Topology

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Assignment №1

2. Let $f : A \to B$ and let $A_i \subset A$ and $B_i \subset B$ for $i = 0$ and $i = 1$. Show that if f^{-1} preserves inclusions, unions, intersections, and differences of sets:

   (c) $f^{-1}(B_0 \cap B_1) = f^{-1}(B_0) \cap f^{-1}(B_1)$.

   To prove that the set $f^{-1}(B_0 \cap B_1)$ is equal to the set $f^{-1}(B_0) \cap f^{-1}(B_1)$, we have to show that $f^{-1}(B_0 \cap B_1) \subset f^{-1}(B_0) \cap f^{-1}(B_1)$ and $f^{-1}(B_0) \cap f^{-1}(B_1) \subset f^{-1}(B_0 \cap B_1)$.

   Case I: $f^{-1}(B_0 \cap B_1) \subset f^{-1}(B_0) \cap f^{-1}(B_1)$

   Let $x \in f^{-1}(B_0 \cap B_1)$. Then $f(x) \in B_0 \cap B_1$. Thus, $f(x) \in B_0$ and $f(x) \in B_1$. From this, we get that $x \in f^{-1}(B_0)$ and $x \in f^{-1}(B_1)$. Therefore, $x \in f^{-1}(B_0) \cap f^{-1}(B_1)$. Hence, if $x \in f^{-1}(B_0 \cap B_1)$, then $x \in f^{-1}(B_0) \cap f^{-1}(B_1)$ which means that $f^{-1}(B_0 \cap B_1) \subset f^{-1}(B_0) \cap f^{-1}(B_1)$. $\square$

   Case II: $f^{-1}(B_0) \cap f^{-1}(B_1) \subset f^{-1}(B_0 \cap B_1)$

   Let $x \in f^{-1}(B_0) \cap f^{-1}(B_1)$. Then $x \in f^{-1}(B_0)$ and $x \in f^{-1}(B_1)$. Thus, $f(x) \in B_0$ and $f(x) \in B_1$. Finally, we have that $f(x) \in B_0 \cap B_1$ which is equivalent to saying $x \in f^{-1}(B_0 \cap B_1)$. Hence, if $x \in f^{-1}(B_0) \cap f^{-1}(B_1)$, then $x \in f^{-1}(B_0 \cap B_1)$ which means that $f^{-1}(B_0) \cap f^{-1}(B_1) \subset f^{-1}(B_0 \cap B_1)$. $\square$

   We have now proven that $f^{-1}(B_0 \cap B_1) \subset f^{-1}(B_0) \cap f^{-1}(B_1)$ and $f^{-1}(B_0) \cap f^{-1}(B_1) \subset f^{-1}(B_0 \cap B_1)$ and thus, $f^{-1}(B_0 \cap B_1) = f^{-1}(B_0) \cap f^{-1}(B_1)$. $\square$

   (g) $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$; show that inequality holds if f is injective.

   Let's first show that $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$ even if f is not injective.

   Let $x \in f(A_0 \cap A_1)$. Then $\exists x' \in A_0 \cap A_1$ such that $f(x') = x$. Now, since $x' \in A_0$ and $x' \in A_1$, we get that $x \in f(A_0)$ and $x \in f(A_1)$ thus, $x \in f(A_0) \cap f(A_1)$. $\square$

Now let's prove that $f(A_0 \cap A_1) = f(A_0) \cap f(A_1)$ if f is injective. We have already shown that independent of whether f is injective or not, $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$. Thus, we just have to show that $f(A_0) \cap f(A_1) \subset f(A_0 \cap A_1)$ if f is injective.

Let $x \in f(A_0) \cap f(A_1)$. Then $x \in f(A_0)$ and $x \in f(A_1)$. Besides, since f is injective, there exists **unique** x' such that $f(x') = x$. Therefore, $x' \in A_0$ and $x' \in A_1$. Finally, we get that $x \in f(A_0 \cap A_1)$. $\square$

5. In general, let us denote the ***identity function*** for a set C by i_C. That is, define $i_C : C \to C$ to be the function given by the rule $i_C(x) = x$ for all $x \in C$. Given $f : A \to B$, we say that a function $g : B \to A$ is a ***left inverse*** for f if $g \circ f = i_A$; and we say that $h : B \to A$ is a ***right inverse*** for f if $f \circ h = i_B$.

   (a) Show that if f has a left inverse, f is injective; and if f has a right inverse, f is surjective.

   Let's first show that if f has a left inverse, then f is injective.

   Suppose, for the sake of contradiction, that $f : A \to B$ is function such that it has a left inverse and that f is not injective. Since f is not injective, there exists $x_0, x_1 \in A$ such that $f(x_0) = f(x_1)$ and $x_0 \neq x_1$. Since f has the left inverse, there exists $g : B \to A$ such that $g \circ f = i_A$. Consider functions $(g \circ f)(x_0)$ and $(g \circ f)(x_1)$. These functions could be rewritten as $g(f(x_0))$ and $g(f(x_1))$. Since $f(x_0) = f(x_1)$, we have that $g(f(x_0)) = g(f(x_1))$. Therefore, we got that $i_A(x_0) = i_A(x_1)$ and thus, $x_0 = x_1$. At last, we have reached the contradiction since initially we assumed that $x_0 \neq x_1$. Hence, if f has a left inverse, then f is injective. $\square$

   Now let's show that if f has a right inverse, then f is surjective.

   Suppose $f : A \to B$ is function such that it has a right inverse. Then there exists $h : B \to A$ such that $f \circ h = i_A$.

   (b) Give an example of a function that has a left inverse but no right inverse.

   Solution to b.

   (c) Give an example of a function that has a right inverse but no left inverse.

   Solution to c.

   (d) Can a function have more than one left inverse? More than one right inverse?

   Solution to d.

   (e) Show that if f has both a left inverse g and a right inverse h, then f is bijective and $g = h = f^{-1}$.

   Solution to e.