Module 7 Notes (MATH-211)

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General Notes (and Definitions)

- Working with Integrals
  A function $f(x)$ is **even** if $f(-x) = f(x)$.
  A function $f(x)$ is **odd** if $f(-x) = -f(x)$.
  Let $a \in \mathbb{R}$ such that $a > 0$ and let f be an integrable function on the interval $[-a, a]$.

  $$\text{If } f \text{ is even, } \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

  $$\text{If } f \text{ is odd, } \int_{-a}^{a} f(x)\,dx = 0$$

  The average value of an integrable function f on the interval $[a, b]$ is

  $$\overline{f} = \frac{1}{b-a}\int_{a}^{b} f(x)\,dx$$

  Let f be continuous on the interval $[a, b]$. There exists a point c in (a, b) such that (Mean Value Theorem)

  $$f(c) = \overline{f} = \frac{1}{b-a}\int_{a}^{b} f(t)\,dx$$

Examples

1. Use symmetry to evaluate integrals

$$\int_{-200}^{200} 2x^5\,dx = 0$$

$$\begin{aligned}
\int_{-2}^{2} \left(x^2 + x^3\right)\,dx &= \int_{-2}^{2} x^2\,dx + \int_{-2}^{2} x^3\,dx && (1)\\
&= 2\int_{0}^{2} x^2\,dx + 0 && (2)\\
&= 2\frac{x^3}{3} && (3)\\
&= \frac{16}{3} && (4)
\end{aligned}$$

2. A derivative calculation

$$s(t) = -16t^2 + 64t$$

$$t = 4$$

$$[0, 4]$$

$$\begin{aligned}
v(t) &= s'(t) && (1)\\
\overline{v} &= \frac{1}{4}\int_0^4 v(t)\,dx && (2)\\
&= \frac{1}{4}\int_0^4 s'(t)\,dx && (3)\\
&= \frac{1}{4}s(t) && (4)\\
&= \frac{1}{4}\left(s(4)-s(0)\right) && (5)\\
&= 0 && (6)
\end{aligned}$$

3. Applying MVT for integrals

$$f(x) = e^x$$

$$[0,2]$$

$$\begin{aligned}
\overline{f} &= \frac{1}{2}\left(\int_0^2 e^x\,dx\right) && (1)\\
&= \frac{e^x}{2} && (2)\\
&= \frac{e^2}{2}-\frac{e^0}{2} && (3)\\
&= \frac{e^2-1}{2} && (4)\\
e^x &= \frac{e^2-1}{2} && (5)\\
\ln e^x &= \ln\frac{e^2-1}{2} && (6)
\end{aligned}$$

Related Exercises

1. (Section 5.4, Exercise 15)

$$\begin{aligned}
\int_{-2}^{2}\left(x^2+x^3\right)\,dx &= \int_{-2}^{2} x^2\,dx + \int_{-2}^{2} x^3\,dx && (1)\\
&= 2\int_0^2 x^2\,dx + 0 && (2)\\
&= 2\frac{x^3}{3} && (3)\\
&= 2\frac{2^3}{3} - 2\frac{0^3}{3} && (4)\\
&= 2\frac{8}{3} && (5)\\
&= \frac{16}{3} && (6)
\end{aligned}$$

2. (Section 5.4, Exercise 16)

$$\int_{-\pi}^{\pi} t^2 \sin t\,dx = 0$$

3. (Section 5.4, Exercise 26)

$$f(x) = x^2+1$$

$$[-2,2]$$

$$
\begin{aligned}
\overline{f} &= \frac{1}{2-(-2)} \int_{-2}^{2} x^2 + 1\, dx && (1)\\
&= \frac{1}{4}\left(\int_{-2}^{2} x^2\, dx + 1 \int_{-2}^{2} x^0\, dx\right) && (2)\\
&= \frac{1}{4}\left(\frac{x^3}{3} + x\right) && (3)\\
&= \frac{1}{4}\left(\int_{-2}^{2} x^2\, dx + \int_{-2}^{2} 1\, dx\right) && (4)\\
&= \frac{1}{4}\left(\frac{2^3}{3} - \frac{(-2)^3}{3} + 2 - (-2)\right) && (5)\\
&= \frac{1}{4}\left(\frac{8}{3} - \frac{-8}{3} + 4\right) && (6)\\
&= \frac{1}{4}\left(\frac{16}{3} + 4\right) && (7)\\
&= \frac{1}{4}\left(\frac{28}{3}\right) && (8)\\
&= \frac{7}{3} && (9)
\end{aligned}
$$

4. (Section 5.4, Exercise 34)

$$f(x) = x^3 - 5x^2 + 30$$

$$[0, 4]$$

$$
\begin{aligned}
\overline{f} &= \frac{1}{4}\left(\int_{0}^{4} \left(x^3 - 5x^2 + 30\right)\, dx\right) && (1)\\
&= \frac{1}{4}\left(\int_{0}^{4} x^3 - 5\int_{0}^{4} x^2 + 30\int_{0}^{4} x^0\right) && (2)\\
&= \frac{1}{4}\left(\frac{x^4}{4} - 5\frac{x^3}{3} + 30x\right) && (3)\\
&= \frac{1}{4}\left(\left(\frac{4^4}{4} - \frac{0^4}{4}\right) - \left(5\frac{4^3}{3} - 5\frac{0^3}{3}\right) + (30(4) - 30(0))\right) && (4)\\
&= \frac{1}{4}\left(64 - \frac{320}{3} + 120\right) && (5)\\
&= \frac{1}{4}\left(\frac{232}{3}\right) && (6)\\
&= \frac{58}{3} && (7)
\end{aligned}
$$

5. (Section 5.4, Exercise 41)

$$f(x) = 1 - \frac{x^2}{a^2}$$

$$[0, a]$$

$$
\begin{aligned}
\overline{f} &= \frac{1}{a}\left(\int_0^a 1-\frac{x^2}{a^2}\,dx\right) && (1)\\
&= \frac{1}{a}\left(\int_0^a 1\,dx-\int_0^a \frac{x^2}{a^2}\,dx\right) && (2)\\
&= \frac{1}{a}\left(x-\frac{1}{a^2}\int_0^a x^2\,dx\right) && (3)\\
&= \frac{1}{a}\left(x-\frac{1}{a^2}\frac{x^3}{3}\right) && (4)\\
&= \frac{1}{a}\left(x-\frac{x^3}{3a^2}\right) && (5)\\
&= \frac{1}{a}\left((a-0)-\frac{1}{a^2}\left(\frac{a^3}{3}-\frac{0^3}{3}\right)\right) && (6)\\
&= \frac{1}{a}\left(a-\frac{a^3}{3a^2}\right) && (7)\\
&= \frac{1}{a}\left(a-\frac{a}{3}\right) && (8)\\
&= \frac{1}{a}\left(\frac{2a}{3}\right) && (9)\\
&= \frac{2}{3} && (10)\\
1-\frac{c^2}{a^2} &= \frac{2}{3} && (11)\\
\frac{c^2}{a^2} &= \frac{1}{3} && (12)\\
c^2 &= \frac{a^2}{3} && (13)\\
c &= \sqrt{\frac{a^2}{3}} && (14)\\
&= \frac{a}{\sqrt{3}} && (15)
\end{aligned}
$$

6. (Section 5.4, Exercise 42)

$$f(x)=\frac{\pi}{4}\sin x$$

$$[0,\pi]$$

$$
\begin{aligned}
\overline{f} &= \frac{1}{\pi}\int_0^\pi \frac{\pi}{4}\sin x && (1)\\
&= \frac{1}{\pi}\frac{\pi}{4}\int_0^\pi \sin x && (2)\\
&= \frac{1}{\pi}\frac{\pi}{4}(-\cos x) && (3)\\
&= \frac{1}{\pi}\frac{\pi}{4}(-\cos\pi+\cos 0) && (4)\\
&= \frac{1}{\pi}\frac{\pi}{4}(1+1) && (5)\\
&= \frac{1}{\pi}\frac{\pi}{2} && (6)\\
&= \frac{1}{2} && (7)\\
\frac{\pi}{4}\sin x &= \frac{1}{2} && (8)\\
\sin x &= \frac{2}{\pi} && (9)\\
\sin^{-1}\sin x &= \sin^{-1}\frac{2}{\pi} && (10)\\
x &= \sin^{-1}\frac{2}{\pi} && (11)
\end{aligned}
$$

A copy of my notes (in LaTeX) are available on my GitHub