
Real Analysis

Assignment №4

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- 2.6.2 (a) Such sequence exists. $a_n = \frac{(-1)^n}{n}$ is a Cauchy sequence that is not monotone since it alternates, but converges to 0.
- (b) Per **Lemma 2.6.3**, such sequence cannot exist.
- (c) Such sequences cannot exist. A divergent monotone sequence implies that the sequence is unbounded. Unbounded and monotone sequence, on the other hand, cannot contain a convergent subsequence. Hence, by **Cauchy Criterion (Theorem 2.6.4)**, it cannot contain a Cauchy subsequence.
- (d) Such sequence exists. Let us define

$$a_n = \begin{cases} n & \text{if } n \in \mathbb{N} \text{ is odd} \\ 0 & \text{if } n \in \mathbb{N} \text{ is even} \end{cases}$$

Then it is easy to see that sequence is unbounded since $\forall k \in \mathbb{N}, a_{2k+1} = 2k+1 > k$. On the other hand the subsequence formed by the even-termed elements is comprised of only zeros and hence, converges to 0. Therefore, the subsequence is Cauchy. Hence, we found an unbounded sequence containing a subsequence that is Cauchy.

- 2.6.3 (a) Since x_n and y_n are both Cauchy sequences, $\forall \epsilon > 0, \exists N_1, N_2 \in \mathbb{N}$ s.t. $|x_{m_1} - x_{n_1}| < \frac{\epsilon}{2}$ and $|y_{m_2} - y_{n_2}| < \frac{\epsilon}{2}$ with $m_1, n_1 \geq N_1$ and $m_2, n_2 \geq N_2$. Then let $N = \max\{N_1, N_2\}$. It follows that $\forall m, n \geq N, |x_m - x_n| < \frac{\epsilon}{2}$ and $|y_m - y_n| < \frac{\epsilon}{2}$. Finally, using the triangle inequality, we get:

$$\begin{aligned} |(x_m + y_m) - (x_n + y_n)| &= |(x_m - x_n) - (y_m - y_n)| \\ &\leq |x_m - x_n| + |y_m - y_n| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Thus, we conclude that $(x_n + y_n)$ is a Cauchy sequence.

□

- (b) Since x_n and y_n are both Cauchy sequence, they are also bounded and hence, $\exists X, Y$ s.t. $\forall n \in \mathbb{N}, |x_n| < X$ and $|y_n| < Y$. Additionally, $\forall \epsilon > 0, \exists N_1, N_2 \in \mathbb{N}$ s.t. $|x_{m_1} - x_{n_1}| < \frac{\epsilon}{2}$ and $|y_{m_2} - y_{n_2}| < \frac{\epsilon}{2}$ with $m_1, n_1 \geq N_1$ and $m_2, n_2 \geq N_2$. Then let $N = \max\{N_1, N_2\}$. We get:

$$\begin{aligned} |x_m y_m - x_n y_n| &= |x_m(y_m - y_n) + y_n(x_m - x_n)| \\ &\leq |x_m||y_m - y_n| + |y_n||x_m - x_n| \\ &< \frac{A\epsilon}{2} + \frac{A\epsilon}{2} = \frac{\epsilon}{2}AB. \end{aligned}$$

Thus, we found $\epsilon' = \frac{\epsilon}{2}AB > 0$ with N s.t. $m, n \geq N$ and $|x_m y_m - x_n y_n| < \epsilon'$. Hence, we conclude that $(x_n y_n)$ is a Cauchy sequence.

□

- 2.7.5 We have to prove that the series $\sum_{n=1}^{\infty} 1/n^p$ converges if and only if $p > 1$. By **Cauchy Condensation Test**, the series converges if and only if $\sum_{n=1}^{\infty} \frac{2^n}{2^{np}} = \sum_{n=1}^{\infty} 2^{n(1-p)}$ converges. Now, we need to get $2^{n(1-p)}$ less than 1 as otherwise, the sequence will diverge (all terms will be different and ≥ 1). $2^{n(1-p)} < 1$ if and only if $n(1-p) < 0$. Since $n \in \mathbb{N}$, we can safely divide both sides of the inequality by n . We get $1-p < 0$ and thus, $p > 1$. Hence, we have proven that the series $\sum_{n=1}^{\infty} 1/n^p$ converges if and only if $p > 1$.

*NOTE: We did not take the classic “prove it directly and prove its converse” approach since every statement used in the proof was if and only if statement. **Cauchy Condensation Test** is if and only if and $2^{n(1-p)} < 1$ when $n(1-p) < 0$ is if and only if.*

2.7.8 (a) True. By **Theorem 2.7.3**, $\sum a_n$ converges absolutely. It follows that $\lim a_n = 0$. Thus, $|a_n|$ is bounded and $\exists B > 0$ s.t. $\forall n \in \mathbb{N}, |a_n| \leq B$. Now, by **Algebraic Limit Theorem for Series (Theorem 2.7.1)**, $\sum B|a_n|$ and $B|a_n| \geq a_n^2$. Finally, per **Comparison Test (Theorem 2.7.4)**, $\sum a_n^2$ converges.

□

(b) False. Let $a_n = b_n = \frac{(-1)^n}{\sqrt{n}}$. Now, $\lim b_n = 0$ and thus, (b_n) converges. Additionally, $\lim \frac{1}{\sqrt{n}} = 0$ and $\frac{1}{\sqrt{n}}$ is decreasing. Hence, by **Alternating Series Test (Theorem 2.7.7)**, we get that $\sum a_n$ converges. Thus, both $\sum a_n$ and $\lim (b_n)$ converge. However, $a_n b_n = \frac{1}{n}$ which is harmonic series and it does not converge.

(c) True. Suppose, for the sake of contradiction, that $\sum a_n$ converges conditionally and $\sum n^2 a_n$ converges. Then, $\lim n^2 a_n = 0$ and $\exists N$ s.t. $\forall n \geq N, |n^2 a_n| < 1$. We then get $|a_n| < \frac{1}{n^2}$. Now, per **Comparison Test (Theorem 2.7.4)**, $\sum a_n$ converges absolutely and we face the contradiction since $\sum a_n$ converges conditionally. Thus, $\sum a_n$ converges conditionally, then $\sum n^2 a_n$ diverges.

□

2.7.9 (a) Suppose $r < r' < 1$. Since $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = r < 1$, let $\epsilon = r' - r > 0$. Then $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, the following is true:

$$\begin{aligned} \left| \left| \frac{a_{n+1}}{a_n} \right| - r \right| &< \epsilon \\ \left| \frac{a_{n+1}}{a_n} \right| &< r + \epsilon \\ \left| \frac{a_{n+1}}{a_n} \right| &< r + r' - r \\ \left| \frac{a_{n+1}}{a_n} \right| &< r' \\ |a_{n+1}| &\leq |a_n| r' \end{aligned}$$

□

(b) Since $|r'| < 1$, $\sum (r')^n$ is a convergent geometric series. Then, by **Algebraic Limit Theorem for Series (Theorem 2.7.1)**, $|a_N| \sum (r')^n$.

□

(c) Notice that $\sum |a_n| = \sum_{n=1}^N |a_n| + \sum_{n=N+1}^{\infty} |a_n|$. Now, it is easy to see that $N \sum_{n=N+1}^{\infty} a_n$ converges by **Comparison Test (Theorem 2.7.4)** since $N \sum_{n=N+1}^{\infty} |a_n| \leq |a_N| \sum_{n=N+1}^{\infty} r'^{n-N}$

(the fact that $|a_N| \sum_{n=N+1}^{\infty} r^{n-N}$ converges was proved in part (b) of this exercise). Hence, $\sum a_n$ converges as well. Finally, as $\sum |a_n|$ converges absolutely, by **Absolute Convergence Test (Theorem 2.7.6)**, $\sum a_n$ converges absolutely as well.

□

2.8.1 From **Section 2.1**, we know

$$(a_{ij}) = \begin{cases} \frac{1}{2^{j-i}} & \text{if } j > i \\ -1 & \text{if } j = i \\ 0 & \text{if } j < i \end{cases}$$

Now, if we set $j = 1$, it is easy to see that $a_{i1} = (-1, \frac{1}{2}, \frac{1}{4}, \dots)$ and excluding the first term (-1) , a_{i1} is a geometric series whose sum converges. Recall that the sum formula for the geometric series is $\frac{b_n q - b_1}{q - 1}$ where b_1 is the first term, b_n is the last term, and q is the quotient/ratio (current element over previous element). Thus, we get:

$$\sum_{i=1}^k a_{i1} = -1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^{n-1}} = -1 + \frac{\frac{1}{2^{n-1}} \times \frac{1}{2} - \frac{1}{2}}{\frac{1}{2} - 1} = -1 + \frac{\frac{1}{2^{n-1}} \times \frac{1}{2} - \frac{1}{2}}{-\frac{1}{2}} = -1 + 1 - \frac{1}{2^{n-1}} = -\frac{1}{2^{n-1}}$$

In general, $\forall j < k$, we have:

$$\sum_{i=1}^k a_{ij} = -1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^{n-j}} = -1 + \frac{\frac{1}{2^{n-j}} \times \frac{1}{2} - \frac{1}{2}}{\frac{1}{2} - 1} = -1 + \frac{\frac{1}{2^{n-j}} \times \frac{1}{2} - \frac{1}{2}}{-\frac{1}{2}} = -1 + 1 - \frac{1}{2^{n-j}} = -\frac{1}{2^{n-j}}$$

Finally, we get:

$$s_{nn} = \sum_{j=1}^n \sum_{i=1}^n a_{ij} = -1 + \sum_{j=1}^{n-1} -\frac{1}{2^{n-j}} = -1 - (1 - \frac{1}{2^{n-1}}) = -2 + \frac{1}{2^{n-1}}$$

Now, since $\lim_{n \rightarrow \infty} \frac{1}{2^{n-1}} = 0$, we get that $\sum_{i,j}^{\infty} a_{ij} = \lim_{n \rightarrow \infty} s_{nn} = \lim_{n \rightarrow \infty} (-2 + \frac{1}{2^{n-1}}) = -2 + \lim_{n \rightarrow \infty} \frac{1}{2^{n-1}} = -2 + 0 = -2$. Hence, we get $s_{nn} = -2$.

The iterated sums would give us $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = -1 + \sum_{i=2}^{\infty} -\frac{1}{2^{i-1}}$. Now, recall that the sum can be computed by the infinite geometric series formula $\frac{b_1}{q-1}$. Finally, we get

$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = -1 + \frac{-\frac{1}{2}}{-\frac{1}{2}} = -1 + 1 = 0$. Similarly, $\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} (-1 + \sum_{i=1}^{\infty} \frac{1}{2^i} = \sum_{j=1}^{\infty} 0 = 0)$.