

Module 2 Notes (MATH-211)

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General Notes (and Definitions)

- Derivatives

- A **derivative** is a new function made up of the slopes of the tangent lines as they change along a curve
- If a curve represents the trajectory of a moving object, the tangent line at a point indicates the direction of motion at that point
- As $x \rightarrow a$, the slope of the secant lines approaches the slope of the tangent line
- Alternative definition for Tangent Line(s): Consider the curve $y = f(x)$ and a secant line intersecting the curve at points $P(a, f(a))$ and $Q(a + h, f(a + h))$, with m_{sec} and m_{tan}

Interval: $(a, a + h)$

$$m_{sec} = \frac{f(a + h) - f(a)}{h}$$

$$m_{tan} = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$

$$y - f(a) = m_{tan}(x - a)$$

- **Definition:** The derivative of f at a , denoted $f'(a)$, is given by either the two following limits, provided the limits exist and a is in the domain of f

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \quad (1)$$

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h} \quad (2)$$

If $f'(a)$ exists, we say that f is **differentiable** at a

- Derivatives as Functions

- The slope of the tangent line of some function f is a function called the derivative of f

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

- If $f'(x)$ exists, we say that f is **differentiable** at x
- If f is differentiable at every point in some open interval I , we say that f is differentiable on I
- For some function f we can denote the derivative of f like such:

$$f'(x) \quad (1)$$

$$\frac{dy}{dx} \quad (2)$$

$$\frac{df}{dx} \quad (3)$$

$$\frac{d}{dx}(f(x)) \quad (4)$$

$$D_x(f(x)) \quad (5)$$

$$y'(x) \quad (6)$$

- When evaluating some derivative f at a , we can use the following:

$$f'(a) \quad (1)$$

$$y'(a) \quad (2)$$

$$\left. \frac{df}{dx} \right|_{x=a} \quad (3)$$

$$\left. \frac{dy}{dx} \right|_{x=a} \quad (4)$$

- If f is differentiable at a , then f is continuous at a
- If f is not continuous at a , then f is not differentiable at a

Examples

1. Instantaneous Velocity

$$s(t) = -16t^2 + 128t + 192$$

$$t = 2$$

$$\lim_{t \rightarrow 2} \frac{s(t) - s(2)}{t - 2} = \lim_{t \rightarrow 2} \frac{(-16t^2 + 128t + 192) - (-16(2^2) + 128(2) + 192)}{t - 2} \quad (1)$$

$$= \lim_{t \rightarrow 2} \frac{(-16t^2 + 128t + 192) - (-64 + 256 + 192)}{t - 2} \quad (2)$$

$$= \lim_{t \rightarrow 2} \frac{(-16t^2 + 128t + 192) - 384}{t - 2} \quad (3)$$

$$= \lim_{t \rightarrow 2} \frac{-16t^2 + 128t - 192}{t - 2} \quad (4)$$

$$= \lim_{t \rightarrow 2} \frac{(t - 2)(-16t + 96)}{t - 2} \quad (5)$$

$$= \lim_{t \rightarrow 2} -16t + 96 \quad (6)$$

$$= -32 + 96 \quad (7)$$

$$= 64 \quad (8)$$

2. Secant Lines

$$y = f(x)$$

Intersection Points: $P(a, f(a))$ and $Q(x, f(x))$

$$\text{Secant Line Slope} = \frac{f(x) - f(a)}{x - a}$$

3. Tangent Lines

$$f(x) = 2x^2 + 4x - 3$$

$$(-1, 5)$$

$$\lim_{x \rightarrow -1} \frac{f(x) - f(-1)}{x - (-1)} = \lim_{x \rightarrow -1} \frac{2x^2 + 4x - 3 - (-5)}{x + 1} \quad (1)$$

$$= \lim_{x \rightarrow -1} \frac{2x^2 + 4x + 2}{x + 1} \quad (2)$$

$$= \lim_{x \rightarrow -1} \frac{(x + 1)(2x + 2)}{x + 1} \quad (3)$$

$$= \lim_{x \rightarrow -1} \frac{(x + 1)(2x + 2)}{x + 1} \quad (4)$$

$$= \lim_{x \rightarrow -1} 2x + 2 \quad (5)$$

$$= 2(-1) + 2 \quad (6)$$

$$= -2 + 2 \quad (7)$$

$$= 0 \quad (8)$$

4. Alternative Tangent Lines

$$f(x) = 5 - x^3$$

$$(2, -3)$$

$$a = 2$$

$$h = -3 - 2 = -5$$

$$\lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \rightarrow 0} \frac{f(2+h) - (-3)}{h} \quad (1)$$

$$= \lim_{h \rightarrow 0} \frac{f(2+h) + 3}{h} \quad (2)$$

$$= \lim_{h \rightarrow 0} \frac{5 - (2+h)^3 + 3}{h} \quad (3)$$

$$= \lim_{h \rightarrow 0} \frac{8 - (2+h)^3}{h} \quad (4)$$

$$= \lim_{h \rightarrow 0} \frac{2^3 - (2+h)^3}{h} \quad (5)$$

$$= \lim_{h \rightarrow 0} \frac{(2 - (2+h))(2^2 + 2(2+h) + (2+h)^2)}{h} \quad (6)$$

$$= \lim_{h \rightarrow 0} \frac{-h(4 + 4 + 2h + h^2 + 4h + 4)}{h} \quad (7)$$

$$= \lim_{h \rightarrow 0} \frac{-h(h^2 + 6h + 12)}{h} \quad (8)$$

$$= \lim_{h \rightarrow 0} -(h^2 + 6h + 12) \quad (9)$$

$$= -12 \quad (10)$$

$$(11)$$

$$y + 3 = -12(x - 2) = -12x + 24$$

$$y = -12x + 21$$

5. Derivative Example

$$f(x) = \sqrt{x-1}$$

$$x = 2$$

$$f(x) = f(2) = \sqrt{2-1} = \sqrt{1} = 1$$

$$(2, 1)$$

$$f'(2) = \lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2} \quad (1)$$

$$= \lim_{x \rightarrow 2} \frac{\sqrt{x-1} - 1}{x - 2} \quad (2)$$

$$= \lim_{x \rightarrow 2} \frac{\sqrt{x-1} - 1}{x - 2} \cdot \frac{\sqrt{x-1} + 1}{\sqrt{x-1} + 1} \quad (3)$$

$$= \lim_{x \rightarrow 2} \frac{x - 2}{(x - 2)(\sqrt{x-1} + 1)} \quad (4)$$

$$= \lim_{x \rightarrow 2} \frac{1}{\sqrt{x-1} + 1} \quad (5)$$

$$= \frac{1}{\sqrt{2-1} + 1} \quad (6)$$

$$= \frac{1}{\sqrt{1} + 1} \quad (7)$$

$$= \frac{1}{1 + 1} \quad (8)$$

$$= \frac{1}{2} \quad (9)$$

$$(10)$$

$$y - 1 = \frac{1}{2}(x - 2) \quad (1)$$

$$y = \frac{1}{2}(x - 2) + 1 \quad (2)$$

$$= \frac{1}{2}x - 1 + 1 \quad (3)$$

$$= \frac{1}{2}x \quad (4)$$

6. Derivative Application Example

$$V(t) = 3t$$

$$V'(12) = \lim_{x \rightarrow 12} \frac{V(x) - V(12)}{x - 12} \quad (1)$$

$$= \lim_{x \rightarrow 12} \frac{3x - 36}{x - 12} \quad (2)$$

$$= \lim_{x \rightarrow 12} \frac{3(x - 12)}{x - 12} \quad (3)$$

$$= \lim_{x \rightarrow 12} 3 \quad (4)$$

$$= 3 \quad (5)$$

$$y - 36 = 3(x - 12) \quad (1)$$

$$y = 3x - 36 + 36 \quad (2)$$

$$= 3x \quad (3)$$

$$(4)$$

7. Find the Derivative

$$f(x) = 4x^2 - 5x + 6$$

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{4(x+h)^2 - 5(x+h) + 6 - (4x^2 - 5x + 6)}{h} \quad (1)$$

$$= \lim_{h \rightarrow 0} \frac{4(x+h)^2 - 5x - 5h + 6 - 4x^2 + 5x - 6}{h} \quad (2)$$

$$= \lim_{h \rightarrow 0} \frac{4(x+h)^2 - 5h - 4x^2}{h} \quad (3)$$

$$= \lim_{h \rightarrow 0} \frac{4h^2 + 8xh - 5h}{h} \quad (4)$$

$$= \lim_{h \rightarrow 0} 4h + 8x - 5 \quad (5)$$

$$= 4(0) + 8x - 5 \quad (6)$$

$$= 8x - 5 \quad (7)$$

8. Calculating a Derivative

$$f(x) = \frac{1}{x}$$

$$\left(-5, -\frac{1}{5}\right)$$

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad (1)$$

$$= \lim_{h \rightarrow 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \quad (2)$$

$$= \lim_{h \rightarrow 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h} \quad (3)$$

$$= \lim_{h \rightarrow 0} \frac{\frac{-h}{x(x+h)}}{h} \quad (4)$$

$$= \lim_{h \rightarrow 0} \frac{-h}{x(x+h)} \cdot \frac{1}{h} \quad (5)$$

$$= \lim_{h \rightarrow 0} \frac{-h}{h(x(x+h))} \quad (6)$$

$$= \lim_{h \rightarrow 0} \frac{-1}{x(x+h)} \quad (7)$$

$$= \frac{-1}{x(x+0)} \quad (8)$$

$$= \frac{-1}{x^2} \quad (9)$$

$$m_{\tan} = \left. \frac{dy}{dx} \right|_{x=-5} = \frac{-1}{(-5)^2} = \frac{-1}{25}$$

$$y - \left(-\frac{1}{5}\right) = \frac{-1}{25}(x - (-5))$$

$$y = \frac{-1}{25}x - \frac{1}{5} - \frac{1}{5} = \frac{-1}{25}x - \frac{2}{5}$$

Related Exercises

1. (Section 3.1, Related Exercise 13)

$$s(t) = -16t^2 + 100t$$

$$a = 1$$

$$\lim_{h \rightarrow 0} \frac{s(a+h) - s(a)}{h} = \lim_{h \rightarrow 0} \frac{s(1+h) - 84}{h} \quad (1)$$

$$= \lim_{h \rightarrow 0} \frac{-16(1+h)^2 + 100(1+h) - 84}{h} \quad (2)$$

$$= \lim_{h \rightarrow 0} \frac{-16(h^2 + 2h + 1) + 100 + 100h - 84}{h} \quad (3)$$

$$= \lim_{h \rightarrow 0} \frac{-16h^2 - 32h - 16 + 100 + 100h - 84}{h} \quad (4)$$

$$= \lim_{h \rightarrow 0} \frac{-16h^2 + 68h}{h} \quad (5)$$

$$= \lim_{h \rightarrow 0} -16h + 68 \quad (6)$$

$$= -16(0) + 68 \quad (7)$$

$$= 68 \quad (8)$$

2. (Section 3.1, Related Exercise 14)

$$s(t) = -16t^2 + 128t + 192$$

$$a = 2$$

$$\lim_{h \rightarrow 0} \frac{s(a+h) - s(a)}{h} = \lim_{h \rightarrow 0} \frac{s(2+h) - 384}{h} \quad (1)$$

$$= \lim_{h \rightarrow 0} \frac{-16(2+h)^2 + 128(2+h) + 192 - 384}{h} \quad (2)$$

$$= \lim_{h \rightarrow 0} \frac{-16(h^2 + 4h + 4) + 128(2+h) + 192 - 384}{h} \quad (3)$$

$$= \lim_{h \rightarrow 0} \frac{-16h^2 - 64h - 64 + 256 + 128h + 192 - 384}{h} \quad (4)$$

$$= \lim_{h \rightarrow 0} \frac{-16h^2 + 64h}{h} \quad (5)$$

$$= \lim_{h \rightarrow 0} -16h + 64 \quad (6)$$

$$= -16(0) + 64 \quad (7)$$

$$= 64 \quad (8)$$

3. (Section 3.1, Related Exercise 17)

$$f(x) = \frac{1}{x}$$

$$P(-1, -1)$$

$$\lim_{x \rightarrow -1} \frac{f(x) - f(-1)}{x - (-1)} = \lim_{x \rightarrow -1} \frac{f(x) - (-1)}{x + 1} \quad (1)$$

$$= \lim_{x \rightarrow -1} \frac{f(x) + 1}{x + 1} \quad (2)$$

$$= \lim_{x \rightarrow -1} \frac{\frac{1}{x} + 1}{x + 1} \quad (3)$$

$$= \lim_{x \rightarrow -1} \frac{\frac{1+x}{x}}{x + 1} \quad (4)$$

$$= \lim_{x \rightarrow -1} \frac{\frac{1+x}{x}}{x + 1} \cdot \frac{x}{x} \quad (5)$$

$$= \lim_{x \rightarrow -1} \frac{1+x}{(x+1)x} \quad (6)$$

$$= \lim_{x \rightarrow -1} \frac{1}{x} \quad (7)$$

$$= \frac{1}{-1} \quad (8)$$

$$= -1 \quad (9)$$

$$y - (-1) = -1(x - (-1))$$

$$y = -1(x + 1) - 1 = -x - 1 - 1 = -x - 2$$

4. (Section 3.1, Related Exercise 18)

$$f(x) = \frac{4}{x^2}$$

$$(-1, 4)$$

$$\lim_{x \rightarrow -1} \frac{f(x) - 4}{x - (-1)} = \lim_{x \rightarrow -1} \frac{f(x) - 4}{x + 1} \quad (1)$$

$$= \lim_{x \rightarrow -1} \frac{\frac{4}{x^2} - 4}{x + 1} \quad (2)$$

$$= \lim_{x \rightarrow -1} \frac{\frac{4 - 4x^2}{x^2}}{x + 1} \quad (3)$$

$$= \lim_{x \rightarrow -1} \frac{\frac{4 - 4x^2}{x^2}}{x + 1} \cdot \frac{x^2}{x^2} \quad (4)$$

$$= \lim_{x \rightarrow -1} \frac{4 - 4x^2}{x^2(x + 1)} \quad (5)$$

$$= \lim_{x \rightarrow -1} \frac{4(1 - x^2)}{x^2(x + 1)} \quad (6)$$

$$= \lim_{x \rightarrow -1} \frac{4(1 - x)(1 + x)}{x^2(x + 1)} \quad (7)$$

$$= \lim_{x \rightarrow -1} \frac{4(1 - x)}{x^2} \quad (8)$$

$$= \frac{4(1 - (-1)^2)}{(-1)^2} \quad (9)$$

$$= \frac{4(1 - 1)}{1} \quad (10)$$

$$= 0 \quad (11)$$

$$y - 4 = 0$$

$$y = 4$$

5. (Section 3.1, Related Exercise 23)

$$f(x) = 3x^2 - 4x$$

$$(1, -1)$$

$$\lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h} = \lim_{h \rightarrow 0} \frac{f(1 + h) - (-1)}{h} \quad (1)$$

$$= \lim_{h \rightarrow 0} \frac{3(1 + h)^2 - 4(1 + h) + 1}{h} \quad (2)$$

$$= \lim_{h \rightarrow 0} \frac{3h^2 + 6h + 3 - 4 - 4h + 1}{h} \quad (3)$$

$$= \lim_{h \rightarrow 0} \frac{3h^2 + 2h + 3 - 4 + 1}{h} \quad (4)$$

$$= \lim_{h \rightarrow 0} \frac{3h^2 + 2h}{h} \quad (5)$$

$$= \lim_{h \rightarrow 0} 3h + 2 \quad (6)$$

$$= 3(0) + 2 \quad (7)$$

$$= 2 \quad (8)$$

$$y - (-1) = 2(x - 1)$$

$$y = 2(x - 1) - 1 = 2x - 2 - 1 = 2x - 3$$

6. (Section 3.1, Related Exercise 27)

$$f(x) = x^3$$

$$(1, 1)$$

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \rightarrow 0} \frac{f(1+h) - 1}{h} \quad (1)$$

$$= \lim_{h \rightarrow 0} \frac{(1+h)^3 - 1}{h} \quad (2)$$

$$= \lim_{h \rightarrow 0} \frac{h^3 + h^2 + 2h + 2h^2 + h + 1 - 1}{h} \quad (3)$$

$$= \lim_{h \rightarrow 0} \frac{h^3 + 2h^2 + 3h}{h} \quad (4)$$

$$= \lim_{h \rightarrow 0} h^2 + 2h + 3 \quad (5)$$

$$= 0^2 + 2(0) + 3 \quad (6)$$

$$= 3 \quad (7)$$

$$y - 1 = 3(x - 1)$$

$$y = 3x - 3 + 1 = 3x - 2$$

7. (Section 3.1, Related Exercise 39)

$$f(x) = \sqrt{2x+1}$$

$$a = 4$$

$$\lim_{x \rightarrow 4} \frac{f(x) - f(a)}{x - a} = \lim_{x \rightarrow 4} \frac{f(x) - 3}{x - 4} \quad (1)$$

$$= \lim_{x \rightarrow 4} \frac{\sqrt{2x+1} - 3}{x - 4} \quad (2)$$

$$= \lim_{x \rightarrow 4} \frac{\sqrt{2x+1} - 3}{x - 4} \cdot \frac{\sqrt{2x+1} + 3}{\sqrt{2x+1} + 3} \quad (3)$$

$$= \lim_{x \rightarrow 4} \frac{2x + 1 - 9}{(x - 4)(\sqrt{2x+1} + 3)} \quad (4)$$

$$= \lim_{x \rightarrow 4} \frac{2(x - 4)}{(x - 4)(\sqrt{2x+1} + 3)} \quad (5)$$

$$= \lim_{x \rightarrow 4} \frac{2}{\sqrt{2x+1} + 3} \quad (6)$$

$$= \frac{2}{\sqrt{9} + 3} \quad (7)$$

$$= \frac{2}{3 + 3} \quad (8)$$

$$= \frac{2}{6} \quad (9)$$

$$= \frac{1}{3} \quad (10)$$

$$y - 3 = \frac{1}{3}(x - 4)$$

$$y = \frac{1}{3}x - \frac{4}{3} + \frac{9}{3} = \frac{1}{3}x + \frac{5}{3}$$

8. (Section 3.1, Related Exercise 40)

$$f(x) = \sqrt{3x}$$

$$a = 12$$

$$\lim_{x \rightarrow 12} \frac{f(x) - f(a)}{x - a} = \lim_{x \rightarrow 12} \frac{f(x) - 6}{x - 12} \quad (1)$$

$$= \lim_{x \rightarrow 12} \frac{\sqrt{3x} - 6}{x - 12} \quad (2)$$

$$= \lim_{x \rightarrow 12} \frac{\sqrt{3x} - 6}{x - 12} \cdot \frac{\sqrt{3x} + 6}{\sqrt{3x} + 6} \quad (3)$$

$$= \lim_{x \rightarrow 12} \frac{3x - 36}{(x - 12)(\sqrt{3x} + 6)} \quad (4)$$

$$= \lim_{x \rightarrow 12} \frac{3(x - 12)}{(x - 12)(\sqrt{3x} + 6)} \quad (5)$$

$$= \lim_{x \rightarrow 12} \frac{3}{\sqrt{3x} + 6} \quad (6)$$

$$= \frac{3}{\sqrt{3(12)} + 6} \quad (7)$$

$$= \frac{3}{\sqrt{36} + 6} \quad (8)$$

$$= \frac{3}{6 + 6} \quad (9)$$

$$= \frac{3}{12} \quad (10)$$

$$= \frac{1}{4} \quad (11)$$

$$y - 6 = \frac{1}{4}(x - 12)$$

$$y = \frac{1}{4}x - 3 + 6 = \frac{1}{4}x + 3$$

9. (Section 3.1, Related Exercise 49)

$$d(t) = 16t^2$$

$$a = 4$$

$$\lim_{x \rightarrow 4} \frac{f(x) - f(a)}{x - a} = \lim_{h \rightarrow 0} \frac{f(x) - 256}{x - 4} \quad (1)$$

$$= \lim_{x \rightarrow 4} \frac{16x^2 - 256}{x - 4} \quad (2)$$

$$= \lim_{x \rightarrow 4} \frac{(16x + 64)(x - 4)}{x - 4} \quad (3)$$

$$= \lim_{x \rightarrow 4} 16x + 64 \quad (4)$$

$$= 16(4) + 64 \quad (5)$$

$$= 64 + 64 \quad (6)$$

$$= 128 \quad (7)$$

10. (Section 3.1, Related Exercise 50)

$$F(x) = \frac{k}{x^2} \text{ where } k \text{ is some constant}$$

$$a = 1$$

$$\lim_{h \rightarrow 0} \frac{F(a+h) - F(a)}{h} = \lim_{h \rightarrow 0} \frac{F(1+h) - \frac{k}{1}}{h} \quad (1)$$

$$= \lim_{h \rightarrow 0} \frac{F(1+h) - \frac{k}{1}}{h} \quad (2)$$

$$= \lim_{h \rightarrow 0} \frac{\frac{k}{(1+h)^2} - \frac{k}{1}}{h} \quad (3)$$

$$= \lim_{h \rightarrow 0} \frac{\frac{k}{(1+h)^2} - \frac{k(1+h)^2}{(1+h)^2}}{h} \quad (4)$$

$$= \lim_{h \rightarrow 0} \frac{\frac{k - k(1+h)^2}{(1+h)^2}}{h} \quad (5)$$

$$= \lim_{h \rightarrow 0} \frac{\frac{k - k(1+h)^2}{(1+h)^2}}{h} \quad (6)$$

$$= \lim_{h \rightarrow 0} \frac{\frac{k - (kh^2 + 2kh + k)}{(1+h)^2}}{h} \quad (7)$$

$$= \lim_{h \rightarrow 0} \frac{\frac{k - kh^2 - 2kh - k}{(1+h)^2}}{h} \quad (8)$$

$$= \lim_{h \rightarrow 0} \frac{\frac{-kh^2 - 2kh}{(1+h)^2}}{h} \quad (9)$$

$$= \lim_{h \rightarrow 0} \frac{-kh^2 - 2kh}{(1+h)^2} \cdot \frac{1}{h} \quad (10)$$

$$= \lim_{h \rightarrow 0} \frac{h(-kh - 2k)}{h(1+h)^2} \quad (11)$$

$$= \lim_{h \rightarrow 0} \frac{-kh - 2k}{(1+h)^2} \quad (12)$$

$$= \frac{-kh - 2k}{(1+h)^2} \quad (13)$$

$$= \frac{-k(0) - 2k}{(1+0)^2} \quad (14)$$

$$= \frac{-2k}{1} \quad (15)$$

$$= -2k \quad (16)$$

11. (Section 3.1, Related Exercise 53)

Hint: Sketch a Secant Line

$$L'(1.5) \approx 4$$

$$L'(a) \approx 0 \text{ where } a \geq 4$$

12. (Section 3.1, Related Exercise 54)

$$D'(60) \approx 0.6$$

$$D'(170) \approx 0$$