This exam is due **Monday, January 18, at 11:00 p.m.** Write your solutions in LaTeX or scan your neatly handwritten solutions and upload a pdf to this folder. You may refer to your notes, homework, and textbook. Ask me for clarification on the wording of problems. **Do not discuss the exam with anyone else or use any outside sources.**

1. (7 pts) Define $f(x) = \begin{cases} x^4 \sin(1/x^2) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

   (a) Is $f(x)$ continuous at 0?

   (b) Is $f(x)$ differentiable at 0?

   (c) Is $f'(x)$ continuous at 0?

   (d) Is $f'(x)$ differentiable at 0?

2. (6 pts) Let f be a differentiable function on $[0, 4]$ with $f(0) = 2,\ f(4) = 1$, and $f'(1) = 2$.

   (a) Show $f(x)$ has a fixed point on $[0, 4]$.

   (b) Show $\exists\, c \in (0, 4)$ such that $f'(c) = 0$.

3. (5 pts) Let $g_n(x) = \dfrac{x^n e^{-x}}{n!}$ on $[0, \infty)$.

   (a) Find the pointwise limit $g(x) = \lim_{n\to\infty} g_n(x)$.

   (b) Does the sequence of functions (g_n) converge uniformly to g on $[0, \infty)$?

4. (7 pts) Let $f_n(x) = xe^{-nx^2}$.

   (a) Compute $f_n'(x)$.

   (b) Find the maximum and minimum values of f_n and where they occur. Sketch $f_n(x)$ for a typical n.

   (c) Find the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$.

   (d) Prove that f_n converges uniformly to f on $\mathbb{R}$.

   (e) Does $f'(x) = \lim_{n\to\infty} f_n'(x)$?

5. (7 pts) Let $g(x) = \sum_{n=1}^{\infty} \frac{\cos(3^n x)}{2^n}$.

   (a) Is g continuous on $\mathbb{R}$?

   (b) Is g is differentiable on $\mathbb{R}$?

6. (6 pts) $\mathbb{Q}$ is countable, therefore it has some enumeration $\{r_1, r_2, r_3, \ldots\}$.
   Define a sequence of functions

$$f_n(x) = \begin{cases} 1/2^n & \text{if} \quad x > r_n \\ 0 & \text{if} \quad x \leq r_n \end{cases}$$

   Prove that $f(x) = \sum_{n=1}^{\infty} f_n(x)$ converges on $\mathbb{R}$, is increasing on $\mathbb{R}$, and is continuous on the irrationals.

7. (12 pts) Let $f(x) = \ln(1 + x)$.

   (a) Use the formula $a_n = f^{(n)}(0)/n!$ to find the Taylor series representation for $f(x)$ centered at 0.

   (b) Find the interval of convergence for the Taylor series.

   (c) Does the Taylor series converge uniformly to f on its interval of convergence?