CSCI-C 241 HW #8
Instructors: Wennstrom & LaRacuente Due Tuesday, March 19th, 2024

1. For each relation, decide if the relation is a function, a partial function but not a function, or not even a partial function. Justify your answer appropriately.
   (a) the relation $R_1 = \{(1,2),(1,3),(2,2),(3,2)\}$ on $A = \{1,2,3\}$
   (b) the relation $R_2 = \{(1,2),(2,2),(3,2)\}$ on $A = \{1,2,3\}$
   (c) the relation $R_3 = \{(a,b),(b,d),(c,a)\}$ on $B = \{a,b,c,d\}$
   (d) the relation $P_2 = \{(x,y) \mid x \cdot y = 120\}$ on $\mathbb{R}$
   (e) the relation $P_3 = \{(x,y) \mid x \cdot y = 120\}$ on $\mathbb{R}^*$, where $\mathbb{R}^* = \{x \mid x \in \mathbb{R} \wedge x \neq 0\}$ is the set of all non-zero real numbers.

2. Answer each question. Justify your answer with a brief explanation, including an example where appropriate.
   (a) Consider the function $f_1 = \{(a,b),(b,d),(c,c),(d,b)\}$ on $B = \{a,b,c,d\}$. Is f_1 one-to-one?
   (b) Is f_1 onto?
   (c) Consider the function $f_2 = \{(a,b),(b,a),(c,c),(d,d)\}$ on $B = \{a,b,c,d\}$. Is f_2 one-to-one?
   (d) Is f_2 onto?
   (e) Consider the function $s_1 : \mathbb{R} \to \mathbb{R}$ defined by $s_1(x) = x^2 - 10$. Is s_1 one-to-one?
   (f) Consider the function $s_2 : [0,\infty) \to [0,\infty)$ defined by $s_2(x) = x^2 + 10$, where $[0,\infty)$ is the set of all non-negative real numbers. Is s_1 one-to-one?
   (g) Is s_2 onto?
   (h) Consider the function $c_1 : \mathbb{R} \to \mathbb{R}$ defined by $c_1(x) = x^3 - 10$. Is c_1 one-to-one?
   (i) Is c_1 onto?
   (j) Consider the function $c_2 : \mathbb{Z} \to \mathbb{Z}$ defined by $c_2(x) = x^3 - 10$. Is c_2 onto?
   (k) Let Str be the set of all strings.
   Consider $d : \mathsf{Str} \to \mathsf{Str}$ defined by $d(s) = s$ with dashes inserted between all characters. So for example $d($`"foobar"`$) =$ `"f-o-o-b-a-r"`, $d($`"12"`$) =$ `"1-2"`, $d($`"X"`$) =$ `"X"` and $d(\varepsilon) = \varepsilon$.
   Is d a function or only a partial function?
   (l) Is d one-to-one?
   (m) Is d onto?
   (n) Consider $f : \mathsf{Str} \to \mathsf{Str}$ defined by $f(s) =$ the first character in s.
   Is f a function or only a partial function?
   (o) Is f one-to-one?

3. Let $A = \{1,2,3\}$ and $B = \{a,b,c,d\}$. Give an example of the following. If you think no such example exists, you must explain why not. You may not use any examples from

elsewhere in this assignment. For parts (a)-(c), you should use set-list notation or a directed graph. For the remaining parts, use function notation.

Hint: Only two of these are impossible.

(a) A function from A to B that is not one-to-one

(b) A function from A to B that is onto

(c) A function from B to A that is one-to-one

(d) A function from $\mathbb{Z}$ to $\mathbb{N}$ that is not onto.

(e) A function on $\mathbb{Z}$ that is one-to-one, but not onto

(f) A function on $\mathbb{Z}$ that is onto, but not one-to-one

(g) A function on $\mathbb{Z}$ that is one-to-one and onto, but is not the identity function (so you can't just write $f(x) = x$).

Warning: The domain and codomain are the set of *integers*, not the set of *real numbers*. So getting an onto function might not be as easy as you think it is.

(h) A function from Str to Str that is one-to-one, but is not onto. (Str is the set of all text strings as usual.)

(i) A function from Str to Str that is onto, but not one-to-one.

(j) A function from Str to $\mathbb{N}$ that is onto.

(k) What does your answer to the previous problem tell you about the cardinalities of Str and $\mathbb{N}$?

(l) **Bonus**: A function from Str to $\mathbb{N}$ that is one-to-one. **Hint**: This one is possible!

(m) There do exist functions from Str to $\mathbb{N}$ that are both one-to-one and onto. What does this tell you about the cardinality of Str?

For the following problems, you should either be writing a proof (following course guidelines) or giving a counterexample and a short explanation of why your counterexample works.

4. (a) Define $k : \mathbb{R} \to \mathbb{R}$ by $k(x) = (x-3)^2$. Prove that k is not one-to-one.

   (b) Define $k_2 : (3, \infty) \to \mathbb{R}$ by $k_2(x) = (x-3)^2$. Recall that $(3, \infty) = \{x \mid x \in \mathbb{R} \wedge x > 3\}$ is the set of all real numbers that are strictly greater than 3.

   Prove that k_2 is one-to-one.

5. Define a function h on the set of strings by $h(s) =$ (s with all dashes removed). So $h($`"A-B-C"`$) =$ `"ABC"` and $h($`"foobar"`$) =$ `"foobar"`.

   (a) Prove that h is not one-to-one.

   (b) Prove that h is not onto.

6. Define the function a from the set of strings to $\mathbb{N}$ by:

$$a(s) = (\text{the sum of all the numeric digits in } s)$$

So for example, $a(\texttt{"10203"}) = 1+0+2+0+3 = 6$, $a(\texttt{"10-plus-62"}) = 1+0+6+2 = 9$, and $a(\texttt{"foobar"}) = 0$.

Warning: a works on *digits* not *numbers*, so $a(\texttt{"12"})$ is 3, not 12.

(a) Prove that a is not one-to-one.

(b) Prove that a is onto.

7. Suppose you had two sets A and B and you wanted to prove that they did *not* have the same cardinality. How would you start this proof? Obviously, you won't be able to finish the proof unless since I didn't tell you which sets A and B are, but you should be able to come up with at least the first two steps of the proof.