Module 5 Notes (MATH-211)

Lillie Donato

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General Notes (and Definitions)

- Maxima and Minima
  Absolute Maximum: Assume a function f is defined on a set D, and $x = c$ is a point in D. Then, $y = f(c)$ is an **absolute maximum value** of f on D if $f(c) \geq f(x)$ for every x in D. Changing the set on which f is defined may change the absolute maximum value.
  Absolute Minimum: Assume a function f is defined on a set D, and $x = c$ is a point in D. Then, $y = f(c)$ is an **absolute minimum value** of f on D if $f(c) \leq f(x)$ for every x in D. Changing the set on which f is defined may change the absolute minimum value.
  Extreme Value Theorem: A function that is continuous on a closed interval is guarenteed to have both an absolute maximum value and an absolute minmum value.
  A discontinuous function, or a function defined on an interval that is not closed, may still have absolute extrema.
  Local Maximum and Minimum Values: Assume $x = c$ is an interior point (not an endpoint) of some interval I in the domain of f. Then, $y = f(c)$ is a **local maximum value** of f if $f(c) \geq f(x)$ for every x in I, and $y = f(c)$ is a **local minimum value** of f if $f(c) \leq f(x)$ for every x in I.
  Critical Points: An interior point $x = c$ of the domain of f is called a **critical point** of f if either $f'(c) = 0$ or $f'(c)$ does not exist.
  Local Extreme Value Theorem: If a function f has a local maximum or a local minimum at a point $x = c$, then either $f'(c) = 0$ or $f'(c)$ does not exist.
  If f has a local extreme, it must occur at a critical point.
  Not every critical point is the location of a local extreme value.
  For a continuous function f on a closed interval $[a, b]$, absolute extremes are guaranteed to exist, and they must occur either at the endpoints of interval or at critical points of f within the interval.

- Mean Value Theorem
  Rolle's Theorem: Let f be a continuous function on a closed interval $[a, b]$ that is differentiable on (a, b), with $f(a) = f(b)$. Then, there is at least one point $x = c$ in (a, b) where $f'(c) = 0$.
  Mean Value Theorem: If f is a continuous function on a closed interval $[a, b]$ that is differentiable on (a, b), then there is at least one point $x = c$ in (a, b) where

  $$f'(c) = \frac{f(b) - f(a)}{b - a}$$

  Zero Derivative Implies Constant Function: If f is differentiable on an open interval I, and $f'(x) = 0$ for all x in I, then f is a constant function on I.
  Function with Equal Derivative Differ by a Constant: If $f'(x) = g'(x)$ for all x in an open interval I, then $f(x) = g(x) + C$ for some constant C.

Examples

1. Locate absolute maxima and minima from a graph
   Absolute Maximum: $f(c)$ and occurs at $x = c$
   Absolute Minimum: None, as the $f(b)$ does not exist

2. Locate local maxima and minima from a graph
   Absolute Min at $(a, f(a))$
   Absolute Max at $(p, f(p))$
   Local Max at $(p, f(p))$
   Local Max at $(r, f(r))$

Local Min at $(q, f(q))$
Local Min at $(s, f(s))$

3. Find critical points of a function

$$f(t) = t^2 - 2\ln\left(t^2 + 1\right)$$
$$f'(t) = \frac{2t\,(t+1)\,(t-1)}{t^2+1}$$

Critical Point at $x = -1$
Critical Point at $x = 0$
Critical Point at $x = 1$

4. Find absolute extremes of a continuous function on a closed interval

$$f(x) = \frac{x}{(x^2+9)^5}$$
$$f'(x) = \frac{-9x^2+9}{(x^2+9)^6}$$
$$[-2, 2]$$
$$f(-2) \approx -0.000005$$
$$f(2) \approx 0.000005$$
$$f(-1) = -0.00001$$
$$f(1) = 0.00001$$

Absolute Min at $(-1, f(-1))$
Absolute Max at $(1, f(1))$

5. Application of finding absolute extreme values

$$P(x) = 2x + \frac{128}{x}$$
$$P'(x) = 2 + \frac{-128}{x^2}$$
$$(0, \infty)$$
$$f(8) = 18$$

Absolute min at $(8, 32)$ or a perimeter of 32 units

6. Verifying Rolle's Theorem

$$f(x) = x^3 - 2x^2 - 8x$$
$$f'(x) = 3x^2 - 4x - 8$$
$$[-2, 4]$$
$$f(-2) = 0 = f(4)$$
$$x = \frac{2+2\sqrt{7}}{3} \approx 2.43$$
$$x = \frac{2-2\sqrt{7}}{3} \approx -1.097$$
$$x = \frac{2 \pm 2\sqrt{7}}{3}$$

7. Verifying the Mean Value Theorem

$$f(x) = x^3 - 2x^2$$
$$f'(x) = 3x^2 - 4x$$
$$[0, 1]$$
$$f'(c) = -1$$
$$(3x-1)(x-1) = 0$$
$$x = \frac{1}{3}$$
$$f\left(\frac{1}{3}\right) = -1$$

8. Application of the Mean Value Theorem

$$\frac{30}{27} = \frac{30}{0.45} \approx 66.667$$
$$66.667 > 60$$

Related Exercises

1. (Section 4.1, Exercise 11)
   Absolute Min at $x = c_2$
   Absolute Max at $x = b$

2. (Section 4.1, Exercise 14)
   Absolute Min at $x = c$
   Absolute Max at $x = b$

3. (Section 4.1, Exercise 15)
   Absolute Max at $x = b$
   Absolute Min at $x = a$
   Local Max at $x = p$
   Local Max at $x = r$
   Local Min at $x = q$
   Local Min at $x = s$

4. (Section 4.1, Exercise 18) Absolute Max at $x = p$
   Absolute Min at $x = u$
   Local Max at $x = p$
   Local Max at $x = r$
   Local Max at $x = t$
   Local Min at $x = q$
   Local Min at $x = s$
   Local Min at $x = u$

5. (Section 4.1, Exercise 35)
$$f(x) = \frac{1}{x} + \ln x$$
$$f'(x) = \frac{x-1}{x^2}$$
   Critical Points at $x = 1$

6. (Section 4.1, Exercise 36)
$$f(t) = t^2 - 2\ln\left(t^2+1\right)$$
$$f'(t) = \frac{2t\,(t+1)\,(t-1)}{t^2+1}$$
   Critical Points at $t = -1$, $t = 0$ and $t = 1$

7. (Section 4.1, Exercise 46)
$$f(x) = x^4 - 4x^3 + 4x^2$$
$$f'(x) = 4x^3 - 12x^2 + 8x$$
$$[-1, 3]$$
$$f(-1) = 9$$
$$f(0) = 0$$
$$f(1) = 1$$
$$f(2) = 0$$
$$f(3) = 9$$
   Absolute Max at $(-1, 9)$ and $(3, 9)$
   Absolute Min at $(0, 0)$ and $(2, 0)$

8. (Section 4.1, Exercise 52)

$$f(x) = 3x^{\frac{2}{3}}$$

$$f'(x) = \frac{2}{x^{\frac{1}{3}}}$$

$$[0, 27]$$

$$f(0) = 0$$

$$f(27) = 27$$

Absolute Min at $(0, 0)$
Absolute Min at $(27, 27)$

9. (Section 4.1, Exercise 73)

$$s(t) = -16t^2 + 64t + 192$$

$$s'(t) = -32t + 64$$

$$0 \leq t \leq 6$$

$$s(0) = 192$$

$$s(2) = 256$$

$$s(6) = 0$$

The stone will reach its maximum height at 2 seconds

A copy of my notes (in LaTeX) are available on my GitHub