HW6 (CSCI-C241)

Lillie Donato

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1. Question One

   (a) Claim: $2x < 2y$ if $x + 5 < y$

   Proof.

   1. Choose two real numbers x and y such that $x + 5 < y$
   2. Since $x + 5 < y$, we know $2x + 10 < 2y$
   3. Because $2y > 2x + 10$, we know $2y > 2x$, for the reason that $10 + 2x > 2x$

   $\square$

   (b) Claim: $\{x \mid \frac{x+1}{2} \geq 4\} \subseteq \{x \mid 2x - 2 > 10\}$

   Proof.

   1. Choose a real number x such that $\dfrac{x+1}{2} \geq 4$
   2. Since $\dfrac{x+1}{2} \geq 4$, we know $x + 1 \geq 8$
   3. Since $x + 1 \geq 8$, we know $x \geq 7$
   4. Since $x \geq 7$, we know $x > 6$
   5. Since $x > 6$, we know $2x > 12$
   6. Since $2x > 12$, we know $2x - 2 > 10$
   7. Since $2x - 2 > 10$, we know $x \in \{x \mid 2x - 2 > 10\}$
   8. Since all the members of $\{x \mid \dfrac{x+1}{2} \geq 4\}$ are members of $\{x \mid 2x - 2 > 10\}$, we know $\{x \mid \dfrac{x+1}{2} \geq 4\} \subseteq \{x \mid 2x - 2 > 10\}$

   $\square$

   (c) Claim: $A \cap B \subseteq (A \cup C) \cap D$

   Proof.

   1. Choose the sets A, B, C, D
   2. Since $B \cup C \subseteq D$, we know $B \subseteq D$
   3. Since A, we know $A \subseteq A$
   4. Since $A \subseteq A$, we know $A \subseteq (A \cup C)$
   4. Since $B \cup C \subseteq D$, we know $C \subseteq D$
   5. Since $A \subseteq (A \cup C)$ and $B \subseteq D$, we know $A \cap B \subseteq (A \cup C) \cap D$

   $\square$

   (d) Claim: $X \subseteq \overline{Y \cap Z}$

Proof.

1. Choose the sets X, Y, Z
2. Choose towards a contradiction $X \subseteq Y \cap Z$
3. Since $(X \cap Y) \subseteq \overline{Z}$, we know $X \subseteq \overline{Z}$
4. Since $X \subseteq \overline{Z}$, we know $X \nsubseteq Z$
5. Since $X \nsubseteq Z$, we know $X \nsubseteq Y \cap Z$
6. Since $X \nsubseteq Y \cap Z$, we know $X \subseteq \overline{Y \cap Z}$

□

(e) Claim: $H \cup (J \cap K) \subseteq (H \cup K) \setminus M$

Proof.

Choose the sets H, J, K, L, M (1)
Case 1: H (2)
Since $H \subseteq \overline{M}$, we know $H \nsubseteq M$ (3)
Since $H \subseteq H$, we know $H \subseteq (H \cup K)$ (4)
Since $H \nsubseteq M$ and $H \subseteq (H \cup K)$, we know $H \subseteq (H \cup K) \setminus M$ (5)
Case 2: $J \cap K$ (6)
Since $J \subseteq L \setminus M$, we know $J \subseteq \overline{M}$ (7)
Since $J \subseteq \overline{M}$, we know $J \nsubseteq M$ (8)
Since $K \subseteq K$, we know $J \cap K \subseteq K$ (9)
Since $J \cap K \subseteq K$, we know $J \cap K \subseteq (H \cup K)$ (10)
Since $J \cap K \subseteq (H \cup K)$ and $J \nsubseteq M$, we know $J \cap K \subseteq (H \cup K) \setminus M$ (11)
In either case of $H \cup (J \cap K)$, we proved that they were a subset of $(H \cup K) \setminus M$, (12)
so $H \cup (J \cap K) \subseteq (H \cup K) \setminus M$ (13)

□

2. Claim: $A \cup B \subseteq B \cap C$ if $A \subseteq C$

The Claim is proven $false$, with the following counter example:

$A = \{1, 2, 3\}$
$B = \{4, 5, 6\}$
$C = \{-3, -2, -1, 0, 1, 2, 3\}$

3. Question Three

(a) $S_n = \{nx \mid x \in \mathbb{Z}\}$
Claim: if $a, b \in S_n$ then $5a - b \in S_n$

Proof.

1. Choose $a, b \in S_n$
2. Since $a \in S_n$, we know $a = nx$, for some $x \in \mathbb{Z}$
3. Since $b \in S_n$, we know $b = ny$, for some $y \in \mathbb{Z}$
4. Since $5a - b = 5(nx) - ny$, we know $5nx - ny$
5. Since $5nx - ny$, we know $n(5x - y)$
6. Let $z = 5x - y$
7. Since x, y are both integers, we know z must be an integer as well

8 Since x, z are both integers, we can write it as nz

9. Because of this, we know $5a - b \in S_n$

□

(b) Claim: The sum of any two rational numbers is rational

Proof.

1. Choose two rational numbers x, y
2. Since x is rational, there exist integers p_1, q_1 where $x = \dfrac{p_1}{q_1}$ and $q_1 \neq 0$
3. Since y is rational, there exist integers p_2, q_2 where $x = \dfrac{p_2}{q_2}$ and $q_2 \neq 0$
4. Since $x + y = \dfrac{p_1}{q_1} + \dfrac{p_2}{q_2}$, we know $x + y = \dfrac{p_1 q_2 + p_2 q_1}{q_1 q_2}$
5. Let $R = p_1 q_2 + p_2 q_1$ and $S = q_1 q_2$
6. Since p_1 and p_2 are integers, and q_1 and q_2 are non-zero integers, we know that R is an integer and S is a non-zero integer
7. Since R is an integer and S is a non-zero integer, we know that $x + y$ is rational

□

(c) $T = \{x + y\sqrt{2} \mid x \in \mathbb{Q} \wedge y \in \mathbb{Q}\}$
$S = \{st \mid s \in T \wedge t \in T\}$
Claim: S = T

Proof.

Choose $s \in T$ and $t \in T$ (1)
Since $s \in T$, we know $s = x_1 + y_1\sqrt{2}$, where $x_1, y_1 \in \mathbb{Q}$ (2)
Since $t \in T$, we know $t = x_2 + y_2\sqrt{2}$, where $x_2, y_2 \in \mathbb{Q}$ (3)
Since $st = (x_1 + y_1\sqrt{2})(x_2 + y_2\sqrt{2})$, we know $st = x_1x_2 + x_1y_2\sqrt{2} + x_2y_1\sqrt{2} + 2y_1y_2$ (4)
Since st, we know $st = (x_1x_2 + 2y_1y_2) + \sqrt{2}(x_1y_2 + x_2y_1)$ (5)
Let $a = x_1x_2 + 2y_1y_2$ and $b = x_1y_2 + x_2y_1$ (6)
Since $x_1, x_2, y_1, y_2 \in \mathbb{Q}$, we know $a, b \in \mathbb{Q}$ (7)
Since $st = a + b\sqrt{2}$ and $a, b \in \mathbb{Q}$, we know $S \subseteq T$ (8)
Choose $t \in T$ where $t = x + y\sqrt{2}$, and $x, y \in \mathbb{Q}$ (9)
Since $t \in T$, we know $t = 1 \cdot t$, where $1 \in \mathbb{Q}$ (10)
Since $t = 1 \cdot t$, we know $t \in S$, so $T \subseteq S$ (11)
Since $S \subseteq T$ and $T \subseteq S$, we know $S = T$ (12)

□

(d) The sum of an irrational number and a rational number is irrational.

Proof.

$$\text{Choose an irrational number } x \text{ and a rational number } y \tag{1}$$
$$\text{Since } x \text{ is irrational, we know } x \neq \frac{p}{q}, \text{ where } p, q \in \mathbb{Z} \tag{2}$$
$$\text{Since } y \text{ is rational, we know } y = \frac{p_1}{q_1}, \text{ where } p_1, q_1 \in \mathbb{Z} \tag{3}$$
$$\text{Let } z = x + y \tag{4}$$
$$\text{Assume towards a contradiction that } z \text{ is rational} \tag{5}$$
$$\text{Since } z \text{ is rational, we know } z = \frac{p_2}{q_2}, \text{ where } p_2, q_2 \in \mathbb{Z} \tag{6}$$
$$\text{We can rewrite } x + y = z \text{ as } x + \frac{p_1}{q_1} = \frac{p_2}{q_2} \tag{7}$$
$$\text{Since } x + \frac{p_1}{q_1} = \frac{p_2}{q_2}, \text{ we know } x = \frac{p_2}{q_2} - \frac{p_1}{q_1} \tag{8}$$
$$\text{Since } x = \frac{p_2 q_1 - p_1 q_2}{q_1 q_2}, \text{ we know } x = \frac{p_1 q_2 - p_2 q_1}{q_1 q_2} \tag{9}$$
$$\text{Since } x = \frac{p_1 q_2 - p_2 q_1}{q_1 q_2}, \text{ we know } x \text{ is rational} \tag{10}$$
$$\text{This is a contradiction to our earlier claim, so } z \text{ must be irrational} \tag{11}$$

□

(e) $S_n = \{nx \mid x \in \mathbb{Z}\}$ Claim: if $ab \notin S_n$ then $a, b \notin S_n$

Proof.

$$\text{Choose } a, b, n \in S_n \text{ and } ab \text{ where } ab \notin S_n \tag{1}$$
$$\text{Assume towards a contradiction that } a \in S_n \tag{2}$$
$$\text{Since } a \in S_n, \text{ we know } a = nx, \text{ where } x \in \mathbb{Z} \tag{3}$$
$$\text{Since } a = nx, \text{ we know } x = \frac{a}{n} \tag{4}$$
$$\text{Since } x \in \mathbb{Z}, \text{ we know } \frac{a}{n} \in \mathbb{Z} \tag{5}$$
$$\text{Since } \frac{a}{n} \in \mathbb{Z} \text{ and } b \in \mathbb{Z}, \text{ we know } \frac{ab}{n} \in \mathbb{Z} \tag{6}$$
$$\text{Since } \frac{ab}{n} \in \mathbb{Z}, \text{ we know } \frac{ab}{n} \cdot n \in S_n \tag{7}$$
$$\text{Since } \frac{ab}{n} \cdot n \in S_n, \text{ we know } ab \in S_n \tag{8}$$
$$\text{This is a contradiction to our earlier claim, so } a \notin S_n \tag{9}$$
$$\text{Assume towards a contradiction that } b \in S_n \tag{10}$$
$$\text{Since } b \in S_n, \text{ we know } b = nx, \text{ where } x \in \mathbb{Z} \tag{11}$$
$$\text{Since } b = nx, \text{ we know } x = \frac{b}{n} \tag{12}$$
$$\text{Since } x \in \mathbb{Z}, \text{ we know } \frac{b}{n} \in \mathbb{Z} \tag{13}$$
$$\text{Since } \frac{b}{n} \in \mathbb{Z} \text{ and } b \in \mathbb{Z}, \text{ we know } \frac{ab}{n} \in \mathbb{Z} \tag{14}$$
$$\text{Since } \frac{ab}{n} \in \mathbb{Z}, \text{ we know } \frac{ab}{n} \cdot n \in S_n \tag{15}$$
$$\text{Since } \frac{ab}{n} \cdot n \in S_n, \text{ we know } ab \in S_n \tag{16}$$
$$\text{This is a contradiction to our earlier claim, so } b \notin S_n \tag{17}$$

□