HW12 (CSCI-C241)

Lillie Donato

23 April 2024

1. $\texttt{first} = a$
   $\texttt{rest} = [b, c, d, e, f]$

2. $\texttt{reverse}(L) = \begin{cases} \texttt{[]} & \text{if } L = \texttt{[]} \\ \texttt{append(reverse(rest), first)} & \text{if } L = \texttt{[first, *rest]} \end{cases}$

3. Claim: $\texttt{len(reverse}(L)) = \texttt{len}(L)$

   Proof.
   (Base Step, $L = \texttt{[]}$):

$$
\begin{align}
\texttt{len(reverse([]))} &= \texttt{len([])} \tag{1} \\
&= 0 \tag{2} \\
&= \texttt{len([])} \tag{3}
\end{align}
$$

   (Inductive Step, $L = \texttt{[first, *rest]}$):
   Assume for some list $\texttt{rest}$, that $\texttt{len(reverse(rest))} = \texttt{len(rest)}$

$$
\begin{align}
\texttt{len(reverse}(L)) &= \texttt{len(reverse([first, *rest]))} && \tag{1} \\
&= \texttt{len(append(reverse(rest), first))} && \text{(By the definition of } \texttt{reverse}) \tag{2} \\
&= \texttt{len(reverse(rest))} + \texttt{len([first])} && \tag{3} \\
&= \texttt{len(reverse(rest))} + 1 && \text{(Because the length of a single item list is 1)} \tag{4} \\
&= \texttt{len(rest)} + 1 && \text{(By the Induction Hypothesis)} \tag{5} \\
&= \texttt{len([first, *rest])} && \text{(By the definition of } \texttt{len}) \tag{6} \\
&= \texttt{len}(L) && \tag{7}
\end{align}
$$

$\square$

4. Claim: Every **X-number** n is divisble by 3 ($\frac{n}{3} \in \mathbb{Z}$)

   Proof.
   (Base Step, $n = 12$):

$$
\begin{align}
\frac{12}{3} &= 4 \tag{1} \\
4 &\in \mathbb{Z} \tag{2}
\end{align}
$$

   (Base Step, $n = 15$):

$$
\begin{align}
\frac{15}{3} &= 5 \tag{1} \\
5 &\in \mathbb{Z} \tag{2}
\end{align}
$$

   (Inductive Step, $n = x + y$, where x, y are both **X-numbers**):
   Assume for some **X-numbers** x, y, that they are divisble by 3

$$
\begin{align}
x &= 3a && \text{(By our Induction Hypothesis and where } a \in \mathbb{Z}) \tag{1} \\
y &= 3b && \text{(By our Induction Hypothesis and where } b \in \mathbb{Z}) \tag{2} \\
x + y &= 3a + 3b && \text{(Where } x, y \text{ are both } \textbf{X-numbers}) \tag{3} \\
&= 3(a + b) && (3(a+b) \text{ is divisible by 3 because 3 and } a + b \in \mathbb{Z}, \text{ so } 3(a+b) \in \mathbb{Z}) \tag{4}
\end{align}
$$

(Inductive Step, $n = x - y$, where x, y are both **X-numbers**):

$$
\begin{aligned}
x &= 3a && \text{(By our Induction Hypothesis and where } a \in \mathbb{Z}) && (1)\\
y &= 3b && \text{(By our Induction Hypothesis and where } b \in \mathbb{Z}) && (2)\\
x - y &= 3a - 3b && \text{(Where } x, y \text{ are both \textbf{X-numbers})} && (3)\\
&= 3(a - b) && (3(a - b) \text{ is divisible by 3 because 3 and } a - b \in \mathbb{Z}, \text{ so } 3(a - b) \in \mathbb{Z}) && (4)
\end{aligned}
$$

$\square$

5. Claim: For any $n \in \mathbb{Z}$ where $n \geq 1$, $L_n = F_{n-1} + F_{n+1}$

Proof.
(Base Step, $n = 1$):

$$
\begin{aligned}
L_1 &= 1 && \text{(By the definition of the Lucas Numbers)} && (1)\\
&= 1 && && (2)\\
&= 1 + 0 && && (3)\\
&= F_1 + F_0 && && (4)\\
&= F_{2-1} + F_{2-2} && && (5)\\
&= 0 + (F_{2-1} + F_{2-2}) && && (6)\\
&= F_0 + F_2 && && (7)\\
&= F_{1-1} + F_{1+1} && && (8)
\end{aligned}
$$

(Inductive Step):
Assume for some $k \in \mathbb{Z}$ where $k \geq 1$, $L_i = F_{i-1} + F_{i+1}$ for all $1 \leq i \leq k$

$$
\begin{aligned}
L_{k+1} &= L_k + L_{k-1} && && (1)\\
&= (F_{k-1} + F_{k+1}) + (F_{k-2} + F_k) && \text{(By the Inductive Hypothesis)} && (2)\\
&= (F_{k-1} + F_{k-2}) + (F_{k+1} + F_k) && && (3)\\
&= F_k + F_{k+2} && && (4)\\
&= F_{(k+1)-1} + F_{(k+1)+1} && && (5)
\end{aligned}
$$

$\square$

6. Claim: Every $n \in \mathbb{Z}$ where $n \geq 1$, n can be written as the sum of powers of 2

Proof.
(Base Step, $n = 1$):

$$
2^0 = 1 \tag{1}
$$

1 can be written in the sum of powers of 2, because any number raised to 0 is 1
(Inductive Step):
Assume for some $k \in \mathbb{Z}$ where $k \geq 1$, i can be written as the sum of power of 2, where $1 \leq i \leq k$

Case 1: k is odd $\hfill(1)$

By our IH, $\dfrac{k+1}{2}$ can be written as a unique sum of powers of 2, $\hfill(2)$

so $\dfrac{k+1}{2} = c_0 \cdot 2^0 + c_1 \cdot 2^1 + c_2 \cdot 2^2 + c_3 \cdot 2^3 + \cdots + c_n \cdot 2^n,$ $\hfill(3)$

where all of the values multiplying the powers of 2 are 0 or 1 $\hfill(4)$

Therefore, $\hfill(5)$

$$k + 1 = 2 \cdot \dfrac{k+1}{2} = c_0 \cdot 2^{0+1} + c_1 \cdot 2^{1+1} + c_2 \cdot 2^{2+1} + c_3 \cdot 2^{3+1} + \cdots + c_n \cdot 2^{n+1}$$ $\hfill(6)$

Case 2: k is even $\hfill(7)$

Since k is even and can be represented in sums of powers of 2 (according to our IH), $\hfill(8)$

then in the sum of powers of 2 for k, 2^0 is not a member $\hfill(9)$

Therefore $k + 1 = k + 2^0$ $\hfill(10)$

In either case of $k + 1$, it can be made by a unique sum of powers of 2, so in general it can be $\hfill(11)$

made by a sum of powers of 2 $\hfill(12)$

$\square$