is divisble by 10 and we get the value of q.

* If number ends on 7, we can pick $r = 7$, and then k is divisble by 10 and we get the value of q.

* If number ends on 8, we can pick $r = 8$, and then k is divisble by 10 and we get the value of q.

* If number ends on 9, we can pick $r = 9$, and then k is divisble by 10 and we get the value of q.

Thus, we considered all the possible cases and proved that any integer can be written in the form $10q + r$ where $0 \leq r \leq 9$.

Q.E.D.

(b)

Show that the square of any integer ends in 0, 1, 4, 5, 6, or 9.

There are 9 numbers on which an integer could end.
These numbers are: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.
Then let's consider the following 10 cases.

* Number ends on 0
* Number ends on 1
* Number ends on 2
* Number ends on 3
* Number ends on 4

* Number ends on 5
* Number ends on 6
* Number ends on 7
* Number ends on 8
* Number ends on 9

If number ends on 0, its square ends on the same number as 0^2 ends on, thus on 0.
If number ends on 0, its square ends on the same number as 1^2 ends on, thus on 1.
If number ends on 0, its square ends on the same number as 2^2 ends on, thus on 4.
If number ends on 0, its square ends on the same number as 3^2 ends on, thus on 9.
If number ends on 0, its square ends on the same number as 4^2 ends on, thus on 6.
If number ends on 0, its square ends on the same number as 5^2 ends on, thus on 5.
If number ends on 0, its square ends on the same number as 6^2 ends on, thus on 6.
If number ends on 0, its square ends on the same number as 7^2 ends on, thus on 9.
If number ends on 0, its square ends on the same number as 8^2 ends on, thus on 4.
If number ends on 0, its square ends on the same number as 9^2 ends on, thus on 1.

Thus, the ending numbers we got are: 0, 1, 4, 9, 6, 5, 6, 9, 4, 1.
Now, if we eliminate the duplicates, we get: 0, 1, 4, 5, 6, and 9.

Q.E.D.

8. Can one form a ten-digit integer by putting a digit between 0 and 9 in the empty boxes in the given table as follows: The digit in the box labeled 0 indicates the number of times 0 appears in the number, the digit in box 1 indicated the number of times 1 appears in the number, etc.?

0	1	2	3	4	5	6	8	9

(For instance, if 9 is placed in box 0, then the remaining boxes must be filled with 0 in order that the nine zeros actually appear. In this case, however, we reach a contradiction since box 9 cannot contain 0 as at least one 9 appears in the number. Thus the desired ten-digit number cannot have 9 as its leftmost digit.)(Hint: Consider the possible ways of filling the box labeled 0.) How many such numbers exist?

Let's consider the possible ways of filling the column labeled 0.
There are 10 possible ways: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

* If 9 is placed in the box under label 0, we reach a contradiction since box 9 cannot contain 0 as at least one 9 appears in the number.
* If 8 is placed in the box under label 0, we have 1 possible ways of picking number under box labeled 9
* If 7 is placed in the box under label 0, we have 2 possible ways of picking number under box labeled 9.
* If 6 is placed in the box under label 0, we have 3 possible ways of picking number under box labeled 9.
* If 5 is placed in the box under label 0, we have 4 possible ways of picking number under box labeled 9.
* If 4 is placed in the box under label 0, we have 5 possible ways of picking number under box labeled 9.
* If 3 is placed in the box under label 0, we have 6 possible ways of picking number under box labeled 9.
* If 2 is placed in the box under label 0, we have 7 possible ways of picking number under box labeled 9.
* If 1 is placed in the box under label 0, we have 8 possible ways of picking number under box labeled 9.
* If 0 is placed in the box under label 0, we have 9 possible ways of picking number under box labeled 9.

PROVE THAT WE CAN BY SHOWING EXAMPLE. Finally, we get that ten-digit number can indeed be formed and there are 999999999 such ten-digit numbers.