Module 7 Notes (MATH-211)

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General Notes (and Definitions)

- Working with Integrals
  A function $f(x)$ is **even** if $f(-x) = f(x)$.
  A function $f(x)$ is **odd** if $f(-x) = -f(x)$.
  Let $a \in \mathbb{R}$ such that $a > 0$ and let f be an integrable function on the interval $[-a, a]$.

$$\text{If } f \text{ is even, } \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

$$\text{If } f \text{ is odd, } \int_{-a}^{a} f(x)\,dx = 0$$

  The average value of an integrable function f on the interval $[a, b]$ is

$$\overline{f} = \frac{1}{b-a}\int_{a}^{b} f(x)\,dx$$

  Let f be continuous on the interval $[a, b]$. There exists a point c in (a, b) such that (Mean Value Theorem)

$$f(c) = \overline{f} = \frac{1}{b-a}\int_{a}^{b} f(t)\,dx$$

- Substitution Rule
  Let $u = g(x)$, where g is differentiable on an interval, and let f be continuous on the corresponding range of g. On that interval,

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

  1. Given an indefinite integral involving a commposite function $f(g(x))$, identify an inner function $u = g(x)$ such that a constant multiple of $g'(x)$ appears in the integrand.
  2. Substitute $u = g(x)$ and $du = g'(x)\,dx$ in the integral.
  3. Evaluate the new indefinite integral with respect to u.
  4. Write the result in terms of x using $u = g(x)$.

  Let $u = g(x)$, where g' is continuous on $[a, b]$, and let f be continuous on the range of g. Then

$$\int_{a}^{b} f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

General formulas for indefinite integrals

$$\begin{aligned}
\int \cos ax\, dx &= \frac{1}{a}\sin ax + C && (1)\\
\int \sin ax\, dx &= -\frac{1}{a}\cos ax + C && (2)\\
\int \sec^2 ax\, dx &= \frac{1}{a}\tan ax + C && (3)\\
\int \csc^2 ax\, dx &= -\frac{1}{a}\cot ax + C && (4)\\
\int \sec ax \tan ax\, dx &= \frac{1}{a}\sec ax + C && (5)\\
\int \csc ax \cot ax\, dx &= -\frac{1}{a}\csc ax + C && (6)\\
\int e^{ax}\, dx &= \frac{1}{a}e^{ax} + C && (7)\\
\int b^x\, dx &= \frac{1}{\ln b}b^x + C, b > 0, b \neq 1 && (8)\\
\int \frac{dx}{a^2 + x^2} &= \frac{1}{a}\tan^{-1}\frac{x}{a} + C && (9)\\
\int \frac{dx}{\sqrt{a^2 - x^2}} &= \sin^{-1}\frac{x}{a} + C, a > 0 && (10)\\
\int \frac{dx}{x\sqrt{x^2 - a^2}} &= \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C, a > 0 && (11)
\end{aligned}$$

Examples

1. Use symmetry to evaluate integrals

$$\int_{-200}^{200} 2x^5\, dx = 0$$

$$\begin{aligned}
\int_{-2}^{2}\left(x^2 + x^3\right)\, dx &= \int_{-2}^{2} x^2\, dx + \int_{-2}^{2} x^3\, dx && (1)\\
&= 2\int_{0}^{2} x^2\, dx + 0 && (2)\\
&= 2\frac{x^3}{3} && (3)\\
&= \frac{16}{3} && (4)
\end{aligned}$$

2. A derivative calculation

$$s(t) = -16t^2 + 64t$$

$$t = 4$$

$$[0, 4]$$

$$\begin{aligned}
v(t) &= s'(t) && (1)\\
\overline{v} &= \frac{1}{4}\int_{0}^{4} v(t)\, dx && (2)\\
&= \frac{1}{4}\int_{0}^{4} s'(t)\, dx && (3)\\
&= \frac{1}{4}s(t) && (4)\\
&= \frac{1}{4}\left(s(4) - s(0)\right) && (5)\\
&= 0 && (6)
\end{aligned}$$

3. Applying MVT for integrals

$$f(x) = e^x$$

$$[0,2]$$

$$\begin{aligned}
\overline{f} &= \frac{1}{2}\left(\int_0^2 e^x \, dx\right) && (1)\\
&= \frac{e^x}{2} && (2)\\
&= \frac{e^2}{2} - \frac{e^0}{2} && (3)\\
&= \frac{e^2-1}{2} && (4)\\
e^x &= \frac{e^2-1}{2} && (5)\\
\ln e^x &= \ln\frac{e^2-1}{2} && (6)
\end{aligned}$$

4. Perfect substitutions in indefinite integrals

$$\begin{aligned}
u &= 4x^3 - 8 && (1)\\
du &= 12x^2 \, dx && (2)\\
\int 12x^2 \left(4x^3-8\right)^5 \, dx &= \int 12x^2 u^5 \, dx && (3)\\
&= \frac{u^6}{6} + C && (4)\\
&= \frac{\left(4x^3-8\right)^6}{6} + C && (5)
\end{aligned}$$

$$\begin{aligned}
u &= \sin t && (1)\\
du &= \cos t \, dt && (2)\\
\int (\cos t)\, e^{\sin t} \, dt &= \int e^u \, du && (3)\\
&= e^u + C && (4)\\
&= e^{\sin t} + C && (5)
\end{aligned}$$

5. Introducting constants when integrating by substitution

$$\begin{aligned}
u &= 6x + 4 && (1)\\
du &= 6 \, dx && (2)\\
dx &= \frac{du}{6} && (3)\\
\int (6x+4)^9 \, dx &= \int \frac{1}{6} \cdot u^9 \, du && (4)\\
&= \frac{1}{6}\int u^8 \, du && (5)\\
&= \frac{1}{6} \cdot \frac{u^9}{9} + C && (6)\\
&= \frac{(6x+4)^9}{54} + C && (7)
\end{aligned}$$

$$\begin{aligned}
u &= \cot x && (1)\\
du &= -\csc^2 x \, dx && (2)\\
\int \cot^2 x \csc^2 x \, dx &= \int -u^2 \, du && (3)\\
&= -\frac{u^3}{3} + C && (4)\\
&= -\frac{\csc^3 x}{3} + C && (5)
\end{aligned}$$

6. Variations on the substitution method

$$\begin{aligned} u &= x-1 && (1)\\ du &= dx && (2)\\ x &= u+1 && (3)\\ \int x\sqrt{x-1}\,dx &= \int (u+1)\sqrt{u}\,du && (4)\\ &= \int u\sqrt{u}+\sqrt{u}\,du && (5)\\ &= \int u^{\frac{3}{2}}+u^{\frac{1}{2}}\,du && (6)\\ &= \frac{2}{5}u^{\frac{5}{2}}+\frac{2}{3}u^{\frac{3}{2}}+C && (7)\\ &= \frac{2}{5}(x-1)^{\frac{5}{2}}+\frac{2}{3}(x-1)^{\frac{3}{2}}+C && (8) \end{aligned}$$

7. Use known formulas to evaluate indefinite integrals

$$\begin{aligned} \int 2e^{-4x}\,dx &= 2\int e^{-4x}\,dx && (1)\\ &= \frac{2}{-4}e^{-4x}+C && (2)\\ &= -\frac{1}{2}e^{-4x}+C && (3)\\ & && (4) \end{aligned}$$

$$\begin{aligned} \int \frac{dx}{\sqrt{36-x^2}} &= \int \frac{dx}{\sqrt{6^2-x^2}} && (1)\\ &= \sin^{-1}\frac{x}{6}+C && (2) \end{aligned}$$

8. Evaluating definite integrals using substitution

$$\begin{aligned} u &= 2^x+4 && (1)\\ du &= 2^x\ln 2\,dx && (2)\\ \frac{1}{\ln 2}du &= 2^x\,dx && (3)\\ \int_1^3 \frac{2^x}{2^x+4}\,dx &= \int_1^3 \frac{1}{u\ln 2}\,du && (4)\\ \int_{g(1)}^{g(3)} \frac{1}{u\ln 2}\,du &= \int_6^{12} \frac{1}{u\ln 2}\,du && (5)\\ &= \frac{1}{\ln 2}\int_6^{12}\frac{du}{u} && (6)\\ &= \frac{1}{\ln 2}\cdot(\ln 12-\ln 6) && (7)\\ &= \frac{\ln 2}{\ln 2} && (8)\\ &= 1 && (9) \end{aligned}$$

$$\begin{aligned} u &= \ln p && (1)\\ du &= \frac{1}{p}\,dx && (2)\\ \int_1^{e^2} \frac{\ln p}{p} &= \int_0^2 u\,du && (3)\\ &= \frac{2^2}{2}-\frac{0^2}{2} && (4)\\ &= \frac{4}{2} && (5)\\ &= 2 && (6) \end{aligned}$$

9. Integrals involving $\cos^2 x$ and $\sin^2 x$

$$\begin{aligned}
u &= 2x && (1)\\
du &= 2\,dx && (2)\\
dx &= \frac{1}{2}du && (3)\\
\sin^2 x &= \frac{1-\cos 2x}{2} && (4)\\
\int_0^\pi \sin^2 x\,dx &= \int_0^\pi \frac{1-\cos 2x}{2}\,dx && (5)\\
&= \frac{1}{2}\int_0^\pi 1-\cos 2x\,dx && (6)\\
&= \frac{1}{2}\left(\int_0^\pi 1\,dx - \int_0^\pi \cos 2x\,dx\right) && (7)\\
&= \frac{1}{2}\left((\pi-0) - \frac{1}{2}\int_0^{2\pi} \cos u\,du\right) && (8)\\
&= \frac{1}{2}\left(\pi - \frac{1}{2}(\sin 2\pi - \sin 0)\right) && (9)\\
&= \frac{1}{2}(\pi - 0) && (10)\\
&= \frac{\pi}{2} && (11)
\end{aligned}$$

Related Exercises

1. (Section 5.4, Exercise 15)

$$\begin{aligned}
\int_{-2}^{2}\left(x^2+x^3\right)\,dx &= \int_{-2}^{2} x^2\,dx + \int_{-2}^{2} x^3\,dx && (1)\\
&= 2\int_0^2 x^2\,dx + 0 && (2)\\
&= 2\frac{x^3}{3} && (3)\\
&= 2\frac{2^3}{3} - 2\frac{0^3}{3} && (4)\\
&= 2\frac{8}{3} && (5)\\
&= \frac{16}{3} && (6)
\end{aligned}$$

2. (Section 5.4, Exercise 16)

$$\int_{-\pi}^{\pi} t^2 \sin t\,dx = 0$$

3. (Section 5.4, Exercise 26)

$$f(x) = x^2 + 1$$

$$[-2, 2]$$

$$\begin{aligned}
\overline{f} &= \frac{1}{2-(-2)}\int_{-2}^{2} x^2+1\,dx && (1)\\
&= \frac{1}{4}\left(\int_{-2}^{2} x^2\,dx + 1\int_{-2}^{2} x^0\,dx\right) && (2)\\
&= \frac{1}{4}\left(\frac{x^3}{3}+x\right) && (3)\\
&= \frac{1}{4}\left(\int_{-2}^{2} x^2\,dx + \int_{-2}^{2} 1\,dx\right) && (4)\\
&= \frac{1}{4}\left(\frac{2^3}{3}-\frac{(-2)^3}{3}+2-(-2)\right) && (5)\\
&= \frac{1}{4}\left(\frac{8}{3}-\frac{-8}{3}+4\right) && (6)\\
&= \frac{1}{4}\left(\frac{16}{3}+4\right) && (7)\\
&= \frac{1}{4}\left(\frac{28}{3}\right) && (8)\\
&= \frac{7}{3} && (9)
\end{aligned}$$

4. (Section 5.4, Exercise 34)

$$f(x) = x^3 - 5x^2 + 30$$

$$[0,4]$$

$$\begin{aligned}
\overline{f} &= \frac{1}{4}\left(\int_{0}^{4}\left(x^3-5x^2+30\right)\,dx\right) && (1)\\
&= \frac{1}{4}\left(\int_{0}^{4} x^3 - 5\int_{0}^{4} x^2 + 30\int_{0}^{4} x^0\right) && (2)\\
&= \frac{1}{4}\left(\frac{x^4}{4}-5\frac{x^3}{3}+30x\right) && (3)\\
&= \frac{1}{4}\left(\left(\frac{4^4}{4}-\frac{0^4}{4}\right)-\left(5\frac{4^3}{3}-5\frac{0^3}{3}\right)+(30(4)-30(0))\right) && (4)\\
&= \frac{1}{4}\left(64-\frac{320}{3}+120\right) && (5)\\
&= \frac{1}{4}\left(\frac{232}{3}\right) && (6)\\
&= \frac{58}{3} && (7)
\end{aligned}$$

5. (Section 5.4, Exercise 41)

$$f(x) = 1 - \frac{x^2}{a^2}$$

$$[0,a]$$

$$
\begin{aligned}
\overline{f} &= \frac{1}{a}\left(\int_0^a 1-\frac{x^2}{a^2}\,dx\right) && (1)\\
&= \frac{1}{a}\left(\int_0^a 1\,dx-\int_0^a \frac{x^2}{a^2}\,dx\right) && (2)\\
&= \frac{1}{a}\left(x-\frac{1}{a^2}\int_0^a x^2\,dx\right) && (3)\\
&= \frac{1}{a}\left(x-\frac{1}{a^2}\frac{x^3}{3}\right) && (4)\\
&= \frac{1}{a}\left(x-\frac{x^3}{3a^2}\right) && (5)\\
&= \frac{1}{a}\left((a-0)-\frac{1}{a^2}\left(\frac{a^3}{3}-\frac{0^3}{3}\right)\right) && (6)\\
&= \frac{1}{a}\left(a-\frac{a^3}{3a^2}\right) && (7)\\
&= \frac{1}{a}\left(a-\frac{a}{3}\right) && (8)\\
&= \frac{1}{a}\left(\frac{2a}{3}\right) && (9)\\
&= \frac{2}{3} && (10)\\
1-\frac{c^2}{a^2} &= \frac{2}{3} && (11)\\
\frac{c^2}{a^2} &= \frac{1}{3} && (12)\\
c^2 &= \frac{a^2}{3} && (13)\\
c &= \sqrt{\frac{a^2}{3}} && (14)\\
&= \frac{a}{\sqrt{3}} && (15)
\end{aligned}
$$

6. (Section 5.4, Exercise 42)

$$f(x)=\frac{\pi}{4}\sin x$$

$$[0,\pi]$$

$$
\begin{aligned}
\overline{f} &= \frac{1}{\pi}\int_0^\pi \frac{\pi}{4}\sin x && (1)\\
&= \frac{1}{\pi}\frac{\pi}{4}\int_0^\pi \sin x && (2)\\
&= \frac{1}{\pi}\frac{\pi}{4}(-\cos x) && (3)\\
&= \frac{1}{\pi}\frac{\pi}{4}(-\cos\pi+\cos 0) && (4)\\
&= \frac{1}{\pi}\frac{\pi}{4}(1+1) && (5)\\
&= \frac{1}{\pi}\frac{\pi}{2} && (6)\\
&= \frac{1}{2} && (7)\\
\frac{\pi}{4}\sin x &= \frac{1}{2} && (8)\\
\sin x &= \frac{2}{\pi} && (9)\\
\sin^{-1}\sin x &= \sin^{-1}\frac{2}{\pi} && (10)\\
x &= \sin^{-1}\frac{2}{\pi} && (11)
\end{aligned}
$$

A copy of my notes (in LaTeX) are available on my GitHub