

Module 1 Notes (MATH-211)

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General Notes (and Definitions)

- Limit Definition(s):

- Simple: The value that the outputs of a function approach as inputs approach a certain value
- Preliminary: Suppose a function f is defined for all x near a except possibly at a . If $f(x)$ is arbitrarily close to L all x sufficiently close (but not equal) to a , we write the following.

$$\lim_{x \rightarrow a} = L$$

- Secant Line: a line passing through two points $(t_0, s(t_0))$ and $(t_1, s(t_1))$. The slope is given by

$$\frac{s(t_1) - s(t_0)}{t_1 - t_0}$$

- Tangent Line: the line passing through $(t_0, s(t_0))$ with slope

$$\lim_{t \rightarrow t_0} \frac{s(t) - s(t_0)}{t - t_0}$$

- One Sided limits:

- Right-hand (Definition): Suppose a function f is defined for all x near a with $x > a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x > a$ we write

$$\lim_{x \rightarrow a^+} f(x) = L$$

- Left-hand (Definition): Suppose a function f is defined for all x near a with $x < a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x < a$ we write

$$\lim_{x \rightarrow a^-} f(x) = L$$

- In order for there to be a double sided limit, we must have:

$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x)$$

- If the limits from sides are not equal, then a the double sided limit, "does not exist"

- Limits can be simplified/solved in an easier way (as compared to numerically/graphically) using Limit Rules/Laws

- Limit Example Types:

- Tangent lines
- Velocity

- Velocity

- Average Velocity

- * The average velocity over some interval $[t_0, t_1]$ is defined as

$$v_{av} = \frac{s(t_1) - s(t_0)}{t_1 - t_0}$$

- Instantaneous Velocity

* The average velocity over some interval $[t_0, t_1]$ is defined as

$$v_{inst} = \lim_{t \rightarrow a} v_{av} = \frac{s(t) - s(a)}{t - a}$$

- Solving Techniques

- Factoring and canceling out
- Using conjugates

* When direct substitution is not possible, you may rationalize the numerator

Limit Rules/Laws

Assume $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist.

The following properties hold where c is a real number, and $n > 0$ is an integer.

- **Sum Rule**

$$\lim_{x \rightarrow a} (f(x) + g(x)) = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$$

- **Difference Rule**

$$\lim_{x \rightarrow a} (f(x) - g(x)) = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$$

- **Constant Multiple Rule**

$$\lim_{x \rightarrow a} (cf(x)) = c \lim_{x \rightarrow a} f(x)$$

- **Product Rule**

$$\lim_{x \rightarrow a} (f(x)g(x)) = (\lim_{x \rightarrow a} f(x))(\lim_{x \rightarrow a} g(x))$$

- **Quotient Rule**

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}, \text{ provided } \lim_{x \rightarrow a} g(x) \neq 0$$

- **Power Rule**

$$\lim_{x \rightarrow a} f(x)^n = (\lim_{x \rightarrow a} f(x))^n$$

- **Root Rule**

$$\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)}, \text{ provided } f(x) > 0, \text{ for } x \text{ near } a, \text{ if } n \text{ is even}$$

- **Polynomials**

A **Polynomial** is defined as A function of the form $x_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ where $n \geq 0$ is an integer If $p(x)$ is a polynomial then:

$$\lim_{x \rightarrow a} p(x) = p(a)$$

If $p(x)$ and $q(x)$ are polynomials and $q(a) \neq 0$ then (Direct Substitution):

$$\lim_{x \rightarrow a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$$

- **The Squeeze Theorem**

Assume for some functions f , g and h that satisfy $f(x) \leq g(x) \leq h(x)$ for x near a (except possibly at $x = a$). If

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$$

then

$$\lim_{x \rightarrow a} g(x) = L$$

As $x \rightarrow a, h(x) \rightarrow L$. Therefore, $g(x) \rightarrow L$. As x approaches a , if f and h approach the same value, so does g .

Examples

- (Describing Limits) As x approaches 3, x^2 approaches 9

$$\lim_{x \rightarrow 3} x^2 = 9$$

- (Common Use) Values that are undefined can still have limits, given a graph G where $f(3) = \text{undefined}$ ($f(3)$ is a hole), the following limit is valid:

$$\lim_{x \rightarrow 3} f(x) = 4$$

- Calculating Limits Numerically:

$$f(x) = \frac{x^3 - 1}{x - 1}$$

0.9	0.99	0.999	0.9999
2.71	2.9701	2.997001	2.99970001

1.1	1.01	1.001	1.0001
3.31	3.0301	3.003001	3.00030001

As x approaches 1, $f(x)$ approaches 3: $\lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1} = 3$

- Calculating One-sided limits:

$$g(x) = \frac{x^3 - 4x}{8|x - 2|}$$

1.9	1.09	1.009	1.0009
-0.92625	-0.9925125	-0.999250125	-0.9999250013

2.1	2.01	2.001	2.0001
1.07625	1.0075125	1.000750125	1.000075001

$$\lim_{x \rightarrow 2} g(x) = \text{Does not exist}$$

$$\lim_{x \rightarrow 2^-} g(x) = -1$$

$$\lim_{x \rightarrow 2^+} g(x) = 1$$

- Calculating piecewise function limits

$$f(x) = \begin{cases} 3 - x & \text{if } x < 2 \\ x - 1 & \text{if } x > 2 \end{cases}$$

$$a = 2$$

1.92	1.99	1.999	1.9999
1.1	1.01	1.001	1.0001

2.1	2.01	2.001	2.0001
1.1	1.01	1.001	1.0001

Explanation: Since $f(2)$ is not defined within the piece wise function, a graph representing this function would have a whole where $x = a$ and have two lines with inverse slopes

$$f(a) = \text{undefined}$$

$$\lim_{x \rightarrow a} f(x) = 1$$

$$\lim_{x \rightarrow a^-} f(x) = 1$$

$$\lim_{x \rightarrow a^+} f(x) = 1$$

- Limit Rules/Laws:

(a) Definitions:

$$\lim_{x \rightarrow 3} f(x) = 2$$

$$\lim_{x \rightarrow 3} g(x) = -1$$

$$\lim_{x \rightarrow 3} h(x) = 6$$

(b) Problems:

i. Sum, Constant Multiple

$$\lim_{x \rightarrow 3} (f(x) + 2g(x)) = \lim_{x \rightarrow 3} f(x) + \lim_{x \rightarrow 3} 2g(x) \quad (1)$$

$$= \lim_{x \rightarrow 3} f(x) + 2(\lim_{x \rightarrow 3} g(x)) \quad (2)$$

$$= 2 + 2(-1) \quad (3)$$

$$= 0 \quad (4)$$

ii. Quotient

$$\lim_{x \rightarrow 3} \frac{h(x)}{g(x)} = \frac{\lim_{x \rightarrow 3} h(x)}{\lim_{x \rightarrow 3} g(x)} \quad (1)$$

$$= \frac{6}{-1} \quad (2)$$

$$= -6 \quad (3)$$

iii. Quotient, Root, Difference

$$\lim_{x \rightarrow 3} \frac{h(x)}{\sqrt{f(x) - g(x)}} = \frac{\lim_{x \rightarrow 3} h(x)}{\lim_{x \rightarrow 3} \sqrt{f(x) - g(x)}} \quad (1)$$

$$= \frac{\lim_{x \rightarrow 3} h(x)}{\sqrt{\lim_{x \rightarrow 3} (f(x) - g(x))}} \quad (2)$$

$$= \frac{\lim_{x \rightarrow 3} h(x)}{\sqrt{\lim_{x \rightarrow 3} f(x) - \lim_{x \rightarrow 3} g(x)}} \quad (3)$$

$$= \frac{6}{\sqrt{2+1}} \quad (4)$$

$$= \frac{6}{\sqrt{3}} \quad (5)$$

$$= 2\sqrt{3} \quad (6)$$

7.

$$\lim_{x \rightarrow 1} \frac{3x^2 - 7x + 1}{x + 2} = \frac{3(1)^2 - 7(1) + 1}{1 + 2} \quad (1)$$

$$= \frac{3 - 7 + 1}{1 + 2} \quad (2)$$

$$= \frac{-3}{3} \quad (3)$$

$$= -1 \quad (4)$$

8.

$$\lim_{x \rightarrow 4} \frac{\left(\frac{1}{x} - \frac{1}{4}\right)}{x - 4} = \lim_{x \rightarrow 4} \frac{\left(\frac{4}{4x} - \frac{x}{4x}\right)}{x - 4} \quad (1)$$

$$= \lim_{x \rightarrow 4} \frac{\left(\frac{4-x}{4x}\right)}{x - 4} \quad (2)$$

$$= \lim_{x \rightarrow 4} \frac{\left(\frac{4-x}{4x}\right)}{\left(\frac{x-4}{1}\right)} \quad (3)$$

$$= \lim_{x \rightarrow 4} \left(\frac{4-x}{4x}\right) \left(\frac{1}{x-4}\right) \quad (4)$$

$$= \lim_{x \rightarrow 4} \frac{4-x}{4x(x-4)} \quad (5)$$

$$= \lim_{x \rightarrow 4} \frac{-(-4+x)}{4x(x-4)} \quad (6)$$

$$= \lim_{x \rightarrow 4} \frac{-(x-4)}{4x(x-4)} \quad (7)$$

$$= \lim_{x \rightarrow 4} \frac{-1}{4x} \quad (8)$$

$$= \lim_{x \rightarrow 4} \frac{-1}{4(4)} \quad (9)$$

$$= -\frac{1}{16} \quad (10)$$

9.

$$\lim_{x \rightarrow 9} \frac{x-9}{\sqrt{x}-3} = \lim_{x \rightarrow 9} \frac{x-9}{\sqrt{x}-3} \cdot \frac{\sqrt{x}+3}{\sqrt{x}+3} \quad (1)$$

$$= \lim_{x \rightarrow 9} \frac{(x-9)(\sqrt{x}+3)}{(\sqrt{x}-3)(\sqrt{x}+3)} \quad (2)$$

$$= \lim_{x \rightarrow 9} \frac{(x-9)(\sqrt{x}+3)}{x-9} \quad (3)$$

$$= \lim_{x \rightarrow 9} \sqrt{x}+3 \quad (4)$$

$$= \sqrt{9}+3 \quad (5)$$

$$= 3+3 \quad (6)$$

$$= 6 \quad (7)$$

10.

$$1 - \frac{x^2}{2} \leq \cos x \leq 1$$

$$\lim_{x \rightarrow 0} \left(1 - \frac{x^2}{2}\right) = 1 - \frac{0^2}{2} \quad (1)$$

$$= 1 - 0 \quad (2)$$

$$= 1 \quad (3)$$

$$= \lim_{x \rightarrow 0} 1 \quad (4)$$

$$\lim_{x \rightarrow 0} \cos x = 1 \quad (\text{By the Squeeze Theorem}) \quad (5)$$

11.

$$\lim_{x \rightarrow 0} \sin x = 0 \quad (\text{By the Squeeze Theorem}) \quad (1)$$

$$\lim_{x \rightarrow 0} \cos x = 1 \quad (\text{By the Squeeze Theorem}) \quad (2)$$

$$\lim_{x \rightarrow 0} \frac{\sin 2x}{\sin x} = \lim_{x \rightarrow 0} \frac{2 \sin x \cos x}{\sin x} \quad (1)$$

$$= \lim_{x \rightarrow 0} 2 \cos x \quad (2)$$

$$= 2 \lim_{x \rightarrow 0} \cos x \quad (3)$$

$$= 2 \cdot 1 \quad (4)$$

$$= 2 \quad (5)$$

12. (Section 2.1, Related Exercise 13):

Hint: use the secant line slope formula

$$s(t) = -16t^2 + 128t$$

(a) $[1, 4]$

$$\frac{256 - 112}{4 - 1} = \frac{144}{3} = 48$$

(b) $[1, 3]$

$$\frac{240 - 112}{3 - 1} = \frac{128}{2} = 64$$

(c) $[1, 2]$

$$\frac{192 - 112}{2 - 1} = \frac{80}{1} = 84$$

(d) $[1, 1 + h]$, where $h > 0$ is a real number

$$\frac{112 + -16h^2 + 128h - 112}{1 + h - 1} = \frac{-16h^2 + 128h}{h} = -16h + 128 = 16(-h + 6)$$

13. (Section 2.1, Related Exercise 15): Hint: we use the slope formula for the secant line, and the relationship is referring to the interval

$$s(t) = -16t^2 + 100t$$

$$\frac{s(t_1) - s(t_0)}{t_1 - t_0} = \frac{s(2) - s(0.5)}{2 - 0.5} \quad (1)$$

$$= \frac{136 - 46}{1.5} \quad (2)$$

$$= \frac{90}{1.5} \quad (3)$$

$$= 60 \quad (4)$$

The slope of this secant line, through the lens of average velocity could be viewed as the average velocity over the interval $[0.5, 2]$

14. (Section 2.1, Related Exercise 17):

$$s(t) = -16t^2 + 128t$$

$[1, 2]$	$[1, 1.5]$	$[1, 1.1]$	$[1, 1.01]$	$[1, 1.001]$
80	88	94.4	95.84	95.984

$$v_{inst} = \lim_{t \rightarrow 1} s(t) = 96$$

15. (Section 2.1, Related Exercise 19):

$$s(t) = -16t^2 + 100t$$

$[2, 3]$	$[2.9, 3]$	$[2.99, 3]$	$[2.999, 3]$	$[2.9999, 3]$
20	5.6	4.16	4.016	4.002

$$v_{inst} = \lim_{t \rightarrow 3} s(t) = 4$$

16. (Section 2.2, Related Exercise 3):

- $h(2) = 5$
- $\lim_{x \rightarrow 2} h(x) = 3$
- $h(4) = \text{Does not exist}$
- $\lim_{x \rightarrow 4} h(x) = 1$
- $\lim_{x \rightarrow 5} h(x) = 2$

17. (Section 2.2, Related Exercise 4):

- $g(0) = 0$
- $\lim_{x \rightarrow 0} g(x) = 1$
- $g(1) = 2$
- $\lim_{x \rightarrow 1} g(x) = 2$

18. (Section 2.2, Related Exercise 7):

$$f(x) = \frac{x^2 - 4}{x - 2}$$

1.9	1.99	1.999	1.9999
3.9	3.99	3.999	3.9999

2.1	2.01	2.001	2.0001
4.1	4.01	4.001	4.0001

$$\lim_{x \rightarrow 2} f(x) = 4$$

19. (Section 2.2, Related Exercise 8):

$$f(x) = \frac{x^3 - 1}{x - 1}$$

0.9	0.99	0.999	0.9999
2.71	2.9701	3.997001	3.99970001

1.1	1.01	1.001	1.0001
3.31	3.0301	3.003001	3.00030001

$$\lim_{x \rightarrow 1} f(x) = 3$$

20. (Section 2.2, Related Exercise 27):

$$f(x) = \frac{x - 2}{\ln |x - 2|}$$

$$\lim_{x \rightarrow 2} f(x) = 2$$

21. (Section 2.2, Related Exercise 28):

$$f(x) = \frac{e^{2x} - 2x - 1}{x^2}$$

$$\lim_{x \rightarrow 0} f(x) = 0$$

22. (Section 2.2, Related Exercise 19):

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x \leq -1 \\ 3 & \text{if } x > -1 \end{cases}$$

$$\lim_{x \rightarrow -1^-} f(x) = 2$$

$$\lim_{x \rightarrow -1^+} f(x) = 3$$

$$\lim_{x \rightarrow -1} f(x) = \text{Does not exist}$$

23. (Section 2.2, Related Exercise 20):

$$f(x) = \begin{cases} 3 - x & \text{if } x < 2 \\ x - 1 & \text{if } x > 2 \end{cases}$$

$$\lim_{x \rightarrow 2^-} f(x) = 1$$

$$\lim_{x \rightarrow 2^+} f(x) = 1$$

$$\lim_{x \rightarrow 2} f(x) = 1$$

24. (Section 2.3, Related Exercise 19):

$$\lim_{x \rightarrow 4} 3x - 7 = 3(4) - 7 = 12 - 7 = 5$$

25. (Section 2.3, Related Exercise 22):

$$\lim_{x \rightarrow 6} 4 = 4$$

26. (Section 2.3, Related Exercise 11): Quotient, Difference

$$\lim_{x \rightarrow 1} \frac{f(x)}{g(x) - h(x)} = \frac{\lim_{x \rightarrow 1} f(x)}{\lim_{x \rightarrow 1} g(x) - h(x)} \quad (1)$$

$$= \frac{\lim_{x \rightarrow 1} f(x)}{\lim_{x \rightarrow 1} g(x) - h(x)} \quad (2)$$

$$= \frac{\lim_{x \rightarrow 1} f(x)}{\lim_{x \rightarrow 1} g(x) - \lim_{x \rightarrow 1} h(x)} \quad (3)$$

$$= \frac{8}{3 - 2} \quad (4)$$

$$= \frac{8}{1} \quad (5)$$

$$= 8 \quad (6)$$

27. (Section 2.3, Related Exercise 12): Root, Sum, Product

$$\lim_{x \rightarrow 1} \sqrt[3]{f(x)g(x) + 3} = \sqrt[3]{\lim_{x \rightarrow 1} f(x)g(x) + 3} \quad (1)$$

$$= \sqrt[3]{\lim_{x \rightarrow 1} f(x)g(x) + 3} \quad (2)$$

$$= \sqrt[3]{\lim_{x \rightarrow 1} f(x)g(x) + \lim_{x \rightarrow 1} 3} \quad (3)$$

$$= \sqrt[3]{\lim_{x \rightarrow 1} f(x) \lim_{x \rightarrow 1} g(x) + \lim_{x \rightarrow 1} 3} \quad (4)$$

$$= \sqrt[3]{8 \cdot 3 + 3} \quad (5)$$

$$= \sqrt[3]{24 + 3} \quad (6)$$

$$= \sqrt[3]{27} \quad (7)$$

$$= 3 \quad (8)$$

28. (Section 2.3, Related Exercise 25):

$$\lim_{x \rightarrow 1} \frac{5x^2 + 6x + 1}{8x - 4} = \frac{5(1^2) + 6(1) + 1}{8(1) - 4} \quad (1)$$

$$= \frac{5 + 6 + 1}{8 - 4} \quad (2)$$

$$= \frac{12}{4} \quad (3)$$

$$= 3 \quad (4)$$

29. (Section 2.3, Related Exercise 26):

$$\lim_{t \rightarrow 3} \sqrt[3]{t^2 - 10} = \sqrt[3]{\lim_{t \rightarrow 3} t^2 - 10} \quad (1)$$

$$= \sqrt[3]{3^2 - 10} \quad (2)$$

$$= \sqrt[3]{9 - 10} \quad (3)$$

$$= \sqrt[3]{-1} \quad (4)$$

$$= -1 \quad (5)$$

30. (Section 2.3, Related Exercise 27):

$$\lim_{p \rightarrow 2} \frac{3p}{\sqrt{4p+1}-1} = \frac{\lim_{p \rightarrow 2} 3p}{\lim_{p \rightarrow 2} \sqrt{4p+1}-1} \quad (1)$$

$$= \frac{3(2)}{\sqrt{\lim_{p \rightarrow 2} 4p+1}-1} \quad (2)$$

$$= \frac{6}{\sqrt{4(2)+1}-1} \quad (3)$$

$$= \frac{6}{\sqrt{8+1}-1} \quad (4)$$

$$= \frac{6}{\sqrt{9}-1} \quad (5)$$

$$= \frac{6}{3-1} \quad (6)$$

$$= \frac{6}{2} \quad (7)$$

$$= 3 \quad (8)$$

31. (Section 2.3, Related Exercise 72):

$$g(x) = \begin{cases} 5x - 15 & \text{if } x < 4 \\ \sqrt{6x+1} & \text{if } x \geq 4 \end{cases}$$

$$\lim_{x \rightarrow 4^-} g(x) = 5$$

$$\lim_{x \rightarrow 4^+} g(x) = 5$$

$$\lim_{x \rightarrow 4} g(x) = 5$$

32. (Section 2.3, Related Exercise 73):

$$g(x) = \begin{cases} x^2 + 1 & \text{if } x < -1 \\ \sqrt{x+1} & \text{if } x \geq -1 \end{cases}$$

$$\lim_{x \rightarrow -1^-} g(x) = 2$$

$$\lim_{x \rightarrow -1^+} g(x) = 0$$

$$\lim_{x \rightarrow -1} g(x) = \text{Does not exist}$$

33. (Section 2.3, Related Exercise 34):

$$\lim_{x \rightarrow 3} \frac{x^2 - 2x - 3}{x - 3} = \lim_{x \rightarrow 3} \frac{(x-3)(x+1)}{x-3} \quad (1)$$

$$= \lim_{x \rightarrow 3} x + 1 \quad (2)$$

$$= 3 + 1 \quad (3)$$

$$= 4 \quad (4)$$

34. (Section 2.3, Related Exercise 41):

$$\lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x - 9} = \lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x - 9} \cdot \frac{\sqrt{x} + 3}{\sqrt{x} + 3} \quad (1)$$

$$= \lim_{x \rightarrow 9} \frac{(\sqrt{x} - 3)(\sqrt{x} + 3)}{(x - 9)(\sqrt{x} + 3)} \quad (2)$$

$$= \lim_{x \rightarrow 9} \frac{x - 9}{(x - 9)(\sqrt{x} + 3)} \quad (3)$$

$$= \lim_{x \rightarrow 9} \frac{1}{\sqrt{x} + 3} \quad (4)$$

$$= \frac{1}{\sqrt{9} + 3} \quad (5)$$

$$= \frac{1}{3 + 3} \quad (6)$$

$$= \frac{1}{6} \quad (7)$$

35. (Section 2.3, Related Exercise 69):

$$\lim_{x \rightarrow 1^+} \frac{x - 1}{\sqrt{x^2 - 1}} = \text{Does not exist}$$

36. (Section 2.3, Related Exercise 70):

$$\lim_{x \rightarrow 1^+} \frac{x - 1}{\sqrt{x^2 - 1}} = \lim_{x \rightarrow 1^+} \frac{x - 1}{\sqrt{x^2 - 1}} \cdot \frac{x + 1}{x + 1} \quad (1)$$

$$= \lim_{x \rightarrow 1^+} \frac{x^2 - 1}{\sqrt{x^2 - 1}(x + 1)} \quad (2)$$

$$= \lim_{x \rightarrow 1^+} \frac{x^2 - 1}{(x^2 - 1)^{\frac{1}{2}}(x + 1)} \quad (3)$$

$$= \lim_{x \rightarrow 1^+} \frac{(x^2 - 1)^{\frac{1}{2}}}{x + 1} \quad (4)$$

$$= \lim_{x \rightarrow 1^+} \frac{\sqrt{x^2 - 1}}{x + 1} \quad (5)$$

$$= \frac{\sqrt{1 - 1}}{1 + 1} \quad (6)$$

$$= \frac{\sqrt{0}}{2} \quad (7)$$

$$= \frac{0}{2} \quad (8)$$

$$= 0 \quad (9)$$

37. (Section 2.3, Related Exercise 95):

$$\frac{2^x - 2^0}{x - 0} = \frac{2^x - 1}{x}$$

-1	-0.1	-0.01	-0.001	-0.0001	-0.00001
0.5	0.6696700846	0.6907504563	0.6929070095	0.6931231585	0.6931447783

$$\lim_{x \rightarrow 0^1} \frac{2^x - 1}{x} = 0.693$$

38. (Section 2.3, Related Exercise 96):

$$\frac{3^x - 3^0}{x - 0} = \frac{3^x - 1}{x}$$

-0.1	-0.01	-0.001	-0.0001
1.040415402	1.092599583	1.098009035	1.098551943

0.0001	0.001	0.01	0.1
1.098672638	1.099215984	1.104669194	1.161231740

$$\lim_{x \rightarrow 0^1} \frac{3^x - 1}{x} = 1.0986$$

39. (Section 2.3, Related Exercise 81):

$$-|x| < 0 < |x| \text{ and } \sin \frac{1}{x} \leq 1, \text{ so } |x| \sin \frac{1}{x} \leq |x| \text{ and } -|x| \sin \frac{1}{x} \geq -|x|$$

$$\lim_{x \rightarrow 0} -|x| = -|0| = 0$$

$$\lim_{x \rightarrow 0} |x| = |0| = 0$$

$$\lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0$$

By the Squeeze Theorem, since $\lim_{x \rightarrow 0} -|x| = \lim_{x \rightarrow 0} |x|$ and the functions are chronologically greater than the last

40. (Section 2.3, Related Exercise 82):

$$\lim_{x \rightarrow 0} 1 - \frac{x^2}{2} = 1 - \frac{0}{2} = 1 - 0 = 1$$

$$\lim_{x \rightarrow 0} 1 = 1$$

$$\lim_{x \rightarrow 0} \cos x = 1$$

By the Squeeze Theorem, since $\lim_{x \rightarrow 0} 1 - \frac{x^2}{2} = \lim_{x \rightarrow 0} 1$ and the functions are chronologically greater than the last

41. (Section 2.3, Related Exercise 60):

$$\lim_{x \rightarrow 0} \frac{\sin 2x}{\sin x} = \lim_{x \rightarrow 0} \frac{2 \sin x \cos x}{\sin x} \quad (1)$$

$$= \lim_{x \rightarrow 0} 2 \cos x \quad (2)$$

$$= 2 \cos 0 \quad (3)$$

$$= 2 \cdot 1 \quad (4)$$

$$= 2 \quad (5)$$

42. (Section 2.3, Related Exercise 61):

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{\cos^2 x - 3 \cos x + 2} = \lim_{x \rightarrow 0} \frac{1}{\cos^2 x - 2 \cos x + 2} \quad (1)$$

$$= \lim_{x \rightarrow 0} \frac{1}{\cos x \cos x - 2 \cos x + 2} \quad (2)$$

$$= \frac{1}{\cos 0 \cos 0 - 2 \cos 0 + 2} \quad (3)$$

$$= \frac{1}{1 \cdot 1 - 2(1) + 2} \quad (4)$$

$$= \frac{1}{1 - 2 + 2} \quad (5)$$

$$= \frac{1}{1} \quad (6)$$

$$= 1 \quad (7)$$