The Topology of Robotic Configuration and Motion Planning

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Abstract

This paper explores the topological approach to the problem of robot motion planning. Particularly, we will discuss the safe way to coordinate automated guided vehicles or AGVs. AGVs are mobile robots which are used extensively in manufacturing facilities. One of the biggest challenges in designing such facility is setting up mobile robot routes to achieve the safe and efficient coordination of robots. The tools and concepts of topology are naturally employed in this planning process. This paper follows the bottom-up approach by first introducing concepts and then building up on these ideas. It is therefore accessible to most of math undegrads.

Configuration Spaces

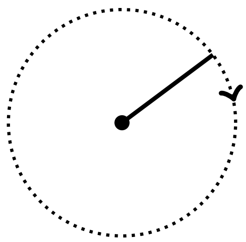
We shall start by introducing the notion of configuration spaces. The idea of configuration spaces come from physics. In classical mechanics, the configuration space is the vector space defined by the generalized coordinates (coordinates that describe the configuration of the physical system). Put it simply, the configuration space is the set of all possible states that could exist in the physical system. For instance, the configuration space of some particle in the room is the set of all points/states of the type (x, y, z) where x, y and z are the coordinates bounded by the room. If the room is a $3 \times 3 \times 3$ cube then we define the configuration space of the particle by

$$C^3(\text{room}) = \{(x, y, z) \mid 0 \leq x, y, z \leq 3\}.$$

In other words, the configuration space of the particle, is all of $3 \times 3 \times 3$ cube (this example obviously assumes that the particle is allowed to move freely in the room).

It appears that the physical notion of configuration spaces is very much connected to that of mathematics. In fact, the idea is the same but rather generalized. To better understand configuration spaces, let us first go through several *classic* examples.

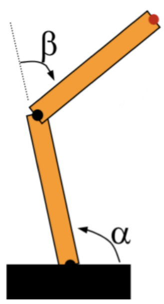
[1] Consider a planar system where we have a rod with a fixed end that can rotate freely. Then it is easy to see that the set of all possible configurations of the rotating rod is a circle.



Circle obtained by the rotational motion of the rod.

In other words, as the rod rotates, it creates the circle around itself with the radius equal to the length of the rod. This configuration space is also referred to as S^1.

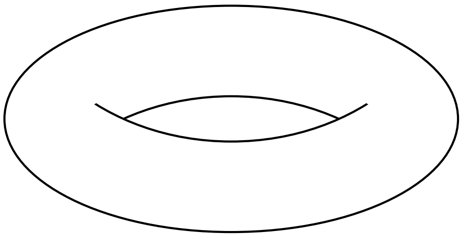
[2] On the other hand, the configuration space of the two-rod system in 3D space where one rod is fixed and the other one is attached to it is a torus.



A two-rod system.

This space is also known as $S^1 \times S^1$ configuration space. We already know that a rod with fixed end generates a circle. In this case, we have two rods:

one attached to the ground and the other one attached to the end of the first one. Then obviously both of the rods can go through a full circle of states and therefore, create a configuration space which geometrically represents a torus.



A torus obtained by the motion of two-rod system.

As of now, this is all we need to know about the configuration spaces. This idea will be very useful once we learn more about other topological concepts.

Topological Spaces

The fundamental idea in topology is that of a topological space. We will use this idea to then introduce and define other important concepts.

Definition. [3] A ***topology*** on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

(1) $\varnothing$ and X are in $\mathcal{T}$.

(2) The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

(3) The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

A set X for which a topology $\mathcal{T}$ has been specified is called a ***topological space***.

Let us first look at some examples. Consider a set $X = \{a, b, c\}$. Then we can define a topology on X by $\mathcal{T} = \{\varnothing, \{a, b, c\}, \{a, b\}, \{c\}\}$. Observe that $\varnothing, X \in \mathcal{T}$ therefore the first criterion is satisfied. It is easy to see that arbitrary unions will be in $\mathcal{T}$ since the only "interesting" case is when we consider $\{a, b\}$ and $\{c\}$, but in this case $\{a, b\} \cup \{c\} = \{a, b, c\} \in \mathcal{T}$. This satisfies the second requirement. Finally, any arbitrary intersection of the finite subcollections of $\mathcal{T}$ is also in $\mathcal{T}$ and therefore, we conclude that $\mathcal{T}$ is indeed a topology on X. Hence, X is a topological space (note that, properly speaking, a topological space is an ordered pair $(X, \mathcal{T})$, but we often omit mentioning $\mathcal{T}$ and say that X is a topological space).

At this point, you might have already noticed that one could always define more than one topology for a given set. In the previous example, sets $\mathcal{P}(x)$ (powerset of x) and $\{\varnothing, X\}$ are also topologies on X called discrete and indiscrete topologies correspondigly. In fact, for any set X, $\mathcal{P}(x)$ and $\{\varnothing, X\}$ will always be two distinct topologies on X.

We will now get acquainted with the notion of the *open set.*

Definition. If X is a topological space with topology $\mathcal{T}$, we say that a subset U of X is an **open set** of X if U belongs to the collection $\mathcal{T}$.

Consider our topology $\mathcal{T} = \{\varnothing, \{a, b, c\}, \{a, b\}, \{c\}\}$ on the topological space $X = \{a, b, c\}$. Then notice that $\varnothing, \{a, b, c\}, \{a, b\}, \{c\} \in \mathcal{T}$ and therefore, all are open sets.

Continuous Functions and Homeomorphisms

The notion of continuous functions is familiar to most high-school students. Most people associate them with a nice-looking monotonically increasing or decreasing functions with no leaps or jumps. There are several definitions for a function continuity. Here is the calculus definition.

Definition. A function $f(x)$ is said to be continuous at the point $x = a$ if

$$\lim_{x \to a} f(x) = f(a).$$

Furthermore, a function is said to be continuous on the interval $[a, b]$ if it is continuous at each point in the interval.

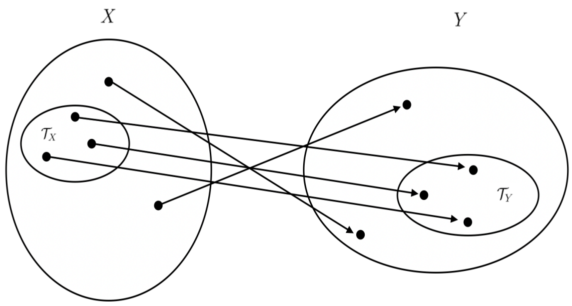
In topology we cannot really use this definition since the definition assumes that one could take a limit of the function. This, however, is sometimes very difficult or nearly impossible when considering functions defined over more abstract sets such as the configuration space of AGV. Now, we shall introduce more general notion of continuity.

Definition. [5] Let X and Y be topological spaces. A function $f : X \to Y$ is said to be **continuous** if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.

It is important to note that in this definition $f^{-1}(V)$ does not refer to the inverse of the function. Therefore, we are not assuming that $f : X \to Y$ is a bijection. $f^{-1}(V)$ refers to the preimage of the function. In other words, the function $f : X \to Y$ over two topological spaces X and Y is continuous if and only if the open sets in the preimage map to the open sets in the image.

This is the case where the visualization might be useful to see how the definition of continuous functions really works. Consider a continuous function

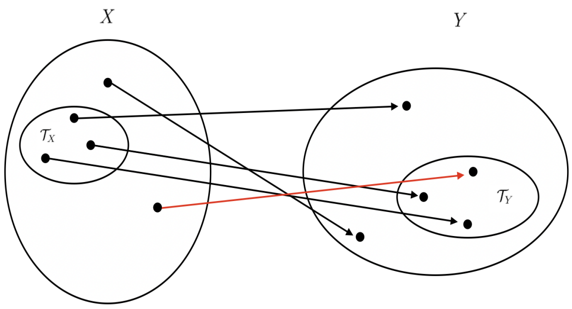
$f : X \to Y$ where X and Y are topological spaces with topologies $\mathcal{T}_X$ and $\mathcal{T}_Y$. Therefore, by the definition, open sets in X map to the open sets in Y.



A continuous function $f : X \to Y$.

Notice that the function shown above is continuous. $\mathcal{T}_X$ and $\mathcal{T}_Y$ are the topologies in X and Y correspondigly and the points from $\mathcal{T}_X$ map to the points in $\mathcal{T}_Y$. Therefore, every open set in Y has a preimage which is also open. In other words, the open sets in Y have the preimages that are also open.

The function below, however, is not continuous as there is one point that is not in $\mathcal{T}_X$ but maps to a point in $\mathcal{T}_Y$ (it is highlighted with the red arrow).



A discontinuous function $f : X \to Y$.

Now that we are familiar with the notion of continuous functions, we shall introduce a new idea, that of homeomorphisms.

Definition. [6] Let X and Y be topological spaces; let $f : X \to Y$ be a bijection. If both the function f and the inverse function

$$f^{-1} : Y \to X$$

are continuous, then f is called a homeomorphism.

As one delves deeper in topology, this definition is then taken further and two objects are said to be homeomorphic if one could be obtained by the continuous deformation of the other. We may not use this notion further in the paper yet, it is a helpful way to think about homeomorphisms.

Connectedness and Path Connectedness

One of the important ideas in topology is that of connectedness. This idea is used extensively in various other fields of mathematics such as graph theory and knot theory. Let us first define connectedness.

Definition. [7] Let X be a topological space. A **separation** of X is a pair U, V of disjoint nonempty open subsets of X whose union is X. The space X is said to be **connected** if there does not exist a separation of X.

Consider a topological space $X = \{a, b, c\}$ with a topology $\mathcal{T} = \{\varnothing, \{a, b, c\}, \{a, b\}, \{c\}\}$. Then it is easy to see that X is a disconnected space. Sets $\{a, b\}$ and $\{c\}$ are open since $\{a, b\}, \{c\} \in \mathcal{T}$. Besides, $\{a, b\} \cap \{c\} = \varnothing$ and $\{a, b\} \cup \{c\} = \{a, b, c\} = X$. Hence, $U = \{a, b\}$ and $V = \{c\}$ is a pair of disjoint open subsets of X whose union is X and therefore U, V is a separation of X. This means that X is a disconnected space.

On the other hand, a topological space $Y = \{a, b\}$ with a topology $\mathcal{T} = \{\varnothing, \{a, b\}, \{a\}\}$ is connected as there is no pair of disjoint nonempty open subsets of Y such that their union is Y. In other words Y has no separation. Note that $\varnothing \cup \{a, b\} = \{a, b\} = Y$, but it must be <u>nonempty</u> sets whose union is Y. Therefore, $\varnothing, \{a, b\}$ is not a separation of Y.

Knowing what it means for a topological space to be connected, we can now introduce the notion of path connectedness.

Definition. [8] Given points x and y of the space X, a **path** in X from x to y is a continuous map $f : [a, b] \to X$ of some closed interval in the real line into X, such that $f(a) = x$ and $f(b) = y$. A space X is said to be ***path connected*** if every pair of points of X can be joined by a path in X.

References

[1] C. Adams and R. Fransoza, "Introduction to Topology", p. 105.

[2] C. Adams and R. Fransoza, "Introduction to Topology", pp. 105 – 106.

[3] J. R. Munkres, "Topology", Second Edition, p. 76

[4] J. R. Munkres, "Topology", Second Edition, p. 76

[5] J. R. Munkres, "Topology", Second Edition, p. 102

[6] J. R. Munkres, "Topology", Second Edition, p. 105

[7] J. R. Munkres, "Topology", Second Edition, p. 148

[8] J. R. Munkres, "Topology", Second Edition, p. 155