Module 2 Notes (MATH-211)

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General Notes (and Definitions)

- Derivatives
  - A **derivative** is a new function made up of the slopes of the tangent lines as they change along a curve
  - If a curve represents the trajectory of a moving object, the tangent line at a point indicates the direction of motion at that point
  - As $x \to a$, the slope of the secant lines approaches the slope of the tangent line
  - Alternative definition for Tangent Line(s): Consider the curve $y = f(x)$ and a secant line intersecting the curve at points $P(a, f(a))$ and $Q(a+h, f(a+h))$, with m_{sec} and m_{tan}

$$\text{Interval:}(a, a+h)$$
$$m_{sec} = \frac{f(a+h) - f(a)}{h}$$
$$m_{tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$
$$y - f(a) = m_{tan}(x - a)$$

  - **Definition**: The derivative of f at a, denoted $f'(a)$, is given by either the two following limits, provided the limits exist and a is in the domain of f

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad (1)$$
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad (2)$$

    If $f'(a)$ exists, we say that f is **differentiable** at a

- Derivatives as Functions
  - The slope of the tangent line of somefunction f is a function called the derivative of f

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

  - If $f'(x)$ exists, we say that f is **differentiable** at x
  - If f is differentiable at every point in some open interval I, we say that f is differentiable on I
  - For some function f we can denote the derivative of f like such:

$$f'(x) \quad (1)$$
$$\frac{dy}{dx} \quad (2)$$
$$\frac{df}{dx} \quad (3)$$
$$\frac{d}{dx}(f(x)) \quad (4)$$
$$D_x(f(x)) \quad (5)$$
$$y'(x) \quad (6)$$

- When evaluating some derivative f at a, we can use the following:

$$f'(a) \tag{1}$$
$$y'(a) \tag{2}$$
$$\left.\frac{df}{dx}\right|_{x=a} \tag{3}$$
$$\left.\frac{dy}{dx}\right|_{x=a} \tag{4}$$

- If f is differentiable at a, then f is continuous at a
- If f is not continuous at a, then f is not differentiable at a

Rules of Differentiation

- Constant Rule

$$\text{If } c \in \mathbb{R}, \text{ then } \frac{d}{dx}(c) = 0$$

- Power Rule

$$\text{If } n \in \mathbb{Z} \text{ and } n > 0, \text{ then } \frac{d}{dx}(x^n) = nx^{n-1}$$

- Derivative of a Root

$$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

- Constant Multiple Rule

$$\text{If } f \text{ is differentiable at } x \text{ and } c \text{ is a constant, then } \frac{d}{dx}(cf(x)) = cf'(x)$$

- Sum Rule

$$\text{If } f \text{ and } g \text{ are differentiable at } x, \text{ then } \frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$$

- Generalized Sum Rule

$$\frac{d}{dx}(f_1(x) + f_2(x) + ... + f_x(x)) = f_1'(x) + f_2'(x) + ... + f_n'(x)$$

- Difference Rule

$$\frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x)$$

- Euler's Number

$$\text{The function } f(x) = e^x \text{ is differentiable for all } x \in \mathbb{R}, \text{ and } \frac{d}{dx}(e^x) = e^x$$

- Higher-order Derivatives
  Assuming $y = f(x)$ can be differentiated as often as necessary, the **second derivative** of f is

$$f''(x) = \frac{d}{dx}(f'(x))$$

  For $n \in \mathbb{Z}$ where $n \geq 1$, the **nth derivative** of f is

$$f^{(n)}(x) = \frac{d}{dx}\left(f^{(n-1)}(x)\right)$$

- Product Rule

$$\text{If } f \text{ and } g \text{ are differentiable at } x, \text{ then } \frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

- Quotient Rule

  If f and g are differentiable at x and $g(x) \neq 0$, then the derivative of $\frac{f}{g}$ at x exists and

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

- Power Rule for Negative Integers

  Proof.

  Choose $n \in \mathbb{Z}$
  Assume that $n < 0$
  Let $m = -n$
  Since $m = -n$ and $n < 0$, we know $m > 0$

$$\begin{aligned}
\frac{d}{dx}(x^n) &= \frac{d}{dx}\left(\frac{1}{x^m}\right) && (1)\\
&= \frac{x^m \cdot 0 - 1 \cdot (mx^{m-1})}{(x^m)^2} && (2)\\
&= \frac{-mx^{m-1}}{x^{2m}} && (3)\\
&= -mx^{(m-1)-2m} && (4)\\
&= -mx^{-m-1} && (5)\\
&= nx^{n-1} && (6)\\
& && (7)
\end{aligned}$$

  Under the assumption that $n \in \mathbb{Z}$ and $n < 0$, we proved $\frac{d}{dx}(x^n) = nx^{n-1}$

  □

Examples

1. Instantaneous Velocity

$$s(t) = -16t^2 + 128t + 192$$
$$t = 2$$

$$\begin{aligned}
\lim_{t\to 2} \frac{s(t) - s(2)}{t-2} &= \lim_{t\to 2} \frac{(-16t^2 + 128t + 192) - (-16(2^2) + 128(2) + 192)}{t-2} && (1)\\
&= \lim_{t\to 2} \frac{(-16t^2 + 128t + 192) - (-64 + 256 + 192)}{t-2} && (2)\\
&= \lim_{t\to 2} \frac{(-16t^2 + 128t + 192) - 384}{t-2} && (3)\\
&= \lim_{t\to 2} \frac{-16t^2 + 128t - 192}{t-2} && (4)\\
&= \lim_{t\to 2} \frac{(t-2)(-16t + 96)}{t-2} && (5)\\
&= \lim_{t\to 2} -16t + 96 && (6)\\
&= -32 + 96 && (7)\\
&= 64 && (8)
\end{aligned}$$

2. Secant Lines

$$y = f(x)$$

Intersection Points: $P(a, f(a))$ and $Q(x, f(x))$

$$\text{Secant Line Slope} = \frac{f(x) - f(a)}{x - a}$$

3. Tangent Lines

$$f(x) = 2x^2 + 4x - 3$$

$$(-1, 5)$$

$$\begin{aligned}
\lim_{x \to -1} \frac{f(x) - f(-1)}{x - (-1)} &= \lim_{x \to -1} \frac{2x^2 + 4x - 3 - (-5)}{x + 1} && (1)\\
&= \lim_{x \to -1} \frac{2x^2 + 4x + 2}{x + 1} && (2)\\
&= \lim_{x \to -1} \frac{(x+1)(2x+2)}{x + 1} && (3)\\
&= \lim_{x \to -1} \frac{(x+1)(2x+2)}{x + 1} && (4)\\
&= \lim_{x \to -1} 2x + 2 && (5)\\
&= 2(-1) + 2 && (6)\\
&= -2 + 2 && (7)\\
&= 0 && (8)
\end{aligned}$$

4. Alternative Tangent Lines

$$f(x) = 5 - x^3$$

$$(2, -3)$$

$$a = 2$$

$$h = -3 - 2 = -5$$

$$\begin{aligned}
\lim_{h \to 0} \frac{f(2+h) - f(2)}{h} &= \lim_{h \to 0} \frac{f(2+h) - (-3)}{h} && (1)\\
&= \lim_{h \to 0} \frac{f(2+h) + 3}{h} && (2)\\
&= \lim_{h \to 0} \frac{5 - (2+h)^3 + 3}{h} && (3)\\
&= \lim_{h \to 0} \frac{8 - (2+h)^3}{h} && (4)\\
&= \lim_{h \to 0} \frac{2^3 - (2+h)^3}{h} && (5)\\
&= \lim_{h \to 0} \frac{(2 - (2+h))(2^2 + 2(2+h) + (2+h)^2)}{h} && (6)\\
&= \lim_{h \to 0} \frac{-h(4 + 4 + 2h + h^2 + 4h + 4)}{h} && (7)\\
&= \lim_{h \to 0} \frac{-h(h^2 + 6h + 12)}{h} && (8)\\
&= \lim_{h \to 0} -(h^2 + 6h + 12) && (9)\\
&= -12 && (10)\\
& && (11)
\end{aligned}$$

$$y + 3 = -12(x - 2) = -12x + 24$$

$$y = -12x + 21$$

5. Derivative Example

$$f(x) = \sqrt{x-1}$$

$$x = 2$$

$$f(x) = f(2) = \sqrt{2-1} = \sqrt{1} = 1$$

$$(2, 1)$$

$$\begin{aligned}
f'(2) &= \lim_{x\to 2} \frac{f(x) - f(2)}{x-2} && (1)\\
&= \lim_{x\to 2} \frac{\sqrt{x-1} - 1}{x-2} && (2)\\
&= \lim_{x\to 2} \frac{\sqrt{x-1} - 1}{x-2} \cdot \frac{\sqrt{x-1}+1}{\sqrt{x-1}+1} && (3)\\
&= \lim_{x\to 2} \frac{x-2}{(x-2)(\sqrt{x-1}+1)} && (4)\\
&= \lim_{x\to 2} \frac{1}{\sqrt{x-1}+1} && (5)\\
&= \frac{1}{\sqrt{2-1}+1} && (6)\\
&= \frac{1}{\sqrt{1}+1} && (7)\\
&= \frac{1}{1+1} && (8)\\
&= \frac{1}{2} && (9)\\
& && (10)
\end{aligned}$$

$$\begin{aligned}
y - 1 &= \frac{1}{2}(x-2) && (1)\\
y &= \frac{1}{2}(x-2) + 1 && (2)\\
&= \frac{1}{2}x - 1 + 1 && (3)\\
&= \frac{1}{2}x && (4)
\end{aligned}$$

6. Derivative Application Example

$$V(t) = 3t$$

$$\begin{aligned}
V'(12) &= \lim_{x\to 12} \frac{V(x) - V(12)}{x-12} && (1)\\
&= \lim_{x\to 12} \frac{3x-36}{x-12} && (2)\\
&= \lim_{x\to 12} \frac{3(x-12)}{x-12} && (3)\\
&= \lim_{x\to 12} 3 && (4)\\
&= 3 && (5)
\end{aligned}$$

$$\begin{aligned}
y - 36 &= 3(x-12) && (1)\\
y &= 3x - 36 + 36 && (2)\\
&= 3x && (3)\\
& && (4)
\end{aligned}$$

7. Find the Derivative

$$f(x) = 4x^2 - 5x + 6$$

$$
\begin{aligned}
\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{4(x+h)^2-5(x+h)+6-(4x^2-5x+6)}{h} && (1)\\
&= \lim_{h\to 0}\frac{4(x+h)^2-5x-5h+6-4x^2+5x-6}{h} && (2)\\
&= \lim_{h\to 0}\frac{4(x+h)^2-5h-4x^2}{h} && (3)\\
&= \lim_{h\to 0}\frac{4h^2+8xh-5h}{h} && (4)\\
&= \lim_{h\to 0} 4h+8x-5 && (5)\\
&= 4(0)+8x-5 && (6)\\
&= 8x-5 && (7)
\end{aligned}
$$

8. Calculating a Derivative

$$f(x) = \frac{1}{x}$$

$$(-5, -\frac{1}{5})$$

$$
\begin{aligned}
\frac{dy}{dx} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} && (1)\\
&= \lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h} && (2)\\
&= \lim_{h\to 0}\frac{\frac{x-(x+h)}{x(x+h)}}{h} && (3)\\
&= \lim_{h\to 0}\frac{\frac{-h}{x(x+h)}}{h} && (4)\\
&= \lim_{h\to 0}\frac{-h}{x(x+h)}\cdot\frac{1}{h} && (5)\\
&= \lim_{h\to 0}\frac{-h}{h(x(x+h))} && (6)\\
&= \lim_{h\to 0}\frac{-1}{x(x+h)} && (7)\\
&= \frac{-1}{x(x+0)} && (8)\\
&= \frac{-1}{x^2} && (9)
\end{aligned}
$$

$$m_{\tan} = \frac{dy}{dx}\bigg|_{x=-5} = \frac{-1}{(-5)^2} = \frac{-1}{25}$$

$$y - (-\frac{1}{5}) = \frac{-1}{25}(x-(-5))$$

$$y = \frac{-1}{25}x - \frac{1}{5} - \frac{1}{5} = \frac{-1}{25}x - \frac{2}{5}$$

9. Constant and Power Rules

$$\frac{d}{dx}\left(x^6\right) = 6x^5$$

$$\frac{d}{dx}\left(x\right) = 1x^0 = 1$$

$$\frac{d}{dx}\left(\pi^2\right) = 0$$

10. Constant Multiple Rule

$$\frac{d}{dx}\left(-4x^9\right) = -4\left(9x^8\right) = -36x^8$$

$$\frac{d}{dt}\left(\frac{2}{5}t^5\right) = \frac{2}{5}\left(5t^4\right) = 2t^4$$

11. Sum and Difference Rules with a Polynomial

$$\frac{d}{dx}\left(6x^5 - \frac{5}{2}x^2 + x + 5\right) = 30x^4 - 5x + 1$$

12. Euler's Number with Derivatives

$$f(x) = 5x + \frac{1}{3}e^x$$

$$\left(0, \frac{1}{3}\right)$$

$$\frac{d}{dx}\left(5x + \frac{1}{3}e^x\right) = 5 + \frac{1}{3}e^x$$

$$y = \frac{16}{3}x + \frac{1}{3}$$

13. Higher-order derivatives

$$f(x) = 3x^4 - 2x^2 + 7x - e^x$$

$$f'(x) = 12x^3 - 4x + 7 - e^x$$

$$f''(x) = 36x^2 - 4 - e^x$$

$$f'''(x) = 72x - e^x$$

$$f^{(4)}(x) = 72 - e^x$$

14. Product Rule

$$\frac{d}{dt}\big((t+1)(t^2 - t + 1)\big) = 1(t^2 - t + 1) + (2t - 1)(t + 1) = 3t^2$$

$$\frac{d}{dx}\left(x^5 e^x\right) = 5x^4(e^x) + e^x(x^5) = e^x\left(x^5 + 5x^4\right)$$

15. Quotient Rule

$$\begin{aligned}
\frac{d}{dx}\left(\frac{x^4 + 5x^2 + x}{\sin x}\right) &= \frac{\sin x(4x^3 + 10x + 1) - (x^4 + 5x^2 + x)\cos x}{\sin^2 x} && (1)\\
&= \frac{(4x^3 + 10x + 1) - (x^4 + 5x^2 + x)\cot x}{\sin x} && (2)
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dx}\left(\frac{2e^x - 1}{3e^x + 1}\right) &= \frac{2e^x(2e^x + 1) - 2e^x(2e^x - 1)}{(2e^x + 1)^2} && (1)\\
&= \frac{2e^x(2e^x + 1 - 2e^x + 1)}{(2e^x + 1)^2} && (2)\\
&= \frac{2e^x(2)}{(2e^x + 1)^2} && (3)\\
&= \frac{4e^x}{(2e^x + 1)^2} && (4)
\end{aligned}$$

16. Tangent Lines with Quotient Rule

$$f(x) = \frac{2x^2}{3x - 1}$$

$$(1, 1)$$

$$f(1) = 1$$

$$f'(1) = \frac{4x(3x - 1) - 3(2x^2)}{(3x - 1)^2} = \frac{2}{4} = \frac{1}{2}$$

$$y = \frac{1}{2}x + \frac{1}{2}$$

17. Power Rule with Negative Powers

$$\begin{aligned}
\frac{d}{dx}\left(\frac{-5}{x^6}\right) &= \frac{d}{dx}\left(-5x^{-6}\right) && (1)\\
&= -5\left(-6x^{-7}\right) && (2)\\
&= 30x^{-7} && (3)\\
&= \frac{1}{30x^7} && (4)
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dp}\left(\frac{2p^8-7}{p^3}\right) &= \frac{d}{dp}2p^5 - 7p^{-3} && (1)\\
&= 10p^4 + 21p^{-4} && (2)\\
&= \frac{10p^4}{21p^4} && (3)
\end{aligned}$$

18. Combining Rules to Evaluate Derivatives

$$f(x) = \left(\frac{x^2+1}{x}\right)e^x = \left(x + x^{-1}\right)e^x$$
$$f'(x) = \left(1 + -x^{-2}\right)e^x + \left(x + x^{-1}\right)e^x$$

Related Exercises

1. (Section 3.1, Related Exercise 13)

$$s(t) = -16t^2 + 100t$$
$$a = 1$$

$$\begin{aligned}
\lim_{h\to 0}\frac{s(a+h)-s(a)}{h} &= \lim_{h\to 0}\frac{s(1+h)-84}{h} && (1)\\
&= \lim_{h\to 0}\frac{-16(1+h)^2+100(1+h)-84}{h} && (2)\\
&= \lim_{h\to 0}\frac{-16(h^2+2h+1)+100+100h-84}{h} && (3)\\
&= \lim_{h\to 0}\frac{-16h^2-32h-16+100+100h-84}{h} && (4)\\
&= \lim_{h\to 0}\frac{-16h^2+68h}{h} && (5)\\
&= \lim_{h\to 0} -16h+68 && (6)\\
&= -16(0)+68 && (7)\\
&= 68 && (8)
\end{aligned}$$

2. (Section 3.1, Related Exercise 14)

$$s(t) = -16t^2 + 128t + 192$$
$$a = 2$$

$$\begin{aligned}
\lim_{h\to 0}\frac{s(a+h)-s(a)}{h} &= \lim_{h\to 0}\frac{s(2+h)-384}{h} && (1)\\
&= \lim_{h\to 0}\frac{-16(2+h)^2+128(2+h)+192-384}{h} && (2)\\
&= \lim_{h\to 0}\frac{-16(h^2+4h+4)+128(2+h)+192-384}{h} && (3)\\
&= \lim_{h\to 0}\frac{-16h^2-64h-64+256+128h+192-384}{h} && (4)\\
&= \lim_{h\to 0}\frac{-16h^2+64h}{h} && (5)\\
&= \lim_{h\to 0} -16h+64 && (6)\\
&= -16(0)+64 && (7)\\
&= 64 && (8)
\end{aligned}$$

3. (Section 3.1, Related Exercise 17)

$$f(x) = \frac{1}{x}$$
$$P(-1,-1)$$

$$\begin{aligned}
\lim_{x\to-1}\frac{f(x)-f(-1)}{x-(-1)} &= \lim_{x\to-1}\frac{f(x)-(-1)}{x+1} && (1)\\
&= \lim_{x\to-1}\frac{f(x)+1}{x+1} && (2)\\
&= \lim_{x\to-1}\frac{\frac{1}{x}+1}{x+1} && (3)\\
&= \lim_{x\to-1}\frac{\frac{1+x}{x}}{x+1} && (4)\\
&= \lim_{x\to-1}\frac{\frac{1+x}{x}}{x+1}\cdot\frac{x}{x} && (5)\\
&= \lim_{x\to-1}\frac{1+x}{(x+1)x} && (6)\\
&= \lim_{x\to-1}\frac{1}{x} && (7)\\
&= \frac{1}{-1} && (8)\\
&= -1 && (9)
\end{aligned}$$

$$y-(-1) = -1(x-(-1))$$
$$y = -1(x+1)-1 = -x-1-1 = -x-2$$

4. (Section 3.1, Related Exercise 18)

$$f(x) = \frac{4}{x^2}$$
$$(-1,4)$$

$$\begin{aligned}
\lim_{x\to-1}\frac{f(x)-4}{x-(-1)} &= \lim_{x\to-1}\frac{f(x)-4}{x+1} && (1)\\
&= \lim_{x\to-1}\frac{\frac{4}{x^2}-4}{x+1} && (2)\\
&= \lim_{x\to-1}\frac{\frac{4-4x^2}{x^2}}{x+1} && (3)\\
&= \lim_{x\to-1}\frac{\frac{4-4x^2}{x^2}}{x+1}\cdot\frac{x^2}{x^2} && (4)\\
&= \lim_{x\to-1}\frac{4-4x^2}{x^2(x+1)} && (5)\\
&= \lim_{x\to-1}\frac{4(1-x^2)}{x^2(x+1)} && (6)\\
&= \lim_{x\to-1}\frac{4(1-x)(1+x)}{x^2(x+1)} && (7)\\
&= \lim_{x\to-1}\frac{4(1-x)}{x^2} && (8)\\
&= \frac{4(1-(-1)^2)}{(-1)^2} && (9)\\
&= \frac{4(1-1)}{1} && (10)\\
&= 0 && (11)
\end{aligned}$$

$$y-4 = 0$$
$$y = 4$$

5. (Section 3.1, Related Exercise 23)

$$f(x) = 3x^2 - 4x$$

$$(1, -1)$$

$$\begin{aligned}
\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} &= \lim_{h\to 0} \frac{f(1+h)-(-1)}{h} && (1)\\
&= \lim_{h\to 0} \frac{3(1+h)^2 - 4(1+h) + 1}{h} && (2)\\
&= \lim_{h\to 0} \frac{3h^2 + 6h + 3 - 4 - 4h + 1}{h} && (3)\\
&= \lim_{h\to 0} \frac{3h^2 + 2h + 3 - 4 + 1}{h} && (4)\\
&= \lim_{h\to 0} \frac{3h^2 + 2h}{h} && (5)\\
&= \lim_{h\to 0} 3h + 2 && (6)\\
&= 3(0) + 2 && (7)\\
&= 2 && (8)
\end{aligned}$$

$$y - (-1) = 2(x - 1)$$

$$y = 2(x-1) - 1 = 2x - 2 - 1 = 2x - 3$$

6. (Section 3.1, Related Exercise 27)

$$f(x) = x^3$$

$$(1, 1)$$

$$\begin{aligned}
\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} &= \lim_{h\to 0} \frac{f(1+h)-1}{h} && (1)\\
&= \lim_{h\to 0} \frac{(1+h)^3 - 1}{h} && (2)\\
&= \lim_{h\to 0} \frac{h^3 + h^2 + 2h + 2h^2 + h + 1 - 1}{h} && (3)\\
&= \lim_{h\to 0} \frac{h^3 + 2h^2 + 3h}{h} && (4)\\
&= \lim_{h\to 0} h^2 + 2h + 3 && (5)\\
&= 0^2 + 2(0) + 3 && (6)\\
&= 3 && (7)
\end{aligned}$$

$$y - 1 = 3(x - 1)$$

$$y = 3x - 3 + 1 = 3x - 2$$

7. (Section 3.1, Related Exercise 39)

$$f(x) = \sqrt{2x+1}$$

$$a = 4$$

$$\begin{aligned}
\lim_{x\to 4}\frac{f(x)-f(a)}{x-a} &= \lim_{x\to 4}\frac{f(x)-3}{x-4} && (1)\\
&= \lim_{x\to 4}\frac{\sqrt{2x+1}-3}{x-4} && (2)\\
&= \lim_{x\to 4}\frac{\sqrt{2x+1}-3}{x-4}\cdot\frac{\sqrt{2x+1}+3}{\sqrt{2x+1}+3} && (3)\\
&= \lim_{x\to 4}\frac{2x+1-9}{(x-4)(\sqrt{2x+1}+3)} && (4)\\
&= \lim_{x\to 4}\frac{2(x-4)}{(x-4)(\sqrt{2x+1}+3)} && (5)\\
&= \lim_{x\to 4}\frac{2}{\sqrt{2x+1}+3} && (6)\\
&= \frac{2}{\sqrt{9}+3} && (7)\\
&= \frac{2}{3+3} && (8)\\
&= \frac{2}{6} && (9)\\
&= \frac{1}{3} && (10)
\end{aligned}$$

$$y-3=\frac{1}{3}(x-4)$$

$$y=\frac{1}{3}x-\frac{4}{3}+\frac{9}{3}=\frac{1}{3}x+\frac{5}{3}$$

8. (Section 3.1, Related Exercise 40)

$$f(x)=\sqrt{3x}$$

$$a=12$$

$$\begin{aligned}
\lim_{x\to 12}\frac{f(x)-f(a)}{x-a} &= \lim_{x\to 12}\frac{f(x)-6}{x-12} && (1)\\
&= \lim_{x\to 12}\frac{\sqrt{3x}-6}{x-12} && (2)\\
&= \lim_{x\to 12}\frac{\sqrt{3x}-6}{x-12}\cdot\frac{\sqrt{3x}+6}{\sqrt{3x}+6} && (3)\\
&= \lim_{x\to 12}\frac{3x-36}{(x-12)(\sqrt{3x}+6)} && (4)\\
&= \lim_{x\to 12}\frac{3(x-12)}{(x-12)(\sqrt{3x}+6)} && (5)\\
&= \lim_{x\to 12}\frac{3}{\sqrt{3x}+6} && (6)\\
&= \frac{3}{\sqrt{3(12)}+6} && (7)\\
&= \frac{3}{\sqrt{36}+6} && (8)\\
&= \frac{3}{6+6} && (9)\\
&= \frac{3}{12} && (10)\\
&= \frac{1}{4} && (11)
\end{aligned}$$

$$y-6=\frac{1}{4}(x-12)$$

$$y=\frac{1}{4}x-3+6=\frac{1}{4}x+3$$

9. (Section 3.1, Related Exercise 49)

$$d(t) = 16t^2$$

$$a = 4$$

$$\begin{aligned}
\lim_{x\to 4}\frac{f(x)-f(a)}{x-a} &= \lim_{h\to 0}\frac{f(x)-256}{x-4} && (1)\\
&= \lim_{x\to 4}\frac{16x^2-256}{x-4} && (2)\\
&= \lim_{x\to 4}\frac{(16x+64)(x-4)}{x-4} && (3)\\
&= \lim_{x\to 4} 16x+64 && (4)\\
&= 16(4)+64 && (5)\\
&= 64+64 && (6)\\
&= 128 && (7)
\end{aligned}$$

10. (Section 3.1, Related Exercise 50)

$$F(x) = \frac{k}{x^2} \text{ where } k \text{ is some constant}$$

$$a = 1$$

$$\begin{aligned}
\lim_{h\to 0}\frac{F(a+h)-F(a)}{h} &= \lim_{h\to 0}\frac{F(1+h)-\frac{k}{1}}{h} && (1)\\
&= \lim_{h\to 0}\frac{F(1+h)-\frac{k}{1}}{h} && (2)\\
&= \lim_{h\to 0}\frac{\frac{k}{(1+h)^2}-\frac{k}{1}}{h} && (3)\\
&= \lim_{h\to 0}\frac{\frac{k}{(1+h)^2}-\frac{k(1+h)^2}{(1+h)^2}}{h} && (4)\\
&= \lim_{h\to 0}\frac{\frac{k-k(1+h)^2}{(1+h)^2}}{h} && (5)\\
&= \lim_{h\to 0}\frac{\frac{k-k(1+h)^2}{(1+h)^2}}{h} && (6)\\
&= \lim_{h\to 0}\frac{\frac{k-(kh^2+2kh+k)}{(1+h)^2}}{h} && (7)\\
&= \lim_{h\to 0}\frac{\frac{k-kh^2-2kh-k}{(1+h)^2}}{h} && (8)\\
&= \lim_{h\to 0}\frac{\frac{-kh^2-2kh}{(1+h)^2}}{h} && (9)\\
&= \lim_{h\to 0}\frac{-kh^2-2kh}{(1+h)^2}\cdot\frac{1}{h} && (10)\\
&= \lim_{h\to 0}\frac{h(-kh-2k)}{h(1+h)^2} && (11)\\
&= \lim_{h\to 0}\frac{-kh-2k}{(1+h)^2} && (12)\\
&= \frac{-kh-2k}{(1+h)^2} && (13)\\
&= \frac{-k(0)-2k}{(1+0)^2} && (14)\\
&= \frac{-2k}{1} && (15)\\
&= -2k && (16)
\end{aligned}$$

11. (Section 3.1, Related Exercise 53)
    Hint: Sketch a Secant Line

$$L'(1.5) \approx 4$$

$$L'(a) \approx 0 \text{ where } a \geq 4$$

12. (Section 3.1, Related Exercise 54)

$$D'(60) \approx 0.6$$

$$D'(170) \approx 0$$

13. (Section 3.2, Related Exercise 23)

$$f(x) = 4x^2 + 1$$

$$a = 2, 4$$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} && (1)\\
&= \lim_{h\to 0} \frac{4(x+h)^2 + 1 - (4x^2 + 1)}{h} && (2)\\
&= \lim_{h\to 0} \frac{4(x^2 + 2xh + h^2) + 1 - 4x^2 - 1}{h} && (3)\\
&= \lim_{h\to 0} \frac{4x^2 + 8xh + 4h^2 - 4x^2}{h} && (4)\\
&= \lim_{h\to 0} \frac{8xh + 4h^2}{h} && (5)\\
&= \lim_{h\to 0} 8x + 4h && (6)\\
&= 8x + 4(0) && (7)\\
&= 8x && (8)
\end{aligned}$$

$$f'(2) = 8(2) = 16$$

$$f'(4) = 8(4) = 32$$

14. (Section 3.2, Related Exercise 24)

$$f(x) = x^2 + 3x$$

$$a = -1, 4$$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} && (1)\\
&= \lim_{h\to 0} \frac{(x+h)^2 + 3(x+h) - (x^2 + 3x)}{h} && (2)\\
&= \lim_{h\to 0} \frac{x^2 + h^2 + xh + 3x + 3h - x^2 - 3x}{h} && (3)\\
&= \lim_{h\to 0} \frac{h^2 + xh + 3h}{h} && (4)\\
&= \lim_{h\to 0} h + x + 3 && (5)\\
&= 0 + x + 3 && (6)\\
&= x + 3 && (7)
\end{aligned}$$

$$f'(-1) = -1 + 3 = 2$$

$$f'(4) = 4 + 3 = 7$$

15. (Section 3.2, Related Exercise 37)

$$f(x) = \sqrt{3x + 1}$$

$$a = 8$$

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} && (1)\\
&= \lim_{h\to 0} \frac{\sqrt{3(x+h)+1}-\sqrt{3x+1}}{h} && (2)\\
&= \lim_{h\to 0} \frac{\sqrt{3(x+h)+1}-\sqrt{3x+1}}{h} \cdot \frac{\sqrt{3(x+h)+1}+\sqrt{3x+1}}{\sqrt{3(x+h)+1}+\sqrt{3x+1}} && (3)\\
&= \lim_{h\to 0} \frac{3(x+h)+1-(3x+1)}{h(\sqrt{3(x+h)+1}+\sqrt{3x+1})} && (4)\\
&= \lim_{h\to 0} \frac{3x+3h+1-3x-1}{h(\sqrt{3(x+h)+1}+\sqrt{3x+1})} && (5)\\
&= \lim_{h\to 0} \frac{3h}{h(\sqrt{3(x+h)+1}+\sqrt{3x+1})} && (6)\\
&= \lim_{h\to 0} \frac{3}{\sqrt{3(x+h)+1}+\sqrt{3x+1}} && (7)\\
&= \frac{3}{\sqrt{3(x+0)+1}+\sqrt{3x+1}} && (8)\\
&= \frac{3}{\sqrt{3x+1}+\sqrt{3x+1}} && (9)\\
&= \frac{3}{2\sqrt{3x+1}} && (10)
\end{aligned}
$$

$$f'(8) = \frac{3}{2\sqrt{3(8)+1}} = \frac{3}{2\sqrt{24+1}} = \frac{3}{2\sqrt{25}} = \frac{3}{2(5)} = \frac{3}{10}$$

$$y - f(8) = f'(8)(x-8)$$

$$y = \frac{3}{10}(x-8)+5 = \frac{3}{10}x - \frac{12}{5} + 5 = \frac{3}{10}x + \frac{13}{5}$$

16. (Section 3.2, Related Exercise 38)

$$f(x) = \sqrt{x+2}$$

$$a = 7$$

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} && (1)\\
&= \lim_{h\to 0} \frac{\sqrt{x+h+2}-\sqrt{x+2}}{h} && (2)\\
&= \lim_{h\to 0} \frac{\sqrt{x+h+2}-\sqrt{x+2}}{h} \cdot \frac{\sqrt{x+h+2}+\sqrt{x+2}}{\sqrt{x+h+2}+\sqrt{x+2}} && (3)\\
&= \lim_{h\to 0} \frac{x+h+2-(x+2)}{h(\sqrt{x+h+2}+\sqrt{x+2})} && (4)\\
&= \lim_{h\to 0} \frac{x+h+2-x-2}{h(\sqrt{x+h+2}+\sqrt{x+2})} && (5)\\
&= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h+2}+\sqrt{x+2})} && (6)\\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+h+2}+\sqrt{x+2}} && (7)\\
&= \frac{1}{\sqrt{x+0+2}+\sqrt{x+2}} && (8)\\
&= \frac{1}{\sqrt{x+2}+\sqrt{x+2}} && (9)\\
&= \frac{1}{2\sqrt{x+2}} && (10)
\end{aligned}
$$

$$f'(7) = \frac{1}{2\sqrt{7+2}} = \frac{1}{2\sqrt{9}} = \frac{1}{2\cdot 3} = \frac{1}{6}$$

$$y - f(7) = f'(7)(x-7)$$

$$y = \frac{1}{6}(x-7) + 3 = \frac{1}{6}x - \frac{7}{6} + 3 = \frac{1}{6}x + \frac{11}{6}$$

17. (Section 3.2, Related Exercise 25)

$$f(x) = \frac{1}{x+1}$$

$$a = -\frac{1}{2}, 5$$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} && (1)\\
&= \lim_{h\to 0} \frac{\frac{1}{x+h+1} - \frac{1}{x+1}}{h} && (2)\\
&= \lim_{h\to 0} \frac{\frac{1}{x+h+1} - \frac{1}{x+1}}{h} && (3)\\
&= \lim_{h\to 0} \frac{\frac{x+1-(x+h+1)}{(x+h+1)(x+1)}}{h} && (4)\\
&= \lim_{h\to 0} \frac{x+1-x-h-1}{h(x+h+1)(x+1)} && (5)\\
&= \lim_{h\to 0} \frac{-h}{h(x+h+1)(x+1)} && (6)\\
&= \lim_{h\to 0} \frac{-1}{(x+h+1)(x+1)} && (7)\\
&= \frac{-1}{(x+0+1)(x+1)} && (8)\\
&= \frac{-1}{(x+1)(x+1)} && (9)\\
&= \frac{-1}{(x+1)^2} && (10)
\end{aligned}$$

$$f'(-\frac{1}{2}) = \frac{-1}{(-\frac{1}{2}+1)^2} = \frac{-1}{(\frac{1}{2})^2} = \frac{-1}{\frac{1}{4}} = -1 \cdot 4 = -4$$

$$f'(5) = \frac{-1}{(5+1)^2} = \frac{-1}{(6)^2} = \frac{-1}{36}$$

18. (Section 3.2, Related Exercise 27)

$$f(t) = \frac{1}{\sqrt{t}}$$

$$a = 9, \frac{1}{4}$$

$$\begin{aligned}
f'(t) &= \lim_{h\to 0}\frac{f(t+h)-f(t)}{h} && (1)\\
&= \lim_{h\to 0}\frac{\frac{1}{\sqrt{t+h}}-\frac{1}{\sqrt{t}}}{h} && (2)\\
&= \lim_{h\to 0}\frac{\frac{\sqrt{t}-\sqrt{t+h}}{\sqrt{t+h}\sqrt{t}}}{h} && (3)\\
&= \lim_{h\to 0}\frac{\frac{\sqrt{t}-\sqrt{t+h}}{\sqrt{t+h}\sqrt{t}}\cdot\frac{\sqrt{t}+\sqrt{t+h}}{\sqrt{t}+\sqrt{t+h}}}{h} && (4)\\
&= \lim_{h\to 0}\frac{t-(t+h)}{h(t\sqrt{t+h}+(t+h)\sqrt{t})} && (5)\\
&= \lim_{h\to 0}\frac{-h}{h(t\sqrt{t+h}+(t+h)\sqrt{t})} && (6)\\
&= \lim_{h\to 0}\frac{-1}{t\sqrt{t+h}+(t+h)\sqrt{t}} && (7)\\
&= \frac{-1}{t\sqrt{t+0}+(t+0)\sqrt{t}} && (8)\\
&= \frac{-1}{t\sqrt{t}+t\sqrt{t}} && (9)\\
&= \frac{-1}{2(t\sqrt{t})} && (10)
\end{aligned}$$

$$f'(9)=\frac{-1}{2\left(9\sqrt{9}\right)}=\frac{-1}{2\,(9\cdot 3)}=\frac{-1}{2\,(27)}=\frac{-1}{54}$$

$$f'\left(\frac{1}{4}\right)=\frac{-1}{2\left(\frac{1}{4}\sqrt{\frac{1}{4}}\right)}=\frac{-1}{2\left(\frac{1}{4}\cdot\frac{1}{2}\right)}=\frac{-1}{2\left(\frac{1}{8}\right)}=\frac{-1}{\frac{2}{8}}=-1\,(4)=-4$$

19. (Section 3.2, Related Exercise 53)
    f is not continuous at $x=1$
    f is not differentiable at $x=1,2$

20. (Section 3.2, Related Exercise 54)
    f is not continuous at $x=1$
    f is not differentiable at $x=1,2$

21. (Section 3.3, Related Exercise 19)
$$\frac{d}{dx}\left(x^5\right)=5x^4$$

22. (Section 3.3, Related Exercise 22)
$$\frac{d}{dx}\left(e^3\right)=0$$

23. (Section 3.3, Related Exercise 23)
$$\frac{d}{dx}\left(5x^3\right)=15x^2$$

24. (Section 3.3, Related Exercise 24)
$$\frac{d}{dx}\left(\frac{5}{6}w^{12}\right)=10w^{11}$$

25. (Section 3.3, Related Exercise 28)
$$\frac{d}{dx}\left(6\sqrt{t}\right)=\frac{3}{\sqrt{t}}$$

26. (Section 3.3, Related Exercise 31)
$$\frac{d}{dx}\left(3x^4+7x\right)=12x^3+7$$

27. (Section 3.3, Related Exercise 33)
$$\frac{d}{dx}\left(10x^4-32x+e^2\right)=40x^3-32$$

28. (Section 3.3, Related Exercise 60)

$$f(x) = x^3 - 4x^2 + 2x - 1$$
$$a = 2$$
$$f'(x) = 3x^2 - 8x + 2$$
$$y = -2x - 1$$

29. (Section 3.3, Related Exercise 61)

$$f(x) = e^x$$
$$a = \ln 3$$
$$f'(x) = e^x$$
$$y = 3x - 3(\ln 3) + 3$$

30. (Section 3.3, Related Exercise 63)

$$f(x) = x^2 - 6x + 5$$
$$f'(x) = 2x - 6$$
$$y = f(x) \text{ is } 0\text{, when } x = 3$$
$$y = f(x) \text{ is } 2\text{, when } x = 4$$

31. (Section 3.3, Related Exercise 64)

$$f(t) = t^3 - 27t + 5$$
$$f'(t) = 3t^2 - 27$$
$$y = f(t) \text{ is } 0\text{, when } x = 3$$
$$y = f(t) \text{ is } 21\text{, when } x = 4$$

32. (Section 3.3, Related Exercise 69)

$$f(x) = 5x^4 + 10x^3 + 3x + 6$$
$$f'(x) = 20x^3 + 30x^2 + 3$$
$$f''(x) = 60x^2 + 60x$$
$$f'''(x) = 120x + 60$$

33. (Section 3.3, Related Exercise 70)

$$f(x) = 3x^2 + 5e^x$$
$$f'(x) = 6x + 5e^x$$
$$f''(x) = 6 + 5e^x$$
$$f'''(x) = 5e^x$$

A copy of my notes (in LaTeX) are available on my GitHub