Homework №8

Author: David Oniani
Instructor: Tommy Occhipinti

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Additional Proof Practice

53. A subset S of $\mathbb{Z}^+$ is called a P_3-**set** if there exists (not necessarily distinct) elements $x, y, z \in S$ such that $x + y + z$ is prime.

    (a) Give some examples of P_3-sets.

    $\{1\}$ because $1 + 1 + 1 = 3$ is a prime.
    $\{2, 3\}$ because $2 + 3 = 5$ is a prime.
    $\{12, 25, 30\}$ because $12 + 25 + 30 = 67$ is a prime.

    (b) Prove or Disprove: If A is a P_3-set, and $A \subseteq B \subseteq \mathbb{Z}^+$, then B is a P_3-set.

    It's right so let's prove it. Since we know that A is a P_3-set, we know there exist elements (not necessarily distinct) x, y and z such that $x + y + z$ is prime. Since $A \subseteq B$, we know that all elements of A are also in B meaning that x, y and z are in B as well and there exist elements x, y, z (which are also in A) such that $x + y + z$ is a prime.

    Q.E.D.

    (c) Prove or Disprove: If S is a P_3-set, then so is $S_{+3} := \{x + 3 \mid x \in S\}$.

    It's false. Counterexample: Let $S = \{1\}$, then we know that S is a P_3-set since $1 + 1 + 1 = 3$ is a prime. However, $S_{+3} = \{4\}$ and $4 + 4 + 4 = 12$ which is certainly composite.

    (d) Prove or Disprove: Every P_3 set contains a prime.

    False. Let $S = \{1\}$, then 1 is not a prime but $1+1+1 = 3$ is a prime.

    (e) Prove or Disprove: The intersection of two P_3-sets is a P_3-set.

    It's false. Let $A = \{1\}$, then A is P_3-set since $1 + 1 + 1 = 3$ is a prime. Let $B = \{2, 3\}$, then $2+3 = 5$ is a prime thus B is also P_3-set. $A \cap B = \emptyset$ which means that there are no elements x, y, z such that $x + y + z$ is a prime and thus, the intersection of two P_3-sets is not necessarily a P_3-set.

(f) Prove or Disprove: Every P_3-set contains an odd integer.

It's true. Suppose, for the sake of contradiction, that S is a P_3-set and it does not contain any odd integers. Thus, it means that all the elements of S are even. Now, since all the elements are even, it means that no matter what 3 elements x, y and z we take, their sum will always be even. On the other hand, the only even prime we have is 2. But unfortunately, there are no three numbers $x, y, z \in \mathbb{Z}^+$ which sum up to 2. The best we can do is $1 + 1 + 1$ which is 3 and is one more than 2. Thus, there is no way to get 2 and otherwise, we won't have 3 elements which sum up to the prime. Hence, we have reached the contradiction and S is a not a P_3-set.

Q.E.D.

(g) Prove or Disprove: Every infinite subset S of $\mathbb{Z}^+$ is a P_3-set.

It's false. Since we already proved that every P_3-set contains an odd integer, we can take a set of all positive even integers which is a subset of $\mathbb{Z}^+$. Let's call this subset E. Then, we know that every element of the subset E is even and sum of any 3 elements (not necessarily distinct) will also be even. However, once again, the only even integer which is a prime is 2 and we cannot get 2 by summing 3 integers which are greater than or equal to 2 (greater than equal because $E = \{2, 4, 6, 8, 10...\}$).

(h) Prove or Disprove: If S is a finite subset of $\mathbb{Z}^+$, then $\mathbb{Z}^+ - S$ is a P_3-set.

It's true. Let's prove it. Since S is a finite set, we know that it cannot contain all the elements of $\mathbb{Z}^+$ because $\mathbb{Z}^+$ is infinite. We already proved that there are infinitely many primes. Then we can find a prime p such that $p - 2 \notin S$. then, we can have a set $L = \{1, p - 2\}$ which is a P_3-set because $1 + 1 + p - 2 = p$ is a prime.

Q.E.D.

54. If a subset S of $\mathbb{Z}^+$ is a P_3-set then the **core** of S is the set

$$\text{core}(S) := \{s \in S \mid S - \{s\} \text{ is not a } P_3\text{-set}\}.$$

What is the core of $S = \{2, 3, 6\}$?

(a) The core of $S = \{2, 3, 6\}$ is $\text{core}(S) = \{2, 3\}$. The reason is that if we take out 2, we are left with 3 and 6 which are both multiples of 3 and any variations of their sums will never be a prime ($3 + 3 + 3$ is a not a prime, $3 + 6 + 3$ is not a prime etc.). If we take out 3, we are left with two even numbers, namely 2 and 6, and still we know that every P_3-set contains an odd integer thus, taking out 3 will leave us with non-P_3-set (any variations of the sums of the even integers will not be even; the only case is 2 but $2 + 2 + 2 = 6$ is the best we can do). On the other hand, if we take out 6, S will still be a P_3-set since $2 + 2 + 3 = 7$ is a prime. Thus, $\text{core}(S) = \{2, 3\}$.

(b) Give an example of a P_3-set whose core is the empty set, or prove none exists.

Here is an example: let $S = \{1\}$, then S is a P_3 set since $1 + 1 + 1 = 3$ is a prime. However, if we take out 1, we have $S = \emptyset$ and there are no x, y, z such that $x + y + z$ is a prime.

(c) Give an example of a P_3-set whose core is infinite, or prove none exists.

There is no such P_3-set. Let's prove it. Suppose, for the sake of contradiction, there exists a P_3 set S such that its core is infinite. Let the core be the set $C = \{c_0, c_1, c_2, c_3 \ldots\}$. Then, we know that if we took out c_0, the set S would not be P_3. Then, since c_0 affected the outcome of whether S is a P_3-set or not, it means that c_0 plays a role in $x + y + z$. Same goes if we took out c_1. The same for c_2, and the same for c_4, c_5... etc. However, since c_0, c_1 and c_2 play a role in the sum, we know that there are at most 3 different elements of the set in the sum as by the definition $x + y + z$ must be a prime. But here we see that infinitely many elements are in this sum and we reached a contradiction. Thus, there are no P_3-set whose core is finite.

(d) Prove or Disprove: If S is a P_3-set then $\text{core}(S)$ is a P_3-set.

It's false. Counterexample: let $S = \{2, 3, 5\}$. Then $\text{core}(S) = \{3\}$ since if we take out 2, $3 + 3 + 5 = 11$ is still a prime and if we take out 5, $2 + 2 + 3 = 7$ is still a prime. However, if we take out 3, all the possible sums are:
$2 + 2 + 2 = 6$ is not a prime
$2 + 2 + 5 = 9$ is not a prime
$2 + 5 + 5 = 12$ is not a prime
$5 + 5 + 5 = 15$ is not a prime

Thus, we end up with a set $\text{core}(S) = \{3\}$ which is not a P_3-set since $3 + 3 + 3 = 9$ is the only sum we can get and it is not a prime.

(e) Prove or Disprove: If S and T are P_3-sets then $\text{core}(S \cup T) \subseteq \text{core}(S) \cap \text{core}(T)$.

It's true, let's prove it. Suppose, for the sake of contradiction, that for some two P_3-sets S_1 and S_2, $\text{core}(S_1) \cap \text{core}(S_2) \subset \text{core}(S_1 \cup S_2)$. Then we know that there exists $x \in \text{core}(S_1 \cup S_2)$ such that $x \notin \text{core}(S_1) \cap \text{core}(S_2)$. Now, since $x \in \text{core}(S_1) \cap \text{core}(S_2)$, it means that $x \in \text{core}(S_1)$ and $x \in \text{core}(S_2)$.

(f) Prove or Disprove: If S and T are P_3-sets with $S \subseteq T$ then we have core(T) $\subseteq$ core(S). 5, 7 5, 7, 9

Blah... 3 EXERCISES LEFT HERE TO FINISH.

55. A subset S of $\mathbb{Z}$ is called threequaline if for every $x, y \in S$ one has $3 \mid (x - y)$.

(a) Prove or Disprove: Every subset of a threequaline set is threequaline.

It's false. Let, for the sake of contradiction, that S is a threequaline set and also suppose that all the subsets of S are threequaline. Then we know that an empty set is a subset of every set and S is also a set thus, the emptyset is also a subset of S. However, an empty set has no elements and we cannot find x, y such that $x - y$ is a multiple of 3 and we reached the contradiction.

Q.E.D.

(b) Prove that if S is threequaline than either every element of S is divisible by 3 or none are.

Suppose, for the sake of contradiction, that S is a threequaline set and there exists two elements x, y such that x is a multiple of 3 and y is not a multiple of 3. Then $x - y$ will not be a multiple of 3 and we have reached the contradiction.

Q.E.D.

(c) Prove that if S is threequaline and r and t are integers, then the set $\{rx + t \mid x \in S\}$ is also threequaline.

Since we know that S is a threequaline set, for every $x, y \in S$, $x - y$ is a multiple of 3. Suppose, z_1, z_2 are some elements of the set S. Then, we know that $z_1 - z_2$ is a multiple of 3. The new set will "transform" these elements into $rz_1 + t$ and $rz_2 + t$. On the other hand, $z_1 - z_2 = rz_1 + t - (rz_2 + t) = r(z_1 - z_2)$ which is a multiple of 3 since z_1, z_2 are the members of S and $z_1 - z_2$ is a multiple of 3.

Q.E.D.

(d) Prove that if S and T are threequaline and $S \cap T \neq \emptyset$ then $S \cup T$ is threequaline.

Let's prove the contrapositive. The contrapositive of the statement is: prove that if S and T are threequaline and $S \cup T$ is not a threequaline, then $S \cap T = \emptyset$. Suppose, S_1 and S_2 are two threequaline sets and also suppose that $S_1 \cup S_2$ is not a threequaline set. Then there exist x, y in $S_1 \cup S_2$ such that $x - y$ is not a multiple of 3. Since S_1 and S_2 are both threequaline, we also know that both x and y cannot be a part the same set whether its S_1 or S_2. Let x be an element of S_1

(we could do it with S_2 as well, it is arbitrary), then we know that S_1 has some element z such that $x - z$ is a multiple of 3. On the other hand, it cannot be in S_2 because then it must also have x then which cannot be the case.
FINISH this!!!

56. A subset S of $\mathbb{R}$ is called **crunched** if there exist integers $m, n \in \mathbb{Z}$ such that for all $x \in S$ we have $m < x < n$.

    NOTE: When I mention lower bound or upper bound, I really mean the smallest element or the biggest element of the set.

    (a) Give some examples of sets that are and are not crunched.

    $S_1 = \{1\}$ is crunched as for all $x \in S$, $0 < x < 2$ ($m = 0, n = 2$).
    $S_2 = \{1, 2, 3\}$ is crunched as for all $x \in S$, $0 < x < 4$ ($m = 0, n = 4$).
    $S_3 = \mathbb{Z}^+$ is not crunched as it has no bounds and we cannot find m, n such that for all $x \in \mathbb{Z}^+, m < x < n$.
    $S_4 = \nvDash, \ntrianglerighteq, \ntriangleleft ...$ (a set of positive even numbers) is not crunched as it has no upper bound and we cannot find n such that for all $x \in \mathbb{Z}^+, m < x < n$ (note: we can find m. m can be any integer that is less than or equal to 1 but n cannot be fixed).

    (b) Prove or Disprove: All crunched sets are finite.

    That's false. Counterexample: let $S = \{1, 1/2, 1/4, ...\}$ (infinite geometric series), then we know that S has an upper bound 1 and the lower bound which is 0. Then, we can say with the great certainty, that for all $x \in S$, $-10 < x < 10$ ($m = -10$, $n = 10$). Thus, crunched sets are not necessarily finite and the initial claim is false.

    (c) Prove or Disprove: All finite sets are crunched.

    It's true. Let S be a finite set. Then it must have a lower bound (the smallest element), let it be k_1 and the upper bound (the biggest element), let it be k_2. Then let $m = k_1 - 1$ and let $n = k_2 + 1$ and we have that for all $x \in S$, $m < x < n$.

    Q.E.D.

    (d) Prove or Disprove: Every subset of a crunched set is crunched

    It's true. Suppose S is a crunched set. Then we know that for all $x \in S$, there exist m, n such that $m < x < n$. Let S_0 be a subset of S. Then, since all the elements of S_0 are also in S_1, we know that all elements of S_0 are between m and n which makes S_0 crunched. Thus, every subset of a crunched set is crunched.

Q.E.D.

(e) Prove or Disprove: The union of two crunched sets is crunched.

Suppose S_1 and S_2 are two crunched sets. Then we know that for all $x_1 \in S_1$, $m_1 < x_1 < n_1$ and for all $x_2 \in S_2$, $m_2 < x_1 < n_2$. Then, for all z in $S_1 \cup S_2, max(m_1, m_2) < z < max(n_1, n_2)$ which means that $S_1 \cup S_2$ is bounded.

Q.E.D.