Hypothesised filter for independent stochastic populations

Jeremie Houssineau and Daniel E. Clark

Abstract

A new filter for independent stochastic populations is studied and detailed, based on recent works introducing the concept of distinguishability in point processes. This filter propagates a set of hypotheses corresponding to individuals in a population, using a version of Bayes' theorem for multi-object systems. Due to the inherent complexity of the general multi-object data association problem, approximations are required to make the filter tractable. These approximations are justified and detailed along with explanations for the modelling choices. The efficiency of this new approach is then demonstrated through a comparison against one of the most popular multi-object filters.

1 Introduction

The existing solutions in the field of multi-object estimation can be decomposed into two classes of multi-object filters. One class consists of "classical" filters, such as the Multiple Hypothesis Tracking or MHT, see e.g. Blackman [1986], that are based on practical generalisations of single-object filters. The strength of these classical filters is their ability to distinguish the objects of interest and to naturally characterise each of them. The other class of filters comprises "point process inspired" approaches, such as the Probability Hypothesis Density (PHD) filter as introduced by Mahler [2003]. These filters successfully propagate global statistics about the population of interest and integrate clutter and appearance of objects in a principled way. However, they do not naturally propagate specific information about objects because of the point process usual assumption of indistinguishability. One of the attempts to overcome this limitation can be found in Vo and Vo [2013], where labelled random finite sets are used. However this approach is computationally expensive.

In this paper, we derive a new multi-object filter, called the Hypothesised filter for Independent Stochastic Population or HISP filter, based on a new framework relying on the introduction of the concept of distinguishability in the theory of point processes. This framework is described in the recent article Houssineau et al. [2013], where a solution for the data association problem is deduced and expressed in terms of association measures. The HISP filter allows to characterise separately the individuals of a population while preserving a sufficiently general modelling of the population dynamics. After introducing some notations in Section 2, we present the main results from Houssineau et al. [2013] along with more detailed explanations and expressions in Section 3. In Section 4, the approximations required for the implementation of the HISP filter are introduced. Finally, a comparison against the PHD filter is carried out on simulated data in Section 5.

2 Notations

The formalism of probability theory is used in this article to facilitate the statement of the results, even though this choice is not the most usual in the area of multi-target tracking, see e.g. Blackman [1986]. Specifically, $\mathcal{P}(E)$ will stand for the set of probability measures on a given measurable space $(E, \mathcal{E})$, and we will write $\mu(f) = \int \mu(\mathrm{d}x) f(x)$ for any $\mu \in \mathcal{P}(E)$ and any bounded measurable function f on $(E, \mathcal{E})$. Also, the relative complement of the set A in the set B will be denoted $B - A$ when $A \subseteq B$ and $B \setminus A$ otherwise. The concatenation of one element x at the end of a sequence $\boldsymbol{x}$ is denoted $\boldsymbol{x}^\frown x$.

Let $\mathfrak{X}$ be the population of interest. At time t, individuals in $\mathfrak{X}$ are described in the extended space $\bar{\mathbf{X}}_t = \{\psi\} \cup \mathbf{X}_t$, where $\mathbf{X}_t$ is a complete separable metric space (c.s.m.s.), e.g. $\mathbb{R}^d$ with $d > 0$, and where ψ represents the individuals with no image in $\mathbf{X}_t$. At time t, the observation process represents individuals of $\mathfrak{X}$ in the extended observation space $\bar{\mathbf{Z}}_t = \{\phi\} \cup \mathbf{Z}_t$, with $\mathbf{Z}_t$ a c.s.m.s. and ϕ the empty observation.

The finite observation set at time t is denoted Z_t and we write $\bar{Z}_t = \{\phi\} \cup Z_t$. Let the set $\bar{Y}_t$ of sequences of observations be defined as

$$\bar{Y}_t = \big\{(z_0, \ldots, z_t) : z_{t'} \in \bar{Z}_{t'}, 0 \leq t' \leq t\big\},$$

and let $\boldsymbol{\phi}_t \in \bar{Y}_t$ be the sequence such that $z_{t'} = \phi$, for any $0 \leq t' \leq t$. According to Pace and Del Moral [2013], elements of $\bar{Y}_t$ are called observation paths up to time t and we also define the set Y_t of non-empty observation paths as $Y_t = \bar{Y}_t - \{\boldsymbol{\phi}_t\}$. The observation path $\boldsymbol{\phi}_t$ represents individuals that have never been detected.

Let t_+ denote the time at which an individual in $\mathfrak{X}$ appeared in the state space. At time t, the time interval of existence of an individual is denoted by $T \subseteq [0, t]$. We assume that an individual cannot generate an observation at time t if it is not in the space $\mathbf{X}_t$. Let $\boldsymbol{y} \in Y_{t-1}$ and $T \subseteq [0, t]$, then the pair $\boldsymbol{x} = (T, \boldsymbol{y})$ defines a potential individual and the objective is to assess whether or not this potential individual is an actual individual in $\mathfrak{X}$.

As the estimation within the HISP filter is concerned with individuals only, the space $\bar{\mathbf{X}}_t$ is further augmented by the point φ to account for the uncertainty of the presence of an individual in $\bar{\mathbf{X}}_t$ and we denote

$$\mathbf{X}_t^+ = \{\varphi, \psi\} \cup \mathbf{X}_t = \{\varphi\} \cup \bar{\mathbf{X}}_t.$$

For any $p_t^{\boldsymbol{x}} \in \mathcal{P}(\mathbf{X}_t^+)$, the scalar $p_t^{\boldsymbol{x}}(\bar{\mathbf{X}}_t)$ is the probability for $\boldsymbol{x}$ to be an individual of $\mathfrak{X}$. The set of all potential individuals at time t, before the update, is denoted X_t. The set of updated potential individuals represented by $\boldsymbol{x} = (T, \boldsymbol{y})$ with $T \subseteq [0, t]$ and $\boldsymbol{y} \in Y_t$ is denoted $\hat{X}_t$.

Let $\mathcal{X}_t$ be the estimated population at time t, as stochastic representations in X_t are assumed to represent no more than one individual, there is a one-to-one relation between $\mathcal{X}_t$ and X_t. On the other hand, possibly many individuals are not observed during the observation process, and there is only a surjection between the representation $\mathcal{Z}_t$ of the population $\mathfrak{X}_t$ and the extended observation set $\bar{Z}_t$.

3 The HISP filter

3.1 Initialisation

At time $t = 0$, no observation has been made available yet so that no individual can be distinguished and the set of individual stochastic representations X_0 is such that $X_0 = \{\boldsymbol{x}_0\}$, with $\boldsymbol{x}_0 = (\{0\}, ())$. The associated law $p_0^{\boldsymbol{x}_0} \in \mathcal{P}(\mathbf{X}_0^+)$ is denoted p_0^{b} as individuals with representation $\boldsymbol{x}_0$ are thought as being newborn individuals at time 0. We assume the number of individuals with representation $\boldsymbol{x}_0$ is driven by the cardinality distribution c_0^{b}.

3.2 Time update

Given the independence of the individuals in the population $\mathfrak{X}$, the law $\hat{p}_t^{\boldsymbol{x}}$ of an individual with representation $\boldsymbol{x} \in \hat{X}_{t-1}$ can be predicted straightforwardly by using the Chapman-Kolmogorov equation with a Markov kernel $M_{t|t-1}$ from $\mathbf{X}_{t-1}^+$ to $\mathbf{X}_t^+$. More formally, for any measurable function f on $\mathbf{X}_t^+$,

$$p_t^{\boldsymbol{x}'}(f) = \hat{p}_{t-1}^{\boldsymbol{x}}(M_{t|t-1}(f|\cdot)),$$

where $\boldsymbol{x}' = ([t_+, t], \boldsymbol{y})$ whenever $\boldsymbol{x} = ([t_+, t-1], \boldsymbol{y})$. As a consequence, there is a one-to-one correspondence between $\hat{X}_{t-1}$ and X_t. The Markov kernel $M_{t|t-1}$ from $\mathbf{X}_{t-1}^+$ to $\mathbf{X}_t^+$ can be decomposed as follows: For any $x \in \mathbf{X}_{t-1}$ and any $x' \in \mathbf{X}_t$,

$$\begin{aligned} M_{t|t-1}(\mathrm{d}x'|x) &= p_{S,t}(x) m_{t|t-1}(\mathrm{d}x'|x), \\ M_{t|t-1}(\psi|x) &= 1 - p_{S,t}(x), \\ M_{t|t-1}(\psi|\psi) &= 1, \quad M_{t|t-1}(\phi|\phi) = 1, \end{aligned}$$

where $m_{t|t-1}$ is a Markov kernel from $\mathbf{X}_{t-1}$ to $\mathbf{X}_t$. The interpretation is that an individual at point x persists to time t with probability $p_{S,t}(x)$. E.g., the probability for $\boldsymbol{x}$ to represent a disappeared individual at time t is

$$p_{t-1}^{\boldsymbol{x}}(M_{t|t-1}(\psi|\cdot)).$$

The birth at time t is modelled by a unique individual stochastic representation $\boldsymbol{x} = (\{t\}, ())$ with law $p_t^{\mathrm{b}} \in \mathcal{P}(\mathbf{X}_t^+)$ with cardinality distribution c_t^{b}. We assume that $p_t^{\mathrm{b}}(\psi) = 0$ as newborn individuals exist almost surely.

3.3 Observation update

At time t, a set Z_t of observations in $\mathbf{Z}_t$ is received. The geometry of the sensor from which these observations are generated is assumed to be known, and we denote π the partition of $\mathbf{Z}_t$ corresponding to the sensor resolution cells. Each resolution cell is represented by a point z_ω in $\mathbf{Z}_t$, which may be the centre of the cell, and the set Z_t^+ is defined as $\{z_\omega \text{ s.t. } \omega \in \pi\}$.

To simplify the association, the newborn individuals are *localised* according to the back-projection of the partition π on $\mathbf{X}_t$, i.e. an individual stochastic representation is induced by each observation $z \in Z_t \subseteq Z_t^+$, and its law is denoted $p_t^{\mathrm{b}_z}$. We assume that the probability of having more than one birth per resolution cell is negligible. The probability $p_t^{\mathrm{b}_z}(\mathbf{X}_t)$ is computed considering the probability of having one newborn individual within the cell.

Similarly, we assume that spurious observations are generated per resolution cell, and that these observations correspond to individuals that are not in the population $\mathfrak{X}$ and have therefore a state ψ in $\mathbf{X}_t^+$. The set of such individuals at time t is denoted $\mathcal{X}_t^{\mathrm{op}}$. We can now introduce the sets X_t^{b} and X_t^{op} of individual stochastic representations that are in one-to-one correspondence with individuals in $\mathcal{X}_t^{\mathrm{b}}$ and $\mathcal{X}_t^{\mathrm{op}}$. Representations $\boldsymbol{x}_t^{\mathrm{b}_z} \in X_t^{\mathrm{b}}$ and $\boldsymbol{x}_t^{\mathrm{op}_z} \in X_t^{\mathrm{op}}$ have the form $(\{t\}, \boldsymbol{y}_z)$ and $(\emptyset, \boldsymbol{y}_z)$ respectively, where $\boldsymbol{y}_z = \boldsymbol{\phi}_{t-1}{}^\frown z$ for any $z \in Z_t$. The set of predicted stochastic representations X_t can then be extended in two different ways:

$$
\begin{aligned}
X_t^z &= X_t \cup \{\boldsymbol{x}_t^{\mathrm{b}_z}, \boldsymbol{x}_t^{\mathrm{op}_z}\}, \qquad \forall z \in \bar{Z}_t, \\
X_t^+ &= X_t \cup X_t^{\mathrm{b}} \cup X_t^{\mathrm{op}}.
\end{aligned}
$$

Data association is described by a bijection ν between $\mathcal{X}_t^+ = \mathcal{X}_t \cup \mathcal{X}_t^{\mathrm{b}} \cup \mathcal{X}_t^{\mathrm{op}}$ and $\mathcal{Z}_t$. As only one newborn individual and one individual in $\mathcal{X}_t^{\mathrm{op}}$ can be associated with each $z \in Z_t$, the association between $\mathcal{X}_t^{\mathrm{b}}$, $\mathcal{X}_t^{\mathrm{op}}$ and Z_t is characterised by the choice of disjoint subsets Z_t^{b} and Z_t^{op} of Z_t. The other observations in Z_t, i.e. the observations in $Z_t^{\mathrm{d}} = Z_t - (Z_t^{\mathrm{b}} \cup Z_t^{\mathrm{op}})$ are associated with representations in X_t in a way described by the injective function $\varsigma : Z_t^{\mathrm{d}} \to X_t$, induced by ν. The image of Z_t^{d} via ς, denoted X_t^{d}, contains the representations of detected individuals. The restriction of ς^{-1} to X_t^{d} is a bijection denoted σ.

The association $\nu : \mathcal{X}_t \to \mathcal{Z}_t$ can then be characterised by the choice of Z_t^{b}, Z_t^{op} and ς, the latter being equivalently expressed as the choice of X_t^{d} and σ. We then consider an association to be the vector $\boldsymbol{a}$ defined

$$
\boldsymbol{a} = (X_t^{\mathrm{d}}, \sigma, Z_t^{\mathrm{b}}),
$$

in the set Adm_t of all admissible associations at time t.

The likelihood function g_t from $\mathbf{X}_t^+$ to $\bar{\mathbf{Z}}_t$ can be decomposed as follows: For any $z \in Z_t$, any $x \in \mathbf{X}_t$ and any $\boldsymbol{x} \in X_t$,

$$
\begin{aligned}
g_t(z|x, \boldsymbol{x}) &= p_{D,t}(x)\ell_t(z|x), \\
g_t(\phi|x, \boldsymbol{x}) &= 1 - p_{D,t}(x), \\
g_t(\phi|\psi, \boldsymbol{x}) &= 1, \quad g_t(\phi|\varphi, \boldsymbol{x}) = 1,
\end{aligned}
$$

where $\ell_t(z|x)$ is a likelihood from $\mathbf{Z}_t$ to $\mathbf{X}_t$ and where $p_{D,t}(x)$ is the probability for an object at point x to generate an observation. For the individuals in $\mathcal{X}_t^{\mathrm{op}}$, the likelihood g_t is defined as

$$g_t(z'|\psi, \mathrm{op}_z) = \delta_z(z') p_t^{\mathrm{fa},z},$$
$$g_t(\phi|\psi, \mathrm{op}_z) = 1 - p_t^{\mathrm{fa},z},$$

where "op_z" denote the representation $\boldsymbol{x} = (\emptyset, \boldsymbol{\phi}_{t-1}{}^\frown z) \in X_t^{\mathrm{op}}$ and where $p_t^{\mathrm{fa},z} \in [0,1]$ is the probability for z to be a false alarm.

An additional assumption regarding birth can be made to lighten the computational cost as in Ristic et al. [2010] and Houssineau and Laneuville [2010]. This is achieved by specifying the likelihood function g_t for newborn individuals as

$$g_t(z'|x, \mathrm{b}_z) = \delta_z(z')\ell_t(z|x),$$
$$g_t(\phi|x, \mathrm{b}_z) = 0,$$

where "b_z" denote the representation $\boldsymbol{x} = (\{t\}, \boldsymbol{\phi}_{t-1}{}^\frown z) \in X_t^{\mathrm{b}}$. This is equivalent to set $p_{D,t} = 1$ for newborn individuals.

For any $z \in \bar{Z}_t$ and any $\boldsymbol{x} \in X_t^+$, we denote the prior probability of association $p_t^{\boldsymbol{x},z} \in \mathcal{P}(\mathbf{X}_t^+)$ expressed as

$$p_t^{\boldsymbol{x},z}(f) = p_t^{\boldsymbol{x}}\big(f\, g_t(z|\cdot, \boldsymbol{x})\big),$$

which represents the prior probability for the individual associated with $\boldsymbol{x}$ to be observed via z. Following Houssineau et al. [2013], the distribution $P_t^{\boldsymbol{a}}$ describing the association under $\boldsymbol{a}$ is found to be

$$P_t^{\boldsymbol{a}}(B_t^{\boldsymbol{a}}) = \prod_{\boldsymbol{x} \in X_t^+} p_t^{\boldsymbol{x},\nu(\boldsymbol{x})}\big(A_t^{\boldsymbol{x},\nu(\boldsymbol{x})}\big),$$

where $B_t^{\boldsymbol{a}} = \bigtimes_{\boldsymbol{x} \in X_t^+} A_t^{\boldsymbol{x},\nu(\boldsymbol{x})}$, and we introduce the marginal

$$P_t = \sum_{\boldsymbol{a} \in \mathrm{Adm}_t} P_t^{\boldsymbol{a}}(B_t^{\boldsymbol{a}}).$$

Equipped with these concepts and notations, we can proceed to the expression of the HISP Bayes' theorem as follows.

Theorem 1 (HISP Bayes' theorem)**.** *For any $z \in \bar{Z}_t$ and for any $\boldsymbol{x} \in X_t^z$, the posterior individual law $\hat{p}_t^{\boldsymbol{x},z} \in \mathcal{P}(\mathbf{X}_t^+)$ representing the probability for the individual associated with $\boldsymbol{x}$ to be observed via z can be expressed in two different ways as*

$$\hat{p}_t^{\boldsymbol{x},z} = \frac{w(\boldsymbol{x},z)p_t^{\boldsymbol{x},z}}{\sum_{\boldsymbol{x}' \in X_t^z} w(\boldsymbol{x}',z)p_t^{\boldsymbol{x}',z}(1)}, \tag{1}$$

and

$$\hat{p}_t^{\boldsymbol{x},z} = \frac{w(\boldsymbol{x},z)p_t^{\boldsymbol{x},z}}{\sum_{z' \in \bar{Z}_t} w(\boldsymbol{x},z')p_t^{\boldsymbol{x},z'}(1)}, \tag{2}$$

where $w(\boldsymbol{x}, z)$ is the joint probability for the individuals associated with $X_t^+ - \{\boldsymbol{x}\}$ to be successfully associated with the observations in $\bar{Z}_t - \{z\}$.

Proof. Let $\mathrm{Adm}_t(\boldsymbol{x}, z)$ be the subset of Adm_t in which $\boldsymbol{x}$ is associated to z. Then the posterior marginal $\hat{p}_t^{\boldsymbol{x},z}$ can be expressed as

$$\hat{p}_t^{\boldsymbol{x},z}(A_t^{\boldsymbol{x},z}) = \frac{1}{P_t} \sum_{\boldsymbol{a} \in \mathrm{Adm}_t(\boldsymbol{x},z)} P_t^{\boldsymbol{a}}\big(B_t^{\boldsymbol{a}}|_{\boldsymbol{x},z}\big), \tag{3}$$

where $B_t^{\boldsymbol{a}}|_{\boldsymbol{x},z}$ is the measurable subset $B_t^{\boldsymbol{a}}$ where $A_t^{\boldsymbol{x},z'} = \emptyset$ if $z' \neq z$ and where only $A_t^{\boldsymbol{x},z}$ is not set to $\mathbf{X}_t^+$ if $z' = z$. The proof can then be decomposed into two parts: (i) prove that there exists a function $w : X_t^+ \times \bar{Z}_t \to [0, 1]$ such that the sum in (3) can be factorised into $p_t^{\boldsymbol{x},z}(A_t^{\boldsymbol{x},z})w(\boldsymbol{x}, z)$, and (ii) prove that P_t can be equivalently expressed as the denominator of either (1) or (2). To prove (i), expend the sum in (3) as follows

$$\sum_{\boldsymbol{a} \in \mathrm{Adm}_t(\boldsymbol{x},z)} P_t^{\boldsymbol{a}}\big(B_t^{\boldsymbol{a}}|_{\boldsymbol{x},z}\big) = p_t^{\boldsymbol{x},z}(A_t^{\boldsymbol{x},z}) \sum_{\boldsymbol{a}' \in \mathrm{Adm}'_t(\boldsymbol{x},z)} P_t^{\boldsymbol{a}'}(1),$$

where $\mathrm{Adm}'_t(\boldsymbol{x}, z)$ is made of the associations $\boldsymbol{a}' = (Z_t^{\mathrm{b}}, Z_t^{\mathrm{op}}, X_t^{\mathrm{d}}, \sigma)$ such that Z_t^{b} is a subset of $Z_t - \{z\}$, and Z_t^{d} and X_t^{d} are subsets of $X_t^+ - \{\boldsymbol{x}\}$ and $Z_t \setminus \{z\}$ respectively. The function $w : X_t^+ \times \bar{Z}_t \to [0, 1]$ is then defined as

$$w(\boldsymbol{x}, z) = \sum_{\boldsymbol{a}' \in \mathrm{Adm}'_t(\boldsymbol{x},z)} P_t^{\boldsymbol{a}'}(1),$$

so that

$$\hat{p}_t^{\boldsymbol{x},z}(A_t^{\boldsymbol{x},z}) = \frac{1}{P_t} p_t^{\boldsymbol{x},z}(A_t^{\boldsymbol{x},z}) w(\boldsymbol{x}, z).$$

To prove (ii), it is sufficient to see that for any $z \in \bar{Z}_t$ and any $\boldsymbol{x} \in X_t^+$

$$\bigcup_{\boldsymbol{x}' \in X_t^z} \mathrm{Adm}_t(\boldsymbol{x}', z) = \bigcup_{z' \in \bar{Z}_t} \mathrm{Adm}_t(\boldsymbol{x}, z') = \mathrm{Adm}_t,$$

since the constraint $\nu(\boldsymbol{y}) = z$ is loosen by summing over the representations in X_t^+ or over all observations in $\bar{Z}_t$. We finally find that

$$P_t = \sum_{\boldsymbol{a} \in \mathrm{Adm}_t} P_t^{\boldsymbol{a}}(1) = \sum_{\boldsymbol{x}' \in X_t^z} w(\boldsymbol{x}', z) p_t^{\boldsymbol{x}',z}(1) = \sum_{z' \in \bar{Z}_t} w(\boldsymbol{x}, z') p_t^{\boldsymbol{x},z'}(1),$$

which allows to prove the desired result. □

Note that HISP Bayes' theorem can be rewritten to make the underlying single-object filters appear. For instance, (1) can be expressed as follows

$$\hat{p}_t^{\boldsymbol{x},z}(\mathrm{d}x) = \frac{w(\boldsymbol{x}, z) p_t^{\boldsymbol{x},z}(1)}{\sum_{\boldsymbol{x}' \in X_t^z} w(\boldsymbol{x}', z) p_t^{\boldsymbol{x}',z}(1)} \times \frac{g_t(z|x) p_t^{\boldsymbol{x}}(\mathrm{d}x)}{\int g_t(z|y) p_t^{\boldsymbol{x}}(\mathrm{d}y)},$$

where the second term on the r.h.s. is the standard Kalman-filter update. An important feature of the HISP filter can also be highlighted: an a posteriori probability of missed-detection can be computed through (2) when $z = \phi$.

4 The HISP filter in practice

To make the HISP filter more tractable, it is now required to find approximations of the joint probabilities $w(\boldsymbol{x}, z)$, for any $\boldsymbol{x} \in X_t^+$ and any $z \in \bar{Z}_t$.

4.1 Approximations

The HISP Bayes' theorem provides an expression of the posterior law $\hat{p}_t^{\boldsymbol{x},z}$ given the prior law $p_t^{\boldsymbol{x},z}$. However, the joint probabilities $w(\boldsymbol{x}, z)$ have not been detailed yet. Also, for practical reasons such as devising approximations, it is convenient to define a factorised form for $w(\boldsymbol{x}, z)$, which can be deduced from the factorisation of P_t given in the following lemma. For the sake of clarity we omit the argument of the marginalised measures and write $p_t^{\boldsymbol{x},z}$ to designate the scalar values $p_t^{\boldsymbol{x},z}(1)$.

Lemma 1. *The probability P_t can be expressed in a factorised form as*

$$P_t = C_t^{\phi}\Big[\prod_{z\in Z_t} C_t^z\Big] \times \Bigg[\sum_{(X_t^{\mathrm{d}},\sigma,\cdot)\in \mathrm{Adm}_t}\Big[\prod_{\boldsymbol{x}\in X_t^{\mathrm{d}}} \frac{p_t^{\boldsymbol{x},z}}{p_t^{\boldsymbol{x},\phi} C_t^z}\Big]\Bigg],$$

where $z = \sigma^{-1}(\boldsymbol{x})$ whenever $\boldsymbol{x}$ is defined, where C_t^{ϕ} is the probability for all the individuals with representations in X_t^+ to be miss-detected, defined as follows

$$C_t^{\phi} = \prod_{\boldsymbol{x}\in X_t^+} p_t^{\boldsymbol{x},\phi},$$

and where C_t^z is defined as

$$C_t^z = \frac{p_t^{\mathrm{b}_z,z}}{p_t^{\mathrm{b}_z,\phi}} + \frac{p_t^{\mathrm{op}_z,z}}{p_t^{\mathrm{op}_z,\phi}}.$$

Proof. The first step in proving the result is to rewrite $P_t^{\boldsymbol{a}}(1)$ in a suitable way:

$$P_t^{\boldsymbol{a}}(1) = \Big[\prod_{\boldsymbol{x}\in X_t^+} p_t^{\boldsymbol{x},\phi}\Big] \times \Big[\prod_{\boldsymbol{x}\in X_t^{\mathrm{d}}} \frac{p_t^{\boldsymbol{x},z}}{p_t^{\boldsymbol{x},\phi}}\Big] \times \Big[\prod_{z\in Z_t^{\mathrm{b}}} \frac{p_t^{\mathrm{b}_z,z}}{p_t^{\mathrm{b}_z,\phi}}\Big] \times \Big[\prod_{z\in Z_t^{\mathrm{op}}} \frac{p_t^{\mathrm{op}_z,z}}{p_t^{\mathrm{op}_z,\phi}}\Big],$$

with $Z_t^{\mathrm{op}} = Z_t - (Z_t^{\mathrm{d}} - Z_t^{\mathrm{b}})$, where Z_t^{d} is the image of X_t^{d} through σ. The second step is to consider P_t as follows

$$P_t = C_t^{\phi} \sum_{(X_t^{\mathrm{d}},\sigma,\cdot)\in \mathrm{Adm}_t} \Bigg[\Big[\prod_{\boldsymbol{x}\in X_t^{\mathrm{d}}} \frac{p_t^{\boldsymbol{x},z}}{p_t^{\boldsymbol{x},\phi}}\Big] \times \Big[\sum_{Z_t^{\mathrm{b}}\subseteq Z_t - Z_t^{\mathrm{d}}} \Big[\prod_{z\in Z_t^{\mathrm{b}}} \frac{p_t^{\mathrm{b}_z,z}}{p_t^{\mathrm{b}_z,\phi}}\Big] \times \Big[\prod_{z\in Z_t^{\mathrm{op}}} \frac{p_t^{\mathrm{op}_z,z}}{p_t^{\mathrm{op}_z,\phi}}\Big]\Big]\Bigg]$$

which can be re-expressed as

$$P_t = C_t^{\phi} \sum_{(X_t^{\mathrm{d}},\sigma,\cdot)\in \mathrm{Adm}_t} \Bigg[\Big[\prod_{\boldsymbol{x}\in X_t^{\mathrm{d}}} \frac{p_t^{\boldsymbol{x},z}}{p_t^{\boldsymbol{x},\phi}}\Big] \times \Big[\prod_{z\in Z_t - Z_t^{\mathrm{d}}} C_t^z\Big]\Bigg],$$

from which the desired result follows easily. $\square$

As the expressions of P_t and $w(\boldsymbol{x}, z)$ are very similar:

$$P_t = \sum_{\boldsymbol{a} \in \mathrm{Adm}_t} P_t^{\boldsymbol{a}}(1) \qquad \text{and} \qquad w(\boldsymbol{x}, z) = \sum_{\boldsymbol{a}' \in \mathrm{Adm}_t(\boldsymbol{x}, z)} P_t^{\boldsymbol{a}'}(1),$$

a factorisation for $w(\boldsymbol{x}, z)$ can be found easily from Lemma 1. This factorisation makes easier the introduction of the following approximations.

Approximation 1. *Let X and Z be subsets of X_t and Z_t respectively, then the approximation* $\mathrm{Approx}_1(X, Z)$ *consists in assuming that*

$$p_t^{\boldsymbol{x},z} p_t^{\boldsymbol{x},z'} \approx 0,$$

for any $\boldsymbol{x} \in X$ and any $z, z' \in Z$.

Considering that $\mathrm{Approx}_1(X, Z)$ holds is equivalent to assuming that two observations in Z are unlikely to be *associated* with the same individual stochastic representation in X. There is an obvious counterpart to this approximation, for which two representations are unlikely to be associated with the same observation. This other approximation is defined as follows.

Approximation 2. *Let X and Z be subsets of X_t and Z_t respectively, then the approximation* $\mathrm{Approx}_2(X, Z)$ *consists in assuming that*

$$p_t^{\boldsymbol{x},z} p_t^{\boldsymbol{x}',z} \approx 0,$$

for any $\boldsymbol{x}, \boldsymbol{x}' \in X$ and any $z \in Z$.

These two approximations allow to factorise further the expressions of P_t and of the joint probabilities $w(\boldsymbol{x}, z)$, with $\boldsymbol{x} \in X_t^+$ and $z \in \bar{Z}_t$.

Proposition 1. *Under approximation* $\mathrm{Approx}_1(X_t, Z_t)$*, the joint probability P_t can be factorised as follows*

$$P_t = C_t \prod_{\boldsymbol{x} \in X_t} \left[p_t^{\boldsymbol{x},\phi} + \sum_{z \in Z_t} \frac{p_t^{\boldsymbol{x},z}}{C_t^z} \right],$$

where

$$C_t = \left[\prod_{\boldsymbol{x} \in X_t^{\mathrm{b}} \cup X_t^{\mathrm{op}}} p_t^{\boldsymbol{x},\phi} \right] \times \left[\prod_{z \in Z_t} C_t^z \right].$$

Also, under $\mathrm{Approx}_2(X_t, Z_t)$*, the joint probability P_t can be factorised as follows*

$$P_t = \left[\prod_{\boldsymbol{x} \in X_t} p_t^{\boldsymbol{x},\phi} \right] \times \left[\prod_{z \in Z_t} \left[C_t^z + \sum_{\boldsymbol{x} \in X_t} \frac{p_t^{\boldsymbol{x},z}}{p_t^{\boldsymbol{x},\phi}} \right] \right].$$

Henceforth, we will focus on Approximation 1 as results for Approximation 2 are very similar. As a consequence of Proposition 1, the joint probability $w(\boldsymbol{x}, z)$ can be re-expressed as follows.

Corollary 1. *For any* $\boldsymbol{x} \in X_t^+$*, any* $z \in \bar{Z}_t$ *let* $X'_t = X_t \setminus \{\boldsymbol{x}\}$ *and* $Z'_t = Z_t \setminus \{z\}$*. Then, under approximation* $\mathrm{Approx}_1(X'_t, Z'_t)$*, the joint probability* $w(\boldsymbol{x}, z)$ *can be factorised as follows*

$$w(\boldsymbol{x}, z) = C'_t(\boldsymbol{x}, z) \prod_{\boldsymbol{x}' \in X'_t} \left[p_t^{\boldsymbol{x}',\phi} + \sum_{z' \in Z'_t} \frac{p_t^{\boldsymbol{x}',z'}}{C_t^{z'}} \right],$$

where

$$C'_t(\boldsymbol{x}, z) = \left[\prod_{\boldsymbol{x}' \in (X_t^{\mathrm{b}} \cup X_t^{\mathrm{op}}) \setminus \{\boldsymbol{x}\}} p_t^{\boldsymbol{x}',\phi} \right] \times \left[\prod_{z' \in Z'_t} C_t^{z'} \right].$$

We now describe the implementation of the HISP filter.

4.2 Implementation of the HISP filter

Both Sequential Monte Carlo (SMC) and Gaussian Mixture (GM) implementation are possible, as in Vo et al. [2005] and Vo and Ma [2006]. Note that the SMC-HISP filter does not require clustering, thus preventing from possible discrepancies and alleviating the computational cost of the filter. The complexity of the computation of the terms w, as described in the Algorithm 1, is in $O\big(|Z_t||X_t|\big)$ at time t. Thus, the HISP filter also has this complexity and is then similar to the PHD filter from this point of view.

Algorithm 1: Computation of $w(\boldsymbol{x}, z)$ for Approximation 1

Data: Probability $p_t^{\boldsymbol{x},z}$ for $\boldsymbol{x} \in X_t^+$ and $z \in \bar{Z}_t$
Result: $w(\boldsymbol{x}, z)$ for $(\boldsymbol{x}, z) \in X_t^+ \times \bar{Z}_t$

```
for x ∈ X_t do
    for z ∈ Z_t do
        Compute S(x, z) = p_t^{x,z} / C_t^z
    end
    Compute T(x, φ) = p_t^{x,φ} + ∑_{z∈Z_t} S(x, z)
    for z ∈ Z_t do
        Compute T(x, z) = T(x, φ) − S(x, z)
    end
end
for z ∈ Z̄_t do
    Compute P(z) = ∏_{x∈X_t} T(x, z)
end
for (x, z) ∈ X_t^+ × Z̄_t do
    Compute w(x, z) = C'_t(x, z) P(z) / T(x, z)
end
```

Some of the terms composing $w(\boldsymbol{x}, z)$ can be simplified in the quotients (1) and (2), reducing the occurrence of computational accuracy issues with large products of possibly small quantities.

5 Results

The performance of the HISP filter is assessed and compared against the PHD filter on one scenario, for different probabilities of detection and false alarm.

We consider a sensor placed at the centre of the coordinate system that delivers range and bearing observations every 4s during 300s. The size of the resolution cell of this sensor is $1^\circ \times 15$m. Considering small fixed random error and bias error, the standard deviation of the observations is $\sigma_r = 6.2$m and $\sigma_\theta = 4.5$mrad for a SNR of 3dB and $\sigma_r = 4.87$m and $\sigma_\theta = 3.5$mrad for a SNR of 5dB. The range r is in $[50\text{m}, 500\text{m}]$ and the bearing θ is in $(-\pi, \pi]$.

The scenario comprises 5 objects with states $X_{t,1}, \dots, X_{t,5}$ at time t, with initial states expressed in $[x, y, v_x, v_y]$ coordinates, with v_x and v_y the velocities in m.s^{-1} along the x- and y-axis, as

$$\begin{aligned}
&X_{0,1} = [-400, -50, 1, 1.1], \quad X_{0,2} = [-50, -300, 0.4, 0.6],\\
&X_{0,3} = [50, -300, -0.4, 0.6], \quad X_{0,4} = [150, 150, -0.2, 0.2],\\
&X_{0,5} = [200, 300, 0.25, -1].
\end{aligned}$$

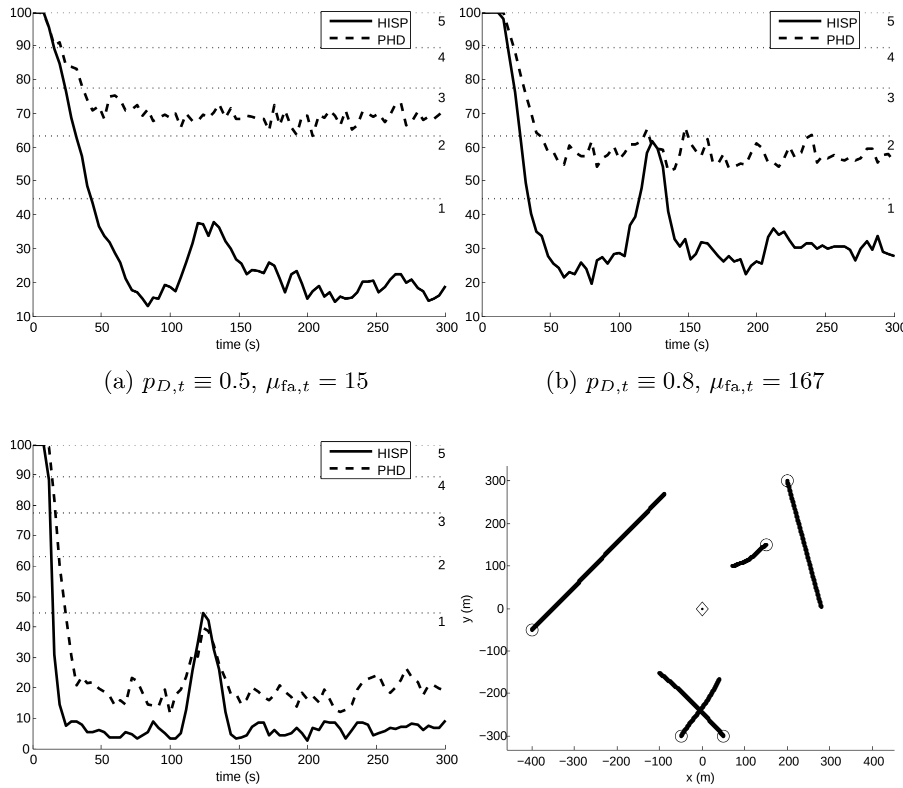
The motion of the objects is driven by a linear model with noise q such that $\mathrm{var}(q) = 0.05\text{m}^2.\text{s}^{-4}$. We assume that the objects never spontaneously disappear so that we set the probability of survival to $p_{S,t} \equiv 1$. The scenario is depicted in Figure 1d. Note that objects 3 and 4 are crossing around $t = 120$s.

Henceforth, we consider a GM-HISP filter based on Approximation 1 where the detected and undetected hypotheses are updated through (1) and (2) respectively. The non-linearity of the observation model is dealt with by an extended Kalman filter. To reduce the computational cost, pruning (with parameter $\tau = 10^{-5}$) and merging (with parameter $d_\mathrm{m} = 4$) are carried out on the set of hypotheses as in Vo and Ma [2006]. For the HISP filter, merging two hypotheses means that these hypotheses are assumed to represent the same object, so that the sum of the probabilities of the merged hypothesis is limited to 1. An hypothesis is considered as "confirmed" if it has a probability of existence above $\tau_\mathrm{c} = 0.99$ or if it was previously confirmed and has a probability of existence above $\tau_\mathrm{uc} = 0.9$.

In the considered scenarios, the probability of birth is assumed to be constant across the state space $\mathbf{X}_t$ and is denoted p_t^b. The average number of false alarm in the observation set at time t is denoted μ_t^fa.

5.1 Case 1: Low probability of detection (3dB)

We set $p_t^\mathrm{b} = 10^{-6}$ and $p_{D,t} \equiv 0.5$ so that $p_t^\mathrm{fa} = 1.34 \times 10^{-3}$ and $\mu_t^\mathrm{fa} \approx 15$. The OSPA distance, defined in Schuhmacher et al. [2008], is averaged over 50 runs and depicted in Figure 1a. The OSPA distance for the HISP filter is below the one of the PHD filter at all time. The performance of the HISP filter decreases when object 3 and 4 cross, which is natural since there is uncertainty on the association.



(a) $p_{D,t} \equiv 0.5$, $\mu_{\mathrm{fa},t} = 15$

(b) $p_{D,t} \equiv 0.8$, $\mu_{\mathrm{fa},t} = 167$

(c) $p_{D,t} \equiv 0.995$, $\mu_{\mathrm{fa},t} = 83$

(d) $\diamond$: sensor, $\circ$: object at $t = 0$

Figure 1: OSPA distance versus time in Case 1 (a), Case 2 (b), Case 3 (c) on the scenario (d) over 50 Monte Carlo runs. HISP filter: solid line. PHD filter: dashed line. The dotted line numbered n represents the OSPA for a cardinality error of n without localisation error.

5.2 Case 2: High probability of false alarm (3dB)

In this case, we set $p_t^{\mathrm{b}} = 5 \times 10^{-7}$ and $p_{D,t} \equiv 0.8$ so that $p_t^{\mathrm{fa}} = 1.54 \times 10^{-2}$ and $\mu_t^{\mathrm{fa}} \approx 167$. The average OSPA distance is depicted in Figure 1b. The PHD filter, which is known to be robust to high probabilities of false alarm, behaves better than in Case 1 even though the HISP filter still outperforms it.

5.3 Case 3: High probability of detection (5dB)

We set $p_t^{\mathrm{b}} = 10^{-6}$ and $p_{D,t} \equiv 0.995$ so that $p_t^{\mathrm{fa}} = 7.67 \times 10^{-3}$ and $\mu_t^{\mathrm{fa}} \approx 83$. The average OSPA distance is depicted in Figure 1c. The performance of the PHD filter approaches the one of the HISP filter and is even slightly higher when objects 3 and 4 cross. This can be due to the higher object state uncertainty in the PHD filter which facilitates merging.

6 Conclusion

A new multi-object filter for independent stochastic populations, called the HISP filter, has been derived and detailed. When studying this filter, it appeared that there is more than one way of using the update equations and that there are different approximations as well as diverse applicable modelling alternatives, even though only one has been studied here. In this sense, the HISP filter can be seen as a general way of approaching the problem of multi-object estimation. The HISP filter allows to characterise each hypothesis separately thus giving a local picture of the underlying multi-object problem while controlling the level of approximation. Its efficiency has been compared with the performance of the PHD filter, since the two filters have equivalent complexities. The results show that the HISP filter outperforms the PHD filter in several cases with different probabilities of false-alarm and different probabilities of detection.

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