HW6 Corrections (CSCI-C241)

Lillie Donato

17 March 2024

- Question One
  - 1d
    Claim: $X \subseteq \overline{Y \cap Z}$

    Proof.

    $$\begin{aligned}
    &\text{Choose the sets } X, Y, Z \text{ and } x \in X && (1)\\
    &\text{Assume } (X \cap Y) \subseteq \overline{Z} && (2)\\
    &\quad\text{Suppose towards a contradiction } x \notin \overline{Y \cap Z} && (3)\\
    &\quad\quad\text{Since } x \notin \overline{Y \cap Z}\text{, we know } x \in Y \cap Z && (4)\\
    &\quad\quad\text{Since } x \in Y \cap Z\text{, we know } x \in Y \text{ and } x \in Z && (5)\\
    &\quad\quad\text{Since } x \in X \text{ and } x \in Y\text{, we know } x \in X \cap Y && (6)\\
    &\quad\quad\text{Since } x \in X \cap Y\text{, we know } x \in \overline{Z} && (7)\\
    &\quad\quad\text{Since } x \in \overline{Z}\text{, we know } x \notin Z && (8)\\
    &\quad\text{Under the assumption of } x \notin \overline{Y \cap Z}\text{, we proved an impossibility of } x \in Z \text{ and } x \notin Z\text{,} && (9)\\
    &\quad\text{therefore } x \in \overline{Y \cap Z} && \\
    &\text{Since } x \in X \text{ and } x \in \overline{Y \cap Z}\text{, we know } X \subseteq \overline{Y \cap Z} && (10)\\
    & && (11)
    \end{aligned}$$

    □

  - 1e
    Claim: $H \cup (J \cap K) \subseteq (H \cup K) \setminus M$

    Proof.

    $$\begin{aligned}
    &\text{Choose the sets } H, J, K, L, M \text{ and } x \in H \cup (J \cap K) && (1)\\
    &\text{Assume } J \subseteq L \setminus M \text{ and } H \subseteq \overline{M} && (2)\\
    &\quad\text{Case 1: } x \in H && (3)\\
    &\quad\quad\text{Since } x \in H \text{ and } H \subseteq \overline{M}\text{, we know } x \in \overline{M} && (4)\\
    &\quad\quad\text{Since } x \in \overline{M}\text{, we know } x \notin M && (5)\\
    &\quad\quad\text{Since } x \in H\text{, we know } x \in H \cup K && (6)\\
    &\quad\quad\text{Since } x \in H \cup K \text{ and } x \notin M\text{, we know } x \in (H \cup K) \setminus M && (7)\\
    &\quad\text{Case 2: } x \in J \cap K && (8)\\
    &\quad\quad\text{Since } x \in J \cap K\text{, we know } x \in J \text{ and } x \in K && (9)\\
    &\quad\quad\text{Since } x \in J \text{ and } J \subseteq L \setminus M\text{, we know } x \in L \setminus M && (10)\\
    &\quad\quad\text{Since } x \in L \setminus M\text{, we know } x \in L \text{ and } x \notin M && (11)\\
    &\quad\quad\text{Since } x \in K\text{, we know } x \in H \cup K && (12)\\
    &\quad\quad\text{Since } x \in H \cup K \text{ and } x \notin M\text{, we know } x \in (H \cup K) \setminus M && (13)\\
    &\text{In either case of } H \cup (J \cap K)\text{, we proved that } x \in (H \cup K) \setminus M && (14)\\
    &\text{Under the assumption of } J \subseteq L \setminus M \text{ and } H \subseteq \overline{M}\text{,} && (15)\\
    &\quad\text{we proved that any member of } H \cup (J \cap K) \text{ is a member of } (H \cup K) \setminus M\text{,} && \\
    &\quad\text{therefore } H \cup (J \cap K) \subseteq (H \cup K) \setminus M &&
    \end{aligned}$$

$\square$

- Question Three
  - 3c
    $T = \{x + y\sqrt{2} \mid x \in \mathbb{Q} \land y \in \mathbb{Q}\}$
    $S = \{st \mid s \in T \land t \in T\}$
    Claim: S = T

    Proof.

    Choose $s \in T$ and $t \in T$ (1)

    Since $s \in T$, we know $s = x_1 + y_1\sqrt{2}$, where $x_1, y_1 \in \mathbb{Q}$ (2)

    Since $t \in T$, we know $t = x_2 + y_2\sqrt{2}$, where $x_2, y_2 \in \mathbb{Q}$ (3)

    Since $st = (x_1 + y_1\sqrt{2})(x_2 + y_2\sqrt{2})$, we know $st = x_1x_2 + x_1y_2\sqrt{2} + x_2y_1\sqrt{2} + 2y_1y_2$ (4)

    Since st, we know $st = (x_1x_2 + 2y_1y_2) + \sqrt{2}(x_1y_2 + x_2y_1)$ (5)

    Let $a = x_1x_2 + 2y_1y_2$ and $b = x_1y_2 + x_2y_1$ (6)

    Since $x_1, x_2, y_1, y_2 \in \mathbb{Q}$, we know $a, b \in \mathbb{Q}$ (7)

    Since $st = a + b\sqrt{2}$ and $a, b \in \mathbb{Q}$, we know $S \subseteq T$ (8)

    Choose $t \in T$ where $t = x + y\sqrt{2}$, and $x, y \in \mathbb{Q}$ (9)

    Since $0, 1 \in \mathbb{Q}$, $0\sqrt{2} = 0$ and $1 + 0 = 1$, we know $1 \in T$ (10)

    Since $t, 1 \in T$ and $t = 1 \cdot t$, we know $t \in S$ (11)

    Since $t \in S$, we know $T \subseteq S$ (12)

    Since $S \subseteq T$ and $T \subseteq S$, we know $S = T$ (13)

    $\square$

  - 3d
    Claim: The sum of an irrational number and a rational number is irrational.

    Proof.

    Choose an irrational number x and a rational number y (1)

    Since y is rational, we know $y = \frac{p_1}{q_1}$, where $p_1, q_1 \in \mathbb{Z}$ and $q_1 \neq 0$ (2)

    Let $z = x + y$ (3)

    Assume towards a contradiction that z is rational (4)

    Since z is rational, we know $z = \frac{p_2}{q_2}$, where $p_2, q_2 \in \mathbb{Z}$ and $q_2 \neq 0$ (5)

    We can rewrite $x + y = z$ as $x + \frac{p_1}{q_1} = \frac{p_2}{q_2}$ (6)

    Since $x + \frac{p_1}{q_1} = \frac{p_2}{q_2}$, we know $x = \frac{p_2}{q_2} - \frac{p_1}{q_1}$ (7)

    Since $x = \frac{p_2}{q_2} - \frac{p_1}{q_1}$, we know $x = \frac{p_2q_1 - p_1q_2}{q_1q_2}$ (8)

    Since $q_1 \neq 0$ and $q_2 \neq 0$, we know $q_1q_2 \neq 0$ (9)

    Since $x = \frac{p_1q_2 - p_2q_1}{q_1q_2}$, we know $x = \frac{p}{q}$ where $p, q \in \mathbb{Z}$ and $q \neq 0$, meaning x is rational (10)

    This is a contradiction to our earlier claim, so z must be irrational, meaning the sum of a irrational number and a rational number is irrational (11)

    $\square$