HW10 (CSCI-C241)

Lillie Donato

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1. Question One
   (a) $1^3 + 2^3 + 3^3 = 1 + 8 + 27 = 36$
   (b) $\frac{1}{3} + \frac{1}{4} = \frac{7}{12}$
   (c) $\sqrt{2}$
   (d) $\frac{1}{1} = 1$
   (e) $\frac{1}{k+1}$
   (f) $\frac{1}{k}$
   (g) $\sum_{i=1}^{k} \frac{1}{i} + \frac{1}{k+1}$
2. Question Two
   (a) Claim: For $n \in \mathbb{N}$, $n \geq 1$, $\sum_{i=1}^{n} 2^{i-1} = 2^n - 1$

   Proof. (induction on n)
   (Base Step, $n = 1$):

$$\begin{aligned}
\sum_{i=1}^{n} 2^{1-1} &= \sum_{i=1}^{n} 2^0 && (1)\\
&= \sum_{i=1}^{n} 1 && (2)\\
&= 1 && (3)\\
&= 2^0 && (4)\\
&= 2^{1-1} && (5)
\end{aligned}$$

   (Inductive Step):
   Assume $\sum_{i=1}^{k} 2^{i-1} = 2^k - 1$ for some $k \geq 1$

$$\begin{aligned}
\sum_{i=1}^{k+1} 2^{i-1} &= \sum_{i=1}^{k} 2^{i-1} + 2^{k+1-1} && (1)\\
&= \sum_{i=1}^{k} 2^{i-1} + 2^k && (2)\\
&= 2^k - 1 + 2^k \quad \text{(by the induction hypothesis, and } 2^k = 2^k\text{)} && (3)\\
&= 2^k + 2^k - 1 && (4)\\
&= 2 \cdot 2^k - 1 && (5)\\
&= 2^{k+1} - 1 && (6)
\end{aligned}$$

   □

   (b) Claim: For all $n \in \mathbb{N}$, $\sum_{i=0}^{n} i! \cdot i = (n+1)! - 1$

Proof. (induction on n)
(Base Step, $n = 0$):

$$\begin{aligned} \sum_{i=0}^{n} 0! \cdot 0 &= \sum_{i=0}^{n} 0 && (1) \\ &= 0 && (2) \\ &= 1 - 1 && (3) \\ &= 1! - 1 && (4) \\ &= (0+1)! - 1 && (5) \end{aligned}$$

(Inductive Step):
Assume $\sum_{i=0}^{k} i! \cdot i = (k+1)! - 1$ for some $k \in \mathbb{N}$

$$\begin{aligned} \sum_{i=0}^{k+1} i! \cdot i &= (\sum_{i=0}^{k} i! \cdot i) + ((k+1)! \cdot (k+1)) && (1) \\ &= (k+1)! - 1 + (k+1)! \cdot (k+1) \qquad \text{(By the induction hypothesis)} && (2) \\ &= (k+1)! + (k+1)! \cdot (k+1) - 1 && (3) \\ &= (k+1)! \cdot (1 + (k+1)) - 1 && (4) \\ &= (k+1)! \cdot (k+2) - 1 && (5) \\ &= (k+2)! - 1 && (6) \\ &= (k+1+1)! - 1 && (7) \end{aligned}$$

$\square$

(c) Claim: For $x \in \mathbb{N}$, $n \in \mathbb{N}$ such that $n =$ the number of digits in x, $x \geq$ the sum of the digits of x

Proof. (induction on n, the number of digits of x)
(Base Step, $n = 1$, x has one digit):

Since x has one digit $x < 10$ (1)
Since $x < 10$ and the number of digits of x is 1, the sum of the digits is x (2)
Since the sum of the digits of x is x, we know $x \geq$ the sum of the number of digits (3)

(Inductive Step):
Assume for some $x_1 \in \mathbb{N}$, $k =$ the number of digits in $x_1, x_1 \geq$ the sum of the digits of x_1

$$\begin{aligned} \text{Let } c &= \text{some natural number} < 10 && (1) \\ \text{Let } x_2 &= x_1 + c \cdot 10^k \qquad (x_2 = x_1 \text{ with } c \text{ added to the beginning, } k+1 \text{ digits}) && (2) \\ \text{So, sum}(x_2) &= \text{sum}(x_1) + c && (3) \\ \text{Therefore, } x_2 &\geq \text{sum}(x_2) \qquad \text{(by the Induction Hypothesis and } c \cdot 10^k \geq c) && (4) \end{aligned}$$

$\square$

3. Question Three
   (a) The minimum value where $f(x) = g(x)$ is 10^{12}
   (b) A value where $f(x) > g(x)$ is 19
4. Claim: For every positive real number a where $a \geq e$, there exists $m \in \mathbb{N}$ such that for all $n \geq m$, $n! > a^n$

Proof.

$$
\begin{aligned}
&\text{Choose } a \in \mathbb{R} \text{ such that } a \geq e && (1)\\
&\text{Suppose } m \in \mathbb{N} \text{ such that } n \geq m && (2)\\
&\quad \text{Since half of the numbers that are being multiplied in } n! \text{ are greater than } \frac{n}{2}, && (3)\\
&\quad \text{we know } \frac{n}{2} \text{ numbers of } n! \text{ are greater than } \frac{n}{2} && \\
&\quad \text{Since } \frac{n}{2} \text{ numbers are greater than } \frac{n}{2}, \text{ we know } n! > \left(\frac{n}{2}\right)^{\frac{n}{2}} && (4)\\
&\quad \text{Since } n! > \frac{n}{2}, \text{ we know } n! > \left(\frac{n}{2}\right)^{\frac{n}{2}} && (5)\\
&\quad \text{Since } n \geq m, \text{ we know } \left(\frac{n}{2}\right)^{\frac{n}{2}} \geq \left(\frac{m}{2}\right)^{\frac{n}{2}} && (6)\\
&\quad \text{Since } \left(\frac{n}{2}\right)^{\frac{n}{2}} \geq \left(\frac{m}{2}\right)^{\frac{n}{2}}, \text{ we know } n! > \left(\frac{m}{2}\right)^{\frac{n}{2}} && (7)\\
&\quad \text{Since } \left(\frac{m}{2}\right)^{\frac{n}{2}} = \left(\sqrt{\frac{m}{2}}\right)^{n}, \text{ we know } n! > \left(\sqrt{\frac{m}{2}}\right)^{n} && (8)\\
&\quad \text{Let } m > 2a^2 && (9)\\
&\quad \text{Since } m > 2a^2, \text{ we know } \frac{m}{2} > a^2 && (10)\\
&\quad \text{Since } \frac{m}{2} > a^2, \text{ we know } \sqrt{\frac{m}{2}} > a && (11)\\
&\quad \text{Since } \sqrt{\frac{m}{2}} > a, \text{ we know } \left(\sqrt{\frac{m}{2}}\right)^{n} > a^n && (12)\\
&\quad \text{Since } \left(\sqrt{\frac{m}{2}}\right)^{n} > a^n, \text{ we know } n! > a^n && (13)
\end{aligned}
$$

□

5. Question Five - Combinatorics

   (a) $10! = 3628800$

   (b) $\frac{50!}{(50-5)!} = 254251200$

   (c) $50^5 = 312500000$

   (d) $\frac{20!}{16!} = 116280$

   (e) $\frac{\frac{20!}{15!}}{5!} = 15504$

6. Question Six

   (a) $52! = 8.0658 \times 10^{67}$ (68 digits)

   (b) I personally do not think every permutation of a 52 deck card has been used. Despite how often cards are used in western culture and for how long they have been, I really do not believe it would be possible for 52! permutations to have been used because of the gigantic number it is.

7. Claim: For any non-empty set A of size n and any integer r with $n \geq r \geq 1$, there are $\frac{n!}{(n-r)!}$ permutations of length r using values taken from A

   Proof. (induction on n)
   (Base Step, $r = 1$):

$$
\begin{aligned}
\frac{n!}{(n-1)!} &= \frac{n \cdot (n-1)!}{(n-1)!} && (1)\\
&= \frac{n}{1} && (2)\\
&= n && (3)
\end{aligned}
$$

(Inductive Step):
Assume there are $\frac{n!}{(n-k)!}$ permutations for some length $k \geq 1$ using values taken from A, a set of length n

$$\begin{aligned}
\frac{n!}{(n-(k+1))!} &= \frac{n!}{(n-k-1)!} && (1)\\
&= \frac{n!}{\frac{(n-k)\cdot(n-k-1)!}{(n-k)}} && (2)\\
&= \frac{n!}{(n-k)!} \cdot (n-k) && (3)\\
& && (4)
\end{aligned}$$

Since there are $\frac{n!}{(n-k)!}$ permutations for some length k (by the Induction Hypothesis), there are $n-k$ possibilities to create a new permutation of length $k+1$ from every permutation of length k, therefore there are $\frac{n!}{(n-k)!} \cdot (n-k)$ permutations for length $k+1$ $\square$