Module 7 Notes (MATH-211)

Lillie Donato

22 July 2024

General Notes (and Definitions)

- Working with Integrals
  A function $f(x)$ is **even** if $f(-x) = f(x)$.
  A function $f(x)$ is **odd** if $f(-x) = -f(x)$.
  Let $a \in \mathbb{R}$ such that $a > 0$ and let f be an integrable function on the interval $[-a, a]$.

$$\text{If } f \text{ is even, } \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

$$\text{If } f \text{ is odd, } \int_{-a}^{a} f(x)\,dx = 0$$

  The average value of an integrable function f on the interval $[a, b]$ is

$$\overline{f} = \frac{1}{b-a}\int_{a}^{b} f(x)\,dx$$

  Let f be continuous on the interval $[a, b]$. There exists a point c in (a, b) such that (Mean Value Theorem)

$$f(c) = \overline{f} = \frac{1}{b-a}\int_{a}^{b} f(t)\,dx$$

- Substitution Rule
  Let $u = g(x)$, where g is differentiable on an interval, and let f be continuous on the corresponding range of g. On that interval,

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

  1. Given an indefinite integral involving a commposite function $f(g(x))$, identify an inner function $u = g(x)$ such that a constant multiple of $g'(x)$ appears in the integrand.
  2. Substitute $u = g(x)$ and $du = g'(x)\,dx$ in the integral.
  3. Evaluate the new indefinite integral with respect to u.
  4. Write the result in terms of x using $u = g(x)$.

  Let $u = g(x)$, where g' is continuous on $[a, b]$, and let f be continuous on the range of g. Then

$$\int_{a}^{b} f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

General formulas for indefinite integrals

$$\begin{aligned}
\int \cos ax\,dx &= \frac{1}{a}\sin ax + C && (1)\\
\int \sin ax\,dx &= -\frac{1}{a}\cos ax + C && (2)\\
\int \sec^2 ax\,dx &= \frac{1}{a}\tan ax + C && (3)\\
\int \csc^2 ax\,dx &= -\frac{1}{a}\cot ax + C && (4)\\
\int \sec ax \tan ax\,dx &= \frac{1}{a}\sec ax + C && (5)\\
\int \csc ax \cot ax\,dx &= -\frac{1}{a}\csc ax + C && (6)\\
\int e^{ax}\,dx &= \frac{1}{a}e^{ax} + C && (7)\\
\int b^x\,dx &= \frac{1}{\ln b}b^x + C, b>0, b\neq 1 && (8)\\
\int \frac{dx}{a^2+x^2} &= \frac{1}{a}\tan^{-1}\frac{x}{a} + C && (9)\\
\int \frac{dx}{\sqrt{a^2-x^2}} &= \sin^{-1}\frac{x}{a} + C, a>0 && (10)\\
\int \frac{dx}{x\sqrt{x^2-a^2}} &= \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C, a>0 && (11)
\end{aligned}$$

Examples

1. Use symmetry to evaluate integrals

$$\int_{-200}^{200} 2x^5\,dx = 0$$

$$\begin{aligned}
\int_{-2}^{2}\left(x^2+x^3\right)\,dx &= \int_{-2}^{2} x^2\,dx + \int_{-2}^{2} x^3\,dx && (1)\\
&= 2\int_0^2 x^2\,dx + 0 && (2)\\
&= 2\frac{x^3}{3} && (3)\\
&= \frac{16}{3} && (4)
\end{aligned}$$

2. A derivative calculation

$$s(t) = -16t^2 + 64t$$

$$t = 4$$

$$[0,4]$$

$$\begin{aligned}
v(t) &= s'(t) && (1)\\
\overline{v} &= \frac{1}{4}\int_0^4 v(t)\,dx && (2)\\
&= \frac{1}{4}\int_0^4 s'(t)\,dx && (3)\\
&= \frac{1}{4}s(t) && (4)\\
&= \frac{1}{4}\left(s(4)-s(0)\right) && (5)\\
&= 0 && (6)
\end{aligned}$$

3. Applying MVT for integrals

$$f(x) = e^x$$
$$[0, 2]$$

$$\begin{aligned}
\overline{f} &= \frac{1}{2}\left(\int_0^2 e^x \, dx\right) && (1)\\
&= \frac{e^x}{2} && (2)\\
&= \frac{e^2}{2} - \frac{e^0}{2} && (3)\\
&= \frac{e^2 - 1}{2} && (4)\\
e^x &= \frac{e^2 - 1}{2} && (5)\\
\ln e^x &= \ln \frac{e^2 - 1}{2} && (6)
\end{aligned}$$

4. Perfect substitutions in indefinite integrals

$$\begin{aligned}
u &= 4x^3 - 8 && (1)\\
du &= 12x^2 \, dx && (2)\\
\int 12x^2 \left(4x^3 - 8\right)^5 \, dx &= \int 12x^2 u^5 \, dx && (3)\\
&= \frac{u^6}{6} + C && (4)\\
&= \frac{\left(4x^3 - 8\right)^6}{6} + C && (5)
\end{aligned}$$

$$\begin{aligned}
u &= \sin t && (1)\\
du &= \cos t \, dt && (2)\\
\int (\cos t) \, e^{\sin t} \, dt &= \int e^u \, du && (3)\\
&= e^u + C && (4)\\
&= e^{\sin t} + C && (5)
\end{aligned}$$

5. Introducting constants when integrating by substitution

$$\begin{aligned}
u &= 6x + 4 && (1)\\
du &= 6 \, dx && (2)\\
dx &= \frac{du}{6} && (3)\\
\int (6x + 4)^9 \, dx &= \int \frac{1}{6} \cdot u^9 \, du && (4)\\
&= \frac{1}{6} \int u^8 \, du && (5)\\
&= \frac{1}{6} \cdot \frac{u^9}{9} + C && (6)\\
&= \frac{(6x + 4)^9}{54} + C && (7)
\end{aligned}$$

$$\begin{aligned}
u &= \cot x && (1)\\
du &= -\csc^2 x \, dx && (2)\\
\int \cot^2 x \csc^2 x \, dx &= \int -u^2 \, du && (3)\\
&= -\frac{u^3}{3} + C && (4)\\
&= -\frac{\csc^3 x}{3} + C && (5)
\end{aligned}$$

6. Variations on the substitution method

$$\begin{aligned}
u &= x-1 && (1)\\
du &= dx && (2)\\
x &= u+1 && (3)\\
\int x\sqrt{x-1}\,dx &= \int (u+1)\sqrt{u}\,du && (4)\\
&= \int u\sqrt{u}+\sqrt{u}\,du && (5)\\
&= \int u^{\frac{3}{2}}+u^{\frac{1}{2}}\,du && (6)\\
&= \frac{2}{5}u^{\frac{5}{2}}+\frac{2}{3}u^{\frac{3}{2}}+C && (7)\\
&= \frac{2}{5}(x-1)^{\frac{5}{2}}+\frac{2}{3}(x-1)^{\frac{3}{2}}+C && (8)
\end{aligned}$$

7. Use known formulas to evaluate indefinite integrals

$$\begin{aligned}
\int 2e^{-4x}\,dx &= 2\int e^{-4x}\,dx && (1)\\
&= \frac{2}{-4}e^{-4x}+C && (2)\\
&= -\frac{1}{2}e^{-4x}+C && (3)\\
& && (4)
\end{aligned}$$

$$\begin{aligned}
\int \frac{dx}{\sqrt{36-x^2}} &= \int \frac{dx}{\sqrt{6^2-x^2}} && (1)\\
&= \sin^{-1}\frac{x}{6}+C && (2)
\end{aligned}$$

8. Evaluating definite integrals using substitution

$$\begin{aligned}
u &= 2^x+4 && (1)\\
du &= 2^x\ln 2\,dx && (2)\\
\frac{1}{\ln 2}\,du &= 2^x\,dx && (3)\\
\int_1^3 \frac{2^x}{2^x+4}\,dx &= \int_1^3 \frac{1}{u\ln 2}\,du && (4)\\
\int_{g(1)}^{g(3)} \frac{1}{u\ln 2}\,du &= \int_6^{12} \frac{1}{u\ln 2}\,du && (5)\\
&= \frac{1}{\ln 2}\int_6^{12}\frac{du}{u} && (6)\\
&= \frac{1}{\ln 2}\cdot(\ln 12-\ln 6) && (7)\\
&= \frac{\ln 2}{\ln 2} && (8)\\
&= 1 && (9)
\end{aligned}$$

$$\begin{aligned}
u &= \ln p && (1)\\
du &= \frac{1}{p}\,dx && (2)\\
\int_1^{e^2}\frac{\ln p}{p} &= \int_0^2 u\,du && (3)\\
&= \frac{2^2}{2}-\frac{0^2}{2} && (4)\\
&= \frac{4}{2} && (5)\\
&= 2 && (6)
\end{aligned}$$

9. Integrals involving $\cos^2 x$ and $\sin^2 x$

$$\begin{aligned}
u &= 2x && (1)\\
du &= 2\,dx && (2)\\
dx &= \frac{1}{2}du && (3)\\
\sin^2 x &= \frac{1-\cos 2x}{2} && (4)\\
\int_0^\pi \sin^2 x\,dx &= \int_0^\pi \frac{1-\cos 2x}{2}\,dx && (5)\\
&= \frac{1}{2}\int_0^\pi 1-\cos 2x\,dx && (6)\\
&= \frac{1}{2}\left(\int_0^\pi 1\,dx - \int_0^\pi \cos 2x\,dx\right) && (7)\\
&= \frac{1}{2}\left((\pi-0) - \frac{1}{2}\int_0^{2\pi}\cos u\,du\right) && (8)\\
&= \frac{1}{2}\left(\pi - \frac{1}{2}(\sin 2\pi - \sin 0)\right) && (9)\\
&= \frac{1}{2}(\pi-0) && (10)\\
&= \frac{\pi}{2} && (11)
\end{aligned}$$

Related Exercises

1. (Section 5.4, Exercise 15)

$$\begin{aligned}
\int_{-2}^{2}\left(x^2+x^3\right)\,dx &= \int_{-2}^{2} x^2\,dx + \int_{-2}^{2} x^3\,dx && (1)\\
&= 2\int_0^2 x^2\,dx + 0 && (2)\\
&= 2\frac{x^3}{3} && (3)\\
&= 2\frac{2^3}{3} - 2\frac{0^3}{3} && (4)\\
&= 2\frac{8}{3} && (5)\\
&= \frac{16}{3} && (6)
\end{aligned}$$

2. (Section 5.4, Exercise 16)

$$\int_{-\pi}^{\pi} t^2 \sin t\,dx = 0$$

3. (Section 5.4, Exercise 26)

$$f(x) = x^2+1$$
$$[-2,2]$$

$$\begin{aligned}
\overline{f} &= \frac{1}{2-(-2)}\int_{-2}^{2} x^2+1\,dx && (1)\\
&= \frac{1}{4}\left(\int_{-2}^{2} x^2\,dx + 1\int_{-2}^{2} x^0\,dx\right) && (2)\\
&= \frac{1}{4}\left(\frac{x^3}{3}+x\right) && (3)\\
&= \frac{1}{4}\left(\int_{-2}^{2} x^2\,dx + \int_{-2}^{2} 1\,dx\right) && (4)\\
&= \frac{1}{4}\left(\frac{2^3}{3}-\frac{(-2)^3}{3}+2-(-2)\right) && (5)\\
&= \frac{1}{4}\left(\frac{8}{3}-\frac{-8}{3}+4\right) && (6)\\
&= \frac{1}{4}\left(\frac{16}{3}+4\right) && (7)\\
&= \frac{1}{4}\left(\frac{28}{3}\right) && (8)\\
&= \frac{7}{3} && (9)
\end{aligned}$$

4. (Section 5.4, Exercise 34)

$$f(x) = x^3 - 5x^2 + 30$$

$$[0,4]$$

$$\begin{aligned}
\overline{f} &= \frac{1}{4}\left(\int_{0}^{4} \left(x^3-5x^2+30\right)\,dx\right) && (1)\\
&= \frac{1}{4}\left(\int_{0}^{4} x^3 - 5\int_{0}^{4} x^2 + 30\int_{0}^{4} x^0\right) && (2)\\
&= \frac{1}{4}\left(\frac{x^4}{4}-5\frac{x^3}{3}+30x\right) && (3)\\
&= \frac{1}{4}\left(\left(\frac{4^4}{4}-\frac{0^4}{4}\right)-\left(5\frac{4^3}{3}-5\frac{0^3}{3}\right)+(30(4)-30(0))\right) && (4)\\
&= \frac{1}{4}\left(64-\frac{320}{3}+120\right) && (5)\\
&= \frac{1}{4}\left(\frac{232}{3}\right) && (6)\\
&= \frac{58}{3} && (7)
\end{aligned}$$

5. (Section 5.4, Exercise 41)

$$f(x) = 1 - \frac{x^2}{a^2}$$

$$[0,a]$$

$$
\begin{aligned}
\overline{f} &= \frac{1}{a}\left(\int_0^a 1-\frac{x^2}{a^2}\,dx\right) && (1)\\
&= \frac{1}{a}\left(\int_0^a 1\,dx-\int_0^a \frac{x^2}{a^2}\,dx\right) && (2)\\
&= \frac{1}{a}\left(x-\frac{1}{a^2}\int_0^a x^2\,dx\right) && (3)\\
&= \frac{1}{a}\left(x-\frac{1}{a^2}\frac{x^3}{3}\right) && (4)\\
&= \frac{1}{a}\left(x-\frac{x^3}{3a^2}\right) && (5)\\
&= \frac{1}{a}\left((a-0)-\frac{1}{a^2}\left(\frac{a^3}{3}-\frac{0^3}{3}\right)\right) && (6)\\
&= \frac{1}{a}\left(a-\frac{a^3}{3a^2}\right) && (7)\\
&= \frac{1}{a}\left(a-\frac{a}{3}\right) && (8)\\
&= \frac{1}{a}\left(\frac{2a}{3}\right) && (9)\\
&= \frac{2}{3} && (10)\\
1-\frac{c^2}{a^2} &= \frac{2}{3} && (11)\\
\frac{c^2}{a^2} &= \frac{1}{3} && (12)\\
c^2 &= \frac{a^2}{3} && (13)\\
c &= \sqrt{\frac{a^2}{3}} && (14)\\
&= \frac{a}{\sqrt{3}} && (15)
\end{aligned}
$$

6. (Section 5.4, Exercise 42)

$$f(x)=\frac{\pi}{4}\sin x$$

$$[0,\pi]$$

$$
\begin{aligned}
\overline{f} &= \frac{1}{\pi}\int_0^\pi \frac{\pi}{4}\sin x && (1)\\
&= \frac{1}{\pi}\frac{\pi}{4}\int_0^\pi \sin x && (2)\\
&= \frac{1}{\pi}\frac{\pi}{4}(-\cos x) && (3)\\
&= \frac{1}{\pi}\frac{\pi}{4}(-\cos\pi+\cos 0) && (4)\\
&= \frac{1}{\pi}\frac{\pi}{4}(1+1) && (5)\\
&= \frac{1}{\pi}\frac{\pi}{2} && (6)\\
&= \frac{1}{2} && (7)\\
\frac{\pi}{4}\sin x &= \frac{1}{2} && (8)\\
\sin x &= \frac{2}{\pi} && (9)\\
\sin^{-1}\sin x &= \sin^{-1}\frac{2}{\pi} && (10)\\
x &= \sin^{-1}\frac{2}{\pi} && (11)
\end{aligned}
$$

7. (Section 5.5, Exercise 17)

$$\begin{aligned}
u &= x^2 - 1 && (1)\\
du &= 2x\,dx && (2)\\
\int 2x\left(x^2-1\right)^{99}\,dx &= \int u^{99}\,du && (3)\\
&= \frac{u^{100}}{100} + C && (4)\\
&= \frac{\left(x^2-1\right)^{100}}{100} + C && (5)
\end{aligned}$$

8. (Section 5.5, Exercise 20)

$$\begin{aligned}
u &= \sqrt{x} + 1 && (1)\\
du &= \frac{1}{2\sqrt{x}}\,dx && (2)\\
\int \frac{\left(\sqrt{x}+1\right)^4}{2\sqrt{x}}\,dx &= \int u^4\,du && (3)\\
&= \frac{u^5}{5} + C && (4)\\
&= \frac{\left(\sqrt{x}+1\right)^5}{5} + C && (5)
\end{aligned}$$

9. (Section 5.5, Exercise 21)

$$\begin{aligned}
u &= x^2 + x && (1)\\
du &= 2x + 1\,dx && (2)\\
\int \left(x^2+x\right)^{10}(2x+1)\,dx &= \int u^{10}\,du && (3)\\
&= \frac{u^{11}}{11} + C && (4)\\
&= \frac{\left(x^2+x\right)^{11}}{11} + C && (5)
\end{aligned}$$

10. (Section 5.5, Exercise 23)

$$\begin{aligned}
u &= x^4 + 16 && (1)\\
du &= 4x^3\,dx && (2)\\
x^3\,dx &= \frac{1}{4}du && (3)\\
\int x^3\left(x^4+16\right)^6\,dx &= \int \frac{1}{4}u^6\,du && (4)\\
&= \frac{1}{4}\int u^6\,du && (5)\\
&= \frac{1}{4}\cdot\frac{u^7}{7} + C && (6)\\
&= \frac{\left(x^4+16\right)^7}{28} + C && (7)
\end{aligned}$$

11. (Section 5.5, Exercise 24)

$$\begin{aligned}
u &= \sin\theta && (1)\\
du &= \cos\theta\,d\theta && (2)\\
\int \sin^{10}\theta\cos\theta\,d\theta &= \int u^{10}\,du && (3)\\
&= \frac{u^{11}}{11} + C && (4)\\
&= \frac{\sin^{11}\theta}{11} + C && (5)
\end{aligned}$$

12. (Section 5.5, Exercise 78)

$$\begin{aligned}
u &= x-2 && (1)\\
x &= u+2 && (2)\\
du &= dx && (3)\\
\int \frac{x}{x-2}\,dx &= \int \frac{u+2}{u}\,du && (4)\\
&= \int \frac{u}{u}\,du + \int \frac{2}{u}\,du && (5)\\
&= u + 2\int \frac{1}{u}\,du + C && (6)\\
&= u + 2\ln|u| + C && (7)\\
&= x - 2 + 2\ln|u| + C && (8)
\end{aligned}$$

13. (Section 5.5, Exercise 79)

$$\begin{aligned}
u &= \sqrt{x-4} && (1)\\
u^2 &= x-4 && (2)\\
x &= u^2+4 && (3)\\
dx &= 2u\,du && (4)\\
\int \frac{x}{\sqrt{x-4}}\,dx &= \int 2u\frac{u^2+4}{u}\,du && (5)\\
&= \int \frac{2u^3+8u}{u}\,du && (6)\\
&= 2\left(\int u^2\,du + \int 4\,du\right) && (7)\\
&= 2\left(\frac{\sqrt{x-4}^3}{3} + 4\sqrt{x-4}\right) + C && (8)\\
&= \frac{2}{3}\sqrt{x-4} + 8\sqrt{x-4} + C && (9)
\end{aligned}$$

14. (Section 5.5, Exercise 15)

$$\begin{aligned}
\int e^{10x}\,dx &= \frac{1}{10}e^{10x} + C && (1)\\
\int \sec 5x \tan 5x\,dx &= \frac{1}{5}\sec 5x + C && (2)\\
\int \sin 7x\,dx &= -\frac{1}{7}\cos 7x + C && (3)\\
\int \cos\frac{x}{7}\,dx &= 7\sin\frac{x}{7} + C && (4)\\
\int \frac{dx}{81+9x^2} &= \int \frac{dx}{9^2+9x^2} && (5)\\
&= \int \frac{dx}{9(9+x^2)} && (6)\\
&= \frac{1}{9}\int \frac{dx}{3^2+x^2} && (7)\\
&= \frac{1}{27}\tan^{-1}\frac{x}{3} + C && (8)\\
\int \frac{dx}{\sqrt{36-x^2}} &= \int \frac{dx}{\sqrt{6^2-x^2}} && (9)\\
&= \sin^{-1}\frac{x}{6} + C && (10)
\end{aligned}$$

15. (Section 5.5, Exercise 16)

$$\begin{aligned}
\int_0^1 10^x\,dx &= \frac{1}{\ln 10}10 - \frac{1}{\ln 10} && (1)\\
&= \frac{9}{\ln 10} && (2)\\
\int_0^{\frac{\pi}{40}} \cos 20x\,dx &= \cos\frac{20\pi}{40} - \cos 20(0) && (3)\\
&= 0-1 && (4)\\
&= -1 && (5)\\
\int_{3\sqrt{2}}^{6} \frac{dx}{x\sqrt{x^2-9}} &= \int_{3\sqrt{2}}^{6} \frac{dx}{x\sqrt{x^2-3^2}} && (6)\\
&= \frac{1}{3}\sec^{-1}\left|\frac{6}{3}\right| - \frac{1}{3}\sec^{-1}\left|\frac{3\sqrt{2}}{3}\right| && (7)\\
&= \frac{1}{3}\sec^{-1}\frac{\pi}{3} - \frac{1}{3}\sec^{-1}\frac{\pi}{4} && (8)\\
\int_0^{\frac{\pi}{16}} \sec^2 4x\,dx &= \frac{1}{4}\tan\frac{4\pi}{16} - \frac{1}{4}\tan 0 && (9)\\
&= \frac{1}{4} - 0 && (10)\\
&= \frac{1}{4} && (11)
\end{aligned}$$

16. (Section 5.5, Exercise 49)

$$\begin{aligned}
u &= 2^x + 4 && (1)\\
du &= 2^x \ln 2\,dx && (2)\\
2^x\,dx &= \frac{1}{\ln 2}\,du && (3)\\
\int_1^3 \frac{2^x}{2^x+4}\,dx &= \frac{1}{\ln 2}\int_6^{12}\frac{1}{u}\,du && (4)\\
&= \frac{1}{\ln 2}(\ln 12 - \ln 6) && (5)\\
&= \frac{1}{\ln 2}\cdot \ln 2 && (6)\\
&= 1 && (7)
\end{aligned}$$

17. (Section 5.5, Exercise 51)

$$\begin{aligned}
u &= \sin\theta && (1)\\
du &= \cos\theta\,d\theta && (2)\\
\int_0^{\frac{\pi}{2}} \sin^2\theta\cos\theta\,d\theta &= \int_0^{\frac{\pi}{2}} u^2\,du && (3)\\
&= \int_0^1 u^2\,du && (4)\\
&= \frac{1^3}{3} - \frac{0^3}{3} && (5)\\
&= \frac{1}{3} && (6)
\end{aligned}$$

18. (Section 5.5, Exercise 64)

$$\begin{aligned}
u &= 3+2e^x && (1)\\
du &= 2e^x\,dx && (2)\\
e^x\,dx &= \frac{1}{2}\,du && (3)\\
\int_0^{\ln 4}\frac{e^x}{3+2e^x}\,dx &= \frac{1}{2}\int_{u=5}^{u=11}\frac{1}{u}\,du && (4)\\
&= \frac{\ln 11-\ln 5}{2} && (5)
\end{aligned}$$

19. (Section 5.5, Exercise 87)

$$\begin{aligned}
u &= 2x && (1)\\
du &= 2\,dx && (2)\\
dx &= \frac{1}{2}\,du && (3)\\
\cos^2 x &= \frac{1+\cos 2x}{2} && (4)\\
\int_{-\pi}^{\pi}\cos^2 x\,dx &= \int_{-\pi}^{\pi}\frac{1}{2}+\frac{\cos 2x}{2}\,dx && (5)\\
&= \frac{1}{2}\int_{-\pi}^{\pi}1+\cos 2x\,dx && (6)\\
&= \frac{1}{2}\left(\int_{-\pi}^{\pi}1\,dx+\frac{1}{2}\int_{-\pi}^{\pi}\cos u\,du\right) && (7)\\
&= \frac{1}{2}\left((\pi-(-\pi))+\frac{1}{2}\int_{u=-2\pi}^{u=2\pi}\cos u\,du\right) && (8)\\
&= \frac{1}{2}\left(2\pi+\frac{1}{2}\left(\sin 2\pi-\sin(-2\pi)\right)\right) && (9)\\
&= \frac{1}{2}\left(2\pi+\frac{1}{2}(0)\right) && (10)\\
&= \frac{2\pi}{2} && (11)\\
&= \pi && (12)
\end{aligned}$$

20. (Section 5.5, Exercise 91)

$$\begin{aligned}
u &= 4\theta && (1)\\
du &= 4\,d\theta && (2)\\
d\theta &= \frac{1}{4}\,du && (3)\\
\sin^2 x &= \frac{1-\cos 2x}{2} && (4)\\
\sin^2 2\theta &= \frac{1-\cos 4\theta}{2} && (5)\\
\int_{-\pi}^{\pi} \sin^2 2\theta\,d\theta &= \int_{-\pi}^{\pi} \frac{1-\cos 4\theta}{2}\,d\theta && (6)\\
&= \int_{-\pi}^{\pi} \frac{1}{2} - \frac{\cos u}{2}\,d\theta && (7)\\
&= \frac{1}{2}\int_{-\pi}^{\pi} 1-\cos u\,d\theta && (8)\\
&= \frac{1}{2}\left(\int_{-\pi}^{\pi} 1\,d\theta - \int_{-\pi}^{\pi} \cos u\,d\theta\right) && (9)\\
&= \frac{1}{2}\left(2\pi - \frac{1}{4}\int_{u=-4\pi}^{u=4\pi} \cos u\,du\right) && (10)\\
&= \frac{1}{2}\left(2\pi - \frac{1}{4}\left(\sin 4\pi - \sin\left(-4\pi\right)\right)\right) && (11)\\
&= \frac{2\pi}{2} && (12)\\
&= \pi && (13)
\end{aligned}$$

A copy of my notes (in $\LaTeX$) are available on my GitHub