Module 1 Notes (MATH-211)

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10 June 2024

General Notes (and Definitions)

- Limit Definition(s):
  - Simple: The value that the outputs of a function approach as inputs approach a certain value
  - Preliminary: Suppose a function f is defined for all x near a except possibly at a. If $f(x)$ is arbitrarily close to L all x sufficiently close (but not equal) to a, we write the following.

$$\lim_{x\to a} = L$$

- Secant Line: a line passing through two points $(t_0, s(t_0))$ and $(t_1, s(t_1))$. The slope is given by

$$\frac{s(t_1) - s(t_0)}{t_1 - t_0}$$

- Tangent Line: the line passing through $(t_0, s(t_0))$ with slope

$$\lim_{t\to t_0} \frac{s(t) - s(t_0)}{t - t_0}$$

- One Sided limits:
  - Right-hand (Definition): Suppose a function f is defined for all x near a with $x > a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x > a$ we write

$$\lim_{x\to a^+} f(x) = L$$

  - Left-hand (Definition): Suppose a function f is defined for all x near a with $x < a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x < a$ we write

$$\lim_{x\to a^-} f(x) = L$$

  - In order for their to be a double sided limit, we must have:

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$$

  - If the limits from sides are not equal, then a the double sided limit, "does not exist"
- Limits can be simplified/solved in an easier way (as compared to numerically/graphically) using Limit Rules/Laws
- Limit Example Types:
  - Tangent lines
  - Velocity
- Velocity
  - Average Veolcity
    * The average velocity over some interval $[t_0, t_1]$ is defined as

$$v_{av} = \frac{s(t_1) - s(t_0)}{t_1 - t_0}$$

- Instantaneous Veolcity
    - The average velocity over some interval $[t_0, t_1]$ is defined as

$$v_{inst} = \lim_{t \to a} v_{av} == \frac{s(t) - s(a)}{t - a}$$

- Solving Techniques
  - Factoring and canceling out
  - Using conjugates
    - When direct substitution is not possible, you may rationalize the numerator
- Infinite Limits: In either case, the limit does not exist (not a real number) if it is infinite
  - Suppose f is defined for all x near a. If $f(x)$ gorws arbitrarily large for all x sufficiently close (but not equal) to a, we write

$$\lim_{x \to a} f(x) = \infty$$

  - If $f(x)$ is negative and gorws arbitrarily large in magnitude for all x sufficiently close (but not equal) to a, we write

$$\lim_{x \to a} f(x) = -\infty$$

  - The line $x = a$ is a vertical asymptote for f if any of the following hold

$$\lim_{x \to a} f(x) = \pm\infty$$

$$\lim_{x \to a^+} f(x) = \pm\infty$$

$$\lim_{x \to a^-} f(x) = \pm\infty$$

  - A vertical asymptote exists at $x = a$ if any one sided limit as $x \to a$ is ∞ or $-\infty$
  - If you have a limit of a rational function, where $p(a) = L \neq 0$ and $q(a) = 0$, then the one sided limits for $\frac{p(x)}{q(x)}$ approach $\pm\infty$

$$\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{L}{0}$$

- Limits as Infinity
  - **Definition**: If $f(x)$ becoomes arbitrarily close to a finite number L for all sufficiently large and positive x, the we write

$$\lim_{x \to \infty} f(x) = L$$

    The definition for

$$\lim_{x \to -\infty} f(x) = M$$

    is analogous.
  - If $\lim_{x \to \infty} f(x) = L$ we say that the function $f(x)$ has a horizontal asymptote at $y = L$
  - If $\lim_{x \to -\infty} f(x) = M$ we say that the function $f(x)$ has a horizontal asymptote at $y = M$
  - **Principle**: If $n > 0$ is an integer then

$$\lim_{x \to \pm\infty} \frac{1}{x^n} = 0$$

  - Suppose $f(x) = \frac{p(x)}{q(x)}$ is a rational function where

$$p(x) = a_m x^m + a_{m-1} x^{x-1} + ... + a_1 x + a_0$$

$$q(x) = b_n x^n + b_{n-1} x^{x-1} + ... + b_1 x + b_0$$

    If the degree of $p(x)$ is less than the degree of $q(x)$ then

$$\lim_{x \to \pm\infty} f(x) = 0$$

If the degree of $p(x)$ equals the degree of $q(x)$ then

$$\lim_{x \to \pm\infty} f(x) = \frac{a_m}{b_n}$$

If the degree of $p(x)$ is greater than the degree of $q(x)$ then

$$\lim_{x \to \pm\infty} f(x) = -\infty \text{ or } \infty$$

If the graph of a function f approaches a line (with finite and nonzero slope) as $x \to \pm\infty$, then that line is a slant asymptote/oblique asymptote of f

- End behaviour for transcendental functions

$$\lim_{x \to \pm\infty} \sin x = \text{Does not exist}$$

$$\lim_{x \to \infty} e^x = \infty$$

$$\lim_{x \to \infty} e^{-x} = \lim_{x \to \infty} \frac{1}{e^x} = 0$$

$$\lim_{x \to -\infty} e^x = 0$$

$$\lim_{x \to -\infty} e^{-x} = \infty$$

$$\lim_{x \to \infty} \ln x = \infty$$

$$\lim_{x \to 0^+} \ln x = -\infty$$

- Continuity

  - **Definition**: A function f is continuous at a if

$$\lim_{x \to a} f(x) = f(a)$$

  - A function f is continuous at a if
    1. $f(a)$ is defined (Removable Discontinuity)
    2. $\lim_{x \to a} f(x)$ exists (Jump Discontinuity)
    3. $\lim x \to af(x) = f(a)$ (Removable Discontinuity)
  - A function f has an **Infinite Discontinuity** at a if the function has a Vertical Asymptote at a
  - Suppose f is a function defined on an interval I. We say that f is continuous on interval I if f is continuous at every point on the interior of I and the following hold:
    1. If a is the the left-hand endpoint of I and a is contained in I then

$$\lim_{x \to a^+} f(x) = f(a) \ (f \text{ is continuous from the right})$$

    2. If b is the the righ-hand endpoint of I and b is contained in I then

$$\lim_{x \to b^-} f(x) = f(b) \ (f \text{ is continuous from the left})$$

  - **Theorem**: All of the following functions are continuous on the intervals where they are defined.
    1. Polynomials (continuous everywhere)
    2. Rational Functions (continuous except where denominator is zero)
    3. Exponential functions
    4. Logarithmic functions
    5. Trigonometric functions
    6. Inverse trigonometric functions
  - **Theorem**: If f and g are continuous at a, then the following functions are also continuous at a. Assume c is a constant and $n > 0$ is an integer.

1. $f + g$
2. $f - g$
3. cf
4. fg
5. $\frac{f}{g}$ provided $g(a) \neq 0$
6. $(f(x))^n$

– **Theorem**:
  1. A polynomial function is continuous for all x
  2. A rational function (a function of the form $\frac{p}{q}$, where p and q are polynomials) is continuous for all x for which $q(x) \neq 0$

– **Theorem**: If g is continuous at a and f is continuous at $g(a)$ then the composite function $f \circ g$ is continuous at a.

– **Theorem**: Assume n is a positive integer. If n is odd then $(f(x))^{1/n}$ is continuous at all points at which f is continuous. If n is even then $(f(x))^{1/n}$ is continuous at all points a at which f is continuous and $f(a) > 0$

– **Intermediate Value Theorem**: Suppose f is continuous on the interval $[a, b]$ and L is a number strictly between $f(a)$ and $f(b)$. Then there exists at least one number c in (a, b) satisfying $f(c) = L$.

Limit Rules/Laws

Assume $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist.

The following properties hold where c is a real number, and $n > 0$ is an integer.

- **Sum Rule**
$$\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$
- **Difference Rule**
$$\lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$
- **Constant Multiple Rule**
$$\lim_{x \to a} (cf(x)) = c \lim_{x \to a} f(x)$$
- **Product Rule**
$$\lim_{x \to a} (f(x)g(x)) = (\lim_{x \to a} f(x))(\lim_{x \to a} g(x))$$
- **Quotient Rule**
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}, \text{ provided } \lim_{x \to a} g(x) \neq 0$$
- **Power Rule**
$$\lim_{x \to a} f(x)^n = (\lim_{x \to a} f(x))^n$$
- **Root Rule**
$$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}, \text{ provided} f(x) > 0, \text{ for } x \text{ near } a, \text{ if } n \text{ is even}$$
- **Polynomials**
A **Polynomial** is defined as A function of the form $x_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$ where $n \geq 0$ is an integer If $p(x)$ is a polynomial then:
$$\lim_{x \to a} p(x) = p(a)$$
If $p(x)$ and $q(x)$ are polynomials and $q(a) \neq 0$ then (Direct Substitution):
$$\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$$

- **The Squeeze Theorem**
  Assume for some functions f, g and h that satisfy $f(x) \leq g(x) \leq h(x)$ for x near a (except possibly at $x = a$). If
  $$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$
  then
  $$\lim_{x \to a} g(x) = L$$
  As $x \to a, h(x) \to L$. Therefore, $g(x) \to L$. As x approaches a, if f and h approach the same value, so does g.

Examples

1. (Describing Limits) As x approaches 3, x^2 approaches 9
   $$\lim_{x \to 3} x^2 = 9$$
2. (Common Use) Values that are undefined can still have limits, given a graph G where $f(3) =$ undefined ($f(3)$ is a hole), the following limit is valid:
   $$\lim_{x \to 3} f(x) = 4$$
3. Calculating Limits Numerically:
   $$f(x) = \frac{x^3 - 1}{x - 1}$$

0.9	0.99	0.999	0.9999
2.71	2.9701	2.997001	2.99970001

1.1	1.01	1.001	1.0001
3.31	3.0301	3.003001	3.00030001

   As x approaches 1, $f(x)$ approaches 3: $\lim\limits_{x \to 1} \frac{x^3-1}{x-1} = 3$

4. Calculating One-sided limits:
   $$g(x) = \frac{x^3 - 4x}{8|x - 2|}$$

1.9	1.09	1.009	1.0009
-0.92625	-0.9925125	-0.999250125	-0.9999250013

2.1	2.01	2.001	2.0001
1.07625	1.0075125	1.000750125	1.000075001

$$\lim_{x \to 2} g(x) = \text{Does not exist}$$
$$\lim_{x \to 2^-} g(x) = -1$$
$$\lim_{x \to 2^+} g(x) = 1$$

5. Calculating piecewise function limits
   $$f(x) = \begin{cases} 3 - x \text{ if } x < 2 \\ x - 1 \text{ if } x > 2 \end{cases}$$
   $$a = 2$$

1.92	1.99	1.999	1.9999
1.1	1.01	1.001	1.0001

2.1	2.01	2.001	2.0001
1.1	1.01	1.001	1.0001

Explanation: Since $f(2)$ is not defined within the piece wise function, a graph representing this function would have a whole where $x = a$ and have two lines with inverse slopes

$$f(a) = \text{undefined}$$
$$\lim_{x\to a} f(x) = 1$$
$$\lim_{x\to a^-} f(x) = 1$$
$$\lim_{x\to a^+} f(x) = 1$$

6. Limit Rules/Laws:

   (a) Definitions:

$$\lim_{x\to 3} f(x) = 2$$
$$\lim_{x\to 3} g(x) = -1$$
$$\lim_{x\to 3} h(x) = 6$$

   (b) Problems:

      i. Sum, Constant Multiple

$$\begin{aligned}
\lim_{x\to 3}(f(x) + 2g(x)) &= \lim_{x\to 3} f(x) + \lim_{x\to 3} 2g(x) && (1)\\
&= \lim_{x\to 3} f(x) + 2(\lim_{x\to 3} g(x)) && (2)\\
&= 2 + 2(-1) && (3)\\
&= 0 && (4)
\end{aligned}$$

      ii. Quotient

$$\begin{aligned}
\lim_{x\to 3} \frac{h(x)}{g(x)} &= \frac{\lim_{x\to 3} h(x)}{\lim_{x\to 3} g(x)} && (1)\\
&= \frac{6}{-1} && (2)\\
&= -6 && (3)
\end{aligned}$$

      iii. Quotient, Root, Difference

$$\begin{aligned}
\lim_{x\to 3} \frac{h(x)}{\sqrt{f(x) - g(x)}} &= \frac{\lim_{x\to 3} h(x)}{\lim_{x\to 3} \sqrt{f(x) - g(x)}} && (1)\\
&= \frac{\lim_{x\to 3} h(x)}{\sqrt{\lim_{x\to 3}(f(x) - g(x))}} && (2)\\
&= \frac{\lim_{x\to 3} h(x)}{\sqrt{\lim_{x\to 3} f(x) - \lim_{x\to 3} g(x)}} && (3)\\
&= \frac{6}{\sqrt{2+1}} && (4)\\
&= \frac{6}{\sqrt{3}} && (5)\\
&= 2\sqrt{3} && (6)
\end{aligned}$$

7.

$$\begin{aligned}
\lim_{x\to 1} \frac{3x^2 - 7x + 1}{x + 2} &= \frac{3(1)^2 - 7(1) + 1}{1 + 2} && (1)\\
&= \frac{3 - 7 + 1}{1 + 2} && (2)\\
&= \frac{-3}{3} && (3)\\
&= -1 && (4)
\end{aligned}$$

8.

$$\begin{aligned}
\lim_{x\to 4} \frac{\left(\frac{1}{x}-\frac{1}{4}\right)}{x-4} &= \lim_{x\to 4} \frac{\left(\frac{4}{4x}-\frac{x}{4x}\right)}{x-4} && (1)\\
&= \lim_{x\to 4} \frac{\left(\frac{4-x}{4x}\right)}{x-4} && (2)\\
&= \lim_{x\to 4} \frac{\left(\frac{4-x}{4x}\right)}{\left(\frac{x-4}{1}\right)} && (3)\\
&= \lim_{x\to 4} \left(\frac{4-x}{4x}\right)\left(\frac{1}{x-4}\right) && (4)\\
&= \lim_{x\to 4} \frac{4-x}{4x(x-4)} && (5)\\
&= \lim_{x\to 4} \frac{-(-4+x)}{4x(x-4)} && (6)\\
&= \lim_{x\to 4} \frac{-(x-4)}{4x(x-4)} && (7)\\
&= \lim_{x\to 4} \frac{-1}{4x} && (8)\\
&= \lim_{x\to 4} \frac{-1}{4(4)} && (9)\\
&= -\frac{1}{16} && (10)
\end{aligned}$$

9.

$$\begin{aligned}
\lim_{x\to 9} \frac{x-9}{\sqrt{x}-3} &= \lim_{x\to 9} \frac{x-9}{\sqrt{x}-3}\cdot\frac{\sqrt{x}+3}{\sqrt{x}+3} && (1)\\
&= \lim_{x\to 9} \frac{(x-9)(\sqrt{x}+3)}{(\sqrt{x}-3)(\sqrt{x}+3)} && (2)\\
&= \lim_{x\to 9} \frac{(x-9)(\sqrt{x}+3)}{x-9} && (3)\\
&= \lim_{x\to 9} \sqrt{x}+3 && (4)\\
&= \sqrt{9}+3 && (5)\\
&= 3+3 && (6)\\
&= 6 && (7)
\end{aligned}$$

10.

$$1-\frac{x^2}{2} \le \cos x \le 1$$

$$\begin{aligned}
\lim_{x\to 0}\left(1-\frac{x^2}{2}\right) &= 1-\frac{0^2}{2} && (1)\\
&= 1-0 && (2)\\
&= 1 && (3)\\
&= \lim_{x\to 0} 1 && (4)\\
\lim_{x\to 0}\cos x &= 1 \qquad \text{(By the Squeeze Theorem)} && (5)
\end{aligned}$$

11.

$$\begin{aligned}
\lim_{x\to 0}\sin x &= 0 \qquad \text{(By the Squeeze Theorem)} && (1)\\
\lim_{x\to 0}\cos x &= 1 \qquad \text{(By the Squeeze Theorem)} && (2)
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to 0}\frac{\sin 2x}{\sin x} &= \lim_{x\to 0}\frac{2\sin x\cos x}{\sin x} && (1)\\
&= \lim_{x\to 0} 2\cos x && (2)\\
&= 2\lim_{x\to 0}\cos x && (3)\\
&= 2\cdot 1 && (4)\\
&= 2 && (5)
\end{aligned}$$

12. Infinite Limits Numerically

$$f(x) = \frac{x}{(x-2)^2}$$

2.1	2.01	2.001	2.0001
210	20100	2001000	200010000

1.9	1.99	1.999	1.9999
190	19900	1999000	199990000

$$\lim_{x\to 2} f(x) = \infty$$

13. Infinite Limits Graphically

$$\lim_{x\to -2^-} h(x) = -\infty$$

$$\lim_{x\to -2^+} h(x) = -\infty$$

$$\lim_{x\to -2} h(x) = -\infty$$

$$\lim_{x\to 3^-} h(x) = \infty$$

$$\lim_{x\to 3^+} h(x) = -\infty$$

$$\lim_{x\to 3} h(x) = \text{Does not exist}$$

14. Infinite Limits Analytically
Hint: Look at the signs of the fractions

$$\frac{x^2-5x+6}{x^4-4x^2} = \frac{(x-3)(x-2)}{x^2(x+2)(x-2)} = \frac{x-3}{x^2(x+2)}$$

$$\lim_{x\to -2^+}\frac{x^2-5x+6}{x^4-4x^2} = \lim_{x\to -2^+}\frac{x-3}{x^2(x+2)} = -\infty$$

$$\lim_{x\to -2^-}\frac{x^2-5x+6}{x^4-4x^2} = \lim_{x\to -2^-}\frac{x-3}{x^2(x+2)} = \infty$$

$$\lim_{x\to -2}\frac{x^2-5x+6}{x^4-4x^2} = \text{Does not exist}$$

15. Infinite Limits Analytically with Square Root

$$\lim_{x\to 1^+}\frac{x+3}{\sqrt{x^2-5x+4}} = \lim_{x\to 1^+}\frac{x+3}{\sqrt{(x-4)(x-1)}} = \text{Does not exist}$$

$$\lim_{x\to 1^-}\frac{x+3}{\sqrt{x^2-5x+4}} = \lim_{x\to 1^-}\frac{x+3}{\sqrt{(x-4)(x-1)}} = \infty$$

$$\lim_{x\to 1}\frac{x+3}{\sqrt{x^2-5x+4}} = \text{Does not exist}$$

16. Infinite Limit with a Trigonometric Function

$$\lim_{\theta\to 0^-}\frac{\sin\theta}{\cos^2\theta-1} = \lim_{\theta\to 0^-}\frac{\sin\theta}{-\sin^2\theta} = \lim_{\theta\to 0^-}\frac{1}{-\sin^2\theta} = \infty$$

17. Locating Veritical Asymptotes

$$f(x) = \frac{x+7}{x^4 - 49x^2} = \frac{x+7}{x^2(x^2-49)} = \frac{x+7}{x^2((x-7)(x+7))} = \frac{1}{x^2(x-7)}$$

Denominator is 0 at $x = 0$, $x = -7$, $x = 7$
$x = -7$ does not fit, as it is connected with $x + 7$, but cancels out
Vertical Asymptotes: $x = 0$, $x = 7$

18. Limits at Infinity

$$\lim_{x\to\infty} 5 + \frac{1}{x} + \frac{10}{x^2} = 5 + 0 + 0 = 5$$

$$\lim_{x\to\infty} 5 = 5$$

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

$$\lim_{x\to\infty} \frac{10}{x^2} = 0$$

19. End behaviour for rational functions (different degrees)
Hint: the degree of the numerator is less than the denominator

$$\lim_{x\to\infty} \frac{6x+1}{2x^2-5x+2} = \lim_{x\to\infty} \frac{\frac{6}{x}+\frac{1}{x^2}}{2-\frac{5}{x}+\frac{2}{x^2}} = \lim_{x\to\infty} \frac{0+0}{2-0+0} = \frac{0}{2} = 0$$

20. End behaviour for rational functions (equal degrees)
Hint: the degree of the numerator is the same as the denominator

$$\lim_{x\to\infty} \frac{6x^2+1}{2x^2-5x+2} = \lim_{x\to\infty} \frac{6+\frac{1}{x^2}}{2-\frac{5}{x}+\frac{2}{x^2}} = \lim_{x\to\infty} \frac{6+0}{2-0+0} = \frac{6}{2} = 3$$

21. End behaviour for rational functions (different degrees)
Hint: the degrees of the numerator is greater than the degree of the denominator

$$\lim_{x\to\infty} \frac{6x^4+1}{2x^2-5x+2} = \lim_{x\to\infty} \frac{6x^2+\frac{1}{x^2}}{2-\frac{5}{x}+\frac{2}{x^2}} = \lim_{x\to\infty} \frac{6x^2+0}{2-0+0} = \frac{\infty}{2} = \infty$$

22. End behaviours for rational functions
Hint: If there is a negative exponent like $2x^{-2}$, we can rewrite that as $\frac{2}{x^2}$
Hint: Keep in mind the direction at which x is changing (increasing or decreasing)

$$\lim_{x\to-\infty} 2x^{-8} + 4x^3 = \lim_{x\to-\infty} \frac{2}{x^8} + 4x^3 = 0 - \infty = -\infty$$

$$\lim_{x\to\infty} \frac{14x^3+3x^2-2x}{21x^3+x^2+2x+1} = \lim_{x\to\infty} \frac{14+\frac{3}{x}-\frac{2}{x^2}}{21+\frac{1}{x}+\frac{2}{x^2}+\frac{1}{x^3}} = \frac{14}{21} = \frac{2}{3}$$

$$\lim_{x\to\infty} \frac{9x^3+x^2-5}{3x^4+4x^2} = \lim_{x\to\infty} \frac{\frac{9}{x}+\frac{1}{x^2}-\frac{5}{x^4}}{3+\frac{4}{x^2}} = \frac{0}{3} = 0$$

23. Asymptotes for a rational function

$$f(x) = \frac{3x^2-7}{x^2+5x}$$

Horizontal Asymptote(s): $y = 3$
Explanation: The end behaviour for this function approaches 3 (on both ends), so there is a single horizontal asymptote

24. End behaviour for algebraic function

$$\begin{aligned}
\lim_{x\to-\infty}\frac{\sqrt{16x^2+x}}{x} &= \lim_{x\to-\infty}\frac{\frac{\sqrt{16x^2+x}}{x}}{\frac{x}{x}} && (1)\\
&= \lim_{x\to-\infty}\frac{\frac{1}{x}\sqrt{16x^2+x}}{1} && (2)\\
&= \lim_{x\to-\infty}\frac{1}{-\sqrt{x^2}}\sqrt{16x^2+x} && (3)\\
&= \lim_{x\to-\infty}-\sqrt{\frac{16x^2}{x^2}+\frac{x}{x^2}} && (4)\\
&= \lim_{x\to-\infty}-\sqrt{16+\frac{1}{x}} && (5)\\
&= \lim_{x\to-\infty}-\sqrt{16} && (6)\\
&= -4 && (7)\\
& && (8)
\end{aligned}$$

25. End behaviour for transcendental function

$$\lim_{x\to\infty}\frac{\sin x}{e^x+\ln x} = \lim_{x\to\infty}\frac{\sin x}{\infty+\infty} = 0$$

Explanation: Since $\sin x$ is bounded between -1 and 1, and the denominator is a very large number, we know as x increases, the function with approach zero

26. Continuity graphically
$x = 1$ (Removable Discontinuity)
$x = 2$ (Jump Discontinuity)
$x = 3$ (Removable Discontinuity)

27. Continuity analytically

$$f(x) = \begin{cases} \frac{x^2-4x+3}{x-3} & \text{if } x \neq 3 \\ 2 & \text{if } x = 3 \end{cases}$$

$$a = 3$$

$$f(3) = 2$$

$$\lim_{x\to3} f(x) = \lim_{x\to3}\frac{x^2-4x+3}{x-3} = \lim_{x\to3}\frac{(x-3)(x-1)}{x-3} = \lim_{x\to3}(x-1) = 2$$

The function f is continuous at a

28. Interval of continuity Hint: this is a composition of a polynomial and power function

$$f(x) = (x^2-1)^{\frac{3}{2}}$$

$f(x)$ is the composition $h \circ g(x)$ where $g(x) = x^2 - 1$, $h(x) = x^{\frac{3}{2}}$
$g(x)$ is continuous everywhere (because its a polynomial)
$h(x)$ is conintuous on $[0, \infty)$
Because this is a compoisite function, $f(x) = h \circ g(x)$ is continuous at a if $g(a) > 0$
The function $f(x)$ is continuous on $(-\infty, -1]$ and $[1, \infty)$

29. Intermediate Value Theorem

$$f(x) = x\ln x - 1$$

Note: $f(x)$ is continuous on $(0, \infty)$
Interval: $(1, e)$

$$f(1) = 1\ln 1 - 1 = 0 - 1 = -1 < 0$$

$$f(e) = e\ln e - 1 = e - 1 > 0$$

By the Intermediate Value Theorem, there is a $c \in (1, e)$ such that $f(c) = 0$
$c\ln c - 1 = 0$, meaning c is a solution to $x\ln x - 1 = 0$

30. (Section 2.1, Related Exercise 13):
Hint: use the secant line slope formula

$$s(t) = -16t^2 + 128t$$

(a) $[1, 4]$

$$\frac{256 - 112}{4 - 1} = \frac{144}{3} = 48$$

(b) $[1, 3]$

$$\frac{240 - 112}{3 - 1} = \frac{128}{2} = 64$$

(c) $[1, 2]$

$$\frac{192 - 112}{2 - 1} = \frac{80}{1} = 84$$

(d) $[1, 1 + h]$, where $h > 0$ is a real number

$$\frac{112 + -16h^2 + 128h - 112}{1 + h - 1} = \frac{-16h^2 + 128h}{h} = -16h + 128 = 16(-h + 6)$$

31. (Section 2.1, Related Exercise 15): Hint: we use the slope formula for the secant line, and the relationship is referring to the interval

$$s(t) = -16^t + 100t$$

$$\begin{aligned}
\frac{s(t_1) - s(t_0)}{t_1 - t_0} &= \frac{s(2) - s(0.5)}{2 - 0.5} && (1)\\
&= \frac{136 - 46}{1.5} && (2)\\
&= \frac{90}{1.5} && (3)\\
&= 60 && (4)
\end{aligned}$$

The slope of this secant line, through the lens of average velocity could be viewed as the average velocity over the interval $[0.5, 2]$

32. (Section 2.1, Related Exercise 17):

$$s(t) = -16t^2 + 128t$$

$[1, 2]$	$[1, 1.5]$	$[1, 1.1]$	$[1, 1.01]$	$[1, 1.001]$
80	88	94.4	95.84	95.984

$$v_{inst} = \lim_{t \to 1} s(t) = 96$$

33. (Section 2.1, Related Exercise 19):

$$s(t) = -16t^2 + 100t$$

$[2, 3]$	$[2.9, 3]$	$[2.99, 3]$	$[2.999, 3]$	$[2.9999, 3]$
20	5.6	4.16	4.016	4.002

$$v_{inst} = \lim_{t \to 3} s(t) = 4$$

34. (Section 2.2, Related Exercise 3):

- $h(2) = 5$
- $\lim_{x \to 2} h(x) = 3$
- $h(4) =$ Does not exist
- $\lim_{x \to 4} h(x) = 1$
- $\lim_{x \to 5} h(x) = 2$

35. (Section 2.2, Related Exercise 4):

- $g(0) = 0$
- $\lim_{x \to 0} g(x) = 1$
- $g(1) = 2$
- $\lim_{x \to 1} g(x) = 2$

36. (Section 2.2, Related Exercise 7):

$$f(x) = \frac{x^2 - 4}{x - 2}$$

1.9	1.99	1.999	1.9999
3.9	3.99	3.999	3.9999

2.1	2.01	2.001	2.0001
4.1	4.01	4.001	4.0001

$$\lim_{x \to 2} f(x) = 4$$

37. (Section 2.2, Related Exercise 8):

$$f(x) = \frac{x^3 - 1}{x - 1}$$

0.9	0.99	0.999	0.9999
2.71	2.9701	3.997001	3.99970001

1.1	1.01	1.001	1.0001
3.31	3.0301	3.003001	3.00030001

$$\lim_{x \to 1} f(x) = 3$$

38. (Section 2.2, Related Exercise 27):

$$f(x) = \frac{x - 2}{\ln |x - 2|}$$

$$\lim_{x \to 2} f(x) = 2$$

39. (Section 2.2, Related Exercise 28):

$$f(x) = \frac{e^{2x} - 2x - 1}{x^2}$$

$$\lim_{x \to 0} f(x) = 0$$

40. (Section 2.2, Related Exercise 19):

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x \leq -1 \\ 3 & \text{if } x > -1 \end{cases}$$

$$\lim_{x \to -1^-} f(x) = 2$$

$$\lim_{x \to -1^+} f(x) = 3$$

$$\lim_{x \to -1} f(x) = \text{Does not exist}$$

41. (Section 2.2, Related Exercise 20):

$$f(x) = \begin{cases} 3 - x & \text{if } x < 2 \\ x - 1 & \text{if } x > 2 \end{cases}$$

$$\lim_{x \to 2^-} f(x) = 1$$

$$\lim_{x \to 2^+} f(x) = 1$$

$$\lim_{x \to 2} f(x) = 1$$

42. (Section 2.3, Related Exercise 19):

$$\lim_{x\to 4} 3x - 7 = 3(4) - 7 = 12 - 7 = 5$$

43. (Section 2.3, Related Exercise 22):

$$\lim_{x\to 6} 4 = 4$$

44. (Section 2.3, Related Exercise 11): Quotient, Difference

$$\begin{aligned}
\lim_{x\to 1} \frac{f(x)}{g(x) - h(x)} &= \frac{\lim_{x\to 1} f(x)}{\lim_{x\to 1} g(x) - h(x)} && (1)\\
&= \frac{\lim_{x\to 1} f(x)}{\lim_{x\to 1} g(x) - h(x)} && (2)\\
&= \frac{\lim_{x\to 1} f(x)}{\lim_{x\to 1} g(x) - \lim_{x\to 1} h(x)} && (3)\\
&= \frac{8}{3-2} && (4)\\
&= \frac{8}{1} && (5)\\
&= 8 && (6)
\end{aligned}$$

45. (Section 2.3, Related Exercise 12): Root, Sum, Product

$$\begin{aligned}
\lim_{x\to 1} \sqrt[3]{f(x)g(x) + 3} &= \sqrt[3]{\lim_{x\to 1} f(x)g(x) + 3} && (1)\\
&= \sqrt[3]{\lim_{x\to 1} f(x)g(x) + 3} && (2)\\
&= \sqrt[3]{\lim_{x\to 1} f(x)g(x) + \lim_{x\to 1} 3} && (3)\\
&= \sqrt[3]{\lim_{x\to 1} f(x) \lim_{x\to 1} g(x) + \lim_{x\to 1} 3} && (4)\\
&= \sqrt[3]{8 \cdot 3 + 3} && (5)\\
&= \sqrt[3]{24 + 3} && (6)\\
&= \sqrt[3]{27} && (7)\\
&= 3 && (8)
\end{aligned}$$

46. (Section 2.3, Related Exercise 25):

$$\begin{aligned}
\lim_{x\to 1} \frac{5x^2 + 6x + 1}{8x - 4} &= \frac{5(1^2) + 6(1) + 1}{8(1) - 4} && (1)\\
&= \frac{5 + 6 + 1}{8 - 4} && (2)\\
&= \frac{12}{4} && (3)\\
&= 3 && (4)
\end{aligned}$$

47. (Section 2.3, Related Exercise 26):

$$\begin{aligned}
\lim_{t\to 3} \sqrt[3]{t^2 - 10} &= \sqrt[3]{\lim_{t\to 3} t^2 - 10} && (1)\\
&= \sqrt[3]{3^2 - 10} && (2)\\
&= \sqrt[3]{9 - 10} && (3)\\
&= \sqrt[3]{-1} && (4)\\
&= -1 && (5)
\end{aligned}$$

48. (Section 2.3, Related Exercise 27):

$$\begin{aligned}
\lim_{p\to 2}\frac{3p}{\sqrt{4p+1}-1} &= \frac{\lim\limits_{p\to 2} 3p}{\lim\limits_{p\to 2}\sqrt{4p+1}-1} && (1)\\
&= \frac{3(2)}{\sqrt{\lim\limits_{p\to 2} 4p+1}-1} && (2)\\
&= \frac{6}{\sqrt{4(2)+1}-1} && (3)\\
&= \frac{6}{\sqrt{8+1}-1} && (4)\\
&= \frac{6}{\sqrt{9}-1} && (5)\\
&= \frac{6}{3-1} && (6)\\
&= \frac{6}{2} && (7)\\
&= 3 && (8)
\end{aligned}$$

49. (Section 2.3, Related Exercise 72):

$$g(x) = \begin{cases} 5x-15 & \text{if } x<4 \\ \sqrt{6x+1} & \text{if } x\geq 4 \end{cases}$$

$$\lim_{x\to 4^-} g(x) = 5$$

$$\lim_{x\to 4^+} g(x) = 5$$

$$\lim_{x\to 4} g(x) = 5$$

50. (Section 2.3, Related Exercise 73):

$$g(x) = \begin{cases} x^2+1 & \text{if } x<-1 \\ \sqrt{x+1} & \text{if } x\geq -1 \end{cases}$$

$$\lim_{x\to -1^-} g(x) = 2$$

$$\lim_{x\to -1^+} g(x) = 0$$

$$\lim_{x\to -1} g(x) = \text{Does not exist}$$

51. (Section 2.3, Related Exercise 34):

$$\begin{aligned}
\lim_{x\to 3}\frac{x^2-2x-3}{x-3} &= \lim_{x\to 3}\frac{(x-3)(x+1)}{x-3} && (1)\\
&= \lim_{x\to 3} x+1 && (2)\\
&= 3+1 && (3)\\
&= 4 && (4)
\end{aligned}$$

52. (Section 2.3, Related Exercise 41):

$$\begin{aligned}
\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9} &= \lim_{x\to 9}\frac{\sqrt{x}-3}{x-9}\cdot\frac{\sqrt{x}+3}{\sqrt{x}+3} && (1)\\
&= \lim_{x\to 9}\frac{(\sqrt{x}-3)(\sqrt{x}+3)}{(x-9)(\sqrt{x}+3)} && (2)\\
&= \lim_{x\to 9}\frac{x-9}{(x-9)(\sqrt{x}+3)} && (3)\\
&= \lim_{x\to 9}\frac{1}{\sqrt{x}+3} && (4)\\
&= \frac{1}{\sqrt{9}+3} && (5)\\
&= \frac{1}{3+3} && (6)\\
&= \frac{1}{6} && (7)
\end{aligned}$$

53. (Section 2.3, Related Exercise 69):

$$\lim_{x\to 1^+}\frac{x-1}{\sqrt{x^2-1}} = \text{Does not exist}$$

54. (Section 2.3, Related Exercise 70):

$$\begin{aligned}
\lim_{x\to 1^+}\frac{x-1}{\sqrt{x^2-1}} &= \lim_{x\to 1^+}\frac{x-1}{\sqrt{x^2-1}}\cdot\frac{x+1}{x+1} && (1)\\
&= \lim_{x\to 1^+}\frac{x^2-1}{\sqrt{x^2-1}(x+1)} && (2)\\
&= \lim_{x\to 1^+}\frac{x^2-1}{(x^2-1)^{\frac{1}{2}}(x+1)} && (3)\\
&= \lim_{x\to 1^+}\frac{(x^2-1)^{\frac{1}{2}}}{x+1} && (4)\\
&= \lim_{x\to 1^+}\frac{\sqrt{x^2-1}}{x+1} && (5)\\
&= \frac{\sqrt{1-1}}{1+1} && (6)\\
&= \frac{\sqrt{0}}{2} && (7)\\
&= \frac{0}{2} && (8)\\
&= 0 && (9)
\end{aligned}$$

55. (Section 2.3, Related Exercise 95):

$$\frac{2^x-2^0}{x-0} = \frac{2^x-1}{x}$$

-1	-0.1	-0.01	-0.001	-0.0001	-0.00001
0.5	0.6696700846	0.6907504563	0.6929070095	0.6931231585	0.6931447783

$$\lim_{x\to 0^1}\frac{2^x-1}{x} = 0.693$$

56. (Section 2.3, Related Exercise 96):

$$\frac{3^x-3^0}{x-0} = \frac{3^x-1}{x}$$

-0.1	-0.01	-0.001	-0.0001
1.040415402	1.092599583	1.098009035	1.098551943

0.0001	0.001	0.01	0.1
1.098672638	1.099215984	1.104669194	1.161231740

$$\lim_{x\to 0^1} \frac{3^x - 1}{x} = 1.0986$$

57. (Section 2.3, Related Exercise 81):
$-|x| < 0 < |x|$ and $\sin\frac{1}{x} \leq 1$, so $|x|\sin\frac{1}{x} \leq |x|$ and $-|x|\sin\frac{1}{x} \geq -|x|$

$$\lim_{x\to 0} -|x| = -|0| = 0$$

$$\lim_{x\to 0} |x| = |0| = 0$$

$$\lim_{x\to 0} x\sin\frac{1}{x} = 0$$

By the Squeeze Theorem, since $\lim_{x\to 0} -|x| = \lim_{x\to 0} |x|$ and the functions are chronologically greater than the last

58. (Section 2.3, Related Exercise 82):

$$\lim_{x\to 0} 1 - \frac{x^2}{2} = 1 - \frac{0}{2} = 1 - 0 = 1$$

$$\lim_{x\to 0} 1 = 1$$

$$\lim_{x\to 0} \cos x = 1$$

By the Squeeze Theorem, since $\lim_{x\to 0} 1 - \frac{x^2}{2} = \lim_{x\to 0} 1$ and the functions are chronologically greater than the last

59. (Section 2.3, Related Exercise 60):

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin 2x}{\sin x} &= \lim_{x\to 0} \frac{2\sin x\cos x}{\sin x} && (1)\\
&= \lim_{x\to 0} 2\cos x && (2)\\
&= 2\cos 0 && (3)\\
&= 2\cdot 1 && (4)\\
&= 2 && (5)
\end{aligned}$$

60. (Section 2.3, Related Exercise 61):

$$\begin{aligned}
\lim_{x\to 0} \frac{1-\cos x}{\cos^2 x - 3\cos x + 2} &= \lim_{x\to 0} \frac{1}{\cos^2 x - 2\cos x + 2} && (1)\\
&= \lim_{x\to 0} \frac{1}{\cos x\cos x - 2\cos x + 2} && (2)\\
&= \frac{1}{\cos 0\cos 0 - 2\cos 0 + 2} && (3)\\
&= \frac{1}{1\cdot 1 - 2(1) + 2} && (4)\\
&= \frac{1}{1-2+2} && (5)\\
&= \frac{1}{1} && (6)\\
&= 1 && (7)
\end{aligned}$$

61. (Section 2.4, Related Exercise 6):

$$f(x) = \frac{x}{(x^2 - 2x - 3)^2}$$

$$\lim_{x\to -1} f(x) = -\infty$$

$$\lim_{x\to 3} f(x) = \infty$$

62. (Section 2.4, Related Exercise 7):

$$\lim_{x\to 1^-} f(x) = \infty$$
$$\lim_{x\to 1^+} f(x) = \infty$$
$$\lim_{x\to 1} f(x) = \infty$$
$$\lim_{x\to 2^-} f(x) = \infty$$
$$\lim_{x\to 2^+} f(x) = -\infty$$
$$\lim_{x\to 2} f(x) = \text{Does not exist}$$

63. (Section 2.4, Related Exercise 8):

$$\lim_{x\to 2^-} g(x) = \infty$$
$$\lim_{x\to 2^+} g(x) = -\infty$$
$$\lim_{x\to 2} g(x) = \text{Does not exist}$$
$$\lim_{x\to 4^-} g(x) = -\infty$$
$$\lim_{x\to 4^+} g(x) = -\infty$$
$$\lim_{x\to 4} g(x) = -\infty$$

64. (Section 2.4, Related Exercise 21):

$$\lim_{x\to 2^+} \frac{1}{x-2} = \infty$$
$$\lim_{x\to 2^-} \frac{1}{x-2} = -\infty$$
$$\lim_{x\to 2} \frac{1}{x-2} = \text{Does not exist}$$

65. (Section 2.4, Related Exercise 22):

$$\lim_{x\to 3^+} \frac{2}{(x-3)^3} = \infty$$
$$\lim_{x\to 3^-} \frac{2}{(x-3)^3} = -\infty$$
$$\lim_{x\to 3} \frac{2}{(x-3)^3} = \text{Does not exist}$$

66. (Section 2.4, Related Exercise 28):

$$\lim_{t\to -2^+} \frac{t^3-5t^2+6t}{t^4-4t^2} = \lim_{t\to -2^+} \frac{t(t-2)(t-3)}{t^2(t^2-4)} = \lim_{t\to -2^+} \frac{t(t-2)(t-3)}{t^2(t-2)(t+2)} = \lim_{t\to -2^+} \frac{t(t-3)}{t^2(t+2)} = \lim_{t\to -2^+} \frac{t^2-3t}{t^3+2t^2} = -\infty$$
$$\lim_{t\to -2^-} \frac{t^3-5t^2+6t}{t^4-4t^2} = \lim_{t\to -2^-} \frac{t(t-2)(t-3)}{t^2(t^2-4)} = \lim_{t\to -2^-} \frac{t(t-2)(t-3)}{t^2(t-2)(t+2)} = \lim_{t\to -2^-} \frac{t(t-3)}{t^2(t+2)} = \lim_{t\to -2^-} \frac{t^2-3t}{t^3+2t^2} = -\infty$$
$$\lim_{t\to -2} \frac{t^3-5t^2+6t}{t^4-4t^2} = \lim_{t\to -2} \frac{t(t-2)(t-3)}{t^2(t^2-4)} = \lim_{t\to -2} \frac{t(t-2)(t-3)}{t^2(t-2)(t+2)} = \lim_{t\to -2} \frac{t(t-3)}{t^2(t+2)} = \lim_{t\to -2} \frac{t^2-3t}{t^3+2t^2} = -\infty$$
$$\lim_{t\to 2} \frac{t^3-5t^2+6t}{t^4-4t^2} = \lim_{t\to 2} \frac{t(t-2)(t-3)}{t^2(t^2-4)} = -\frac{1}{8}$$

67. (Section 2.4, Related Exercise 31): Remember, if you are able to solve by direct substitution after canceling terms (where the denominator does not equal zero), that's your answer

$$\lim_{x\to 0} \frac{x-3}{x^4-9x^2} = \lim_{x\to 0} \frac{x-3}{x^2(x-3)(x+3)} = \lim_{x\to 0} \frac{1}{x^2(x+3)} = \lim_{x\to 0} \frac{1}{x^3+3x^2} = \infty$$
$$\lim_{x\to 3} \frac{x-3}{x^4-9x^2} = \lim_{x\to 3} \frac{x-3}{x^2(x-3)(x+3)} = \lim_{x\to 3} \frac{1}{x^2(x+3)} = \lim_{x\to -3} \frac{1}{x^3+3x^2} = \frac{1}{54}$$
$$\lim_{x\to -3} \frac{x-3}{x^4-9x^2} = \lim_{x\to -3} \frac{x-3}{x^2(x-3)(x+3)} = \lim_{x\to -3} \frac{1}{x^2(x+3)} = \lim_{x\to -3} \frac{1}{x^3+3x^2} = \text{Does not exist}$$

68. (Section 2.4, Related Exercise 45):

$$f(x) = \frac{x-5}{x^2-25} = \frac{x-5}{(x-5)(x+5)} = \frac{1}{x+5}$$

Vertical Asymptotes: $x = -5$

$$\lim_{x\to 5} f(x) = \lim_{x\to 5} \frac{1}{x+5} = \frac{1}{5+5} = \frac{1}{10}$$

$$\lim_{x\to -5^-} f(x) = \lim_{x\to -5^-} \frac{1}{x+5} = -\infty$$

$$\lim_{x\to -5^+} f(x) = \lim_{x\to -5^+} \frac{1}{x+5} = \infty$$

69. (Section 2.4, Related Exercise 46):

$$f(x) = \frac{x+7}{x^4-49x^2} = \frac{x+7}{x^2(x^2-49)} = \frac{x+7}{x^2(x+7)(x-7)} = \frac{1}{x^2(x-7)} = \frac{1}{x^3-7x^2}$$

Vertical Asymptotes: $x = 0$, $x = 7$, $x = -7$

$$\lim_{x\to 7^-} f(x) = \lim_{x\to 7^-} \frac{1}{x^3-6x^2} = -\infty$$

$$\lim_{x\to 7^+} f(x) = \lim_{x\to 7^+} \frac{1}{x^3-6x^2} = \infty$$

$$\lim_{x\to -7} f(x) = \lim_{x\to -7} \frac{1}{x^3-7x^2} = \text{Does not exist}$$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1}{x^3-7x^2} = -\infty$$

70. (Section 2.4, Related Exercise 39):

$$\lim_{\theta\to 0^+} \csc\theta = \infty$$

71. (Section 2.4, Related Exercise 40):

$$\lim_{x\to 0^-} \csc x = -\infty$$

72. (Section 2.5, Related Exercise 10)

$$\lim_{x\to\infty} 5 + \frac{1}{x} + \frac{10}{x^2} = 5 + 0 + 0 = 5$$

73. (Section 2.5, Related Exercise 19)

$$\lim_{x\to\infty} \frac{\cos x^5}{\sqrt{x}} = 0$$

74. (Section 2.5, Related Exercise 21)

$$\lim_{x\to\infty} 3x^1 2 - 9x^7 = \infty$$

75. (Section 2.5, Related Exercise 23)

$$\lim_{x\to -\infty} -3x^1 6 + 2 = -\infty$$

76. (Section 2.5, Related Exercise 38)

$$f(x) = \frac{3x^2-7}{x^2+5x}$$

Horizontal Asymptote: $y = 3$

77. (Section 2.5, Related Exercise 41)

$$f(x) = \frac{3x^3-7}{x^4+5x^2}$$

Horizontal Asymptote: $y = 0$

78. (Section 2.5, Related Exercise 43)

$$f(x) = \frac{40x^5 + x^2}{16x^4 - 2x}$$

Horizontal Asymptote: None
Explanation: Since the limit is infinity, there is no horizontal asymptote.

79. (Section 2.5, Related Exercise 51)

$$f(x) = \frac{x^2 - 3}{x + 6}$$

Slant Asymptote: $y = x - 6$
Vertical Asymptote: $x = -6$

80. (Section 2.5, Related Exercise 52)

$$f(x) = \frac{x^2 - 1}{x + 2}$$

Slant Asymptote: $y = x - 2$
Vertical Asymptote: $x = -2$

81. (Section 2.5, Related Exercise 46)

$$f(x) = \frac{\sqrt{x^2 + 1}}{2x + 1}$$

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{\sqrt{x^2+1}}{2x+1} && (1)\\
&= \lim_{x\to\infty} \frac{\sqrt{\frac{x^2}{x^2} + \frac{1}{x^2}}}{\frac{2x}{x} + \frac{1}{x}} && (2)\\
&= \lim_{x\to\infty} \frac{\sqrt{1 + \frac{1}{x^2}}}{2 + \frac{1}{x}} && (3)\\
&= \frac{\sqrt{1+0}}{2+0} && (4)\\
&= \frac{\sqrt{1}}{2} && (5)\\
&= \frac{1}{2} && (6)
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to-\infty} f(x) &= \lim_{x\to-\infty} \frac{-\sqrt{x^2+1}}{2x+1} && (1)\\
&= \lim_{x\to-\infty} \frac{-\sqrt{\frac{x^2}{x^2} + \frac{1}{x^2}}}{\frac{2x}{x} + \frac{1}{x}} && (2)\\
&= \frac{-\sqrt{1+0}}{2+0} && (3)\\
&= \frac{-1}{2} && (4)\\
&= -\frac{1}{2} && (5)
\end{aligned}$$

Horizontal Asymptotes: $y = \frac{1}{2}$, $y = -\frac{1}{2}$

82. (Section 2.5, Related Exercise 47)

$$f(x) = \frac{4x^3 + 1}{2x^3 + \sqrt{16x^6 + 1}}$$

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{4x^3+1}{2x^3+\sqrt{16x^6+1}} && (1)\\
&= \lim_{x\to\infty} \frac{\frac{4x^3}{x^3}+\frac{1}{x^3}}{\frac{2x^3}{x^3}+\sqrt{\frac{16x^6}{x^6}+\frac{1}{x^6}}} && (2)\\
&= \frac{4+0}{2+\sqrt{16+0}} && (3)\\
&= \frac{4}{2+\sqrt{16}} && (4)\\
&= \frac{4}{2+4} && (5)\\
&= \frac{4}{6} && (6)\\
&= \frac{2}{3} && (7)
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to-\infty} f(x) &= \lim_{x\to-\infty} \frac{4x^3+1}{2x^3-\sqrt{16x^6+1}} && (1)\\
&= \lim_{x\to-\infty} \frac{\frac{4x^3}{x^3}+\frac{1}{x^3}}{\frac{2x^3}{x^3}-\sqrt{\frac{16x^6}{x^6}+\frac{1}{x^6}}} && (2)\\
&= \frac{4+0}{2-\sqrt{16+0}} && (3)\\
&= \frac{4}{2-\sqrt{16}} && (4)\\
&= \frac{4}{2-4} && (5)\\
&= \frac{4}{-2} && (6)\\
&= -2 && (7)
\end{aligned}$$

Horizontal Asymptotes: $y = \frac{2}{3}$, $y = -2$

83. (Section 2.5, Related Exercise 57)

$$f(x) = -3e^{-x}$$

$$\lim_{x\to\infty} -3e^{-x} = 3(0) = 0$$

$$\lim_{x\to-\infty} -3e^{-x} = 3(\infty) = \infty$$

Horizontal Asymptotes: $y = 0$

84. (Section 2.5, Related Exercise 59)

$$f(x) = 1 - \ln x$$

$$\lim_{x\to\infty} (1-\ln x) = 1-\infty = -\infty$$

$$\lim_{x\to 0^+} (1-\ln x) = 1+\infty = \infty$$

85. (Section 2.5, Related Exercise 62)

$$f(x) = \frac{50}{e^{2x}}$$

$$\lim_{x\to\infty} \frac{50}{e^{2x}} = \frac{50}{\infty} = 0$$

$$\lim_{x\to-\infty} \frac{50}{e^{2x}} = \text{Does not exist}$$

86. (Section 2.6, Related Exercise 5)
$x = 1$ (Removable Discontinuity)
$x = 2$ (Removable Discontinuity)
$x = 3$ (Jump Discontinuity)

87. (Section 2.6, Related Exercise 6)
$x = 1$ (Removable Discontinuity)
$x = 2$ (Jump Discontinuity)
$x = 3$ (Removable Discontinuity)

88. (Section 2.6, Related Exercise 17)

$$f(x) = \frac{2x^2 + 3x + 1}{x^2 + 5x}$$

$$a = -5$$

$$f(a) = f(-5) = \frac{2(-5)^2 + 3(-5) + 1}{(-5)^2 + 5(-5)} = \frac{2(25) - 15 + 1}{25 - 25} = \frac{50 - 15 + 1}{0} = \frac{36}{0} = \text{undefined}$$

f is not continuous at a as $f(a)$ is undefined.

89. (Section 2.6, Related Exercise 22)

$$f(x) = \begin{cases} \frac{x^2-4x+3}{x-3} & \text{if } x \neq 3 \\ 2 & \text{if } x = 3 \end{cases}$$

$$a = 3$$

$$f(a) = f(3) = 2$$

$$\lim_{x\to a} f(x) = \lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{x^2 - 4x + 3}{x - 3} = \lim_{x\to 3} \frac{(x-1)(x-3)}{x - 3} = \lim_{x\to 3} (x - 1) = 2$$

f is continuous at a

90. (Section 2.6, Related Exercise 26)
$(-\infty, \infty)$

91. (Section 2.6, Related Exercise 27)
$(-\infty, -3)$
$(-3, 3)$
$(3, \infty)$
Explanation: Because of the denominator, this function will be undefined when x is -3 or 3

92. (Section 2.6, Related Exercise 31)

$$\lim_{x\to 0} (x^8 - 3x^6 - 1)^{40} = (0 - 0 - 1)^{40} = -1^{40} = 1$$

93. (Section 2.6, Related Exercise 32)

$$\lim_{x\to 2} \left(\frac{3}{2x^5 - 4x^2 - 50}\right)^4 = \left(\frac{3}{2(2)^5 - 4(2)^2 - 50}\right)^4 = \left(\frac{3}{64 - 16 - 50}\right)^4 = \left(\frac{3}{-2}\right)^4 = \left(-\frac{3}{2}\right)^4 = \frac{81}{16}$$

94. (Section 2.6, Related Exercise 33)

$$\begin{aligned} \lim_{x\to 4} \sqrt{\frac{x^3 - 2x^2 - 8x}{x - 4}} &= \lim_{x\to 4} \sqrt{\frac{x(x^2 - 2x - 8)}{x - 4}} && (1) \\ &= \lim_{x\to 4} \sqrt{\frac{x(x-4)(x+2)}{x - 4}} && (2) \\ &= \lim_{x\to 4} \sqrt{x(x+2)} && (3) \\ &= \lim_{x\to 4} \sqrt{x^2 + 2x} && (4) \\ &= \sqrt{4^2 + 2(4)} && (5) \\ &= \sqrt{16 + 8} && (6) \\ &= \sqrt{24} && (7) \\ &= 2\sqrt{6} && (8) \end{aligned}$$

95. (Section 2.6, Related Exercise 34)

$$\lim_{x\to 4} \frac{t - 4}{\sqrt{t} - 2} = \lim_{x\to 4} \frac{t - 4}{\sqrt{t} - 2} \cdot \frac{\sqrt{t} + 2}{\sqrt{t} + 2} = \lim_{x\to 4} \frac{(t-4)(\sqrt{t} + 2)}{t - 4} = \lim_{x\to 4} \sqrt{t} + 2 = \sqrt{4} + 2 = 2 + 2 = 4$$

96. (Section 2.6, Related Exercise 39)

$$f(x) = \begin{cases} 2x & \text{if } x < 1 \\ x^2 + 3x & \text{if } x \geq 1 \end{cases}$$
$$a = 1$$
$$f(a) = f(1) = 1^2 + 3(1) = 1 + 3 = 4$$
$$\lim_{x \to 1^-} f(x) = 2$$
$$\lim_{x \to 1^+} f(x) = 4$$
$$\lim_{x \to 1} = \text{Does not exist}$$

The function f is continuous from the right to a.
The function f is continuous in the following intervals:

- $(-\infty, 1)$
- $[1, \infty)$

97. (Section 2.6, Related Exercise 40)

$$f(x) = \begin{cases} x^3 + 4x + 1 & \text{if } x \geq 0 \\ 2x^3 & \text{if } x > 0 \end{cases}$$
$$a = 0$$
$$f(a) = f(0) = 0^3 + 4(0) + 1 = 1$$
$$\lim_{x \to 0^-} f(x) = 1$$
$$\lim_{x \to 0^+} f(x) = 0$$
$$\lim_{x \to 0^+} f(x) = \text{Does not exist}$$

The function f is continuous from the right of a.
The function f is continuous in the following intervals:

- $(-\infty, 0)$
- $[0, \infty)$

98. (Section 2.6, Related Exercise 44)

$$f(x) = \sqrt{x^2 - 3x + 2}$$

The function f is continuous at $(-\infty, 1)$ and $(2, \infty)$

99. (Section 2.6, Related Exercise 45)

$$f(x) = \sqrt[3]{x^2 - 2x - 3}$$

The function f is continuous at $(-\infty, \infty)$

100. (Section 2.6, Related Exercise 62)

$$f(x) = e^{\sqrt{x}}$$

The function f is continuous at $[0, \infty)$

$$\lim_{x \to 4} f(x) = e^{\sqrt{4}} = e^2$$
$$\lim_{x \to 0^+} f(x) = e^0 = 1$$

101. (Section 2.6, Related Exercise 63)

$$f(x) = \frac{1 + \sin x}{\cos x} = \frac{1}{\cos x} + \frac{\sin x}{\cos x} = \sec x + \tan x$$

The function f is continuous at $\left(\frac{n\pi}{2}, \frac{(n+2)\pi}{2}\right)$

$$\lim_{x \to \pi/2^-} f(x) = \infty$$
$$\lim_{x \to 4\pi/3} f(x) = \frac{1}{\cos x} + \frac{\sin x}{\cos x} = \frac{1}{-0.5} + \frac{\sin x}{-0.5} = -2 + \left(-\frac{\sqrt{3}}{2} \div -0.5\right) = -2 + \left(-\frac{\sqrt{3}}{2} \cdot -2\right) = \sqrt{3} - 2$$

102. (Section 2.6, Related Exercise 67)

$$2x^3 + x - 2 = 0$$

$$(-1, 1)$$

$$f(-1) = 2(-1^3) + (-1) - 2 = -2 - 1 - 2 = -5$$

$$f(1) = 2(1^3) + 1 - 2 = 2 + 1 - 2 = 1$$

$$-5 < 0 < 1$$

$$x \approx 0.835$$

103. (Section 2.6, Related Exercise 75)

$$A(r) = 5000\left(1 + \frac{r}{12}\right)^{120} = 7000$$

$$(0, 0.08)$$

$$A(0) = 5000\left(1 + \frac{0}{12}\right)^{120} = 5000(1+0)^{120} = 5000$$

$$A(0.08) = 5000\left(1 + \frac{0.08}{12}\right)^{120} \approx 5000(1{+}0.006667)^{120} \approx 5000(1.006667)^{120} \approx 5000(2.21964) \approx 11098.20117$$

$$5000 < 7000 < 11098.20117$$

$$r \approx 0.034$$

A copy of my notes (in LaTeX) are available on my GitHub