HW9 (CSCI-C241)

Lillie Donato

26 March 2024

1. Question One
   (a) $\forall x((A(x) \lor B(x)) \to (C(x) \land \neg D(x)))$
   (b) $\neg \exists x(A(x) \land B(x))$
2. Question Two
   (a) $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_5, \mathcal{U}_6, \mathcal{U}_7$
   (b) $\mathcal{U}_1, \mathcal{U}_6, \mathcal{U}_7, \mathcal{U}_8$
   (c) $\mathcal{U}_7$
   (d) $\mathcal{U}_3, \mathcal{U}_4, \mathcal{U}_8$
   (e) $\mathcal{U}_2, \mathcal{U}_3, \mathcal{U}_4, \mathcal{U}_5$
   (f) $\mathcal{U}_2, \mathcal{U}_3, \mathcal{U}_4, \mathcal{U}_5$
   (g) $\mathcal{U}_3, \mathcal{U}_4, \mathcal{U}_8$
   (h) Part (h)
      i. $\neg \forall x(L(x) \to V(x))$ and $\exists x(L(x) \land \neg V(x))$
      ii. $\forall x(L(x) \to \neg V(x))$ and $\neg \exists x(L(x) \land V(x))$
3. Question Three
   (a)
   $$\begin{tikzcd} \square \arrow[d] & \bigcirc \arrow[l] \\ \triangle \arrow[r] & \bigstar \arrow[u] \end{tikzcd}$$
   (b)
   $$\begin{tikzcd} \square \arrow[dr] & \bigcirc \arrow[d] \\ \triangle \arrow[r] & \bigstar \arrow[loop below] \end{tikzcd}$$
   (c) $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$
   (d) $\mathcal{M}_2$
   (e) $\mathcal{M}_{10}$
4. Question Four
   (a) $\forall x \forall y \forall z(\exists w R(x, w) \land ((R(x, y) \land R(x, z)) \to y = z))$
5. Question Five
   (a) $\forall x(A(x) \to ((\exists y(B(y) \land R(x, y))) \land (\forall y \forall z((B(y) \land B(z)) \to ((R(x, y) \land R(x, z)) \to y = z)))))$
6. Question Six
   (a) $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$
   (b) $n - 1$

(c) $n+1$

(d) n

(e) $(k+1)\cdot k!$

7. Question Seven

(a) Claim: $n \geq 2$, $3^n > 2^{n+1}$

Proof. (induction on n)
(Base Step, $n = 2$):

$$\begin{aligned} 3^2 &= 9 && (1)\\ &> 8 && (2)\\ &= 2^3 && (3)\\ &= 2^{2+1} && (4) \end{aligned}$$

(Inductive Step):
Assume $3^k > 2^{k+1}$ for some $k \geq 2$

$$\begin{aligned} 3^{k+1} &= 3^k \cdot 3^1 && (1)\\ &= 3^k \cdot 3 && (2)\\ &> 2^{k+1} \cdot 2 \quad \text{(by the induction hypothesis, and } 3 > 2) && (3)\\ &= 2^{k+1+1} && (4) \end{aligned}$$

□

(b) Claim: $n \geq 9$, $3^n < (n-1)!$

Proof. (induction on n)
(Base Step, $n = 9$):

$$\begin{aligned} 3^9 &= 19683 && (1)\\ &< 40320 && (2)\\ &= 8! && (3)\\ &= (9-1)! && (4) \end{aligned}$$

(Inductive Step):
Assume $3^k < (k-1)!$ for some $k \geq 9$

$$\begin{aligned} 3^{k+1} &= 3^k \cdot 3 && (1)\\ &< (k-1)! \cdot k \quad \text{(by the induction hypothesis, and } k > 3) && (2)\\ &= k! && (3)\\ &= (k-1+1)! && (4) \end{aligned}$$

□

(c) Claim: $n \geq 2$, $3^n > n^2$

Proof. (induction on n)
(Base Step, $n = 2$):

$$\begin{aligned} 3^2 &= 9 && (1)\\ &> 4 && (2)\\ &= 2^2 && (3) \end{aligned}$$

(Inductive Step):
Assume $3^k > k^2$ for some $k \geq 2$

$$\begin{aligned}
3^{k+1} &= 3^k \cdot 3 && (1)\\
&> k^2 \cdot 3 \qquad \text{(by the induction hypothesis)} && (2)\\
&= k^2 + k^2 + k^2 && (3)\\
&\geq k^2 + 2k + 1 \qquad (k^2 \geq 2k > 1 \text{ for all } k \geq 2) && (4)\\
&= (k+1)(k+1) && (5)\\
&= (k+1)^2 && (6)
\end{aligned}$$

□

(d) Claim: $n^2 - 3n$ is even for all $n \in \mathbb{N}$

Proof. (induction on n)
(Base Step, $n = 0$):

$$\begin{aligned}
0^2 - 3(0) &= 0 - 0 && (1)\\
&= 0 && (2)
\end{aligned}$$

0 is even
(Inductive Step):
Assume $k^2 - 3k$ is even for all $k \in \mathbb{N}$
$k^2 - 3k = 2c_1$

$$\begin{aligned}
(k+1)^2 - 3(k+1) &= (k+1)(k+1) - 3k - 3 && (1)\\
&= k^2 + 2k + 1 - 3k - 3 && (2)\\
&= (k^2 - 3k) + (2k - 2) && (3)\\
&= (k^2 - 3k) + 2(k - 1) && (4)\\
&= 2c_1 + 2(k-1) \qquad \text{(by the induction hypothesis)} && (5)\\
&= 2c_1 + 2c_2 \qquad (\text{Let } c_2 = k - 1) && (6)\\
&= 2(c_1 + c_2) && (7)\\
&= 2c_3 \qquad (\text{Let } c_3 = c_1 + c_2) && (8)\\
& 2c_3 \text{ is even} \qquad (2n \text{ is even, where } n \in \mathbb{N}) && (9)\\
& && (10)
\end{aligned}$$

Since $2c_3$ is even, $(k+1)^2 - 3(k+1)$ is even □

(e) Claim: There are 2^n binary string of length n for all $n \in \mathbb{N}$

Proof. (induction on n)
(Base Step, $n = 0$):

$$2^0 = 1 \qquad (1)$$

There is only one possible binary string, of length 0, the empty string.
(Inductive Step):
Assume for length k, there are 2^k binary strings

For every binary string of length k, there are 2^k binary strings, (1)
for each binary string, there are two new possible binary strings of length $k+1$, (2)
So, by the induction hypothesis there are $2^k \cdot 2$ binary strings of length $k+1$ (3)

□

(f) Claim: For any set of characters, with the length of a and any $n \in \mathbb{N}$, there are a^n possible strings of length n that explicitly use the said alphabet

Proof. (induction on n)
(Base Step, $n = 0$):

$$a^0 = 1 \tag{1}$$

Regardles of the possible characters, there is only one possible string, of length 0, the empty string.
(Inductive Step):
Assume for length k, there are a^k strings, where a is total amount of different characters

$$\begin{aligned}
&\text{For every string of length } k\text{, there are } a \text{ possible different characters,} && (1)\\
&\qquad \text{and } a^k \text{ possible strings} && (2)\\
&\text{For every string of length } k+1\text{, with the same } a \text{ possible characters,} && (3)\\
&\qquad \text{for each string, there would } a \text{ possible new strings of length } k+1 && (4)\\
&\text{So, by the induction hypothesis there are } a^k \cdot a \text{ strings of length } k+1 && (5)
\end{aligned}$$

□