HW11 (CSCI-C241)

Lillie Donato

16 April 2024

1. Claim: For any graph G, if every pair of vertices in G has one and only one path between them, then G is a tree.

   Proof.

   Choose a graph G and assume every pair of vertices in G has a single path between them (1)
   Since every pair of vertices in G has a single path between them, then we can conclude (2)
   that every pair is connected
   Since every pair of vertices in G is connected, we can conclude G is connected (3)
   Suppose towards a contradiction that G is not a tree (4)
   Since G is not a tree and G is connected, we can conclude G has a cycle (5)
   Since G has a cycle, we can conclude that there is some path (6)
   $P = (V, a_1, a_2, \ldots, a_{n-1}, a_n, W, b_1, b_2, \ldots, b_{n-1}, b_n, V)$ for some pair of vertices (V, W),
   where all vertices in the path $(a_1, a_2, \ldots, a_{n-1}, a_n)$ are different from the vertices
   in the path $(b_1, b_2, \ldots, b_{n-1}, b_n)$ (7)
   Therefore, there are two distinct paths from V to W which contradicts our (8)
   assumption that every pair of vertices has one and only one path between them (9)
   Under the assumption that G is not a tree, we proved an impossibility of there being (10)
   2 distinct paths from one vertex to another, so G must be a tree (11)

   □

2. Claim: For any tree t, if you remove one edge from the tree, the resulting graph will not be connected.
   Define: $\texttt{nd}(t) =$ the number of nodes in t

   Proof. (induction on $\texttt{nd}(t)$)
   (Base Step, $\texttt{nd}(t) = 2$):

   Since $\texttt{nd}(t) = 2$, there are two nodes in some tree t, meaning there is some edge E that connects (1)
   2 nodes V and W (2)
   If E is removed from t, then V and W are no longer connected, meaning t is no longer connected (3)

   (Inductive Step):
   Assume for some tree with k nodes, if one edge is removed, then the resulting graph will no longer be connected

   Let $t =$ some tree with k nodes (1)
   Let $t_1 = t$ with some node n added to it, connected by some edge e, meaning it has $n + 1$ nodes (2)
   If e is removed, n is no longer connected to any other node in t_1, meaning t_1 is not connected (3)
   If any other edge in t_1 is removed, it must be an edge in t, meaning by our Induction Hypothesis (4)
   t_1 will no longer be connected (5)

   □

3. Question Three

(a) $\texttt{ed}(T) = \begin{cases} 0 & \text{if } T = \texttt{[]} \\ \texttt{ed}(T_1) + \texttt{ed}(T_2) + 2 & \text{if } T = \texttt{[}T_1, T_2\texttt{]} \text{ for FBT's } T_1 \text{ and } T_2 \end{cases}$

(b) Claim: For full every binary tree t, $\texttt{nd}(t) = \texttt{ed}(t) + 1$

Proof.
(Base Step: $t = \texttt{[]}$)

$$\begin{aligned}
\texttt{nd}(t) &= 1 && (1)\\
&= 0 + 1 \quad \text{(Because a tree with a single node has no children and no edges)} && (2)\\
&= \texttt{ed}(t) + 1 && (3)\\
& && (4)
\end{aligned}$$

(Inductive Step):
Assume for some FBT's T_1 and T_2, $\texttt{nd}(T_1) = \texttt{ed}(T_1) + 1$ and $\texttt{nd}(T_2) = \texttt{ed}(T_2) + 1$

Let $T = \texttt{[}T_1, T_2\texttt{]}$ (1)

By the recursive definition of $\texttt{nd}$ and $\texttt{ed}$, we know $\texttt{nd}(T) = \texttt{nd}(T_1) + \texttt{nd}(T_2) + 1$ and (2)

$\texttt{ed}(T) = \texttt{ed}(T_1) + \texttt{ed}(T_2) + 2$ (3)

$\texttt{nd}(T) = \texttt{nd}(T_1) + \texttt{nd}(T_2) + 1 = \texttt{ed}(T_1) + 1 + \texttt{ed}(T_2) + 1 + 1$ (By the IH) (4)

$\texttt{ed}(T_1) + 1 + \texttt{ed}(T_2) + 1 + 1 = \texttt{ed}(T_1) + \texttt{ed}(T_2) + 2 + 1 = \texttt{ed}(T) + 1$ (5)

□

(c) $log_2 n$

(d) 2^d

(e) $log_c n$

(f) c^d

(g) Claim: Prove c^d
Define: $\texttt{dp}(t) =$ the depth of t

Proof. (induction on $\texttt{dp}(t)$)
(Base Step, $\texttt{dp}(t) = 0$)

$c^0 = 1$ (If the depth of a tree is 0, then it must have a single node, its only leaf) (1)

(Inductive Step):
Assume for some tree t with at most c children and a depth k, the largest possible number of leaves are c^k

Since t has at most c^k leaves, a tree with a depth of $k+1$ and at most c children, (1)

would have c children for every leaf, a total of $c^k \cdot c$ leaves (2)

$c^k \cdot c = c^{k+1}$ (3)

□