Module 3 Notes (MATH-211)

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General Notes (and Definitions)

- The Chain Rule
  Suppose $y = f(u)$ is differentiable at $u = g(x)$ and $u = g(x)$ is differentiable at x. The composite function $y = f(g(x))$ is differentiable at x, and its derivative can be expressed in two equivalent ways.

$$\frac{dy}{dx} = \frac{dy}{dy} \cdot \frac{du}{dx} \tag{1}$$

$$\frac{d}{dx}\left(f\left(g\left(x\right)\right)\right) = f'\left(g\left(x\right)\right) \cdot g'\left(x\right) \tag{2}$$

  Application of the Chain Rule (Assume the differentiable function $y = f(g(x))$ is given):

  1. Identify an outer function f and an inner function g, and let $u = g(x)$.
  2. Replace $g(x)$ with u to express y in terms of u:

$$y = f(g(x)) = f(u)$$

  3. Calculate the product

$$\frac{dy}{du} \cdot \frac{du}{dx}$$

  4. Replace u with $g(x)$ in $\frac{dy}{du}$ to obtain $\frac{dy}{dx}$

  If g is differentiable for all x in its domain and $p \in \mathbb{R}$,

$$\frac{d}{dx}\left(\left(g\left(x\right)\right)^p\right) = p\left(g\left(x\right)\right)^{p-1} g'\left(x\right)$$

- Implicit Differentiation
  When we are unable to solve for y explicitly, we treat y as a function of x ($y = y(x)$) and apply the Chain Rule:

$$y' = \frac{dy}{dx}$$

$$\frac{d}{dx}y^n = ny^{n-1}\frac{dy}{dx}$$

- Derivatives of Logarithmic and Exponential Functions

$$\frac{d}{dx}\left(\ln x\right) = \frac{1}{x}, \text{ for } x > 0$$

$$\frac{d}{dx}\left(\ln|x|\right) = \frac{1}{x}, \text{ for } x \neq 0$$

  If u is differentiable at x and $u(x) \neq 0$, then

$$\frac{d}{dx}\left(\ln|u(x)|\right) = \frac{u'(x)}{u(x)}$$

  If $b > 0$ and $b \neq 1$, then for all x,

$$\frac{d}{dx}\left(b^x\right) = b^x \ln b$$

  General Power Rule:

$$\text{For } p \in \mathbb{R} \text{ and for } x > 0,\ \frac{d}{dx}\left(x^p\right) = px^{p-1}$$

Furthermore, if u is a positive differentiable function on its domain, then

$$\frac{d}{dx}\left(u\left(x\right)^{p}\right) = p\left(u\left(x\right)\right)^{p-1}\cdot u'\left(x\right)$$

Functions of the form $f(x) = (g(x))^{h(x)}$, where both g and h are nonconstant functions, are neither exponential function nor power functions (they are sometimes called **tower functions**). To compute their derivatives, we use the identity $b^x = e^{x\ln b}$ to rewrite f with base e:

$$f(x) = (g(x))^{h(x)} = e^{h(x)\ln g(x)}$$

If $b > 0$ and $b \neq 1$, then

$$\frac{d}{dx}\left(\log_b x\right) = \frac{1}{x\ln b}, \text{ for } x > 0$$
$$\frac{d}{dx}\left(\log_b |x|\right) = \frac{1}{x\ln b}, \text{ for } x \neq 0$$

Useful Properties of Logarithms

$$\begin{aligned} \ln xy &= \ln x + \ln y && (1)\\ \ln\left(\frac{x}{y}\right) &= \ln x - \ln y && (2)\\ \ln x^z &= z\ln x && (3) \end{aligned}$$

- Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}\left(\sec^{-1}x\right) = \begin{cases} \frac{1}{x\sqrt{x^2-1}} & \text{if } x > 1\\ -\frac{1}{x\sqrt{x^2-1}} & \text{if } x < 1 \end{cases}$$

$$\begin{aligned} \frac{d}{dx}\left(\sin^{-1}x\right) &= \frac{1}{\sqrt{1-x^2}}, \text{ for } -1 < x < 1 && (1)\\ \frac{d}{dx}\left(\cos^{-1}x\right) &= -\frac{1}{\sqrt{1-x^2}}, \text{ for } -1 < x < 1 && (2)\\ \frac{d}{dx}\left(\tan^{-1}x\right) &= \frac{1}{1+x^2}, \text{ for } -\infty < x < \infty && (3)\\ \frac{d}{dx}\left(\cot^{-1}x\right) &= -\frac{1}{1+x^2}, \text{ for } -\infty < x < \infty && (4)\\ \frac{d}{dx}\left(\sec^{-1}x\right) &= \frac{1}{|x|\sqrt{x^2-1}}, \text{ for } |x| > 1 && (5)\\ \frac{d}{dx}\left(\csc^{-1}x\right) &= -\frac{1}{|x|\sqrt{x^2-1}}, \text{ for } |x| > 1 && (6)\\ & && (7) \end{aligned}$$

Let f be differentiable and have an inverse on an interval I. If x_0 is a point of I at which $f'(x_0) \neq 0$, then f^{-1} is differentiable at $y_0 = f(x_0)$ and

$$\left(f^{-1}\right)'(y_0) = \frac{1}{f'\left(x_0\right)}, \text{ where } y_0 = f\left(x_0\right)$$

- Related Rates
  Procedure

  1. Read the problem carefully, making a sketch to organize the given information. Identify the rates that are given and the rate that is to be determined.
  2. Write one or more equations that express the basic relationships among the variables.
  3. Introduce rates of change by differentiating the appropriate equation(s) with respect to time t.
  4. Substitute known values and solve for the desired quantity.
  5. Check that units are consistent and the answer is reasonable. (For example, does it have the correct sign?)

Examples

1. The Chain Rule

$$\begin{aligned}
y &= \left(5x^2 + 11x\right)^{\frac{4}{3}} && (1)\\
u &= 5x^2 + 11x && (2)\\
f &= u^{\frac{4}{3}} && (3)\\
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} && (4)\\
&= \frac{4}{3} u^{\frac{1}{3}} \cdot 10x + 11 && (5)\\
&= \frac{4}{3}\left(5x^2 + 11x\right)^{\frac{1}{3}} \cdot 10x + 11 && (6)\\
& && (7)
\end{aligned}$$

$$\begin{aligned}
y &= e^{4x^2+1} && (1)\\
u &= 4x^2 + 1 && (2)\\
y &= e^u && (3)\\
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} && (4)\\
&= e^u \cdot 8x && (5)\\
&= e^{4x^2+1} \cdot 8x && (6)\\
&= 8xe^{4x^2+1} && (7)
\end{aligned}$$

2. The Chain Rule (with Tables)

$$h(x) = f(g(x))$$

$$\begin{aligned}
y &= f(g(x)) && (1)\\
u &= g(x) && (2)\\
y &= f(u) && (3)\\
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} && (4)\\
&= f(u) \cdot g'(x) && (5)\\
&= f'(g(x)) \cdot g'(x) && (6)
\end{aligned}$$

$$\begin{aligned}
h'(1) &= f'(g(1)) \cdot g'(1) && (7)\\
&= f'(4) \cdot 9 && (8)\\
&= 7 \cdot 9 && (9)\\
&= 63 && (10)
\end{aligned}$$

$$\begin{aligned}
h'(2) &= f'(g(2)) \cdot g'(2) && (11)\\
&= f'(1) \cdot 7 && (12)\\
&= -6 \cdot 7 && (13)\\
&= -42 && (14)
\end{aligned}$$

$$\begin{aligned}
h'(3) &= f'(g(3)) \cdot g'(3) && (15)\\
&= f'(5) \cdot 3 && (16)\\
&= 2 \cdot 3 && (17)\\
&= 6 && (18)
\end{aligned}$$

$$k(x) = g(g(x))$$

$$
\begin{aligned}
y &= g(g(x)) && (1)\\
u &= g(x) && (2)\\
y &= g(u) && (3)\\
\frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx} && (4)\\
&= g'(u)\cdot g'(x) && (5)\\
&= g'(g(x))\cdot g'(x) && (6)
\end{aligned}
$$

$$
\begin{aligned}
k'(3) &= g'(g(3))\cdot g'(3) && (7)\\
&= g'(5)\cdot 3 && (8)\\
&= -5\cdot 3 && (9)\\
&= -15 && (10)
\end{aligned}
$$

$$
\begin{aligned}
k'(1) &= g'(g(1))\cdot g'(1) && (11)\\
&= g'(4)\cdot 9 && (12)\\
&= -1\cdot 9 && (13)\\
&= -9 && (14)
\end{aligned}
$$

$$
\begin{aligned}
k'(5) &= g'(g(5))\cdot g'(5) && (15)\\
&= g'(3)\cdot -5 && (16)\\
&= 3\cdot -5 && (17)\\
&= -15 && (18)
\end{aligned}
$$

3. The Chain Rule (All Forms)

$$
\begin{aligned}
y &= \sqrt[3]{2x^2-x-5} && (1)\\
u &= 2x^2-x-5 && (2)\\
y &= \sqrt[3]{u} && (3)\\
y' &= u^{-\frac{2}{3}}\cdot 4x-1 && (4)\\
&= \frac{1}{3}\left(2x^2-x-5\right)^{-\frac{2}{3}}\cdot 4x-1 && (5)
\end{aligned}
$$

$$
\begin{aligned}
y &= \csc(\tan t) && (1)\\
u &= \tan t && (2)\\
y &= \csc u && (3)\\
y' &= -\csc u\cot u\cdot\sec^2 t && (4)\\
&= -\csc(\tan t)\cot(\tan t)\cdot\sec^2 t && (5)
\end{aligned}
$$

4. The Chain Rule (Nested)

$$
\begin{aligned}
y &= \tan(\sin e^x) && (1)\\
u_2 &= e^x && (2)\\
u_1 &= \sin u_2 && (3)\\
y &= \tan u_1 && (4)\\
y' &= \sec^2(\sin e^x)\cdot\cos e^x\cdot e^x && (5)
\end{aligned}
$$

5. The Chain Rule (Combination of Rules)

$$\begin{aligned} y &= \left(\frac{e^x}{x+1}\right)^8 && (1)\\ y' &= 8\left(\frac{e^x}{x+1}\right)^7 \cdot \frac{xe^x}{(x+1)^2} && (2)\\ &= 8\frac{e^{7x}}{(x+1)^7} \cdot \frac{xe^x}{(x+1)^2} && (3)\\ &= \frac{8xe^{8x}}{(x+1)^9} && (4) \end{aligned}$$

6. Implicit Differentiation

$$\begin{aligned} x^4 + y^4 &= 2 && (1)\\ 4x^3 + 4y^3\frac{dy}{dx} &= 0 && (2)\\ 4y^3\frac{dy}{dx} &= -4x^3 && (3)\\ \frac{dy}{dx} &= \frac{-4x^3}{4y^3} && (4)\\ &= \frac{-x^3}{y^3} && (5) \end{aligned}$$

$$\left.\frac{dy}{dx}\right|_{(1,-1)} = \frac{-(1)^3}{(-1)^3} = \frac{-1}{-1} = 1$$

7. Implicit Differentiation (Finding y)

$$\begin{aligned} y &= y = xe^y && (1)\\ y' &= e^y + xe^y y' && (2)\\ y' - y'xe^y &= e^y && (3)\\ y'(1 - xe^y) &= e^y && (4)\\ y' &= \frac{e^y}{1 - xe^y} && (5) \end{aligned}$$

8. Implicit Differentiation (Tangent Line)

$$\begin{aligned} \cos(x-y) + \sin y &= \sqrt{2} && (1)\\ (-\sin(x-y))(1 - 1y') + \cos y\,(y') &= 0 && (2)\\ -\sin(x-y) + y'\sin(x-y) + y'\cos y &= 0 && (3)\\ y'(\sin(x-y) + \cos y) &= \sin(x-y) && (4)\\ y' &= \frac{\sin(x-y)}{\sin(x-y) + \cos y} && (5)\\ \left.y'\right|_{\left(\frac{\pi}{2},\frac{\pi}{4}\right)} &= \frac{\sin\left(\frac{\pi}{2} - \frac{\pi}{2}\right)}{\sin\left(\frac{\pi}{2} - \frac{\pi}{2}\right) + \cos\frac{\pi}{4}} && (6)\\ &= \frac{1}{2} && (7)\\ y &= \frac{1}{2}x && (8) \end{aligned}$$

9. Implicit Differentiation (Higher Order)

$$\begin{aligned}
x^4+y^4 &= 64 && (1)\\
4x^3+4y^3\frac{dy}{dx} &= 0 && (2)\\
4y^3\frac{dy}{dx} &= -4x^3 && (3)\\
\frac{dy}{dx} &= \frac{-4x^3}{4y^3} && (4)\\
&= \frac{-x^3}{y^3} && (5)\\
\frac{d^2y}{dx^2} &= \frac{-3x^2y^3-\left(-x^3 3y^2\frac{dy}{dx}\right)}{\left(y^3\right)^2} && (6)\\
&= \frac{-3x^2y^3+3x^3y^2\frac{dy}{dx}}{y^6} && (7)\\
&= \frac{-3x^2y^3+3x^3y^2\frac{-x^3}{y^3}}{y^6} && (8)\\
&= \frac{-3x^2y^3+\frac{-3x^6y^2}{y^3}}{y^6} && (9)\\
&= \frac{-3x^2y^3-\frac{3x^6}{y}}{y^6} && (10)\\
&= \frac{\frac{-3x^2y^4-3x^6}{y}}{y^6} && (11)\\
&= \frac{-3x^2y^4-3x^6}{y^7} && (12)
\end{aligned}$$

10. Derivatives with $\ln x$

$$\begin{aligned}
y &= \ln 2x^8 && (1)\\
\frac{dy}{dx} &= \frac{1}{2x^8}\cdot 16x^7 && (2)\\
&= \frac{16x^7}{2x^8} && (3)\\
&= \frac{8}{x} && (4)
\end{aligned}$$

$$\begin{aligned}
y &= x^2\left(1-\ln x^2\right) && (1)\\
\frac{dy}{dx} &= 2x\left(1-\ln x^2\right)+x^2\left(-\frac{2x}{x^2}\right) && (2)\\
&= 2x-2x\ln x^2-2x && (3)\\
&= -2x\ln x^2 && (4)
\end{aligned}$$

11. Derivatives with b^x

$$\begin{aligned}
y &= 2^{2x} && (1)\\
\frac{dy}{dx} &= 2^{2x}\ln 2\cdot 2 && (2)\\
&= 2^{2x+1}\ln 2 && (3)
\end{aligned}$$

$$\begin{aligned}
f(x) &= 7^{-x}\cos x && (1)\\
\frac{dy}{dx} &= -7^{-x}\ln 7\cos x-7^{-x}\sin x && (2)\\
&= -7^{-x}\left(\ln 7\cos x+\sin x\right) && (3)
\end{aligned}$$

12. Derivatives with the General Power Rule

$$\begin{aligned} y &= x^e && (1)\\ \frac{dy}{dx} &= ex^{e-1} && (2) \end{aligned}$$

$$\begin{aligned} f(x) &= \left(x^3+3^x\right)^\pi && (1)\\ \frac{dy}{dx} &= \pi\left(x^3+3^x\right)^{\pi-1}\cdot\left(3x^2+3^x\ln 3\right) && (2) \end{aligned}$$

13. Derivatives with Tower Functions

$$\begin{aligned} g(x) &= x^{\ln x} && (1)\\ &= e^{\ln x \ln x} && (2)\\ &= e^{(\ln x)^2} && (3)\\ g'(x) &= e^{(\ln x)^2}\cdot\frac{2\ln x}{x} && (4)\\ g'(e) &= e^{(\ln e)^2}\cdot\frac{2\ln e}{e} && (5)\\ &= \frac{2e}{e} && (6)\\ &= 2 && (7)\\ y &= 2x-2e+e && (8)\\ &= 2x-e && (9) \end{aligned}$$

14. Derivatives of Logarithmic Functions

$$\begin{aligned} y &= \log_7 5x && (1)\\ \frac{dy}{dx} &= \frac{5}{5x\ln 7} && (2)\\ &= \frac{1}{x\ln 7} && (3) \end{aligned}$$

$$\begin{aligned} y &= \log(\log x) && (1)\\ \frac{dy}{dx} &= \frac{1}{\log x\ln 10}\cdot\frac{1}{x\ln 10} && (2)\\ &= \frac{1}{x\log x\ln(10)^2} && (3) \end{aligned}$$

15. Logarithmic Differentiation

$$\begin{aligned} f(x) &= (\cos x)^{\sec x} && (1)\\ \ln f(x) &= \ln\left((\cos x)^{\sec x}\right) && (2)\\ &= \sec x\cdot\ln(\cos x) && (3)\\ \frac{f'(x)}{f(x)} &= \sec x\cdot\tan x\cdot\ln(\cos x)+\sec x\cdot\frac{-\sin x}{\cos x} && (4)\\ &= \sec x\cdot\tan x\cdot\ln(\cos x)+\sec x\cdot(-\tan x) && (5)\\ &= \tan x\sec x\left(\ln(\cos x)-1\right) && (6)\\ f'(x) &= f(x)\tan x\sec x\left(\ln(\cos x)-1\right) && (7)\\ &= (\cos x)^{\sec x}\tan x\sec x\left(\ln(\cos x)-1\right) && (8)\\ & && (9) \end{aligned}$$

16. Derivatives with $\sin^{-1} x$

$$\begin{aligned} \frac{d}{dx}\left(\sin^{-1}(\ln x)\right) &= \frac{1}{\sqrt{1-(\ln x)^2}}\cdot\frac{1}{x} && (1)\\ &= \frac{1}{x\sqrt{1-(\ln x)^2}} && (2) \end{aligned}$$

$$\begin{aligned}
\frac{d}{dx}\left(\sin^{-1}\left(e^{-2x}\right)\right) &= \frac{1}{\sqrt{1-e^{-4x}}}\cdot e^{-2x}\cdot -2 \quad (1)\\
&= \frac{-2e^{-2x}}{\sqrt{1-e^{-4x}}} \quad (2)\\
& \quad (3)
\end{aligned}$$

17. Finding the Tangent Line of Inverse Trigonometric Functions

$$\begin{aligned}
f(x) &= \cos^{-1} x^2 \quad (1)\\
\left(\frac{1}{\sqrt{2}}, \frac{\pi}{3}\right) & \quad (2)\\
f'(x) &= -\frac{2x}{\sqrt{1-x^4}} \quad (3)\\
f'\left(\frac{1}{\sqrt{2}}\right) &= -\frac{\frac{2}{\sqrt{2}}}{\sqrt{1-\frac{1}{\sqrt{2}}^4}} \quad (4)\\
&= -\frac{\sqrt{2}}{\sqrt{1-\frac{1}{4}}} \quad (5)\\
&= -\frac{\sqrt{2}}{\sqrt{\frac{3}{4}}} \quad (6)\\
&= -\frac{\sqrt{2}}{\frac{\sqrt{3}}{2}} \quad (7)\\
&= -\frac{2\sqrt{2}}{\sqrt{3}} \quad (8)\\
&= \frac{-4}{\sqrt{6}} \quad (9)\\
y &= \frac{-4}{\sqrt{6}}x + \frac{2}{\sqrt{3}} + \frac{\pi}{3} \quad (10)
\end{aligned}$$

18. Application of Derivatives of Inverse Trigonometric Functions

$$\begin{aligned}
\tan\theta &= \frac{400}{x} \quad (1)\\
\theta &= \tan^{-1}\frac{400}{x} \quad (2)\\
\frac{d\theta}{dx} &= \frac{-400}{x^2\left(1+\left(\frac{400}{x}\right)^2\right)} \quad (3)\\
\left.\frac{d\theta}{dx}\right|_{x=500} &= \frac{-400}{500^2\left(1+\left(\frac{400}{500}\right)^2\right)} \quad (4)\\
&= \frac{-400}{500^2\left(1+\left(\frac{4}{5}\right)^2\right)} \quad (5)\\
&= \frac{-400}{500^2\cdot\frac{41}{25}} \quad (6)\\
&= -0.000976 \quad (7)
\end{aligned}$$

19. Derivatives of Inverse Functions

$$f(x) = x^3 + 3$$

If $(2,-1)$ is on the graph of $f^{-1}(x)$, then $(-1,2)$ is on the graph of $f(x)$.

$$\left(f^{-1}\right)(2) = \frac{1}{f'(-1)} = \frac{1}{3(-1)^2} = \frac{1}{3}$$

20. Related Rates (Geometric Ideas)
Hint: we want the decreasing length

Hint (2): use the pythagorean theorem

$$
\begin{aligned}
A &= x^2 && (1)\\
\frac{dA}{dt} &= 2x \cdot \frac{dx}{dt} && (2)\\
f'(5) &= 2(5) \cdot -1 && (3)\\
&= -10 \text{ m/s}^2 && (4)\\
L^2 &= x^2 + x^2 && (5)\\
&= 2x^2 && (6)\\
L &= \sqrt{2x^2} && (7)\\
&= \sqrt{2}x && (8)\\
\frac{dL}{dt} &= \sqrt{2} \cdot -1 && (9)\\
&= -\sqrt{2} \text{ m/s} && (10)
\end{aligned}
$$

21. Related Rates (Distance Formula)
    Hint: use the distance formula

$$
\begin{aligned}
A(h) &= 20h && (1)\\
B(h) &= 15h && (2)\\
A(0.5) &= 20(0.5) && (3)\\
&= 10 && (4)\\
B(0.5) &= 15(0.5) && (5)\\
&= 7.5 && (6)\\
L^2 &= x^2 + y^2 && (7)\\
2L\frac{dL}{dt} &= 2x\frac{dx}{dt} + 2y\frac{dy}{dt} && (8)\\
\frac{dL}{dt} &= \frac{2x\frac{dx}{dt} + 2y\frac{dy}{dt}}{2L} && (9)\\
&= \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{L} && (10)\\
&= \frac{10(20) + 7.5(15)}{\sqrt{10^2 + 7.5^2}} && (11)\\
&= \frac{200 + 112.5}{\sqrt{100 + 56.25}} && (12)\\
&= \frac{312.5}{\sqrt{156.25}} && (13)\\
&= \frac{312.5}{\sqrt{156.25}} && (14)\\
&= \frac{312.5}{12.5} && (15)\\
&= 25 && (16)
\end{aligned}
$$

22. Related Rates (Cylinders & Cones)

$$
\begin{aligned}
V &= \pi r^2 h && (1)\\
\frac{dh}{dt} &= -0.25 && (2)\\
\frac{dv}{dt} &= \pi r^2 \cdot \frac{dh}{dt} && (3)\\
&= \pi 2^2 \cdot -0.25 && (4)\\
&= -\pi && (5)
\end{aligned}
$$

23. Related Rates (Trigonometry)

$$\begin{aligned}
\frac{dy}{dt} &= 20 && (1)\\
\tan\theta &= \frac{y}{300} && (2)\\
\theta &= \tan^{-1}\frac{y}{300} && (3)\\
\frac{d\theta}{dt} &= \frac{1}{1+\left(\frac{y}{300}\right)^2}\cdot\frac{1}{300}\cdot\frac{dy}{dt} && (4)\\
\left.\frac{d\theta}{dt}\right|_{y=400} &= \frac{20}{300\left(1+\frac{16}{9}\right)} && (5)\\
&= 0.024 && (6)
\end{aligned}$$

Related Exercises

1. (Section 3.7 Exercise 15)

$$\begin{aligned}
y &= (3x+7)^{10} && (1)\\
u &= 3x+7 && (2)\\
f(u) &= u^{10} && (3)\\
y' &= 10u^9\cdot 3 && (4)\\
&= 10\,(3x+7)^9\cdot 3 && (5)\\
&= 30\,(3x+7)^9 && (6)
\end{aligned}$$

2. (Section 3.7 Exercise 17)

$$\begin{aligned}
y &= \sin^5 x && (1)\\
u &= \sin x && (2)\\
f(u) &= u^5 && (3)\\
y' &= 5u^4\cdot\cos x && (4)\\
&= 5\sin^4 x\cos x && (5)
\end{aligned}$$

3. (Section 3.7 Exercise 28)

$$\begin{aligned}
y &= \left(x^2+2x+7\right)^8 && (1)\\
u &= x^2+2x+7 && (2)\\
f(u) &= u^8 && (3)\\
y' &= 8u^7\cdot(2x+2) && (4)\\
&= 8\left(x^2+2x+7\right)^7\cdot(2x+2) && (5)\\
&= (16x+16)\left(x^2+2x+7\right)^7 && (6)
\end{aligned}$$

4. (Section 3.7 Exercise 29)

$$\begin{aligned}
y &= \sqrt{10x+1} && (1)\\
u &= 10x+1 && (2)\\
f(u) &= \sqrt{u} && (3)\\
y' &= \frac{1}{2\sqrt{u}}\cdot 10 && (4)\\
&= \frac{1}{2\sqrt{10x+1}}\cdot 10 && (5)\\
&= \frac{10}{2\sqrt{10x+1}} && (6)\\
&= \frac{5}{\sqrt{10x+1}} && (7)
\end{aligned}$$

5. (Section 3.7 Exercise 41)

$$\begin{aligned} y &= \sqrt[4]{\frac{2x}{4x-3}} && (1)\\ u &= \frac{2x}{4x-3} && (2)\\ f(u) &= \sqrt[4]{u} && (3)\\ y' &= \frac{1}{4}u^{-\frac{3}{4}} \cdot -\frac{6}{(4x-3)^2} && (4)\\ &= \frac{1}{4}\left(\frac{2x}{4x-3}\right)^{-\frac{3}{4}} \cdot -\frac{6}{(4x-3)^2} && (5)\\ &= -\frac{6}{4\,(4x-3)^2}\left(\frac{2x}{4x-3}\right)^{-\frac{3}{4}} && (6) \end{aligned}$$

6. (Section 3.7 Exercise 23)

$$\begin{aligned} y &= \tan 5x^2 && (1)\\ u &= 5x^2 && (2)\\ f(u) &= \tan u && (3)\\ y' &= \sec^2 u \cdot 10x && (4)\\ &= \sec^2 5x^2 \cdot 10x && (5)\\ &= 10x \sec^2 5x^2 && (6)\\ & && (7) \end{aligned}$$

7. (Section 3.7 Exercise 24)

$$\begin{aligned} y &= \sin \frac{x}{4} && (1)\\ u &= \frac{x}{4} && (2)\\ f(u) &= \sin u && (3)\\ y' &= \cos u \cdot \frac{4}{16} && (4)\\ &= \cos \frac{x}{4} \cdot \frac{1}{4} && (5)\\ &= \frac{1}{4} \cos \frac{x}{4} && (6) \end{aligned}$$

8. (Section 3.7 Exercise 45)

$$\begin{aligned} y &= (2x^6 - 3x^3 + 3)^{25} && (1)\\ u &= 2x^6 - 3x^3 + 3 && (2)\\ f(u) &= u^{25} && (3)\\ y' &= 25\,(u)^{24} \cdot 12x^5 - 9x^2 && (4)\\ &= 25\left(2x^6 - 3x^3 + 3\right)^{24} \cdot 12x^5 - 9x^2 && (5)\\ &= 25\left(12x^5 - 9x^2\right)\left(2x^6 - 3x^3 + 3\right)^{24} && (6) \end{aligned}$$

9. (Section 3.7 Exercise 46)

$$\begin{aligned} y &= (\cos x + 2\sin x)^8 && (1)\\ u &= \cos x + 2\sin x && (2)\\ f(u) &= u^8 && (3)\\ y' &= 8u^7 \cdot (-\sin x + 2\cos x) && (4)\\ &= 8\,(\cos x + 2\sin x)^7 \cdot (-\sin x + 2\cos x) && (5)\\ &= 8(-\sin x + 2\cos x)\,(\cos x + 2\sin x)^7 && (6)\\ & && (7) \end{aligned}$$

10. (Section 3.7 Exercise 53)

$$\begin{aligned} y &= \sin\left(\sin\left(e^x\right)\right) && (1)\\ y' &= \cos\left(\sin e^x\right)\cos e^x e^x && (2) \end{aligned}$$

11. (Section 3.7 Exercise 54)

$$\begin{aligned} y &= \sin^2 e^{3x+1} && (1)\\ y' &= 6\sin e^{3x+1}\cos e^{3x+1} && (2) \end{aligned}$$

12. (Section 3.7 Exercise 68)

$$\begin{aligned} y &= \left(\frac{3x}{4x+2}\right)^5 && (1)\\ y' &= 5u^4 \cdot \frac{12x+6-12x}{(4x+2)^2} && (2)\\ &= 5\left(\frac{3x}{4x+2}\right)^4 \cdot \frac{6}{(4x+2)^2} && (3)\\ &= \frac{30}{(4x+2)^2}\left(\frac{3x}{4x+2}\right)^4 && (4) \end{aligned}$$

13. (Section 3.7 Exercise 69)

$$\begin{aligned} y &= \left((x+2)\left(x^2+1\right)\right)^4 && (1)\\ y' &= 4u^3 \cdot x^2+1+2x^2+4x && (2)\\ &= 4\left(3x^2+4x+1\right)\left((x+2)\left(x^2+1\right)\right)^3 && (3)\\ &= 4\left(3x+1\right)\left(x+1\right)\left((x+2)\left(x^2+1\right)\right)^3 && (4) \end{aligned}$$

14. (Section 3.8 Exercise 13)

$$\begin{aligned} x^4+y^4 &= 2 && (1)\\ (1,-1) & && (2)\\ 4x^3+4y^3\frac{dy}{dx} &= 0 && (3)\\ 4y^3\frac{dy}{dx} &= -4x^3 && (4)\\ \frac{dy}{dx} &= \frac{-4x^3}{4y^3} && (5)\\ &= \frac{-x^3}{y^3} && (6)\\ \left.\frac{dy}{dx}\right|_{(1,-1)} &= \frac{-(1^3)}{(-1)^3} && (7)\\ &= \frac{-1}{-1} && (8)\\ &= 1 && (9) \end{aligned}$$

15. (Section 3.8 Exercise 15)

$$\begin{aligned} y^2 &= 4x && (1)\\ (1,2) & && (2)\\ 2y\frac{dy}{dx} &= 4 && (3)\\ \frac{dy}{dx} &= \frac{4}{2y} && (4)\\ &= \frac{2}{y} && (5)\\ \left.\frac{dy}{dx}\right|_{(1,2)} &= \frac{2}{2} && (6)\\ &= 1 && (7) \end{aligned}$$

16. (Section 3.8 Exercise 31)

$$\begin{aligned}
\sin xy &= x + y && (1)\\
\cos xy \cdot \left(y + x\frac{dy}{dx}\right) &= 1 + \frac{dy}{dx} && (2)\\
y\cos xy + \frac{dy}{dx}x\cos xy &= 1 + \frac{dy}{dx} && (3)\\
\frac{dy}{dx}x\cos xy - \frac{dy}{dx} &= 1 - y\cos xy && (4)\\
\frac{dy}{dx}(x\cos xy - 1) &= 1 - y\cos xy && (5)\\
\frac{dy}{dx} &= \frac{1 - y\cos xy}{x\cos xy - 1} && (6)
\end{aligned}$$

17. (Section 3.8 Exercise 33)

$$\begin{aligned}
\cos y^2 + x &= e^y && (1)\\
-\frac{dy}{dx}2y\sin y^2 + 1 &= \frac{dy}{dx}e^y && (2)\\
\frac{dy}{dx}2y\sin y^2 + \frac{dy}{dx}e^y &= 1 && (3)\\
\frac{dy}{dx}\left(2y\sin y^2 + e^y\right) &= 1 && (4)\\
\frac{dy}{dx} &= \frac{1}{2y\sin y^2 + e^y} && (5)
\end{aligned}$$

18. (Section 3.8 Exercise 47)

$$\begin{aligned}
x^2 + xy + y^2 &= 7 && (1)\\
(2,1) & && (2)\\
2x + y + \frac{dx}{dy}x + \frac{dx}{dy}2y &= 0 && (3)\\
\frac{dx}{dy}(x + 2y) &= -2x - y && (4)\\
\frac{dx}{dy} &= \frac{-2x - y}{x + 2y} && (5)\\
\left.\frac{dy}{dx}\right|_{(2,1)} &= \frac{-2(2) - 1}{2 + 2(1)} && (6)\\
&= \frac{-4 - 1}{2 + 2} && (7)\\
&= \frac{-5}{4} && (8)\\
y &= \frac{-5}{4}x - 2\frac{-5}{4} + 1 && (9)\\
&= \frac{-5}{4}x + \frac{5}{2} + \frac{2}{2} && (10)\\
&= \frac{-5}{4}x + \frac{7}{2} && (11)
\end{aligned}$$

19. (Section 3.8 Exercise 48)

$$\begin{aligned}
x^4 - x^2y + y^4 &= 1 && (1)\\
(-1,1) & && (2)\\
4x^3 - 2xy - \frac{dy}{dx}x^2 + \frac{dy}{dx}4y^3 &= 0 && (3)\\
\frac{dy}{dx}\left(-x^2 + 4y^3\right) &= 2xy - 4x^3 && (4)\\
\frac{dy}{dx} &= \frac{2xy - 4x^3}{4y^3 - x^2} && (5)\\
\left.\frac{dy}{dx}\right|_{(-1,1)} &= \frac{2(-1)(1) - 4(-1)^3}{4(1)^3 - (-1)^2} && (6)\\
&= \frac{-2+4}{4+1} && (7)\\
&= \frac{2}{5} && (8)\\
y &= \frac{2}{5}x + \frac{2}{5} + 1 && (9)\\
&= \frac{2}{5}x + \frac{7}{5} && (10)
\end{aligned}$$

20. (Section 3.8 Exercise 25)

$$\begin{aligned}
x\sqrt[3]{y} + y &= 10 && (1)\\
(1,8) & && (2)\\
\sqrt[3]{y} + \frac{dy}{dx}\left(\frac{x}{3y^{\frac{2}{3}}}\right) + \frac{dy}{dx} &= 0 && (3)\\
\frac{dy}{dx}\left(\frac{x}{3y^{\frac{2}{3}}} + 1\right) &= -\sqrt[3]{y} && (4)\\
\frac{dy}{dx} &= \frac{-3y^{\frac{2}{3}}\sqrt[3]{y}}{x + 3y^{\frac{2}{3}}} && (5)\\
&= \frac{-3y}{3y^{\frac{2}{3}} + x} && (6)\\
\left.\frac{dy}{dx}\right|_{(1,8)} &= \frac{-3(8)}{3(8)^{\frac{2}{3}} + 1} && (7)\\
&= \frac{-24}{3(4)+1} && (8)\\
&= \frac{-24}{13} && (9)
\end{aligned}$$

21. (Section 3.8 Exercise 26)

$$
\begin{aligned}
(x+y)^{\frac{2}{3}} &= y && (1)\\
(4,4) & && (2)\\
\frac{2}{3}(x+y)^{-\frac{1}{3}} \cdot \left(1+\frac{dy}{dx}\right) &= \frac{dy}{dx} && (3)\\
\frac{2}{3}(x+y)^{-\frac{1}{3}} + \frac{dy}{dx}\frac{2}{3}(x+y)^{-\frac{1}{3}} &= \frac{dy}{dx} && (4)\\
\frac{2}{3}(x+y)^{-\frac{1}{3}} &= \frac{dy}{dx} - \frac{dy}{dx}\frac{2}{3}(x+y)^{-\frac{1}{3}} && (5)\\
\frac{dy}{dx}\left(1-\frac{2}{3}(x+y)^{-\frac{1}{3}}\right) &= \frac{2}{3}(x+y)^{-\frac{1}{3}} && (6)\\
\frac{dy}{dx} &= \frac{\frac{2}{3}(x+y)^{-\frac{1}{3}}}{1-\frac{2}{3}(x+y)^{-\frac{1}{3}}} && (7)\\
\left.\frac{dy}{dx}\right|_{(4,4)} &= \frac{\frac{2}{3}(4+4)^{-\frac{1}{3}}}{1-\frac{2}{3}(4+4)^{-\frac{1}{3}}} && (8)\\
&= \frac{\frac{2}{3}\frac{1}{2}}{1-\frac{2}{3}\frac{1}{2}} && (9)\\
&= \frac{\frac{1}{2}}{-\frac{1}{2}} && (10)\\
&= -1 && (11)
\end{aligned}
$$

22. (Section 3.8 Exercise 51)

$$
\begin{aligned}
x+y^2 &= 1 && (1)\\
1+\frac{dy}{dx}2y &= 0 && (2)\\
\frac{dy}{dx}2y &= -1 && (3)\\
\frac{dy}{dx} &= \frac{-1}{2y} && (4)\\
\frac{d^2y}{dx^2} &= \frac{-1}{2y} && (5)\\
&= \frac{-1}{2}\frac{1}{y} && (6)\\
&= \frac{-1}{2}\frac{dy}{dx}\frac{-1}{y^2} && (7)\\
&= \frac{-1}{2}\frac{-1}{2y}\frac{-1}{y^2} && (8)\\
&= \frac{-1}{4y^3} && (9)
\end{aligned}
$$

23. (Section 3.8 Exercise 52)

$$\begin{aligned}
2x^2 + y^2 &= 4 && (1)\\
4x + 2y\frac{dy}{dx} &= 0 && (2)\\
2y\frac{dy}{dx} &= -4x && (3)\\
\frac{dy}{dx} &= \frac{-4x}{2y} && (4)\\
&= \frac{-2x}{y} && (5)\\
\frac{d^2y}{dx^2} &= \frac{-2x}{y} && (6)\\
&= -2x\frac{1}{y} && (7)\\
&= -2x\frac{dy}{dx}\frac{-1}{y^2} && (8)\\
&= -2x\frac{-2x}{y}\frac{-1}{y^2} && (9)\\
&= \frac{-4x^2}{y^3} && (10)
\end{aligned}$$

24. (Section 3.9 Exercise 15)

$$\begin{aligned}
y &= \ln 7x && (1)\\
y' &= \frac{1}{7x}\cdot 7 && (2)\\
&= \frac{7}{7x} && (3)\\
&= \frac{1}{x} && (4)
\end{aligned}$$

25. (Section 3.9 Exercise 16)

$$\begin{aligned}
y &= x^2 \ln x && (1)\\
y' &= 2x\ln x + \frac{x^2}{x} && (2)\\
&= 2x\ln x + x && (3)
\end{aligned}$$

26. (Section 3.9 Exercise 19)

$$\begin{aligned}
y &= \ln|\sin x| && (1)\\
y' &= \frac{\cos x}{\sin x} && (2)\\
&= \cot x && (3)
\end{aligned}$$

27. (Section 3.9 Exercise 37)

$$\begin{aligned}
y &= 8^x && (1)\\
y' &= 8^x \ln 8 && (2)
\end{aligned}$$

28. (Section 3.9 Exercise 39)

$$\begin{aligned}
y &= 5\cdot 4^x && (1)\\
y' &= 5\cdot 4^x \ln 4 && (2)
\end{aligned}$$

29. (Section 3.9 Exercise 10)

$$\frac{d}{dx}\left(x^e + e^x\right) = ex^{e-1} + e^x \quad (1)$$

30. (Section 3.9 Exercise 33)

$$\begin{aligned} y &= x^e && (1) \\ y' &= ex^{e-1} && (2) \end{aligned}$$

31. (Section 3.9 Exercise 35)

$$\begin{aligned} y &= (2^x+1)^\pi && (1) \\ y' &= \pi(2^x+1)^{\pi-1}\cdot 2^x \ln 2 && (2) \end{aligned}$$

32. (Section 3.9 Exercise 49)

$$\begin{aligned} f(x) &= x^{\cos x} && (1) \\ &= e^{\cos x \ln x} && (2) \\ a &= \frac{\pi}{2} && (3) \\ f'(x) &= e^{\cos x \ln x}\cdot\left(-\sin x\ln x + \frac{\cos x}{x}\right) && (4) \\ &= x^{\cos x}\left(-\sin x \ln x + \frac{\cos x}{x}\right) && (5) \\ f'\left(\frac{\pi}{2}\right) &= \left(\frac{\pi}{2}\right)^0\left(-1\cdot\ln\frac{\pi}{2}+\frac{0}{\pi}\right) && (6) \\ &= -\ln\frac{\pi}{2} && (7) \end{aligned}$$

33. (Section 3.9 Exercise 59)

$$\begin{aligned} f(x) &= x^{\sin x} && (1) \\ &= e^{\sin x \ln x} && (2) \\ f'(x) &= e^{\sin x \ln x}\cdot\left(\cos x\ln x + \frac{\sin x}{x}\right) && (3) \\ f'(1) &= e^0\cdot\left(\cos 1 \ln 1 + \frac{\sin 1}{1}\right) && (4) \\ &= \cos 1\cdot 0 + \sin 1 && (5) \\ &= \sin 1 && (6) \\ y &= x\sin 1 - \sin 1 + 1 && (7) \end{aligned}$$

34. (Section 3.9 Exercise 63)

$$\begin{aligned} y &= 4\log_3\left(x^2-1\right) && (1) \\ y' &= 4\cdot\frac{1}{(x^2-1)\ln 3}\cdot 2x && (2) \\ &= \frac{8x}{(x^2-1)\ln 3} && (3) \\ & && (4) \end{aligned}$$

35. (Section 3.9 Exercise 64)

$$\begin{aligned} y &= \log_{10} x && (1) \\ y' &= \frac{1}{x\ln 10} && (2) \end{aligned}$$

36. (Section 3.9 Exercise 77)

$$
\begin{aligned}
f(x) &= \frac{(x+1)^{10}}{(2x-4)^8} && (1)\\
\ln f(x) &= \ln \frac{(x+1)^{10}}{(2x-4)^8} && (2)\\
&= \ln (x+1)^{10} - \ln (2x-4)^8 && (3)\\
&= 10 \ln x+1 - 8 \ln 2x-4 && (4)\\
\frac{d}{dx}(\ln f(x)) &= \frac{f'(x)}{f(x)} && (5)\\
\frac{f'(x)}{f(x)} &= 10\frac{1}{x+1} - 8\frac{1}{2x-4}2 && (6)\\
&= \frac{10}{x+1} - \frac{16}{2x-4} && (7)\\
&= \frac{10}{x+1} - \frac{8}{x-2} && (8)\\
f'(x) &= f(x)\left(\frac{10}{x+1} - \frac{8}{x-2}\right) && (9)\\
&= \frac{(x+1)^{10}}{(2x-4)^8}\left(\frac{10}{x+1} - \frac{8}{x-2}\right) && (10)\\
& && (11)
\end{aligned}
$$

37. (Section 3.9 Exercise 80)

$$
\begin{aligned}
f(x) &= \frac{\tan^{10} x}{(5x+3)^6} && (1)\\
\ln f(x) &= \ln \frac{\tan^{10} x}{(5x+3)^6} && (2)\\
&= \ln \tan^{10} x - \ln (5x+3)^6 && (3)\\
&= 10 \ln \tan x - 6 \ln 5x+3 && (4)\\
\frac{d}{dx}(\ln f(x)) &= \frac{f'(x)}{f(x)} && (5)\\
\frac{f'(x)}{f(x)} &= 10\frac{1}{\tan x}\sec^2 x - 6\frac{1}{5x+3}5 && (6)\\
&= \frac{10 \sec^2 x}{\tan x} - \frac{30}{5x+3} && (7)\\
&= \frac{10 \sec x}{\sin x} - \frac{30}{5x+3} && (8)\\
f'(x) &= f(x)\left(\frac{10 \sec x}{\sin x} - \frac{30}{5x+3}\right) && (9)\\
&= \frac{\tan^{10} x}{(5x+3)^6}\left(\frac{10 \sec x}{\sin x} - \frac{30}{5x+3}\right) && (10)
\end{aligned}
$$

38. (Section 3.10 Exercise 13)

$$
\begin{aligned}
f(x) &= \sin^{-1} 2x && (1)\\
f'(x) &= \frac{2}{\sqrt{1-(2x)^2}} && (2)\\
&= \frac{2}{\sqrt{1-4x^2}} && (3)
\end{aligned}
$$

39. (Section 3.10 Exercise 15)

$$\begin{aligned} f(w) &= \cos\left(\sin^{-1} 2w\right) && (1)\\ f'(w) &= -\sin\left(\sin^{-1} 2w\right) \cdot \frac{1}{\sqrt{1-(2w)^2}} \cdot 2 && (2)\\ &= -\frac{2\sin\left(\sin^{-1} 2w\right)}{\sqrt{1-4w^2}} && (3)\\ &= -\frac{4w}{\sqrt{1-4w^2}} && (4) \end{aligned}$$

40. (Section 3.10 Exercise 27)

$$\begin{aligned} f(w) &= w^2 - \tan^{-1} w^2 && (1)\\ f'(w) &= 2w - \frac{2w}{1+w^4} && (2)\\ &= \frac{2w^5}{1+w^4} && (3) \end{aligned}$$

41. (Section 3.10 Exercise 41)

$$\begin{aligned} f(x) &= \tan^{-1} 2x && (1)\\ f'(x) &= \frac{2}{1+4x^2} && (2)\\ f'\left(\frac{1}{2}\right) &= 1 && (3)\\ y &= x - \frac{1}{2} + \frac{\pi}{4} && (4) \end{aligned}$$

42. (Section 3.10 Exercise 45)

$$\begin{aligned} \tan\theta &= \frac{150}{x} && (1)\\ \theta &= \tan^{-1}\frac{150}{x} && (2)\\ \frac{d\theta}{dx} &= -\frac{150}{x^2\left(1+\left(\frac{150}{x}\right)^2\right)} && (3)\\ \left.\frac{d\theta}{dx}\right|_{x=500} &= -0.00055 && (4) \end{aligned}$$

43. (Section 3.10 Exercise 7)

(a)

$$\left(f^{-1}\right)'(4) = \frac{1}{f'(0)} = \frac{1}{2}$$

(b)

$$\left(f^{-1}\right)'(6) = \frac{1}{f'(1)} = \frac{2}{3}$$

(c)

$$\left(f^{-1}\right)'(1) = \text{Undeterminable}$$

(d)

$$f'(1) = \frac{3}{2}$$

44. (Section 3.10 Exercise 8)

(a)

$$f'(f(0)) = 2$$

(b)

$$\left(f^{-1}\right)'(0) = \frac{1}{f'(-4)} = \frac{1}{5}$$

(c)

$$\left(f^{-1}\right)'(1) = \frac{1}{f'(-2)} = \frac{1}{4}$$

(d)

$$\left(f^{-1}\right)'(f(4)) = \frac{1}{f'(4)} = 1$$

A copy of my notes (in LaTeX) are available on my GitHub