Module 6 Notes (MATH-211)

Lillie Donato

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General Notes (and Definitions)

- L'Hôpital's Rule
  Indeterminate Form: An expression involving two components where the limit cannot be determined by evaluating the limits of the individual components.
  L'Hôpital's Rule: Suppose f and g are differentiable functions on an open interval I containing the point $x = a$, with $g'(x) \neq 0$ on I when $x \neq a$.
  If $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$ has any of the indeterminate forms: $\frac{0}{0}$, $\frac{\infty}{\infty}$, $-\frac{\infty}{\infty}$, then

  $$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

  provided that one of the following is the case:

  $$\lim_{x\to a} \frac{f'(x)}{g'(x)} \in \mathbb{R}$$

  $$\lim_{x\to a} \frac{f'(x)}{g'(x)} = \infty$$

  $$\lim_{x\to a} \frac{f'(x)}{g'(x)} = -\infty$$

  L'Hôpital's Rule is still valid if $x \to a$ is replaced by any of $x \to a^+$, $x \to a^-$, $x \to \infty$, or $x \to -\infty$. In the last two of these cases, there must be a greatest x-value beyond which both f and g are differentiable at every point.
  Exponential Indeterminate forms: 1^∞, 0^0, ∞^0
  Method for evaluating limits of indeterminate forms 1^∞, 0^0, ∞^0:
  Assume that $L = \lim\limits_{x\to a} f(x)^{g(x)}$ has one of these indeterminate forms.

  1. Use the fact that the natural logarithm and natural exponential functions are inverses to write

     $$L = \lim_{x\to a} e^{\ln\left(f(x)^{g(x)}\right)}$$

  2. Use the power property of logarithm arguments to write

     $$L = \lim_{x\to a} e^{g(x)\ln(f(x))}$$

  3. Use continuity of the exponential function to write

     $$L = e^{\lim\limits_{x\to a} g(x)\ln(f(x))}$$

  4. Rewrite multiplication as division by the reciprocal:

     $$L = e^{\lim\limits_{x\to a}\left(\frac{\ln(f(x))}{\frac{1}{g(x)}}\right)}$$

  5. Use L'Hôpital's Rule to evaluate this limit expression

  Growth Rates: Suppose f and g are functions with $\lim\limits_{x\to\infty} f(x) = \infty$ and $\lim\limits_{x\to\infty} g(x) = \infty$

1. If one of the following are true, f **grows faster than** g, and we use the notation $f \gg g$

$$\begin{aligned} \lim_{x\to\infty} \frac{g(x)}{f(x)} &= 0 \qquad (1) \\ \lim_{x\to\infty} \frac{f(x)}{g(x)} &= \infty \qquad (2) \end{aligned}$$

2. f **and** g **have comparable growth rates**, if there is some non-zero finite number M such that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = M$$

Ranked Growth Rates as $x \to \infty$
For any base $b > 1$, and for any positive numbers p, q, r, and s

$$\ln^q x \ll x^p \ll x^p \ln^r x \ll x^{p+s} \ll b^x \ll x^x$$

- Antiderivatives
Antiderivative: A function F is an antiderivative of another function f on an interval I if for all x in I:

$$F'(x) = f(x)$$

Family of Antiderivatives: Let $F(x)$ be any antiderivative of $f(x)$ on an interval I. Then all antiderivatives of f on I have the form $F(x) + C$, where C is an arbitrary constant.
Differential Equations: Any equation involving an unknown function and its derivatives
  - Infinite family of solutions
  - No two solutions from the family pass through the same point
  - Given an initial condition $f(a) = b$, we can identify the particular family member that solves the given problem by solving for C

- Approximating Areas Under Curves
  - If we know the velocity function of a moving object, what can we learn about its position function?
  - Given an object with velocity function $v(t)$, the displacement of the moving object over the interval $[a, b]$ is the area between the velocity curve and the t-axis from $t = a$ to $t = b$.
  - Because objects do not necessarily move at a constant velocity, we can extend this idea to positive velocities that change over an interval of time.
  - The strategy is to divide the time interval into many subintervals, approximate the velocity on each subinterval with a constant velocity, calculate the individual displacements and sum the results.

Riemann Sums
  - Suppose $f(x)$ is continuous and non-negative on $[a, b]$.
  - Goal is to approximate the area of the region R bounded by the graph of $f(x)$ and the x-axis from $x = a$ to $x = b$.
  - Divide $[a, b]$ into n subintervals $[x_0, x_1], [x_1, x_2], ..., [x_{n-1}, x_n]$ where $a = x_0, b = x_n$.
  - The length of each subinterval is $\Delta x = \frac{b-a}{n}$
  - **Regular Partition**: Suppose $[a, b]$ is a closed interval containing n subintervals

  $$[x_0, x_1], [x_1, x_2], ..., [x_{n-1}, x_n]$$

  of equal length $\Delta x = \frac{b-a}{n}$, with $a = x_0$ and $b = x_n$. The endpoints $x_0, x_1, x_2, ..., x_{n-1}, x_n$ of the subintervals are called **grid points**, and they create a **regular partition** of the interval $[a, b]$. In general the kth grid point is

  $$x_k = a + k\Delta x, \text{ for } k = 0, 1, 2, ..., n$$

  - In the kth subinterval $[x_{k-1}, x_k]$, choose any point x_k^* and build a rectangle whose height is $f(x_k^*)$.
  - The area of the rectangle of the kth subinterval is

  $$\text{height} \cdot \text{base} = f(x_k^*)\Delta x, \text{ where } k = 1, 2, ..., n$$

- Summing the areas of these rectangles, we obtain an approximation to the area of R, which is called a **Riemann sum**:
$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + ... + f(x_n^*)\Delta x$$
- Three notable Riemann sums are the left, right, and midpoint Riemann sums.

Riemann Sum: Suppose f is defined on a closed interval $[a, b]$, which is divided into n subintervals of equal length Δx. If x_k^* is any point in the kth subinterval $[x_{k-1}, x_k]$, for $k = 1, 2, ..., n$, then

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + ... + f(x_n^*)\Delta x$$

is called a **Riemann sum** for f on $[a, b]$. This sum is called

- a **left Riemann sum** if x_k^* is the left endpoint of $[x_{k-1}, x_k]$
- a **right Riemann sum** if x_k^* is the right endpoint of $[x_{k-1}, x_k]$
- a **midpoint Riemann sum** if x_k^* is the midpoint of $[x_{k-1}, x_k]$

Summation notation (Σ):

- Working with Riemann sums is cumbersome when n is large
- We introduce sigma (summation) notation as a shorthand:
$$1 + 2 + ... + 49 + 50 = \sum_{k=1}^{50} k$$
- The symbol Σ (sigma) stands for sum
- k is the index, and takes on all integer values from $k = 1$ to $k = 50$
- The expression immediately following Σ, the summand, is evaluated for each k, and the resulting values are summed
- The index is a dummy variable, and it does not matter which symbol is chosen for the index:
$$\sum_{k=1}^{99} k = \sum_{n=1}^{99} n = \sum_{p-1}^{99} p$$
- Two Properties of Sums and Sigma Notation
  1. Constant Multiple Rule:
  $$\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$$
  2. Addition Rule:
  $$\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$
- **Theorem**: Sums of Power of Integers
  Let $n \in \mathbb{Z}$ such that $n > 0$ and $c \in \mathbb{R}$
$$\sum_{k=1}^{n} c = cn \quad (1)$$
$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad (2)$$
$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \quad (3)$$
$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4} \quad (4)$$

Left, Right, and Midpoint Riemann Sums in Sigma Notation:
Suppose f is defined on a closed interval $[a, b]$, which is divided into subintervals of equal length Δx. If x_k^* is a point in the kth subinterval $[x_{k-1}, x_k]$, for $k = 1, 2, ..., n$, then the **Riemann sum** for f on $[a, b]$ is

$$\sum_{k=1}^{n} f(x_k^*)\Delta x$$

Three cases arise in practice

- $\sum\limits_{k=1}^{n} f(x_k^*)\Delta x$ is a **left Riemann sum** if $x_k^* = a + (k-1)\Delta x$
- $\sum\limits_{k=1}^{n} f(x_k^*)\Delta x$ is a **right Riemann sum** if $x_k^* = a + k\Delta x$
- $\sum\limits_{k=1}^{n} f(x_k^*)\Delta x$ is a **midpoint Riemann sum** if $x_k^* = a + (k - \frac{1}{2})\Delta x$

- Definite Integrals
  Net Area: Consider the region R bounded by the graph of a continuous function f and the x-axis between $x = a$ and $x = b$. The **net area** of R is the sum of the area of the parts of R that lie above the x-axis minus the sum of the areas of the parts of R that lie below the x-axis on $[a, b]$.

  - Where $f(x) < 0$, Riemann sums approximate the negative of the area of the region bounded by the curve
  - On the interval $[a, b]$, we get positive, and negative contributions to the Riemann sum where $f(x)$ is negative
  - Riemann sums approximate the area of the regions that lie above the x-axis minus the area of the regions that lie below the x-axis
  - The difference is called the **net area**; it can be positive, negative, or zero

$$area_{net} = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

  A **general partition** of $[a, b]$ consists of the n subintervals

$$[x_0, x_1], [x_1, x_2], ..., [x_{n-1}, x_n]$$

  where $x_0 = a$and $x_n = b$. The length of the kth subinterval is $\Delta x_k = x_k - k_{k-1}$, for $k = 1, 2, ..., n$. We let x_k^* be any point in the subinterval $[x_{k-1}, x_k]$.
  General Riemann Sum: Suppose $[x_0, x_1], [x_1, x_2], ..., [x_{n-1}, x_n]$ are subintervals of $[a, b]$ with

$$a = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = b$$

  Let $x\Delta x_k$ be the length of the subinterval $[x_{k-1}, x_k]$ and let x_k^* be any point in $[x_{k-1}, x_k]$, for $k = 1, 2, ...n$
  If f is defined on $[a, b]$, the sum

$$\sum_{k=1}^{n} f(x_k^*)\Delta x_k = f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + ... + f(x_n^*)\Delta x_n$$

  is called a **general Riemann sum** for f on $[a, b]$
  Definite Integral: A function f defined on $[a, b]$ is **integrable** on $[a, b]$ if $\lim\limits_{\Delta x\to 0} \sum\limits_{k=1}^{n} f(x_k^*)\Delta x_k$ exists and is unique over all partitions of $[a, b]$ and all choices of x_k^* on a parition. This limit is the **definite integral of f from a to b**, which we rite

$$\int_a^b f(x)dx = \lim_{\Delta x\to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

  Integrable Functions: If f is continuous on $[a, b]$ or bounded on $[a, b]$ with a finite number of discontinuities, then f is integrale on $[a, b]$.
  Let f and g be integrable function on $[a, b]$, where $b > a$

  1. If $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x)\,dx \geq 0$
  2. If $f(x) \geq g(x)$ on $[a, b]$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$
  3. If $m \leq f(x) \leq M$, then $m(b - a) \leq \int_a^b f(x)\,dx \leq M(b - a)$

Antiderivative Rules

- Power Rule
  If $p \neq -1$ and C is an arbitrary constant:
$$\int x^p dx = \frac{x^{p+1}}{p+1} + C$$

- Integral of x^{-1}
$$\int x^{-1} dx = \int \frac{1}{x} dx = \ln|x| + C$$

- Constant Multiple and Sum Rules
  If $c \in \mathbb{R}$:
$$\int cf(x)dx = c\int f(x)dx$$
$$\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$$

- Integral of e^x
$$\int e^x dx = e^x + C$$

- Integral of $\frac{1}{x}$
$$\int \frac{1}{x}\, dx = \ln|x| + C$$

Trigonometric (and inverse) Integrals

$$\int \cos(x)dx = \sin x + C \tag{1}$$
$$\int \sin(x)dx = -\cos x + C \tag{2}$$
$$\int \sec^2(x)dx = \tan x + C \tag{3}$$
$$\int \csc^2(x)dx = -\cot x + C \tag{4}$$
$$\int \sec(x)\tan(x)dx = \sec x + C \tag{5}$$
$$\int \csc(x)\cot(x)dx = -\csc x + C \tag{6}$$
$$\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x + C \tag{7}$$
$$\int \frac{1}{1+x^2}dx = \tan^{-1} x + C \tag{8}$$
$$\int \frac{1}{x\sqrt{x^2-1}}dx = \sec^{-1}|x| + C \tag{9}$$

Properties of Definite Integrals

Let f and g be integrable functions on an interval that contains a, b, and p

$$\begin{aligned}
\int_a^a f(x)\,dx &= 0 &(1)\\
\int_b^a f(x)\,dx &= -\int_a^b f(x)\,dx &(2)\\
\int_a^b (f(x)\pm g(x))\,dx &= \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx &(3)\\
\int_a^b cf(x)\,dx &= c\int_a^b f(x)\,dx, \text{ for any constant } c &(4)\\
\int_a^b f(x)\,dx &= \int_a^p f(x)\,dx + \int_p^b f(x)\,dx &(5)
\end{aligned}$$

The function $|f|$ is integrable on $[a, b]$, and $\int\limits_a^b |f(x)|\,dx$ is the sum of the areas of the regions bounded by the graph of f and the x-axis on $[a, b]$.

Examples

1. Use L'Hôpital's Rule to evaluate a limit with indeterminate form $\frac{0}{0}$

$$\begin{aligned}
\lim_{x\to 0} \frac{e^x - x - 1}{5x^2} &= \lim_{x\to 0} \frac{e^x - 1}{10x} &(1)\\
&= \lim_{x\to 0} \frac{e^x}{10} &(2)\\
&= \frac{e^0}{10} &(3)\\
&= \frac{1}{10} &(4)
\end{aligned}$$

2. Use L'Hôpital's Rule to evaluate a limit with indeterminate form $\frac{\infty}{\infty}$

$$\begin{aligned}
\lim_{x\to 0^+} \frac{1-\ln x}{1+\ln x} &= \lim_{x\to 0^+} \frac{-\frac{1}{x}}{\frac{1}{x}} &(1)\\
&= \lim_{x\to 0^+} \frac{-\frac{1}{x}}{\frac{1}{x}} &(2)\\
&= \frac{-1}{1} &(3)\\
&= -1 &(4)
\end{aligned}$$

3. Use L'Hôpital's Rule to evaluate a limit with indeterminate form $0 \cdot \infty$

$$\begin{aligned}
\lim_{x\to 1^-} (1-x)\tan\left(\frac{\pi x}{2}\right) &= \lim_{x\to 1^-} \frac{(1-x)}{\cot\left(\frac{\pi x}{2}\right)} &(1)\\
&= \lim_{x\to 1^-} \frac{-1}{-\frac{\pi}{2}\csc^2\left(\frac{\pi x}{2}\right)} &(2)\\
&= \lim_{x\to 1^-} \frac{2}{\pi}\sin^2\left(\frac{\pi x}{2}\right) &(3)\\
&= \frac{2}{\pi} &(4)
\end{aligned}$$

4. Use L'Hôpital's Rule to evaluate a limit with exponential indeterminate form

$$\begin{aligned}
\lim_{x\to 0^+} x^{\tan x} &= e^{\lim\limits_{x\to 0^+} \frac{\ln x}{\frac{1}{\tan x}}} && (1)\\
&= e^{\lim\limits_{x\to 0^+} \frac{\ln x}{\cot x}} && (2)\\
&= e^{\lim\limits_{x\to 0^+} \frac{1}{-x\csc^2 x}} && (3)\\
&= e^{\lim\limits_{x\to 0^+} \frac{-\sin^2 x}{x}} && (4)\\
&= e^{\lim\limits_{x\to 0^+} \frac{-2\sin x\cos x}{1}} && (5)\\
&= e^{\lim\limits_{x\to 0^+} -2\sin x\cos x} && (6)\\
&= e^0 && (7)\\
&= 1 && (8)
\end{aligned}$$

5. Compare the growth rates of functions

$$f(x) = x^2 \ln x$$
$$g(x) = x \ln^2 x$$

$$\begin{aligned}
\lim_{x\to\infty} \frac{f(x)}{g(x)} &= \lim_{x\to\infty} \frac{x^2 \ln x}{x \ln^2 x} && (1)\\
&= \lim_{x\to\infty} \frac{x}{\ln x} && (2)\\
&= \lim_{x\to\infty} \frac{1}{\frac{1}{x}} && (3)\\
&= \lim_{x\to\infty} x && (4)\\
&= \infty && (5)
\end{aligned}$$

Since $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = \infty$, $f \gg g$

6. Use knowledge of derivatives to find antiderivatives

$$f(x) = -4\cos x - x$$
$$F(x) = -4\sin x - \frac{1}{2}x^2$$
$$F'(x) = -4\cos x - x$$
$$\int (-4\cos x - x)dx = -4\sin x - \frac{1}{2}x^2 + C$$

7. Determine indefinite integrals using antiderivative rules

$$\int \frac{3}{x^4} + 2 - 3x^2 dx = \int 3x^{-4} + 2 - 3x^2 dx = \frac{-1}{x^3} + 2x - x^3 + C$$

8. Rewrite an indefinite integral to find an antiderivative

$$\begin{aligned}
\int \frac{2 + 3\cos y}{\sin^2 y} dy &= \int 2\csc^2 y + 3\cot y \csc y dy && (1)\\
&= 2\int \csc^2 y dy + 3\int \cot y \csc y dy && (2)\\
&= -2\cot y - 3\csc y + C && (3)
\end{aligned}$$

9. Solve an initial value problem

$$f'(u) = 4\cos u - 4\sin u$$
$$f(\pi) = 0$$
$$f(u) = 4\int \cos u\, du - 4\int \sin u\, du = 4\sin u + 4\cos u + C$$

$$\begin{aligned}
4\sin\pi + 4\cos\pi + C &= 0 && (1)\\
0 - 4 + C &= 0 && (2)\\
C &= 4 && (3)
\end{aligned}$$

$$f(u) = 4\sin u + 4\cos u + 4$$

10. Application of differential equations to linear motion

$$a(t) = 2 + 3\sin t$$

$$v(0) = 1$$

$$s(0) = 10$$

$$\begin{aligned}
v(t) &= \int 2 + 3\sin t\, dt && (1)\\
&= 2\int t^0\, dt + 3\int \sin t\, dt && (2)\\
&= 2t - 3\cos t + C && (3)\\
2(0) - 3\cos 0 + C &= 1 && (4)\\
-3 + C &= 1 && (5)\\
C &= 4 && (6)\\
v(t) &= 2t - 3\cos t + 4 && (7)\\
s(t) &= \int 2t - 3\cos t + 4\, dt && (8)\\
&= 2\int t\, dt - 3\int \cos t\, dt + 4\int t^0\, dt && (9)\\
&= 2\frac{t^2}{2} - 3\sin t + 4t + C && (10)\\
&= t^2 - 3\sin t + 4t + C && (11)\\
0^2 - 3\sin 0 + 4(0) + C &= 10 && (12)\\
C &= 10 && (13)\\
s(t) &= t^2 - 3\sin t + 4t + 10 && (14)
\end{aligned}$$

11. Approximating displacement

$$v = \sqrt{10t}$$

$$1 \le t \le 7$$

(a)

$$n = 3$$

$$[1, 3], [3, 5], [5, 7]$$

$$2, 4, 6$$

$$d \approx 2v(2) + 2v(4) + 2v(6) = 2\sqrt{20} + 2\sqrt{40} + 2\sqrt{60} \approx 37.085$$

(b)

$$n = 6$$

$$1.5, 2.5, 3.5, 4.5, 5.5, 6.5$$

$$\begin{aligned}
d &\approx v(1.5) + v(2.5) + v(3.5) + v(4.5) + v(5.5) + v(6.5) && (1)\\
&= \sqrt{15} + \sqrt{25} + \sqrt{35} + \sqrt{45} + \sqrt{55} + \sqrt{65} && (2)\\
&\approx 36.976 && (3)
\end{aligned}$$

12. Left and right Riemann sums (1)

$$f(x) = x + 1$$

$$[1, 6]$$

$$n = 5$$

(a)

$$\begin{aligned}(1+1)+(2+1)+(3+1)+(4+1)+(5+1) &= 2+3+4+5+6 && (1)\\ &= 20 && (2)\end{aligned}$$

(b)

$$\begin{aligned}(2+1)+(3+1)+(4+1)+(5+1)+(6+1) &= 3+4+5+6+7 && (1)\\ &= 25 && (2)\end{aligned}$$

13. Left and right Riemann sums (2)

$$f(x) = 9 - x$$

$$[3, 8]$$

$$n = 5$$

$$\Delta x = \frac{8-3}{5} = \frac{5}{5} = 1$$

(a) Grid Points:

$$\begin{aligned}x_0 &= 3 && (1)\\ x_1 &= 4 && (2)\\ x_2 &= 5 && (3)\\ x_3 &= 6 && (4)\\ x_4 &= 7 && (5)\\ x_5 &= 8 && (6)\end{aligned}$$

(b) Riemann Sums

$$\begin{aligned}(9-3)+(9-4)+(9-5)+(9-6)+(9-7) &= 6+5+4+3+2 && (1)\\ &= 20 \text{ (Overestimation)} && (2)\end{aligned}$$

$$\begin{aligned}(9-4)+(9-5)+(9-6)+(9-7)+(9-8) &= 5+4+3+2+1 && (1)\\ &= 15 \text{ (Underestimation)} && (2)\end{aligned}$$

14. Midpoint Riemann sum

$$f(x) = 100 - x^2$$

$$[0, 10]$$

$$n = 5$$

$$\Delta x = 2$$

$$\begin{aligned}2f(1)+2f(3)+2f(5)+2f(7)+2f(9) &= 2(99)+2(91)+2(75)+2(51)+2(19) && (1)\\ &= 198+182+150+102+38 && (2)\\ &= 670 && (3)\end{aligned}$$

15. Riemann sums from tables

$$[0, 2]$$

$$n = 4$$

$$\Delta x = \frac{1}{2}$$

$$\begin{aligned}\frac{5}{2}+\frac{3}{2}+\frac{2}{2}+\frac{1}{2} &= 5.5 && (1)\\ \frac{3}{2}+\frac{2}{2}+\frac{1}{2}+\frac{1}{2} &= 3.5 && (2)\end{aligned}$$

16. Computing Riemann sums for large values of n

$$f(x) = x^2 + 1$$
$$[-1, 1]$$
$$n = 50$$

$$\begin{aligned}
\sum_{k=1}^{50} 0.04 f\left(-1 + 0.04\left(k-1\right)\right) &= \sum_{k=1}^{50} 0.04\left(\left(-1 + 0.04\left(k-1\right)\right)^2 + 1\right) && (1)\\
&\approx 2.6672 && (2)
\end{aligned}$$

$$\begin{aligned}
\sum_{k=1}^{50} 0.04 f\left(-1 + 0.04k\right) &= \sum_{k=1}^{50} 0.04\left(\left(-1 + 0.04k\right)^2 + 1\right) && (3)\\
&\approx 2.6672 && (4)
\end{aligned}$$

$$\begin{aligned}
\sum_{k-1}^{50} 0.04 f\left(-1 + 0.04\left(k - \frac{1}{2}\right)\right) &= \sum_{k-1}^{50} 0.04\left(\left(-1 + 0.04\left(k - \frac{1}{2}\right)\right)^2 + 1\right) && (5)\\
&\approx 2.6664 && (6)
\end{aligned}$$

17. Approximating net area

$$f(x) = 4 - 2x$$
$$[0, 4]$$
$$n = 4$$
$$\Delta x = 1$$

$$\begin{aligned}
4 + 2 + 0 - 2 &= 4 && (1)\\
2 + 0 - 2 - 4 &= -4 && (2)\\
3 + 1 - 1 - 3 &= 0 && (3)
\end{aligned}$$

18. Identifying definite integrals as limits of sums

$$\lim_{\Delta \to 0} \sum_{k=1}^{n} \left(4 - x_k^{*2}\right) \Delta x_k$$
$$\int_0^2 \left(4 - x^2\right)\, dx$$

19. Using geometry to evaluate definite integrals

$$\int_0^4 (8 - 2x)\, dx = \frac{4 \cdot 8}{2} = 16$$

20. Definite integrals from graphs

$$\begin{aligned}
\int_0^b f(x)\, dx &= 16 - 5 && (1)\\
&= 11 && (2)\\
\int_0^c f(x)\, dx &= 16 - 5 + 11 && (3)\\
&= 22 && (4)\\
\int_a^0 f(x)\, dx &= -\int_0^a f(x)\, dx && (5)\\
&= -16 && (6)\\
\int_0^c |f(x)|\, dx &= 16 + 5 + 11 && (7)\\
&= 32 && (8)
\end{aligned}$$

21. Using properties of integrals

$$\int_1^4 f(x)\,dx = 8$$
$$\int_1^6 f(x)\,dx = 5$$

$$\begin{aligned}
\int_1^4 (-3f(x))\ dx &= -3\int_1^4 f(x)\,dx && (1)\\
&= -3\cdot 8 && (2)\\
&= -24 && (3)\\
\int_1^4 3f(x)\,dx &= 3\int_1^4 f(x)\,dx && (4)\\
&= 3\cdot 8 && (5)\\
&= 24 && (6)\\
\int_6^4 12f(x)\,dx &= -\int_4^6 12f(x)\,dx && (7)\\
&= -12\int_4^6 f(x)\,dx && (8)\\
&= -12\,(5-8) && (9)\\
&= 36 && (10)\\
\int_4^6 3f(x)\,dx &= 3\int_4^6 f(x)\,dx && (11)\\
&= 3\,(5-8) && (12)\\
&= -9 && (13)
\end{aligned}$$

22. Use definition of definite integral

$$\begin{aligned}
\int_0^2 (2x+1)\ dx &= \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{4k}{n}+1\right)\frac{2}{n} && (1)\\
&= \lim_{n\to\infty}\frac{2}{n}\sum_{k=1}^{n}\left(\frac{4k}{n}+1\right) && (2)\\
&= \lim_{n\to\infty}\frac{2}{n}\left(\sum_{k=1}^{n}\frac{4k}{n}+\sum_{k=1}^{n}1\right) && (3)\\
&= \lim_{n\to\infty}\frac{2}{n}\left(\frac{4}{n}\sum_{k=1}^{n}k+\sum_{k=1}^{n}1\right) && (4)\\
&= \lim_{n\to\infty}\frac{2}{n}\left(\frac{4}{n}\left(\frac{n(n+1)}{2}\right)+n\right) && (5)\\
&= \lim_{n\to\infty}\frac{2}{n}\left(\frac{4n(n+1)}{2n}+n\right) && (6)\\
&= \lim_{n\to\infty}\frac{2}{n}\left(2(n+1)+n\right) && (7)\\
&= \lim_{n\to\infty}\frac{2}{n}\left(2n+2+n\right) && (8)\\
&= \lim_{n\to\infty}\frac{2}{n}\left(3n+2\right) && (9)\\
&= \lim_{n\to\infty}\frac{6n+4}{n} && (10)\\
&= \lim_{n\to\infty}6+\frac{4}{n} && (11)\\
&= 6+0 && (12)\\
&= 6 && (13)
\end{aligned}$$

Related Exercises

1. (Section 4.7, Exercise 17)

$$\begin{aligned}
\lim_{x\to 2}\frac{x^2-2x}{x^2-6x+8} &= \lim_{x\to 2}\frac{2x-2}{2x-6} && (1)\\
&= \frac{2(2)-2}{2(2)-6} && (2)\\
&= \frac{4-2}{4-6} && (3)\\
&= \frac{2}{-2} && (4)\\
&= -1 && (5)
\end{aligned}$$

2. (Section 4.7, Exercise 18)

$$\begin{aligned}
\lim_{x\to -1}\frac{x^4+x^3+2x+2}{x+1} &= \lim_{x\to -1}\frac{4x^3+3x^2+2}{1} && (1)\\
&= \lim_{x\to -1} 4x^3+3x^2+2 && (2)\\
&= 4(-1)^3+3(-1)^2+2 && (3)\\
&= -4+3+2 && (4)\\
&= 1 && (5)
\end{aligned}$$

3. (Section 4.7, Exercise 36)

$$\begin{aligned}
\lim_{x\to 0}\frac{e^x-x-1}{5x^2} &= \lim_{x\to 0}\frac{e^x-1}{10x} && (1)\\
&= \lim_{x\to 0}\frac{e^x}{10} && (2)\\
&= \frac{e^0}{10} && (3)\\
&= \frac{1}{10} && (4)
\end{aligned}$$

4. (Section 4.7, Exercise 39)

$$\begin{aligned}
\lim_{x\to 0}\frac{e^x-\sin x-1}{x^4+8x^3+12x^2} &= \lim_{x\to 0}\frac{e^x-\cos x}{4x^3+24x^2+24x} && (1)\\
&= \lim_{x\to 0}\frac{e^x+\sin x}{12x^2+48x+24} && (2)\\
&= \frac{e^0+\sin 0}{12(0)^2+48(0)+24} && (3)\\
&= \frac{1+0}{24} && (4)\\
&= \frac{1}{24} && (5)
\end{aligned}$$

5. (Section 4.7, Exercise 38)

$$\begin{aligned}
\lim_{x\to \infty}\frac{e^{3x}}{3e^{3x}+5} &= \lim_{x\to \infty}\frac{3e^{3x}}{9e^{3x}} && (1)\\
&= \lim_{x\to \infty}\frac{1}{3}\cdot\frac{e^{3x}}{e^{3x}} && (2)\\
&= \lim_{x\to \infty}\frac{1}{3} && (3)\\
&= \frac{1}{3} && (4)
\end{aligned}$$

6. (Section 4.7, Exercise 51)

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^2 - \ln\frac{2}{x}}{3x^2+2x} &= \lim_{x\to\infty} \frac{2x+\frac{1}{x}}{6x+2} && (1)\\
&= \lim_{x\to\infty} \frac{2-\frac{1}{x^2}}{6} && (2)\\
&= \frac{2-0}{6} && (3)\\
&= \frac{2}{6} && (4)\\
&= \frac{1}{3} && (5)
\end{aligned}$$

7. (Section 4.7, Exercise 53)

$$\begin{aligned}
\lim_{x\to 0} x\csc x &= \lim_{x\to 0} \frac{x}{\sin x} && (1)\\
&= \lim_{x\to 0} \frac{1}{\cos x} && (2)\\
&= \frac{1}{\cos 0} && (3)\\
&= \frac{1}{1} && (4)\\
&= 1 && (5)
\end{aligned}$$

8. (Section 4.7, Exercise 63)

$$\begin{aligned}
\lim_{x\to\infty} \left(x^2 - \sqrt{x^4+16x^2}\right) &= \lim_{x\to\infty} \left(x^2 - \sqrt{x^4\left(1+\frac{16}{x^2}\right)}\right) && (1)\\
&= \lim_{x\to\infty} \left(x^2 - x^2\sqrt{1+\frac{16}{x^2}}\right) && (2)\\
&= \lim_{x\to\infty} x^2\left(1 - \sqrt{1+\frac{16}{x^2}}\right) && (3)\\
&= \lim_{x\to\infty} \frac{1-\sqrt{1+\frac{16}{x^2}}}{\frac{1}{x^2}} && (4)\\
&= \lim_{x\to\infty} \frac{\frac{16}{x^3}}{\frac{-2}{x^3}\sqrt{1+\frac{16}{x^2}}} && (5)\\
&= \lim_{x\to\infty} \frac{\frac{16}{x^3}\cdot\frac{x^3}{-2}}{\sqrt{1+\frac{16}{x^2}}} && (6)\\
&= \lim_{x\to\infty} \frac{\frac{16}{-2}\cdot\frac{x^3}{x^3}}{\sqrt{1+\frac{16}{x^2}}} && (7)\\
&= \lim_{x\to\infty} \frac{-8}{\sqrt{1+\frac{16}{x^2}}} && (8)\\
&= \frac{-8}{\sqrt{1+0}} && (9)\\
&= \frac{-8}{1} && (10)\\
&= -8 && (11)
\end{aligned}$$

9. (Section 4.7, Exercise 64)

$$
\begin{align}
\lim_{x\to\infty}\left(x-\sqrt{x^2+4x}\right) &= \lim_{x\to\infty}\left(x-\sqrt{x^2\left(1+\frac{4}{x}\right)}\right) \tag{1}\\
&= \lim_{x\to\infty}\left(x-x\sqrt{1+\frac{4}{x}}\right) \tag{2}\\
&= \lim_{x\to\infty} x\left(1-\sqrt{1+\frac{4}{x}}\right) \tag{3}\\
&= \lim_{x\to\infty} \frac{1-\sqrt{1+\frac{4}{x}}}{\frac{1}{x}} \tag{4}\\
&= \lim_{x\to\infty} \frac{\frac{2}{x^2}}{\frac{-1}{x^2}\sqrt{1+\frac{4}{x}}} \tag{5}\\
&= \lim_{x\to\infty} \frac{\frac{2}{x^2}\cdot\frac{x^2}{-1}}{\sqrt{1+\frac{4}{x}}} \tag{6}\\
&= \lim_{x\to\infty} \frac{\frac{2}{-1}\cdot\frac{x^2}{x^2}}{\sqrt{1+\frac{4}{x}}} \tag{7}\\
&= \lim_{x\to\infty} \frac{-2}{\sqrt{1+\frac{4}{x}}} \tag{8}\\
&= \frac{-2}{\sqrt{1+0}} \tag{9}\\
&= -2 \tag{10}
\end{align}
$$

10. (Section 4.7, Exercise 75)

$$
\begin{align}
\lim_{x\to 0^+} x^{2x} &= e^{\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{2x}}} \tag{1}\\
&= e^{\lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{2x^2}}} \tag{2}\\
&= e^{\lim_{x\to 0^+} \frac{1}{x}\cdot\frac{2x^2}{-1}} \tag{3}\\
&= e^{\lim_{x\to 0^+} -\frac{2x^2}{x}} \tag{4}\\
&= e^{\lim_{x\to 0^+} -2x} \tag{5}\\
&= e^{-2(0)} \tag{6}\\
&= e^{0} \tag{7}\\
&= 1 \tag{8}
\end{align}
$$

11. (Section 4.7, Exercise 76)

$$
\begin{align}
\lim_{x\to 0}(1+4x)^{\frac{3}{x}} &= e^{\lim_{x\to 0^+} \frac{\ln(1+4x)}{\frac{1}{\frac{3}{x}}}} \tag{1}\\
&= e^{\lim_{x\to 0^+} \frac{\ln(1+4x)}{\frac{x}{3}}} \tag{2}\\
&= e^{\lim_{x\to 0^+} \frac{\frac{4}{(1+4x)}}{\frac{1}{3}}} \tag{3}\\
&= e^{\lim_{x\to 0^+} \frac{4}{(1+4x)}\cdot\frac{3}{1}} \tag{4}\\
&= e^{\lim_{x\to 0^+} \frac{12}{(1+4x)}} \tag{5}\\
&= e^{\frac{12}{1}} \tag{6}\\
&= e^{12} \tag{7}
\end{align}
$$

12. (Section 4.7, Exercise 96)

$$f(x) = x^2 \ln x \tag{1}$$
$$g(x) = \ln^2 x \tag{2}$$

$$\begin{aligned}
\lim_{x\to\infty} \frac{f(x)}{g(x)} &= \lim_{x\to\infty} \frac{x^2 \ln x}{\ln^2 x} && (1)\\
&= \lim_{x\to\infty} \frac{x^2}{\ln x} && (2)\\
&= \lim_{x\to\infty} \frac{2x}{\frac{1}{x}} && (3)\\
&= \lim_{x\to\infty} 2x^2 && (4)\\
&= \infty && (5)
\end{aligned}$$

$$f \gg g$$

13. (Section 4.7, Exercise 100)

$$f(x) = x^2 \ln x \tag{1}$$
$$g(x) = x^3 \tag{2}$$

$$\begin{aligned}
\lim_{x\to\infty} \frac{f(x)}{g(x)} &= \lim_{x\to\infty} \frac{x^2 \ln x}{x^3} && (1)\\
&= \lim_{x\to\infty} \frac{\ln x}{x} && (2)\\
&= \lim_{x\to\infty} \frac{\frac{1}{x}}{1} && (3)\\
&= \lim_{x\to\infty} \frac{1}{x} && (4)\\
&= \frac{1}{\infty} \neq \infty && (5)
\end{aligned}$$

$$g \gg f$$

14. (Section 4.7, Exercise 95)

$$f(x) = x^{10} \tag{1}$$
$$g(x) = e^{0.01x} \tag{2}$$

$$\begin{aligned}
\lim_{x\to\infty}\frac{f(x)}{g(x)} &= \lim_{x\to\infty}\frac{x^{10}}{e^{0.01x}} && (1)\\
&= \lim_{x\to\infty}\frac{10x^9}{0.01e^{0.01x}} && (2)\\
&= \lim_{x\to\infty}\frac{90x^8}{0.01^2e^{0.01x}} && (3)\\
&= \lim_{x\to\infty}\frac{7200x^7}{0.01^3e^{0.01x}} && (4)\\
&= \lim_{x\to\infty}\frac{50400x^6}{0.01^4e^{0.01x}} && (5)\\
&= \lim_{x\to\infty}\frac{302400x^5}{0.01^5e^{0.01x}} && (6)\\
&= \lim_{x\to\infty}\frac{1512000x^4}{0.01^6e^{0.01x}} && (7)\\
&= \lim_{x\to\infty}\frac{6048000x^3}{0.01^7e^{0.01x}} && (8)\\
&= \lim_{x\to\infty}\frac{18144000x^2}{0.01^8e^{0.01x}} && (9)\\
&= \lim_{x\to\infty}\frac{36288000x}{0.01^9e^{0.01x}} && (10)\\
&= \lim_{x\to\infty}\frac{36288000}{0.01^{10}e^{0.01x}} && (11)\\
&= \frac{36288000}{\infty} \neq \infty && (12)\\
& g \gg f
\end{aligned}$$

15. (Section 4.7, Exercise 101)

$$\begin{aligned}
f(x) &= x^{20} && (1)\\
g(x) &= 1.00001^x && (2)
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to\infty}\frac{f(x)}{g(x)} &= \lim_{x\to\infty}\frac{x^{20}}{1.00001^x} && (1)\\
&= \frac{2432902008176640000}{\infty} \neq \infty && (2)\\
& g \gg f
\end{aligned}$$

16. (Section 4.9, Exercise 12)

$$f(x) = 11x^{10}$$

$$\begin{aligned}
\int 11x^{10}\,dx &= 11\int x^{10}\,dx && (1)\\
&= 11\frac{x^{11}}{11} + C && (2)\\
&= x^{11} + C && (3)
\end{aligned}$$

17. (Section 4.9, Exercise 13)

$$f(x) = 2\sin x + 1$$

$$\begin{aligned}
\int 2\sin x + 1\,dx &= 2\int \sin x + \int x^0 && (1)\\
&= -2\cos x + x + C && (2)
\end{aligned}$$

18. (Section 4.9, Exercise 24)

$$\begin{aligned}
\int 3u^{-2} - 4u^2 + 1\,du &= 3\int u^{-2}\,du - 4\int u^2\,du + \int u^0\,du && (1)\\
&= 3\frac{u^{-1}}{-1} - 4\frac{u^3}{3} + u + C && (2)\\
&= \frac{-3}{u} - \frac{4u^3}{3} + u + C && (3)
\end{aligned}$$

19. (Section 4.9, Exercise 25)

$$\begin{aligned}
\int 4\sqrt{x} - \frac{4}{\sqrt{x}}\,dx &= 4\int x^{\frac{1}{2}}\,dx - 4\int x^{-\frac{1}{2}}\,dx && (1)\\
&= 4\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - 4\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C && (2)\\
&= \frac{8x^{\frac{3}{2}}}{3} - 8x^{\frac{1}{2}} + C && (3)\\
&= \frac{8x\sqrt{x}}{3} - 8\sqrt{x} + C && (4)
\end{aligned}$$

20. (Section 4.9, Exercise 31)

$$\begin{aligned}
\int (3x+1)(4-x)\,dx &= \int -3x^2 + 11x + 4\,dx && (1)\\
&= -3\int x^2\,dx + 11\int x\,dx + 4\int x^0\,dx && (2)\\
&= -3\frac{x^3}{3} + 11\frac{x^2}{2} + 4x + C && (3)\\
&= -x^3 + \frac{11x^2}{2} + 4x + C && (4)
\end{aligned}$$

21. (Section 4.9, Exercise 35)

$$\begin{aligned}
\int \frac{4x^4 - 6x^2}{x}\,dx &= \int 4x^3 - 6x\,dx && (1)\\
&= 4\int x^3\,dx - 6\int x\,dx && (2)\\
&= 4\frac{x^4}{4} - 6\frac{x^2}{2} + C && (3)\\
&= x^4 - 3x^2 + C && (4)
\end{aligned}$$

22. (Section 4.9, Exercise 40)

$$\begin{aligned}
\int \left(\csc^2\theta + 1\right) d\theta &= \int \csc^2\theta\,d\theta + \int \theta^0\,d\theta && (1)\\
&= -\cot\theta + \theta + C && (2)
\end{aligned}$$

23. (Section 4.9, Exercise 41)

$$\begin{aligned}
\int \frac{2 + 3\cos y}{\sin^2 y}\,dy &= \int \left(\frac{2}{\sin^2 y} + \frac{3\cos y}{\sin^2 y}\right) dy && (1)\\
&= \int \left(2\csc^2 y + 3\cot y\csc y\right) dy && (2)\\
&= 2\int \csc^2 y\,dy + 3\int \cot y\csc y\,dy && (3)\\
&= -2\cot y - 3\csc y + C && (4)
\end{aligned}$$

24. (Section 4.9, Exercise 47)

$$\begin{aligned}
\int \left(3t^2 + 2\csc^2 t\right) dt &= \int 3t^2\,dt + \int 2\csc^2 t\,dt && (1)\\
&= 3\int t^2\,dt + 2\int \csc^2 t\,dt && (2)\\
&= 3\frac{t^3}{3} - 2\cot t + C && (3)\\
&= t^3 - 2\cot t + C && (4)
\end{aligned}$$

25. (Section 4.9, Exercise 45)

$$\begin{aligned}\int \left(sec^2\theta + \sec\theta\tan\theta\right) d\theta &= \int sec^2\theta\, d\theta + \int \sec\theta\tan\theta\, d\theta && (1)\\ &= \tan\theta + \sec\theta + C && (2)\end{aligned}$$

26. (Section 4.9, Exercise 50)

$$\begin{aligned}\int \frac{\csc^3 x + 1}{\csc x}\, dx &= \int \frac{\csc^3 x}{\csc x} + \frac{1}{\csc x}\, dx && (1)\\ &= \int \csc^2 x + \sin x\, dx && (2)\\ &= \int \csc^2 x\, dx + \int \sin x\, dx && (3)\\ &= -\cot x + -\cos x + C && (4)\end{aligned}$$

27. (Section 4.9, Exercise 51)

$$\begin{aligned}\int \frac{1}{2y}\, dy &= \frac{1}{2}\int \frac{1}{y}\, dy && (1)\\ &= \frac{1}{2}\ln|y| + C && (2)\\ &= \frac{\ln|y|}{2} + C && (3)\end{aligned}$$

28. (Section 4.9, Exercise 53)

$$\begin{aligned}\int \frac{6}{\sqrt{4-4x^2}}\, dx &= \int \frac{6}{\sqrt{4\left(1-x^2\right)}}\, dx && (1)\\ &= \int \frac{6}{2\sqrt{1-x^2}}\, dx && (2)\\ &= 3\int \frac{1}{\sqrt{1-x^2}}\, dx && (3)\\ &= 3\sin^{-1} x + C && (4)\end{aligned}$$

29. (Section 4.9, Exercise 59)

$$\begin{aligned}\int \frac{t+1}{t}\, dt &= \int \left(\frac{t}{t} + \frac{1}{t}\right) dt && (1)\\ &= \int \frac{t}{t}\, dt + \int \frac{1}{t}\, dt && (2)\\ &= \int t^0\, dt + \ln|t| + C && (3)\\ &= t + \ln|t| + C && (4)\end{aligned}$$

30. (Section 4.9, Exercise 61)

$$\begin{aligned}\int e^{x+2}\, dx &= \int e^2 e^x\, dx && (1)\\ &= e^2\int e^x\, dx && (2)\\ &= e^2 e^x && (3)\\ &= e^{x+2} && (4)\end{aligned}$$

31. (Section 4.9, Exercise 54)

$$\begin{aligned}\int \frac{v^3+v+1}{1+v^2}\, dv &= \int \frac{v^3+v}{1+v^2} + \frac{1}{1+v^2}\, dv && (1)\\ &= \int v + \frac{1}{1+v^2}\, dv && (2)\\ &= \int v\, dv + \int \frac{1}{1+v^2}\, dv && (3)\\ &= \frac{v^2}{2} + \tan^{-1} v + C && (4)\end{aligned}$$

32. (Section 4.9, Exercise 62)

$$\begin{aligned}
\int \frac{10t^5-3}{t}\,dt &= \int\left(\frac{10t^5}{t}-\frac{3}{t}\right)dt & (1)\\
&= \int 10t^4\,dt - \int \frac{3}{t}\,dt & (2)\\
&= 10\int t^4\,dt - 3\int \frac{1}{t}\,dt & (3)\\
&= 10\frac{t^5}{5} - 3\ln|t| + C & (4)\\
&= 2t^5 - 3\ln|t| + C & (5)
\end{aligned}$$

33. (Section 4.9, Exercise 78)

$$\begin{aligned}
g'(x) &= 7x^6 - 4x^3 + 12 & (1)\\
g(1) &= 24 & (2)\\
g(x) &= \int 7x^6 - 4x^3 + 12\,dx & (3)\\
&= 7\int x^6\,dx - 4\int x^3\,dx + 12\int x^0\,dx & (4)\\
&= 7\frac{x^7}{7} - 4\frac{x^4}{4} + 12x + C & (5)\\
&= x^7 - x^4 + 12x + C & (6)\\
(1)^7 - (1)^4 + 12(1) + C &= 24 & (7)\\
1 - 1 + 12 + C &= 24 & (8)\\
12 + C &= 24 & (9)\\
C &= 12 & (10)\\
g(x) &= x^7 - x^4 + 12x + 12 & (11)
\end{aligned}$$

34. (Section 4.9, Exercise 83)

$$\begin{aligned}
y'(t) &= \frac{3}{t} + 6 & (1)\\
y(1) &= 8, t > 0 & (2)\\
y(t) &= \int\left(\frac{3}{t}+6\right)dt & (3)\\
&= 3\int \frac{1}{t}\,dt + 6\int t^0\,dt & (4)\\
&= 3\ln|t| + 6t + C & (5)\\
3\ln 1 + 6(1) + C &= 8 & (6)\\
6 + C &= 8 & (7)\\
C &= 2 & (8)\\
y(t) &= 3\ln|t| + 6t + 2 & (9)
\end{aligned}$$

35. (Section 4.9, Exercise 105)

$$\begin{aligned}
v(t) &= \sin t & (1)\\
s(0) &= 0 & (2)\\
s(t) &= \int \sin t\,dt & (3)\\
&= -\cos t + C & (4)\\
-\cos 0 + C &= 0 & (5)\\
-1 + C &= 0 & (6)\\
C &= 1 & (7)\\
s(t) &= -\cos t + 1 & (8)
\end{aligned}$$

$$
\begin{aligned}
V(t) &= \cos t \quad &(1)\\
S(0) &= 0 &(2)\\
S(t) &= \int \cos t \, dt &(3)\\
&= \sin t + C &(4)\\
\sin 0 + C &= 0 &(5)\\
C &= 0 &(6)\\
S(t) &= \sin t &(7)
\end{aligned}
$$

36. (Section 4.9, Exercise 106)

$$
\begin{aligned}
v(t) &= e^t &(1)\\
s(0) &= 0 &(2)\\
s(t) &= \int e^t \, dt &(3)\\
&= e^t + C &(4)\\
e^0 + C &= 0 &(5)\\
1 + C &= 0 &(6)\\
C &= -1 &(7)\\
s(t) &= e^t - 1 &(8)
\end{aligned}
$$

$$
\begin{aligned}
V(t) &= 2 + \cos t &(1)\\
S(0) &= 3 &(2)\\
S(t) &= \int 2 + \cos t \, dt &(3)\\
&= \int 2 \, dt + \int \cos t \, dt &(4)\\
&= 2 \int t^0 \, dt + \sin t + C &(5)\\
&= 2t + \sin t + C &(6)\\
2(0) + \sin 0 + C &= 3 &(7)\\
C &= 3 &(8)
\end{aligned}
$$

37. (Section 5.1, Exercise 3)

$$4(40) + 4(70) = 440$$

$$2(30) + 2(50) + 2(80) + 2(40) = 400$$

38. (Section 5.1, Exercise 15)

$$v = 3t^2 + 1$$

$$0 \le t \le 4$$

$$
\begin{aligned}
f(0.5) + f(1.5) + f(2.5) + f(3.5) &= 1.75 + 7.75 + 19.75 + 37.75 &(1)\\
&= 67 &(2)\\
\frac{f(0.25)}{2} + \frac{f(0.75)}{2} + \frac{f(1.25)}{2} + \frac{f(1.75)}{2} + \frac{f(2.25)}{2} + \frac{f(2.75)}{2} + \frac{f(3.25)}{2} + \frac{f(3.75)}{2} &= 67.75 &(3)
\end{aligned}
$$

39. (Section 5.1, Exercise 16)

$$v = \sqrt{10t}$$

$$1 \le t \le 7$$

(a)

$$n = 3$$

$$[1,3], [3,5], [5,7]$$

$$2, 4, 6$$

$$d \approx 2v(2) + 2v(4) + 2v(6) = 2\sqrt{20} + 2\sqrt{40} + 2\sqrt{60} \approx 37.085$$

(b)

$$n = 6$$

$$1.5, 2.5, 3.5, 4.5, 5.5, 6.5$$

$$\begin{aligned} d &\approx v(1.5) + v(2.5) + v(3.5) + v(4.5) + v(5.5) + v(6.5) && (1) \\ &= \sqrt{15} + \sqrt{25} + \sqrt{35} + \sqrt{45} + \sqrt{55} + \sqrt{65} && (2) \\ &\approx 36.976 && (3) \end{aligned}$$

40. (Section 5.1, Exercise 23)

$$f(x) = x + 1$$

$$[1, 6]$$

$$n = 5$$

(a)

$$\begin{aligned} (1+1) + (2+1) + (3+1) + (4+1) + (5+1) &= 2 + 3 + 4 + 5 + 6 && (1) \\ &= 20 && (2) \end{aligned}$$

(b)

$$\begin{aligned} (2+1) + (3+1) + (4+1) + (5+1) + (6+1) &= 3 + 4 + 5 + 6 + 7 && (1) \\ &= 25 && (2) \end{aligned}$$

41. (Section 5.1, Exercise 24)

$$f(x) = \frac{1}{x}$$

$$[1, 5]$$

$$n = 4$$

$$\Delta x = 1$$

$$\begin{aligned} \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} &= \frac{25}{12} && (1) \\ \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} &= \frac{77}{60} && (2) \end{aligned}$$

42. (Section 5.1, Exercise 29)

$$f(x) = x^2 - 1$$

$$[2, 4]$$

$$n = 4$$

$$\Delta x = \frac{1}{2}$$

$$[2, 2.5], [2.5, 3], [3, 3.5], [3.5, 4]$$

$$\begin{aligned} \frac{3}{2} + \frac{5.25}{2} + \frac{8}{2} + \frac{11.25}{2} &= 13.75 && (1) \\ \frac{5.25}{2} + \frac{8}{2} + \frac{11.25}{2} + \frac{15}{2} &= 19.75 && (2) \end{aligned}$$

43. (Section 5.1, Exercise 33)

$$f(x) = 100 - x^2$$

$$[0, 10]$$

$$n = 5$$

$$\Delta x = 2$$

$$\begin{aligned} 2\,(99) + 2\,(91) + 2\,(75) + 2\,(51) + 2\,(19) &= 198 + 182 + 150 + 102 + 38 && (1) \\ &= 670 && (2) \end{aligned}$$

44. (Section 5.1, Exercise 34)

$$f(t) = \cos\frac{t}{2}$$
$$[0, \pi]$$
$$n = 4$$
$$\Delta x = \frac{\pi}{4}$$

$$\frac{\pi}{4}\left(\cos\frac{\pi}{16}\right) + \frac{\pi}{4}\left(\cos\frac{3\pi}{16}\right) + \frac{\pi}{4}\left(\cos\frac{5\pi}{16}\right) + \frac{\pi}{4}\left(\cos\frac{7\pi}{16}\right) = \frac{704}{105} \tag{1}$$
$$\approx 6.704761905 \tag{2}$$

45. (Section 5.1, Exercise 39)

$$f(x) = \sqrt{x}$$
$$[1, 3]$$
$$n = 4$$
$$\Delta x = \frac{1}{2}$$

$$\frac{\sqrt{1.25}}{2} + \frac{\sqrt{1.75}}{2} + \frac{\sqrt{2.25}}{2} + \frac{\sqrt{2.75}}{2} \approx 2.853 \tag{1}$$

46. (Section 5.1, Exercise 43)

$$n = 4$$
$$[0, 2]$$
$$\Delta x = \frac{1}{2}$$

$$\frac{5}{2} + \frac{3}{2} + \frac{2}{2} + \frac{1}{2} = 5.5 \tag{1}$$
$$\frac{3}{2} + \frac{2}{2} + \frac{1}{2} + \frac{1}{2} = 3.5 \tag{2}$$

47. (Section 5.1, Exercise 44)

$$n = 8$$
$$[1, 5]$$
$$\Delta x = \frac{1}{2}$$

$$\frac{0}{2} + \frac{2}{2} + \frac{3}{2} + \frac{2}{2} + \frac{2}{2} + \frac{1}{2} + \frac{0}{2} + \frac{2}{2} = 6 \tag{1}$$
$$\frac{2}{2} + \frac{3}{2} + \frac{2}{2} + \frac{2}{2} + \frac{1}{2} + \frac{0}{2} + \frac{2}{2} + \frac{3}{2} = 7.5 \tag{2}$$

48. (Section 5.1, Exercise 51)

$$f(x) = 3\sqrt{x}$$
$$[0, 4]$$
$$n = 40$$
$$\Delta x = \frac{1}{10}$$

$$\sum_{k=1}^{40} \frac{3}{10}\sqrt{\frac{k-1}{10}} \approx 15.681 \tag{1}$$
$$\sum_{k=1}^{40} \frac{3}{10}\sqrt{\frac{k}{10}} \approx 16.281 \tag{2}$$
$$\sum_{k=1}^{40} \frac{3}{10}\sqrt{\frac{k-\frac{1}{2}}{10}} \approx 16.005 \tag{3}$$

49. (Section 5.1, Exercise 52)

$$f(x) = x^2 + 1$$

$$[-1, 1]$$

$$n = 50$$

$$\Delta x = \frac{1}{25}$$

$$\sum_{k=1}^{50} 0.04 \left((-1 + 0.04\,(k-1))^2 + 1 \right) \approx 2.6672 \tag{1}$$

$$\sum_{k=1}^{50} 0.04 \left((-1 + 0.04k)^2 + 1 \right) \approx 2.6672 \tag{2}$$

$$\sum_{k=1}^{50} 0.04 \left(\left(-1 + 0.04 \left(k - \frac{1}{2} \right) \right)^2 + 1 \right) \approx 2.6664 \tag{3}$$

A copy of my notes (in LaTeX) are available on my GitHub