Topology

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Assignment №5

Section 31

1. Show that if X is regular, every pair of points of X have neighborhoods whose closures are disjoint.

   Since X is regular, by the definition, $\forall x, y \in X \; \exists U, V$ with $x \in U, y \in V$ and $U \cap V = \varnothing$ (with U and V being open sets). Now, recall that X is regular if and only if given a point x of X and a neighborhood U of x, there is a neighborhood V of x such that $\bar{V} \subset U$ (**Lemma 31.1 (a)**). Then, according to **Lemma 31.1 (a)**, $\exists U', V'$ such that $U' \subset U$ and $V' \subset V$. Now, because $U \cap V = \varnothing$, it follows that $\bar{U} \cap \bar{V} = \varnothing$. $\square$

2. Show that if X is normal, every pair of disjoint closed sets have neighborhoods whose closures are disjoint.

   Let A and B be disjoint closed sets. Then by the definition of normality, $\exists U, V$ such that $U \cap V = \varnothing, A \subset U,$ and $B \subset V$ (with U and V being open sets). Now, recall that X normal if and only if given a closed set A and an open set U containing A, there is an open set V containing A such that $\bar{V} \subset U$ (**Lemma 31.1 (b)**). Then it follows from **Lemma 31.1 (b)** that $\exists U', V'$ with $A \subset U'$ and $B \subset V'$ such that $\bar{U}' \subset U$ and $\bar{V}' \subset V$. Now, since $U \cap V = \varnothing$, we get $\bar{U}' \cap \bar{V}' = \varnothing$. $\square$

3. Show that every order topology is regular.

Suppose that X is an ordered set. Let $x \in X$ and let $U = (a, b)$ be the neighborhood of x. Also, let $A = (a, x)$ and $B = (x, b)$. Now, according to the **Lemma 31.1 (a)**, X is regular if and only if given a point x of X and a neighborhood U of x, there is a neighborhood V of X such that $\bar{V} \subset U$. Then it follows that we have the following four cases:

1. If $u \in A$ and $v \in B$, then $x \in (u, v) \subset \overline{(u, v)} \subset [u, v] \subset (a, b)$.
2. If $A = B = \varnothing$, then $(a, b) = \{x\}$ is both open and closed (since every order topology is Hausdorff)
3. If $A = \varnothing$ and $v \in B$, then $x \in (a, v) \subset [x, v) \subset \overline{[x, v)} \subset [x, v] \subset (a, b)$.
4. If $B = \varnothing$ and $u \in A$, then $x \in (u, b) \subset (u, x] \subset \overline{(u, x]} \subset [u, x] \subset (a, b)$.

Finally, we have considered all the cases and have exhaustively shown that a closed subspace of a normal space is normal. $\square$

Section 31

1. Show that a closed subspace of a normal space is normal.

Suppose that Y is a closed subspace of the normal space X. Now, recall that every simply ordered set is a Hausdorff space in the order topology; The product of two Hausdorff spaces is a Hausdorff space; A subspace of a Hausdorff space is a Hausdorff space. (**Theorem 17.11**). Then according to the **Theorem 17.11**, Y is Hausdorff. Now let A and B be disjoint closed subspaces of Y. Since A and B are closed in X, they can be separated in X by open sets U and V (with $U \cap V = \varnothing$). Then $U \cap Y$ and $V \cap Y$ are open sets in Y separating A and B. Therefore, Y is normal. $\square$