CSCI-C 241 HW #12
Instructors: Wennstrom & LaRacuente Due Tuesday, April 23rd, 2024

Note that **this assignment is not eligible for resubmission**. You may turn it in up until Friday, 4/26 (for the usual late penalty), but make sure you've done it correctly the first time.

Recall the following definitions from the lectures and labs:

A **linked list** (of values of type X) is one of:

- [ ] (the empty list)
- $[\mathsf{first}, *\mathsf{rest}]$ where first is a value of type X and rest is a list of values of type X (the first item in the list is first and the remaining items in the list are in rest)

$$\mathsf{len}(L) = \begin{cases} 0 & \text{if } L = [\,] \\ \mathsf{len}(\mathsf{rest}) + 1 & \text{if } L = [\mathsf{first}, *\mathsf{rest}] \end{cases}$$

$$\mathsf{append}(L, x) = \begin{cases} \big[x, *[\,]\big] & \text{if } L = [\,] \\ \big[\mathsf{first}, *\mathsf{append}(\mathsf{rest}, x)\big] & \text{if } L = [\mathsf{first}, *\mathsf{rest}] \end{cases}$$

Recall the result we proved in the lab:

- For any list L and any value x, $\mathsf{len}\big(\mathsf{append}(L, x)\big) = \mathsf{len}(L) + 1$.

1. Suppose $L = [a, b, c, d, e, f]$. If we define first and rest so that $L = [\mathsf{first}, *\mathsf{rest}]$, what value does first have? What value does rest have?

2. Give a recursive definition of the $\mathsf{reverse}$ function on lists that takes a list and returns the same list in the reverse order. So for example $\mathsf{reverse}([1, 2, 3, 4]) = [4, 3, 2, 1]$.

   Hint: Use the append function from above.

3. Use induction to prove that for every list L, $\mathsf{len}\big(\mathsf{reverse}(L)\big) = \mathsf{len}(L)$.

   Hint: Feel free to use the claim that you proved about append and len in the labs (and that I included at the top of this assignment).

4. **(Bonus)**: Consider the following recursive data definition:

   Definition. An **X-number** is one of:

   - 12, or
   - 15, or
   - $x + y$, where x and y are X-numbers
   - $x - y$, where x and y are X-numbers

   Use structural induction on this definition to prove that every X-number is divisible by 3.

   Hints:

- How many base cases are there in this data definition? How does that impact the number of base steps in the induction proof?
- How many recursive cases are there in this data definition? How does that impact the number of induction steps in the induction proof?
- How many already-existing X-numbers are referred to in each of the recursive cases? How does that impact the number of induction hypotheses you need to assume at the start of each induction step?

5. Recall the definition of the Fibonacci-Hemachandra numbers we used in class:

$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{if } n > 1 \end{cases}$$

   The **Lucas numbers** are a similar sequence of integers, defined as follows:

$$L_n = \begin{cases} 2 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ L_{n-1} + L_{n-2} & \text{if } n > 1 \end{cases}$$

   Using strong induction, prove that for any positive integer n, $L_n = F_{n-1} + F_{n+1}$.

6. A **power of 2** is a number that can be written in the form 2^n for some $n \in \mathbb{N}$. The fact that we can write any number in binary is dependent upon the fact that any integer greater than or equal to 1 can be written as a sum of powers of 2. So for example, $42 = 2^5 + 2^3 + 2^1$ and $65537 = 2^{16} + 2^0$.

   Using strong induction, prove that every integer $n \geq 1$ can be written as the sum of powers of 2.