Propositional Logic

English

Premise Tags (if x, when x, while x, where x, in the case that x)

Conclusion Tags (only if x, only when x, only in the case that x, x is a necessary condition)

Not implication (therefore, thus, hence, so)

Causality is not either (because, since, due to)

Other implication:

$P \to Q$

"if P, then Q", "Q if P", "P only if Q", "P is a sufficient condition for Q", "Q is a necessary condition for P"

$P \leftrightarrow Q$: "P is a necessary and sufficient condition for Q"

"P is a sufficient condition for Q": $P \to Q$

"P is a necessary condition for Q": $Q \to P$

$\neg, \wedge, \vee, \oplus, \to$

Contradictions (Always FALSE)

Tautology (Always TRUE)

Contingency: TRUE (at least once) and FALSE (at least once)

Existential Claims: Prove by giving an example that has some property (Truth Assignment)

Universal Claims: Prove by showing everything (in some category) has some property (Truth Table)

Consistency: A set of formulas is consistent if and only if, there is an assignment that satisfies all of the formulas in the set.

Provable with a truth assignment, Disprovable with a truth table that shows there's no assignment that satisfies all formulas

Logical Equivalency ($\equiv$):

The formulas are logically equivalent if and only if every assignment that satisfies one formula also satisfies the other, vice versa.

Provable with Matching Truth Tables, Disprovable with a truth assignment

Validity

Valid if and only if every assignment that satisfies the premise also satisfies the conclusion

Invalid if and only there exists an assignment that satisfies all the premises but not the conclusion

Equivalence Proofs Rules

Commutative Laws: $p \wedge q \equiv q \wedge p$, $p \vee q \equiv q \vee p$

Distributive Laws: $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$, $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

De Morgan's Laws: $\neg(p \wedge q) \equiv \neg p \vee \neg q$, $\neg(p \vee q) \equiv \neg p \wedge \neg q$

Associative Laws: $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$, $p \vee (q \vee r) \equiv p \vee (q \vee r)$

Material Implication: $p \to q \equiv \neg p \vee q$

Transitive Property: If $x = y$ and $y = z$, then $x = z$, If $p \equiv q$ and $q \equiv r$, then $p \equiv r$, If $p \vdash q$ and $q \vdash r$, then $p \vdash r$

Double Negation: $\neg\neg p \equiv p$

Idempotence Laws: $p \wedge p \equiv p$, $p \vee p \equiv p$

Absorption: $p \vee (p \wedge q) \equiv p$, $p \wedge (p \vee q) \equiv p$

Currying: $A \to (B \to C) \equiv (A \wedge B) \to C$

Contrapositive: $p \to q \equiv \neg q \to \neg p$

Bi-Conditional: $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$, $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

Tautology ($\top$): $A \vee \neg A \equiv \top$, $\top \wedge p \equiv p$, $\top \vee p \equiv \top$

Contradiction ($\bot$): $A \wedge \neg A \equiv \bot$, $\bot \wedge p \equiv \bot$, $\bot \vee p \equiv p$

Informal Proofs:

Proving an Existential Claim: Give an example (and show that the example works)

Proving a Universal Claim: "Choose" a generic example and prove that it works

"Choose a member" of A ("and try to prove it is a member of B", to prove $A \subseteq B$)

Rules:

Universal - Introduction ("Direct Proof")

To prove that every $\star$ has property P, we "choose" a generic $\star$ and then prove that this variable has property P

To prove "every $\star$ is", start by writing: "Choose a $\star$ x"

Universal - "Elimination" (Application)

If you know that "every $\star$ has property P" and you know that "x is a $\star$", then you can conclude x has property P

Existential - Introduction

To prove "there is a $\star$ with property P"

Give an example of a $\star$

Prove that said example has property P

Existential - Elimination

If you already know "there is a $\star$ with property P"

"So there is a $\star$ with property P. Call it x"

"Let a be a $\star$ with property P or Hence some $\star$ M exists that has property P"

Give a name to the $\star$ with property P and use that name to prove something else

Modus Tollens: $p \to q, \neg q \vdash \neg p$

Universal - Negation: To prove a universal claim is False, give a counter example

Existential - Negation: To prove an existential claim is False, Proof by Contradiction

Numbers:

Natural Numbers: $\mathbb{N} = \{0, 1, 2, 3, 4, ...\}$

Integers: $\mathbb{Z} = \{..., -2, -1, 0, 1, 2\} = \{n \mid n \in \mathbb{N} \vee -n \in \mathbb{N}\}$

Rational Numbers: Numbers that can be written as a ratio of integers

$\mathbb{Q} = \{\frac{p}{q} \mid p \in \mathbb{Z} \wedge q \in \mathbb{Z} \wedge q \neq 0\}$

A number x is rational if and only if, there exist integers p and q, such that $x = \frac{p}{q}$ and $q \neq 0$

Real Numbers: $\mathbb{R}$ = every # on the number line/with a decimal expansion (finite, infinite, pattern, no pattern)

Complex Numbers: ($\mathbb{C}$)

Sets:

Set Operations

Union $(A \cup B)$ $(\{x \mid x \in A \vee x \in B\})$

Intersection $(A \cap B)$ $(\{x \mid x \in A \wedge x \in B\})$

Complement $(\overline{A})$ $(\{x \mid x \notin A\})$

Relative Complement/Set Subtraction $(A \setminus B = A - B = \{x \mid x \in A \wedge x \notin B\})$

Subsets: A set A is a subset of a set B if and only if every member of A is also a member of B

We write $A \subseteq B$ to mean "A is a subset of B"

A is a proper (or strict) subset of B if and only if A is a subset of B and $A \neq B$

We write $A \subsetneq B$ to mean "A is a proper subset of B"

Powersets: The powerset of A is the set whose members are the subsets of A

We write $\mathcal{P}(A)$ to mean "the powerset of A"

Thm: For a finite set A, $\mid \mathcal{P}(A) \mid = 2^{|A|}$

Supersets: If A is a subset of B, then B is a superset of A

We write $B \supseteq A$ to mean "B is a superset of A"

Relations/Properties:

Cartesian Product: $A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$

Fact: The set of all ordered pairs where the first component is a member of A and the second component is a member of B

Fact: For any finite sets A and B, $\mid A \times B \mid = |A| \cdot |B|$

Relations: A set of ordered pairs where the first component is a member of A and the second component is a member of B is called a relation from A to B

The Cartesian Product $A \times B$ is the biggest possible relation from A to B

Every relation from A to B is a subset of $A \times B$

Notation — $(2, b) \in R$, $R(2, b)$ ("prefix" notation), $2Rb$ ("infix" notation)

Properties of Relations

A relation R on a set A is reflexive if and only if **for every** $x \in A$, $R(x, x)$ — (everything is related to itself)

A relation R on a set A is antireflexive if and only if **for every** $x \in A$, $\neg R(x, x)$ — (nothing is related to itself)

A relation R on a set A is symmetric if and only if **for all** $x, y \in A$, if $R(x, y)$, then $R(y, x)$

A relation R on a set A is antisymmetric if and only if **for all** $x, y \in A$, if $R(x, y)$ and $R(y, x)$, then $x = y$

if $R(x, y)$ and $x \neq y$, then $\neg R(y, x)$

A relation R on a set A is transitive if and only if **for all** $x, y, z \in A$, if $R(x, y)$ and $R(y, z)$, then $R(x, z)$

Functions — $f(x) = |x|$ (Function Notation):

Uniqueness: A relation R from a set A to a set B is a function if and only if **for every** $x \in A$ and $y, z \in B$, if $R(x, y)$ and $R(x, z)$, then $y = z$

An input maps to an input, or doesn't (nothing more than one) & An input will not map to more than one output

Existence: **For every** $x \in A$, there exists an $y \in B$ with $R(x, y)$ — Every input is mapped to at least one output

Not a function, if not Uniqueness

Partial Function, if Uniqueness and not Existence

Total Function, if both, "total function" = "function"

A function $f \colon A \to B$ is **one-to-one** (an injection) if and only if for every $x, y \in A$, if $f(x) = f(y)$, then $x = y$

Two different inputs cannot map to the same output

A function $f \colon A \to B$ is **onto** (a surjection) if and only if for every $y \in B$ there is an $x \in A$ with $f(x) = y$

Every member of the codomain must be mapped to by at least one member of the domain

A function that is a **bijection** is both **one-to-one** and **onto**

Cardinality (Functions):

Bijection $(A \to B)$: $|A| = |B|$

One-to-One $(A \to B)$: $|A| \leq |B|$

Onto $(A \to B)$: $|A| \geq |B|$

First Order Logic (FOL): $\forall$ (For All) & $\exists$ (There exists)

Strong Induction: Assume for some integer $k \geq 1$, i ... for all $1 \leq i \leq k$

Combinatorics

Repeats - n^r where $|\texttt{result}| = r$ and $|\texttt{set}| = n$

Permutations (Ordered) - $P(n, r) = nPr = \frac{n!}{(n-r)!}$ where $|\texttt{result}| = r$ and $|\texttt{set}| = n$

Combinations - $\frac{{}_nP_r}{r!} = \frac{\frac{n!}{(n-r)!}}{r!} = \frac{n!}{(n-r)! \cdot r!}$ where $|\texttt{result}| = r$ and $|\texttt{set}| = n$

Growth Rates: factorial $(n!)$ > exponential (c^n) > polynomial (n^c) > linear (n) > roots $(\sqrt{n})$ > logarithmic $(log_b n)$ > constant (c)

Graph Theory

Two vertices are **neighbours** or **adjacent** if there is an edge between them (in an undirected graph)

If there is an edge in a directed graph $V \to W$, W is the **child** of V and V is the **parent** of W

The **degree** of a vertex is the number of neighbours

A **path** from a vertices $V \to W$ is a sequence of edges $V \to V_1 \to ... \to W$ starting at V and ending at W with no repeated edges

A **trail** is a path that allows repeated vertices

Vertices (V, W) are **connected** (there is a path from $V \to W$) and a graph is **connected** (any two vertices are **connected**)

A **walk** is a path that can have repeated edges and/or vertices

A **cycle** is a trail (not a walk) from a vertex to itself where only the first and last vertices are the same and there is at least one edge