HW12 (CSCI-C241)

Lillie Donato

23 April 2024

1. $\texttt{first} = a$
   $\texttt{rest} = \texttt{[}b, c, d, e, f\texttt{]}$

2. $\texttt{reverse}(L) = \begin{cases} \texttt{[]} & \text{if } L = \texttt{[]} \\ \texttt{append}(\texttt{reverse}(\texttt{rest}), \texttt{first}) & \text{if } L = \texttt{[first, *rest]} \end{cases}$

3. Claim: $\texttt{len}(\texttt{reverse}(L)) = \texttt{len}(L)$

   Proof.
   (Base Step, $L = \texttt{[]}$):

$$\begin{aligned} \texttt{len}(\texttt{reverse}(\texttt{[]})) &= \texttt{len}(\texttt{[]}) && (1) \\ &= 0 && (2) \\ &= \texttt{len}(\texttt{[]}) && (3) \end{aligned}$$

   (Inductive Step, $L = \texttt{[first, *rest]}$):
   Assume for some list $\texttt{rest}$, that $\texttt{len}(\texttt{reverse}(\texttt{rest})) = \texttt{len}(\texttt{rest})$

$$\begin{aligned} \texttt{len}(\texttt{reverse}(L)) &= \texttt{len}(\texttt{reverse}(\texttt{[first, *rest]})) && (1) \\ &= \texttt{len}(\texttt{append}(\texttt{reverse}(\texttt{rest}), \texttt{first})) \quad \text{(By the definition of } \texttt{reverse}\text{)} && (2) \\ &= \texttt{len}(\texttt{reverse}(\texttt{rest})) + \texttt{len}(\texttt{[first]}) && (3) \\ &= \texttt{len}(\texttt{reverse}(\texttt{rest})) + 1 \quad \text{(Because the length of a single item list is 1)} && (4) \\ &= \texttt{len}(\texttt{rest}) + 1 \quad \text{(By the Induction Hypothesis)} && (5) \\ &= \texttt{len}(\texttt{[first, *rest]}) \quad \text{(By the definition of } \texttt{len}\text{)} && (6) \\ &= \texttt{len}(L) && (7) \end{aligned}$$

   $\square$

4. Claim: Every **X-number** n is divisble by 3 ($\frac{n}{3} \in \mathbb{Z}$)

   Proof.
   (Base Step, $n = 12$):

$$\begin{aligned} \frac{12}{3} &= 4 && (1) \\ 4 &\in \mathbb{Z} && (2) \end{aligned}$$

   (Base Step, $n = 15$):

$$\begin{aligned} \frac{15}{3} &= 5 && (1) \\ 5 &\in \mathbb{Z} && (2) \end{aligned}$$

   (Inductive Step, $n = x + y$, where x, y are both **X-numbers**):
   Assume for some **X-numbers** x, y, that they are divisble by 3

$$\begin{aligned} x &= 3a \quad \text{(By our Induction Hypothesis and where } a \in \mathbb{Z}\text{)} && (1) \\ y &= 3b \quad \text{(By our Induction Hypothesis and where } b \in \mathbb{Z}\text{)} && (2) \\ x + y &= 3a + 3b \quad \text{(Where } x, y \text{ are both } \textbf{X-numbers}\text{)} && (3) \\ &= 3(a + b) \quad (3(a+b) \text{ is divisible by 3 because 3 and } a + b \in \mathbb{Z}\text{, so } 3(a+b) \in \mathbb{Z}) && (4) \end{aligned}$$

(Inductive Step, $n = x - y$, where x, y are both **X-numbers**):

$$
\begin{aligned}
x &= 3a \qquad \text{(By our Induction Hypothesis and where } a \in \mathbb{Z}) && (1)\\
y &= 3b \qquad \text{(By our Induction Hypothesis and where } b \in \mathbb{Z}) && (2)\\
x - y &= 3a - 3b \qquad \text{(Where } x, y \text{ are both } \textbf{X-numbers}) && (3)\\
&= 3(a - b) \qquad (3(a-b) \text{ is divisible by 3 because 3 and } a - b \in \mathbb{Z}, \text{ so } 3(a-b) \in \mathbb{Z}) && (4)
\end{aligned}
$$

□

5. Claim: For any $n \in \mathbb{Z}$ where $n \geq 1$, $L_n = F_{n-1} + F_{n+1}$

   Proof.
   (Base Step, $n = 1$):

$$
\begin{aligned}
L_1 &= 1 \qquad \text{(By the definition of the Lucas Numbers)} && (1)\\
&= 1 + 0 && (2)\\
&= F_1 + F_0 && (3)\\
&= F_{2-1} + F_{2-2} && (4)\\
&= 0 + (F_{2-1} + F_{2-2}) && (5)\\
&= F_0 + F_2 && (6)\\
&= F_{1-1} + F_{1+1} && (7)
\end{aligned}
$$

   (Inductive Step):
   Assume for some $k \in \mathbb{Z}$ where $k \geq 1$, $L_i = F_{i-1} + F_{i+1}$ for all $1 \leq i \leq k$

$$
\begin{aligned}
L_{k+1} &= L_k + L_{k-1} && (1)\\
&= (F_{k-1} + F_{k+1}) + (F_{k-2} + F_k) \qquad \text{(By the Inductive Hypothesis)} && (2)\\
&= (F_{k-1} + F_{k-2}) + (F_{k+1} + F_k) && (3)\\
&= F_k + F_{k+2} && (4)\\
&= F_{(k+1)-1} + F_{(k+1)+1} && (5)
\end{aligned}
$$

□

6. Claim: Every $n \in \mathbb{Z}$ where $n \geq 1$, n can be written as the sum of powers of 2

   Proof.
   (Base Step, $n = 1$):

$$
2^0 = 1 \qquad (1)
$$

   1 can be written in the sum of powers of 2, because any number raised to 0 is 1
   (Inductive Step):
   Assume for some $k \in \mathbb{Z}$ where $k \geq 1$, i can be written as the sum of power of 2, where $1 \leq i \leq k$

Case 1: k is odd (1)

By our IH, $\dfrac{k+1}{2}$ can be written as a unique sum of powers of 2, (2)

so $\dfrac{k+1}{2} = c_0 \cdot 2^0 + c_1 \cdot 2^1 + c_2 \cdot 2^2 + c_3 \cdot 2^3 + \cdots + c_n \cdot 2^n$,

where all of the values multiplying the powers of 2 are 0 or 1

Therefore, (3)

$$k + 1 = 2 \cdot \frac{k+1}{2} = c_0 \cdot 2^{0+1} + c_1 \cdot 2^{1+1} + c_2 \cdot 2^{2+1} + c_3 \cdot 2^{3+1} + \cdots + c_n \cdot 2^{n+1}$$

Case 2: k is even (4)

Since k is even and can be represented in sums of powers of 2 (according to our IH), then in the sum of powers of 2 for k, 2^0 is not a member (5)

Therefore $k + 1 = k + 2^0$ (6)

In either case of $k + 1$, it can be made by a unique sum of powers of 2, so in general it can be made by a sum of powers of 2 (7)

□