

# HW9 (CSCI-C241)

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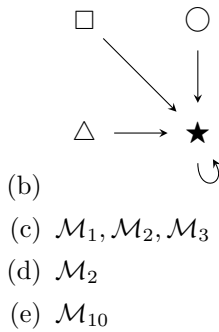
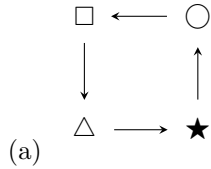
## 1. Question One

- (a)  $\forall x((A(x) \vee B(x)) \rightarrow (C(x) \wedge \neg D(x)))$
- (b)  $\neg \exists x(A(x) \wedge B(x))$

## 2. Question Two

- (a)  $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_5, \mathcal{U}_6, \mathcal{U}_7$
- (b)  $\mathcal{U}_1, \mathcal{U}_6, \mathcal{U}_7, \mathcal{U}_8$
- (c)  $\mathcal{U}_7$
- (d)  $\mathcal{U}_3, \mathcal{U}_4, \mathcal{U}_8$
- (e)  $\mathcal{U}_2, \mathcal{U}_3, \mathcal{U}_4, \mathcal{U}_5$
- (f)  $\mathcal{U}_2, \mathcal{U}_3, \mathcal{U}_4, \mathcal{U}_5$
- (g)  $\mathcal{U}_3, \mathcal{U}_4, \mathcal{U}_8$
- (h) Part (h)
  - i.  $\neg \forall x(L(x) \rightarrow V(x))$  and  $\exists x(L(x) \wedge \neg V(x))$
  - ii.  $\forall x(L(x) \rightarrow \neg V(x))$  and  $\neg \exists x(L(x) \wedge V(x))$

## 3. Question Three



- (c)  $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$
- (d)  $\mathcal{M}_2$
- (e)  $\mathcal{M}_{10}$

## 4. Question Four

- (a)  $\forall x \forall y \forall z (\exists w R(x, w) \wedge ((R(x, y) \wedge R(x, z)) \rightarrow y = z))$

## 5. Question Five

- (a)  $\forall x(A(x) \rightarrow ((\exists y(B(y) \wedge R(x, y))) \wedge (\forall y \forall z((B(y) \wedge B(z)) \rightarrow ((R(x, y) \wedge R(x, z)) \rightarrow y = z))))$

## 6. Question Six

- (a)  $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$
- (b)  $n - 1$

- (c)  $n + 1$
- (d)  $n$
- (e)  $(k + 1) \cdot k!$

7. Question Seven

- (a) Claim:  $n \geq 2, 3^n > 2^{n+1}$

*Proof.* (induction on  $n$ )  
 (Base Step,  $n = 2$ ):

$$3^2 = 9 \quad (1)$$

$$> 8 \quad (2)$$

$$= 2^3 \quad (3)$$

$$= 2^{2+1} \quad (4)$$

(Inductive Step):  
 Assume  $3^k > 2^{k+1}$  for some  $k \geq 2$

$$3^{k+1} = 3^k \cdot 3^1 \quad (1)$$

$$= 3^k \cdot 3 \quad (2)$$

$$> 2^{k+1} \cdot 2 \quad (\text{by the induction hypothesis, and } 3 > 2) \quad (3)$$

$$= 2^{k+1+1} \quad (4)$$

□

- (b) Claim:  $n \geq 9, 3^n < (n - 1)!$

*Proof.* (induction on  $n$ )  
 (Base Step,  $n = 9$ ):

$$3^9 = 19683 \quad (1)$$

$$< 40320 \quad (2)$$

$$= 8! \quad (3)$$

$$= (9 - 1)! \quad (4)$$

(Inductive Step):  
 Assume  $3^k < (k - 1)!$  for some  $k \geq 9$

$$3^{k+1} = 3^k \cdot 3 \quad (1)$$

$$< (k - 1)! \cdot k \quad (\text{by the induction hypothesis, and } k > 3) \quad (2)$$

$$= k! \quad (3)$$

$$= (k - 1 + 1)! \quad (4)$$

□

- (c) Claim:  $n \geq 2, 3^n > n^2$

*Proof.* (induction on  $n$ )  
 (Base Step,  $n = 2$ ):

$$3^2 = 9 \quad (1)$$

$$> 4 \quad (2)$$

$$= 2^2 \quad (3)$$

(Inductive Step):

Assume  $3^k > k^2$  for some  $k \geq 2$

$$3^{k+1} = 3^k \cdot 3 \quad (1)$$

$$> k^2 \cdot 3 \quad (\text{by the induction hypothesis}) \quad (2)$$

$$= k^2 + k^2 + k^2 \quad (3)$$

$$\geq k^2 + 2k + 1 \quad (k^2 \geq 2k > 1 \text{ for all } k \geq 2) \quad (4)$$

$$= (k+1)(k+1) \quad (5)$$

$$= (k+1)^2 \quad (6)$$

□

(d) Claim:  $n^2 - 3n$  is even for all  $n \in \mathbb{N}$

*Proof.* (induction on  $n$ )

(Base Step,  $n = 0$ ):

$$0^2 - 3(0) = 0 - 0 \quad (1)$$

$$= 0 \quad (2)$$

0 is even

(Inductive Step):

Assume  $k^2 - 3k$  is even for all  $k \in \mathbb{N}$

$$k^2 - 3k = 2c_1$$

$$(k+1)^2 - 3(k+1) = (k+1)(k+1) - 3k - 3 \quad (1)$$

$$= k^2 + 2k + 1 - 3k - 3 \quad (2)$$

$$= (k^2 - 3k) + (2k - 2) \quad (3)$$

$$= (k^2 - 3k) + 2(k - 1) \quad (4)$$

$$= 2c_1 + 2(k - 1) \quad (\text{by the induction hypothesis}) \quad (5)$$

$$= 2c_1 + 2c_2 \quad (\text{Let } c_2 = k - 1) \quad (6)$$

$$= 2(c_1 + c_2) \quad (7)$$

$$= 2c_3 \quad (\text{Let } c_3 = c_1 + c_2) \quad (8)$$

$$2c_3 \text{ is even} \quad (2n \text{ is even, where } n \in \mathbb{N}) \quad (9)$$

$$(10)$$

Since  $2c_3$  is even,  $(k+1)^2 - 3(k+1)$  is even

□

(e) Claim: There are  $2^n$  binary string of length  $n$  for all  $n \in \mathbb{N}$

*Proof.* (induction on  $n$ )

(Base Step,  $n = 0$ ):

$$2^0 = 1 \quad (1)$$

There is only one possible binary string, of length 0, the empty string.

(Inductive Step):

Assume for length  $k$ , there are  $2^k$  binary strings

For every binary string of length  $k$ , there are  $2^k$  binary strings, (1)

for each binary string, there are two new possible binary strings of length  $k+1$ , (2)

So, by the induction hypothesis there are  $2^k \cdot 2$  binary strings of length  $k+1$  (3)

□

(f) Claim: For any set of characters, with the length of  $a$  and any  $n \in \mathbb{N}$ , there are  $a^n$  possible strings of length  $n$  that explicitly use the said alphabet

*Proof.* (induction on  $n$ )

(Base Step,  $n = 0$ ):

$$a^0 = 1 \tag{1}$$

Regardless of the possible characters, there is only one possible string, of length 0, the empty string.

(Inductive Step):

Assume for length  $k$ , there are  $a^k$  strings, where  $a$  is total amount of different characters

For every string of length  $k$ , there are  $a$  possible different characters, (1)

and  $a^k$  possible strings (2)

For every string of length  $k + 1$ , with the same  $a$  possible characters, (3)

for each string, there would  $a$  possible new strings of length  $k + 1$  (4)

So, by the induction hypothesis there are  $a^k \cdot a$  strings of length  $k + 1$  (5)

□