Topology

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January 16, 2019

Assignment №3

Section 18

2. Suppose that $f : X \to Y$ is continuous. If x is a limit point of of the subset A of X, is it necessarily true that $f(x)$ is a limit point of $f(A)$?

   It is not. Consider the constant continuous function $f : \mathbb{R} \to \mathbb{R} : x \mapsto 0$. Then 0 is the limit point of A, however, $f(0) = 0$ is not a limit point of $f(A) = \{0\}$ since there is no neighborhood of 0 that intersects $\{0\}$ at point other than 0.

5. Show that the subspace (a, b) of $\mathbb{R}$ is homeomorphic with $(0, 1)$ and the subspace $[a, b]$ of $\mathbb{R}$ is homeomorphic with $[0, 1]$.

   Recall that a homeomorphism is a bijective and continuous function whose inverse is also continuous. Therefore, all we have to do here is to find bijective and continuous function(s) which would map (a, b) to $(0, 1)$ in the first case and $[a, b]$ to $[0, 1]$ in the second case.

   Let's first show that the subspace (a, b) of $\mathbb{R}$ is homeomorphic with $(0, 1)$.

   Consider the function $f : (a, b) \to (0, 1) : x \mapsto \dfrac{x-a}{b-a}$ (note that $a \neq b$; otherwise, (a, b) would not be an interval). Then notice that it is both injective and surjective hence is a bijection. Besides, it is also a continuous function (it can be verified using the limit definition of continuity). The inverse of f is a function f^{-1} : $(0, 1) \to x \mapsto (a, b) : (b - a)x + a$ which is obviously bijective and also continuous (once again, can be verified using the limit definition of continuity). Finally, we have that the subspace (a, b) of $\mathbb{R}$ is homeomorphic with $(0, 1)$. $\square$

   Now, let's show that the subspace $[a, b]$ of $\mathbb{R}$ is homeomorphic with $[0, 1]$. Let's take the exact same function f but let's reconstruct it in the way that it maps

$[a, b]$ to $[0, 1]$. We have, $f : [a, b] \to [0, 1] : x \mapsto \dfrac{x-a}{b-a}$. Once again, this is a continuous bijective function whose inverse is also continuous and therefore the subspace $[a, b]$ of $\mathbb{R}$ is homeomorphic with $[0, 1]$. $\square$

Section 19

3. Prove theorem 19.4.

<u>**Theorem 19.4**</u>

"If each X_α is a Hausdorff space, then $\prod X_\alpha$ is a Hausdorff space in both the box and product topologies."

Since the box topology is finer than the product topology (follows from the **Theorem 19.1**), it is sufficient to prove that the theorem holds under the box topology. Let x and y be two distinct elements in $\prod_{\alpha \in J} X_\alpha$ such that every X_α is Hausdorff. Now, since x and y are distinct, there exists at least one coordinate such that $x_i \neq y_i$. Therefore, for each $x_i \neq y_i$, there exist open neighborhoods U_i and V_i such that $x_i \in U_i$ and $y_i \in V_i$ with U_i and V_i being the subsets of X and $U_i \cap V_i = \emptyset$. Then define open neighborhoods U and V in $\prod_{\alpha \in J} X_\alpha$ by $\prod_{\alpha \in J} U_\alpha$ and $\prod_{\alpha \in J} V_\alpha$ respectively. Then we have:

$$U \cap V = \prod_{\alpha \in J} (U_\alpha \cap V_\alpha) = (U_1 \cap V_1) \times (U_2 \cap V_2) \times (U_3 \cap V_3) \times ... \times (\emptyset) \times ... = \emptyset$$

Finally, since $U \cap V = \emptyset$, we got that $\prod_{\alpha \in J} X_\alpha$ is Hausdorff. Hence, if each X_α is a Hausdorff space, then $\prod X_\alpha$ is a Hausdorff space in both the box and product topologies. $\square$

7. Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences that are "eventually zero", that is, all sequences $(x_1, x_2, ...)$ such that $x_i \neq \emptyset$ for only finitely many values of i. What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in the box and product topologies? Justify your answer.

In the box topology, the closure of $\mathbb{R}^\infty$ is $\mathbb{R}^\infty$. In other words, $\mathbb{R}^\infty$ is closed. To prove this, it is sufficient to show that $\mathbb{R}^\omega - \mathbb{R}^\infty$ is open. Let $(x_n)_{n=1}^\infty \in \mathbb{R}^\omega - \mathbb{R}^\infty$. Then we want to show that there exists an open set U such that $(x_n)_{n=1}^\infty \in U$ and $U \subset \mathbb{R}^\omega - \mathbb{R}^\infty$. Now, lets define U in the following way: if $x_n = 0$, then $U_n = \mathbb{R}$ and if $x_n \neq 0$, then $U_n = \mathbb{R} - \{0\}$ and $U = \prod_{n=1}^{\infty} U_n$. Notice that all U_n are open as we have defined them and hence U is open in the box topology. Now also notice that $(x_n)_{n=1}^\infty \in \mathbb{R}^\omega - \mathbb{R}^\infty$. Finally, we have that $(x_n)_{n=1}^\infty \in U \subset \mathbb{R}^\omega - \mathbb{R}^\infty$ which means that $\mathbb{R}^\omega - \mathbb{R}^\infty$ is open and therefore $\mathbb{R}^\infty$ is closed. Hence, $\overline{\mathbb{R}^\infty} = \mathbb{R}^\infty$.

In the product topology, the closure of $\mathbb{R}^\infty$ is $\mathbb{R}^\omega$. Let $(x_n)_{n=1}^\infty \in \mathbb{R}^\omega$ and let U be open in $\mathbb{R}^\omega$ with $(x_n)_{n=1}^\infty \in U$. Now, because U is open in the product topology, $U = \prod_{n=1}^{\infty} U_n$ where $U_n = \mathbb{R}$ for all but finitely many $n \in \mathbb{Z}^+$. For $n \in \mathbb{Z}^+$ where $U_n \neq \mathbb{R}$, $U_n = (a_n, b_n)$ where $a_n < b_n$. Now, let's define $(y_n)_{n=1}^\infty \in U$ in the following way: if $U_n = \mathbb{R}$, then $y_n = 0$ and otherwise, $y_n \in U_n = (a_n, b_n)$. Notice that $(y_n)_{n=1}^\infty \in \mathbb{R}^\infty$. Thus, $\forall (x_n)_{n=1}^\infty \in R^\omega$, and $\forall U$ such that $(x_n)_{n=1}^\infty \in U$, we have $U \cap R^\infty \neq \emptyset$ since $(y_n)_{n=1}^\infty \in U \cap \mathbb{R}^\infty$. Finally, recall that $x \in \overline{A}$ if and only if $\forall U$ such that $x \in U$ and U is open, we have $U \cap A \neq \emptyset$. Hence, by the definition, we have that $\overline{R^\infty} = R^\omega$.

Section 20

2. Show that $\mathbb{R} \times \mathbb{R}$ in the dictionary order topology is metrizable.

Let us define the function $d : (\mathbb{R} \times \mathbb{R}) \times (\mathbb{R} \times \mathbb{R})$ in the following way:

$$d(x_1 \times x_2, y_1 \times y_2) = \begin{cases} 1 \text{ if } x_1 \neq y_1 \\ \inf\{|x_2 - y_2|, 1\} \text{ if } x_1 = y_1 \end{cases}$$

Let's first show that d is indeed a metric.

(1) As $d(x_1 \times x_2, y_1 \times y_2)$ is either 1 or $\inf\{|x_2 - y_2|, 1\}$, it is always greater than or equal to zero $\forall x_1, x_2, y_1, y_2 \in \mathbb{R}$. Also, notice that the distance is 0 if and only if we have the case $x_1 = y_1$ with $x_2 = y_2$. In other words, it happens if and only if $x_1 \times x_2 = y_1 \times y_2$.

(2) Notice that $d(x_1 \times x_2, y_1 \times y_2) = d(y_1 \times y_2, x_1 \times x_2)$. If we have $d(x_1 \times x_2, y_1 \times y_2)$, we are comparing x_1 with y_1 and if we have $d(y_1 \times y_2, x_1 \times x_2)$, we compare y_1 with x_1 and the values of the function are obviously the same. In other words, if we would describe a function as a relation, then it would be symmetric.

(3) Notice that the triangle inequality holds since we can show that $\forall x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{R}$, $d(x_1 \times x_2, y_1 \times y_2) + d(y_1 \times y_2, z_1 \times z_2) \geq d(x_1 \times x_2, z_1 \times z_2)$. Without a loss of generality, the cases below will exhaust all the possibilities.

If $x_1 \neq y_1 \neq z_1$, then we get $1 + 1 > 1$.
If $x_1 = y_1 \neq z_1$, we get $1 + 1 - k > 1 - l$ where $0 < k, l < 1$.
If $x_1 \neq y_1 = z_1$, we get $1 - k + 1 > 1 - l$ where $0 < k, l < 1$
If $x_1 = y_1 = z_1$, we get $\inf\{|x_2 - y_2|, 1\} + \inf\{|y_2 - z_2|, 1\} \geq \inf\{|x_2 - y_2|, 1\}$

Now, obviously all of the inequalities are true. The last one is true as well (it is similar to the inequality $|x_1 - x_2| + |x_2 - x_3| \geq |x_1 - x_3|$ which is true $\forall x_1, x_2, x_3 \in \mathbb{R}$). Therefore, d is indeed a metric.

The basis of the dictionary order topology on $\mathbb{R} \times \mathbb{R}$ consists of all sets $(a \times b, a \times d)$ with $b < d$. Now, let U be such a basis element and let $x \in U$. Then $\exists \epsilon \in (0, 1)$

such that $(x_2-\epsilon, x_2+\epsilon) \in (b,d)$. Let $x_1 = a$. Then set $(x_1 \times (x_2-\epsilon), x_1 \times (x_2+\epsilon))$ equals to the epsilon ball centered at x that is contained in U. If $y \in B_d(x,\epsilon)$, then $y_1 = x_1$ since $\epsilon < 1$. Thus, $d(x,y) = |x_2 - y_2| < \epsilon$ and $y_1 \times y_2 \in U$ by the definition of ϵ. Hence, we got that the metric topology induced by d is finer than the dictionary order topology.

Now, let $B = B_d(x,\epsilon)$. Then if $\epsilon \geq 1$, $B = \mathbb{R} \times \mathbb{R}$ which is open in the dictionary order topology. On the other hand, if $\epsilon \in (0,1)$, $B = (x_1 \times x_2, x_1 \times y_2)$. However, B is a basis element in the dictionary order topology which means that the dictionary order topology is finer than the metric topology induced by d.

Finally, since we first showed that the metric topology induced by d is finer than the dictionary order topology and then showed that the dictionary order topology is finer than the metric topology induced by d, we have effectively shown that the metric topology induced by d and the dictionary order topology are equal. Therefore, the dictionary order topology is indeed metrizable. $\square$

5. Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences that are eventually zero. What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in the uniform topology? Justify your answer.

   $\overline{R^\infty} = \{(x_i) \mid \lim_{i\to\infty} x_i = 0\}$.

   Justification

   If $(x_i) \in \overline{\mathbb{R}^\infty}$, then $\forall\epsilon$ such that $0 < \epsilon < 1$, the intersection $R^\infty \cap B_{\bar{p}}((x_i),\epsilon) \neq 0$. Now, let $(y_i) \in R^\infty \cap B_{\bar{p}}((x_i),\epsilon)$. Then for some N, we have $y_n = 0 \; \forall n > N$. Since $\bar{p}((x_i),(y_i)) < \epsilon$, we have $|x_n| < \epsilon \; \forall n > N$. From this, we get that $x_n \to 0$. Now, if $x_n \to 0$, then $\forall \epsilon > 0$, $\exists N$ such that $|x_n| < \epsilon/2 \; \forall n > N$. Let $y_n = x_n$ for $n \leq N$ and $y_n = 0$ for $n > N$. Then notice that $\bar{p}((x_i),(y_i)) < \epsilon$. Therefore, $R^\infty \cap B_{\bar{p}}((x_n),\epsilon) \neq 0$. Finally, since we picked arbitrary ϵ, we have $(x_n) \in \overline{\mathbb{R}^\infty}$.

Section 21

6. Define $f_n : [0,1] \to \mathbb{R}$ by the equation $f_n(x) = x^n$. Show that the sequence $(f_n(x))$ converges for each $x \in [0,1]$, but that the sequence (f_n) does not converge uniformly.

   Let's define the functions $f(x)$ in the following way:

$$f(x) = \begin{cases} 0 \text{ if } x \neq 1 \\ 1 \text{ if } x = 1 \end{cases}$$

   Now it is easy to see that $\forall x \; f_n(x) \to f(x)$. If $x = 1$, $f(1) = 1^n = 1$ for all n. And if $x \in [0,1)$ with x being fixed, we have $f_n(x) = x^n$ which is a monotonically decreasing function of n converging to 0 as we can write it $e^{-\ln(1/x)\times n}$ for $1/x > 1$ (if $x = 0$, we, have $f(n) = 0^n = 0$). $\square$

Let's now show that the sequence (f_n) does not converge uniformly. Because f_n is continuous $\forall n \in \mathbb{Z}^+$, if f_n converges to f uniformly, then according to the **Theorem 21.6**, it follows that f is also continuous which is false since it is not continuous at the point $x = 1$. $\square$

9. Let $f_n : \mathbb{R} \to \mathbb{R}$ be the function

$$f_n(x) = \frac{1}{n^3[x - (1/n)]^2 + 1}$$

See Figure 21.1. Let $f : \mathbb{R} \to \mathbb{R}$ be the zero function.

(a) Show that $f_n(x) \to f(x)$ for each $x \in \mathbb{R}$.

Let's fix x, then we have:

$$\lim_{x\to\infty} f_n(x) = \lim_{x\to\infty} \frac{1}{n^3[x - (1/n)]^2 + 1} \to 0$$

$\square$

(b) Show that f_n does not converge uniformly to f. (This shows that the converse of Theorem 21.6 does not hold; the limit function f may be continuous even though the convergence is not uniform).

The sequence $f_n(x)$ does not converge uniformly. If it did for $\epsilon = \frac{1}{2}$ there would be an N so that for $n > N$ we would have $f_n(x) < \frac{1}{2}$ for all x. However, if $x + n = 1/n$, then $f_n(x_n) = 1$ for all n, but when $n > N$, we face a contradiction.