This exam is due **Monday, December 14 at 11:00 p.m.** Write your solutions in LaTeXor scan your neatly handwritten solutions and upload your pdf to this folder. You may refer to your notes, homework, and textbook. Ask me for clarification on the wording of problems. **Do not discuss the exam with anyone else or use any outside sources.**

1. (7 pts) Use the definition of convergence of a sequence to prove that

$$\lim_{n\to\infty} \frac{6n+1}{3n+2} = 2.$$

2. (9 pts) Recall that $A \times B = \{(a, b) : a \in A,\ b \in B\}$. Prove or disprove:

   (a) The product of two countable sets is countable.

   (b) A finite product of countable sets is countable.

   (c) A countable product of countable sets is countable.

3. (6 pts) Given a sequence of real numbers (a_n), define

$$p_n = \begin{cases} a_n & \text{if } a_n > 0 \\ 0 & \text{if } a_n \leq 0 \end{cases} \qquad \text{and} \qquad q_n = \begin{cases} a_n & \text{if } a_n < 0 \\ 0 & \text{if } a_n \geq 0 \end{cases}$$

   Prove: If $\sum a_n$ converges conditionally, then both $\sum p_n$ and $\sum q_n$ diverge.

   Hint: Before you prove this in general, consider a specific example:
   If $a_n = (-1)^{n+1}/n$, what are (p_n) and (q_n)?

4. (9 pts) Write a recursive definition for the sequence $\left(\sqrt{5},\ \sqrt{5\sqrt{5}},\ \sqrt{5\sqrt{5\sqrt{5}}},\ \ldots\right)$, prove that it converges, and find the limit.

5. (9 pts) Suppose (x_n) is a bounded sequence of real numbers such that every convergent subsequence of (x_n) converges to the same value L. Show that (x_n) must converge to L.

6. (10 pts) Consider the open interval $A = (0, 1) = \{x \in \mathbb{R} : \ 0 < x < 1\}$ and the open square $B = (0, 1) \times (0, 1) = \{(x, y) \in \mathbb{R}^2 : \ 0 < x < 1, \ 0 < y < 1\}$.

   (a) Construct a 1-to-1 function $f\colon A \to B$. Is f onto?

   (b) Construct a 1-to-1 function $g\colon B \to A$. Is g onto?

   The Schröeder-Bernstein Theorem then implies $A \sim B$.
   (See Exercise 1.5.11. You do not have to prove it.)

   (c) Prove that $A \sim \mathbb{R}$.

   (d) Now use these results to prove that $\mathbb{R} \sim \mathbb{R}^2$.