Module 6 Notes (MATH-211)

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General Notes (and Definitions)

- L'Hôpital's Rule
  Indeterminate Form: An expression involving two components where the limit cannot be determined by evaluating the limits of the individual components.
  L'Hôpital's Rule: Suppose f and g are differentiable functions on an open interval I containing the point $x = a$, with $g'(x) \neq 0$ on I when $x \neq a$.
  If $\lim\limits_{x \to a} \frac{f(x)}{g(x)}$ has any of the indeterminate forms: $\frac{0}{0}$, $\frac{\infty}{\infty}$, $-\frac{\infty}{\infty}$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

  provided that one of the following is the case:

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} \in \mathbb{R}$$

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = \infty$$

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = -\infty$$

  L'Hôpital's Rule is still valid if $x \to a$ is replaced by any of $x \to a^+$, $x \to a^-$, $x \to \infty$, or $x \to -\infty$. In the last two of these cases, there must be a greatest x-value beyond which both f and g are differentiable at every point.
  Exponential Indeterminate forms: 1^∞, 0^0, ∞^0
  Method for evaluating limits of indeterminate forms 1^∞, 0^0, ∞^0:
  Assume that $L = \lim\limits_{x \to a} f(x)^{g(x)}$ has one of these indeterminate forms.

  1. Use the fact that the natural logarithm and natural exponential functions are inverses to write

$$L = \lim_{x \to a} e^{\ln\left(f(x)^{g(x)}\right)}$$

  2. Use the power property of logarithm arguments to write

$$L = \lim_{x \to a} e^{g(x)\ln(f(x))}$$

  3. Use continuity of the exponential function to write

$$L = e^{\lim\limits_{x \to a} g(x)\ln(f(x))}$$

  4. Rewrite multiplication as division by the reciprocal:

$$L = e^{\lim\limits_{x \to a}\left(\frac{\ln(f(x))}{\frac{1}{g(x)}}\right)}$$

  5. Use L'Hôpital's Rule to evaluate this limit expression

  Growth Rates: Suppose f and g are functions with $\lim\limits_{x \to \infty} f(x) = \infty$ and $\lim\limits_{x \to \infty} g(x) = \infty$

1. If one of the following are true, f **grows faster than** g, and we use the notation $f \gg g$

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0 \tag{1}$$
$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty \tag{2}$$

2. f **and** g **have comparable growth rates**, if there is some non-zero finite number M such that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = M$$

Ranked Growth Rates as $x \to \infty$
For any base $b > 1$, and for any positive numbers p, q, r, and s

$$\ln^q x \ll x^p \ll x^p \ln^r x \ll x^{p+s} \ll b^x \ll x^x$$

- Antiderivatives
  Antiderivative: A function F is an antiderivative of another function f on an interval I if for all x in I:

  $$F'(x) = f(x)$$

  Family of Antiderivatives: Let $F(x)$ be any antiderivative of $f(x)$ on an interval I. Then all antiderivatives of f on I have the form $F(x) + C$, where C is an arbitrary constant.
  Differential Equations: Any equation involving an unknown function and its derivatives
  - Infinite family of solutions
  - No two solutions from the family pass through the same point
  - Given an initial condition $f(a) = b$, we can identify the particular family member that solves the given problem by solving for C

Antiderivative Rules

- Power Rule
  If $p \neq -1$ and C is an arbitrary constant:

  $$\int x^p dx = \frac{x^{p+1}}{p+1} + C$$

- Integral of x^{-1}

  $$\int x^{-1} dx = \int \frac{1}{x} dx = \ln|x| + C$$

- Constant Multiple and Sum Rules
  If $c \in \mathbb{R}$:

  $$\int cf(x)dx = c \int f(x)dx$$
  $$\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$$

- Integral of e^x

  $$\int e^x dx = e^x + C$$

Trigonometric (and inverse) Integrals

$$\int \cos(x)dx = \sin x + C \quad (1)$$
$$\int \sin(x)dx = -\cos x + C \quad (2)$$
$$\int \sec^2(x)dx = \tan x + C \quad (3)$$
$$\int \csc^2(x)dx = -\cot x + C \quad (4)$$
$$\int \sec(x)\tan(x)dx = \sec x + C \quad (5)$$
$$\int \csc(x)\cot(x)dx = -\csc x + C \quad (6)$$
$$\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x + C \quad (7)$$
$$\int \frac{1}{1+x^2}dx = \tan^{-1} x + C \quad (8)$$
$$\int \frac{1}{x\sqrt{x^2-1}}dx = \sec^{-1}|x| + C \quad (9)$$

Examples

1. Use L'Hôpital's Rule to evaluate a limit with indeterminate form $\frac{0}{0}$

$$\begin{aligned} \lim_{x\to 0} \frac{e^x - x - 1}{5x^2} &= \lim_{x\to 0} \frac{e^x - 1}{10x} && (1) \\ &= \lim_{x\to 0} \frac{e^x}{10} && (2) \\ &= \frac{e^0}{10} && (3) \\ &= \frac{1}{10} && (4) \end{aligned}$$

2. Use L'Hôpital's Rule to evaluate a limit with indeterminate form $\frac{\infty}{\infty}$

$$\begin{aligned} \lim_{x\to 0^+} \frac{1-\ln x}{1+\ln x} &= \lim_{x\to 0^+} \frac{-\frac{1}{x}}{\frac{1}{x}} && (1) \\ &= \lim_{x\to 0^+} \frac{-\frac{1}{x}}{\frac{1}{x}} && (2) \\ &= \frac{-1}{1} && (3) \\ &= -1 && (4) \end{aligned}$$

3. Use L'Hôpital's Rule to evaluate a limit with indeterminate form $0 \cdot \infty$

$$\begin{aligned} \lim_{x\to 1^-} (1-x)\tan\left(\frac{\pi x}{2}\right) &= \lim_{x\to 1^-} \frac{(1-x)}{\cot\left(\frac{\pi x}{2}\right)} && (1) \\ &= \lim_{x\to 1^-} \frac{-1}{-\frac{\pi}{2}\csc^2\left(\frac{\pi x}{2}\right)} && (2) \\ &= \lim_{x\to 1^-} \frac{2}{\pi}\sin^2\left(\frac{\pi x}{2}\right) && (3) \\ &= \frac{2}{\pi} && (4) \end{aligned}$$

4. Use L'Hôpital's Rule to evaluate a limit with exponential indeterminate form

$$\begin{aligned}
\lim_{x\to 0^+} x^{\tan x} &= e^{\lim\limits_{x\to 0^+} \frac{\ln x}{\frac{1}{\tan x}}} && (1)\\
&= e^{\lim\limits_{x\to 0^+} \frac{\ln x}{\cot x}} && (2)\\
&= e^{\lim\limits_{x\to 0^+} \frac{1}{-x\csc^2 x}} && (3)\\
&= e^{\lim\limits_{x\to 0^+} \frac{-\sin^2 x}{x}} && (4)\\
&= e^{\lim\limits_{x\to 0^+} \frac{-2\sin x\cos x}{1}} && (5)\\
&= e^{\lim\limits_{x\to 0^+} -2\sin x\cos x} && (6)\\
&= e^0 && (7)\\
&= 1 && (8)
\end{aligned}$$

5. Compare the growth rates of functions

$$f(x) = x^2\ln x$$
$$g(x) = x\ln^2 x$$

$$\begin{aligned}
\lim_{x\to\infty}\frac{f(x)}{g(x)} &= \lim_{x\to\infty}\frac{x^2\ln x}{x\ln^2 x} && (1)\\
&= \lim_{x\to\infty}\frac{x}{\ln x} && (2)\\
&= \lim_{x\to\infty}\frac{1}{\frac{1}{x}} && (3)\\
&= \lim_{x\to\infty} x && (4)\\
&= \infty && (5)
\end{aligned}$$

Since $\lim_{x\to\infty}\frac{f(x)}{g(x)} = \infty$, $f \gg g$

6. Use knowledge of derivatives to find antiderivatives

$$f(x) = -4\cos x - x$$
$$F(x) = -4\sin x - \frac{1}{2}x^2$$
$$F'(x) = -4\cos x - x$$
$$\int (-4\cos x - x)dx = -4\sin x - \frac{1}{2}x^2 + C$$

7. Determine indefinite integrals using antiderivative rules

$$\int \frac{3}{x^4} + 2 - 3x^2 dx = \int 3x^{-4} + 2 - 3x^2 dx = \frac{-1}{x^3} + 2x - x^3 + C$$

8. Rewrite an indefinite integral to find an antiderivative

$$\begin{aligned}
\int \frac{2+3\cos y}{\sin^2 y}dy &= \int 2\csc^2 y + 3\cot y\csc y dy && (1)\\
&= 2\int \csc^2 y dy + 3\int \cot y\csc y dy && (2)\\
&= -2\cot y - 3\csc y + C && (3)
\end{aligned}$$

9. Solve an initial value problem

$$f'(u) = 4\cos u - 4\sin u$$
$$f(\pi) = 0$$
$$f(u) = 4\int \cos u\,du - 4\int \sin u\,du = 4\sin u + 4\cos u + C$$

$$\begin{aligned} 4\sin\pi + 4\cos\pi + C &= 0 && (1)\\ 0 - 4 + C &= 0 && (2)\\ C &= 4 && (3) \end{aligned}$$

$$f(u) = 4\sin u + 4\cos u + 4$$

10. Application of differential equations to linear motion

$$a(t) = 2 + 3\sin t$$

$$v(0) = 1$$

$$s(0) = 10$$

$$\begin{aligned} v(t) &= \int 2 + 3\sin t\,dt && (1)\\ &= 2\int t^0\,dt + 3\int \sin t\,dt && (2)\\ &= 2t - 3\cos t + C && (3)\\ 2(0) - 3\cos 0 + C &= 1 && (4)\\ -3 + C &= 1 && (5)\\ C &= 4 && (6)\\ v(t) &= 2t - 3\cos t + 4 && (7)\\ s(t) &= \int 2t - 3\cos t + 4\,dt && (8)\\ &= 2\int t\,dt - 3\int \cos t\,dt + 4\int t^0\,dt && (9)\\ &= 2\frac{t^2}{2} - 3\sin t + 4t + C && (10)\\ &= t^2 - 3\sin t + 4t + C && (11)\\ 0^2 - 3\sin 0 + 4(0) + C &= 10 && (12)\\ C &= 10 && (13)\\ s(t) &= t^2 - 3\sin t + 4t + 10 && (14) \end{aligned}$$

Related Exercises

1. (Section 4.7, Exercise 17)

$$\begin{aligned} \lim_{x\to 2}\frac{x^2 - 2x}{x^2 - 6x + 8} &= \lim_{x\to 2}\frac{2x - 2}{2x - 6} && (1)\\ &= \frac{2(2) - 2}{2(2) - 6} && (2)\\ &= \frac{4 - 2}{4 - 6} && (3)\\ &= \frac{2}{-2} && (4)\\ &= -1 && (5) \end{aligned}$$

2. (Section 4.7, Exercise 18)

$$\begin{aligned} \lim_{x\to -1}\frac{x^4 + x^3 + 2x + 2}{x + 1} &= \lim_{x\to -1}\frac{4x^3 + 3x^2 + 2}{1} && (1)\\ &= \lim_{x\to -1} 4x^3 + 3x^2 + 2 && (2)\\ &= 4(-1)^3 + 3(-1)^2 + 2 && (3)\\ &= -4 + 3 + 2 && (4)\\ &= 1 && (5) \end{aligned}$$

3. (Section 4.7, Exercise 36)

$$\begin{aligned}
\lim_{x\to 0}\frac{e^x-x-1}{5x^2} &= \lim_{x\to 0}\frac{e^x-1}{10x} &(1)\\
&= \lim_{x\to 0}\frac{e^x}{10} &(2)\\
&= \frac{e^0}{10} &(3)\\
&= \frac{1}{10} &(4)
\end{aligned}$$

4. (Section 4.7, Exercise 39)

$$\begin{aligned}
\lim_{x\to 0}\frac{e^x-\sin x-1}{x^4+8x^3+12x^2} &= \lim_{x\to 0}\frac{e^x-\cos x}{4x^3+24x^2+24x} &(1)\\
&= \lim_{x\to 0}\frac{e^x+\sin x}{12x^2+48x+24} &(2)\\
&= \frac{e^0+\sin 0}{12(0)^2+48(0)+24} &(3)\\
&= \frac{1+0}{24} &(4)\\
&= \frac{1}{24} &(5)
\end{aligned}$$

5. (Section 4.7, Exercise 38)

$$\begin{aligned}
\lim_{x\to \infty}\frac{e^{3x}}{3e^{3x}+5} &= \lim_{x\to \infty}\frac{3e^{3x}}{9e^{3x}} &(1)\\
&= \lim_{x\to \infty}\frac{1}{3}\cdot\frac{e^{3x}}{e^{3x}} &(2)\\
&= \lim_{x\to \infty}\frac{1}{3} &(3)\\
&= \frac{1}{3} &(4)
\end{aligned}$$

6. (Section 4.7, Exercise 51)

$$\begin{aligned}
\lim_{x\to \infty}\frac{x^2-\ln\frac{2}{x}}{3x^2+2x} &= \lim_{x\to \infty}\frac{2x+\frac{1}{x}}{6x+2} &(1)\\
&= \lim_{x\to \infty}\frac{2-\frac{1}{x^2}}{6} &(2)\\
&= \frac{2-0}{6} &(3)\\
&= \frac{2}{6} &(4)\\
&= \frac{1}{3} &(5)
\end{aligned}$$

7. (Section 4.7, Exercise 53)

$$\begin{aligned}
\lim_{x\to 0} x\csc x &= \lim_{x\to 0}\frac{x}{\sin x} &(1)\\
&= \lim_{x\to 0}\frac{1}{\cos x} &(2)\\
&= \frac{1}{\cos 0} &(3)\\
&= \frac{1}{1} &(4)\\
&= 1 &(5)
\end{aligned}$$

8. (Section 4.7, Exercise 63)

$$\begin{aligned}
\lim_{x\to\infty}\left(x^2-\sqrt{x^4+16x^2}\right) &= \lim_{x\to\infty}\left(x^2-\sqrt{x^4\left(1+\frac{16}{x^2}\right)}\right) && (1)\\
&= \lim_{x\to\infty}\left(x^2-x^2\sqrt{1+\frac{16}{x^2}}\right) && (2)\\
&= \lim_{x\to\infty} x^2\left(1-\sqrt{1+\frac{16}{x^2}}\right) && (3)\\
&= \lim_{x\to\infty}\frac{1-\sqrt{1+\frac{16}{x^2}}}{\frac{1}{x^2}} && (4)\\
&= \lim_{x\to\infty}\frac{\frac{16}{x^3}}{\frac{-2}{x^3}\sqrt{1+\frac{16}{x^2}}} && (5)\\
&= \lim_{x\to\infty}\frac{\frac{16}{x^3}\cdot\frac{x^3}{-2}}{\sqrt{1+\frac{16}{x^2}}} && (6)\\
&= \lim_{x\to\infty}\frac{\frac{16}{-2}\cdot\frac{x^3}{x^3}}{\sqrt{1+\frac{16}{x^2}}} && (7)\\
&= \lim_{x\to\infty}\frac{-8}{\sqrt{1+\frac{16}{x^2}}} && (8)\\
&= \frac{-8}{\sqrt{1+0}} && (9)\\
&= \frac{-8}{1} && (10)\\
&= -8 && (11)
\end{aligned}$$

9. (Section 4.7, Exercise 64)

$$\begin{aligned}
\lim_{x\to\infty}\left(x-\sqrt{x^2+4x}\right) &= \lim_{x\to\infty}\left(x-\sqrt{x^2\left(1+\frac{4}{x}\right)}\right) && (1)\\
&= \lim_{x\to\infty}\left(x-x\sqrt{1+\frac{4}{x}}\right) && (2)\\
&= \lim_{x\to\infty} x\left(1-\sqrt{1+\frac{4}{x}}\right) && (3)\\
&= \lim_{x\to\infty}\frac{1-\sqrt{1+\frac{4}{x}}}{\frac{1}{x}} && (4)\\
&= \lim_{x\to\infty}\frac{\frac{2}{x^2}}{\frac{-1}{x^2}\sqrt{1+\frac{4}{x}}} && (5)\\
&= \lim_{x\to\infty}\frac{\frac{2}{x^2}\cdot\frac{x^2}{-1}}{\sqrt{1+\frac{4}{x}}} && (6)\\
&= \lim_{x\to\infty}\frac{\frac{2}{-1}\cdot\frac{x^2}{x^2}}{\sqrt{1+\frac{4}{x}}} && (7)\\
&= \lim_{x\to\infty}\frac{-2}{\sqrt{1+\frac{4}{x}}} && (8)\\
&= \frac{-2}{\sqrt{1+0}} && (9)\\
&= -2 && (10)
\end{aligned}$$

10. (Section 4.7, Exercise 75)

$$\lim_{x\to 0^+} x^{2x} = e^{\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{2x}}} \quad (1)$$
$$= e^{\lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{2x^2}}} \quad (2)$$
$$= e^{\lim_{x\to 0^+} \frac{1}{x}\cdot\frac{2x^2}{-1}} \quad (3)$$
$$= e^{\lim_{x\to 0^+} -\frac{2x^2}{x}} \quad (4)$$
$$= e^{\lim_{x\to 0^+} -2x} \quad (5)$$
$$= e^{-2(0)} \quad (6)$$
$$= e^{0} \quad (7)$$
$$= 1 \quad (8)$$

11. (Section 4.7, Exercise 76)

$$\lim_{x\to 0} (1+4x)^{\frac{3}{x}} = e^{\lim_{x\to 0^+} \frac{\ln(1+4x)}{\frac{1}{\frac{3}{x}}}} \quad (1)$$
$$= e^{\lim_{x\to 0^+} \frac{\ln(1+4x)}{\frac{x}{3}}} \quad (2)$$
$$= e^{\lim_{x\to 0^+} \frac{\frac{4}{(1+4x)}}{\frac{1}{3}}} \quad (3)$$
$$= e^{\lim_{x\to 0^+} \frac{4}{(1+4x)}\cdot\frac{3}{1}} \quad (4)$$
$$= e^{\lim_{x\to 0^+} \frac{12}{(1+4x)}} \quad (5)$$
$$= e^{\frac{12}{1}} \quad (6)$$
$$= e^{12} \quad (7)$$

12. (Section 4.7, Exercise 96)

$$f(x) = x^2 \ln x \quad (1)$$
$$g(x) = \ln^2 x \quad (2)$$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^2 \ln x}{\ln^2 x} \quad (1)$$
$$= \lim_{x\to\infty} \frac{x^2}{\ln x} \quad (2)$$
$$= \lim_{x\to\infty} \frac{2x}{\frac{1}{x}} \quad (3)$$
$$= \lim_{x\to\infty} 2x^2 \quad (4)$$
$$= \infty \quad (5)$$

$$f \gg g$$

13. (Section 4.7, Exercise 100)

$$f(x) = x^2 \ln x \quad (1)$$
$$g(x) = x^3 \quad (2)$$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^2 \ln x}{x^3} \quad (1)$$
$$= \lim_{x\to\infty} \frac{\ln x}{x} \quad (2)$$
$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1} \quad (3)$$
$$= \lim_{x\to\infty} \frac{1}{x} \quad (4)$$
$$= \frac{1}{\infty} \neq \infty \quad (5)$$

$$g \gg f$$

14. (Section 4.7, Exercise 95)

$$\begin{aligned} f(x) &= x^{10} & (1) \\ g(x) &= e^{0.01x} & (2) \end{aligned}$$

$$\begin{aligned}
\lim_{x\to\infty} \frac{f(x)}{g(x)} &= \lim_{x\to\infty} \frac{x^{10}}{e^{0.01x}} & (1) \\
&= \lim_{x\to\infty} \frac{10x^9}{0.01e^{0.01x}} & (2) \\
&= \lim_{x\to\infty} \frac{90x^8}{0.01^2 e^{0.01x}} & (3) \\
&= \lim_{x\to\infty} \frac{7200x^7}{0.01^3 e^{0.01x}} & (4) \\
&= \lim_{x\to\infty} \frac{50400x^6}{0.01^4 e^{0.01x}} & (5) \\
&= \lim_{x\to\infty} \frac{302400x^5}{0.01^5 e^{0.01x}} & (6) \\
&= \lim_{x\to\infty} \frac{1512000x^4}{0.01^6 e^{0.01x}} & (7) \\
&= \lim_{x\to\infty} \frac{6048000x^3}{0.01^7 e^{0.01x}} & (8) \\
&= \lim_{x\to\infty} \frac{18144000x^2}{0.01^8 e^{0.01x}} & (9) \\
&= \lim_{x\to\infty} \frac{36288000x}{0.01^9 e^{0.01x}} & (10) \\
&= \lim_{x\to\infty} \frac{36288000}{0.01^{10} e^{0.01x}} & (11) \\
&= \frac{36288000}{\infty} \neq \infty & (12)
\end{aligned}$$

$$g \gg f$$

15. (Section 4.7, Exercise 101)

$$\begin{aligned} f(x) &= x^{20} & (1) \\ g(x) &= 1.00001^x & (2) \end{aligned}$$

$$\begin{aligned}
\lim_{x\to\infty} \frac{f(x)}{g(x)} &= \lim_{x\to\infty} \frac{x^{20}}{1.00001^x} & (1) \\
&= \frac{2432902008176640000}{\infty} \neq \infty & (2)
\end{aligned}$$

$$g \gg f$$

A copy of my notes (in LaTeX) are available on my GitHub