CSCI-C 241 HW #7
Instructors: Wennstrom & LaRacuente Due Tuesday, March 5th, 2024

1. Let $A = \{1, 2, 3\}$. Write $A \times A$ in set-list notation.

2. Define the relation R from the set $B = \{1, 2, 3, 3, 4, 5, 6, 7\}$ to the set $C = \{\texttt{"a"}, \texttt{"b"}, \texttt{"c"}\}$ as follows: $R = \{(1, \texttt{"b"}), (2, \texttt{"a"}), (2, \texttt{"b"}), (2, \texttt{"c"}), (3, \texttt{"a"}), (5, \texttt{"b"}), (5, \texttt{"c"}), (6, \texttt{"a"}), (7, \texttt{"c"})\}$.

   For these first few parts, answer true or false. No justification is needed.

   (a) $(2, \texttt{"b"}) \in R$
   (b) $(\texttt{"a"}, 6) \in B \times C$
   (c) $(3, \texttt{"c"}) \in B \times C$
   (d) $R(5, \texttt{"b"})$
   (e) $R(\texttt{"b"}, 1)$
   (f) $R(4, \texttt{"c"})$
   (g) $6\ R\ \texttt{"c"}$
   (h) $6\ R\ \texttt{"a"}$
   (i) R relates 7 to $\texttt{"c"}$
   (j) R relates 7 to $\texttt{"a"}$

   Calculate the following numbers.

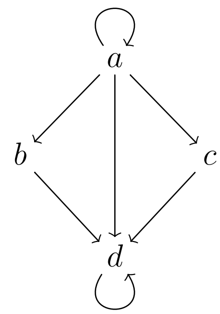
   (k) $|B \times C| = ?$
   (l) $|R| = ?$
   (m) What is the domain of R?
   (n) What is the codomain of R?
   (o) What is the smallest possible relation you can have from B to C?
   (p) What is the biggest possible relation you can have from B to C?

3. For each relation, answer the following questions:

   i Is the relation reflexive?
   ii Is the relation anti-reflexive?
   iii Is the relation symmetric?
   iv Is the relation anti-symmetric?
   v Is the relation transitive?

   If you answer "yes", then you must give a brief explanation why. If you answer "no", then you must give a counterexample that proves that the property does not hold.

   (a) The relation R on $\{a, b, c, d\}$ defined by the following directed graph:



(b) The relation S on $\{1, 2, 3, 4\}$ given by:

$$S = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (2,4), (3,1), (3,3)\}$$

(c) The relation T on $\{a, b, c, d\}$ given by:

$$T = \{(a,b), (a,c), (a,d), (b,c), (c,d), (d,c), (d,d)\}$$

(d) The relation Q on $\mathbb{Z}$ defined by $Q = \{(n, m) \mid n + m \text{ is odd}\}$.

(e) The relation C on the set of text strings, where:

$$C = \{(s, t) \mid s \text{ is a proper substring of } t\}$$

(A string is a **substring** of another string when it appears inside of the other string. So for example, `"shear"` is a substring of `"misheard"`. A **proper** substring is just a substring that isn't equal to the entire string. So `"foo"` is a substring of `"foo"`, but it is not a *proper* substring. But `"foo"` *is* a proper substring of `"foobar"`.)

(f) The relation A on the set of all formulas of propositional logic defined by:

$$A = \{(p, q) \mid p \equiv \neg q\}$$

So for example, $A(P \leftrightarrow Q, P \oplus Q)$, which is true because $P \leftrightarrow Q \equiv \neg(P \oplus Q)$, and also $A\big(\neg(P \wedge Q), P \wedge Q\big)$, which is true because $\neg(P \wedge Q) \equiv \neg(P \wedge Q)$.

4. Consider the relation X on $\mathcal{P}(\mathbb{N})$ defined by:

$$X = \{(A, B) \mid A \text{ and } B \text{ have at least one member in common}\}$$

Keep in mind that the domain is not numbers, but *sets of numbers*. So for example, $X\big(\{1,2,3\}, \{2,4,6\}\big)$ is true because $\{1,2,3\}$ and $\{2,4,6\}$ share the element 2. Similarly $X\big(\{1,2,3\}, \{1,2,3,4,5\}\big)$ is also true, but $X\big(\{1,2,3\}, \{4,5,6\}\big)$ is false.

(a) Is X reflexive, antireflexive, both, or neither? Justify your answer with a brief explanation, including counterexamples where appropriate.

(b) Is X symmetric, antisymmetric, both, or neither? Justify your answer with a brief explanation, including counterexamples where appropriate.

(c) Is X transitive? Justify your answer with a brief explanation, including counterexamples where appropriate.

5. Let M be a relation on the natural numbers $\mathbb{N}$ defined by $M = \{(m, n) \mid m \text{ is a multiple of } n\}$. Recall that a number x is a **multiple** of y iff there exists an integer k with $x = yk$.

   Answer these questions.

   (a) What is the domain of M?

   (b) What is the codomain of M?

   For the next part of this problem, you'll be proving that M is a **partial order**. A **partial order** is any relation that is reflexive, *anti*symmetric, and transitive.

   (c) Explain why M is *reflexive*. You don't need to provide a full proof.

   (d) Give an informal proof that M is *anti-symmetric*.

   You may use the following fact: for any two natural numbers m and n, if $m \cdot n = 1$, then $m = 1$ and $n = 1$.

   (Note that M is only anti-symmetric because the domain is the *natural numbers*. If we switched to the domain of *integers*, then things would be completely different. So if your proof doesn't take this into account, you've missed something.)

   (e) Give an informal proof that M is *transitive*.

6. Consider the relation E_5 on $\mathbb{Z}$ defined as follows: $E_5 = \{(x, y) \mid x - y \text{ is a multiple of } 5\}$.

   Answer true or false for the following statements. No justification needed.

   (a) $E_5(25, 40)$

   (b) $(15, 39) \in E_5$

   (c) E_5 relates 23 to 18.

   (d) $4 \; E_5 \; 9$

   (e) $E_5(12, -3)$

   (f) $E_5(23, 23)$

   For the rest of this problem, you'll be proving that E_5 is an **equivalence relation**. An **equivalence relation** is any relation that is reflexive, symmetric, and transitive.

   (g) Start by giving an informal proof that E_5 is *reflexive*. This will be a pretty short proof.

   (h) Next, give an informal proof that E_5 is *symmetric*.

   (i) Finally, give an informal proof that E_5 is *transitive*.

7. A relation R from a set A to a set B has property F if and only if for every $x \in A$ and every $y, z \in B$, if $R(x, y)$ and $R(x, z)$, then $y = z$. I just made up this property definition; you aren't expected to already know about it.

(a) Consider the relation Q from $\{a, b, c\}$ to $\{1, 2, 3, 4\}$ given by:

$$Q = \{(a, 1), (a, 2), (b, 3), (c, 4)\}$$

Does Q have property F? Justify your answer with a brief explanation, including counterexamples if appropriate.

(b) Consider the relation L from Str (the set of text strings) to $\mathbb{N}$, given by $L = \{(s, n) : |s| = n\}$.

Does L have property F? Justify your answer with a brief explanation, including counterexamples if appropriate.

(c) Consider the relation I from $\mathbb{R}$ to $\mathbb{R}$ given by $I = \{(x, y) \mid x \cdot y = 1\}$.

Prove that I has property F.

8. Give an example of a relation that fits the given requirements. If you think that there is no such relation, you must explain why. In each case, you may *not* use any of the relations that have appeared earlier on this assignment.

(a) A relation on $\{a, b, c, d\}$ that is not reflexive, not antireflexive, and not transitive. You must use set-list notation for your answer.

(b) A relation on $\{a, b, c, d\}$ that is not symmetric and not anti-symmetric. You must use a directed graph to define this relation.

(c) A relation on Str (the set of all text strings) that is symmetric, but not transitive. You must use set-builder notation.

(d) An anti-symmetric relation on Str. You must use set-builder notation.

(e) An infinite relation on $\mathbb{Z}$ that is not reflexive and not anti-reflexive. (The requirement that the relation be infinite means you'll have to use set-builder notation to define this relation.)

(f) A relation on Prop (the set of all formulas of propositional logic) that is reflexive, symmetric, and transitive. (Relations that have all three of these properties are called **equivalence relations**.)

The fact that the relation has to be reflexive and that the domain is infinite means that the relation will have to be infinite, and therefore you'll have to use set-builder notation to define this relation.

(g) An infinite relation on Prop that is *anti*-reflexive, *anti*-symmetric, and transitive. You must use set-builder notation to define this relation. (The requirement that the relation be infinite means you'll have to use set-builder notation to define this relation.)

(h) **Bonus:** A relation on $\{p \mid p \text{ is a student at IU}\}$ that is reflexive, anti-symmetric, and transitive. (Such a relation is called a **partial order**.)