CSCI-C 241 HW #4
Instructors: Wennstrom & LaRacuente Due Tuesday, February 6th, 2023

1. Here are some attempts at proofs inspired by answers that students have given in previous semesters. For each proof, I would like you to explain which steps (if any) are logically incorrect and why. Note that I am not asking you to to explain how they "should" have written the proof.

   I've included some proofs where every step is completely correct, even though the proofs include weird strategies, false starts, or other oddities. These proofs may not be *good* proofs, and you might know of a *better* strategy, but that doesn't make the proofs *wrong*. As long as every step correctly applies a valid inference rule to the available formulas, then you should state that the proof is correct.

   I've included line numbers just to make it easier to talk about the different steps.

   To give you an idea of what I'm looking for, here's an example of a proof and the kind of answer I want from you:

   Claim. $(P \land Q) \to R, Q \land P \vdash R$

   Proof.
   1. Assume $(P \land Q) \to R$ and $Q \land P$.
   2. Since $(P \land Q) \to R$, we can conclude $P \to R$. ($\land$-Elim.)
   3. From $Q \land P$, we know P. ($\land$-Elim.)
   4. Because we have $P \to R$ and P, we know R. (Appl.) □

   Answer: Line 2 is wrong because $\land$ isn't the main connective of $(P \land Q) \to R$ and so you can't use $\land$-Elim. on it.

   Finally, note that there are no mistakes in format, phrasing, citations, or other aspects of presentation, so don't worry about that sort of thing. Just pay attention to what formulas and/or subproofs are being used, what rule is being used, what formula is being concluded, and whether that rule can be used on those formulas/subproofs to deduce that formula.

   (a)

   Claim. $(P \lor Q) \to \neg R, R \vdash \neg P$

   Proof.
   1. Assume $(P \lor Q) \to \neg R$ and R.
   2. Suppose towards a contradiction that P.
   3. Knowing P tells us $P \lor Q$ (Weak.)
   4. Because $P \lor Q$ is true, we can apply $(P \lor Q) \to \neg R$ to get $\neg R$. (Appl.)
   5. We assumed P and proved $\neg R$, which contradicts our earlier assumption R, and therefore $\neg P$. (contrad.) □

   (b)

   Claim. $Y \land Z, X \to \neg Y \vdash \neg(X \land Z)$

Proof.

1. Assume $Y \wedge Z$ and $X \to \neg Y$.
2. Suppose towards a contradiction that X is true.
3. Because X and $X \to \neg Y$ hold, so does $\neg Y$. (Appl.)
4. Since $Y \wedge Z$, we have Y and Z ($\wedge$-Elim.)
5. From X and Z, we can conclude $X \wedge Z$ ($\wedge$-Intro.)
6. We have $X \wedge Z$, and we proved both $\neg Y$ and Y, which is not possible. So therefore $\neg(X \wedge Z)$. (contrad.) ☐

(c)

Claim. $\neg G \wedge \neg H \vdash G \to H$

Proof.

1. Assume $\neg G \wedge \neg H$.
2. Since $\neg G \wedge \neg H$, we also know both $\neg G$ and $\neg H$. ($\wedge$-Elim.)
3. Assume G.
4. Suppose towards a contradiction that $\neg H$ is also true.
5. We already know that both G and $\neg G$ are true, which is impossible.
6. I assumed $\neg H$ and proved an impossibility, so therefore $\neg\neg H$ must hold. (Contrad.)
7. Because $\neg\neg H$, we can conclude H. (Dbl. Neg.)
8. Assuming G, I proved H, and therefore $G \to H$. (Dir. Pf) ☐

(d)

Claim. $Y \wedge Z, X \to \neg Y \vdash \neg(X \wedge Z)$

Proof.

1. Assume $Y \wedge Z$ and $X \to \neg Y$.
2. From $Y \wedge Z$, we can derive Y and Z ($\wedge$-Elim.)
3. Suppose towards a contradiction that X is true.
4. From X and $X \to \neg Y$, we get $\neg Y$. (Appl.)
5. We assumed X and proved $\neg Y$. But that contradicts our earlier statement Y, and hence $\neg X$. (contrad.)
6. Suppose towards a contradiction that $X \wedge Z$ is true.
7. Since $X \wedge Z$, we get X. ($\wedge$-Elim.)
8. Assuming $X \wedge Z$, we got X and $\neg X$, which can't both be true. Therefore we have $\neg(X \wedge Z)$. (contrad.) ☐

2. Some of the following claims are true and some are false. If the claim is true, prove it by giving a semi-formal Natural Deduction proof. If the claim is false, prove this by giving a truth assignment.

   Hint: Start by trying to write a proof. If you get stuck, then switch to trying to find a proof assignment to disprove the claim.

Remember that when writing semi-formal proofs for this class, you must follow the guidelines we discussed in the lectures. You can also find those guidelines in the lecture notes, so please use those as a reference. If your proofs do not follow those guidelines **they will receive 0 points**, and you will be asked to resubmit the assignment.

(a) $P \wedge Q \vdash \neg(P \to \neg Q)$

(b) $\vdash \neg\big((P \wedge Q) \wedge (P \to \neg Q)\big)$

(c) $A \to \neg B, \neg\neg B \vdash \neg(A \to B)$

(d) $(W \wedge X) \to \neg Y, X \vdash \neg(W \wedge Y)$

(e) $(W \wedge X) \to \neg Y \vdash X \to \neg(W \wedge Y)$

(f) $U \to V, \neg V \vdash \neg(U \wedge W)$

(g) $U \to V, \neg V \vdash \neg U \wedge \neg W$

(h) $U \to V, W \to V, \neg V \vdash \neg U \wedge \neg W$

3. Answer the following questions, and give a brief explanation (one or two sentences should be enough) to justify your answer. (You do *not* need to write a proof.)

   (a) True or False: Every assignment that satisfies both A and $\neg A$ also satisfies $A \wedge \neg A$.

   (b) Every assignment that satisfies both A and $\neg A$ also satisfies $B \leftrightarrow (X \oplus \neg Q)$.

   (c) Is the following argument valid?

   $$\begin{array}{l} P \leftrightarrow Q \\ \neg Q \wedge P \\ \hline Q \wedge \neg P \end{array}$$

   (d) Is it possible to have an valid argument where the conclusion is a contradiction?

   (e) Is it possible for a contradiction to be equivalent to another formula?

4. **Bonus**: Give a semi-formal natural deduction proof of the following claim: $\vdash P \vee \neg P$

   Hint: This is a bonus problem because the limitations of Natural Deduction make it a lot harder to prove than you'd think it would be. You might want to start by using proof by contradiction to try and prove $\neg\neg(P \vee \neg P)$.

5. For each of the following pairs of formulas, determine whether the two formulas are logically equivalent or not. If they are not equivalent, say that they are not equivalent and give a truth assignment that proves this. If they are equivalent, say that they are equivalent and write an equivalence proof to prove this. (Please follow the guidelines for equivalence proofs as laid out in the lecture notes.) You should not need to create an entire table for any of these problems.

   (a) $(A \to B) \to C$ and $(\neg A \to C) \wedge (B \to C)$

   (b) $(A \wedge B) \to C$ and $(A \to C) \wedge (B \to C)$

   (c) $\neg\big((W \wedge \neg X) \to (\neg Y \vee Z)\big)$ and $(\neg W \vee X) \wedge (Y \wedge \neg Z)$

(d) $\neg\big((W \wedge \neg X) \rightarrow (\neg Y \vee Z)\big)$ and $(Y \wedge \neg Z) \wedge (W \wedge \neg X)$

(e) $P \wedge (\neg Q \rightarrow R)$ and $(P \rightarrow \neg R) \rightarrow (P \wedge Q)$

(f) **Bonus:** $(M \rightarrow N) \wedge (\neg M \rightarrow N)$ and N.

To solve this problem, you'll have to use some equivalence rules and symbols that we did not discuss in the lectures. You can read about them in the lecture notes.