# Final Review (MATH-211)

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## Applications of Derivatives (M05)

- Maxima and Minima
  **Absolute Maximum:** Assume a function $f$ is defined on a set $D$, and $x = c$ is a point in $D$. Then, $y = f(c)$ is an **absolute maximum value** of $f$ on $D$ if $f(c) \geq f(x)$ for every $x$ in $D$. Changing the set on which $f$ is defined may change the absolute maximum value.
  **Absolute Minimum**: Assume a function $f$ is defined on a set $D$, and $x = c$ is a point in $D$. Then, $y = f(c)$ is an **absolute minimum value** of $f$ on $D$ if $f(c) \leq f(x)$ for every $x$ in $D$. Changing the set on which $f$ is defined may change the absolute minimum value.
  **Extreme Value Theorem**: A function that is continuous on a closed interval is guarenteed to have both an absolute maximum value and an absolute minmum value.
  A discontinuous function, or a function defined on an interval that is not closed, may still have absolute extrema.
  **Local Maximum and Minimum Values**: Assume $x = c$ is an interior point (not an endpoint) of some interval $I$ in the domain of $f$. Then, $y = f(c)$ is a **local maximum value** of $f$ if $f(c) \geq f(x)$ for every $x$ in $I$, and $y = f(c)$ is a **local minimum value** of $f$ if $f(c) \leq f(x)$ for every $x$ in $I$.
  **Critical Points**: An interior point $x = c$ of the domain of $f$ is called a **critical point** of $f$ if either $f'(c) = 0$ or $f'(c)$ does not exist.
  **Local Extreme Value Theorem**: If a function $f$ has a local maximum or a local minimum at a point $x = c$, then either $f'(c) = 0$ or $f'(c)$ does not exist.
  If $f$ has a local extreme, it must occur at a critical point.
  Not every critical point is the location of a local extreme value.
  For a continuous function $f$ on a closed interval $[a, b]$, absolute extremes are guaranteed to exist, and they must occur either at the endpoints of interval or at critical points of $f$ within the interval.

- Mean Value Theorem
  **Rolle's Theorem**: Let $f$ be a continuous function on a closed interval $[a, b]$ that is differentiable on $(a, b)$, with $f(a) = f(b)$. Then, there is at least one point $x = c$ in $(a, b)$ where $f'(c) = 0$.
  **Mean Value Theorem**: If $f$ is a continuous function on a closed interval $[a, b]$ that is differentiable on $(a, b)$, then there is at least one point $x = c$ in $(a, b)$ where

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

  **Zero Derivative Implies Constant Function**: If $f$ is differentiable on an open interval $I$, and $f'(x) = 0$ for all $x$ in $I$, then $f$ is a constant function on $I$.
  **Function with Equal Derivative Differ by a Constant**: If $f'(x) = g'(x)$ for all $x$ in an open interval $I$, then $f(x) = g(x) + C$ for some constant $C$.

- What Derivatives Tell Us
  **Increasing and Decreasing Functions**: Suppose a function $f$ is defined on an interval $I$. We say $f$ is **increasing** on $I$ if $f(x_2) > f(x_1)$ whenever $x_1$ and $x_2$ are in $I$ and $x_2 > x_1$, and we say $f$ is decreasing on $I$ if $f(x_2) < f(x_1)$ whenever $x_1$ and $x_2$ are in $I$ and $x_2 > x_1$.
  **Test for Interals of Increase and Decrease**: Suppose a function $f$ is defined on an interval $I$, and differentiable inside $I$. If $f'(x) > 0$ at all interior points of $I$, then $f$ is increasing on $I$; If $f'(x) < 0$ at all interior points of $I$, then $f$ is decreasing on $I$.
  **First Derivative Test**: Assume $f$ is continuous on an interval containing a critical point $c$, and that $f$ is differentiable on an interval containing $c$ (except possible at $c$ itself). Under these conditions:

  - If $f'$ changes sign from positive to negative as $x$ increases through $c$, then $f$ has a local maximum at $c$.

- If $f'$ changes sign from negative to positive as $x$ increases through $c$, then $f$ has a local minimum at $c$.
- If $f'$ is positive on both sides of $c$, or negative on both sides of $c$, then $f$ has no local extreme value at $c$.

**One local extremum implies absolute extremum**: Suppose $f$ is continuous on an interval $I$ that contains exactly one local extremum $x = c$.

- If $f$ has a local max at $c$, then $f(c)$ is the absolute max of $f$ on $I$.
- If $f$ has a local min at $c$, then $f(c)$ is the absolute min of $f$ on $I$.

**Concavity**: Suppose a function $f$ is twice differentiable on an open interval $I$.

- If $f'$ is increasing on $I$, then $f$ is **concave up** on $I$, and $f'' > 0$ on $I$.
- If $f'$ is decreasing on $I$, then $f$ is **concave down** on $I$, and $f'' < 0$ on $I$.

**Inflection Point**: Suppose a function $f$ is twice differentiable on an open interval $I$. If $f$ is continuous at a point $c$ in $I$ and $f$ changes concavity at $c$, then $f$ has an **inflection point** at $c$.
**Second Derivative Test**: Assume $f''$ is continuous on an open interval containing $x = c$, with $f'(c) = 0$. Under these conditions:

- If $f''(c) > 0$, then $f$ has a local minimum at $c$.
- If $f''(c) < 0$, then $f$ has a local maximum at $c$.
- If $f''(c) = 0$, then the test is inconclusive; $f$ may have a local minimum, a local maximum, or neither of these at $x = c$.

- Graphing Functions
  **Graphing guidelines for a function $f(x)$**:

  1. **Identify the domain of $f$, or intervals of interest.** You need to find out on which intervals the function should be graphed.
  2. **Consider symmetry.** It can be helpful to determine if the function is even, odd, or neither.
  3. **Find formulas for the first and second derivatives of $f$.**
  4. **Find all critical points and possible inflection points.** Within the domain of $f$, critical points are points at which $f' = 0$ or $f'$DNE, and possible inflection points are points at which $f'' = 0$ or $f''$DNE.
  5. **Find intervals on which $f$ is increasing or decreasing, and intervals on which $f$ is concave up or concave down.** Together with discontinuities of $f$, use the critical points of $f$ to make a sign graph for $f'$, and use the possible inflection points of $f$ to make a sign graph for $f''$.
  6. **Identify local extrema and inflection points.** You can get this information from the sign graphs you already made for $f'$ and $f''$. To help graph $f$, you need both the $x$ and $y$-coordinates of these points.
  7. **Locate asymptotes and determine end behaviour.** Vertical asymptotes often occur at zeros of the denominator of $f$. Determine the end behaviour by evaluating limits of $f$ as $x \to \pm\infty$; if either limit exists, $f$ has a horizontal asymptote.
  8. **Find the $x$ and $y$ intercepts of $f$.**
  9. **Plot the graph on an appropriate window.** Be sure that your graph is scaled to clearly show all the important details of the function.

- Optimization
  Goal: Find absolute max/min of a given function called the **objective function**
  New: Applied problems can introduce **constraints** (restrictions) on the variables. This could change the results of the optimization of the objective function.
  Guidelines:

  1. Read the problem carefully, organize the information in a picture, and identify the variables.
  2. Identify the function to be optimized (the objective function), and write this function in terms of the variables in the problem.
  3. Identify all the constraints, and write each of them in terms of the variables in the problem.

4. Use the constraints to rewrite the objective function in terms of only one variable.
5. Identify the appropriate interval of interest for the remaining variable.
6. Use calculus methods to find the absolute maximum and/or absolute minimum value of the constrained objective function on the interval of interest, possible including at endpoints.

## L'Hôpital's Rule and Techniques of Integration (M06)

- L'Hôpital's Rule

  **Indeterminate Form**: An expression involving two components where the limit cannot be determined by evaluating the limits of the individual components.

  **L'Hôpital's Rule**: Suppose $f$ and $g$ are differentiable functions on an open interval $I$ containing the point $x = a$, with $g'(x) \neq 0$ on $I$ when $x \neq a$.

  If $\lim_{x\to a} \frac{f(x)}{g(x)}$ has any of the indeterminate forms: $\frac{0}{0}$, $\frac{\infty}{\infty}$, $-\frac{\infty}{\infty}$, then

  $$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

  provided that one of the following is the case:

  $$\lim_{x\to a} \frac{f'(x)}{g'(x)} \in \mathbb{R}$$

  $$\lim_{x\to a} \frac{f'(x)}{g'(x)} = \infty$$

  $$\lim_{x\to a} \frac{f'(x)}{g'(x)} = -\infty$$

  L'Hôpital's Rule is still valid if $x \to a$ is replaced by any of $x \to a^+$, $x \to a^-$, $x \to \infty$, or $x \to -\infty$. In the last two of these cases, there must be a greatest $x$-value beyond which both $f$ and $g$ are differentiable at every point.

  **Exponential Indeterminate forms**: $1^\infty$, $0^0$, $\infty^0$

  **Method for evaluating limits of indeterminate forms $1^\infty$, $0^0$, $\infty^0$:**

  Assume that $L = \lim_{x\to a} f(x)^{g(x)}$ has one of these indeterminate forms.

  1. Use the fact that the natural logarithm and natural exponential functions are inverses to write

     $$L = \lim_{x\to a} e^{\ln\left(f(x)^{g(x)}\right)}$$

  2. Use the power property of logarithm arguments to write

     $$L = \lim_{x\to a} e^{g(x)\ln(f(x))}$$

  3. Use continuity of the exponential function to write

     $$L = e^{\lim_{x\to a} g(x)\ln(f(x))}$$

  4. Rewrite multiplication as division by the reciprocal:

     $$L = e^{\lim_{x\to a}\left(\frac{\ln(f(x))}{\frac{1}{g(x)}}\right)}$$

  5. Use L'Hôpital's Rule to evaluate this limit expression

  **Growth Rates**: Suppose $f$ and $g$ are functions with $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$

  1. If one of the following are true, $f$ **grows faster than** $g$, and we use the notation $f \gg g$

     $$\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0 \tag{1}$$

     $$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty \tag{2}$$

2. $f$ **and** $g$ **have comparable growth rates**, if there is some non-zero finite number $M$ such that

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = M$$

**Ranked Growth Rates as** $x \to \infty$
For any base $b > 1$, and for any positive numbers $p$, $q$, $r$, and $s$

$$\ln^q x \ll x^p \ll x^p \ln^r x \ll x^{p+s} \ll b^x \ll x^x$$

- Antiderivatives
  **Antiderivative**: A function $F$ is an antiderivative of another function $f$ on an interval $I$ if for all $x$ in $I$:

  $$F'(x) = f(x)$$

  **Family of Antiderivatives**: Let $F(x)$ be any antiderivative of $f(x)$ on an interval $I$. Then all antiderivatives of $f$ on $I$ have the form $F(x) + C$, where $C$ is an arbitrary constant.
  **Differential Equations**: Any equation involving an unknown function and its derivatives

  – Infinite family of solutions
  – No two solutions from the family pass through the same point
  – Given an initial condition $f(a) = b$, we can identify the particular family member that solves the given problem by solving for $C$

- Approximating Areas Under Curves

  – If we know the velocity function of a moving object, what can we learn about its position function?
  – Given an object with velocity function $v(t)$, the displacement of the moving object over the interval $[a, b]$ is the area between the velocity curve and the $t$-axis from $t = a$ to $t = b$.
  – Because objects do not necessarily move at a constant velocity, we can extend this idea to positive velocities that change over an interval of time.
  – The strategy is to divide the time interval into many subintervals, approximate the velocity on each subinterval with a constant velocity, calculate the individual displacements and sum the results.

  **Riemann Sums**

  – Suppose $f(x)$ is continuous and non-negative on $[a, b]$.
  – Goal is to approximate the area of the region $R$ bounded by the graph of $f(x)$ and the $x$-axis from $x = a$ to $x = b$.
  – Divide $[a, b]$ into $n$ subintervals $[x_0, x_1], [x_1, x_2], ..., [x_{n-1}, x_n]$ where $a = x_0, b = x_n$.
  – The length of each subinterval is $\Delta x = \frac{b-a}{n}$
  – **Regular Partition**: Suppose $[a, b]$ is a closed interval containing $n$ subintervals

  $$[x_0, x_1], [x_1, x_2], ..., [x_{n-1}, x_n]$$

  of equal length $\Delta x = \frac{b-a}{n}$, with $a = x_0$ and $b = x_n$. The endpoints $x_0, x_1, x_2, ..., x_{n-1}, x_n$ of the subintervals are called **grid points**, and they create a **regular partition** of the interval $[a, b]$. In general the $k$th grid point is

  $$x_k = a + k\Delta x, \text{ for } k = 0, 1, 2, ..., n$$

  – In the $k$th subinterval $[x_{k-1}, x_k]$, choose any point $x_k^*$ and build a rectangle whose height is $f(x_k^*)$.
  – The area of the rectangle of the $k$th subinterval is

  $$\text{height} \cdot \text{base} = f(x_k^*)\Delta x, \text{ where } k = 1, 2, ..., n$$

  – Summing the areas of these rectangles, we obtain an approximation to the area of $R$, which is called a **Riemann sum**:

  $$f(x_1^*)\Delta x + f(x_2^*)\Delta x + ... + f(x_n^*)\Delta x$$

  – Three notable Riemann sums are the left, right, and midpoint Riemann sums.

**Riemann Sum**: Suppose $f$ is defined on a closed interval $[a,b]$, which is divided into $n$ subintervals of equal length $\Delta x$. If $x_k^*$ is any point in the $k$th subinterval $[x_{k-1},x_k]$, for $k=1,2,...,n$, then

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + ... + f(x_n^*)\Delta x$$

is called a **Riemann sum** for $f$ on $[a,b]$. This sum is called

- a **left Riemann sum** if $x_k^*$ is the left endpoint of $[x_{k-1},x_k]$
- a **right Riemann sum** if $x_k^*$ is the right endpoint of $[x_{k-1},x_k]$
- a **midpoint Riemann sum** if $x_k^*$ is the midpoint of $[x_{k-1},x_k]$

**Summation notation ($\Sigma$)**:

- Working with Riemann sums is cumbersome when $n$ is large
- We introduce sigma (summation) notation as a shorthand:

$$1+2+...+49+50=\sum_{k=1}^{50} k$$

- The symbol $\Sigma$ (sigma) stands for sum
- $k$ is the index, and takes on all integer values from $k=1$ to $k=50$
- The expression immediately following $\Sigma$, the summand, is evaluated for each $k$, and the resulting values are summed
- The index is a dummy variable, and it does not matter which symbol is chosen for the index:

$$\sum_{k=1}^{99} k=\sum_{n=1}^{99} n=\sum_{p-1}^{99} p$$

- Two Properties of Sums and Sigma Notation
  1. Constant Multiple Rule:
  $$\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$$
  2. Addition Rule:
  $$\sum_{k=1}^{n} (a_k+b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$
- **Theorem**: Sums of Power of Integers
  Let $n \in \mathbb{Z}$ such that $n>0$ and $c \in \mathbb{R}$

$$\sum_{k=1}^{n} c = cn \quad (1)$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad (2)$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \quad (3)$$

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4} \quad (4)$$

**Left, Right, and Midpoint Riemann Sums in Sigma Notation**:
Suppose $f$ is defined on a closed interval $[a,b]$, which is divided into subintervals of equal length $\Delta x$. If $x_k^*$ is a point in the $k$th subinterval $[x_{k-1},x_k]$, for $k=1,2,...,n$, then the **Riemann sum** for $f$ on $[a,b]$ is

$$\sum_{k=1}^{n} f(x_k^*)\Delta x$$

Three cases arise in practice

- $\sum_{k=1}^{n} f(x_k^*)\Delta x$ is a **left Riemann sum** if $x_k^* = a + (k-1)\Delta x$
- $\sum_{k=1}^{n} f(x_k^*)\Delta x$ is a **right Riemann sum** if $x_k^* = a + k\Delta x$
- $\sum_{k=1}^{n} f(x_k^*)\Delta x$ is a **midpoint Riemann sum** if $x_k^* = a + (k-\frac{1}{2})\Delta x$

- Definite Integrals
  **Net Area**: Consider the region $R$ bounded by the graph of a continuous function $f$ and the $x$-axis between $x=a$ and $x=b$. The **net area** of $R$ is the sum of the area of the parts of $R$ that lie above the $x$-axis minus the sum of the areas of the parts of $R$ that lie below the $x$-axis on $[a,b]$.
  - Where $f(x) < 0$, Riemann sums approximate the negative of the area of the region bounded by the curve
  - On the interval $[a,b]$, we get positive, and negative contributions to the Riemann sum where $f(x)$ is negative
  - Riemann sums approximate the area of the regions that lie above the $x$-axis minus the area of the regions that lie below the $x$-axis
  - The difference is called the **net area**; it can be positive, negative, or zero

$$area_{net} = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

A **general partition** of $[a,b]$ consists of the $n$ subintervals

$$[x_0, x_1], [x_1, x_2], ..., [x_{n-1}, x_n]$$

where $x_0 = a$and $x_n = b$. The length of the $k$th subinterval is $\Delta x_k = x_k - k_{k-1}$, for $k = 1, 2, ..., n$. We let $x_k^*$ be any point in the subinterval $[x_{k-1}, x_k]$.
**General Riemann Sum**: Suppose $[x_0, x_1], [x_1, x_2], ..., [x_{n-1}, x_n]$ are subintervals of $[a,b]$ with

$$a = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = b$$

Let $x\Delta x_k$ be the length of the subinterval $[x_{k-1}, x_k]$ and let $x_k^*$ be any point in $[x_{k-1}, x_k]$, for $k = 1, 2, ...n$
If $f$ is defined on $[a,b]$, the sum

$$\sum_{k=1}^{n} f(x_k^*)\Delta x_k = f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + ... + f(x_n^*)\Delta x_n$$

is called a **general Riemann sum** for $f$ on $[a,b]$
**Definite Integral**: A function $f$ defined on $[a,b]$ is **integrable** on $[a,b]$ if $\lim_{\Delta x\to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$ exists and is unique over all partitions of $[a,b]$ and all choices of $x_k^*$ on a parition. This limit is the **definite integral of $f$ from a to b**, which we rite

$$\int_a^b f(x)dx = \lim_{\Delta x\to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

**Integrable Functions**: If $f$ is continuous on $[a,b]$ or bounded on $[a,b]$ with a finite number of discontinuities, then $f$ is integrale on $[a,b]$.
Let $f$ and $g$ be integrable function on $[a,b]$, where $b > a$

1. If $f(x) \geq 0$ on $[a,b]$, then $\int_a^b f(x)\,dx \geq 0$
2. If $f(x) \geq g(x)$ on $[a,b]$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$
3. If $m \leq f(x) \leq M$, then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$

- Fundamental Theorem of Calculus
  Area Functions: Let $f$ be a continuous function, for $t \geq a$. The **area function for $f$ with left endpoint $a$** is

$$A(x) = \int_a^x f(t)\,dt$$

where $x \geq a$. The area function gives the net area of the region bounded by the graph of $f$ and the $t$-axis on the interval $[a, x]$.
If $f$ is continuous on $[a, b]$, then the area function

$$A(x) = \int_a^x f(t)\,dt, \text{ for } a \leq x \leq b,$$

is continuous on $[a, b]$ and differentiable on $(a, b)$. The area function satisfies $A'(x) = f(x)$. Equivalently,

$$A'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x),$$

which means that the area function of $f$ is an antiderivative of $f$ on $[a, b]$.
If $f$ is continuous on $[a, b]$ and $F$ is any antiderivative of $f$ on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

# Applications of Integration (M07)

- Working with Integrals
  A function $f(x)$ is **even** if $f(-x) = f(x)$.
  A function $f(x)$ is **odd** if $f(-x) = -f(x)$.
  Let $a \in \mathbb{R}$ such that $a > 0$ and let $f$ be an integrable function on the interval $[-a, a]$.

  $$\text{If } f \text{ is even, } \int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$$

  $$\text{If } f \text{ is odd, } \int_{-a}^a f(x)\,dx = 0$$

  The average value of an integrable function $f$ on the interval $[a, b]$ is

  $$\overline{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

  Let $f$ be continuous on the interval $[a, b]$. There exists a point $c$ in $(a, b)$ such that (Mean Value Theorem)

  $$f(c) = \overline{f} = \frac{1}{b-a}\int_a^b f(t)\,dx$$

- Substitution Rule
  Let $u = g(x)$, where $g$ is differentiable on an interval, and let $f$ be continuous on the corresponding range of $g$. On that interval,

  $$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

  1. Given an indefinite integral involving a commposite function $f(g(x))$, identify an inner function $u = g(x)$ such that a constant multiple of $g'(x)$ appears in the integrand.
  2. Substitute $u = g(x)$ and $du = g'(x)\,dx$ in the integral.
  3. Evaluate the new indefinite integral with respect to $u$.
  4. Write the result in terms of $x$ using $u = g(x)$.

  Let $u = g(x)$, where $g'$ is continuous on $[a, b]$, and let $f$ be continuous on the range of $g$. Then

  $$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

- Velocity and Net Change
  **Position, Velocity, Displacement, and Distance**:

  1. The **position** of an object moving along a line at time $t$, denoted $s(t)$, is the location of the object relative to the origin.

2. The **velocity** of an object at time $t$ is $v(t) = s'(t)$.
3. The **displacement** of the object between $t = a$ and $t = b > a$ is

$$s(b) - s(a) = \int_a^b v(t)\,dt$$

4. The **distance traveled** by the object between $t = a$ and $t = b > a$ is

$$\int_a^b |v(t)|\,dt$$

where $|v(t)|$ is the **speed** of the object at time $t$.

**Theorem: Position from Velocity**
Given the velocity $v(t)$ of an object moving along a line and its initial position $s(0)$, the position function of the object for future times $t \geq 0$ is

$$s(t) = s(0) + \int_0^t v(x)\,dx$$

**Theorem: Velocity from Acceleration**
Given the acceleration $a(t)$ of an object moving along a line and its initial velocity $v(0)$, the velocity of the object for future times $t \geq 0$ is

$$v(t) = v(0) + \int_0^t a(x)\,dx$$

**Theorem: Net Change and Future Value**
Suppose a quantity $Q$ changes over time at a known rate $Q'$. Then the **net change** in $Q$ between $t = a$ and $t = b > a$ is

$$Q(b) - Q(a) = \int_a^b Q'(t)\,dt$$

Given the initial value $Q(0)$, the **future value** of $Q$ at time $t \geq 0$ is

$$Q(t) = Q(0) + \int_0^t Q'(x)\,dx$$

- Area Between Curves
**Area of a Region Between Two Curves**:
Suppose that $f$ and $g$ are continuous functions with $f(x) \geq g(x)$ on the interval $[a, b]$. The area of the region bounded by the graphs of $f$ and $g$ on $[a, b]$ is

$$A = \int_a^b (f(x) - g(x))\,dx$$

**Area of a Region Between Two Curves with Respect to $y$**:
Suppose that $f$ and $g$ are continuous functions with $f(y) \geq g(y)$ on the interval $[c, d]$. The area of the region bounded by the graphs $x = f(y)$ and $x = g(y)$ on $[c, d]$ is

$$A = \int_c^d (f(y) - g(y))\,dy$$

- Volume by Slicing
**General Slicing Method**:
Suppose a solid object extends from $x = a$ to $x = b$ and the cross section of the solid perpendicular to the $x$-axis has an area given by a function $A$ that is integrable on $[a, b]$. The volume of the solid is

$$V = \int_a^b A(x)\,dx$$

**Disk Method about the $x$-Axis**:
Let $f$ be continuous with $f(x) \geq 0$ on the interval $[a, b]$. If the region $R$ bounded by the graph of $f$, the $x$-axis, and the lines $x = a$ and $x = b$ is revolved about the $x$-axis, the volume of the resulting solid of revolution is

$$V = \int_a^b \pi\, f(x)^2\,dx$$

**Washer Method about the $x$-Axis**:
Let $f$ and $g$ be continuous functions with $f(x) \geq g(x) \geq 0$ on $[a, b]$. Let $R$ be the region bounded by $y = f(x)$, $y = g(x)$, and the lines $x = a$ and $x = b$. When $R$ is revolved about the $x$-axis, the volume of the resulting solid of revolution is

$$V = \int_a^b \pi \left( f(x)^2 - g(x)^2 \right) \, dx$$

**Disk and Washer Methods about the $y$-Axis**:
Let $p$ and $q$ be continuous functions with $p(y) \geq q(y) \geq 0$ on $[c, d]$. Let $R$ be the region bounded by $x = p(y)$, $x = q(y)$, and the lines $y = c$ and $y = d$. When $R$ is revolved about the $y$-axis, the volume of the resulting solid of revolution is given by

$$V = \int_c^d \pi \left( p(y)^2 - q(y)^2 \right) \, dy$$

If $q(y) = 0$, the disk method results:

$$V = \int_c^d \pi \, p(y)^2 \, dy$$

- Volume by Shells
  **Volume by the Shell Method**:
  Let $f$ and $g$ be continuous functions with $f(x) \geq g(x)$ on $[a, b]$. If $R$ is the region bounded by the curves $y = f(x)$ and $y = g(x)$ between the lines $x = a$ and $x = b$, the volume of the solid generated when $R$ is revolved about the $y$-axis is

$$V = \int_a^b 2\pi \, x \, (f(x) - g(x)) \, dx$$

# Antiderivative Rules

- Power Rule
  If $p \neq -1$ and $C$ is an arbitrary constant:

$$\int x^p dx = \frac{x^{p+1}}{p+1} + C$$

- Integral of $x^{-1}$

$$\int x^{-1} dx = \int \frac{1}{x} dx = \ln |x| + C$$

- Constant Multiple and Sum Rules
  If $c \in \mathbb{R}$:

$$\int cf(x)dx = c \int f(x)dx$$

$$\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$$

- Integral of $e^x$

$$\int e^x dx = e^x + C$$

- Integral of $\frac{1}{x}$

$$\int \frac{1}{x} \, dx = \ln |x| + C$$

## Trigonometric (and inverse) Integrals

$$\int \cos(x)dx = \sin x + C \tag{1}$$

$$\int \sin(x)dx = -\cos x + C \tag{2}$$

$$\int \sec^2(x)dx = \tan x + C \tag{3}$$

$$\int \csc^2(x)dx = -\cot x + C \tag{4}$$

$$\int \sec(x)\tan(x)dx = \sec x + C \tag{5}$$

$$\int \csc(x)\cot(x)dx = -\csc x + C \tag{6}$$

$$\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x + C \tag{7}$$

$$\int \frac{1}{1+x^2}dx = \tan^{-1} x + C \tag{8}$$

$$\int \frac{1}{x\sqrt{x^2-1}}dx = \sec^{-1}|x| + C \tag{9}$$

## Properties of Definite Integrals

Let $f$ and $g$ be integrable functions on an interval that contains $a$, $b$, and $p$

$$\int_a^a f(x)\,dx = 0 \tag{1}$$

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx \tag{2}$$

$$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx \tag{3}$$

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx, \text{ for any constant } c \tag{4}$$

$$\int_a^b f(x)\,dx = \int_a^p f(x)\,dx + \int_p^b f(x)\,dx \tag{5}$$

The function $|f|$ is integrable on $[a,b]$, and $\int\limits_a^b |f(x)|\,dx$ is the sum of the areas of the regions bounded by the graph of $f$ and the $x$-axis on $[a,b]$.

## General formulas for indefinite integrals

$$\int \cos ax\, dx = \frac{1}{a}\sin ax + C \tag{1}$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax + C \tag{2}$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax + C \tag{3}$$

$$\int \csc^2 ax\, dx = -\frac{1}{a}\cot ax + C \tag{4}$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax + C \tag{5}$$

$$\int \csc ax \cot ax\, dx = -\frac{1}{a}\csc ax + C \tag{6}$$

$$\int e^{ax}\, dx = \frac{1}{a}e^{ax} + C \tag{7}$$

$$\int b^x\, dx = \frac{1}{\ln b}b^x + C, b > 0, b \neq 1 \tag{8}$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C \tag{9}$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C, a > 0 \tag{10}$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C, a > 0 \tag{11}$$

A copy of my notes (in LaTeX) are available on my GitHub