Real Analysis

Assignment №5

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3.2.2 (a) The limit points of A are 1 and -1. The limit points of B are all numbers in the closed interval $[0, 1]$.

Limit points of A are $1, -1$ as $\lim_{n\to\infty} A = 1, -1$.

Limit points of B are all numbers in the closed interval $[0, 1]$ since between any two rationals there are infinitely many irrationals, vice versa. Hence, the limit points of B form the closed interval $[0, 1]$.

(b) A **is neither closed nor open**. B **is neither closed nor open**. This is due to the fact that between any two rational numbers, there exists infinitely many irrational numbers.

(c) All points in A except for 1 and -1 are isolated. Due to the fact that between any two rational numbers, there exists infinitely many rational numbers, **B has no isolated points**.

(d) $\overline{A} = A \cup \{1, -1\} = \left\{(-1)^n + \dfrac{2}{n} \mid n = 1, 2, 3, \dots\right\} \cup \{-1\}$.

$\overline{B} = A \cup [0, 1] = \{x \in \mathbb{Q} \mid 0 < x < 1\} \cup [0, 1] = [0, 1]$.

3.2.6 (a) This is false.

Let $S = (-\infty, \sqrt{3}) \cup (\sqrt{3}, +\infty)$. Then the union of two open sets $(-\infty, \sqrt{3})$ and $(\sqrt{3}, +\infty)$ is also open. Notice that S contains all numbers in $\mathbb{R}$ except for $\sqrt{3}$ which is irrational. Hence, S contains all rational numbers. However, it does not contain $\sqrt{3}$ which means that it does not contain all real numbers.

$\square$

(b) This is false.

Let us define $S_n = [n, \infty)$. Then we get $\bigcap_n S_n = \emptyset$.

(c) This is true.

Let S be a nonempty open set. Then, as S is not empty, $\exists x$ s.t. $x \in S$. Now, since S is also open, $\exists \epsilon > 0$ s.t. $V_\epsilon(x) \subseteq S$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, $\exists r \in \mathbb{Q}$ s.t. $x - \epsilon < r < x + \epsilon$. Finally, we get $r \in V_\epsilon(x) \subseteq S$.

$\square$

(d) This is false.

Consider $S = \{\frac{1}{n} + \sqrt{3} \mid n \in \mathbb{N}\} \cup \{\sqrt{3}\}$. Then, S is a bounded infinite closed set, however, every number in S is irrational.

$\square$

(e) This is true.

The Cantor set is the intersection of closed sets. Now, since the arbitrary intersection of closed sets is closed (proven during the class period, can be proved by taking complements and applying De Morgan's law), the Cantor set must be closed.

3.2.8 (a) $\overline{A \cup B}$ **is sometimes open**. If we set $A = B = \mathbb{R}$, then $A \cup B$ is open. On the other hand, if $A = B = [0, 1]$, then $\overline{A \cup B} = [0, 1]$ which means that $\overline{A \cup B}$ is not open.

$\overline{A \cup B}$ **is definitely closed** since for any set S, $\overline{S}$ is definitely closed.

$\overline{A \cup B}$ **is sometimes both open and closed (aka *clopen*)**. If $A = B = \mathbb{R}$, then $\overline{A \cup B} = \mathbb{R}$ which is both open and closed. However, if we set $A = B = [0, 1]$, then $\overline{A \cup B} = [0, 1]$ which is not open and thus, $\overline{A \cup B}$ is not both open and closed.

$\overline{A \cup B}$ **can never be neither open nor closed** as $\overline{A \cup B}$ is always closed.

(b) $\overline{A \setminus B}$ **is definitely open** since $\overline{A \setminus B} = \overline{A \cap B^c}$. Then, since B is closed, its complement is open. Finally, as both A and B^c are open, $\overline{A \setminus B}$ is open too.

$\overline{A \setminus B}$ **is sometimes closed**. If we let $A = \mathbb{R}$ and $B = \emptyset$, then $A \setminus B = \mathbb{R}$ is closed. However, if we let $A = (0, 5)$ and $B = [1, 6]$, then $A \setminus B = (0, 1)$ which is not closed.

$\overline{A \setminus B}$ **is sometimes both open and closed (aka *clopen*)**. If we let $A = \mathbb{R}$ and $B = \emptyset$, then $A \setminus B = \mathbb{R}$ is closed. Hence, in this case, $\overline{A \setminus B}$ is both open and closed (open since we showed in (a) that it is always open). However, if we let $A = (0, 5)$ and $B = [1, 6]$, then $A \setminus B = (0, 1)$ is not closed.

$\overline{A \setminus B}$ **can never be neither open nor closed** as $\overline{A \setminus B}$ is always open.

(c) $(A^c \cup B)^c$ **is definitely open**. This is the case since if A is open, A^c is closed. Now, as B is closed $A^c \cup B$ is also closed. Hence, $(A^c \cup B)^c$ is open.

$(A^c \cup B)^c$ **is sometimes closed**. If we let $A = B = \mathbb{R}$, then $(A^c \cup B)^c = \emptyset$ which is closed. On the other hand, if we let $A = (0, 1)$ and $B = \emptyset$, then $(A^c \cup B) = (0, 1)$ which is not closed.

$(A^c \cup B)^c$ **is sometimes both open and closed (aka *clopen*)**. If we let $A = \mathbb{R}$ and $B = \emptyset$, then $A \setminus B = \mathbb{R}$ is closed. Hence, in this case, $\overline{A \setminus B}$ is both open and closed (open since we showed in (a) that it is always open). However, if we let $A = (0, 5)$ and

$B = [1, 6]$, then $A \setminus B = (0, 1)$ is not closed.

$(A^c \cup B)^c$ **can never be neither open nor closed** as $(A^c \cup B)^c$ always open.

(d) Notice that $(A \cap B) \cup (A^c \cap B) = B$.

$(A \cap B) \cup (A^c \cap B) = B$ **is sometimes open**. If $B = \mathbb{R}$ then it is open. However, if $B = [0, 1]$, it is not open.

$(A \cap B) \cup (A^c \cap B) = B$**, by definition, is definitely closed**.

$(A \cap B) \cup (A^c \cap B) = B$ **is sometimes both open and closed (aka *clopen*)**. If $B = \mathbb{R}$, then it is both open and closed. However, if $B = [0, 1]$, then it is not open (but it is still open as it is always open).

$(A \cap B) \cup (A^c \cap B) = B$ **can never be neither open nor closed** as $(A^c \cup B)^c$ always open.

(e) Notice that since A is open, A^c is closed and thus, $\overline{A^c} = A^c$. Hence, we get $\overline{A}^c \cap A^c = \overline{A}^c$.

$\overline{A}^c \cap A^c = \overline{A}^c$ **is definitely open**. This is by definition. As A is open, $\overline{A}$ is closed and its complement $\overline{A}^c$ must be open.

$\overline{A}^c \cap A^c = \overline{A}^c$ **is sometimes closed**. If we let $A = \emptyset$, then $\overline{A}^c = \mathbb{R}$ is closed (and open as well). However, if $A = (0, 1)$, then $\overline{A}^c = (-\infty, 0) \cup (1, +\infty)$ which is not closed.

$\overline{A}^c \cap A^c = \overline{A}^c$ **is sometimes both open and closed (aka *clopen*)**. If we let $A = \emptyset$, then $\overline{A}^c = \mathbb{R}$ which is both open and closed. However, if $A = (0, 1)$, then $\overline{A}^c = (-\infty, 0) \cup (1, +\infty)$ which is not closed.

$\overline{A}^c \cap A^c = \overline{A}^c$ **can never be neither open nor closed** as $(A^c \cup B)^c$ always open.

3.2.14 (a) Let us first show that E is closed if and only if $\overline{E} = E$. We will first prove this directly and then prove its converse.

Suppose that E is closed. Then E must contain all of its limit points. Let us denote the set of all limit points of E as L. Then it follows that $L \subseteq E$ but $\overline{E} = E \cup L$. Thus, $\overline{E} = E$.

□

Conversely, suppose that $\overline{E} = E$. Then it follows that E contains all of its limit points since $\overline{E}$ contains all of the limit points of E. Hence, E must be closed.

□

Finally, we have shown that E is closed if and only if $\overline{E} = E$.

□

Let us now show that E is open if and only if $E^\circ = E$. Similarly, we will first prove this statement directly and then prove its convers.

Suppose E is open. Then $\forall x \in E, \exists V_\epsilon(x) \subseteq E$. It follows that $x \in E^\circ$ and thus, $E \subseteq E^\circ$. On the other hand, by definition, $E^\circ \subseteq E$. Hence, $E^\circ = E$

□

Conversely suppose $E^\circ = E$. Then, by definition, since E° is open, E must be too.

□

Finally, we have shown that E is open if and only if $E^\circ = E$.

□

(b) Let us first show that $\overline{E}^c = (E^c)^\circ$. In order to prove this, we first show that $\overline{E}^c \subseteq (E^c)^\circ$ and then show that $(E^c)^\circ \subseteq \overline{E}^c$.

Let $x \in \overline{E}^c$. Then, as $\overline{E}^c$ is open, $\exists V_\epsilon(x) \subseteq \overline{E}^c$. Now, since $E \subseteq \overline{E}$, it follows that $\overline{E}^c \subseteq \overline{E^c}$.

Now, let $x \in (E^c)^\circ$. Then $\exists V_\epsilon(x) \subseteq \overline{E}^c \subseteq E^c$. It follows that $V_\epsilon(x) \cap E = \emptyset$.

Now, notice that we $V_\epsilon(x) \cap \overline{E} = \emptyset$. To prove this, suppose, for the sake of contradiction, that $V_\epsilon(x) \cap \overline{E} \neq \emptyset$. Then $\exists y \in \overline{E}$ s.t. $y \in V_\epsilon(x)$ (with $V_\epsilon(x)$ being open). Then there must exists some ϵ-neighborhood of y that is contained in $V_\epsilon(x)$. However, ϵ-neighborhood of y contains points of E which contradicts $V_\epsilon(x) \cap E = \emptyset$ (which we have already shown). Therefore, $V_\epsilon(x) \cap \overline{E} = \emptyset$. Hence, $V_\epsilon(x) \subseteq \overline{E}^c$ and it follows that $(E^c)^\circ \subseteq \overline{E}^c$.

Finally, we have shown both $\overline{E}^c \subseteq (E^c)^\circ$ and then show that $(E^c)^\circ \subseteq \overline{E}^c$. Hence, $\overline{E}^c = (E^c)^\circ$.

$\square$

Let us now show that $\overline{E}^c = (E^c)^\circ$. Recall that we have already shown $\overline{E}^c = (E^c)^\circ$. If we simply substitute E with E^c in this equality, we get $\overline{E^c}^c = ((E^c)^c)^\circ = E^\circ$. Taking the complement of both sides gives us $\overline{E^c} = \overline{E^\circ}^c$.

$\square$