# Module 1 Notes (MATH-211)

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## General Notes (and Definitions)

- Limit Definition(s):
  - Simple: The value that the outputs of a function approach as inputs approach a certain value
  - Preliminary: Suppose a function $f$ is defined for all $x$ near $a$ except possibly at $a$. If $f(x)$ is arbitrarily close to $L$ all $x$ sufficiently close (but not equal) to $a$, we write the following.

$$\lim_{x \to a} = L$$

- Secant Line: a line passing through two points $(t_0, s(t_0))$ and $(t_1, s(t_1))$. The slope is given by

$$\frac{s(t_1) - s(t_0)}{t_1 - t_0}$$

- Tangent Line: the line passing through $(t_0, s(t_0))$ with slope

$$\lim_{t \to t_0} \frac{s(t) - s(t_0)}{t - t_0}$$

- One Sided limits:
  - Right-hand (Definition): Suppose a function $f$ is defined for all $x$ near $a$ with $x > a$. If $f(x)$ is arbitrarily close to $L$ for all $x$ sufficiently close to $a$ with $x > a$ we write

$$\lim_{x \to a^+} f(x) = L$$

  - Left-hand (Definition): Suppose a function $f$ is defined for all $x$ near $a$ with $x < a$. If $f(x)$ is arbitrarily close to $L$ for all $x$ sufficiently close to $a$ with $x < a$ we write

$$\lim_{x \to a^-} f(x) = L$$

  - In order for their to be a double sided limit, we must have:

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

  - If the limits from sides are not equal, then a the double sided limit, "does not exist"
- Limits can be simplified/solved in an easier way (as compared to numerically/graphically) using Limit Rules/Laws
- Limit Example Types:
  - Tangent lines
  - Velocity
- Velocity
  - Average Veolcity
    * The average velocity over some interval $[t_0, t_1]$ is defined as

$$v_{av} = \frac{s(t_1) - s(t_0)}{t_1 - t_0}$$

- Instantaneous Veolcity
    - The average velocity over some interval $[t_0, t_1]$ is defined as

$$v_{inst} = \lim_{t \to a} v_{av} == \frac{s(t) - s(a)}{t - a}$$

- Solving Techniques
  - Factoring and canceling out
  - Using conjugates
    - When direct substitution is not possible, you may rationalize the numerator
- Infinite Limits: In either case, the limit does not exist (not a real number) if it is infinite
  - Suppose $f$ is defined for all $x$ near $a$. If $f(x)$ gorws arbitrarily large for all $x$ sufficiently close (but not equal) to $a$, we write

$$\lim_{x \to a} f(x) = \infty$$

  - If $f(x)$ is negative and gorws arbitrarily large in magnitude for all $x$ sufficiently close (but not equal) to $a$, we write

$$\lim_{x \to a} f(x) = -\infty$$

  - The line $x = a$ is a vertical asymptote for $f$ if any of the following hold

$$\lim_{x \to a} f(x) = \pm\infty$$

$$\lim_{x \to a^+} f(x) = \pm\infty$$

$$\lim_{x \to a^-} f(x) = \pm\infty$$

  - A vertical asymptote exists at $x = a$ if any one sided limit as $x \to a$ is $\infty$ or $-\infty$
  - If you have a limit of a rational function, where $p(a) = L \neq 0$ and $q(a) = 0$, then the one sided limits for $\frac{p(x)}{q(x)}$ approach $\pm\infty$

$$\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{L}{0}$$

- Limits as Infinity
  - **Definition**: If $f(x)$ becoomes arbitrarily close to a finite number $L$ for all sufficiently large and positive $x$, the we write

$$\lim_{x \to \infty} f(x) = L$$

    The definition for

$$\lim_{x \to -\infty} f(x) = M$$

    is analogous.
  - If $\lim_{x \to \infty} f(x) = L$ we say that the function $f(x)$ has a horizontal asymptote at $y = L$
  - If $\lim_{x \to -\infty} f(x) = M$ we say that the function $f(x)$ has a horizontal asymptote at $y = M$
  - **Principle**: If $n > 0$ is an integer then

$$\lim_{x \to \pm\infty} \frac{1}{x^n} = 0$$

  - Suppose $f(x) = \frac{p(x)}{q(x)}$ is a rational function where

$$p(x) = a_m x^m + a_{m-1} x^{x-1} + ... + a_1 x + a_0$$

$$q(x) = b_n x^n + b_{n-1} x^{x-1} + ... + b_1 x + b_0$$

    If the degree of $p(x)$ is less than the degree of $q(x)$ then

$$\lim_{x \to \pm\infty} f(x) = 0$$

If the degree of $p(x)$ equals the degree of $q(x)$ then

$$\lim_{x \to \pm\infty} f(x) = \frac{a_m}{b_n}$$

If the degree of $p(x)$ is greater than the degree of $q(x)$ then

$$\lim_{x \to \pm\infty} f(x) = -\infty \text{ or } \infty$$

– End behaviour for transcendental functions

$$\lim_{x \to \pm\infty} \sin x = \text{Does not exist}$$

$$\lim_{x \to \infty} e^x = \infty$$

$$\lim_{x \to \infty} e^{-x} = \lim_{x \to \infty} \frac{1}{e^x} = 0$$

$$\lim_{x \to -\infty} e^x = 0$$

$$\lim_{x \to \infty} e^{-x} = \infty$$

$$\lim_{x \to \infty} \ln x = \infty$$

$$\lim_{x \to 0^+} \ln x = -\infty$$

# Limit Rules/Laws

Assume $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist.

The following properties hold where $c$ is a real number, and $n > 0$ is an integer.

- **Sum Rule**

$$\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

- **Difference Rule**

$$\lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$

- **Constant Multiple Rule**

$$\lim_{x \to a} (cf(x)) = c \lim_{x \to a} f(x)$$

- **Product Rule**

$$\lim_{x \to a} (f(x)g(x)) = (\lim_{x \to a} f(x))(\lim_{x \to a} g(x))$$

- **Quotient Rule**

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}, \text{ provided } \lim_{x \to a} g(x) \neq 0$$

- **Power Rule**

$$\lim_{x \to a} f(x)^n = (\lim_{x \to a} f(x))^n$$

- **Root Rule**

$$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}, \text{ provided} f(x) > 0, \text{ for } x \text{ near } a, \text{ if } n \text{ is even}$$

- **Polynomials**
  A **Polynomial** is defined as A function of the form $x_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$ where $n \geq 0$ is an integer If $p(x)$ is a polynomial then:

$$\lim_{x \to a} p(x) = p(a)$$

  If $p(x)$ and $q(x)$ are polynomials and $q(a) \neq 0$ then (Direct Substitution):

$$\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$$

- **The Squeeze Theorem**
  Assume for some functions $f$, $g$ and $h$ that satisfy $f(x) \leq g(x) \leq h(x)$ for $x$ near $a$ (except possibly at $x = a$). If
  $$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$
  then
  $$\lim_{x \to a} g(x) = L$$
  As $x \to a, h(x) \to L$. Therefore, $g(x) \to L$. As $x$ approaches $a$, if $f$ and $h$ approach the same value, so does $g$.

# Examples

1. (Describing Limits) As $x$ approaches 3, $x^2$ approaches 9
   $$\lim_{x \to 3} x^2 = 9$$
2. (Common Use) Values that are undefined can still have limits, given a graph $G$ where $f(3) =$ undefined ($f(3)$ is a hole), the following limit is valid:
   $$\lim_{x \to 3} f(x) = 4$$
3. Calculating Limits Numerically:
   $$f(x) = \frac{x^3 - 1}{x - 1}$$

| | | | |
|---|---|---|---|
| 0.9 | 0.99 | 0.999 | 0.9999 |
| 2.71 | 2.9701 | 2.997001 | 2.99970001 |

| | | | |
|---|---|---|---|
| 1.1 | 1.01 | 1.001 | 1.0001 |
| 3.31 | 3.0301 | 3.003001 | 3.00030001 |

   As $x$ approaches 1, $f(x)$ approaches 3: $\lim_{x \to 1} \frac{x^3-1}{x-1} = 3$

4. Calculating One-sided limits:
   $$g(x) = \frac{x^3 - 4x}{8|x - 2|}$$

| | | | |
|---|---|---|---|
| 1.9 | 1.09 | 1.009 | 1.0009 |
| -0.92625 | -0.9925125 | -0.999250125 | -0.9999250013 |

| | | | |
|---|---|---|---|
| 2.1 | 2.01 | 2.001 | 2.0001 |
| 1.07625 | 1.0075125 | 1.000750125 | 1.000075001 |

$$\lim_{x \to 2} g(x) = \text{Does not exist}$$
$$\lim_{x \to 2^-} g(x) = -1$$
$$\lim_{x \to 2^+} g(x) = 1$$

5. Calculating piecewise function limits
   $$f(x) = \begin{cases} 3 - x \text{ if } x < 2 \\ x - 1 \text{ if } x > 2 \end{cases}$$
   $$a = 2$$

| | | | |
|---|---|---|---|
| 1.92 | 1.99 | 1.999 | 1.9999 |
| 1.1 | 1.01 | 1.001 | 1.0001 |

| | | | |
|---|---|---|---|
| 2.1 | 2.01 | 2.001 | 2.0001 |
| 1.1 | 1.01 | 1.001 | 1.0001 |

**Explanation**: Since $f(2)$ is not defined within the piece wise function, a graph representing this function would have a whole where $x = a$ and have two lines with inverse slopes

$$f(a) = \text{undefined}$$
$$\lim_{x \to a} f(x) = 1$$
$$\lim_{x \to a^-} f(x) = 1$$
$$\lim_{x \to a^+} f(x) = 1$$

6. Limit Rules/Laws:

   (a) Definitions:

$$\lim_{x \to 3} f(x) = 2$$
$$\lim_{x \to 3} g(x) = -1$$
$$\lim_{x \to 3} h(x) = 6$$

   (b) Problems:

      i. Sum, Constant Multiple

$$\begin{aligned}
\lim_{x \to 3} (f(x) + 2g(x)) &= \lim_{x \to 3} f(x) + \lim_{x \to 3} 2g(x) && (1)\\
&= \lim_{x \to 3} f(x) + 2(\lim_{x \to 3} g(x)) && (2)\\
&= 2 + 2(-1) && (3)\\
&= 0 && (4)
\end{aligned}$$

      ii. Quotient

$$\begin{aligned}
\lim_{x \to 3} \frac{h(x)}{g(x)} &= \frac{\lim_{x \to 3} h(x)}{\lim_{x \to 3} g(x)} && (1)\\
&= \frac{6}{-1} && (2)\\
&= -6 && (3)
\end{aligned}$$

      iii. Quotient, Root, Difference

$$\begin{aligned}
\lim_{x \to 3} \frac{h(x)}{\sqrt{f(x) - g(x)}} &= \frac{\lim_{x \to 3} h(x)}{\lim_{x \to 3} \sqrt{f(x) - g(x)}} && (1)\\
&= \frac{\lim_{x \to 3} h(x)}{\sqrt{\lim_{x \to 3} (f(x) - g(x))}} && (2)\\
&= \frac{\lim_{x \to 3} h(x)}{\sqrt{\lim_{x \to 3} f(x) - \lim_{x \to 3} g(x)}} && (3)\\
&= \frac{6}{\sqrt{2+1}} && (4)\\
&= \frac{6}{\sqrt{3}} && (5)\\
&= 2\sqrt{3} && (6)
\end{aligned}$$

7.

$$\begin{aligned}
\lim_{x \to 1} \frac{3x^2 - 7x + 1}{x + 2} &= \frac{3(1)^2 - 7(1) + 1}{1 + 2} && (1)\\
&= \frac{3 - 7 + 1}{1 + 2} && (2)\\
&= \frac{-3}{3} && (3)\\
&= -1 && (4)
\end{aligned}$$

8.

$$\begin{aligned}
\lim_{x\to 4}\frac{\left(\frac{1}{x}-\frac{1}{4}\right)}{x-4} &= \lim_{x\to 4}\frac{\left(\frac{4}{4x}-\frac{x}{4x}\right)}{x-4} && (1)\\
&= \lim_{x\to 4}\frac{\left(\frac{4-x}{4x}\right)}{x-4} && (2)\\
&= \lim_{x\to 4}\frac{\left(\frac{4-x}{4x}\right)}{\left(\frac{x-4}{1}\right)} && (3)\\
&= \lim_{x\to 4}\left(\frac{4-x}{4x}\right)\left(\frac{1}{x-4}\right) && (4)\\
&= \lim_{x\to 4}\frac{4-x}{4x(x-4)} && (5)\\
&= \lim_{x\to 4}\frac{-(-4+x)}{4x(x-4)} && (6)\\
&= \lim_{x\to 4}\frac{-(x-4)}{4x(x-4)} && (7)\\
&= \lim_{x\to 4}\frac{-1}{4x} && (8)\\
&= \lim_{x\to 4}\frac{-1}{4(4)} && (9)\\
&= -\frac{1}{16} && (10)
\end{aligned}$$

9.

$$\begin{aligned}
\lim_{x\to 9}\frac{x-9}{\sqrt{x}-3} &= \lim_{x\to 9}\frac{x-9}{\sqrt{x}-3}\cdot\frac{\sqrt{x}+3}{\sqrt{x}+3} && (1)\\
&= \lim_{x\to 9}\frac{(x-9)(\sqrt{x}+3)}{(\sqrt{x}-3)(\sqrt{x}+3)} && (2)\\
&= \lim_{x\to 9}\frac{(x-9)(\sqrt{x}+3)}{x-9} && (3)\\
&= \lim_{x\to 9}\sqrt{x}+3 && (4)\\
&= \sqrt{9}+3 && (5)\\
&= 3+3 && (6)\\
&= 6 && (7)
\end{aligned}$$

10.

$$1-\frac{x^2}{2}\le \cos x\le 1$$

$$\begin{aligned}
\lim_{x\to 0}\left(1-\frac{x^2}{2}\right) &= 1-\frac{0^2}{2} && (1)\\
&= 1-0 && (2)\\
&= 1 && (3)\\
&= \lim_{x\to 0}1 && (4)\\
\lim_{x\to 0}\cos x &= 1 \qquad \text{(By the Squeeze Theorem)} && (5)
\end{aligned}$$

11.

$$\begin{aligned}
\lim_{x\to 0}\sin x &= 0 \qquad \text{(By the Squeeze Theorem)} && (1)\\
\lim_{x\to 0}\cos x &= 1 \qquad \text{(By the Squeeze Theorem)} && (2)
\end{aligned}$$

$$\lim_{x\to 0}\frac{\sin 2x}{\sin x} = \lim_{x\to 0}\frac{2\sin x\cos x}{\sin x} \quad (1)$$
$$= \lim_{x\to 0} 2\cos x \quad (2)$$
$$= 2\lim_{x\to 0}\cos x \quad (3)$$
$$= 2\cdot 1 \quad (4)$$
$$= 2 \quad (5)$$

12. Infinite Limits Numerically

$$f(x) = \frac{x}{(x-2)^2}$$

| | | | |
|---|---|---|---|
| 2.1 | 2.01 | 2.001 | 2.0001 |
| 210 | 20100 | 2001000 | 200010000 |

| | | | |
|---|---|---|---|
| 1.9 | 1.99 | 1.999 | 1.9999 |
| 190 | 19900 | 1999000 | 199990000 |

$$\lim_{x\to 2} f(x) = \infty$$

13. Infinite Limits Graphically

$$\lim_{x\to -2^-} h(x) = -\infty$$
$$\lim_{x\to -2^+} h(x) = -\infty$$
$$\lim_{x\to -2} h(x) = -\infty$$
$$\lim_{x\to 3^-} h(x) = \infty$$
$$\lim_{x\to 3^+} h(x) = -\infty$$
$$\lim_{x\to 3} h(x) = \text{Does not exist}$$

14. Infinite Limits Analytically
Hint: Look at the signs of the fractions

$$\frac{x^2-5x+6}{x^4-4x^2} = \frac{(x-3)(x-2)}{x^2(x+2)(x-2)} = \frac{x-3}{x^2(x+2)}$$
$$\lim_{x\to -2^+}\frac{x^2-5x+6}{x^4-4x^2} = \lim_{x\to -2^+}\frac{x-3}{x^2(x+2)} = -\infty$$
$$\lim_{x\to -2^-}\frac{x^2-5x+6}{x^4-4x^2} = \lim_{x\to -2^-}\frac{x-3}{x^2(x+2)} = \infty$$
$$\lim_{x\to -2}\frac{x^2-5x+6}{x^4-4x^2} = \text{Does not exist}$$

15. Infinite Limits Analytically with Square Root

$$\lim_{x\to 1^+}\frac{x+3}{\sqrt{x^2-5x+4}} = \lim_{x\to 1^+}\frac{x+3}{\sqrt{(x-4)(x-1)}} = \text{Does not exist}$$
$$\lim_{x\to 1^-}\frac{x+3}{\sqrt{x^2-5x+4}} = \lim_{x\to 1^-}\frac{x+3}{\sqrt{(x-4)(x-1)}} = \infty$$
$$\lim_{x\to 1}\frac{x+3}{\sqrt{x^2-5x+4}} = \text{Does not exist}$$

16. Infinite Limit with a Trigonometric Function

$$\lim_{\theta\to 0^-}\frac{\sin\theta}{\cos^2\theta-1} = \lim_{\theta\to 0^-}\frac{\sin\theta}{-\sin^2\theta} = \lim_{\theta\to 0^-}\frac{1}{-\sin^2\theta} = \infty$$

17. Locating Veritical Asymptotes

$$f(x) = \frac{x+7}{x^4 - 49x^2} = \frac{x+7}{x^2(x^2-49)} = \frac{x+7}{x^2((x-7)(x+7))} = \frac{1}{x^2(x-7)}$$

Denominator is 0 at $x = 0$, $x = -7$, $x = 7$
$x = -7$ does not fit, as it is connected with $x + 7$, but cancels out
Vertical Asymptotes: $x = 0$, $x = 7$

18. Limits at Infinity

$$\lim_{x\to\infty} 5 + \frac{1}{x} + \frac{10}{x^2} = 5 + 0 + 0 = 5$$
$$\lim_{x\to\infty} 5 = 5$$
$$\lim_{x\to\infty} \frac{1}{x} = 0$$
$$\lim_{x\to\infty} \frac{10}{x^2} = 0$$

19. End behaviour for rational functions (different degrees)
Hint: the degree of the numerator is less than the denominator

$$\lim_{x\to\infty} \frac{6x+1}{2x^2-5x+2} = \lim_{x\to\infty} \frac{\frac{6}{x}+\frac{1}{x^2}}{2-\frac{5}{x}+\frac{2}{x^2}} = \lim_{x\to\infty} \frac{0+0}{2-0+0} = \frac{0}{2} = 0$$

20. End behaviour for rational functions (equal degrees)
Hint: the degree of the numerator is the same as the denominator

$$\lim_{x\to\infty} \frac{6x^2+1}{2x^2-5x+2} = \lim_{x\to\infty} \frac{6+\frac{1}{x^2}}{2-\frac{5}{x}+\frac{2}{x^2}} = \lim_{x\to\infty} \frac{6+0}{2-0+0} = \frac{6}{2} = 3$$

21. End behaviour for rational functions (different degrees)
Hint: the degrees of the numerator is greater than the degree of the denominator

$$\lim_{x\to\infty} \frac{6x^4+1}{2x^2-5x+2} = \lim_{x\to\infty} \frac{6x^2+\frac{1}{x^2}}{2-\frac{5}{x}+\frac{2}{x^2}} = \lim_{x\to\infty} \frac{6x^2+0}{2-0+0} = \frac{\infty}{2} = \infty$$

22. End behaviours for rational functions
Hint: If there is a negative exponent like $2x^{-2}$, we can rewrite that as $\frac{2}{x^2}$
Hint: Keep in mind the direction at which $x$ is changing (increasing or decreasing)

$$\lim_{x\to-\infty} 2x^{-8} + 4x^3 = \lim_{x\to-\infty} \frac{2}{x^8} + 4x^3 = 0 - \infty = -\infty$$

$$\lim_{x\to\infty} \frac{14x^3+3x^2-2x}{21x^3+x^2+2x+1} = \lim_{x\to\infty} \frac{14+\frac{3}{x}-\frac{2}{x^2}}{21+\frac{1}{x}+\frac{2}{x^2}+\frac{1}{x^3}} = \frac{14}{21} = \frac{2}{3}$$

$$\lim_{x\to\infty} \frac{9x^3+x^2-5}{3x^4+4x^2} = \lim_{x\to\infty} \frac{\frac{9}{x}+\frac{1}{x^2}-\frac{5}{x^4}}{3+\frac{4}{x^2}} = \frac{0}{3} = 0$$

23. Asymptotes for a rational function

$$f(x) = \frac{3x^2-7}{x^2+5x}$$

Horizontal Asymptote(s): $y = 3$
**Explanation**: The end behaviour for this function approaches 3 (on both ends), so there is a single horizontal asymptote

24. End heaviour for gebraic function

$$\begin{aligned}
\lim_{x\to-\infty}\frac{\sqrt{16x^2+x}}{x} &= \lim_{x\to-\infty}\frac{\frac{\sqrt{16x^2+x}}{x}}{\frac{x}{x}} && (1)\\
&= \lim_{x\to-\infty}\frac{\frac{1}{x}\sqrt{16x^2+x}}{1} && (2)\\
&= \lim_{x\to-\infty}\frac{1}{-\sqrt{x^2}}\sqrt{16x^2+x} && (3)\\
&= \lim_{x\to-\infty}-\sqrt{\frac{16x^2}{x^2}+\frac{x}{x^2}} && (4)\\
&= \lim_{x\to-\infty}-\sqrt{16+\frac{1}{x}} && (5)\\
&= \lim_{x\to-\infty}-\sqrt{16} && (6)\\
&= -4 && (7)\\
& && (8)
\end{aligned}$$

25. End behaviour for transcendental function

$$\lim_{x\to\infty}\frac{\sin x}{e^x+\ln x}=\lim_{x\to\infty}\frac{\sin x}{\infty+\infty}=0$$

**Explanation**: Since $\sin x$ is bounded between $-1$ and 1, and the denominator is a very large number, we know as $x$ increases, the function with approach zero

26. (Section 2.1, Related Exercise 13):
Hint: use the secant line slope formula

$$s(t)=-16t^2+128t$$

(a) $[1,4]$

$$\frac{256-112}{4-1}=\frac{144}{3}=48$$

(b) $[1,3]$

$$\frac{240-112}{3-1}=\frac{128}{2}=64$$

(c) $[1,2]$

$$\frac{192-112}{2-1}=\frac{80}{1}=84$$

(d) $[1,1+h]$, where $h>0$ is a real number

$$\frac{112+-16h^2+128h-112}{1+h-1}=\frac{-16h^2+128h}{h}=-16h+128=16(-h+6)$$

27. (Section 2.1, Related Exercise 15): Hint: we use the slope formula for the secant line, and the relationship is referring to the interval

$$s(t)=-16^t+100t$$

$$\begin{aligned}
\frac{s(t_1)-s(t_0)}{t_1-t_0} &= \frac{s(2)-s(0.5)}{2-0.5} && (1)\\
&= \frac{136-46}{1.5} && (2)\\
&= \frac{90}{1.5} && (3)\\
&= 60 && (4)
\end{aligned}$$

The slope of this secant line, through the lens of average velocity could be viewed as the average velocity over the interval $[0.5,2]$

28. (Section 2.1, Related Exercise 17):

$$s(t) = -16t^2 + 128t$$

| | | | | |
|---|---|---|---|---|
| $[1,2]$ | $[1,1.5]$ | $[1,1.1]$ | $[1,1.01]$ | $[1,1.001]$ |
| 80 | 88 | 94.4 | 95.84 | 95.984 |

$$v_{inst} = \lim_{t\to 1} s(t) = 96$$

29. (Section 2.1, Related Exercise 19):

$$s(t) = -16t^2 + 100t$$

| | | | | |
|---|---|---|---|---|
| $[2,3]$ | $[2.9,3]$ | $[2.99,3]$ | $[2.999,3]$ | $[2.9999,3]$ |
| 20 | 5.6 | 4.16 | 4.016 | 4.002 |

$$v_{inst} = \lim_{t\to 3} s(t) = 4$$

30. (Section 2.2, Related Exercise 3):

- $h(2) = 5$
- $\lim_{x\to 2} h(x) = 3$
- $h(4) =$ Does not exist
- $\lim_{x\to 4} h(x) = 1$
- $\lim_{x\to 5} h(x) = 2$

31. (Section 2.2, Related Exercise 4):

- $g(0) = 0$
- $\lim_{x\to 0} g(x) = 1$
- $g(1) = 2$
- $\lim_{x\to 1} g(x) = 2$

32. (Section 2.2, Related Exercise 7):

$$f(x) = \frac{x^2-4}{x-2}$$

| | | | |
|---|---|---|---|
| 1.9 | 1.99 | 1.999 | 1.9999 |
| 3.9 | 3.99 | 3.999 | 3.9999 |

| | | | |
|---|---|---|---|
| 2.1 | 2.01 | 2.001 | 2.0001 |
| 4.1 | 4.01 | 4.001 | 4.0001 |

$$\lim_{x\to 2} f(x) = 4$$

33. (Section 2.2, Related Exercise 8):

$$f(x) = \frac{x^3-1}{x-1}$$

| | | | |
|---|---|---|---|
| 0.9 | 0.99 | 0.999 | 0.9999 |
| 2.71 | 2.9701 | 3.997001 | 3.99970001 |

| | | | |
|---|---|---|---|
| 1.1 | 1.01 | 1.001 | 1.0001 |
| 3.31 | 3.0301 | 3.003001 | 3.00030001 |

$$\lim_{x\to 1} f(x) = 3$$

34. (Section 2.2, Related Exercise 27):

$$f(x) = \frac{x-2}{\ln|x-2|}$$

$$\lim_{x\to 2} f(x) = 2$$

35. (Section 2.2, Related Exercise 28):

$$f(x) = \frac{e^{2x} - 2x - 1}{x^2}$$

$$\lim_{x\to 0} f(x) = 0$$

36. (Section 2.2, Related Exercise 19):

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x \le -1 \\ 3 & \text{if } x > -1 \end{cases}$$

$$\lim_{x\to -1^-} f(x) = 2$$

$$\lim_{x\to -1^+} f(x) = 3$$

$$\lim_{x\to -1} f(x) = \text{Does not exist}$$

37. (Section 2.2, Related Exercise 20):

$$f(x) = \begin{cases} 3 - x & \text{if } x < 2 \\ x - 1 & \text{if } x > 2 \end{cases}$$

$$\lim_{x\to 2^-} f(x) = 1$$

$$\lim_{x\to 2^+} f(x) = 1$$

$$\lim_{x\to 2} f(x) = 1$$

38. (Section 2.3, Related Exercise 19):

$$\lim_{x\to 4} 3x - 7 = 3(4) - 7 = 12 - 7 = 5$$

39. (Section 2.3, Related Exercise 22):

$$\lim_{x\to 6} 4 = 4$$

40. (Section 2.3, Related Exercise 11): Quotient, Difference

$$\begin{aligned}
\lim_{x\to 1} \frac{f(x)}{g(x) - h(x)} &= \frac{\lim_{x\to 1} f(x)}{\lim_{x\to 1} g(x) - h(x)} && (1) \\
&= \frac{\lim_{x\to 1} f(x)}{\lim_{x\to 1} g(x) - h(x)} && (2) \\
&= \frac{\lim_{x\to 1} f(x)}{\lim_{x\to 1} g(x) - \lim_{x\to 1} h(x)} && (3) \\
&= \frac{8}{3-2} && (4) \\
&= \frac{8}{1} && (5) \\
&= 8 && (6)
\end{aligned}$$

41. (Section 2.3, Related Exercise 12): Root, Sum, Product

$$\begin{aligned}
\lim_{x\to 1} \sqrt[3]{f(x)g(x)+3} &= \sqrt[3]{\lim_{x\to 1} f(x)g(x)+3} && (1)\\
&= \sqrt[3]{\lim_{x\to 1} f(x)g(x)+3} && (2)\\
&= \sqrt[3]{\lim_{x\to 1} f(x)g(x)+\lim_{x\to 1} 3} && (3)\\
&= \sqrt[3]{\lim_{x\to 1} f(x)\lim_{x\to 1} g(x)+\lim_{x\to 1} 3} && (4)\\
&= \sqrt[3]{8\cdot 3+3} && (5)\\
&= \sqrt[3]{24+3} && (6)\\
&= \sqrt[3]{27} && (7)\\
&= 3 && (8)
\end{aligned}$$

42. (Section 2.3, Related Exercise 25):

$$\begin{aligned}
\lim_{x\to 1} \frac{5x^2+6x+1}{8x-4} &= \frac{5(1^2)+6(1)+1}{8(1)-4} && (1)\\
&= \frac{5+6+1}{8-4} && (2)\\
&= \frac{12}{4} && (3)\\
&= 3 && (4)
\end{aligned}$$

43. (Section 2.3, Related Exercise 26):

$$\begin{aligned}
\lim_{t\to 3} \sqrt[3]{t^2-10} &= \sqrt[3]{\lim_{t\to 3} t^2-10} && (1)\\
&= \sqrt[3]{3^2-10} && (2)\\
&= \sqrt[3]{9-10} && (3)\\
&= \sqrt[3]{-1} && (4)\\
&= -1 && (5)
\end{aligned}$$

44. (Section 2.3, Related Exercise 27):

$$\begin{aligned}
\lim_{p\to 2} \frac{3p}{\sqrt{4p+1}-1} &= \frac{\lim_{p\to 2} 3p}{\lim_{p\to 2} \sqrt{4p+1}-1} && (1)\\
&= \frac{3(2)}{\sqrt{\lim_{p\to 2} 4p+1}-1} && (2)\\
&= \frac{6}{\sqrt{4(2)+1}-1} && (3)\\
&= \frac{6}{\sqrt{8+1}-1} && (4)\\
&= \frac{6}{\sqrt{9}-1} && (5)\\
&= \frac{6}{3-1} && (6)\\
&= \frac{6}{2} && (7)\\
&= 3 && (8)
\end{aligned}$$

45. (Section 2.3, Related Exercise 72):

$$g(x) = \begin{cases} 5x-15 & \text{if } x<4 \\ \sqrt{6x+1} & \text{if } x\geq 4 \end{cases}$$

$$\lim_{x\to 4^-} g(x) = 5$$
$$\lim_{x\to 4^+} g(x) = 5$$
$$\lim_{x\to 4} g(x) = 5$$

46. (Section 2.3, Related Exercise 73):

$$g(x) = \begin{cases} x^2+1 & \text{if } x < -1 \\ \sqrt{x+1} & \text{if } x \geq -1 \end{cases}$$

$$\lim_{x\to -1^-} g(x) = 2$$
$$\lim_{x\to -1^+} g(x) = 0$$
$$\lim_{x\to -1} g(x) = \text{Does not exist}$$

47. (Section 2.3, Related Exercise 34):

$$\begin{aligned}
\lim_{x\to 3} \frac{x^2-2x-3}{x-3} &= \lim_{x\to 3} \frac{(x-3)(x+1)}{x-3} && (1)\\
&= \lim_{x\to 3} x+1 && (2)\\
&= 3+1 && (3)\\
&= 4 && (4)
\end{aligned}$$

48. (Section 2.3, Related Exercise 41):

$$\begin{aligned}
\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} &= \lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} \cdot \frac{\sqrt{x}+3}{\sqrt{x}+3} && (1)\\
&= \lim_{x\to 9} \frac{(\sqrt{x}-3)(\sqrt{x}+3)}{(x-9)(\sqrt{x}+3)} && (2)\\
&= \lim_{x\to 9} \frac{x-9}{(x-9)(\sqrt{x}+3)} && (3)\\
&= \lim_{x\to 9} \frac{1}{\sqrt{x}+3} && (4)\\
&= \frac{1}{\sqrt{9}+3} && (5)\\
&= \frac{1}{3+3} && (6)\\
&= \frac{1}{6} && (7)
\end{aligned}$$

49. (Section 2.3, Related Exercise 69):

$$\lim_{x\to 1^+} \frac{x-1}{\sqrt{x^2-1}} = \text{Does not exist}$$

50. (Section 2.3, Related Exercise 70):

$$\begin{aligned}
\lim_{x\to 1^+} \frac{x-1}{\sqrt{x^2-1}} &= \lim_{x\to 1^+} \frac{x-1}{\sqrt{x^2-1}} \cdot \frac{x+1}{x+1} && (1)\\
&= \lim_{x\to 1^+} \frac{x^2-1}{\sqrt{x^2-1}(x+1)} && (2)\\
&= \lim_{x\to 1^+} \frac{x^2-1}{(x^2-1)^{\frac{1}{2}}(x+1)} && (3)\\
&= \lim_{x\to 1^+} \frac{(x^2-1)^{\frac{1}{2}}}{x+1} && (4)\\
&= \lim_{x\to 1^+} \frac{\sqrt{x^2-1}}{x+1} && (5)\\
&= \frac{\sqrt{1-1}}{1+1} && (6)\\
&= \frac{\sqrt{0}}{2} && (7)\\
&= \frac{0}{2} && (8)\\
&= 0 && (9)
\end{aligned}$$

51. (Section 2.3, Related Exercise 95):

$$\frac{2^x - 2^0}{x-0} = \frac{2^x - 1}{x}$$

| | | | | | |
|---|---|---|---|---|---|
| -1 | -0.1 | -0.01 | -0.001 | -0.0001 | -0.00001 |
| 0.5 | 0.6696700846 | 0.6907504563 | 0.6929070095 | 0.6931231585 | 0.6931447783 |

$$\lim_{x\to 0^1} \frac{2^x-1}{x} = 0.693$$

52. (Section 2.3, Related Exercise 96):

$$\frac{3^x - 3^0}{x-0} = \frac{3^x - 1}{x}$$

| | | | |
|---|---|---|---|
| -0.1 | -0.01 | -0.001 | -0.0001 |
| 1.040415402 | 1.092599583 | 1.098009035 | 1.098551943 |

| | | | |
|---|---|---|---|
| 0.0001 | 0.001 | 0.01 | 0.1 |
| 1.098672638 | 1.099215984 | 1.104669194 | 1.161231740 |

$$\lim_{x\to 0^1} \frac{3^x-1}{x} = 1.0986$$

53. (Section 2.3, Related Exercise 81):
$-|x| < 0 < |x|$ and $\sin\frac{1}{x} \leq 1$, so $|x|\sin\frac{1}{x} \leq |x|$ and $-|x|\sin\frac{1}{x} \geq -|x|$

$$\lim_{x\to 0} -|x| = -|0| = 0$$

$$\lim_{x\to 0} |x| = |0| = 0$$

$$\lim_{x\to 0} x\sin\frac{1}{x} = 0$$

By the Squeeze Theorem, since $\lim_{x\to 0} -|x| = \lim_{x\to 0} |x|$ and the functions are chronologically greater than the last

54. (Section 2.3, Related Exercise 82):

$$\lim_{x\to 0} 1 - \frac{x^2}{2} = 1 - \frac{0}{2} = 1 - 0 = 1$$

$$\lim_{x\to 0} 1 = 1$$

$$\lim_{x\to 0} \cos x = 1$$

By the Squeeze Theorem, since $\lim_{x\to 0} 1 - \frac{x^2}{2} = \lim_{x\to 0} 1$ and the functions are chronologically greater than the last

55. (Section 2.3, Related Exercise 60):

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin 2x}{\sin x} &= \lim_{x\to 0} \frac{2\sin x \cos x}{\sin x} && (1)\\
&= \lim_{x\to 0} 2\cos x && (2)\\
&= 2\cos 0 && (3)\\
&= 2 \cdot 1 && (4)\\
&= 2 && (5)
\end{aligned}$$

56. (Section 2.3, Related Exercise 61):

$$\begin{aligned}
\lim_{x\to 0} \frac{1-\cos x}{\cos^2 x - 3\cos x + 2} &= \lim_{x\to 0} \frac{1}{\cos^2 x - 2\cos x + 2} && (1)\\
&= \lim_{x\to 0} \frac{1}{\cos x \cos x - 2\cos x + 2} && (2)\\
&= \frac{1}{\cos 0 \cos 0 - 2\cos 0 + 2} && (3)\\
&= \frac{1}{1 \cdot 1 - 2(1) + 2} && (4)\\
&= \frac{1}{1 - 2 + 2} && (5)\\
&= \frac{1}{1} && (6)\\
&= 1 && (7)
\end{aligned}$$

57. (Section 2.4, Related Exercise 6):

$$f(x) = \frac{x}{(x^2 - 2x - 3)^2}$$

$$\lim_{x\to -1} f(x) = -\infty$$

$$\lim_{x\to 3} f(x) = \infty$$

58. (Section 2.4, Related Exercise 7):

$$\lim_{x\to 1^-} f(x) = \infty$$

$$\lim_{x\to 1^+} f(x) = \infty$$

$$\lim_{x\to 1} f(x) = \infty$$

$$\lim_{x\to 2^-} f(x) = \infty$$

$$\lim_{x\to 2^+} f(x) = -\infty$$

$$\lim_{x\to 2} f(x) = \text{Does not exist}$$

59. (Section 2.4, Related Exercise 8):

$$\lim_{x\to 2^-} g(x) = \infty$$

$$\lim_{x\to 2^+} g(x) = -\infty$$

$$\lim_{x\to 2} g(x) = \text{Does not exist}$$

$$\lim_{x\to 4^-} g(x) = -\infty$$

$$\lim_{x\to 4^+} g(x) = -\infty$$

$$\lim_{x\to 4} g(x) = -\infty$$

60. (Section 2.4, Related Exercise 21):

$$\lim_{x\to 2^+} \frac{1}{x-2} = \infty$$
$$\lim_{x\to 2^-} \frac{1}{x-2} = -\infty$$
$$\lim_{x\to 2} \frac{1}{x-2} = \text{Does not exist}$$

61. (Section 2.4, Related Exercise 22):

$$\lim_{x\to 3^+} \frac{2}{(x-3)^3} = \infty$$
$$\lim_{x\to 3^-} \frac{2}{(x-3)^3} = -\infty$$
$$\lim_{x\to 3} \frac{2}{(x-3)^3} = \text{Does not exist}$$

62. (Section 2.4, Related Exercise 28):

$$\lim_{t\to -2^+} \frac{t^3-5t^2+6t}{t^4-4t^2} = \lim_{t\to -2^+} \frac{t(t-2)(t-3)}{t^2(t^2-4)} = \lim_{t\to -2^+} \frac{t(t-2)(t-3)}{t^2(t-2)(t+2)} = \lim_{t\to -2^+} \frac{t(t-3)}{t^2(t+2)} = \lim_{t\to -2^+} \frac{t^2-3t}{t^3+2t^2} = -\infty$$
$$\lim_{t\to -2^-} \frac{t^3-5t^2+6t}{t^4-4t^2} = \lim_{t\to -2^-} \frac{t(t-2)(t-3)}{t^2(t^2-4)} = \lim_{t\to -2^-} \frac{t(t-2)(t-3)}{t^2(t-2)(t+2)} = \lim_{t\to -2^-} \frac{t(t-3)}{t^2(t+2)} = \lim_{t\to -2^-} \frac{t^2-3t}{t^3+2t^2} = -\infty$$
$$\lim_{t\to -2} \frac{t^3-5t^2+6t}{t^4-4t^2} = \lim_{t\to -2} \frac{t(t-2)(t-3)}{t^2(t^2-4)} = \lim_{t\to -2} \frac{t(t-2)(t-3)}{t^2(t-2)(t+2)} = \lim_{t\to -2} \frac{t(t-3)}{t^2(t+2)} = \lim_{t\to -2} \frac{t^2-3t}{t^3+2t^2} = -\infty$$
$$\lim_{t\to 2} \frac{t^3-5t^2+6t}{t^4-4t^2} = \lim_{t\to 2} \frac{t(t-2)(t-3)}{t^2(t^2-4)} = -\frac{1}{8}$$

63. (Section 2.4, Related Exercise 31): Remember, if you are able to solve by direct substitution after canceling terms (where the denominator does not equal zero), that's your answer

$$\lim_{x\to 0} \frac{x-3}{x^4-9x^2} = \lim_{x\to 0} \frac{x-3}{x^2(x-3)(x+3)} = \lim_{x\to 0} \frac{1}{x^2(x+3)} = \lim_{x\to 0} \frac{1}{x^3+3x^2} = \infty$$
$$\lim_{x\to 3} \frac{x-3}{x^4-9x^2} = \lim_{x\to 3} \frac{x-3}{x^2(x-3)(x+3)} = \lim_{x\to 3} \frac{1}{x^2(x+3)} = \lim_{x\to -3} \frac{1}{x^3+3x^2} = \frac{1}{54}$$
$$\lim_{x\to -3} \frac{x-3}{x^4-9x^2} = \lim_{x\to -3} \frac{x-3}{x^2(x-3)(x+3)} = \lim_{x\to -3} \frac{1}{x^2(x+3)} = \lim_{x\to -3} \frac{1}{x^3+3x^2} = \text{Does not exist}$$

64. (Section 2.4, Related Exercise 45):

$$f(x) = \frac{x-5}{x^2-25} = \frac{x-5}{(x-5)(x+5)} = \frac{1}{x+5}$$

Vertical Asymptotes: $x = -5$

$$\lim_{x\to 5} f(x) = \lim_{x\to 5} \frac{1}{x+5} = \frac{1}{5+5} = \frac{1}{10}$$
$$\lim_{x\to -5^-} f(x) = \lim_{x\to -5^-} \frac{1}{x+5} = -\infty$$
$$\lim_{x\to -5^+} f(x) = \lim_{x\to -5^+} \frac{1}{x+5} = \infty$$

65. (Section 2.4, Related Exercise 46):

$$f(x) = \frac{x+7}{x^4-49x^2} = \frac{x+7}{x^2(x^2-49)} = \frac{x+7}{x^2(x+7)(x-7)} = \frac{1}{x^2(x-7)} = \frac{1}{x^3-7x^2}$$

Vertical Asymptotes: $x = 0$, $x = 7$, $x = -7$

$$\lim_{x\to 7^-} f(x) = \lim_{x\to 7^-} \frac{1}{x^3-6x^2} = -\infty$$

$$\lim_{x\to 7^+} f(x) = \lim_{x\to 7^+} \frac{1}{x^3 - 6x^2} = \infty$$

$$\lim_{x\to -7} f(x) = \lim_{x\to -7} \frac{1}{x^3 - 7x^2} = \text{Does not exist}$$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1}{x^3 - 7x^2} = -\infty$$

66. (Section 2.4, Related Exercise 39):

$$\lim_{\theta\to 0^+} \csc\theta = \infty$$

67. (Section 2.4, Related Exercise 40):

$$\lim_{x\to 0^-} \csc x = -\infty$$

A copy of my notes (in LaTeX) are available on my GitHub