HW10 Corrections (CSCI-C241)

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- Question Four
  - Claim: For every positive real number a where $a \geq e$, there exists $m \in \mathbb{N}$ such that for all $n \geq m$, $n! > a^n$

    Proof.

$$\begin{aligned}
&\text{Choose } a \in \mathbb{R} \text{ such that } a \geq e && (1)\\
&\text{Suppose } m \in \mathbb{N} \text{ such that } n \geq m && (2)\\
&\qquad \text{Since half of the numbers that are being multiplied in } n! \text{ are greater than } \frac{n}{2}, && (3)\\
&\qquad \text{we know } \frac{n}{2} \text{ numbers of } n! \text{ are greater than } \frac{n}{2} &&\\
&\qquad \text{Since } \frac{n}{2} \text{ numbers are greater than } \frac{n}{2}, \text{ we know } n! > \left(\frac{n}{2}\right)^{\frac{n}{2}} && (4)\\
&\qquad \text{Since } n! > \frac{n}{2}, \text{ we know } n! > \left(\frac{n}{2}\right)^{\frac{n}{2}} && (5)\\
&\qquad \text{Since } n \geq m, \text{ we know } \left(\frac{n}{2}\right)^{\frac{n}{2}} \geq \left(\frac{m}{2}\right)^{\frac{n}{2}} && (6)\\
&\qquad \text{Since } \left(\frac{n}{2}\right)^{\frac{n}{2}} \geq \left(\frac{m}{2}\right)^{\frac{n}{2}}, \text{ we know } n! > \left(\frac{m}{2}\right)^{\frac{n}{2}} && (7)\\
&\qquad \text{Since } \left(\frac{m}{2}\right)^{\frac{n}{2}} = \left(\sqrt{\frac{m}{2}}\right)^{n}, \text{ we know } n! > \left(\sqrt{\frac{m}{2}}\right)^{n} && (8)\\
&\qquad \text{Let } m = 2a^2 && (9)\\
&\qquad \text{Since } m = 2a^2, \text{ we know } \frac{m}{2} = a^2 && (10)\\
&\qquad \text{Since } \frac{m}{2} = a^2, \text{ we know } \sqrt{\frac{m}{2}} = a && (11)\\
&\qquad \text{Since } \sqrt{\frac{m}{2}} = a, \text{ we know } \left(\sqrt{\frac{m}{2}}\right)^{n} = a^n && (12)\\
&\qquad \text{Since } n! > \left(\sqrt{\frac{m}{2}}\right)^{n} \text{ and } \left(\sqrt{\frac{m}{2}}\right)^{n} = a^n, \text{ we know } n! > a^n && (13)
\end{aligned}$$

    □

- Question Seven
  - Claim: For any non-empty set A of size n and any integer r with $n \geq r \geq 1$, there are $\frac{n!}{(n-r)!}$ permutations of length r using values taken from A

    Proof. (induction on n)
    (Base Step, $r = 1$):

$$\begin{aligned}
\frac{n!}{(n-1)!} &= \frac{n \cdot (n-1)!}{(n-1)!} && (1)\\
&= \frac{n}{1} && (2)\\
&= n && (3)
\end{aligned}$$

For a permutation of length 1 for some set of length n, since said permutation would have a single item, this item could be any item of the set of length n.
(Inductive Step):
Assume there are $\frac{n!}{(n-k)!}$ permutations for some length $k \geq 1$ using values taken from A, a set of length n

$$\begin{aligned}
\frac{n!}{(n-k)!} \cdot (n-k) &= \frac{n!}{\frac{(n-k)\cdot(n-k-1)!}{(n-k)}} && (1)\\
&= \frac{n!}{(n-k-1)!} && (2)\\
&= \frac{n!}{(n-(k+1))!} && (3)
\end{aligned}$$

Since there are $\frac{n!}{(n-k)!}$ permutations for some length k (by the Induction Hypothesis), there are $n-k$ possibilities to create a new permutation of length $k+1$ from every permutation of length k, therefore there are $\frac{n!}{(n-k)!} \cdot (n-k)$ permutations for length $k+1$ $\square$