Topology

Author: David Oniani
Instructor: Dr. Eric Westlund

January 11, 2019

Assignment №2

Section 13

7. Consider the following topologies on $\mathbb{R}$:

$$\mathcal{T}_1 = \text{the standard topology},$$
$$\mathcal{T}_2 = \text{the topology of } \mathbb{R}_K,$$
$$\mathcal{T}_3 = \text{the finite complement topology},$$
$$\mathcal{T}_4 = \text{the upper limit topology, having all sets } (a, b] \text{ as basis},$$
$$\mathcal{T}_5 = \text{the topology having all sets } (-\infty, a] = \{x \mid x < a\} \text{ as basis.}$$

Determine, for each of these topologies, which of the other it contains.

From **Lemma 13.4**, we know that $\mathcal{T}_2$ is strictly finer than $\mathcal{T}_1$.

The finite complement topology will look like $(-\infty, x_0) \cup (x_0, x_1) \cup ....(x_{n-1}, +\infty) \cup (x_n, +\infty)$. Then it is easy to notice that $\mathcal{T}_1$ is strictly finer than $\mathcal{T}_3$ (since $\mathcal{T}_3$ is an open set in $\mathcal{T}_1$; also $(2, 3)$ is open in standard topology but not in finite complement topology).

Now, let $B = (a, b)$ be the element in the basis of $\mathcal{T}_1$. Let $x \in B$. Then, we can find element $(a, x]$ in the upper limit topology that clearly contains element x. Hence, the upper limit topology is finer standard topology. In fact $\mathcal{T}_4$ is strictly finer than $\mathcal{T}_1$ since $(2, 3]$ is not open in $\mathcal{T}_1$.

Hence, as of now, we have the relationship $\mathcal{T}_3 \subsetneq \mathcal{T}_1 \subsetneq \mathcal{T}_2, \mathcal{T}_4$.

Let's now find out the relationship between $\mathcal{T}_2$ and $\mathcal{T}_4$.

The upper limit topology is finer than the topology of $\mathbb{R}_K$. To show this let $B = (a, b) - K$ be the element in the basis of $\mathcal{T}_2$. Then let $x \in B$. If $x < 0$, then we can find element $(a, x]$ which is in the basis of the $\mathcal{T}_4$. In this case, it is easy to notice that $x \in (a, x] \in B$. On the other hand, if $x \geq 0$, then let k be the smallest integer such that $\frac{1}{k} < x$. We have $B \cap (\frac{1}{n}, x] = (s, x]$ where $s = a$ if $a > \frac{1}{k}$ and $s = \frac{1}{k}$ if $\frac{1}{k} > a$. It follows that $x \in (s, x] \subset B$, and we get that $(x, s]$ is a basis element for $\mathcal{T}_4$. Hence, we got that $\mathcal{T}_2 \subset \mathcal{T}_4$.

Now our relationship looks a bit better: $\mathcal{T}_3 \subsetneq \mathcal{T}_1 \subsetneq \mathcal{T}_2 \subset \mathcal{T}_4$.

Let's now find the relationship between $\mathcal{T}_5$ and $\mathcal{T}_1$. Let $B = (-\infty, a]$ the element in the basis for $\mathcal{T}_5$. Then let $x \in B$. Notice that $B = \bigcup_{i=1}^{+\infty}(-i, a)$. Knowing that any topology is closed under union, we then know that $B \in \mathcal{T}_1$. Thus, we got that $\mathcal{T}_5 \subset \mathcal{T}_1$. Now, notice that there is no element e such that $e \in (-\infty, 2) \subset (2, 3)$ hence, $\mathcal{T}_1$ is strictly finer than $\mathcal{T}_5$ and $\mathcal{T}_5 \subsetneq \mathcal{T}_1$.

As of now, our relationship is $\mathcal{T}_3, \mathcal{T}_5 \subsetneq \mathcal{T}_1 \subsetneq \mathcal{T}_2 \subset \mathcal{T}_4$.

Finally, let's find out the relationship between $\mathcal{T}_3$ and $\mathcal{T}_5$. They are not comparable! (I looked at the previous exercise which asks for similar but not exactly the same question/proof – **Exercise 6**). Consider two open sets $\mathbb{R} - \{2\}$ and $(-\infty, 4)$ from $\mathcal{T}_3$ and $\mathcal{T}_5$ correspondingly. Then both open sets contain the point 3, however, neither of these open sets contain the open set from the other topology that contains 3. Hence, there is no way to find out whether $\mathcal{T}_3$ is finer than $\mathcal{T}_5$, vice versa.

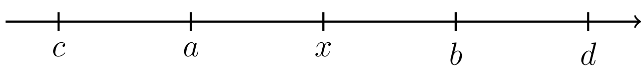
Finally, we have established the relationship $\mathcal{T}_3, \mathcal{T}_5 \subsetneq \mathcal{T}_1 \subsetneq \mathcal{T}_2 \subset \mathcal{T}_4$ with $\mathcal{T}_3$ and $\mathcal{T}_5$ being impossible to compare.

8. (a) Apply Lemma 13.2 to show that the countable collection

$$\mathcal{B} = \{(a, b) \mid a < b, a \text{ and } b \text{ rational}\}$$

is a basis that generates the standard topology on $\mathbb{R}$.

Suppose that (c, d) is the element in the basis of the standard topology. Let $x \in (c, d)$, $c < x < d$. Then $\exists a, b \in \mathbb{Q}$ such that $c < a < x < b < d$. Below is the graphical representation of how the sets would look like (it is obviously not necessary, just learning how to to use the `tikz` package ☺).



We got $x \in (a, b) \subset (c, d)$ with $(c, d) \in \mathcal{B}$. In other words, $\forall x \in B_{\mathbb{R}}$ (where $B_{\mathbb{R}}$ is the open set of $\mathbb{R}$), $\exists C$ such that $x \in C \subset B_{\mathbb{R}}$. We can now apply **Lemma 13.2** and claim that $\mathcal{B}$ is the basis that generates the standard topology on $\mathbb{R}$. $\square$

(b) Show that the collection

$$\mathcal{C} = \{[a, b) \mid a < b, a \text{ and } b \text{ are rational}\}$$

is a basis that generates a topology diffferent from the lower limit topology on $\mathbb{R}$.

Consider the point $\sqrt{5}$ in the open set $[\sqrt{5}, 5)$ of the lower limit topology. Then, since $\mathcal{C}$ can only generate open sets of the topology of the type (a, b) where $a < b \ \wedge a, b \in \mathbb{Q}$ or unions of such sets, it is clear that there is no basis element in $\mathcal{C}$ which would contain $\sqrt{5}$ and be a subset of $[\sqrt{5}, 5)$. In fact, the topology generated by $\mathcal{C}$ is strictly coarser than the lower limit topology on $\mathbb{R}$. $\square$

Section 16

3. Consider the set $Y = [-1, 1]$ as a subspace of $\mathbb{R}$. Which of the following sets are open in Y? Which are open in $\mathbb{R}$?

$$\begin{aligned}
A &= \{x \mid \frac{1}{2} < |x| < 1\}, \\
B &= \{x \mid \frac{1}{2} < |x| \leq 1\}, \\
C &= \{x \mid \frac{1}{2} \leq |x| < 1\}, \\
D &= \{x \mid \frac{1}{2} \leq |x| \leq 1\}, \\
E &= \{x \mid 0 < |x| < 1 \text{ and } 1/x \notin \mathbb{Z}^+\}.
\end{aligned}$$

Let's check the openness in Y and $\mathbb{R}$ one by one.

First consider the set $A = \{x \mid \frac{1}{2} < |x| < 1\}$. Notice that A is the union of open intervals since is open in $\mathbb{R}$. Because $Y \subset \mathbb{R}$, A is also open in Y.

Consider the set $B = \{x \mid \frac{1}{2} < |x| \leq 1\}$. We have $1 \in B$, but for any $\delta > 0$, $(1, \delta) \not\subset B$. Hence, B is not open in $\mathbb{R}$. Now notice that $B = Y \ \cap \ ((-2, -\frac{1}{2}) \cup (\frac{1}{2}, 2)) = [-1, -\frac{1}{2}) \ \cup \ (\frac{1}{2}, 1]$. Now, since, $[-1, -\frac{1}{2}) \ \cup \ (\frac{1}{2}, 1]$ is open, B is open in Y.

Now, let's take a look at the set $C = \{x \mid \frac{1}{2} \leq |x| < 1\}$. Suppose, for the sake of contradiction, that C is open in Y. Then $\exists U \in Y$ such that $C = U \cap Y$. In other words, $\exists \delta > 0$ such that $(\frac{1}{2}, \delta) \in U$. Then there would also exist $\delta\prime$ such that $(\frac{1}{2}, \delta\prime) \in C$ which is false. Thus, we have reached the contradiction and the set C is not open in Y. It follows that C is also not open in $\mathbb{R}$.

Consider the set $D = \{x \mid \frac{1}{2} \leq |x| \leq 1\}$. Suppose, for the sake of contradiction, that D is open in Y. Then $\exists U \in Y$ such that $D = U \cap Y$. In other words, $\exists \delta > 0$ such that $(\frac{1}{2}, \delta) \in U$. Then there would also exist $\delta\prime$ such that $(\frac{1}{2}, \delta\prime) \in D$ which is false. Thus, we have reached the contradiction and the set D is not open in Y. It follows that D is also not open in $\mathbb{R}$.

E is open in both Y and $\mathbb{R}$ since it is the union of open intervals.

Finally, we got that A and E are open in both Y and $\mathbb{R}$. The set B is open only in Y. And sets C and D are not open in Y or $\mathbb{R}$.

4. A map $f : X \to Y$ is said to be an **open map** if for every open set U of X, the set $f(U)$ is open in Y. Show that $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are open maps.

At first, let us show that $\pi_1 : X \times Y \to X$ is an open map. Let $U \subset X \times Y$ be an open set and let $x \in \pi_1(U)$. Then $\exists y$ such that $x \times y \in U$. Now, because U is open, there is a basis set $A \times B \in U$ such that $x \times y \in U$. Now, since $A \times B$ is a basis set, A is open in X. Besides, $x \in A = \pi_1(A \times B) \subset \pi_1(U)$. Hence, $\pi_1(U)$ is open. Therefore, $\pi_1 : X \times Y \to X$ is an open map. $\square$

Now, let's show that $\pi_2 : X \times Y \to Y$ is an open map. Let $V \subset X \times Y$ be an open set and let $y \in \pi_1(V)$. Then $\exists x$ such that $x \times y \in V$. Now, because V is open, there is a basis set $A \times B \in V$ such that $x \times y \in V$. Now, since $A \times B$ is a basis set, B is open in Y. Besides, $y \in B = \pi_1(A \times B) \subset \pi_1(V)$. Hence, $\pi_2(V)$ is open. Therefore, $\pi_2 : X \times Y \to Y$ is an open map. $\square$

6. Show that the countable collection

$$\{(a, b) \times (c, d) \mid a < b \text{ and } c < d, \text{and } a, b, c, d \text{ are rational}\}$$

is a basis for $\mathbb{R}^2$.

For simplicity, let's call this set S. Thus, $S = \{(a, b) \times (c, d) \mid a < b \text{ and } c < d, \text{and } a, b, c, d \text{ are rational}\}$.

Suppose that $(a, b) \times (c, d)$ is the element in the basis of the topology for $\mathbb{R}^2$. Let $x \in (a, b) \times (c, d)$. Then $\exists e, f, g, h \in \mathbb{Q}$ with $e < f$ and $g < h$ such that $(e, f) \times (g, h) \subset (a, b) \times (c, d)$. In other words, $\forall x \in B_{\mathbb{R}^2}$ (where $B_{\mathbb{R}^2}$ is the open set in $\mathbb{R}^2$), $\exists C$ such that $x \in C \subset B^2_{\mathbb{R}}$. We can now apply **Lemma 13.2** and claim that S is the basis that generates the topology on $\mathbb{R}^2$. $\square$

10. Let $I = [0, 1]$. Compare the product topology on $I \times I$, the dictionary order topology on $I \times I$, and the topology $I \times I$ inherits the subspace of $\mathbb{R} \times \mathbb{R}$ in the dictionary order topology.

Let's first compare the product topology on $I \times I$ and the dictionary order topology on $I \times I$. CONTINUE HERE!