HW10 (CSCI-C241)

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1. Question One
   (a) $1^3 + 2^3 + 3^3 = 1 + 8 + 27 = 36$
   (b) $\frac{1}{3} + \frac{1}{4} = \frac{7}{12}$
   (c) $\sqrt{2}$
   (d) $\frac{1}{1} = 1$
   (e) $\frac{1}{k+1}$
   (f) $\frac{1}{k}$
   (g) $\sum_{i=1}^{k} \frac{1}{i} + \frac{1}{k+1}$
2. Question Two
   (a) Claim: For $n \in \mathbb{N}$, $n \geq 1$, $\sum_{i=1}^{n} 2^{i-1} = 2^n - 1$

   Proof. (induction on n)
   (Base Step, $n = 1$):

$$\begin{aligned} \sum_{i=1}^{n} 2^{1-1} &= \sum_{i=1}^{n} 2^0 && (1)\\ &= \sum_{i=1}^{n} 1 && (2)\\ &= 1 && (3)\\ &= 2^0 && (4)\\ &= 2^{1-1} && (5) \end{aligned}$$

   (Inductive Step):
   Assume $\sum_{i=1}^{k} 2^{i-1} = 2^k - 1$ for some $k \geq 1$

$$\begin{aligned} \sum_{i=1}^{k+1} 2^{i-1} &= \sum_{i=1}^{k} 2^{i-1} + 2^{k+1-1} && (1)\\ &= \sum_{i=1}^{k} 2^{i-1} + 2^k && (2)\\ &= 2^k - 1 + 2^k \quad \text{(by the induction hypothesis, and } 2^k = 2^k\text{)} && (3)\\ &= 2^k + 2^k - 1 && (4)\\ &= 2 \cdot 2^k - 1 && (5)\\ &= 2^{k+1} - 1 && (6) \end{aligned}$$

   □

   (b) Claim: For all $n \in \mathbb{N}$, $\sum_{i=0}^{n} i! \cdot i = (n+1)! - 1$

Proof. (induction on n)
(Base Step, $n = 0$):

$$\begin{aligned}
\sum_{i=0}^{n} 0! \cdot 0 &= \sum_{i=0}^{n} 0 && (1)\\
&= 0 && (2)\\
&= 1 - 1 && (3)\\
&= 1! - 1 && (4)\\
&= (0+1)! - 1 && (5)
\end{aligned}$$

(Inductive Step):
Assume $\sum_{i=0}^{k} i! \cdot i = (k+1)! - 1$ for some $k \in \mathbb{N}$

$$\begin{aligned}
\sum_{i=0}^{k+1} i! \cdot i &= (\sum_{i=0}^{k} i! \cdot i) + ((k+1)! \cdot (k+1)) && (1)\\
&= (k+1)! - 1 + (k+1)! \cdot (k+1) \qquad \text{(By the induction hypothesis)} && (2)\\
&= (k+1)! + (k+1)! \cdot (k+1) - 1 && (3)\\
&= (k+1)! \cdot (1 + (k+1)) - 1 && (4)\\
&= (k+1)! \cdot (k+2) - 1 && (5)\\
&= (k+2)! - 1 && (6)\\
&= (k+1+1)! - 1 && (7)
\end{aligned}$$

□

(c) Claim: For $n \in \mathbb{N}$, $n \geq$ the sum of the number of digits in n

Proof. (induction on the number of digits of n)
(Base Step, n has one digit):

$$\begin{aligned}
&\text{Since } n \text{ has one digit } n < 10 && (1)\\
&\text{Since } n < 10 \text{ and the number of digits of } n = 1, \text{ the sum of the digits is } n && (2)
\end{aligned}$$

(Inductive Step): □

3. Question Three
   (a) The minimum value where $f(x) = g(x)$ is 10^{12}
   (b) A value where $f(x) > g(x)$ is 19
4. Claim: For every positive real number a where $a \geq e$, there exists $m \in \mathbb{N}$ such that for all $n \geq m$, $n! > a^n$

Proof.

$$\begin{aligned}
&\text{Choose } a \in \mathbb{R} \text{ such that } a \geq 0 \text{ and } a \geq e && (1)\\
&\text{There exists some } m \in \mathbb{N} \text{ where } m > 2a^2 && (2)\\
&\text{Choose } n \in \mathbb{N} \text{ where } n \geq m && (3)\\
&\quad \text{Since half of the numbers that are being multiplied in } n! \text{ are greater than } \frac{n}{2}, && (4)\\
&\quad \text{we know } \frac{n}{2} \text{ numbers of } n! \text{ are greater than } \frac{n}{2} &&\\
&\quad \text{Since } \frac{n}{2} \text{ numbers are greater than } \frac{n}{2}, \text{ we know } n! > \left(\frac{n}{2}\right)^{\frac{n}{2}} && (5)\\
&\quad \text{Since } n! > \frac{n}{2}, \text{ we know } n! > \left(\frac{n}{2}\right)^{\frac{n}{2}} && (6)\\
&\quad \text{Since } n \geq m, \text{ we know } \left(\frac{n}{2}\right)^{\frac{n}{2}} \geq \left(\frac{m}{2}\right)^{\frac{n}{2}} && (7)\\
&\quad \text{Since } \left(\frac{n}{2}\right)^{\frac{n}{2}} \geq \left(\frac{m}{2}\right)^{\frac{n}{2}}, \text{ we know } n! > \left(\frac{m}{2}\right)^{\frac{n}{2}} && (8)\\
&\quad \text{Since } \left(\frac{m}{2}\right)^{\frac{n}{2}} = \left(\sqrt{\frac{m}{2}}\right)^{n}, \text{ we know } n! > \left(\sqrt{\frac{m}{2}}\right)^{n} && (9)\\
&\quad \text{Since } m > 2a^2, \text{ we know } \frac{m}{2} > a^2 && (10)\\
&\quad \text{Since } \frac{m}{2} > a^2, \text{ we know } \sqrt{\frac{m}{2}} > a && (11)\\
&\quad \text{Since } \sqrt{\frac{m}{2}} > a, \text{ we know } \left(\sqrt{\frac{m}{2}}\right)^{n} > a^n && (12)\\
&\quad \text{Since } \left(\sqrt{\frac{m}{2}}\right)^{n} > a^n, \text{ we know } n! > a^n && (13)
\end{aligned}$$

□

5. Question Five - Combinatorics
   (a) $10! = 3628800$
   (b) $\frac{50!}{(50-5)!} = 254251200$
   (c) $50^5 = 312500000$
   (d) $\frac{20!}{16!} = 116280$
   (e) $\frac{\frac{20!}{15!}}{5!} = 15504$
6. Question Six
   (a) $52! = 8.0658 \times 10^{67}$ (68 digits)
   (b) I personally do not think every permutation of a 52 deck card has been used. Despite how often cards are used in western culture and for how long they have been, I really do not believe it would be possible for 52! permutations to have been used because of the gigantic number it is.
7. Claim: For any non-empty set A of size n and any integer r with $n \geq r \geq 1$, there are $\frac{n!}{(n-r)!}$ permutations of length r using values taken from A

   Proof. (induction on n)
   (Base Step, $r = 1$):

$$\begin{aligned}
\frac{n!}{(n-1)!} &= \frac{n \cdot (n-1)!}{(n-1)!} && (1)\\
&= \frac{n}{1} && (2)\\
&= n && (3)
\end{aligned}$$

(Inductive Step):
Assume there are $\frac{n!}{(n-k)!}$ permutations for some length $k \geq 1$ using values taken from A, a set of length n

$$\begin{aligned}
\frac{n!}{(n-(k+1))!} &= \frac{n!}{(n-k-1)!} && (1)\\
&= \frac{n!}{\frac{(n-k)\cdot(n-k-1)!}{(n-k)}} && (2)\\
&= \frac{n!}{(n-k)!}\cdot(n-k) && (3)\\
& && (4)
\end{aligned}$$

Since there are $\frac{n!}{(n-k)!}$ permutations for some length k (by the Induction Hypothesis), there are $n-k$ possibilities for every permutation for some length $k+1$, therefore there are $\frac{n!}{(n-k)!}\cdot(n-k)$ permutations for length $k+1$ $\square$