The Topology of Robotic Configuration and Motion Planning

*The paper is written in scope of the independent study course with Dr. Eric Westlund.

David Oniani

Luther College

oniada01@luther.edu

January 30, 2019

Abstract

This paper explores the topological approach to the problem of robot motion planning. Particularly, we will discuss the safe way to coordinate automated guided vehicles or AGVs. AGVs are mobile robots which are used extensively in manufacturing facilities. Since these robots are costly and cannot tolerate collisions, one of the biggest challenges in designing such facility is setting up mobile robot routes to achieve the safe and efficient coordination of robots. The tools and concepts of topology are naturally employed in this planning process. This paper follows the bottom-up approach by first introducing concepts and then building up on these ideas. It does not assume any background neither in topology nor in robotics and is therefore accessible to most undergraduate mathematics students with the knowledge of set theory.

Configuration Spaces

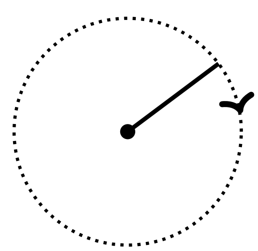
We shall start by introducing the notion of configuration spaces. The idea of configuration spaces come from physics. In classical mechanics, the configuration space is the vector space defined by the generalized coordinates (coordinates that describe the configuration of the physical system). Put it simply, the configuration space is the set of all possible states that could exist in the physical system. For instance, the configuration space of some particle in the room is the set of all points/states of the type (x, y, z) where x, y and z are the coordinates bounded by the room. If the room is a $3 \times 3 \times 3$ cube then we define the configuration space of the particle by

$$C^3(\text{room}) = \{(x, y, z) \mid 0 \leq x, y, z \leq 3\}.$$

In other words, the configuration space of the particle, is all of $3 \times 3 \times 3$ cube (this example obviously assumes that the particle is allowed to move freely in the room). The configuration space of the same particle in a spherical room, however, would be the set of all points that are in a sphere.

It appears that the physical notion of configuration spaces is very much connected to that of mathematics. In fact, the idea is the same but rather generalized. To better understand configuration spaces, let us first go through several *classic* examples.

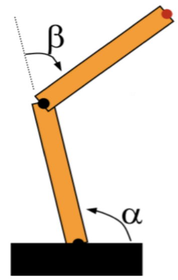
[1] Consider a planar system where we have a rod with a fixed end that can rotate freely. Then it is easy to see that the space of all possible configurations of the rotating rod is a circle.



Circle obtained by the rotational motion of the rod.

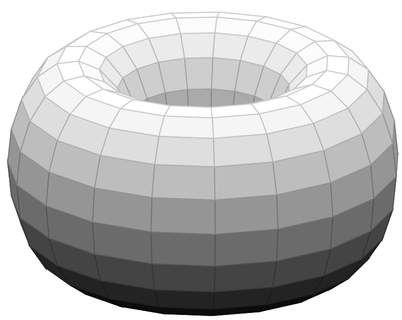
In other words, as the rod rotates, it creates the circle around itself with the radius equal to the length of the rod. This configuration space is also known to as S^1.

[2] On the other hand, the configuration space of the two-rod system in 3D space where one rod is fixed and the other one is attached to it (also known as a 2R robot) is a torus. This space is also known as $S^1 \times S^1$ configuration space.



A two-rod system ($2R$ robot).

We already know that a rod with fixed end generates a circle. In this case, we have two rods: one attached to the ground and the other one attached to the end of the first one. Then obviously both of the rods can go through a full circle of states and therefore, create a configuration space which geometrically represents a torus.



A torus obtained by the motion of two-rod system.

As of now, this is all we need to know about the configuration spaces. This idea will be very useful once we learn more about other topological concepts.

Topological Spaces

The fundamental idea in topology is that of a topological space. We will use this idea to then introduce and define other important concepts.

Definition. [3] A ***topology*** on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

(1) $\varnothing$ and X are in $\mathcal{T}$.

(2) The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

(3) The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

A set X for which a topology $\mathcal{T}$ has been specified is called a ***topological space***.

Let us first look at some examples. Consider a set $X = \{a, b, c\}$. Then we can define a topology on X by $\mathcal{T} = \{\varnothing, \{a, b, c\}, \{a, b\}, \{c\}\}$. Observe that $\varnothing, X \in \mathcal{T}$ therefore the first criterion is satisfied. It is easy to see that arbitrary unions will be in $\mathcal{T}$ since the only "interesting" case is when we consider $\{a, b\}$ and $\{c\}$, but in this case $\{a, b\} \cup \{c\} = \{a, b, c\} \in \mathcal{T}$. This satisfies the second requirement. Finally, any arbitrary intersection of the finite subcollections of $\mathcal{T}$ is also in $\mathcal{T}$ and therefore, we conclude that $\mathcal{T}$ is indeed a topology on X. Hence, X is a topological space (note that, properly speaking, a topological space is an ordered pair $(X, \mathcal{T})$, but we often omit mentioning $\mathcal{T}$ and say that X is a topological space).

At this point, you might have already noticed that one could always define more than one topology for a given set. In the previous example, sets $\mathcal{P}(x)$ (powerset of x) and $\{\varnothing, X\}$ are also topologies on X called *discrete* and *indiscrete* topologies. In fact, for any set X, $\mathcal{P}(x)$ and $\{\varnothing, X\}$ will always be two distinct topologies on X.

We will now get acquainted with the notion of the *open set*.

Definition. [4] If X is a topological space with topology $\mathcal{T}$, we say that a subset U of X is an **open set** of X if U belongs to the collection $\mathcal{T}$.

Consider our topology $\mathcal{T} = \{\varnothing, \{a, b, c\}, \{a, b\}, \{c\}\}$ on the topological space $X = \{a, b, c\}$. Then notice that $\varnothing, \{a, b, c\}, \{a, b\}, \{c\} \in \mathcal{T}$ and therefore, all are open sets.

Continuous Functions and Homeomorphisms

The notion of continuous functions is familiar to most high-school students. Most people associate them with a nice-looking monotonically increasing or decreasing functions with no leaps or jumps. There are several definitions for a function continuity. Here is the calculus definition.

Definition. [5] A function $f(x)$ is continuous at a point $x = c$ if and only if it meets the following three conditions:

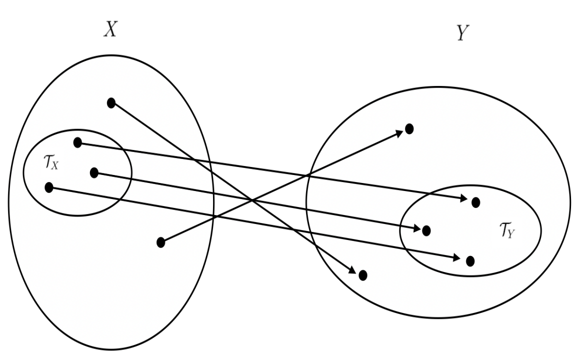
1. $f(c)$ exists (c lies in the domain of f).
2. $\lim_{x \to c} f(x)$ exists (f has a limit as $x \to c$).
3. $\lim_{x \to c} f(x) = f(c)$ (the limit equals to the function value).

In topology we cannot use this definition since the definition assumes that one could take a limit of the function. This, however, is sometimes very difficult or nearly impossible when considering functions defined over more abstract sets such as the configuration space of AGV. Now, we shall introduce more general notion of continuity.

Definition. [6] Let X and Y be topological spaces. A function $f : X \to Y$ is said to be **continuous** if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.

It is important to note that in this definition $f^{-1}(V)$ does not refer to the inverse of the function. Therefore, we are not assuming that $f : X \to Y$ is a bijection. $f^{-1}(V)$ refers to the preimage (the elements of the domain that map to some elements in the codomain) of the function. In other words, the function $f : X \to Y$ over two topological spaces X and Y is continuous if and only if every open set in the image is mapped by an open set from the preimage.

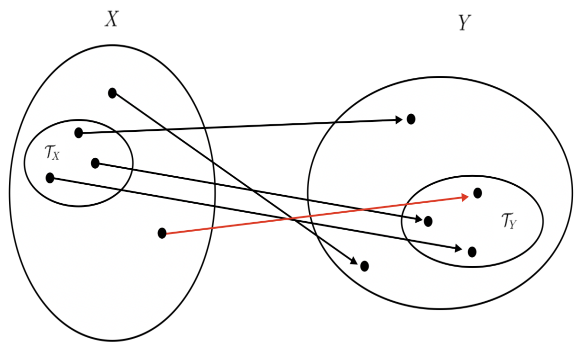
This is the case where the visualization might be useful to see how the definition of continuous functions really works. Consider a continuous function $f : X \to Y$ where X and Y are topological spaces with topologies $\mathcal{T}_X$ and $\mathcal{T}_Y$.



A continuous function $f : X \to Y$.

In the figure above, the big blobs X and Y are topological spaces and the smaller blobs $\mathcal{T}_X$ and $\mathcal{T}_Y$ are the topologies on X and Y correspondingly. By the definition, the elements in $\mathcal{T}_X$ or $\mathcal{T}_Y$ are open sets. Then notice that the function $f : X \to Y$ shown above is continuous. All the points which are in $\mathcal{T}_Y$ have a preimage in $\mathcal{T}_X$. Note that there are points in Y that do not have a preimage in $\mathcal{T}_X$, but it is of no importance to the continuity of f since it must be open sets in Y (not just any set) that must have an open preimage.

The function below, however, is not continuous as there is one point that is not in $\mathcal{T}_X$ but maps to a point in $\mathcal{T}_Y$ (it is highlighted with the red arrow).



A discontinuous function $f : X \to Y$.

Now that we are familiar with the notion of continuous functions, we shall introduce a new concept, that of homeomorphism.

Definition. [7] Let X and Y be topological spaces; let $f : X \to Y$ be a bijection. If both the function f and the inverse function

$$f^{-1} : Y \to X$$

are continuous, then f is called a homeomorphism.

The important fact about homeomorphisms is that they preserve the *topological structure*. Roughly speaking, a topological space is a geometric object. We say that two objects are homeomorphic if one can be obtained from the other by continuous deformation (vice versa). From the topological viewpoint, homeomorphism implies equality and therefore, two objects are considered the same if they are homeomorphic to each other.

Connectedness and Path Connectedness

One of the important ideas in topology is that of connectedness. This idea is used extensively in various other fields of mathematics such as graph theory and knot theory. Let us first define connectedness.

Definition. [8] Let X be a topological space. A **separation** of X is a pair U, V of disjoint nonempty open subsets of X whose union is X. The space X is said to be **connected** if there does not exist a separation of X.

Consider a topological space $X = \{a, b, c\}$ with a topology $\mathcal{T} = \{\varnothing, \{a, b, c\}, \{a, b\}, \{c\}\}$. Then it is easy to see that X is a disconnected space. Sets $\{a, b\}$ and $\{c\}$ are open since $\{a, b\}, \{c\} \in \mathcal{T}$. Besides, $\{a, b\} \cap \{c\} = \varnothing$ and $\{a, b\} \cup \{c\} = \{a, b, c\} = X$ with $\{a, b\}, \{c\} \neq \varnothing$. Hence, $U = \{a, b\}$ and $V = \{c\}$ is a pair of disjoint open subsets of X whose union is X and therefore U, V is a separation of X. This means that X is a disconnected space.

On the other hand, a topological space $Y = \{a, b\}$ with a topology $\mathcal{T} = \{\varnothing, \{a, b\}, \{a\}\}$ is a connected space as there is no pair of disjoint nonempty open subsets of Y such that their union is Y. In other words Y has no separation. Note that $\varnothing \cup \{a, b\} = \{a, b\} = Y$, but it must be <u>nonempty</u> sets whose union is Y. Therefore, $\varnothing, \{a, b\}$ is not a separation of Y.

Knowing what it means for a topological space to be connected, we can now introduce the notion of path connectedness.

Definition. [9] Given points x and y of the space X, a **path** in X from x to y is a continuous map $f : [a, b] \to X$ of some closed interval in the real line into X, such that $f(a) = x$ and $f(b) = y$. A space X is said to be ***path connected*** if every pair of points of X can be joined by a path in X.

Path connected spaces are certainly related to the connected spaces. [10] In fact, one could easily verify that path connectedness implies connectedness. Therefore, every path-connected space is connected.

There is an important theorem on path connected spaces which implies that a product of two path connected spaces is path connected. Let us prove this theorem.

Proof. Suppose that X and Y are path-connected spaces. Let $x_1 \times x_2, y_1 \times y_2 \in X \times Y$. Notice that $X \times y_1$ is homeomorphic to X and thus, is path-connected. Therefore, there exists a continuous function $f : [0, 1] \to X \times y_1$ with $f(0) = x_1 \times y_1$ and $f(1) = x_2 \times y_2$. Besides, $x_2 \times Y$ is homeomorphic to Y and thus, is path-connected. Therefore, there exists a continuous function g such that $g : [0, 1] \to x_2 \times Y$ with $g(0) = x_2 \times y_1$ and $g(1) = x_2 \times y_2$. Let's now define a function h in the following way:

$$h(x) = \begin{cases} f(\frac{x}{2}) \text{ if } 0 \leq x \leq \frac{1}{2} \\ f(\frac{x}{2} + \frac{1}{2}) \text{ if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

The according to the [11] **the pasting lemma**, h is continuous. Besides, $h(0) = f(\frac{0}{2}) = f(0) = x_1 \times y_1$ and $h(1) = g(\frac{1}{2} + \frac{1}{2}) = g(1) = x_2 \times y_2$. Therefore, h is a path from $x_1 \times y_1$ to $x_2 \times y_2$ and because $x_1 \times y_1$ and $x_2 \times y_2$ are arbitrary, we have that $X \times Y$ is path-connected. □

Now, because the product of two spaces is path connected, we can generalize this notion to n path connected spaces $X_1, X_2, \ldots, X_N$ and say that the product $X_1 \times X_2 \times \cdots \times X_N$ is also path connected. The proof of this using induction is almost trivial so we will not go through it in this paper.

Configuration Spaces Revisited - Robotic Configurations

In previous sections, we went through what is called a configuration space. Yet, we did not have a precise definition for it since it varies from field to field. Let us now define what a configuration space means in the robotics context.

Definition. [12] A ***configuration*** of a robot is a specification of the position of all points of a robot. The ***configuration space*** of a robot is the space of all configurations of a robot.

Suppose that we have a robot R that can move freely on a line L. Then we can model the robot as a point with a coordinate x_R. The configuration space for this robot is

$$C^1(L) = \{x_R \mid x_R \in L\} = L = L^1.$$

with every point in the room being a unique configuration of the robot. The dimension of C (in this case 1), is also called ***degree(s) of freedom***. In this case, the configuration space C has 1 degree of freedom.

What if we had 2 robots (say R_1 and R_2) that can move freely? Then the configuration space would be C^2 which can be represented as the set

$$C^2(L) = \{(x_{R_1}, x_{R_2}) \mid x_{R_1}, x_{R_2} \in L\} = L \times L = L^2.$$

The result here is somewhat natural. If we have one robot moving on a line, we have a configuration space L. If there are two robots, the configuration changes from a single point to a 2-tuple ((x_{R_1}, x_{R_2}) in the case) which yields a configuration space equal to L^2.

In general, the configuration space for n robots $R_1, R_2, \ldots, R_n$ that can move freely on a line L is

$$C^n(L) = \{(x_{R_1}, x_{R_2}, \ldots, x_{R_n}) \mid x_{R_1}, x_{R_2}, \ldots, x_{R_n} \in L\} = \underbrace{L \times L \times \ldots\ldots L}_{n \text{ times}} = L^n.$$

In fact, we could generalize this notion even further. Instead of having n robots moving on a line L, consider n robots moving on some k-dimensional space L^k. Then each configuration will be a tuple consisting of n number of k-tuples and the configuration space of all n robots will be a set of all such tuples. Therefore, we have

$$\begin{aligned} C^n(L^k) = \{((x_{R_{1,1}}, x_{R_{1,2}}, \ldots, x_{R_{1,k}}), \ldots, (x_{R_{n,1}}, \ldots, x_{R_{n,k}})) \mid\ & (x_{R_{1,1}}, x_{R_{1,2}}, \ldots, x_{R_{1,k}}), \\ & \ldots, \\ & (x_{R_{n,1}}, \ldots, x_{R_{n,k}}) \in L^K\} \\ & = \underbrace{L^k \times L^k \times \cdots \times L^k}_{n \text{ times}} \\ & = L^{nk}. \end{aligned}$$

where $(x_{R_{1,1}}, x_{R_{1,1}}, \ldots, x_{R_{1,k}})$ is the coordinate of R_1, $(x_{R_{2,1}}, x_{R_{2,1}}, \ldots, x_{R_{2,k}})$ is the coordinate of R_2, etc.

Safe Robotic Configurations

At this point, we have enough machinery to understand safe robotic configurations. Let us model each robot with a point that moves through a topological space represeting the robot routes in the factory.

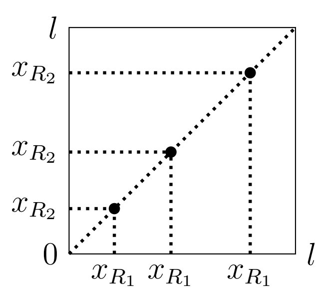
Consider two robots R_1 and R_2 moving through the line represented by L (L is obviously a topological space). Then the configuration space for these robots is

$$C = \{(x_{R_1}, x_{R_2}\} \mid x_{R_1}, x_{R_2} \in L\} = L \times L = L^2.$$

Now, since x_{R_1} and x_{R_2} represent the coordinates of the robots, we cannot allow them to be the same. In other words, if $x_{R_1} = x_{R_2}$, we have a collision. We therefore modify our configuration space C to eliminate all the configurations where $x_{R_1} = x_{R_2}$. Let us call this new configuration space SC (a safe configuration space) with

$$SC = \{(x_{R_1}, x_{R_2}) \mid x_{R_1}, x_{R_2} \in \mathbb{R}, x_{R_1} \neq x_{R_2}\}.$$

It is really helpful to think about this configuration space geometrically. Particularly, let's think about this space as a small chunk of the XOY coordinate system. To do this, we will need to do some transformations. Since L is a line we might as well represent it as a closed interval $L = [0, l]$ where l is the length of the line L and therefore $l \in \mathbb{R}$. Now we can say that both robots R_1 and R_2 are moving through the interval $[0, l]$. Let us now consider the motion of R_1 and R_2 independently. To do this, we make a copy of the interval $[0, l]$, put one interval as an X axis and the other one as the Y axis.



The configuration space C as a chunk of XOY coordinate system.

From the picture above, it is easy to see that the points where $x_{R_1} = x_{R_2}$ are the points on the **diagonal** of the square bounded by interval $[0, l]$ on X axis and its copy on the Y axis. We denote this diagonal by Δ. Then we can simply say that

$$SC = C - \Delta.$$

In the general case with n robots and the topological space X, it is easy to see that

$$SC^n = \underbrace{X \times X \times \cdots \times X}_{n \text{ times}} - \Delta \text{ where } \Delta = \{(x_{R_1}, x_{R_2}, \ldots, x_{R_n}) \mid x_{R_i} = x_{R_j} \text{ for some } i \neq j\}.$$

We are basically taking the product of n copies of X because we have n robots and account for the position of each robot independently. Every configuration in this configuration space will look like $(x_{R_1}, x_{R_2}, \ldots x_{R_n})$ and we eliminate each configuration for which there exists x_{R_i} and x_{R_j} such that $x_{R_i} = x_{R_j}$ with $i \neq j$. This means that we do not allow any two robots to have the same coordinates in any configuration which ensures that there are no collisions.

Safe Robotic Relocations

Now that we know about safe robotic configurations, let us model the relocation of n robots in the space X. To do this, we need to use a path in $SC^n(X)$. Suppose that initially the robots had the configuration $(I_1, I_2, \ldots, I_n)$ where I_j is the initial position of the j-th robot. Then we define what is called the **relocation** of these n robots.

Definition. [13] We define a ***relocation*** of the n robots from the initial configuration $I = (I_1, \ldots, I_n)$ to the finial configuration $F = (F_1, \ldots, F_n)$ to be a path $p : [0, 1] \to SC^n(X)$ such that $p(0) = I$ and $P(1) = F$.

Since the robots are mapped to the configuration in $SC^n(X)$, by the definition of SC, there is no way that any two robots will occupy the same coordinate and therefore, the map will ensure that there are no collisions.

We can now introduce the notion of **attainability** of configurations using which we will define some other important concept.

Definition. [14] Given two configurations of the robots, $M = (M_1, \ldots, M_n)$ and $N = (N_1, \ldots, N_n)$, we say that N is ***attainable*** from M if there is a relocation of the robots from M to N.

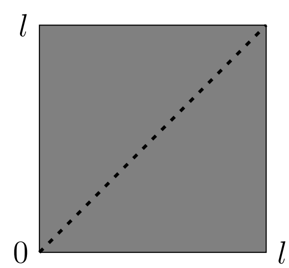
Now let's apply the definition of path connectedness that we covered a few chapter ago. Notice that if the safe configuration space $SC^n(X)$ is path connected, then every configuration is attainable from every other one and therefore, robots can move freely in the space. In this case, we say that the robots are **freely transportable in** X.

The notion of free transportability is of the utmost important to robotics. We will spend the rest of the paper exploring the free transportability under various robotic configurations.

Examples

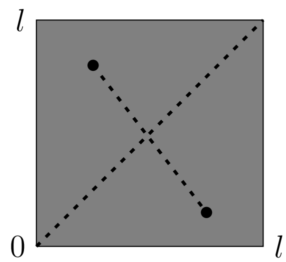
1

Consider two robots R_1 and R_2 on the line L that we have discussed previously. We know that the safe configuration space for two robots is the square without the diagonal. Note that we consider the motion of R_1 robot on the horizontal line segment $[0, l]$ and the motion of R_2 robot on the vertical $[0, l]$ line segment.



A safe configuration space for two robots moving on the line L.

Then this safe configuration space is not path connected (recall that path connectedness means that every pair of points of SC can be joined by a path in SC). Observe that the safe configuration space SC is the union of two sets $U_1 = \{(x_{R_1}, x_{R_2} \mid x_{R_1}, x_{R_2} \in L, x_{R_1} < x_{R_2}\}$ and $U_2 = \{(x_{R_1}, x_{R_2} \mid x_{R_1}, x_{R_2} \in L, x_{R_2} < x_{R_1}\}$ where U_1 is the set in which R_1 is to the left of R_2 and U_2 is the set where R_2 is to the left of R_1. Notice that one of the sets has all the configurations that lie above the excluded diagonal and the other one has all the configurations that lie below the excluded diagonal. It is easy to show that this safe configuration space is not path connected. Take one point on the one side of the diagonal and the other one on the other side (in other words, consider one configuration from the set U_1 and the other one from the set U_2). If one would connect these points, there would not be a way to avoid intersection of the diagonal.



An attempt to connect two points from one side of the excluded diagonal to the other.

Thus, since the safe configuration space SC is not path connected and by the definition of free transportability, R_1 and R_2 are not freely transportable in L.

2

Consider two robots R_1 and R_2 on the configuration space S^1 from the first section. It is interesting to know what would the configuration space look like in this case. We know that the configuration space S^1 is a circle. Then we consider the motion of R_1 and R_2 independently. Let us now apply the same trick as we did with two robots on the line L. But now, since a circle is a 2D object, we need to do it in the 3D space. Therefore, it is helpful to think about $XOYZ$ coordinate system. We put a circle for R_1 on the XOY plane and the circle for R_2 on the circle for R_1 circle in the way that that these two circles share only one point. For simplicity, we will refer to these circles as R_1 and R_2 in all further correspondence. The picture below is the visualization of our approach.

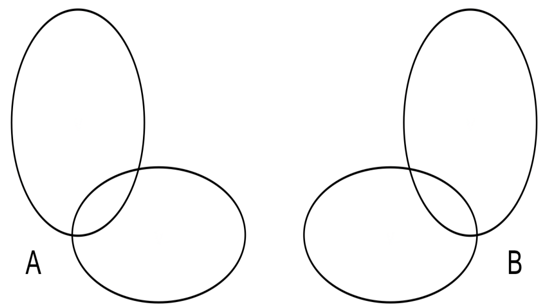
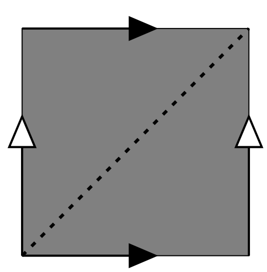


Figure A shows the first position of the circle and picture B the other one after sliding.

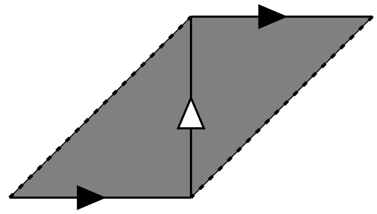
Recall that each configuration looks like (x_{R_1}, x_{R_2}). Then to get all the configurations we keep the circle on the bottom (on the XOY plane) fixed and slide the circle on the XOZ plane one point at a time. Each of these slides will generate a set of configurations which would be a set of all points of upper circle paired with a single point on the bottom circle. We will continue this motion exhaustively, until there are no points on the bottom circle. Figure B shows the picture when we are halfway through. Geometrically, however, it is easy to see that the shape we get is a torus ([15] in fact, the configuration space for n robots on the circle is the n-tori).

Obviously, we also want to know what is the safe configuration space for these two robots. Here we will use another trick. Notice that we can glue the edges of a square to obtain a torus. Therefore, we can "unfold" the torus to get the square. We can represent the torus as a square with opposite sides identified.



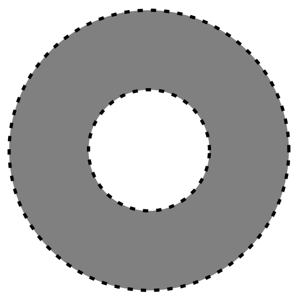
Torus as a square with opposite sides identified.

The picture above shows the *safe configuration space* which is the square minus the diagonal line. We can decompose this square into two "semi-open" triangles and put them together to get a parallelogram.



Parallelogram without left and right sides constructed from the square without diagonal.

Then it is easy to see that the space shown above is homeomorphic to the space $(0, 1) \times S^1$ which is an open annulus (we can continuously deform this parallelogram to get an open annulus).



An open annulus.

Notice $(0, 1)$ and S^1 are both path connected and therefore, as we have proven in the previous section, the product of two path connected spaces is connected and $(0, 1) \times S^1$ is path connected. Finally, we conclude that the safe configuration space $SC^2(S^1)$ is also path connected and therefore, robots R_1 and R_2 are freely transportable in the space S^1.

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