Real Analysis

Assignment №12

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7.2.1 Let f be a bounded function on $[a,b]$ and let P be an arbitraty partition of $[a,b]$. Now, let A be a collection of all possible partitions of $[a,b]$. Then $U(f) = \inf\{U(f,Q) \mid Q \in A\}$ and it follows by **Lemma 7.2.4** that $L(f,A) \leq \inf\{U(f,Q) \mid Q \in A\} = U(f)$. Thus, $U(f) \geq L(f,P)$. Now, we have $\forall A' \in A, L(f,A') \leq U(f)$ which means that $U(f)$ is an upper bound for $\{L(f,A') \mid A' \in A\}$. Finally, we get $L(f) = \{L(f,A') \mid A' \in A\} \leq U(f)$. Hence, we have proven **Lemma 7.2.6**.

□

7.2.2 (a) Notice that the function f is decreasing on interval $[1,4]$. Then we have $m_k = \inf\{f(x) \mid x \in [x_{k-1}, x_k]\} = f(x)$ and $M_k = \sup\{f(x) \mid x \, in[x_{k-1}, x_k]\} = f(x_{k-1})$. We have:

$$L(f,P) = f\left(\frac{3}{2}\right) \times \left(\frac{3}{2} - 1\right) + f(2) \times (2 - \frac{3}{2}) + f(4)(4-2) = \frac{13}{12}$$

With the similar line of reasoning, we get:

$$U(f,P) = f(1) \times (\frac{3}{2} - 1) + f\left(\frac{3}{2}\right) \times (2 - \frac{3}{2}) + f(2)(4-2) = \frac{11}{6}$$

Finally, we have $L(f,P) = \frac{13}{12}, U(f,P) = \frac{11}{6}$, and $U(f,P) - L(f,P) = \frac{3}{4}$.

(b) Notice that interval $[2,3]$ contributes $f(2)(4-2)-f(4)(4-2) = \frac{1}{2}$ to $U(f,P)-L(f,P)$. If the point 3 is added to the partition, the subpartition contribution will be $f(2)+f(3)-f(3)-f(4) = f(2)-f(4) = \frac{1}{4}$. Hence, when we add add the point 3 to the partition, $U(f,P)-L(f,P)$ decreases by $\frac{1}{2}-\frac{1}{4} = \frac{1}{4}$ and we get $U(f,P)-L(f,P) = \frac{3}{4}-\frac{1}{4} = \frac{1}{2}$.

(c) Let $P' = \left\{1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, \ldots\right\}$. Then we have $U(f,P')-L(f,P') = \frac{3}{8}$ which is less than $\frac{2}{5}$.

7.3.1 (a) Let P be a partition of $[0,1]$ s.t. it is comprised of points $0 = x_0 < x_1 < \cdots < x_n = 1$. Then, $\forall[x_{k-1}, x_k], m_k(h) = \inf\{h(t) : t \in [x_{k-1}, x_k\} = 1$. It follows that $L(h,P) = \sum_{k=1}^{n} m_k(h)\Delta x_k = \sum_{k=1}^{n} 1 \times \Delta x_k = \sum_{k=1}^{n} \Delta x_k = 1$.

(b) Let $P = \{0, \frac{19}{20}, 1\}$. Then we get:

$$U(h,P) = \sup_{i\in[0,\frac{19}{20}]} h(i) \times \frac{19}{20} + \sup_{i\in[\frac{19}{20},1]} h(i) \times \frac{1}{20} = \frac{19}{20} + \frac{2}{20} = 1 + \frac{1}{20} < 1 + \frac{1}{10}$$

(c) Let $\epsilon > 0$ be given. Choose $k \in \mathbb{R}$ s.t. $0 < 1 - k\epsilon < \epsilon$ and define $P_\epsilon = \{0, k\epsilon, 1\}$. We then have:

$$U(h,P_\epsilon) = \sup_{i\in[0,k\epsilon]} h(i) \times k\epsilon + \sup_{i\in[k\epsilon,1]} h(i) \times (1-k\epsilon) = 2 - k\epsilon = 1 + (1-k\epsilon) < 1 + \epsilon$$

7.3.3 Notice that $L(f,P) = 0$ for any partition P. Now, take a partition P_n comprised of points of the form $x_i = \frac{i}{n^2}$ (with $x_0 = 0$). Then the length of the interval is $\Delta x_i = \frac{1}{n^2}$ and we have:

$$U(f,P_n) = \frac{1}{n^2} \times (1+1+1+\cdots+1) + \frac{1}{n^2} \times \sup\{f(j) \mid f \leq \frac{1}{n}\} = \frac{1}{n} + \frac{1}{n^2}$$

Now, let $\epsilon > 0$ be given. Then it is easy to see that $\exists N$ s.t. $\frac{1}{N} + \frac{1}{N^2} < \epsilon$. Hence, $\forall n \geq N, U(f,P_n) < \epsilon$ and we have $U(f,P) = 0$. Finally, it follows that $\int_0^1 = L(f,P) = U(f,P) = 0$. Therefore, f is integrable on $[0,1]$ and $\int_0^1 f = 0$.

7.4.3 (a) Not true.

Counterxample: Let us define function f as follows:

$$f(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q} \\ -1 \text{ if } x \notin \mathbb{Q} \end{cases}$$

Then $|f|$ is integrable on $[0,1]$, but f is not integrable on $[0,1]$.

(b) Not true.

Counterexample: Let us define function g as follows:

$$f(x) = \begin{cases} \frac{1}{n} \text{ if } x = \frac{1}{n}, n \in \mathbb{N} \\ 0 \text{ otherwise} \end{cases}$$

Then $\forall x_n = \frac{1}{n}, g(x_n) > 0$ $(n \in \mathbb{N})$, however, $\int_0^1 = g(x) = 0$.

(c) This is true.

Proof:

If g is continuous on $[a, b]$ and if $g(y_0) > 0$, then $\exists \delta > 0$ s.t. $\forall (y_0 - \delta, y_0 + \delta), g(x) > 0$. Pick $m = \inf\{g(y) \mid y_0 - \delta \leq y \leq y_0 + \delta\}$. Then we have:

$$\int_a^b g \geq \int_{y_0-\delta}^{y_0+\delta} g \geq 2m\delta > 0$$

Hence, if g is continuous on $[a, b]$ and $g(x) \geq 0$ with $g(y_0) > 0$ for at least one point $y_0 \in [a, b]$, then $\int_a^b g > 0$.

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