Module 6 Notes (MATH-211)

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General Notes (and Definitions)

- L'Hôpital's Rule
  Indeterminate Form: An expression involving two components where the limit cannot be determined by evaluating the limits of the individual components.
  L'Hôpital's Rule: Suppose f and g are differentiable functions on an open interval I containing the point $x = a$, with $g'(x) \neq 0$ on I when $x \neq a$.
  If $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$ has any of the indeterminate forms: $\frac{0}{0}$, $\frac{\infty}{\infty}$, $-\frac{\infty}{\infty}$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

  provided that one of the following is the case:

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} \in \mathbb{R}$$

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = \infty$$

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = -\infty$$

  L'Hôpital's Rule is still valid if $x \to a$ is replaced by any of $x \to a^+$, $x \to a^-$, $x \to \infty$, or $x \to -\infty$. In the last two of these cases, there must be a greatest x-value beyond which both f and g are differentiable at every point.
  Exponential Indeterminate forms: 1^∞, 0^0, ∞^0
  Method for evaluating limits of indeterminate forms 1^∞, 0^0, ∞^0:
  Assume that $L = \lim\limits_{x\to a} f(x)^{g(x)}$ has one of these indeterminate forms.

  1. Use the fact that the natural logarithm and natural exponential functions are inverses to write
$$L = \lim_{x\to a} e^{\ln\left(f(x)^{g(x)}\right)}$$
  2. Use the power property of logarithm arguments to write
$$L = \lim_{x\to a} e^{g(x)\ln(f(x))}$$
  3. Use continuity of the exponential function to write
$$L = e^{\lim\limits_{x\to a} g(x)\ln(f(x))}$$
  4. Rewrite multiplication as division by the reciprocal:
$$L = e^{\lim\limits_{x\to a}\left(\frac{\ln(f(x))}{\frac{1}{g(x)}}\right)}$$
  5. Use L'Hôpital's Rule to evaluate this limit expression

  Growth Rates: Suppose f and g are functions with $\lim\limits_{x\to\infty} f(x) = \infty$ and $\lim\limits_{x\to\infty} g(x) = \infty$

1. If one of the following are true, f **grows faster than** g, and we use the notation $f \gg g$

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0 \qquad (1)$$
$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty \qquad (2)$$

2. f **and** g **have comparable growth rates**, if there is some non-zero finite number M such that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = M$$

Ranked Growth Rates as $x \to \infty$
For any base $b > 1$, and for any positive numbers p, q, r, and s

$$\ln^q x \ll x^p \ll x^p \ln^r x \ll x^{p+s} \ll b^x \ll x^x$$

Examples

1. Use L'Hôpital's Rule to evaluate a limit with indeterminate form $\frac{0}{0}$

$$\begin{aligned}
\lim_{x\to 0} \frac{e^x - x - 1}{5x^2} &= \lim_{x\to 0} \frac{e^x - 1}{10x} && (1)\\
&= \lim_{x\to 0} \frac{e^x}{10} && (2)\\
&= \frac{e^0}{10} && (3)\\
&= \frac{1}{10} && (4)
\end{aligned}$$

2. Use L'Hôpital's Rule to evaluate a limit with indeterminate form $\frac{\infty}{\infty}$

$$\begin{aligned}
\lim_{x\to 0^+} \frac{1 - \ln x}{1 + \ln x} &= \lim_{x\to 0^+} \frac{-\frac{1}{x}}{\frac{1}{x}} && (1)\\
&= \lim_{x\to 0^+} \frac{-\frac{1}{x}}{\frac{1}{x}} && (2)\\
&= \frac{-1}{1} && (3)\\
&= -1 && (4)
\end{aligned}$$

3. Use L'Hôpital's Rule to evaluate a limit with indeterminate form $0 \cdot \infty$

$$\begin{aligned}
\lim_{x\to 1^-} (1-x)\tan\left(\frac{\pi x}{2}\right) &= \lim_{x\to 1^-} \frac{(1-x)}{\cot\left(\frac{\pi x}{2}\right)} && (1)\\
&= \lim_{x\to 1^-} \frac{-1}{-\frac{\pi}{2}\csc^2\left(\frac{\pi x}{2}\right)} && (2)\\
&= \lim_{x\to 1^-} \frac{2}{\pi}\sin^2\left(\frac{\pi x}{2}\right) && (3)\\
&= \frac{2}{\pi} && (4)
\end{aligned}$$

4. Use L'Hôpital's Rule to evaluate a limit with exponential indeterminate form

$$\begin{aligned}
\lim_{x\to 0^+} x^{\tan x} &= e^{\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\tan x}}} && (1)\\
&= e^{\lim_{x\to 0^+} \frac{\ln x}{\cot x}} && (2)\\
&= e^{\lim_{x\to 0^+} \frac{\frac{1}{x}}{-\csc^2 x}} && (3)\\
&= e^{\lim_{x\to 0^+} \frac{-\sin^2 x}{x}} && (4)\\
&= e^{\lim_{x\to 0^+} \frac{-2\sin x\cos x}{1}} && (5)\\
&= e^{\lim_{x\to 0^+} -2\sin x\cos x} && (6)\\
&= e^0 && (7)\\
&= 1 && (8)
\end{aligned}$$

5. Compare the growth rates of functions

$$f(x) = x^2 \ln x$$
$$g(x) = x \ln^2 x$$

$$\begin{aligned}
\lim_{x\to\infty} \frac{f(x)}{g(x)} &= \lim_{x\to\infty} \frac{x^2 \ln x}{x \ln^2 x} && (1)\\
&= \lim_{x\to\infty} \frac{x}{\ln x} && (2)\\
&= \lim_{x\to\infty} \frac{1}{\frac{1}{x}} && (3)\\
&= \lim_{x\to\infty} x && (4)\\
&= \infty && (5)
\end{aligned}$$

Since $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$, $f \gg g$

Related Exercises

1. (Section 4.7, Exercise 17)

$$\begin{aligned}
\lim_{x\to 2} \frac{x^2-2x}{x^2-6x+8} &= \lim_{x\to 2} \frac{2x-2}{2x-6} && (1)\\
&= \frac{2(2)-2}{2(2)-6} && (2)\\
&= \frac{4-2}{4-6} && (3)\\
&= \frac{2}{-2} && (4)\\
&= -1 && (5)
\end{aligned}$$

2. (Section 4.7, Exercise 18)

$$\begin{aligned}
\lim_{x\to -1} \frac{x^4+x^3+2x+2}{x+1} &= \lim_{x\to -1} \frac{4x^3+3x^2+2}{1} && (1)\\
&= \lim_{x\to -1} 4x^3+3x^2+2 && (2)\\
&= 4(-1)^3+3(-1)^2+2 && (3)\\
&= -4+3+2 && (4)\\
&= 1 && (5)
\end{aligned}$$

3. (Section 4.7, Exercise 36)

$$\begin{aligned}
\lim_{x\to 0} \frac{e^x - x - 1}{5x^2} &= \lim_{x\to 0} \frac{e^x - 1}{10x} && (1)\\
&= \lim_{x\to 0} \frac{e^x}{10} && (2)\\
&= \frac{e^0}{10} && (3)\\
&= \frac{1}{10} && (4)
\end{aligned}$$

4. (Section 4.7, Exercise 39)

$$\begin{aligned}
\lim_{x\to 0} \frac{e^x - \sin x - 1}{x^4 + 8x^3 + 12x^2} &= \lim_{x\to 0} \frac{e^x - \cos x}{4x^3 + 24x^2 + 24x} && (1)\\
&= \lim_{x\to 0} \frac{e^x + \sin x}{12x^2 + 48x + 24} && (2)\\
&= \frac{e^0 + \sin 0}{12(0)^2 + 48(0) + 24} && (3)\\
&= \frac{1+0}{24} && (4)\\
&= \frac{1}{24} && (5)
\end{aligned}$$

5. (Section 4.7, Exercise 38)

$$\begin{aligned}
\lim_{x\to \infty} \frac{e^{3x}}{3e^{3x} + 5} &= \lim_{x\to \infty} \frac{3e^{3x}}{9e^{3x}} && (1)\\
&= \lim_{x\to \infty} \frac{1}{3} \cdot \frac{e^{3x}}{e^{3x}} && (2)\\
&= \lim_{x\to \infty} \frac{1}{3} && (3)\\
&= \frac{1}{3} && (4)
\end{aligned}$$

6. (Section 4.7, Exercise 51)

$$\begin{aligned}
\lim_{x\to \infty} \frac{x^2 - \ln \frac{2}{x}}{3x^2 + 2x} &= \lim_{x\to \infty} \frac{2x + \frac{1}{x}}{6x + 2} && (1)\\
&= \lim_{x\to \infty} \frac{2 - \frac{1}{x^2}}{6} && (2)\\
&= \frac{2-0}{6} && (3)\\
&= \frac{2}{6} && (4)\\
&= \frac{1}{3} && (5)
\end{aligned}$$

7. (Section 4.7, Exercise 53)

$$\begin{aligned}
\lim_{x\to 0} x \csc x &= \lim_{x\to 0} \frac{x}{\sin x} && (1)\\
&= \lim_{x\to 0} \frac{1}{\cos x} && (2)\\
&= \frac{1}{\cos 0} && (3)\\
&= \frac{1}{1} && (4)\\
&= 1 && (5)
\end{aligned}$$

8. (Section 4.7, Exercise 63)

$$\begin{aligned}
\lim_{x\to\infty}\left(x^2-\sqrt{x^4+16x^2}\right) &= \lim_{x\to\infty}\left(x^2-\sqrt{x^4\left(1+\frac{16}{x^2}\right)}\right) && (1)\\
&= \lim_{x\to\infty}\left(x^2-x^2\sqrt{1+\frac{16}{x^2}}\right) && (2)\\
&= \lim_{x\to\infty} x^2\left(1-\sqrt{1+\frac{16}{x^2}}\right) && (3)\\
&= \lim_{x\to\infty}\frac{1-\sqrt{1+\frac{16}{x^2}}}{\frac{1}{x^2}} && (4)\\
&= \lim_{x\to\infty}\frac{\frac{16}{x^3}}{\frac{-2}{x^3}\sqrt{1+\frac{16}{x^2}}} && (5)\\
&= \lim_{x\to\infty}\frac{\frac{16}{x^3}\cdot\frac{x^3}{-2}}{\sqrt{1+\frac{16}{x^2}}} && (6)\\
&= \lim_{x\to\infty}\frac{\frac{16}{-2}\cdot\frac{x^3}{x^3}}{\sqrt{1+\frac{16}{x^2}}} && (7)\\
&= \lim_{x\to\infty}\frac{-8}{\sqrt{1+\frac{16}{x^2}}} && (8)\\
&= \frac{-8}{\sqrt{1+0}} && (9)\\
&= \frac{-8}{1} && (10)\\
&= -8 && (11)
\end{aligned}$$

9. (Section 4.7, Exercise 64)

$$\begin{aligned}
\lim_{x\to\infty}\left(x-\sqrt{x^2+4x}\right) &= \lim_{x\to\infty}\left(x-\sqrt{x^2\left(1+\frac{4}{x}\right)}\right) && (1)\\
&= \lim_{x\to\infty}\left(x-x\sqrt{1+\frac{4}{x}}\right) && (2)\\
&= \lim_{x\to\infty} x\left(1-\sqrt{1+\frac{4}{x}}\right) && (3)\\
&= \lim_{x\to\infty}\frac{1-\sqrt{1+\frac{4}{x}}}{\frac{1}{x}} && (4)\\
&= \lim_{x\to\infty}\frac{\frac{2}{x^2}}{\frac{-1}{x^2}\sqrt{1+\frac{4}{x}}} && (5)\\
&= \lim_{x\to\infty}\frac{\frac{2}{x^2}\cdot\frac{x^2}{-1}}{\sqrt{1+\frac{4}{x}}} && (6)\\
&= \lim_{x\to\infty}\frac{\frac{2}{-1}\cdot\frac{x^2}{x^2}}{\sqrt{1+\frac{4}{x}}} && (7)\\
&= \lim_{x\to\infty}\frac{-2}{\sqrt{1+\frac{4}{x}}} && (8)\\
&= \frac{-2}{\sqrt{1+0}} && (9)\\
&= -2 && (10)
\end{aligned}$$

10. (Section 4.7, Exercise 75)

$$\begin{aligned}
\lim_{x\to 0^+} x^{2x} &= e^{\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{2x}}} && (1)\\
&= e^{\lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{2x^2}}} && (2)\\
&= e^{\lim_{x\to 0^+} \frac{1}{x}\cdot\frac{2x^2}{-1}} && (3)\\
&= e^{\lim_{x\to 0^+} -\frac{2x^2}{x}} && (4)\\
&= e^{\lim_{x\to 0^+} -2x} && (5)\\
&= e^{-2(0)} && (6)\\
&= e^{0} && (7)\\
&= 1 && (8)
\end{aligned}$$

11. (Section 4.7, Exercise 76)

$$\begin{aligned}
\lim_{x\to 0} (1+4x)^{\frac{3}{x}} &= e^{\lim_{x\to 0^+} \frac{\ln(1+4x)}{\frac{1}{\frac{3}{x}}}} && (1)\\
&= e^{\lim_{x\to 0^+} \frac{\ln(1+4x)}{\frac{x}{3}}} && (2)\\
&= e^{\lim_{x\to 0^+} \frac{\frac{4}{(1+4x)}}{\frac{1}{3}}} && (3)\\
&= e^{\lim_{x\to 0^+} \frac{4}{(1+4x)}\cdot\frac{3}{1}} && (4)\\
&= e^{\lim_{x\to 0^+} \frac{12}{(1+4x)}} && (5)\\
&= e^{\frac{12}{1}} && (6)\\
&= e^{12} && (7)
\end{aligned}$$

12. (Section 4.7, Exercise 96)

$$\begin{aligned}
f(x) &= x^2 \ln x && (1)\\
g(x) &= \ln^2 x && (2)
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to\infty} \frac{f(x)}{g(x)} &= \lim_{x\to\infty} \frac{x^2 \ln x}{\ln^2 x} && (1)\\
&= \lim_{x\to\infty} \frac{x^2}{\ln x} && (2)\\
&= \lim_{x\to\infty} \frac{2x}{\frac{1}{x}} && (3)\\
&= \lim_{x\to\infty} 2x^2 && (4)\\
&= \infty && (5)
\end{aligned}$$

$$f \gg g$$

13. (Section 4.7, Exercise 100)

$$\begin{aligned}
f(x) &= x^2 \ln x && (1)\\
g(x) &= x^3 && (2)
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to\infty} \frac{f(x)}{g(x)} &= \lim_{x\to\infty} \frac{x^2 \ln x}{x^3} && (1)\\
&= \lim_{x\to\infty} \frac{\ln x}{x} && (2)\\
&= \lim_{x\to\infty} \frac{\frac{1}{x}}{1} && (3)\\
&= \lim_{x\to\infty} \frac{1}{x} && (4)\\
&= \frac{1}{\infty} \neq \infty && (5)
\end{aligned}$$

$$g \gg f$$

14. (Section 4.7, Exercise 95)

$$\begin{aligned} f(x) &= x^{10} &(1)\\ g(x) &= e^{0.01x} &(2) \end{aligned}$$

$$\begin{aligned}
\lim_{x\to\infty} \frac{f(x)}{g(x)} &= \lim_{x\to\infty} \frac{x^{10}}{e^{0.01x}} &(1)\\
&= \lim_{x\to\infty} \frac{10x^9}{0.01e^{0.01x}} &(2)\\
&= \lim_{x\to\infty} \frac{90x^8}{0.01^2e^{0.01x}} &(3)\\
&= \lim_{x\to\infty} \frac{7200x^7}{0.01^3e^{0.01x}} &(4)\\
&= \lim_{x\to\infty} \frac{50400x^6}{0.01^4e^{0.01x}} &(5)\\
&= \lim_{x\to\infty} \frac{302400x^5}{0.01^5e^{0.01x}} &(6)\\
&= \lim_{x\to\infty} \frac{1512000x^4}{0.01^6e^{0.01x}} &(7)\\
&= \lim_{x\to\infty} \frac{6048000x^3}{0.01^7e^{0.01x}} &(8)\\
&= \lim_{x\to\infty} \frac{18144000x^2}{0.01^8e^{0.01x}} &(9)\\
&= \lim_{x\to\infty} \frac{36288000x}{0.01^9e^{0.01x}} &(10)\\
&= \lim_{x\to\infty} \frac{36288000}{0.01^{10}e^{0.01x}} &(11)\\
&= \frac{36288000}{\infty} \neq \infty &(12)
\end{aligned}$$

$$g \gg f$$

15. (Section 4.7, Exercise 101)

$$\begin{aligned} f(x) &= x^{20} &(1)\\ g(x) &= 1.00001^{x} &(2) \end{aligned}$$

$$\begin{aligned}
\lim_{x\to\infty} \frac{f(x)}{g(x)} &= \lim_{x\to\infty} \frac{x^{20}}{1.00001^{x}} &(1)\\
&= \frac{2432902008176640000}{\infty} \neq \infty &(2)
\end{aligned}$$

$$g \gg f$$

A copy of my notes (in LaTeX) are available on my GitHub