Real Analysis

Assignment №2

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2.2.1 For instance, the sequence $(1, 0, 1, 0, 1, 0, \dots)$ is vercongent. The sequence verconges at 2 since for $\epsilon = 4, \forall N \in \mathbb{N}, n \geq N$ implies $|x_n - 2| \leq 1 < \epsilon$. This vercongent sequence is also a divergent sequence. Thus, divergent vercongent sequences exist. Notice that the sequence verconges at 3 as well since $\forall x \in (1, 0, 1, 0, 1, 0, \dots), |x_n - 3| \leq 1 < \epsilon (\epsilon = 4)$. Therefore, the sequence also verconges to two different values. This definition describes **bounded sequences.**

2.2.2 (a) Let $N = \lceil \frac{1}{4\epsilon} \rceil$. Then, $\forall n \geq N$ and $\forall \epsilon > 0$, $|\frac{2n+1}{5n+4} - \frac{2}{5}| < \frac{1}{5n} < \frac{1}{5N} < \epsilon$.

NOTE: $\frac{1}{5N} \leq \frac{4\epsilon}{5} < \epsilon$.

Hence, $\lim \frac{2n+1}{5n+4} = \frac{2}{5}$.

□

(b) Let $N = \lceil \frac{3}{\epsilon} \rceil$. Then, $\forall n \geq N$ and $\forall \epsilon > 0$, $|\frac{2n^2}{n^3+3} - 0| = \frac{2n^2}{n^3+3} < \frac{2}{n} < \frac{2}{N} < \epsilon$.

NOTE: $\frac{2}{N} \leq \frac{2\epsilon}{3} < \epsilon$.

Hence, $\lim \frac{2n^2}{n^3+3} = 0$.

□

(c) Let $N = \lceil \frac{2}{\epsilon^3} \rceil$. Then, $\forall n \geq N$ and $\forall \epsilon > 0$, $|\frac{\sin n^2}{\sqrt[3]{n}} - 0| = \frac{\sin n^2}{\sqrt[3]{n}} \leq \frac{1}{\sqrt[3]{n}} < \frac{1}{\sqrt[3]{N}} < \epsilon$.

NOTE: $\frac{1}{\sqrt[3]{N}} \leq \frac{\epsilon}{\sqrt[3]{2}} < \epsilon$.

Hence, $\lim \frac{\sin n^2}{\sqrt[3]{n}} = 0$.

□

2.2.7 (a) It is frequently in $\{1\}$ since it alternates, but it is not eventually in $\{1\}$. Let us now prove this statement. We first prove that it is *frequently* in $\{1\}$ and then prove that it cannot be *eventually* in $\{1\}$.

$\forall N \in \mathbb{N}, \exists n = 2N$ with $(-1)^{2N} = 1 \in \{1\}$. Hence, the sequence is frequently in $\{1\}$

□

On the other hand, $\forall N \in \mathbb{N}, \exists n = 2N + 1$ with $(-1)^{2N+1} = -1 \notin \{1\}$. Hence, the sequence is not eventually in $\{1\}$

□

Therefore, the sequence $(-1)^n$ is frequently, but not eventually in $\{1\}$.

(b) Eventually is certainly stronger. Eventually imples frequently. This is true since if $\exists N \in \mathbb{N}$ s.t $\forall n \geq N, a_n \in A$ also implies that $\forall N \in \mathbb{N}, \exists n \geq N$ s.t. $a_n \in A$. Put it simply, $\forall$ statement is stronger than $\exists$ statement since it generalizes and applies to all numbers from a certain point while satisfying exists condition only requires finding a single case.

(c) We need to use *eventually.* Here is an alternate rephrasing of Definition 2.2.3B:

"A sequence (a_n) converges to a real number a if $\forall \epsilon > 0$, the sequence is eventually in the ϵ-neighborhood of a."

(d) It is frequently in $(1.9, 2.1)$. This is the case since $\forall N \in \mathbb{N}, \exists n \geq N$ s.t. $a_n \in (1.9, 2.1)$ (as the number of 2s is infinite).

On the other hand, it is not necessarily eventually in $(1.9, 2.1)$. A counterexample would be a sequence $(2, 0, 2, 0, \dots)$. The sequence is frequently in $(1.9, 2.1)$ (as will be all sequences with infinite number of 2s, but is not eventually in $(1.9, 2.1)$ as $2s$ and $0s$ alternate and $\forall N \in \mathbb{N}, \exists n = 2N$ with $0 \neq 2$.

2.3.3 $\forall \epsilon > 0, \exists N_1, N_2 \in \mathbb{N}$ s.t. $\forall n_1 \geq N_1$ and $\forall n_2 \geq N_2, |x_{n_1} - l| < \epsilon$ and $|z_{n_2} - l| < \epsilon$. Then, let us define $N = \max(N_1, N_2)$. It follows that $\forall n \geq N, |x_n - l| < \epsilon$ and $|z_n - l| < \epsilon$. We get $-\epsilon < x_n - l < \epsilon$ and $-\epsilon < z_n - l < \epsilon$. After adding l to all three sides of the inequalities, we then get $l - \epsilon < x_n < l + \epsilon$ and $l - \epsilon < z_n < l + \epsilon$. Now, since we know that $x_n \leq y_n \leq z_n$, we have $l - \epsilon < x_n \leq y_n$ and $y_n \leq z_n < l + \epsilon$. Therefore, $l - \epsilon < y_n < l + \epsilon$ and it follows that $|y_n - l| < \epsilon$. Hence, $\lim y_n = l$.

□

2.3.7 (a) Let $(x_n) = (-1, -1, -1 \dots)$ and $(y_n) = (1, 1, 1, \dots)$. Then both x_n and y_n diverge, but their sum $(x_n + y_n) = (0, 0, 0, \dots)$ converges to 0. Hence, such sequences do exist.

(b) Such sequences cannot exist. Suppose, for the sake of contradiction, that $(x_n), (x_n + y_n)$ are convergent and (y_n) is divergent. Then, since $y_n = (x_n + y_n) - (x_n)$, it follows by **Algebraic Limit Theorem** that y_n is also convergent and we face a contradiction. Hence, such sequences do not exist.

(c) Let $b_n = \dfrac{1}{n}$. Then b_n converges to 0 with $b_n \neq 0 \forall n \in \mathbb{N}$. However, $1/b_n = 1/(1/n) = n$ which is a divergent sequence (sequence (n) diverges). In other words, since (n) is not bounded, it is divergent by **Theorem 2.3.2**. Hence, such sequence does exist.

(d) Such sequences cannot exist. Suppose, for the sake of contradiction, that (a_n) and (b_n) are unbounded and convergent sequences respectively with $(a_n - b_n)$ being a bounded sequence. Since b_n is convergent (by **Theorem 2.3.2, it is also has to be bounded**), let its bound be B and let $(a_n - b_n)$ be bounded by D. Then it follows that $\forall n \in \mathbb{N}, |a_n| \leq |a_n - b_n| \leq |a_n - b_n| + |b_n| \leq D + B$. Thus, we got that (a_n) is bounded too and we face the contradiction. Hence, such sequences do not exist.

(e) Let $(a_n) = (0, 0, 0, \dots)$ and let $(b_n) = (1, -1, 1, \dots)$. Then (a_n) and $(a_n b_n)$ both converge to 0, but (b_n) is divergent. Hence, such sequences do exist.

2.3.13 (a) Notice that $a_{mn} = \dfrac{m}{m+n} = \dfrac{\frac{m}{m}}{\frac{m}{m} + \frac{n}{m}} = \dfrac{1}{1 + \frac{n}{m}}$.

Then we have $\lim_{m\to\infty} a_{mn} = 1$ and $\lim_{n\to\infty} \left(\lim_{m\to\infty} a_{mn}\right) = \lim_{n\to\infty} 1 = 1$.
Similarly, $\lim_{n\to\infty} a_{mn} = 0$ and $\lim_{m\to\infty} \left(\lim_{n\to\infty} a_{mn}\right) = \lim_{m\to\infty} 0 = 0$.
Finally, we got that $\lim_{n\to\infty} \left(\lim_{m\to\infty} a_{mn}\right) = 1$ and $\lim_{m\to\infty} \left(\lim_{n\to\infty} a_{mn}\right) = 0$