

Module 2 Notes (MATH-211)

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General Notes (and Definitions)

- Derivatives

- A **derivative** is a new function made up of the slopes of the tangent lines as they change along a curve
- If a curve represents the trajectory of a moving object, the tangent line at a point indicates the direction of motion at that point
- As $x \rightarrow a$, the slope of the secant lines approaches the slope of the tangent line
- Alternative definition for Tangent Line(s): Consider the curve $y = f(x)$ and a secant line intersecting the curve at points $P(a, f(a))$ and $Q(a + h, f(a + h))$, with m_{sec} and m_{tan}

Interval: $(a, a + h)$

$$m_{sec} = \frac{f(a + h) - f(a)}{h}$$

$$m_{tan} = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$

$$y - f(a) = m_{tan}(x - a)$$

- **Definition:** The derivative of f at a , denoted $f'(a)$, is given by either the two following limits, provided the limits exist and a is in the domain of f

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \quad (1)$$

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h} \quad (2)$$

If $f'(a)$ exists, we say that f is **differentiable** at a

- Derivatives as Functions

- The slope of the tangent line of some function f is a function called the derivative of f

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

- If $f'(x)$ exists, we say that f is **differentiable** at x
- If f is differentiable at every point in some open interval I , we say that f is differentiable on I
- For some function f we can denote the derivative of f like such:

$$f'(x) \quad (1)$$

$$\frac{dy}{dx} \quad (2)$$

$$\frac{df}{dx} \quad (3)$$

$$\frac{d}{dx}(f(x)) \quad (4)$$

$$D_x(f(x)) \quad (5)$$

$$y'(x) \quad (6)$$

- When evaluating some derivative f at a , we can use the following:

$$f'(a) \tag{1}$$

$$y'(a) \tag{2}$$

$$\left. \frac{df}{dx} \right|_{x=a} \tag{3}$$

$$\left. \frac{dy}{dx} \right|_{x=a} \tag{4}$$

- If f is differentiable at a , then f is continuous at a
- If f is not continuous at a , then f is not differentiable at a

Rules of Differentiation

- Constant Rule

$$\text{If } c \in \mathbb{R}, \text{ then } \frac{d}{dx}(c) = 0$$

- Power Rule

$$\text{If } n \in \mathbb{Z} \text{ and } n > 0, \text{ then } \frac{d}{dx}(x^n) = nx^{n-1}$$

- Derivative of a Root

$$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

- Constant Multiple Rule

$$\text{If } f \text{ is differentiable at } x \text{ and } c \text{ is a constant, then } \frac{d}{dx}(cf(x)) = cf'(x)$$

- Sum Rule

$$\text{If } f \text{ and } g \text{ are differentiable at } x, \text{ then } \frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$$

- Generalized Sum Rule

$$\frac{d}{dx}(f_1(x) + f_2(x) + \dots + f_n(x)) = f'_1(x) + f'_2(x) + \dots + f'_n(x)$$

- Difference Rule

$$\frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x)$$

- Euler's Number

$$\text{The function } f(x) = e^x \text{ is differentiable for all } x \in \mathbb{R}, \text{ and } \frac{d}{dx}(e^x) = e^x$$

- Higher-order Derivatives

Assuming $y = f(x)$ can be differentiated as often as necessary, the **second derivative** of f is

$$f''(x) = \frac{d}{dx}(f'(x))$$

For $n \in \mathbb{Z}$ where $n \geq 1$, the **nth derivative** of f is

$$f^{(n)}(x) = \frac{d}{dx}(f^{(n-1)}(x))$$

- Product Rule

$$\text{If } f \text{ and } g \text{ are differentiable at } x, \text{ then } \frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

- Quotient Rule

If f and g are differentiable at x and $g(x) \neq 0$, then the derivative of $\frac{f}{g}$ at x exists and

$$\frac{d}{dx} \left(\frac{f(x)}{g(x)} \right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

- Power Rule for Negative Integers

Proof.

Choose $n \in \mathbb{Z}$

Assume that $n < 0$

Let $m = -n$

Since $m = -n$ and $n < 0$, we know $m > 0$

$$\frac{d}{dx} (x^n) = \frac{d}{dx} \left(\frac{1}{x^m} \right) \quad (1)$$

$$= \frac{x^m \cdot 0 - 1 \cdot (mx^{m-1})}{(x^m)^2} \quad (2)$$

$$= \frac{-mx^{m-1}}{x^{2m}} \quad (3)$$

$$= -mx^{(m-1)-2m} \quad (4)$$

$$= -mx^{-m-1} \quad (5)$$

$$= -nx^{n-1} \quad (6)$$

$$(7)$$

Under the assumption that $n \in \mathbb{Z}$ and $n < 0$, we proved $\frac{d}{dx} (x^n) = nx^{n-1}$

□

Examples

1. Instantaneous Velocity

$$s(t) = -16t^2 + 128t + 192$$

$$t = 2$$

$$\lim_{t \rightarrow 2} \frac{s(t) - s(2)}{t - 2} = \lim_{t \rightarrow 2} \frac{(-16t^2 + 128t + 192) - (-16(2^2) + 128(2) + 192)}{t - 2} \quad (1)$$

$$= \lim_{t \rightarrow 2} \frac{(-16t^2 + 128t + 192) - (-64 + 256 + 192)}{t - 2} \quad (2)$$

$$= \lim_{t \rightarrow 2} \frac{(-16t^2 + 128t + 192) - 384}{t - 2} \quad (3)$$

$$= \lim_{t \rightarrow 2} \frac{-16t^2 + 128t - 192}{t - 2} \quad (4)$$

$$= \lim_{t \rightarrow 2} \frac{(t - 2)(-16t + 96)}{t - 2} \quad (5)$$

$$= \lim_{t \rightarrow 2} -16t + 96 \quad (6)$$

$$= -32 + 96 \quad (7)$$

$$= 64 \quad (8)$$

2. Secant Lines

$$y = f(x)$$

Intersection Points: $P(a, f(a))$ and $Q(x, f(x))$

$$\text{Secant Line Slope} = \frac{f(x) - f(a)}{x - a}$$

3. Tangent Lines

$$f(x) = 2x^2 + 4x - 3$$

$$(-1, 5)$$

$$\lim_{x \rightarrow -1} \frac{f(x) - f(-1)}{x - (-1)} = \lim_{x \rightarrow -1} \frac{2x^2 + 4x - 3 - (-5)}{x + 1} \quad (1)$$

$$= \lim_{x \rightarrow -1} \frac{2x^2 + 4x + 2}{x + 1} \quad (2)$$

$$= \lim_{x \rightarrow -1} \frac{(x + 1)(2x + 2)}{x + 1} \quad (3)$$

$$= \lim_{x \rightarrow -1} \frac{(x + 1)(2x + 2)}{x + 1} \quad (4)$$

$$= \lim_{x \rightarrow -1} 2x + 2 \quad (5)$$

$$= 2(-1) + 2 \quad (6)$$

$$= -2 + 2 \quad (7)$$

$$= 0 \quad (8)$$

4. Alternative Tangent Lines

$$f(x) = 5 - x^3$$

$$(2, -3)$$

$$a = 2$$

$$h = -3 - 2 = -5$$

$$\lim_{h \rightarrow 0} \frac{f(2 + h) - f(2)}{h} = \lim_{h \rightarrow 0} \frac{f(2 + h) - (-3)}{h} \quad (1)$$

$$= \lim_{h \rightarrow 0} \frac{f(2 + h) + 3}{h} \quad (2)$$

$$= \lim_{h \rightarrow 0} \frac{5 - (2 + h)^3 + 3}{h} \quad (3)$$

$$= \lim_{h \rightarrow 0} \frac{8 - (2 + h)^3}{h} \quad (4)$$

$$= \lim_{h \rightarrow 0} \frac{2^3 - (2 + h)^3}{h} \quad (5)$$

$$= \lim_{h \rightarrow 0} \frac{(2 - (2 + h))(2^2 + 2(2 + h) + (2 + h)^2)}{h} \quad (6)$$

$$= \lim_{h \rightarrow 0} \frac{-h(4 + 4 + 2h + h^2 + 4h + 4)}{h} \quad (7)$$

$$= \lim_{h \rightarrow 0} \frac{-h(h^2 + 6h + 12)}{h} \quad (8)$$

$$= \lim_{h \rightarrow 0} -(h^2 + 6h + 12) \quad (9)$$

$$= -12 \quad (10)$$

$$(11)$$

$$y + 3 = -12(x - 2) = -12x + 24$$

$$y = -12x + 21$$

5. Derivative Example

$$f(x) = \sqrt{x-1}$$

$$x = 2$$

$$f(x) = f(2) = \sqrt{2-1} = \sqrt{1} = 1$$

$$(2, 1)$$

$$f'(2) = \lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2} \quad (1)$$

$$= \lim_{x \rightarrow 2} \frac{\sqrt{x-1} - 1}{x - 2} \quad (2)$$

$$= \lim_{x \rightarrow 2} \frac{\sqrt{x-1} - 1}{x - 2} \cdot \frac{\sqrt{x-1} + 1}{\sqrt{x-1} + 1} \quad (3)$$

$$= \lim_{x \rightarrow 2} \frac{x - 2}{(x - 2)(\sqrt{x-1} + 1)} \quad (4)$$

$$= \lim_{x \rightarrow 2} \frac{1}{\sqrt{x-1} + 1} \quad (5)$$

$$= \frac{1}{\sqrt{2-1} + 1} \quad (6)$$

$$= \frac{1}{\sqrt{1} + 1} \quad (7)$$

$$= \frac{1}{1 + 1} \quad (8)$$

$$= \frac{1}{2} \quad (9)$$

$$(10)$$

$$y - 1 = \frac{1}{2}(x - 2) \quad (1)$$

$$y = \frac{1}{2}(x - 2) + 1 \quad (2)$$

$$= \frac{1}{2}x - 1 + 1 \quad (3)$$

$$= \frac{1}{2}x \quad (4)$$

6. Derivative Application Example

$$V(t) = 3t$$

$$V'(12) = \lim_{x \rightarrow 12} \frac{V(x) - V(12)}{x - 12} \quad (1)$$

$$= \lim_{x \rightarrow 12} \frac{3x - 36}{x - 12} \quad (2)$$

$$= \lim_{x \rightarrow 12} \frac{3(x - 12)}{x - 12} \quad (3)$$

$$= \lim_{x \rightarrow 12} 3 \quad (4)$$

$$= 3 \quad (5)$$

$$y - 36 = 3(x - 12) \quad (1)$$

$$y = 3x - 36 + 36 \quad (2)$$

$$= 3x \quad (3)$$

$$(4)$$

7. Find the Derivative

$$f(x) = 4x^2 - 5x + 6$$

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{4(x+h)^2 - 5(x+h) + 6 - (4x^2 - 5x + 6)}{h} \quad (1)$$

$$= \lim_{h \rightarrow 0} \frac{4(x+h)^2 - 5x - 5h + 6 - 4x^2 + 5x - 6}{h} \quad (2)$$

$$= \lim_{h \rightarrow 0} \frac{4(x+h)^2 - 5h - 4x^2}{h} \quad (3)$$

$$= \lim_{h \rightarrow 0} \frac{4h^2 + 8xh - 5h}{h} \quad (4)$$

$$= \lim_{h \rightarrow 0} 4h + 8x - 5 \quad (5)$$

$$= 4(0) + 8x - 5 \quad (6)$$

$$= 8x - 5 \quad (7)$$

8. Calculating a Derivative

$$f(x) = \frac{1}{x}$$

$$\left(-5, -\frac{1}{5}\right)$$

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad (1)$$

$$= \lim_{h \rightarrow 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \quad (2)$$

$$= \lim_{h \rightarrow 0} \frac{\frac{x - (x+h)}{x(x+h)}}{h} \quad (3)$$

$$= \lim_{h \rightarrow 0} \frac{\frac{-h}{x(x+h)}}{h} \quad (4)$$

$$= \lim_{h \rightarrow 0} \frac{-h}{x(x+h)} \cdot \frac{1}{h} \quad (5)$$

$$= \lim_{h \rightarrow 0} \frac{-h}{h(x(x+h))} \quad (6)$$

$$= \lim_{h \rightarrow 0} \frac{-1}{x(x+h)} \quad (7)$$

$$= \frac{-1}{x(x+0)} \quad (8)$$

$$= \frac{-1}{x^2} \quad (9)$$

$$m_{\text{tan}} = \frac{dy}{dx} \Big|_{x=-5} = \frac{-1}{(-5)^2} = \frac{-1}{25}$$

$$y - \left(-\frac{1}{5}\right) = \frac{-1}{25}(x - (-5))$$

$$y = \frac{-1}{25}x - \frac{1}{5} - \frac{1}{5} = \frac{-1}{25}x - \frac{2}{5}$$

9. Constant and Power Rules

$$\frac{d}{dx} (x^6) = 6x^5$$

$$\frac{d}{dx} (x) = 1x^0 = 1$$

$$\frac{d}{dx} (\pi^2) = 0$$

10. Constant Multiple Rule

$$\frac{d}{dx} (-4x^9) = -4(9x^8) = -36x^8$$

$$\frac{d}{dt} \left(\frac{2}{5}t^5\right) = \frac{2}{5}(5t^4) = 2t^4$$

11. Sum and Difference Rules with a Polynomial

$$\frac{d}{dx} \left(6x^5 - \frac{5}{2}x^2 + x + 5 \right) = 30x^4 - 5x + 1$$

12. Euler's Number with Derivatives

$$f(x) = 5x + \frac{1}{3}e^x$$

$$\left(0, \frac{1}{3} \right)$$

$$\frac{d}{dx} \left(5x + \frac{1}{3}e^x \right) = 5 + \frac{1}{3}e^x$$

$$y = \frac{16}{3}x + \frac{1}{3}$$

13. Higher-order derivatives

$$f(x) = 3x^4 - 2x^2 + 7x - e^x$$

$$f'(x) = 12x^3 - 4x + 7 - e^x$$

$$f''(x) = 36x^2 - 4 - e^x$$

$$f'''(x) = 72x - e^x$$

$$f^{(4)}(x) = 72 - e^x$$

14. Product Rule

$$\frac{d}{dt} ((t+1)(t^2-t+1)) = 1(t^2-t+1) + (2t-1)(t+1) = 3t^2$$

$$\frac{d}{dx} (x^5 e^x) = 5x^4(e^x) + e^x(x^5) = e^x(x^5 + 5x^4)$$

15. Quotient Rule

$$\frac{d}{dx} \left(\frac{x^4 + 5x^2 + x}{\sin x} \right) = \frac{\sin x(4x^3 + 10x + 1) - (x^4 + 5x^2 + x) \cos x}{\sin^2 x} \quad (1)$$

$$= \frac{(4x^3 + 10x + 1) - (x^4 + 5x^2 + x) \cot x}{\sin x} \quad (2)$$

$$\frac{d}{dx} \left(\frac{2e^x - 1}{3e^x + 1} \right) = \frac{2e^x(2e^x + 1) - 2e^x(2e^x - 1)}{(2e^x + 1)^2} \quad (1)$$

$$= \frac{2e^x(2e^x + 1 - 2e^x + 1)}{(2e^x + 1)^2} \quad (2)$$

$$= \frac{2e^x(2)}{(2e^x + 1)^2} \quad (3)$$

$$= \frac{4e^x}{(2e^x + 1)^2} \quad (4)$$

16. Tangent Lines with Quotient Rule

$$f(x) = \frac{2x^2}{3x - 1}$$

$$(1, 1)$$

$$f(1) = 1$$

$$f'(1) = \frac{4x(3x - 1) - 3(2x^2)}{(3x - 1)^2} = \frac{2}{4} = \frac{1}{2}$$

$$y = \frac{1}{2}x + \frac{1}{2}$$

17. Power Rule with Negative Powers

$$\frac{d}{dx} \left(\frac{-5}{x^6} \right) = \frac{d}{dx} (-5x^{-6}) \quad (1)$$

$$= -5(-6x^{-7}) \quad (2)$$

$$= 30x^{-7} \quad (3)$$

$$= \frac{1}{30x^7} \quad (4)$$

$$\frac{d}{dp} \left(\frac{2p^8 - 7}{p^3} \right) = \frac{d}{dp} 2p^5 - 7p^{-3} \quad (1)$$

$$= 10p^4 + 21p^{-4} \quad (2)$$

$$= \frac{10p^4}{21p^4} \quad (3)$$

18. Combining Rules to Evaluate Derivatives

$$f(x) = \left(\frac{x^2 + 1}{x} \right) e^x = (x + x^{-1}) e^x$$

$$f'(x) = (1 + -x^{-2}) e^x + (x + x^{-1}) e^x$$

Related Exercises

1. (Section 3.1, Related Exercise 13)

$$s(t) = -16t^2 + 100t$$

$$a = 1$$

$$\lim_{h \rightarrow 0} \frac{s(a+h) - s(a)}{h} = \lim_{h \rightarrow 0} \frac{s(1+h) - 84}{h} \quad (1)$$

$$= \lim_{h \rightarrow 0} \frac{-16(1+h)^2 + 100(1+h) - 84}{h} \quad (2)$$

$$= \lim_{h \rightarrow 0} \frac{-16(h^2 + 2h + 1) + 100 + 100h - 84}{h} \quad (3)$$

$$= \lim_{h \rightarrow 0} \frac{-16h^2 - 32h - 16 + 100 + 100h - 84}{h} \quad (4)$$

$$= \lim_{h \rightarrow 0} \frac{-16h^2 + 68h}{h} \quad (5)$$

$$= \lim_{h \rightarrow 0} -16h + 68 \quad (6)$$

$$= -16(0) + 68 \quad (7)$$

$$= 68 \quad (8)$$

2. (Section 3.1, Related Exercise 14)

$$s(t) = -16t^2 + 128t + 192$$

$$a = 2$$

$$\lim_{h \rightarrow 0} \frac{s(a+h) - s(a)}{h} = \lim_{h \rightarrow 0} \frac{s(2+h) - 384}{h} \quad (1)$$

$$= \lim_{h \rightarrow 0} \frac{-16(2+h)^2 + 128(2+h) + 192 - 384}{h} \quad (2)$$

$$= \lim_{h \rightarrow 0} \frac{-16(h^2 + 4h + 4) + 128(2+h) + 192 - 384}{h} \quad (3)$$

$$= \lim_{h \rightarrow 0} \frac{-16h^2 - 64h - 64 + 256 + 128h + 192 - 384}{h} \quad (4)$$

$$= \lim_{h \rightarrow 0} \frac{-16h^2 + 64h}{h} \quad (5)$$

$$= \lim_{h \rightarrow 0} -16h + 64 \quad (6)$$

$$= -16(0) + 64 \quad (7)$$

$$= 64 \quad (8)$$

3. (Section 3.1, Related Exercise 17)

$$f(x) = \frac{1}{x}$$

$$P(-1, -1)$$

$$\lim_{x \rightarrow -1} \frac{f(x) - f(-1)}{x - (-1)} = \lim_{x \rightarrow -1} \frac{f(x) - (-1)}{x + 1} \quad (1)$$

$$= \lim_{x \rightarrow -1} \frac{f(x) + 1}{x + 1} \quad (2)$$

$$= \lim_{x \rightarrow -1} \frac{\frac{1}{x} + 1}{x + 1} \quad (3)$$

$$= \lim_{x \rightarrow -1} \frac{\frac{1+x}{x}}{x + 1} \quad (4)$$

$$= \lim_{x \rightarrow -1} \frac{\frac{1+x}{x}}{x + 1} \cdot \frac{x}{x} \quad (5)$$

$$= \lim_{x \rightarrow -1} \frac{1 + x}{(x + 1)x} \quad (6)$$

$$= \lim_{x \rightarrow -1} \frac{1}{x} \quad (7)$$

$$= \frac{1}{-1} \quad (8)$$

$$= -1 \quad (9)$$

$$y - (-1) = -1(x - (-1))$$

$$y = -1(x + 1) - 1 = -x - 1 - 1 = -x - 2$$

4. (Section 3.1, Related Exercise 18)

$$f(x) = \frac{4}{x^2}$$

$$(-1, 4)$$

$$\lim_{x \rightarrow -1} \frac{f(x) - 4}{x - (-1)} = \lim_{x \rightarrow -1} \frac{f(x) - 4}{x + 1} \quad (1)$$

$$= \lim_{x \rightarrow -1} \frac{\frac{4}{x^2} - 4}{x + 1} \quad (2)$$

$$= \lim_{x \rightarrow -1} \frac{\frac{4-4x^2}{x^2}}{x + 1} \quad (3)$$

$$= \lim_{x \rightarrow -1} \frac{\frac{4-4x^2}{x^2}}{x + 1} \cdot \frac{x^2}{x^2} \quad (4)$$

$$= \lim_{x \rightarrow -1} \frac{4 - 4x^2}{x^2(x + 1)} \quad (5)$$

$$= \lim_{x \rightarrow -1} \frac{4(1 - x^2)}{x^2(x + 1)} \quad (6)$$

$$= \lim_{x \rightarrow -1} \frac{4(1 - x)(1 + x)}{x^2(x + 1)} \quad (7)$$

$$= \lim_{x \rightarrow -1} \frac{4(1 - x)}{x^2} \quad (8)$$

$$= \frac{4(1 - (-1)^2)}{(-1)^2} \quad (9)$$

$$= \frac{4(1 - 1)}{1} \quad (10)$$

$$= 0 \quad (11)$$

$$y - 4 = 0$$

$$y = 4$$

5. (Section 3.1, Related Exercise 23)

$$f(x) = 3x^2 - 4x$$

$$(1, -1)$$

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \rightarrow 0} \frac{f(1+h) - (-1)}{h} \quad (1)$$

$$= \lim_{h \rightarrow 0} \frac{3(1+h)^2 - 4(1+h) + 1}{h} \quad (2)$$

$$= \lim_{h \rightarrow 0} \frac{3h^2 + 6h + 3 - 4 - 4h + 1}{h} \quad (3)$$

$$= \lim_{h \rightarrow 0} \frac{3h^2 + 2h + 3 - 4 + 1}{h} \quad (4)$$

$$= \lim_{h \rightarrow 0} \frac{3h^2 + 2h}{h} \quad (5)$$

$$= \lim_{h \rightarrow 0} 3h + 2 \quad (6)$$

$$= 3(0) + 2 \quad (7)$$

$$= 2 \quad (8)$$

$$y - (-1) = 2(x - 1)$$

$$y = 2(x - 1) - 1 = 2x - 2 - 1 = 2x - 3$$

6. (Section 3.1, Related Exercise 27)

$$f(x) = x^3$$

$$(1, 1)$$

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \rightarrow 0} \frac{f(1+h) - 1}{h} \quad (1)$$

$$= \lim_{h \rightarrow 0} \frac{(1+h)^3 - 1}{h} \quad (2)$$

$$= \lim_{h \rightarrow 0} \frac{h^3 + h^2 + 2h + 2h^2 + h + 1 - 1}{h} \quad (3)$$

$$= \lim_{h \rightarrow 0} \frac{h^3 + 2h^2 + 3h}{h} \quad (4)$$

$$= \lim_{h \rightarrow 0} h^2 + 2h + 3 \quad (5)$$

$$= 0^2 + 2(0) + 3 \quad (6)$$

$$= 3 \quad (7)$$

$$y - 1 = 3(x - 1)$$

$$y = 3x - 3 + 1 = 3x - 2$$

7. (Section 3.1, Related Exercise 39)

$$f(x) = \sqrt{2x + 1}$$

$$a = 4$$

$$\lim_{x \rightarrow 4} \frac{f(x) - f(a)}{x - a} = \lim_{x \rightarrow 4} \frac{f(x) - 3}{x - 4} \quad (1)$$

$$= \lim_{x \rightarrow 4} \frac{\sqrt{2x+1} - 3}{x - 4} \quad (2)$$

$$= \lim_{x \rightarrow 4} \frac{\sqrt{2x+1} - 3}{x - 4} \cdot \frac{\sqrt{2x+1} + 3}{\sqrt{2x+1} + 3} \quad (3)$$

$$= \lim_{x \rightarrow 4} \frac{2x + 1 - 9}{(x - 4)(\sqrt{2x+1} + 3)} \quad (4)$$

$$= \lim_{x \rightarrow 4} \frac{2(x - 4)}{(x - 4)(\sqrt{2x+1} + 3)} \quad (5)$$

$$= \lim_{x \rightarrow 4} \frac{2}{\sqrt{2x+1} + 3} \quad (6)$$

$$= \frac{2}{\sqrt{9} + 3} \quad (7)$$

$$= \frac{2}{3 + 3} \quad (8)$$

$$= \frac{2}{6} \quad (9)$$

$$= \frac{1}{3} \quad (10)$$

$$y - 3 = \frac{1}{3}(x - 4)$$

$$y = \frac{1}{3}x - \frac{4}{3} + \frac{9}{3} = \frac{1}{3}x + \frac{5}{3}$$

8. (Section 3.1, Related Exercise 40)

$$f(x) = \sqrt{3x}$$

$$a = 12$$

$$\lim_{x \rightarrow 12} \frac{f(x) - f(a)}{x - a} = \lim_{x \rightarrow 12} \frac{f(x) - 6}{x - 12} \quad (1)$$

$$= \lim_{x \rightarrow 12} \frac{\sqrt{3x} - 6}{x - 12} \quad (2)$$

$$= \lim_{x \rightarrow 12} \frac{\sqrt{3x} - 6}{x - 12} \cdot \frac{\sqrt{3x} + 6}{\sqrt{3x} + 6} \quad (3)$$

$$= \lim_{x \rightarrow 12} \frac{3x - 36}{(x - 12)(\sqrt{3x} + 6)} \quad (4)$$

$$= \lim_{x \rightarrow 12} \frac{3(x - 12)}{(x - 12)(\sqrt{3x} + 6)} \quad (5)$$

$$= \lim_{x \rightarrow 12} \frac{3}{\sqrt{3x} + 6} \quad (6)$$

$$= \frac{3}{\sqrt{3(12)} + 6} \quad (7)$$

$$= \frac{3}{\sqrt{36} + 6} \quad (8)$$

$$= \frac{3}{6 + 6} \quad (9)$$

$$= \frac{3}{12} \quad (10)$$

$$= \frac{1}{4} \quad (11)$$

$$y - 6 = \frac{1}{4}(x - 12)$$

$$y = \frac{1}{4}x - 3 + 6 = \frac{1}{4}x + 3$$

9. (Section 3.1, Related Exercise 49)

$$d(t) = 16t^2$$

$$a = 4$$

$$\lim_{x \rightarrow 4} \frac{f(x) - f(a)}{x - a} = \lim_{h \rightarrow 0} \frac{f(x) - 256}{x - 4} \quad (1)$$

$$= \lim_{x \rightarrow 4} \frac{16x^2 - 256}{x - 4} \quad (2)$$

$$= \lim_{x \rightarrow 4} \frac{(16x + 64)(x - 4)}{x - 4} \quad (3)$$

$$= \lim_{x \rightarrow 4} 16x + 64 \quad (4)$$

$$= 16(4) + 64 \quad (5)$$

$$= 64 + 64 \quad (6)$$

$$= 128 \quad (7)$$

10. (Section 3.1, Related Exercise 50)

$$F(x) = \frac{k}{x^2} \text{ where } k \text{ is some constant}$$

$$a = 1$$

$$\lim_{h \rightarrow 0} \frac{F(a + h) - F(a)}{h} = \lim_{h \rightarrow 0} \frac{F(1 + h) - \frac{k}{1}}{h} \quad (1)$$

$$= \lim_{h \rightarrow 0} \frac{F(1 + h) - \frac{k}{1}}{h} \quad (2)$$

$$= \lim_{h \rightarrow 0} \frac{\frac{k}{(1+h)^2} - \frac{k}{1}}{h} \quad (3)$$

$$= \lim_{h \rightarrow 0} \frac{\frac{k}{(1+h)^2} - \frac{k(1+h)^2}{(1+h)^2}}{h} \quad (4)$$

$$= \lim_{h \rightarrow 0} \frac{\frac{k - k(1+h)^2}{(1+h)^2}}{h} \quad (5)$$

$$= \lim_{h \rightarrow 0} \frac{\frac{k - k(1+h)^2}{(1+h)^2}}{h} \quad (6)$$

$$= \lim_{h \rightarrow 0} \frac{\frac{k - (kh^2 + 2kh + k)}{(1+h)^2}}{h} \quad (7)$$

$$= \lim_{h \rightarrow 0} \frac{\frac{k - kh^2 - 2kh - k}{(1+h)^2}}{h} \quad (8)$$

$$= \lim_{h \rightarrow 0} \frac{\frac{-kh^2 - 2kh}{(1+h)^2}}{h} \quad (9)$$

$$= \lim_{h \rightarrow 0} \frac{-kh^2 - 2kh}{(1+h)^2} \cdot \frac{1}{h} \quad (10)$$

$$= \lim_{h \rightarrow 0} \frac{h(-kh - 2k)}{h(1+h)^2} \quad (11)$$

$$= \lim_{h \rightarrow 0} \frac{-kh - 2k}{(1+h)^2} \quad (12)$$

$$= \frac{-kh - 2k}{(1+h)^2} \quad (13)$$

$$= \frac{-k(0) - 2k}{(1+0)^2} \quad (14)$$

$$= \frac{-2k}{1} \quad (15)$$

$$= -2k \quad (16)$$

11. (Section 3.1, Related Exercise 53)
Hint: Sketch a Secant Line

$$L'(1.5) \approx 4$$

$$L'(a) \approx 0 \text{ where } a \geq 4$$

12. (Section 3.1, Related Exercise 54)

$$D'(60) \approx 0.6$$

$$D'(170) \approx 0$$

13. (Section 3.2, Related Exercise 23)

$$f(x) = 4x^2 + 1$$

$$a = 2, 4$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad (1)$$

$$= \lim_{h \rightarrow 0} \frac{4(x+h)^2 + 1 - (4x^2 + 1)}{h} \quad (2)$$

$$= \lim_{h \rightarrow 0} \frac{4(x^2 + 2xh + h^2) + 1 - 4x^2 - 1}{h} \quad (3)$$

$$= \lim_{h \rightarrow 0} \frac{4x^2 + 8xh + 4h^2 - 4x^2}{h} \quad (4)$$

$$= \lim_{h \rightarrow 0} \frac{8xh + 4h^2}{h} \quad (5)$$

$$= \lim_{h \rightarrow 0} 8x + 4h \quad (6)$$

$$= 8x + 4(0) \quad (7)$$

$$= 8x \quad (8)$$

$$f'(2) = 8(2) = 16$$

$$f'(4) = 8(4) = 32$$

14. (Section 3.2, Related Exercise 24)

$$f(x) = x^2 + 3x$$

$$a = -1, 4$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad (1)$$

$$= \lim_{h \rightarrow 0} \frac{(x+h)^2 + 3(x+h) - (x^2 + 3x)}{h} \quad (2)$$

$$= \lim_{h \rightarrow 0} \frac{x^2 + h^2 + xh + 3x + 3h - x^2 - 3x}{h} \quad (3)$$

$$= \lim_{h \rightarrow 0} \frac{h^2 + xh + 3h}{h} \quad (4)$$

$$= \lim_{h \rightarrow 0} h + x + 3 \quad (5)$$

$$= 0 + x + 3 \quad (6)$$

$$= x + 3 \quad (7)$$

$$f'(-1) = -1 + 3 = 2$$

$$f'(4) = 4 + 3 = 7$$

15. (Section 3.2, Related Exercise 37)

$$f(x) = \sqrt{3x+1}$$

$$a = 8$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad (1)$$

$$= \lim_{h \rightarrow 0} \frac{\sqrt{3(x+h)+1} - \sqrt{3x+1}}{h} \quad (2)$$

$$= \lim_{h \rightarrow 0} \frac{\sqrt{3(x+h)+1} - \sqrt{3x+1}}{h} \cdot \frac{\sqrt{3(x+h)+1} + \sqrt{3x+1}}{\sqrt{3(x+h)+1} + \sqrt{3x+1}} \quad (3)$$

$$= \lim_{h \rightarrow 0} \frac{3(x+h)+1 - (3x+1)}{h(\sqrt{3(x+h)+1} + \sqrt{3x+1})} \quad (4)$$

$$= \lim_{h \rightarrow 0} \frac{3x+3h+1-3x-1}{h(\sqrt{3(x+h)+1} + \sqrt{3x+1})} \quad (5)$$

$$= \lim_{h \rightarrow 0} \frac{3h}{h(\sqrt{3(x+h)+1} + \sqrt{3x+1})} \quad (6)$$

$$= \lim_{h \rightarrow 0} \frac{3}{\sqrt{3(x+h)+1} + \sqrt{3x+1}} \quad (7)$$

$$= \frac{3}{\sqrt{3(x+0)+1} + \sqrt{3x+1}} \quad (8)$$

$$= \frac{3}{\sqrt{3x+1} + \sqrt{3x+1}} \quad (9)$$

$$= \frac{3}{2\sqrt{3x+1}} \quad (10)$$

$$f'(8) = \frac{3}{2\sqrt{3(8)+1}} = \frac{3}{2\sqrt{24+1}} = \frac{3}{2\sqrt{25}} = \frac{3}{2(5)} = \frac{3}{10}$$

$$y - f(8) = f'(8)(x - 8)$$

$$y = \frac{3}{10}(x - 8) + 5 = \frac{3}{10}x - \frac{12}{5} + 5 = \frac{3}{10}x + \frac{13}{5}$$

16. (Section 3.2, Related Exercise 38)

$$f(x) = \sqrt{x+2}$$

$$a = 7$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad (1)$$

$$= \lim_{h \rightarrow 0} \frac{\sqrt{x+h+2} - \sqrt{x+2}}{h} \quad (2)$$

$$= \lim_{h \rightarrow 0} \frac{\sqrt{x+h+2} - \sqrt{x+2}}{h} \cdot \frac{\sqrt{x+h+2} + \sqrt{x+2}}{\sqrt{x+h+2} + \sqrt{x+2}} \quad (3)$$

$$= \lim_{h \rightarrow 0} \frac{x+h+2 - (x+2)}{h(\sqrt{x+h+2} + \sqrt{x+2})} \quad (4)$$

$$= \lim_{h \rightarrow 0} \frac{x+h+2-x-2}{h(\sqrt{x+h+2} + \sqrt{x+2})} \quad (5)$$

$$= \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{x+h+2} + \sqrt{x+2})} \quad (6)$$

$$= \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h+2} + \sqrt{x+2}} \quad (7)$$

$$= \frac{1}{\sqrt{x+0+2} + \sqrt{x+2}} \quad (8)$$

$$= \frac{1}{\sqrt{x+2} + \sqrt{x+2}} \quad (9)$$

$$= \frac{1}{2\sqrt{x+2}} \quad (10)$$

$$f'(7) = \frac{1}{2\sqrt{7+2}} = \frac{1}{2\sqrt{9}} = \frac{1}{2 \cdot 3} = \frac{1}{6}$$

$$y - f(7) = f'(7)(x - 7)$$

$$y = \frac{1}{6}(x - 7) + 3 = \frac{1}{6}x - \frac{7}{6} + 3 = \frac{1}{6}x + \frac{11}{6}$$

17. (Section 3.2, Related Exercise 25)

$$f(x) = \frac{1}{x+1}$$

$$a = -\frac{1}{2}, 5$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad (1)$$

$$= \lim_{h \rightarrow 0} \frac{\frac{1}{x+h+1} - \frac{1}{x+1}}{h} \quad (2)$$

$$= \lim_{h \rightarrow 0} \frac{\frac{1}{x+h+1} - \frac{1}{x+1}}{h} \quad (3)$$

$$= \lim_{h \rightarrow 0} \frac{\frac{x+1-(x+h+1)}{(x+h+1)(x+1)}}{h} \quad (4)$$

$$= \lim_{h \rightarrow 0} \frac{x+1-x-h-1}{h(x+h+1)(x+1)} \quad (5)$$

$$= \lim_{h \rightarrow 0} \frac{-h}{h(x+h+1)(x+1)} \quad (6)$$

$$= \lim_{h \rightarrow 0} \frac{-1}{(x+h+1)(x+1)} \quad (7)$$

$$= \frac{-1}{(x+0+1)(x+1)} \quad (8)$$

$$= \frac{-1}{(x+1)(x+1)} \quad (9)$$

$$= \frac{-1}{(x+1)^2} \quad (10)$$

$$f'(-\frac{1}{2}) = \frac{-1}{(-\frac{1}{2}+1)^2} = \frac{-1}{(\frac{1}{2})^2} = \frac{-1}{\frac{1}{4}} = -1 \cdot 4 = -4$$

$$f'(5) = \frac{-1}{(5+1)^2} = \frac{-1}{(6)^2} = \frac{-1}{36}$$

18. (Section 3.2, Related Exercise 27)

$$f(t) = \frac{1}{\sqrt{t}}$$

$$a = 9, \frac{1}{4}$$

$$f'(t) = \lim_{h \rightarrow 0} \frac{f(t+h) - f(t)}{h} \quad (1)$$

$$= \lim_{h \rightarrow 0} \frac{\frac{1}{\sqrt{t+h}} - \frac{1}{\sqrt{t}}}{h} \quad (2)$$

$$= \lim_{h \rightarrow 0} \frac{\frac{\sqrt{t} - \sqrt{t+h}}{\sqrt{t+h}\sqrt{t}}}{h} \quad (3)$$

$$= \lim_{h \rightarrow 0} \frac{\frac{\sqrt{t} - \sqrt{t+h}}{\sqrt{t+h}\sqrt{t}} \cdot \frac{\sqrt{t} + \sqrt{t+h}}{\sqrt{t} + \sqrt{t+h}}}{h} \quad (4)$$

$$= \lim_{h \rightarrow 0} \frac{t - (t+h)}{h(t\sqrt{t+h} + (t+h)\sqrt{t})} \quad (5)$$

$$= \lim_{h \rightarrow 0} \frac{-h}{h(t\sqrt{t+h} + (t+h)\sqrt{t})} \quad (6)$$

$$= \lim_{h \rightarrow 0} \frac{-1}{t\sqrt{t+h} + (t+h)\sqrt{t}} \quad (7)$$

$$= \frac{-1}{t\sqrt{t+0} + (t+0)\sqrt{t}} \quad (8)$$

$$= \frac{-1}{t\sqrt{t} + t\sqrt{t}} \quad (9)$$

$$= \frac{-1}{2(t\sqrt{t})} \quad (10)$$

$$f'(9) = \frac{-1}{2(9\sqrt{9})} = \frac{-1}{2(9 \cdot 3)} = \frac{-1}{2(27)} = \frac{-1}{54}$$

$$f'\left(\frac{1}{4}\right) = \frac{-1}{2\left(\frac{1}{4}\sqrt{\frac{1}{4}}\right)} = \frac{-1}{2\left(\frac{1}{4} \cdot \frac{1}{2}\right)} = \frac{-1}{2\left(\frac{1}{8}\right)} = \frac{-1}{\frac{2}{8}} = -1(4) = -4$$

19. (Section 3.2, Related Exercise 53)

f is not continuous at $x = 1$

f is not differentiable at $x = 1, 2$

20. (Section 3.2, Related Exercise 54)

f is not continuous at $x = 1$

f is not differentiable at $x = 1, 2$

21. (Section 3.3, Related Exercise 19)

$$\frac{d}{dx}(x^5) = 5x^4$$

22. (Section 3.3, Related Exercise 22)

$$\frac{d}{dx}(e^3) = 0$$

23. (Section 3.3, Related Exercise 23)

$$\frac{d}{dx}(5x^3) = 15x^2$$

24. (Section 3.3, Related Exercise 24)

$$\frac{d}{dx}\left(\frac{5}{6}w^{12}\right) = 10w^{11}$$

25. (Section 3.3, Related Exercise 28)

$$\frac{d}{dx}(6\sqrt{t}) = \frac{3}{\sqrt{t}}$$

26. (Section 3.3, Related Exercise 31)

$$\frac{d}{dx}(3x^4 + 7x) = 12x^3 + 7$$

27. (Section 3.3, Related Exercise 33)

$$\frac{d}{dx}(10x^4 - 32x + e^2) = 40x^3 - 32$$

28. (Section 3.3, Related Exercise 60)

$$f(x) = x^3 - 4x^2 + 2x - 1$$

$$a = 2$$

$$f'(x) = 3x^2 - 8x + 2$$

$$y = -2x - 1$$

29. (Section 3.3, Related Exercise 61)

$$f(x) = e^x$$

$$a = \ln 3$$

$$f'(x) = e^x$$

$$y = 3x - 3(\ln 3) + 3$$

30. (Section 3.3, Related Exercise 63)

$$f(x) = x^2 - 6x + 5$$

$$f'(x) = 2x - 6$$

$$y = f(x) \text{ is } 0, \text{ when } x = 3$$

$$y = f(x) \text{ is } 2, \text{ when } x = 4$$

31. (Section 3.3, Related Exercise 64)

$$f(t) = t^3 - 27t + 5$$

$$f'(t) = 3t^2 - 27$$

$$y = f(t) \text{ is } 0, \text{ when } x = 3$$

$$y = f(t) \text{ is } 21, \text{ when } x = 4$$

32. (Section 3.3, Related Exercise 69)

$$f(x) = 5x^4 + 10x^3 + 3x + 6$$

$$f'(x) = 20x^3 + 30x^2 + 3$$

$$f''(x) = 60x^2 + 60x$$

$$f'''(x) = 120x + 60$$

33. (Section 3.3, Related Exercise 70)

$$f(x) = 3x^2 + 5e^x$$

$$f'(x) = 6x + 5e^x$$

$$f''(x) = 6 + 5e^x$$

$$f'''(x) = 5e^x$$

34. (Section 3.4, Related Exercise 19)

$$\frac{d}{dx} (3x^4(2x^2 - 1)) = 12x^3(2x^2 - 1) + 3x^4(4x) = 36x^5 - 12x^3$$

35. (Section 3.4, Related Exercise 20)

$$\frac{d}{dx} (6x - 2xe^x) = 6 - 2(e^x + 2xe^x) = 6 - 2e^x - 2xe^x$$

36. (Section 3.4, Related Exercise 22)

$$\frac{d}{dx} \left(\frac{x^3 - 4x^2 + x}{x - 2} \right) = \frac{(3x^2 - 8x + 1)(x - 2) - (x^3 - 4x^2 + x)}{(x - 2)^2} \quad (1)$$

$$= \frac{3x^3 - 6x^2 - 8x^2 + 16x + x - 2 - x^3 + 4x^2 - x}{(x - 2)^2} \quad (2)$$

$$= \frac{2x^3 + -10x^2 + 16x - 2}{(x - 2)^2} \quad (3)$$

37. (Section 3.4, Related Exercise 27)

$$\frac{d}{dx}(xe^{-x}) = \frac{d}{dx}\left(\frac{x}{e^x}\right) \quad (1)$$

$$= \frac{e^x - xe^x}{(e^x)^2} \quad (2)$$

$$= \frac{e^x(1-x)}{(e^x)^2} \quad (3)$$

$$= \frac{1-x}{e^x} \quad (4)$$

$$= e^{-x}(1-x) \quad (5)$$

38. (Section 3.4, Related Exercise 61)

$$f(x) = \frac{x+5}{x-1}$$

$$a = 3$$

$$\frac{d}{dx}\left(\frac{x+5}{x-1}\right) = \frac{(x-1) - (x+5)}{(x-1)^2} \quad (1)$$

$$= \frac{x-1-x-5}{(x-1)^2} \quad (2)$$

$$= \frac{-6}{(x-1)^2} \quad (3)$$

$$y = -\frac{3}{2}x + \frac{17}{2}$$

39. (Section 3.4, Related Exercise 62)

$$f(x) = \frac{2x^2}{3x-1}$$

$$a = 1$$

$$\frac{d}{dx}\left(\frac{2x^2}{3x-1}\right) = \frac{4x(3x-1) - 3(2x^2)}{(3x-1)^2} \quad (1)$$

$$= \frac{12x^2 - 4x - 6x^2}{(3x-1)^2} \quad (2)$$

$$= \frac{6x^2 - 4x}{(3x-1)^2} \quad (3)$$

$$y = \frac{1}{2}x + \frac{1}{2}$$

40. (Section 3.4, Related Exercise 28)

$$\frac{d}{dx}(e^x \sqrt[3]{x}) = ? \quad (1)$$

$$(2)$$

41. (Section 3.4, Related Exercise 39)

$$\frac{d}{dx}(3x^{-9}) = ? \quad (1)$$

$$(2)$$

42. (Section 3.4, Related Exercise 43)

$$\frac{d}{dt}\left(\frac{t^3 + 3t^2 + t}{t^3}\right) = ? \quad (1)$$

$$(2)$$

43. (Section 3.4, Related Exercise 51)

$$\frac{d}{dw} \left(\frac{w^{\frac{5}{3}}}{w^{\frac{5}{3}} + 1} \right) = ? \quad (1)$$

(2)

44. (Section 3.4, Related Exercise 45)

$$\frac{d}{dx} \left(\frac{(x+1)e^x}{x-2} \right) = ? \quad (1)$$

(2)

45. (Section 3.4, Related Exercise 46)

$$\frac{d}{dx} \left(\frac{(x-1)(2x^2-1)}{(x^3-1)} \right) = ? \quad (1)$$

(2)