Module 6 Notes (MATH-211)

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General Notes (and Definitions)

- L'Hôpital's Rule
  Indeterminate Form: An expression involving two components where the limit cannot be determined by evaluating the limits of the individual components.
  L'Hôpital's Rule: Suppose f and g are differentiable functions on an open interval I containing the point $x = a$, with $g'(x) \neq 0$ on I when $x \neq a$.
  If $\lim\limits_{x \to a} \frac{f(x)}{g(x)}$ has any of the indeterminate forms: $\frac{0}{0}$, $\frac{\infty}{\infty}$, $-\frac{\infty}{\infty}$, then

  $$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

  provided that one of the following is the case:

  $$\lim_{x \to a} \frac{f'(x)}{g'(x)} \in \mathbb{R}$$

  $$\lim_{x \to a} \frac{f'(x)}{g'(x)} = \infty$$

  $$\lim_{x \to a} \frac{f'(x)}{g'(x)} = -\infty$$

  L'Hôpital's Rule is still valid if $x \to a$ is replaced by any of $x \to a^+$, $x \to a^-$, $x \to \infty$, or $x \to -\infty$. In the last two of these cases, there must be a greatest x-value beyond which both f and g are differentiable at every point.
  Exponential Indeterminate forms: 1^∞, 0^0, ∞^0
  Method for evaluating limits of indeterminate forms 1^∞, 0^0, ∞^0:
  Assume that $L = \lim\limits_{x \to a} f(x)^{g(x)}$ has one of these indeterminate forms.

  1. Use the fact that the natural logarithm and natural exponential functions are inverses to write

     $$L = \lim_{x \to a} e^{\ln\left(f(x)^{g(x)}\right)}$$

  2. Use the power property of logarithm arguments to write

     $$L = \lim_{x \to a} e^{g(x)\ln(f(x))}$$

  3. Use continuity of the exponential function to write

     $$L = e^{\lim\limits_{x \to a} g(x)\ln(f(x))}$$

  4. Rewrite multiplication as division by the reciprocal:

     $$L = e^{\lim\limits_{x \to a}\left(\frac{\ln(f(x))}{\frac{1}{g(x)}}\right)}$$

  5. Use L'Hôpital's Rule to evaluate this limit expression

  Growth Rates: Suppose f and g are functions with $\lim\limits_{x \to \infty} f(x) = \infty$ and $\lim\limits_{x \to \infty} g(x) = \infty$

1. If one of the following are true, f **grows faster than** g, and we use the notation $f \gg g$

$$\begin{aligned} \lim_{x\to\infty} \frac{g(x)}{f(x)} &= 0 \qquad (1) \\ \lim_{x\to\infty} \frac{f(x)}{g(x)} &= \infty \qquad (2) \end{aligned}$$

2. f **and** g **have comparable growth rates**, if there is some non-zero finite number M such that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = M$$

Ranked Growth Rates as $x \to \infty$
For any base $b > 1$, and for any positive numbers p, q, r, and s

$$\ln^q x \ll x^p \ll x^p \ln^r x \ll x^{p+s} \ll b^x \ll x^x$$

Examples

1. Use L'Hôpital's Rule to evaluate a limit with indeterminate form $\frac{0}{0}$

$$\begin{aligned} \lim_{x\to 0} \frac{e^x - x - 1}{5x^2} &= \lim_{x\to 0} \frac{e^x - 1}{10x} \qquad (1) \\ &= \lim_{x\to 0} \frac{e^x}{10} \qquad (2) \\ &= \frac{e^0}{10} \qquad (3) \\ &= \frac{1}{10} \qquad (4) \end{aligned}$$

2. Use L'Hôpital's Rule to evaluate a limit with indeterminate form $\frac{\infty}{\infty}$

$$\begin{aligned} \lim_{x\to 0^+} \frac{1 - \ln x}{1 + \ln x} &= \lim_{x\to 0^+} \frac{-\frac{1}{x}}{\frac{1}{x}} \qquad (1) \\ &= \lim_{x\to 0^+} \frac{-\frac{1}{x}}{\frac{1}{x}} \qquad (2) \\ &= \frac{-1}{1} \qquad (3) \\ &= -1 \qquad (4) \end{aligned}$$

3. Use L'Hôpital's Rule to evaluate a limit with indeterminate form $0 \cdot \infty$

$$\begin{aligned} \lim_{x\to 1^-} (1-x)\tan\left(\frac{\pi x}{2}\right) &= \lim_{x\to 1^-} \frac{(1-x)}{\cot\left(\frac{\pi x}{2}\right)} \qquad (1) \\ &= \lim_{x\to 1^-} \frac{-1}{-\frac{\pi}{2}\csc^2\left(\frac{\pi x}{2}\right)} \qquad (2) \\ &= \lim_{x\to 1^-} \frac{2}{\pi}\sin^2\left(\frac{\pi x}{2}\right) \qquad (3) \\ &= \frac{2}{\pi} \qquad (4) \end{aligned}$$

4. Use L'Hôpital's Rule to evaluate a limit with exponential indeterminate form

$$\begin{aligned}
\lim_{x\to 0^+} x^{\tan x} &= e^{\lim\limits_{x\to 0^+} \frac{\ln x}{\frac{1}{\tan x}}} && (1)\\
&= e^{\lim\limits_{x\to 0^+} \frac{\ln x}{\cot x}} && (2)\\
&= e^{\lim\limits_{x\to 0^+} \frac{1}{-x\csc^2 x}} && (3)\\
&= e^{\lim\limits_{x\to 0^+} \frac{-\sin^2 x}{x}} && (4)\\
&= e^{\lim\limits_{x\to 0^+} \frac{-2\sin x\cos x}{1}} && (5)\\
&= e^{\lim\limits_{x\to 0^+} -2\sin x\cos x} && (6)\\
&= e^0 && (7)\\
&= 1 && (8)
\end{aligned}$$

5. Compare the growth rates of functions

$$f(x) = x^2 \ln x$$
$$g(x) = x \ln^2 x$$

$$\begin{aligned}
\lim_{x\to\infty} \frac{f(x)}{g(x)} &= \lim_{x\to\infty} \frac{x^2 \ln x}{x \ln^2 x} && (1)\\
&= \lim_{x\to\infty} \frac{x}{\ln x} && (2)\\
&= \lim_{x\to\infty} \frac{1}{\frac{1}{x}} && (3)\\
&= \lim_{x\to\infty} x && (4)\\
&= \infty && (5)
\end{aligned}$$

Since $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = \infty$, $f \gg g$

Related Exercises

1. Example

A copy of my notes (in LaTeX) are available on my GitHub