CSCI-C 241 HW #10
Instructors: Wennstrom & LaRacuente Due Tuesday, April 9th, 2024

1. Answer the following questions.

   When you are asked to write X in terms of Y, we mean that you should write an equation of the form $X =$ (some expression involving Y). So for example, if I asked you to "write 2^{k+1} in terms of 2^k", you would write something like: $2^{k+1} = 2^k \cdot 2$.

   (a) $\sum_{i=1}^{3} i^3 = ?$

   (b) $\sum_{i=3}^{4} \frac{1}{i} = ?$

   (c) $\sum_{i=2}^{2} \sqrt{i} = ?$

   (d) What is the first term in $\sum_{i=1}^{k+1} \frac{1}{i}$?

   (e) What is the last term in $\sum_{i=1}^{k+1} \frac{1}{i}$?

   (f) What is the second-to-last term in $\sum_{i=1}^{k+1} \frac{1}{i}$?

   (g) Write $\sum_{i=1}^{k+1} \frac{1}{i}$ in terms of $\sum_{i=1}^{k} \frac{1}{i}$.

2. Use numerical induction to prove the following claims:

   (a) For all natural numbers $n \geq 1$, $\sum_{i=1}^{n} 2^{i-1} = 2^n - 1$.

   (b) For all $n \in \mathbb{N}$, $\sum_{i=0}^{n} i! \cdot i = (n+1)! - 1$.

   (c) **Bonus**: For any natural number n, n is greater than or equal to the sum of its digits in base 10.

   Hint: You should be inducting not directly on n, but on the *number of digits* in n. Recall that if you write the number n (with d digits) in base 10 as $n_{d-1}n_{d-2}\cdots n_2n_1n_0$, then $n = \sum_{i=0}^{d-1} = n_0 \cdot 10^0 + n_1 \cdot 10^1 + \cdots + n_{d-1} \cdot 10^{d-2} + n_d \cdot 10^{d-1}$.

   So for example, for the number $n = 65537$, the number of digits is $d = 5$, with $n_0 = 7$, $n_1 = 3$, $n_2 = 5$, $n_3 = 5$, and $n_{d-1} = n_{5-1} = n_4 = 6$, so $65537 = 7 + 30 + 500 + 5000 + 60000 = 7 \cdot 10^0 + 3 \cdot 10^1 + 5 \cdot 10^2 + 5 \cdot 10^3 + 6 \cdot 10^4$.

3. (a) Let f and g be functions on positive real numbers respectively defined by $f(x) = 5,000,000\sqrt{x}$, and $g(x) = 5x$. What is the minimum value M such that for all inputs $z > M$, $f(z) < g(z)$?

   Hint 1: If you're having trouble understanding what this question is even asking: We learned in the lectures that a linear function (like $g(x) = 5x$) grows faster than a square-root function (like $f(x) = 5,000,000\sqrt{x}$). That means that even though for small values of x, $f(x)$ might be bigger than $g(x)$, $g(x)$ will "eventually" be bigger than $f(x)$. Or in other words, for "sufficiently large" z, $g(z) > f(z)$. What we're asking for here is the specific input value M at which $g(z)$ becomes bigger than $f(z)$ (and stays bigger).

Hint 2: You may find it helpful to define $z = \sqrt{M}$, and write an equation in terms of z.

Note: this is actually the sort of problem that comes up when attempting to determine whether a quantum computer would be able to search for a solution to a problem faster than would an ordinary (classical) computer. One computes the *crossover size*, which is how big an input problem must be for an asymptotically smaller function to overcome smaller constants.

(b) Let f and g be functions on positive real numbers respectively defined by $f(x) = 2^x$, and $g(x) = 1,000x^2$. Find *a* value M such that for every $z \geq M$, $f(z) > g(z)$.

Hint: Unlike the previous question, we don't need you to find the *minimal* such value M, just for you to find *any* value M such that for all inputs bigger than M, f is bigger than g.

4. Recall that $e \approx 2.71828\cdots$ (sometimes called the **exponential constant**, **Napier's constant**, or **Euler's number**, not to be confused with **Euler's constant**, which is something completely different).

   Prove the following claim: for every positive real number $a \geq e$, there exists a natural number m such that for all $n \geq m$, $n! > a^n$. There's more than one way to prove this claim. We've provided hints for two different methods; you can use whatever method you like.

   Hint: You do not need induction for either of these methods.

   Method 1 Hint: try getting n to be greater than or equal to a power of a. You may use the fact that $n! > (n/e)^n$.

   Method 2 Hint: Half of the numbers being multiplied in $n!$ are bigger than $\frac{n}{2}$. If you focus on those and ignore the other half, you might be able to make some progress.

   This method 's slightly trickier to get the wording and details correct, but it doesn't need the assumption $a \geq e$ and it doesn't require using the fact about $(n/e)^n$ from the previous problem.

5. Calculate the following numbers:

   (a) If you have a playlist with 10 songs in it, how many different orders can you put those 10 songs in?

   (b) If you have a collection of 50 songs and you want to make a 5-song playlist (order matters, no repeated songs), how many different playlists can you make?

   (c) If you have a collection of 50 songs and you want to make a 5-song playlist (order matters, repeated songs are allowed), how many different playlists can you make?

(d) If you have a club with 20 members and you need to choose four of them: one to be president, one to be vice president, one to be treasurer, and one to be secretary, how many different ways are of doing this?

(e) If you have a club with 20 members and you need to choose five of them to be on a committee (there's no difference between the five positions on the committee), how many ways are there of doing this?

6. The most common deck of Western playing cards has 52 cards in it. There are 52! different ways to shuffle such a deck of cards. (As in there are 52! different permutations of 52 objects.)

   (a) Enter 52! into your favorite calculator or calculator app. How many digits does your answer have?

   (b) People have been using this deck of cards for hundreds of years and there are currently billions of decks of playing cards in the world. Do you think it's plausible that every possible permutation of cards has occurred somewhere in the world? Why or why not?

7. In class, we proved that if you have a set with n members, there are $n!$ permutations that use all n members of that set. We showed you a formula for $P(n, r) = \frac{n!}{(n-r)!}$, the number of permutations of length r taken from a set with n elements, but we did not prove that this formula works.

   Give a proof using induction on r of the following claim. (You must use induction on r to get credit for this problem.)

   Claim. For any non-empty set A of size n and any integer r with $n \geq r \geq 1$, there are $\frac{n!}{(n-r)!}$ permutations of length r using values taken from A.

   Hint 1: n is a fixed value for this proof. So while you will be looking at different values for r in the base step and induction step, n will stay the same throughout.

   Hint 2: In class, the proof we did (in at least some of the lectures) that "there are $n!$ permutations of length n" involved counting how many different positions you could place a new value. That strategy likely won't work out nicely here. Instead, I recommend using an approach similar to how we proved that "there are a^n strings of length n using characters from an alphabet of size a."

8. Read the following proof of the claim that for any two vertices a and b in a tree, there is one and only one path between them.

   Proof. Let T be a tree and choose two vertices a and b. Since all trees are connected, there must exist at least one path from a to b. The vertices in this path are all distinct from each other because this is a path.

   Suppose towards a contradiction that there is a different path from a to b. Again, all the vertices are distinct from each other.

We don't necessarily know that the vertices in the first path are *all* different from the vertices in the second path, but at least *some* of them must be different. If they're not all different, shrink down the two paths to find a pair of subpaths that start and end with the same vertices but don't have any other vertices in common. (It's possible these paths might just have one vertex that is different, but that's okay.) Name the vertices in these two paths $(v_0, v_1, v_2, \ldots, v_n)$ and $(w_0, w_1, w_2, \ldots, w_m)$, where $v_0 = w_0$ and $v_n = w_n$, and where all of the other vertices are distinct. (Note that v_0 and v_n are different because you can't have two distinct paths between a vertex and itself.)

The trail $(v_0, v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_m)$ is a cycle because $v_0 = w_m$ are the only vertices that are the same and there's at least one edge in this trail because $v_0 \neq v_n$. But T is a tree, so there can't be any cycles in it.

Therefore there cannot exist more than one path from a to b. $\square$

Give a similar proof of the converse claim:

Claim. For any graph G, if every pair of vertices in G has one and only one path between them, then G is a tree.

9. Give a proof of the following claim, using induction on the number of vertices in the graph:

   Claim. For any tree T, if you remove one edge from the tree, the resulting graph will not be connected.