Real Analysis

Assignment №13

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7.5.8 (a) $L_1 = \int_1^1 \frac{1}{x} = 0$. $L(x)$ is differentiable since $\frac{1}{t}$ is continuous and it follows by **Theorem 7.5.1 (Fundamental Theorem of Calculus) (part (ii))** that $L(x)$ is differentiable with $L(x)' = \frac{1}{x}$.

(b) Keeping y constant, we have:

$$\frac{d}{dx}L(xy) = yL'(xy) = y \times \frac{1}{xy} = \frac{1}{x}$$

Now, integrating with respect to x get us the following:

$$L(xy) = \int_1^x \frac{1}{t}dt + c(y) \qquad (c(y) \text{ is a function of only } y)$$

If we now differentiate with respect to y, we get:

$$\frac{1}{y} = c'(y) \tag{1}$$

$$c(y) = \int_1^y \frac{1}{t}dt \tag{2}$$

Finally, we have:

$$L(xy) = \int_0^x \frac{1}{t}dt + \int_0^y \frac{1}{t}dt = L(x) + L(y)$$

7.6.1 (a) It follows by the **density property** that every subinterval of any partition has an irrational number in it. Hence, the infinum on this interval is 0. Thus, for any partition P, we have $L(t, P) = 0$.

$\square$

(b) The set of points $\geq \epsilon/2$ are:

$$\begin{aligned} x &= 0 \\ x &= \frac{1}{1} \\ x &= \frac{1}{2} \\ x &= \frac{1}{3} \\ &\dots \\ x &= \frac{1}{\lfloor \frac{2}{\epsilon} \rfloor} \end{aligned}$$

Thus, the size of $D_{\frac{\epsilon}{2}}$ is $\lfloor \frac{2}{\epsilon} + 1 \rfloor$.

(c) Pick the following partition:

$$\left\{0, \frac{1}{\lfloor 2/\epsilon \rfloor}\right\} \cup \left\{V_{\frac{\epsilon^2}{9}}(x)\right\}$$

Then we have:

$$\begin{aligned} U(t, P_\epsilon) &= \frac{\epsilon}{2} \cdot 1 + \left[\left(\lfloor \frac{2}{\epsilon} \rfloor + 1\right) \cdot \frac{\epsilon^2}{9}\right] \\ &= \frac{\epsilon}{2} + \frac{\epsilon^2}{3} \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{3} && \text{(for } \epsilon < 1\text{)} \\ &< \epsilon \end{aligned}$$

Since $\sup U(t, P) = 1$, showing this $\forall \epsilon \geq 1$ is trivial. Hence, any partition will work for $\epsilon \geq 1$. Finally, we have constructed a partition P_ϵ of $[0, 1]$ s.t. $U(t, P_\epsilon) < \epsilon$.

7.6.3 Let $S = \{s_1, s_2, s_3, \dots\}$ be an arbitary countable set. Then $\forall \epsilon > 0$ pick the sequence of intervals $I_n = \left[s_n - \frac{\epsilon}{2^{n+1}}, s_n + \frac{\epsilon}{2^{n+1}}\right]$. Notice that $|I_n| = \frac{\epsilon}{2^n}$ and $\{s_1, s_2, s_3, \dots\} \subseteq \bigcup_{n=1}^{\infty} I_n$. Now, define $I = \bigcup_{n=1}^{\infty} I_n$ and we have $|I| = \frac{\epsilon}{2} + \frac{\epsilon}{2^2} + \frac{\epsilon}{2^3} + \cdots = \frac{\frac{\epsilon}{2}}{1 - \frac{1}{2}} = \frac{\frac{\epsilon}{2}}{\frac{1}{2}} = \epsilon$. Hence, we got that $\forall \epsilon > 0, |S| < \epsilon$ and thus, S has the measure of 0.

$\square$