Real Analysis Exams

Exam №4

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1. (a) Notice that we have:

$$\limsup a_n = \lim_{n\to\infty} \left(4 + \frac{1}{n}\right) \cos\left(\frac{n\pi}{4}\right) = 4 \times 1 = 4$$

Similarly, $\liminf a_n = 4 \times (-1) = -4$. Hence, $\limsup a_n = 4$ and $\liminf a_n = -4$.

(b) Notice that the subsequence $a_k = \left(4 + \frac{1}{8k}\right) \cos(2\pi k)$ ($n = 8k$ with $k \in \mathbb{N}$) converges to 0 since $\cos(2\pi k) = 0$ and thus, every $a_k = \left(4 + \frac{1}{8k}\right) \times 0 = 0$.

(c) This set will be $A = (-4, 4)$.

2. Let $\epsilon > 0$ be given, x_0 be fixed, and let $\delta = \min\left(1, \dfrac{\epsilon}{5(1+2|x_0|)}\right)$. Then for $|x - x_0| < \delta$ we have:

$$\begin{aligned}
|f(x) - f(x_0)| &= |5x^2 + 3 - 5x_0^2 - 3| \\
&= |5(x^2 - x_0^2)| \\
&= 5|x - x_0||x - x_0 + 2x_0| \\
&< 5\delta(|x - x_0| + |2x_0|) \\
&< 5\delta(\delta + 2|x_0|) \\
&\leq 5\frac{\epsilon}{5(1+2|x_0|)}(\delta + 2|x_0|) \\
&\leq \frac{\epsilon}{1+2|x_0|}(1 + 2|x_0|) \\
&= \epsilon \qquad\qquad (\text{Thus, } |f(x) - f(x_0)| < \epsilon)
\end{aligned}$$

Hence, $f(x) = 5x^2 + 3$ is continuous at each point $x_0 \in \mathbb{R}$.

□

3. (a) Counterexample: let us define

$$f_n : [0,1] \to \mathbb{R} : x \mapsto \begin{cases} n \text{ if } 0 < x < \frac{1}{n} \\ 0 \text{ if } x = 0 \text{ or } \frac{1}{n} \leq x \leq 1 \end{cases}$$

Then notice that $\int_0^1 f_n = 1$ and the pointwise limit of f_n is $f(x) = 0$ ($x \in [0,1]$) for which we have $\int_0^1 f = 0$. Hence, we have $\int_0^1 f_n \to 1 \neq \int_0^1 f = 0$. Finally, we get that every f_n is Riemann-integrable, but f is not.

(b) As $f_n \to f$ uniformly, pick n_1 s.t. the following holds:

$$|f_{n_1}(x) - f(x)| < \frac{\epsilon}{3 \cdot (b-a)}$$

Now, since every f_n is integrable, take n_2 such that

$$|U(f_{n_1}, P_{n_2}) - L(f_{n_1}, P_{n_2})| < \frac{\epsilon}{3}.$$

Now, choose $n = \max(n_1, n_2)$,

Notice that

$$\begin{aligned}
|U(f,P_n)-U(f_n,P_n)| &\le \sum_{x_k}|f(x_k)-f_n(x_k)|\Delta x_k \\
&< \sum_{x_k}\frac{\epsilon}{3(b-a)}\Delta x_k \\
&= \frac{\epsilon}{3(b-a)}\sum_{x_k}\Delta x_k \\
&= \frac{\epsilon}{3(b-a)}(b-a) \\
&= \frac{\epsilon}{3}
\end{aligned}$$

Now, notice that over $[x_k, x_{k+1}]$, $|\sup f(x) - \sup f_n(x)| \le |f_n(x) - f(x)|$ (since every point of f_n is close to f).

A similar results holds for

$$|L(f,P_{n_2})-L(f_n,P_{n_2})| < \frac{\epsilon}{3}$$

Hence, we have:

$$\begin{aligned}
|U(f,P_n)-L(f,P_n)| &\le \Big|U(f,P_n)-U(f_n,P_n)+U(f_n,P_n)-L(f_n,P_n)-\Big(L(f,P_n)-L(f_n,P_n)\Big)\Big| \\
&\le |U(f,P_n)-U(f_n,P_n)|+|U(f_n,P_n)-L(f_n,P_n)|+|L(f,P_n)-L(f_n,P_n)| \\
&< \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}=\epsilon
\end{aligned}$$

Finally, we got that if $f_n \to f$ uniformly on $[a,b]$, then f is integrable on $[a,b]$.

□

(c) We have already shown in (b) part of the exercise that f is integrable on $[a,b]$. We now need to show that the following holds:

$$\lim_{n\to\infty}\int_a^b f_n = \int_a^b f$$

It follows by the uniform convergence of f_n that

$$\begin{aligned}
|f_n - f| &< \epsilon & &\implies f-\epsilon < f_n < f+\epsilon & (1)\\
\int_a^b\Big(f-\epsilon\Big) < \int_a^b f_n &< \int_a^b\Big(f+\epsilon\Big) & &\implies \Big|\int_a^b f_n - \int_a^b f\Big| < \epsilon(b-a) & (2)
\end{aligned}$$

Hence, we can make the difference $|\int_a^b f_n - \int_a^b f|$ arbitrarily small and we get $\lim_{n\to\infty}\int_a^b f_n = \int_a^b f$.

□

4. Yes, G is differentiable at 1. In order to show this, we must show that the following limit exists:

$$\lim_{x \to 1} \frac{G(x) - G(1)}{x - 1}$$

Notice that the antiderivative A of function g can be calcuated as:

$$A = \begin{cases} x \text{ if } x \neq 1 \\ 2x \text{ if } x = 1 \end{cases}$$

We have:

$$G(1) = \int_0^1 g = A'(1) - A'(0) = 2 - 1 = 1$$

Plugging in the values into the limit formula above, we get:

$$\lim_{x \to 1} \frac{\int_0^x g - 1}{x - 1}$$

Now, clearly $\lim_{x \to 1^-} \frac{1-1}{x-1} = 0$ and $\lim_{x \to 1^+} \frac{1-1}{x-1} = 0$. Hence, $\lim_{x \to 1} \frac{\int_0^x g - 1}{x-1} = 0$ and always exists. Finally, we got that G is differentiable at 1.

□

5. No, H is not differentiable at 1. In order to show this, we must show that the following limit does not exist:

$$\lim_{x \to 1} \frac{H(x) - H(1)}{x - 1}$$

Notice that the antiderivative A of function h can be calcuated as:

$$A = \begin{cases} x \text{ if } x < 1 \\ 2x \text{ if } x \geq 1 \end{cases}$$

We have:

$$G(1) = \int_0^1 g = A'(1) - A'(0) = 2 - 1 = 1$$

Plugging in the values into the limit formula above, we get:

$$\lim_{x \to 1} \frac{\int_0^x h - 1}{x - 1}$$

Now, clearly $\lim_{x \to 1^-} \frac{1-1}{x-1} = 0$ and $\lim_{x \to 1^+} \frac{1-2}{x-1} \neq 0$. Hence, $\lim_{x \to 1^-} \neq \lim_{x \to 1^+}$ and the two-sided limits are not the same which implies that the limit does not exist. Finally, we got that H is not differentiable at 1.

□

6. Let $A(t)$ be the antiderivative of $t^2\sin(t^2)$. We have:

$$\begin{aligned}\frac{d}{dx}\int_0^{x^2} t^2\sin(t^2)dt &= \frac{d}{dx}A(x^2) - A(0)\\ &= 2xA'(x^2)\\ &= 2x \times (x^2)^2\sin((x^2)^2\\ &= 2x^5\sin(x^4)\end{aligned}$$

7. Since f is continuous on $[a,b]$, it follows that f achieves both the absolute maximum M and the absolute minimum m in $[a,b]$. Suppose, without a loss of generality, that these points are c_1 and c_2 with $c_1 < c_2$. We then have:

$$m(b-a) \leq \int_a^b f \leq M(b-a) \tag{3}$$

$$m \leq \frac{1}{b-a}\int_a^b f \leq M \tag{4}$$

Now, it follows by **Intermediate Value Theorem** that $\exists c \in (c_1, c_2) \subset [a,b]$ s.t. $f(c) = \frac{1}{b-a}\int_a^b f$.

□

8. Suppose, for the sake of contradiction, that f has Generalized Riemann integrals Q_1 and Q_2 with $Q_1 \neq Q_2$ and let $\epsilon > 0$ be given. Then,.it follows that $\exists\delta_1(x)$ s.t. $\forall\delta_1(x)$-fine tagged partitions, $|R(f,P) - Q_1| < \frac{\epsilon}{2}$. Similarly, $\exists\delta_2(x)$ s.t. $\forall\delta_2(x)$-fine tagged partitions, $|R(f,P) - Q_2| < \frac{\epsilon}{2}$. Now, let $\delta(x) = \min(\delta_1(x), \delta_2(x))$. It follows by **Theorem 8.1.5** that there exists a tagged partition $(P, \{c_k\})$ s.t. it is both $\delta_1(x)$-fine and $\delta_2(x)$-fine. We have:

$$|Q_1 - Q_2| \leq |Q_1 - R(f,P)| + |R(f,P) - Q_2| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Hence, we got that $Q_1 = Q_2$ and we face a contradiction since we have assumed that $Q_1 \neq Q_2$. Finally, we conclude that if f has a generalized Riemann integral on $[a,b]$, then the value of the integral $\int_a^b f$ is unique.

□

9. (a) Placeholder.

   (b) Placeholder.

   (c) Placeholder.

(d) Placeholder.

(e) Placeholder.

(f) Placeholder.