CSCI-C 241 HW #9
Instructors: Wennstrom & LaRacuente Due Tuesday, March 26th, 2024

1. For this problem, let A, B, C, and D be sets, and let the universe be some set $\mathcal{U}$ that is a superset of A, B, C, and D. (In other words, the universe includes all the members of these three sets, and possibly some others.)

   Use $A(x)$ to mean "$x \in A$", $B(x)$ to mean "$x \in B$", $C(x)$ to mean "$x \in C$", and $D(x)$ to mean "$x \in D$".

   As an example, if we were to ask you to translate the statement "There is a member of $A \cap B$ into a formula of first-order logic, you would write "$\exists x\big(A(x) \wedge B(x)\big)$.

   (a) Translate the statement "$A \cup B \subseteq C \setminus D$" into a formula of first-order logic.

   (b) Translate the statement "$A \cap B = \varnothing$" into a formula of first-order logic.

2. For each of the parts of this problem, the universes will be sets of letters. Let $V(x)$ mean "x is a vowel," and $L(x)$ mean "x is lowercase." (These letters are case sensitive.)

   Use these universes to answer the following questions:

   $\mathcal{U}_1 = \{$ e, A, i, E $\}$
   $\mathcal{U}_2 = \{$ e, A, j, T $\}$
   $\mathcal{U}_3 = \{$ B, d, j, T $\}$
   $\mathcal{U}_4 = \{$ E, A, j, T $\}$
   $\mathcal{U}_5 = \{$ e, a, j, T $\}$
   $\mathcal{U}_6 = \{$ e, A, I, T $\}$
   $\mathcal{U}_7 = \{$ e, a, i, u $\}$
   $\mathcal{U}_8 = \{$ E, A, B, T $\}$

   (a) For which universes ($\mathcal{U}_1$–$\mathcal{U}_8$) is the formula $\exists x\big(L(x) \wedge V(x)\big)$ true?

   (b) For which universes is the formula $\forall x\big(L(x) \rightarrow V(x)\big)$ true?

   (c) For which universes is the formula $\forall x\big(L(x) \wedge V(x)\big)$ true?

   (d) For which universes is the formula $\neg\exists x\big(L(x) \wedge V(x)\big)$ true?

   (e) For which universes is the formula $\neg\forall x\big(L(x) \rightarrow V(x)\big)$ true?

   (f) For which universes is the formula $\exists x\big(L(x) \wedge \neg V(x)\big)$ true?

   (g) For which universes is the formula $\forall x\big(L(x) \rightarrow \neg V(x)\big)$ true?

   (h) Are any of the formulas in the previous parts of this problem equivalent to each other? If so, which ones?

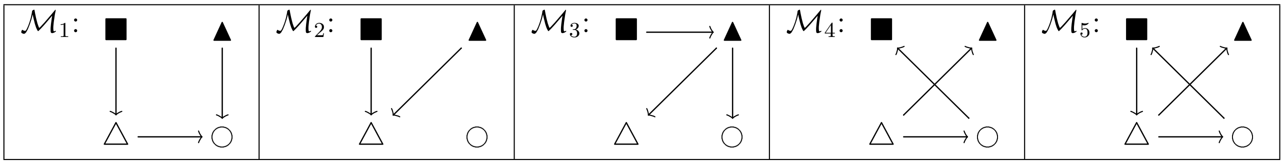
3. For this problem, the universes will be small collections of shapes with arrows drawn between some of the shapes. $S(x)$ means "x is solid," $C(x)$ means "x is a circle," and $P(x, y)$ means "There is an arrow pointing from x to y."

   If you think it is not possible to solve the problem, explain why not.

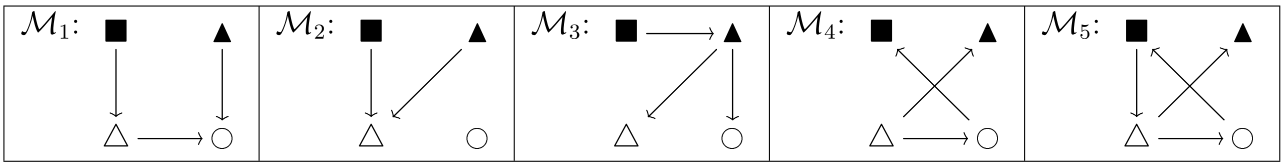
   (a) Create a drawing where $\forall x \exists y P(x, y)$ is true, but $\exists y \forall x P(x, y)$ is not.

   (b) Create a drawing where $\exists y \forall x P(x, y)$ is true, but $\forall x \exists y P(x, y)$ is not.

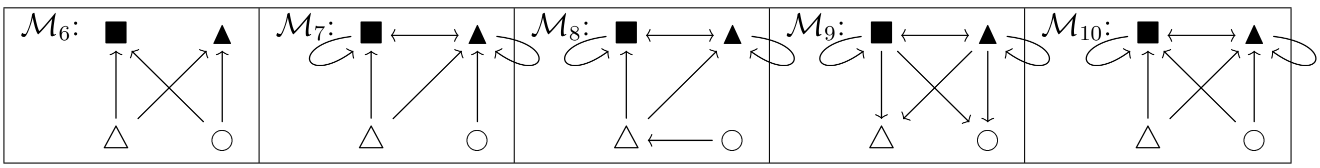
(c) For which of the following drawings is $\forall x\big(S(x)\to\exists y P(x,y)\big)$ true? (For this problem and the next few problems, there may be more than one correct answer, and you need to list *all* of the correct answers.)



(d) For which of the following drawings is $\exists y\forall x\big(S(x)\to P(x,y)\big)$ true?



(e) For which of the following drawings is $\forall x\forall y\big(S(y)\to P(x,y)\big)$ true?



4. Suppose that A is a set and R is a relation on A. For this problem, you can use $R(x,y)$ to mean "$(x,y)\in R$". The universe is A. You can use the symbol $=$ as part of your formula.

   Write the claim "R is a function from A to B," as a formula of first-order logic.

   Hint: Try dividing this up into two smaller formulas, one for the "uniqueness" property and one for the "existence" property, and then think of how to combine them to make one big formula.

5. **Bonus:** Suppose that A and B are sets and R is a relation from A to B. For this problem, you can use $A(x)$ to mean "$x\in A$", $B(x)$ to mean "$x\in B$", and $R(x,y)$ to mean "$(x,y)\in R$". The universe is some set $\mathcal{U}$ that contains both A and B as subsets. You can use the symbol $=$ as part of your formula.

   Write the claim "R is a function from A to B," as a formula of first-order logic.

   (Note that you'll have to mention the domain and codomain explicitly this time.)

Reminder: I know you've seen factorials in your other math classes, but in case you've forgotten, the **factorial** of a number n (written $n!$) is the product of all the integers from 1 up to n. So for example, $4! = 1\cdot 2\cdot 3\cdot 4 = 24$. For the same reason that $2^0 = 1$, we also have $0! = 1$.

6. Answer the following questions.

   When you are asked to write X in terms of Y, we mean that you should write an equation of the form $X =$ (some expression involving Y). So for example, if I asked you to "write 2^{k+1} in terms of 2^k", you would write something like: $2^{k+1} = 2^k\cdot 2$.

(a) $5! = ?$

(b) The first factor in $n!$ is 1. The second factor in $n!$ is 2. The last factor in $n!$ is n. What is the second-to-last factor in $n!$?

(c) What is the last factor in $(n+1)!$?

(d) What is the second-to-last factor in $(n+1)!$?

(e) Write $(k+1)!$ in terms of $k!$.

7. Use numerical induction to prove the following claims:

   (a) For all natural numbers n, if $n \geq 2$, then $3^n > 2^{n+1}$.

   (b) For all natural numbers $n \geq 9$, $3^n < (n-1)!$.

   Feel free to use a calculator for the base case.

   (c) For all natural numbers $n \geq 2$, $3^n > n^2$.

   Hint: The algebra on this one is tricky, so don't be disheartened if you need help finishing up the proof, and more importantly, don't make up rules to try and "finish" the proof.

   (d) For all $n \in \mathbb{N}$, $n^2 - 3n$ is even.

   (Note: It is possible to prove this *without* induction. But for this assignment, you *must* provide an induction proof.)

   (e) For any natural number n, there are 2^n binary strings of length n.

   A **binary string** is a string consisting of only `0`s and `1`s. (Note that the empty string ε counts as a binary string.

   (f) For any alphabet with a characters in it, and for any $n \in \mathbb{N}$, there are a^n strings of length n that use that alphabet.

   By **alphabet**, we just mean a set of characters for the strings. So for binary strings, the alphabet would be $\{\mathtt{0}, \mathtt{1}\}$ and for alphanumeric strings $(a = 2)$, the alphabet would be $\{\mathtt{a}, \mathtt{b}, \ldots, \mathtt{z}, \mathtt{A}, \mathtt{B}, \ldots, \mathtt{Z}, \mathtt{0}, \mathtt{1}, \ldots, \mathtt{9}\}$ $(a = 62)$.