HW6 Corrections (CSCI-C241)

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- Question One
  - 1d
    Claim: $X \subseteq \overline{Y \cap Z}$

    Proof.

    Choose the sets X, Y, Z and $x \in X$ (1)

    Assume $(X \cap Y) \subseteq \overline{Z}$ (2)

    Suppose towards a contradiction $x \notin \overline{Y \cap Z}$ (3)

    Since $x \notin \overline{Y \cap Z}$, we know $x \in Y \cap Z$ (4)

    Since $x \in Y \cap Z$, we know $x \in Y$ and $x \in Z$ (5)

    Since $x \in X$ and $x \in Y$, we know $x \in X \cap Y$ (6)

    Since $x \in X \cap Y$, we know $x \in \overline{Z}$ (7)

    Since $x \in \overline{Z}$, we know $x \notin Z$ (8)

    Under the assumption of $x \notin \overline{Y \cap Z}$, we proved an impossibility of $x \in Z$ and $x \notin Z$, (9)
    therefore $x \in \overline{Y \cap Z}$

    Since $x \in X$ and $x \in \overline{Y \cap Z}$, we know $X \subseteq \overline{Y \cap Z}$ (10)

    (11)

    □

  - 1e
    Claim: $H \cup (J \cap K) \subseteq (H \cup K) \setminus M$

    Proof.

    Choose the sets H, J, K, L, M and $x \in H \cup (J \cap K)$ (1)

    Assume $J \subseteq L \setminus M$ and $H \subseteq \overline{M}$ (2)

    Case 1: $x \in H$ (3)

    Since $x \in H$ and $H \subseteq \overline{M}$, we know $x \in \overline{M}$ (4)

    Since $x \in \overline{M}$, we know $x \notin M$ (5)

    Since $x \in H$, we know $x \in H \cup K$ (6)

    Since $x \in H \cup K$ and $x \notin M$, we know $x \in (H \cup K) \setminus M$ (7)

    Case 2: $x \in J \cap K$ (8)

    Since $x \in J \cap K$, we know $x \in J$ and $x \in K$ (9)

    Since $x \in J$ and $J \subseteq L \setminus M$, we know $x \in L \setminus M$ (10)

    Since $x \in L \setminus M$, we know $x \in L$ and $x \notin M$ (11)

    Since $x \in K$, we know $x \in H \cup K$ (12)

    Since $x \in H \cup K$ and $x \notin M$, we know $x \in (H \cup K) \setminus M$ (13)

    In either case of $H \cup (J \cap K)$, we proved that $x \in (H \cup K) \setminus M$ (14)

    Under the assumption of $J \subseteq L \setminus M$ and $H \subseteq \overline{M}$, (15)
    we proved that any member of $H \cup (J \cap K)$ is a member of $(H \cup K) \setminus M$,
    therefore $H \cup (J \cap K) \subseteq (H \cup K) \setminus M$

$\square$

- Question Three
  - 3c
    $T = \{x + y\sqrt{2} \mid x \in \mathbb{Q} \land y \in \mathbb{Q}\}$
    $S = \{st \mid s \in T \land t \in T\}$
    Claim: S = T

    Proof.

    $$\begin{aligned}
    &\text{Choose } s \in T \text{ and } t \in T && (1)\\
    &\quad \text{Since } s \in T\text{, we know } s = x_1 + y_1\sqrt{2}\text{, where } x_1, y_1 \in \mathbb{Q} && (2)\\
    &\quad \text{Since } t \in T\text{, we know } t = x_2 + y_2\sqrt{2}\text{, where } x_2, y_2 \in \mathbb{Q} && (3)\\
    &\quad \text{Since } st = (x_1 + y_1\sqrt{2})(x_2 + y_2\sqrt{2})\text{, we know } st = x_1x_2 + x_1y_2\sqrt{2} + x_2y_1\sqrt{2} + 2y_1y_2 && (4)\\
    &\quad \text{Since } st\text{, we know } st = (x_1x_2 + 2y_1y_2) + \sqrt{2}(x_1y_2 + x_2y_1) && (5)\\
    &\quad \text{Let } a = x_1x_2 + 2y_1y_2 \text{ and } b = x_1y_2 + x_2y_1 && (6)\\
    &\quad \text{Since } x_1, x_2, y_1, y_2 \in \mathbb{Q}\text{, we know } a, b \in \mathbb{Q} && (7)\\
    &\quad \text{Since } st = a + b\sqrt{2} \text{ and } a, b \in \mathbb{Q}\text{, we know } S \subseteq T && (8)\\
    &\text{Choose } t \in T \text{ where } t = x + y\sqrt{2}\text{, and } x, y \in \mathbb{Q} && (9)\\
    &\quad \text{Since } 0, 1 \in \mathbb{Q}\text{, } 0\sqrt{2} = 0 \text{ and } 1 + 0 = 1\text{, we know } 1 \in T && (10)\\
    &\quad \text{Since } t, 1 \in T \text{ and } t = 1 \cdot t\text{, we know } t \in S && (11)\\
    &\quad \text{Since } t \in S\text{, we know } T \subseteq S && (12)\\
    &\text{Since } S \subseteq T \text{ and } T \subseteq S\text{, we know } S = T && (13)
    \end{aligned}$$

    $\square$

  - 3d
    Claim: The sum of an irrational number and a rational number is irrational.

    Proof.

    $$\begin{aligned}
    &\text{Choose an irrational number } x \text{ and a rational number } y && (1)\\
    &\quad \text{Since } y \text{ is rational, we know } y = \frac{p_1}{q_1}\text{, where } p_1, q_1 \in \mathbb{Z} \text{ and } q_1 \neq 0 && (2)\\
    &\quad \text{Let } z = x + y && (3)\\
    &\quad \text{Assume towards a contradiction that } z \text{ is rational} && (4)\\
    &\qquad \text{Since } z \text{ is rational, we know } z = \frac{p_2}{q_2}\text{, where } p_2, q_2 \in \mathbb{Z} \text{ and } q_2 \neq 0 && (5)\\
    &\qquad \text{We can rewrite } x + y = z \text{ as } x + \frac{p_1}{q_1} = \frac{p_2}{q_2} && (6)\\
    &\qquad \text{Since } x + \frac{p_1}{q_1} = \frac{p_2}{q_2}\text{, we know } x = \frac{p_2}{q_2} - \frac{p_1}{q_1} && (7)\\
    &\qquad \text{Since } x = \frac{p_2}{q_2} - \frac{p_1}{q_1}\text{, we know } x = \frac{p_2q_1 - p_1q_2}{q_1q_2} && (8)\\
    &\qquad \text{Since } q_1 \neq 0 \text{ and } q_2 \neq 0\text{, we know } q_1q_2 \neq 0 && (9)\\
    &\qquad \text{Since } x = \frac{p_1q_2 - p_2q_1}{q_1q_2}\text{, we know } x = \frac{p}{q} \text{ where } p, q \in \mathbb{Z} \text{ and } q \neq 0\text{,} && (10)\\
    &\qquad \text{meaning } x \text{ is rational} &&\\
    &\quad \text{This is a contradiction to our earlier claim, so } z \text{ must be irrational,} && (11)\\
    &\quad \text{meaning the sum of a irrational number and a rational number is irrational} &&
    \end{aligned}$$

    $\square$