Chaos and Strange Attractors

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Introduction

Overview

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- Thesis

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- Thesis
- Review
  - Chaotic Systems
  - Fractal Geometry
  - Nonstrange Nonchaotic Attractors
- Strange Nonchaotic Attractors

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  - Lyapunov exponents & Kaplan-Yorke Dimension
  - Characteristics of Chaotic Systems

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  - Lyapunov exponents & Kaplan-Yorke Dimension
  - Characteristics of Chaotic Systems
- Conclusion

Thesis

The purpose of this presentation is to use fractal geometry and chaos to determine what separates strange chaotic attractors from other attractors.

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For a discrete system:

$$\lambda = \frac{1}{n} \sum_{i=1}^{n} \log_2 \left| \frac{d_i}{d_0} \right| . \tag{2}$$

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$$\lambda_1 \leq 0 \Leftrightarrow non - chaotic \tag{4}$$

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$$D_{KY} = j + \frac{\lambda_1 + \lambda_2 + ... + \lambda_j}{|\lambda_{j+1}|}. \tag{5}$$

Where j is the largest integer for which $\lambda_1 + \lambda_2 + ... + \lambda_j \geq 0$

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If D_{KY} is a positive non-integer, it is a fractal. However, it is possible for an attractor to have an integer dimension and still have fractal properties.

Nonstrange Nonchaotic Attractors

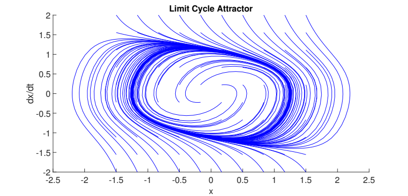

Figure 1: Phase Plane for Van der Pol Oscillator for $a = b = c = d = 1$

The Van der Pol oscillator can be defined by the following set of differential equations:

$$\frac{dx}{dt} = y \tag{6}$$

$$\frac{dy}{dt} = \frac{-1}{cd}(x + by^3 - ay) \tag{7}$$

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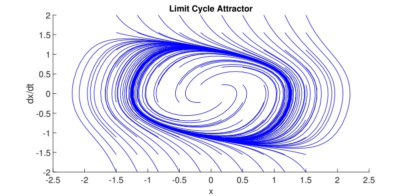

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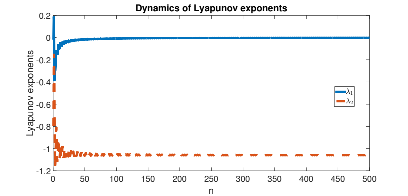

Figure 2: Lyapunov exponents for Van der Pol Oscillator

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Figure 2 shows that the Lyapunov exponents converge at $\lambda_1 = 0$ and $\lambda_2 = -1.06$. The largest Lyapunov exponent indicates a nonchaotic system. Using the Kaplan-Yorke Dimension we get:

$$D_{KY} = 1 + \frac{0}{|-1.06|} = 1$$

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Strange is sometimes used to name attractors that show chaotic behavior, which is misleading. Strangeness is not dependent on chaos, thus strange attractors do not have to be chaotic. In a strange system, orbits on an attractor are non-periodic. Meaning that any point in the attracting set is never visited more than once, while some regions are never visited. These sets of points have a fractal nature.

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The GPOY model is a discrete system defined by the equations:

$$x_{n+1} = 2\sigma \tanh(x_n) \cos(2\pi\theta_n) \tag{8}$$

$$\theta_{n+1} = \theta_n + \omega \mod 1 \tag{9}$$

Strange Nonchaotic Attractors

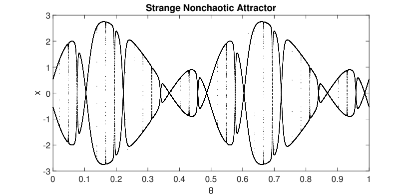

Figure 3: Single orbit of the GOPY model for $\sigma = 1.5$ and $\omega = \frac{\sqrt{5}-1}{2}$.

The Lyapunov exponents for the GPOY model are $\lambda_1 = 0$ and $\lambda_2 = -1.53$. The Kaplan-Yorke dimension is $D_{KY} = 1$. Despite the integer dimension, the GOPY is a fractal because it is nonperiodic. The model is nonperiodic because ω is an irrational number.

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One example of a chaotic attractor is the Lorenz system.
Edward Lorenz developed a model for atmospheric convection using the equations of the form:

$$\frac{dx}{dt} = \sigma(y - x) \tag{10}$$

$$\frac{dy}{dt} = x(\rho - z) - y \tag{11}$$

$$\frac{dy}{dt} = xy - \beta z \tag{12}$$

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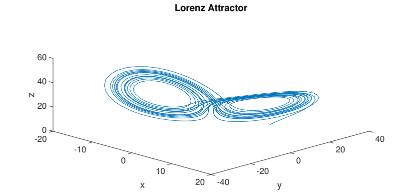
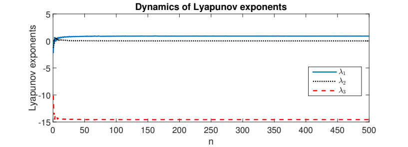


(a)



(b)

Figure 4: (a) Single orbit of the Lorenz attractor. (b) Lyapunov exponents for the Lorenz attractor

Figure 4a shows the Lorenz attractor with $\rho = 28, \sigma = 10$, and $\beta = \frac{8}{3}$ as parameters, and [10 10 10] as the initial vector. As the orbit converges and diverges, it gives the attractor its unique butterfly shape.

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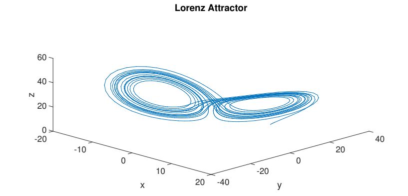

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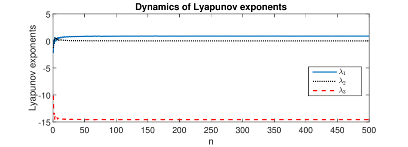

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- We can also show that the Lorenz attractor is chaotic because it has the characteristics of a chaotic system.

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Given that there are no random numbers or probabilities involved, we know that our system is deterministic. If the same initial state is used, the Lorenz attractor will always produce the same output.

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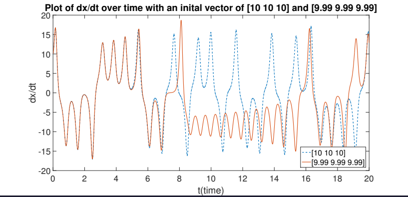

Figure 5: Plot of the rate of change in x over time for different initial conditions.

Figure 5 gives the rate of change in x over time for the initial vectors [10, 10, 10] and [9.99, 9.99, 9.99].

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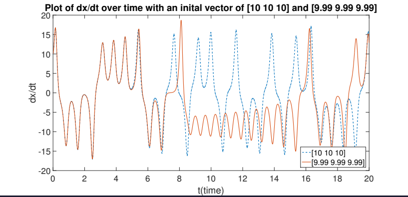

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- The two different trajectories also show that the system is nonlinear.

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- Thus, the Lorenz attractor is a strange chaotic attractor.

Conclusion

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Conclusion

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- Furthermore, in studying fractal geometry we know that orbits that are non-periodic exhibit fractal properties. We can also use the Kaplan-Yorke dimension to classify attractors with fractal dimensions.
- In studying distinguishing properties of attractors, we are able to see how fractal geometry and chaos theory is related.

Questions