Real Analysis

Assignment №7

Instructor: Dr. Eric Westlund

David Oniani

Luther College

oniada01@luther.edu

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4.2.5 (a) Prove that $\lim_{x\to 2}(3x+4) = 10$.

Let $\epsilon > 0$ be given and let $\delta = \dfrac{\epsilon}{3}$. Then $0 < |x-2| < \delta$ and we have that $|x-2| < \dfrac{\epsilon}{3}$ and it follows that $|3x-6| < \epsilon$. Now, notice that $3x-6 = 3x+4-10$ and thus, $|(3x+4)-10| < \epsilon$. Hence, $\lim_{x\to 2}(3x+4) = 10$.

$\square$

(b) Prove that $\lim_{x\to 0} x^3 = 0$.

Let $\epsilon > 0$ be given and let $\delta = \sqrt[3]{\epsilon}$. Then $0 < |x-0| < \delta$ and we have that $|x| < \sqrt[3]{\epsilon}$. Now, notice that $|x|^3 = |x^3|$. and thus, $|x^3 - 0| < \epsilon$. Hence, $\lim_{x\to 0} x^3 = 0$.

$\square$

(c) Prove that $\lim_{x\to 2}(x^2+x-1) = 5$.

Let $\epsilon > 0$ be given and let $|x-2| < \delta$. Then $|x-2| < \delta$, $|x+3| < 5+\delta$. We have $0 < |x^2+x-6| < \delta$. Notice that $x^2+x-6 = (x-2)(x+3)$. Then $|(x-2)(x+3)| < \delta(\delta+5)$.

Notice that the equation $\delta(\delta+5) = \epsilon$ has a discriminant $\mathbb{D} = 25+4\epsilon > 0$ and the equation always has at least one solution since $\epsilon > 0$. Thus, $\exists\delta$ s.t. $\delta(\delta+5) < \epsilon$ and it follows that $|(x-2)(x+3)| < \epsilon$. Finally, we have that $|(x-2)(x+3)| = |(x^2+x-1)-5| < \epsilon$. Hence, $\lim_{x\to 2}(x^2+x-1) = 5$.

$\square$

(d) Let $\epsilon > 0$ be given and let $|x-3| < \delta$. Then notice that $\left|\frac{1}{x} - \frac{1}{3}\right| = \left|\frac{x-3}{3x}\right| < \frac{\delta}{3(\delta+3)}$. Now, the equation $\frac{\delta}{3(\delta+3)} = \epsilon$ always has at least one solution as the discriminant $\mathbb{D} = 9\epsilon^2 + 36\epsilon > 0$ and thus, $\forall\epsilon > 0, \exists\delta$ s.t. $\frac{\delta}{3(\delta+3)} < \epsilon$. Finally, we get that $\left|\frac{1}{x} - \frac{1}{3}\right| < \epsilon$. Hence, $\lim_{x\to 3}\frac{1}{x} = \frac{1}{3}$.

$\square$

4.2.8 (a) $\lim_{x\to 2}\frac{|x-2|}{x-2}$ does not exist.

To show this, let $x_n = \frac{1}{n} + n$ and let $y_n = -\frac{1}{n} + 2$ with $n \in \mathbb{N}$..

Then we have $\frac{|x_n-2|}{x_n-2} = \frac{\frac{1}{n}}{\frac{1}{n}} = 1$ and $\frac{|y_n-2|}{y_n-2} = \frac{\frac{1}{n}}{-\frac{1}{n}} = -1$. Now, **by Corollary 4.2.5**, the limit does not exist.

$\square$

(b) $\lim_{x\to\frac{7}{4}}\frac{|x-2|}{x-2} = -1$.

Notice that $\lim_{x\to\frac{7}{4}}|x-2| = 0.25$ and $\lim_{x\to\frac{7}{4}}(x-2) = -0.25$. Then, **per Algebraic Limit Theorem**, we get that $\lim_{x\to\frac{7}{4}}\frac{|x-2|}{x-2} = -1$.

$\square$

(c) $\lim x \to 0(-1)^{[[\frac{1}{x}]]}$ does not exist.

To show this, let $x_n = \frac{1}{2n}$ and let $y_n = \frac{1}{2n+1}$ with $n \in \mathbb{N}$.

Then we have $(-1)^{[[\frac{1}{x}]]} = (-1)^{[[2n]]} = (-1)^{2n} = 1$ and $(-1)^{[[\frac{1}{y}]]} = (-1)^{[[2n+1]]} = (-1)^{2n+1} = -1$. Now, **by Corollary 4.2.5**, the limit does not exist.

$\square$

(d) $\lim_{x\to 0}\sqrt[3]{x}(-1)^{[[\frac{1}{x}]]} = 0$.

Let $\epsilon > 0$ be given. Then let $\delta = \epsilon^3$. We have $|x-0| < \delta$ and it follows that $|\sqrt[3]{x}| < \epsilon$. We get $|\sqrt[3]{x}(-1)^{[[\frac{1}{x}]]}| - 0 = |\sqrt[3]{x}(-1)^{[[\frac{1}{x}]]}| = |\sqrt[3]{x}| < \epsilon$. Hence, $\lim_{x\to 0}\sqrt[3]{x}(-1)^{[[\frac{1}{x}]]} = 0$.

$\square$

4.2.9 (a) Let $M > 0$ be given and let $\delta = \dfrac{1}{\sqrt{M}}$. Then, if $|x| < \delta$, we have $\dfrac{1}{x^2} > \dfrac{1}{\delta^2}$ and it follows that $\dfrac{1}{x^2} > M$. Hence, we showed that $\lim_{x\to 0} \dfrac{1}{x^2} = \infty$.

$\square$

(b) The definition would read as follows:

"We say $\lim_{x\to\infty} f(x) = L$ if $\forall\epsilon > 0, \exists M > 0$ s.t. if $x > M$ we have $|f(x) - L| < \epsilon$."

Let us now prove that $\lim_{x\to\infty} \dfrac{1}{x} = 0$. Let $\epsilon > 0$ be given and let $M = \dfrac{1}{\epsilon}$. Then if $x > M$, we have $x > \dfrac{1}{\epsilon}$. We have $\dfrac{1}{x} = |\dfrac{1}{x} - 0| < \epsilon$. Hence, $\lim_{x\to\infty} \dfrac{1}{x} = 0$.

$\square$

(c) The definition would read as follows:

"$\lim_{x\to\infty} f(x) = \infty$ if $\forall M > 0, \exists N > 0$ s.t. $\forall x > N, f(x) > M$."

For instance, $\lim_{x\to\infty} x = \infty$ is one example. In this case, given $M > 0$, we can pick $N = M$.

4.2.11 Let us first $\forall c \in \mathbb{R}$ define $S_\delta(c) = \{x \in \mathbb{R} \mid 0 < |x - c| < \delta\}$

Let $\epsilon > 0$ be given. Then, since $\lim_{x\to c} f(x) = L$, by definition, $\exists\delta_1 > 0$ s.t. $\forall x \in S_{\delta_1}(c), |f(x) - L| < \epsilon$. Similarly, we can also find δ_2 s.t. $\forall x \in S_{\delta_2}(c), |h(x) - L| < \epsilon$. Then let $\delta = \min(\delta_1, \delta_2)$. We get that $\forall x \in S_\delta(c)$, we have $L - \epsilon < f(x) \le g(x) \le h(x) < L + \epsilon$. And finally, it follows that $|g(x) - L| < \epsilon$. Hence, $\lim_{x\to c} g(x) = L$.

$\square$

4.3.3 (a) As g is continuous at $f(c)$, it follows that $\forall\epsilon > 0, \exists\delta' > 0$ s.t. if $|f(x) - f(c)|$, we have $|g(f(c)) - g(f(x))| < \epsilon$. Now, as f is continuous at c, $\forall\delta' > 0$, we get $\exists\delta > 0$ s.t. if $|x - c| < \delta$, we get $|f(x) - f(c)| < \delta'$. Hence, $g \circ f$ is continuous at c.

$\square$

(b) Let $(x_n) \to c$. Then, as f is continuous at c, we have that $(f(x_n)) \to f(c)$. We get that $(f(x_n))$ is a convergent sequence with $(f(x_n)) \to f(c)$. Since g is continuous at $f(c)$, we get $(g(f(x_n))) \to g(f(c))$ and thus, $\lim_{x\to c} g(f(x)) = g(f(c))$.

$\square$

4.3.7 (a) First consider an arbitrary $r \in \mathbb{Q}$. Now, recall that $\mathbb{I}$ is dense in $\mathbb{R}$. Then, due to the density property, there exists a sequence $(x_n) \subseteq \mathbb{I}$ s.t. $(x_n) \to r$. Then it follows that $g(x_n) = 0$ for all $n \in N$ with $g(r) = 1$. Since $\lim g(x_n) = 0 \neq g(r)$, **by Corollary 4.3.3 (Criterion for Discontinuity)**, we conclude that $g(x)$ is not continuous at $r \in \mathbb{Q}$.

Let us now consider an arbitrary $i \in \mathbb{I}$. Recall that $\mathbb{Q}$ is dense in $\mathbb{R}$. Then, due to the density property, we can find a sequence $(y_n) \subseteq \mathbb{Q}$ s.t. $(y_n) \to i$. This time $g(y_n) = 1$ for all $n \in N$ with $g(i) = 0$. Now, since $\lim g(y_n) = 1 \neq g(i)$, we can conclude that g is not continuous at i.

Now, since $g(x)$ is not continuous at any $r \in \mathbb{Q}$ as well as at any $i \in \mathbb{I}$, we conclude that Dirichlet's function is nowhere continuous on $\mathbb{R}$.

□

(b) Consider an arbitrary rational number $r \in \mathbb{Q}$. Then, notice that $t(r) \neq 0$. Now, since $\mathbb{I}$ is dense in $\mathbb{Q}$, there exists a sequence $(x_n) \subseteq I$ s.t. $(x_n) \to r$. It follows that $n \in N, t(x_n) = 0$ with $t(r) \neq 0$. Thus $\lim t(x_n) \neq t(r)$ and $t(x)$ is not continuous at r. Hence, Thomae's function fails to be continuous at every rational point.

□

(c) Let $c \in \mathbb{I}$ be an arbitrary irrational number. Given $\epsilon > 0$, set $T = \{x \in \mathbb{R} \mid t(x) \geq \epsilon\}$. If $x \in T$, then x is a rational number of the form $x = \frac{m}{n}$ with $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ where $n \leq \frac{1}{\epsilon}$. The restriction on the size of n implies that the intersection of T with the interval $[c–1, c+1]$ is finite. In a finite set, all points are isolated, so we can pick a neighborhood $V_\delta(c)$ around c such that $x \in V_\delta(c)$ implies $x \notin T$. But if $x \notin T$, then $t(x) < \epsilon$, i.e., $t(x) \in V_\delta(0) = V_\delta(t(c))$. Finally, **by Theorem 4.3.2 (iii)**, we conclude that $t(x)$ is continuous at c.

□

4.3.8 (a) It is true that $g(1) \geq 0$.

Suppose, for the sake of contradiction, that g is continuous on $\mathbb{R}$ and $g(1) < 0$. Then $\exists \epsilon = |g(1)|$ s.t. $|g(x) - g(1) \geq \epsilon|$ for every choice $\delta = |x - c|$. And we face a contradiction since we assumed that g is continuous at $x = 1 \in \mathbb{R}$. Hence, $g(1) \geq 0$ as well.

□

(b) It is true that $g(x) = 0$ for all $x \in \mathbb{R}$.

Let $x \in \mathbb{R}$ be given. Recall that $\mathbb{Q}$ is dense in $\mathbb{R}$. Then, there exists a sequence of rational numbers (s_n) s.t. $(s_n) \to x$. Now, as $g(x)$ is continuous at x, **per Theorem 4.3.2**, if follows that $g(x) = \lim_{n\to\infty} g(s_n) = 0$ and thus $g(x) = 0$ for all $x \in \mathbb{R}$.

□

(c) It is true that $g(x)$ is strictly positive for uncountably many points.

Let $c = g(x_0) > 0$. Now, since $g(x)$ continuous at x_0, $\exists \delta > 0$ s.t. $|f(x) - f(x_0)| < \dfrac{c}{2}$ for all $|x - x_0| < \delta$. Hence, $\forall |x - x_0|$ we get $-\dfrac{c}{2} < f(x) - c < \dfrac{c}{2}$ and thus, $\dfrac{c}{2} < f(x) < \dfrac{3c}{2}$. Hence, we get $\forall |x - x_0| < \delta, f(x) > 0$ and it follows that $g(x)$ is strictly positive for uncountably many points.

$\square$