Homework №12

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87. To prove that a relation is an equivalence relation, we must show that the relation is reflexive, symmetric, and transitive.

    I. Showing that $\sim$ is reflexive.

    $x \sim x = x + 2x = 3x$ Thus, $\sim$ is reflexive.

    II. Showing that $\sim$ is symmetric.

    Suppose that $x \sim y$, then $x + 2y = 3k$ where $k \in \mathbb{Z}$. Then, if we solve the equation for x, we get $x = 3k - 2y$. Now, consider $y + 2x$. Let's substitute x with $3k - 2y$. We get, $y + 6k - 4y = 6k - 3y = 3 \times (2k - y)$. Hence, if $x + 2y$ is divisible by 3, then $y + 2x$ is also divisible by 3 and the relation is symmetric.

    III. Showing that $\sim$ is transitive.

    Suppose that $x \sim y$ and $y \sim z$. Then $x + 2y = 3k$ where $k \in \mathbb{Z}$ and $y + 2z = 3l$ where $l \in \mathbb{Z}$. Consider the relation on the variables x and z. The relation is $x \sim z = x + 2z$. Now, from the first equation, let's substitute x and from the second one, substitute z. We get that $x = 3k - 2y$ and $z = \dfrac{3l - y}{2}$. Finally, we get: $x + 2z = 3k - 2y + 2 \times \dfrac{3l - y}{2} = 3k - 2y + 3l - y = 3k + 3l - 3y = 3 \times (k + l - 1)$ and $3 \times (k + l - 1)$ is clearly a multiple of 3. Hence, we got that $\sim$ is transitive.

    Now, we proved that the relation $\sim$ is reflexive, symmetric, and transitive and thus, the relation $\sim$ is the equivalence relation. $\square$

88. (a) $\Xi(S)$ is a relation on $\mathcal{P}(S)$ which is a set of ordered pairs of elements of $\mathcal{P}(S)$ such that the intersection of the elements of each of the ordered pairs is not equal to the empty set.

(b) It is symmetric. Suppose $(X, Y) \in R$. Then we know that $X \cap Y \neq \emptyset$. Therefore, we also know that $Y \cap X \neq \emptyset$. Hence, we got that if $(X, Y) \in R$, then $(Y, X) \in R$ and thus, the relation is symmetric.

(c) It is not transitive. Let $S = \{1, 2, 3, 4\}$. Let's take three tuples: $(\{1, 2\}, \{2, 3\}$, $(\{2, 3\}, \{4, 3\}$, and $(\{1, 2\}, \{4, 3\})$. Then $\{1, 2\} \cap \{2, 3\} \neq \emptyset$ as well as $\{2, 3\} \cap \{4, 3\} \neq \emptyset$, however, $\{1, 2\} \cap \{4, 3\} = \emptyset$. Hence, the relation is not transitive.

(d) It is not reflexive. Suppose we have a set S. Then, we know that $\mathcal{P}(S) \times \mathcal{P}(S)$ contains the ordered pair $\{\emptyset, \emptyset\}$. Consequently, we have that $\emptyset \cap \emptyset = \emptyset$ and thus, the relation is not reflexive.

Bookwork

4.2

2. $\in$, $\notin$, $\subset$, $=$, $\mathcal{P}$

4. (a) It is not. Consider tuples (a, b) and (b, a). For the relation R to be transitive, since $(a, b) \in R$ and $(b, a) \in R$, it must be the case that $(a, a) \in R$. However, $(a, a) \notin R$. Thus, the relation is not transitive. $\square$

   (b) i. It is. Suppose that $x - y = q_1$ and $y - z = q_2$ where $q_1, q_2 \in \mathbb{Q}$. Let's then sum those two equations up, and we get $x - y + y - z = q_1 + q_2$ and finally $x - z = q_1 + q_2$. Hence, we got that $x - z$ is a sum of two rational numbers and thus is rational itself. Therefore, the relation R is transitive. $\square$

   ii. It is not. Consider $x = \sqrt{2}$, $y = 1$, and $z = \sqrt{2} + 1$. Then $x - y = \sqrt{2} - 1$ thus is irrational and $y - z = 1 - (\sqrt{2} + 1) = -\sqrt{2}$ hence, is also irrational. However, $x - z = \sqrt{2} - (\sqrt{2} + 1) = \sqrt{2} - \sqrt{2} - 1 = -1$ which is rational. Therefore, the relation R is not transitive. $\square$

   iii. It is not. Consider $x = 1$, $y = 2$, and $z = 4$. Then $|x - y| = 1$ and $|y - z| = 2$. However, $|x - z| = 3 > 2$. Hence, the relation R is not transitive. $\square$

12. (a) It is reflexive. Suppose that $x \in R \cap S$. Then $x \in R$ and $x \in S$. Since R and S are reflexive, $(x, x) \in R$ and $(x, x) \in S$. Therefore, $(x, x) \in R \cap S$. $\square$

   (b) It is reflexive. Suppose that $x \in R \cup S$. Then, without a loss of generality, let $x \in S$. Now, since S is reflexive, $(x, x) \in S$ and thus, $(x, x) \in R \cup S$. $\square$

   (e) It is transitive. Suppose that $(x, y), (y, z) \in R \cap S$. Then $(x, y), (y, z) \in R$ and $(x, y), (y, z) \in S$. Since R, S are transitive, $(x, z) \in R$ and $(x, z) \in S$. Finally, $(x, z) \in R \cap S$ and $R \cap S$ is transitive. $\square$

   (f) It is transitive. Suppose that $(x, y), (y, z) \in R \cup S$. Then, without a loss of generality, $(x, y), (y, z) \in R$. Since R is transitive, $(x, z) \in R$ and $(x, z) \in R \cup S$. Therefore, $R \cup S$ is transitive. $\square$

4.4

1. (a) Yes, it is an equivalence relation since it satisfies all the criteria: reflexive, transitive, symmetric.

   1. It is reflexive, since $(a, a),\ (b, b),\ (c, c) \in R$.

   2. It is transitive. $(a, a), (a, c) \in R$ and $(a, c) \in R$; $(c, a), (a, a) \in R$ and $(c, a) \in R$; $(c, c), (c, a) \in R$ and $(c, a) \in R$; $(a, c), (c, c) \in R$ and $(a, c) \in R$.

   3. It is symmetric. $(a, a) \in R$ and $(a, a) \in R$; $(b, b) \in R$ and $(b, b) \in R$; $(c, c) \in R$ and $(c, c) \in R$; $(a, c) \in R$ and $(c, a) \in R$; $(c, a) \in R$ and $(a, c) \in R$.

   (b) No, it is not since $(b, a), (a, c) \in R$ but $(b, c) \notin R$ (it is not transitive).

3. It is not. For a relation to be the equivalence relation, it must be reflexive, transitive, symmetric. It is not transitive since if we have three lines x, y, z in the euclidean space and if $x \perp y$ and $y \perp z$, then $y \not\perp z$ (because $y \parallel z$). As a side note, it is not reflexive either since the line cannot be perpendicular to itself.

9 (a) Equivalence relation R such that xRy if $x, y \leq 0$ or $x, y > 0$.

   (b) Equivalence relation R such that xRy if $x, y < 0$, $x, y = 0$ or $x, y > 0$.

13. To show that $\equiv_2$ is an equivalence relation, we must show that it is reflexive, symmetric, and transitive.

    1. It is reflexive because if $(a, b) \equiv_2 (a, b)$, then $a - a = 0$ is even and $b - b = 0$ is also even.
    2. It is symmetric because if $(a, b) \equiv_2 (c, d)$, then $a - c = 2k$ is even and $b - d = 2l$ where $k, l \in \mathbb{Z}$. Hence, $(c, d) \equiv_2 (a, b)$ because $c - a = 2 \times (-k)$ and $d - b = 2 \times (-l)$
    3. It is transitive because if $(a, b) \equiv_2 (c, d)$ and $(c, d) \equiv_2 (e, f)$, it means that $a - c = 2k,\ b - d = 2l,\ c - e = 2m$ and $d - f = 2n$ where $k, l, m, n \in \mathbb{Z}$. Then, we

have that $a - e = (a - c) + (c - e) = 2k + 2m = 2 \times (k + m)$. On the other hand, $b - f = (b - d) + (d - f) = 2l + 2n = 2 \times (l + n)$. Hence, we got that if $(a, b) \equiv_2 (c, d)$ and $(c, d) \equiv_2 (e, f)$, then $(a, b) \equiv_2 (e, f)$ and therefore, the relation is transitive.

Thus, we have now proven that the relation $\equiv_2$ is reflexive, symmetric, and transitive thus, $\equiv_2$ is an equivalence relation.

A partition of $\equiv_2$ is $\{(a, b), (c, d) \in \mathbb{Z} \times \mathbb{Z} \mid (a, c \bmod 2 = 0 \text{ or } a, c \bmod 2 = 1) \text{ and } (b, d \bmod 2 = 0 \text{ or } b, d \bmod 2 = 1)\}$.

23. $\{(1, 2), (2, 1), (3, 3), (4, 5), (5, 4)\}$

30. To prove that the relation is the equivalence relation, we must prove that it is reflexive, symmetric, and transitive.

    It is reflexive because if $f \sim f$, then $f' = f'$ which is true since given the particular value, the function and then its derivative is the same (the function cannot have more than one output for a single input).

    It is symmetric because if $f \sim g$, it means that $f' = g'$ and since we are dealing with values, we know that if $x = y$, then $y = x$ and the same applied for the derivatives here. Hence, if $f' = g'$, then $g' = f'$. Finally, we got that if $f \sim g$, then $g \sim f$ and the function is symmetric.

    It is transitive because if $f' \sim g'$ and $g' \sim h'$, then $f' = g'$ and $g' = h'$ and thus $f' = g' = h'$ and $f' = h'$. Hence, we got that if $f' \sim g'$ and $g' \sim h'$, then $f' \sim h'$ and the relation is transitive.

    Thus, we have proven that the relation is reflexive, symmetric, and transitive and hence, is the equivalence relation.

31. (a) To show that the relation is the equivalence relation, we must prove that it is reflexive, symmetric, and transitive.

    It is reflexive because if $A_i \sim A_i$, then $|A_i| = |A_i|$ and obviously, $|A_i| = |A_i|$ because the size of the set is constant. Hence, we have shown that the relation is reflexive.

    It is symmetric because if $A_1 \sim A_2$, it means that $|A_1| = |A_2|$ and thus

$|A_2| = |A_1|$. Hence, it is symmetric.

It is transitive because if $A_1 \sim A_2$ and $A_2 \sim A_3$, it means that $|A_1| = |A_2|$ and $|A_2| = |A_3|$. Therefore, $|A_1| = |A_2| = |A_3|$ and we get that $|A_1| = |A_3|$. Thus, we have proven that if $A_1 \sim A_2$ and $A_2 \sim A_3$, then $A_1 \sim A_3$ and the relation is transitive.

At this point, we have shown that the relation is all three: reflexive, symmetric, and transitive and thus, is the equivalence relation.

(b) We have 5 equivalence classes.
The first equivalence class will have all the sets of the size 0 (empty set).
The second equivalence class will have all the sets of the size 1.
The third equivalence class will have all the sets of the size 2.
The four equivalence class will have all the sets of the size 3.
The five equivalence class will have all the sets of the size 4.

36. (a) To prove that the relation is the equivalence relation, we must prove that the relation is reflexive, symmetric, and transitive.

Suppose we have a number $n \in A$ which has a highest dividing power of two equal to 2^k. Then, we know that the highest dividing power of 2 of the same number is the same and thus $n \sim n$. Hence, relation is reflexive.

Suppose we have a number $m, n \in A$ which have a highest dividing power of two equal to 2^k. Then we know that $m \sim n$. On the other hand $n \sim m$ too since the numbers did not change and accordingly, the highest dividing power of 2 is the same. Thus, the relation is symmetric.

Suppose we have a number $m, n, o \in A$ where m, n have the same highest dividing power of 2 and n, o have the same highest dividing power of 2. For m, n let it be 2^k and for m, o let it be 2^l. Then we know that m has the same highest dividing factor as n which is 2^k. And from the second assumption, the n has the highest dividing factor of 2^l. Thus, $2^l = 2^k$ and $k = l$. Finally, we have that all of the three m, n, o have the same highest dividing power of 2 and we have proven that if mRn and nRo, then mRo. Thus, the relation is transitive.

At this point, we have proven that the relation is reflexive, symmetric, and transitive. Thus, the relation is the equivalence relation.

(b) $\{1, 2, 3\}$. For all of these number, the highest dividing power of 2 is 1.

(c) The set is:

$$\{\{-2^1-1, -2^1, -2^0, 2^0, 2^1, 2^1+1\}, \{-2^2, -2^2+1, 2^2, 2^2+1\}, \{-2^3, -2^3+1, 2^3, 2^3+1\},$$
$$\{-2^4, -2^4+1, 2^4+1\}...\{-2^n, -2^n+1, 2^n, 2^n+1\}...\}$$

Notice that only the first element of the set has 6 elements - the rest of the element sets contain only 4.