Module 2 Notes (MATH-211)

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General Notes (and Definitions)

- Derivatives
  - A **derivative** is a new function made up of the slopes of the tangent lines as they change along a curve
  - If a curve represents the trajectory of a moving object, the tangent line at a point indicates the direction of motion at that point
  - As $x \to a$, the slope of the secant lines approaches the slope of the tangent line
  - Alternative definition for Tangent Line(s): Consider the curve $y = f(x)$ and a secant line intersecting the curve at points $P(a, f(a))$ and $Q(a+h, f(a+h))$, with m_{sec} and m_{tan}

$$\text{Interval:}(a, a+h)$$

$$m_{sec} = \frac{f(a+h) - f(a)}{h}$$

$$m_{tan} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

$$y - f(a) = m_{tan}(x - a)$$

  - **Definition**: The derivative of f at a, denoted $f'(a)$, is given by either the two following limits, provided the limits exist and a is in the domain of f

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \tag{1}$$

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \tag{2}$$

  If $f'(a)$ exists, we say that f is **differentiable** at a

- Derivatives as Functions
  - The slope of the tangent line of somefunction f is a function called the derivative of f

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

  - If $f'(x)$ exists, we say that f is **differentiable** at x
  - If f is differentiable at every point in some open interval I, we say that f is differentiable on I
  - For some function f we can denote the derivative of f like such:

$$f'(x) \tag{1}$$

$$\frac{dy}{dx} \tag{2}$$

$$\frac{df}{dx} \tag{3}$$

$$\frac{d}{dx}(f(x)) \tag{4}$$

$$D_x(f(x)) \tag{5}$$

$$y'(x) \tag{6}$$

- When evaluating some derivative f at a, we can use the following:

$$f'(a) \tag{1}$$
$$y'(a) \tag{2}$$
$$\left.\frac{df}{dx}\right|_{x=a} \tag{3}$$
$$\left.\frac{dy}{dx}\right|_{x=a} \tag{4}$$

- If f is differentiable at a, then f is continuous at a
- If f is not continuous at a, then f is not differentiable at a

• Trigonometric Derivatives

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$
$$\lim_{x\to 0}\frac{\cos x - 1}{x} = 0$$
$$\frac{d}{dx}(\sin x) = \cos x$$
$$\frac{d}{dx}(\cos x) = -\sin x$$
$$\frac{d}{dx}(\tan x) = \sec^2 x$$
$$\frac{d}{dx}(\cot x) = -\csc^2 x$$
$$\frac{d}{dx}(\sec x) = \sec x \tan x$$
$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

• Derivatives as Rate of Change

- Secant Lines give average velocities

$$v_{av} = \frac{f(a+\Delta t) - f(a)}{\Delta t}$$

- Tangent Line gives instantaneous velocity

$$v(a) = \lim_{\Delta t\to 0}\frac{f(a+\Delta t) - f(a)}{\Delta t} = f'(a)$$

- Velocity, speed, and acceleration
Suppose and object moves along a line with position $s = f(t)$

$$\text{the } \textbf{velocity} \text{ at time } t \text{ is } v = \frac{ds}{dt} = f'(t)$$
$$\text{the } \textbf{speed} \text{ at time } t \text{ is } |v| = |f'(t)|$$
$$\text{the } \textbf{acceleration} \text{ at time } t \text{ is } a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = f''(t)$$

- Average and marginal cost

The **cost function** $C(x)$ gives the cost to produce the first x items in a manufacturing process

$$\text{The } \textbf{average cost} \text{ to produce } x \text{ items is } \overline{C}(x) = \frac{C(x)}{x}$$

The **marginal cost** $C'(x)$ is the approximate cost to produce one additional item after producing x items

- Elasticity

$$E(p) = \frac{dD}{dp}\frac{p}{D} \text{ where } D = f(p)$$

Rules of Differentiation

- Constant Rule

$$\text{If } c \in \mathbb{R}, \text{ then } \frac{d}{dx}(c) = 0$$

- Power Rule

$$\text{If } n \in \mathbb{Z} \text{ and } n > 0, \text{ then } \frac{d}{dx}(x^n) = nx^{n-1}$$

- Derivative of a Root

$$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

- Constant Multiple Rule

$$\text{If } f \text{ is differentiable at } x \text{ and } c \text{ is a constant, then } \frac{d}{dx}(cf(x)) = cf'(x)$$

- Sum Rule

$$\text{If } f \text{ and } g \text{ are differentiable at } x, \text{ then } \frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$$

- Generalized Sum Rule

$$\frac{d}{dx}(f_1(x) + f_2(x) + ... + f_x(x)) = f_1'(x) + f_2'(x) + ... + f_n'(x)$$

- Difference Rule

$$\frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x)$$

- Euler's Number

$$\text{The function } f(x) = e^x \text{ is differentiable for all } x \in \mathbb{R}, \text{ and } \frac{d}{dx}(e^x) = e^x$$

- Higher-order Derivatives
  Assuming $y = f(x)$ can be differentiated as often as necessary, the **second derivative** of f is

$$f''(x) = \frac{d}{dx}(f'(x))$$

  For $n \in \mathbb{Z}$ where $n \geq 1$, the **nth derivative** of f is

$$f^{(n)}(x) = \frac{d}{dx}\left(f^{(n-1)}(x)\right)$$

- Product Rule

$$\text{If } f \text{ and } g \text{ are differentiable at } x, \text{ then } \frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

- Quotient Rule

$$\text{If } f \text{ and } g \text{ are differentiable at } x \text{ and } g(x) \neq 0, \text{ then the derivative of } \frac{f}{g} \text{ at } x \text{ exists and}$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

- Power Rule for Negative Integers

  Proof.

  Choose $n \in \mathbb{Z}$

  Assume that $n < 0$

  Let $m = -n$

  Since $m = -n$ and $n < 0$, we know $m > 0$

$$
\begin{aligned}
\frac{d}{dx}(x^n) &= \frac{d}{dx}\left(\frac{1}{x^m}\right) && (1)\\
&= \frac{x^m \cdot 0 - 1 \cdot (mx^{m-1})}{(x^m)^2} && (2)\\
&= \frac{-mx^{m-1}}{x^{2m}} && (3)\\
&= -mx^{(m-1)-2m} && (4)\\
&= -mx^{-m-1} && (5)\\
&= nx^{n-1} && (6)\\
& && (7)
\end{aligned}
$$

Under the assumption that $n \in \mathbb{Z}$ and $n < 0$, we proved $\frac{d}{dx}(x^n) = nx^{n-1}$

$\square$

Examples

1. Instantaneous Velocity

$$s(t) = -16t^2 + 128t + 192$$
$$t = 2$$

$$
\begin{aligned}
\lim_{t \to 2} \frac{s(t) - s(2)}{t - 2} &= \lim_{t \to 2} \frac{(-16t^2 + 128t + 192) - (-16(2^2) + 128(2) + 192)}{t - 2} && (1)\\
&= \lim_{t \to 2} \frac{(-16t^2 + 128t + 192) - (-64 + 256 + 192)}{t - 2} && (2)\\
&= \lim_{t \to 2} \frac{(-16t^2 + 128t + 192) - 384}{t - 2} && (3)\\
&= \lim_{t \to 2} \frac{-16t^2 + 128t - 192}{t - 2} && (4)\\
&= \lim_{t \to 2} \frac{(t-2)(-16t + 96)}{t - 2} && (5)\\
&= \lim_{t \to 2} -16t + 96 && (6)\\
&= -32 + 96 && (7)\\
&= 64 && (8)
\end{aligned}
$$

2. Secant Lines

$$y = f(x)$$

Intersection Points: $P(a, f(a))$ and $Q(x, f(x))$

$$\text{Secant Line Slope} = \frac{f(x) - f(a)}{x - a}$$

3. Tangent Lines

$$f(x) = 2x^2 + 4x - 3$$
$$(-1, 5)$$

$$\begin{aligned}
\lim_{x\to -1}\frac{f(x)-f(-1)}{x-(-1)} &= \lim_{x\to -1}\frac{2x^2+4x-3-(-5)}{x+1} && (1)\\
&= \lim_{x\to -1}\frac{2x^2+4x+2}{x+1} && (2)\\
&= \lim_{x\to -1}\frac{(x+1)(2x+2)}{x+1} && (3)\\
&= \lim_{x\to -1}\frac{(x+1)(2x+2)}{x+1} && (4)\\
&= \lim_{x\to -1} 2x+2 && (5)\\
&= 2(-1)+2 && (6)\\
&= -2+2 && (7)\\
&= 0 && (8)
\end{aligned}$$

4. Alternative Tangent Lines

$$f(x) = 5 - x^3$$

$$(2, -3)$$

$$a = 2$$

$$h = -3 - 2 = -5$$

$$\begin{aligned}
\lim_{h\to 0}\frac{f(2+h)-f(2)}{h} &= \lim_{h\to 0}\frac{f(2+h)-(-3)}{h} && (1)\\
&= \lim_{h\to 0}\frac{f(2+h)+3}{h} && (2)\\
&= \lim_{h\to 0}\frac{5-(2+h)^3+3}{h} && (3)\\
&= \lim_{h\to 0}\frac{8-(2+h)^3}{h} && (4)\\
&= \lim_{h\to 0}\frac{2^3-(2+h)^3}{h} && (5)\\
&= \lim_{h\to 0}\frac{(2-(2+h))(2^2+2(2+h)+(2+h)^2)}{h} && (6)\\
&= \lim_{h\to 0}\frac{-h(4+4+2h+h^2+4h+4)}{h} && (7)\\
&= \lim_{h\to 0}\frac{-h(h^2+6h+12)}{h} && (8)\\
&= \lim_{h\to 0} -(h^2+6h+12) && (9)\\
&= -12 && (10)\\
& && (11)
\end{aligned}$$

$$y + 3 = -12(x-2) = -12x + 24$$

$$y = -12x + 21$$

5. Derivative Example

$$f(x) = \sqrt{x-1}$$

$$x = 2$$

$$f(x) = f(2) = \sqrt{2-1} = \sqrt{1} = 1$$

$$(2, 1)$$

$$
\begin{aligned}
f'(2) &= \lim_{x\to 2}\frac{f(x)-f(2)}{x-2} && (1)\\
&= \lim_{x\to 2}\frac{\sqrt{x-1}-1}{x-2} && (2)\\
&= \lim_{x\to 2}\frac{\sqrt{x-1}-1}{x-2}\cdot\frac{\sqrt{x-1}+1}{\sqrt{x-1}+1} && (3)\\
&= \lim_{x\to 2}\frac{x-2}{(x-2)(\sqrt{x-1}+1)} && (4)\\
&= \lim_{x\to 2}\frac{1}{\sqrt{x-1}+1} && (5)\\
&= \frac{1}{\sqrt{2-1}+1} && (6)\\
&= \frac{1}{\sqrt{1}+1} && (7)\\
&= \frac{1}{1+1} && (8)\\
&= \frac{1}{2} && (9)\\
& && (10)
\end{aligned}
$$

$$
\begin{aligned}
y-1 &= \frac{1}{2}(x-2) && (1)\\
y &= \frac{1}{2}(x-2)+1 && (2)\\
&= \frac{1}{2}x-1+1 && (3)\\
&= \frac{1}{2}x && (4)
\end{aligned}
$$

6. Derivative Application Example

$$V(t) = 3t$$

$$
\begin{aligned}
V'(12) &= \lim_{x\to 12}\frac{V(x)-V(12)}{x-12} && (1)\\
&= \lim_{x\to 12}\frac{3x-36}{x-12} && (2)\\
&= \lim_{x\to 12}\frac{3(x-12)}{x-12} && (3)\\
&= \lim_{x\to 12} 3 && (4)\\
&= 3 && (5)
\end{aligned}
$$

$$
\begin{aligned}
y-36 &= 3(x-12) && (1)\\
y &= 3x-36+36 && (2)\\
&= 3x && (3)\\
& && (4)
\end{aligned}
$$

7. Find the Derivative

$$f(x) = 4x^2 - 5x + 6$$

$$\begin{aligned}
\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{4(x+h)^2-5(x+h)+6-(4x^2-5x+6)}{h} && (1)\\
&= \lim_{h\to 0}\frac{4(x+h)^2-5x-5h+6-4x^2+5x-6}{h} && (2)\\
&= \lim_{h\to 0}\frac{4(x+h)^2-5h-4x^2}{h} && (3)\\
&= \lim_{h\to 0}\frac{4h^2+8xh-5h}{h} && (4)\\
&= \lim_{h\to 0}4h+8x-5 && (5)\\
&= 4(0)+8x-5 && (6)\\
&= 8x-5 && (7)
\end{aligned}$$

8. Calculating a Derivative

$$f(x)=\frac{1}{x}$$

$$(-5,-\frac{1}{5})$$

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} && (1)\\
&= \lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h} && (2)\\
&= \lim_{h\to 0}\frac{\frac{x-(x+h)}{x(x+h)}}{h} && (3)\\
&= \lim_{h\to 0}\frac{\frac{-h}{x(x+h)}}{h} && (4)\\
&= \lim_{h\to 0}\frac{-h}{x(x+h)}\cdot\frac{1}{h} && (5)\\
&= \lim_{h\to 0}\frac{-h}{h(x(x+h))} && (6)\\
&= \lim_{h\to 0}\frac{-1}{x(x+h)} && (7)\\
&= \frac{-1}{x(x+0)} && (8)\\
&= \frac{-1}{x^2} && (9)
\end{aligned}$$

$$m_{\tan}=\left.\frac{dy}{dx}\right|_{x=-5}=\frac{-1}{(-5)^2}=\frac{-1}{25}$$

$$y-(-\frac{1}{5})=\frac{-1}{25}(x-(-5))$$

$$y=\frac{-1}{25}x-\frac{1}{5}-\frac{1}{5}=\frac{-1}{25}x-\frac{2}{5}$$

9. Constant and Power Rules

$$\frac{d}{dx}\left(x^6\right)=6x^5$$

$$\frac{d}{dx}\left(x\right)=1x^0=1$$

$$\frac{d}{dx}\left(\pi^2\right)=0$$

10. Constant Multiple Rule

$$\frac{d}{dx}\left(-4x^9\right)=-4\left(9x^8\right)=-36x^8$$

$$\frac{d}{dt}\left(\frac{2}{5}t^5\right)=\frac{2}{5}\left(5t^4\right)=2t^4$$

11. Sum and Difference Rules with a Polynomial

$$\frac{d}{dx}\left(6x^5 - \frac{5}{2}x^2 + x + 5\right) = 30x^4 - 5x + 1$$

12. Euler's Number with Derivatives

$$f(x) = 5x + \frac{1}{3}e^x$$

$$\left(0, \frac{1}{3}\right)$$

$$\frac{d}{dx}\left(5x + \frac{1}{3}e^x\right) = 5 + \frac{1}{3}e^x$$

$$y = \frac{16}{3}x + \frac{1}{3}$$

13. Higher-order derivatives

$$f(x) = 3x^4 - 2x^2 + 7x - e^x$$
$$f'(x) = 12x^3 - 4x + 7 - e^x$$
$$f''(x) = 36x^2 - 4 - e^x$$
$$f'''(x) = 72x - e^x$$
$$f^{(4)}(x) = 72 - e^x$$

14. Product Rule

$$\frac{d}{dt}\big((t+1)(t^2 - t + 1)\big) = 1(t^2 - t + 1) + (2t - 1)(t + 1) = 3t^2$$

$$\frac{d}{dx}\left(x^5 e^x\right) = 5x^4(e^x) + e^x(x^5) = e^x\left(x^5 + 5x^4\right)$$

15. Quotient Rule

$$\begin{aligned}
\frac{d}{dx}\left(\frac{x^4 + 5x^2 + x}{\sin x}\right) &= \frac{\sin x(4x^3 + 10x + 1) - (x^4 + 5x^2 + x)\cos x}{\sin^2 x} && (1)\\
&= \frac{(4x^3 + 10x + 1) - (x^4 + 5x^2 + x)\cot x}{\sin x} && (2)
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dx}\left(\frac{2e^x - 1}{3e^x + 1}\right) &= \frac{2e^x(2e^x + 1) - 2e^x(2e^x - 1)}{(2e^x + 1)^2} && (1)\\
&= \frac{2e^x(2e^x + 1 - 2e^x + 1)}{(2e^x + 1)^2} && (2)\\
&= \frac{2e^x(2)}{(2e^x + 1)^2} && (3)\\
&= \frac{4e^x}{(2e^x + 1)^2} && (4)
\end{aligned}$$

16. Tangent Lines with Quotient Rule

$$f(x) = \frac{2x^2}{3x - 1}$$

$$(1, 1)$$

$$f(1) = 1$$

$$f'(1) = \frac{4x(3x - 1) - 3(2x^2)}{(3x - 1)^2} = \frac{2}{4} = \frac{1}{2}$$

$$y = \frac{1}{2}x + \frac{1}{2}$$

17. Power Rule with Negative Powers

$$\begin{aligned}\frac{d}{dx}\left(\frac{-5}{x^6}\right) &= \frac{d}{dx}\left(-5x^{-6}\right) && (1)\\ &= -5\left(-6x^{-7}\right) && (2)\\ &= 30x^{-7} && (3)\\ &= \frac{1}{30x^7} && (4)\end{aligned}$$

$$\begin{aligned}\frac{d}{dp}\left(\frac{2p^8-7}{p^3}\right) &= \frac{d}{dp}2p^5-7p^{-3} && (1)\\ &= 10p^4+21p^{-4} && (2)\\ &= \frac{10p^4}{21p^4} && (3)\end{aligned}$$

18. Combining Rules to Evaluate Derivatives

$$f(x)=\left(\frac{x^2+1}{x}\right)e^x=\left(x+x^{-1}\right)e^x$$

$$f'(x)=\left(1+-x^{-2}\right)e^x+\left(x+x^{-1}\right)e^x$$

19. Evaluating Trigonometric Limits

$$\lim_{x\to0}\frac{\sin 3x}{x}=3$$

$$\lim_{x\to0}\frac{\tan 7x}{\sin x}=\lim_{x\to0}\frac{\sin 7x}{\cos 7x\sin x}=\lim_{x\to0}\frac{\sin 7x}{\sin x}\cdot\frac{1}{\cos 7x}=7\cdot1=1$$

20. Finding Trigonometric Derivatives

$$\frac{d}{dx}(x\sin x) = \sin x+x\cos x \qquad (1)$$

$$\begin{aligned}\frac{d}{dx}\left(\frac{\cos x}{e^x}\right) &= \frac{-\sin xe^x-\cos xe^x}{\left(e^x\right)^2} && (1)\\ &= \frac{e^x(-\sin x-\cos x)}{\left(e^x\right)^2} && (2)\\ &= \frac{-\sin x-\cos x}{e^x} && (3)\\ & && (4)\end{aligned}$$

21. Finding Other Trigonometric Derivatives

$$\begin{aligned}e^x\csc x &= e^x\csc x-\csc x\cot xe^x && (1)\\ &= e^x\csc x(1-\cot x) && (2)\end{aligned}$$

22. Higher-order Trigonometric Derivatives

$$\begin{aligned}\frac{d}{dx}\left(\frac{d}{dx}(\tan x)\right) &= \frac{d}{dx}\left(\sec^2 x\right) && (1)\\ &= \frac{d}{dx}(\sec x\sec x) && (2)\\ &= \sec^2 x\tan x+\sec^2 x\tan x && (3)\\ &= 2\left(\sec^2 x\tan x\right) && (4)\end{aligned}$$

23. Velocity

$$\frac{600}{1.5}=400$$

$$\frac{-300}{1}=-300$$

The velocity at $2.75\le t\le 4.75$, where the plane is probably stationary at minneapolis.
$v(6)=-400$, because they are getting closer to seattle

24. Position, velocity, and acceleration
    Hint: the object is stationary when $v(t) = 0$

$$s = -t^2 + 4t - 3$$

$$0 \leq t \leq 5$$

$$v(t) = s'(t) = -2t + 4$$

$$\begin{aligned} -2t + 4 &= 0 && (1)\\ -2t &= -4 && (2)\\ t &= 2 && (3) \end{aligned}$$

The object is stationary after 2 seconds.

$$a(t) = f''(t) = v'(t) = -2$$

After 1 second, the acceleration is -2
The acceleration when the velocity is zero ($t = 2$), is -2

25. Growth Rates

$$p_{av} = \frac{p(6) - p(0)}{6} = \frac{7.1788 - 6.73}{6} = \frac{0.4488}{6} = 0.0748$$

$$p'(t) = 0.0156t + 0.028$$

$$p'(1) = 0.0436$$

$$p'(5) = 0.106$$

26. Average and marginal cost

$$C(x) = -0.04x^2 + 100x + 800$$

$$\overline{C}(x) = \frac{-0.04x^2 + 100x + 800}{x} = -0.04x + 100 + \frac{800}{x}$$

$$C'(x) = -0.08x + 100$$

$$\overline{C}(500) = -0.04(500) + 100 + \frac{800}{500} = -20 + 100 + 1.6 = 81.6$$

$$C'(500) = -0.08(500) + 100 = 60$$

Related Exercises

1. (Section 3.1, Related Exercise 13)

$$s(t) = -16t^2 + 100t$$

$$a = 1$$

$$\begin{aligned} \lim_{h \to 0} \frac{s(a+h) - s(a)}{h} &= \lim_{h \to 0} \frac{s(1+h) - 84}{h} && (1)\\ &= \lim_{h \to 0} \frac{-16(1+h)^2 + 100(1+h) - 84}{h} && (2)\\ &= \lim_{h \to 0} \frac{-16(h^2 + 2h + 1) + 100 + 100h - 84}{h} && (3)\\ &= \lim_{h \to 0} \frac{-16h^2 - 32h - 16 + 100 + 100h - 84}{h} && (4)\\ &= \lim_{h \to 0} \frac{-16h^2 + 68h}{h} && (5)\\ &= \lim_{h \to 0} -16h + 68 && (6)\\ &= -16(0) + 68 && (7)\\ &= 68 && (8) \end{aligned}$$

2. (Section 3.1, Related Exercise 14)

$$s(t) = -16t^2 + 128t + 192$$

$$a = 2$$

$$\begin{aligned}
\lim_{h\to 0} \frac{s(a+h)-s(a)}{h} &= \lim_{h\to 0} \frac{s(2+h)-384}{h} && (1)\\
&= \lim_{h\to 0} \frac{-16(2+h)^2+128(2+h)+192-384}{h} && (2)\\
&= \lim_{h\to 0} \frac{-16(h^2+4h+4)+128(2+h)+192-384}{h} && (3)\\
&= \lim_{h\to 0} \frac{-16h^2-64h-64+256+128h+192-384}{h} && (4)\\
&= \lim_{h\to 0} \frac{-16h^2+64h}{h} && (5)\\
&= \lim_{h\to 0} -16h+64 && (6)\\
&= -16(0)+64 && (7)\\
&= 64 && (8)
\end{aligned}$$

3. (Section 3.1, Related Exercise 17)

$$f(x) = \frac{1}{x}$$

$$P(-1,-1)$$

$$\begin{aligned}
\lim_{x\to -1} \frac{f(x)-f(-1)}{x-(-1)} &= \lim_{x\to -1} \frac{f(x)-(-1)}{x+1} && (1)\\
&= \lim_{x\to -1} \frac{f(x)+1}{x+1} && (2)\\
&= \lim_{x\to -1} \frac{\frac{1}{x}+1}{x+1} && (3)\\
&= \lim_{x\to -1} \frac{\frac{1+x}{x}}{x+1} && (4)\\
&= \lim_{x\to -1} \frac{\frac{1+x}{x}}{x+1}\cdot\frac{x}{x} && (5)\\
&= \lim_{x\to -1} \frac{1+x}{(x+1)x} && (6)\\
&= \lim_{x\to -1} \frac{1}{x} && (7)\\
&= \frac{1}{-1} && (8)\\
&= -1 && (9)
\end{aligned}$$

$$y-(-1) = -1(x-(-1))$$

$$y = -1(x+1)-1 = -x-1-1 = -x-2$$

4. (Section 3.1, Related Exercise 18)

$$f(x) = \frac{4}{x^2}$$

$$(-1,4)$$

$$\begin{aligned}
\lim_{x\to -1} \frac{f(x)-4}{x-(-1)} &= \lim_{x\to -1} \frac{f(x)-4}{x+1} && (1)\\
&= \lim_{x\to -1} \frac{\frac{4}{x^2}-4}{x+1} && (2)\\
&= \lim_{x\to -1} \frac{\frac{4-4x^2}{x^2}}{x+1} && (3)\\
&= \lim_{x\to -1} \frac{\frac{4-4x^2}{x^2}}{x+1} \cdot \frac{x^2}{x^2} && (4)\\
&= \lim_{x\to -1} \frac{4-4x^2}{x^2(x+1)} && (5)\\
&= \lim_{x\to -1} \frac{4(1-x^2)}{x^2(x+1)} && (6)\\
&= \lim_{x\to -1} \frac{4(1-x)(1+x)}{x^2(x+1)} && (7)\\
&= \lim_{x\to -1} \frac{4(1-x)}{x^2} && (8)\\
&= \frac{4(1-(-1)^2)}{(-1)^2} && (9)\\
&= \frac{4(1-1)}{1} && (10)\\
&= 0 && (11)
\end{aligned}$$

$$y-4=0$$

$$y=4$$

5. (Section 3.1, Related Exercise 23)

$$f(x)=3x^2-4x$$

$$(1,-1)$$

$$\begin{aligned}
\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} &= \lim_{h\to 0} \frac{f(1+h)-(-1)}{h} && (1)\\
&= \lim_{h\to 0} \frac{3(1+h)^2-4(1+h)+1}{h} && (2)\\
&= \lim_{h\to 0} \frac{3h^2+6h+3-4-4h+1}{h} && (3)\\
&= \lim_{h\to 0} \frac{3h^2+2h+3-4+1}{h} && (4)\\
&= \lim_{h\to 0} \frac{3h^2+2h}{h} && (5)\\
&= \lim_{h\to 0} 3h+2 && (6)\\
&= 3(0)+2 && (7)\\
&= 2 && (8)
\end{aligned}$$

$$y-(-1)=2(x-1)$$

$$y=2(x-1)-1=2x-2-1=2x-3$$

6. (Section 3.1, Related Exercise 27)

$$f(x)=x^3$$

$$(1,1)$$

$$
\begin{aligned}
\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} &= \lim_{h\to 0}\frac{f(1+h)-1}{h} && (1)\\
&= \lim_{h\to 0}\frac{(1+h)^3-1}{h} && (2)\\
&= \lim_{h\to 0}\frac{h^3+h^2+2h+2h^2+h+1-1}{h} && (3)\\
&= \lim_{h\to 0}\frac{h^3+2h^2+3h}{h} && (4)\\
&= \lim_{h\to 0} h^2+2h+3 && (5)\\
&= 0^2+2(0)+3 && (6)\\
&= 3 && (7)
\end{aligned}
$$

$$y-1=3(x-1)$$

$$y=3x-3+1=3x-2$$

7. (Section 3.1, Related Exercise 39)

$$f(x)=\sqrt{2x+1}$$

$$a=4$$

$$
\begin{aligned}
\lim_{x\to 4}\frac{f(x)-f(a)}{x-a} &= \lim_{x\to 4}\frac{f(x)-3}{x-4} && (1)\\
&= \lim_{x\to 4}\frac{\sqrt{2x+1}-3}{x-4} && (2)\\
&= \lim_{x\to 4}\frac{\sqrt{2x+1}-3}{x-4}\cdot\frac{\sqrt{2x+1}+3}{\sqrt{2x+1}+3} && (3)\\
&= \lim_{x\to 4}\frac{2x+1-9}{(x-4)(\sqrt{2x+1}+3)} && (4)\\
&= \lim_{x\to 4}\frac{2(x-4)}{(x-4)(\sqrt{2x+1}+3)} && (5)\\
&= \lim_{x\to 4}\frac{2}{\sqrt{2x+1}+3} && (6)\\
&= \frac{2}{\sqrt{9}+3} && (7)\\
&= \frac{2}{3+3} && (8)\\
&= \frac{2}{6} && (9)\\
&= \frac{1}{3} && (10)
\end{aligned}
$$

$$y-3=\frac{1}{3}(x-4)$$

$$y=\frac{1}{3}x-\frac{4}{3}+\frac{9}{3}=\frac{1}{3}x+\frac{5}{3}$$

8. (Section 3.1, Related Exercise 40)

$$f(x)=\sqrt{3x}$$

$$a=12$$

$$\begin{aligned}
\lim_{x\to 12}\frac{f(x)-f(a)}{x-a} &= \lim_{x\to 12}\frac{f(x)-6}{x-12} && (1)\\
&= \lim_{x\to 12}\frac{\sqrt{3x}-6}{x-12} && (2)\\
&= \lim_{x\to 12}\frac{\sqrt{3x}-6}{x-12}\cdot\frac{\sqrt{3x}+6}{\sqrt{3x}+6} && (3)\\
&= \lim_{x\to 12}\frac{3x-36}{(x-12)(\sqrt{3x}+6)} && (4)\\
&= \lim_{x\to 12}\frac{3(x-12)}{(x-12)(\sqrt{3x}+6)} && (5)\\
&= \lim_{x\to 12}\frac{3}{\sqrt{3x}+6} && (6)\\
&= \frac{3}{\sqrt{3(12)}+6} && (7)\\
&= \frac{3}{\sqrt{36}+6} && (8)\\
&= \frac{3}{6+6} && (9)\\
&= \frac{3}{12} && (10)\\
&= \frac{1}{4} && (11)
\end{aligned}$$

$$y-6=\frac{1}{4}(x-12)$$

$$y=\frac{1}{4}x-3+6=\frac{1}{4}x+3$$

9. (Section 3.1, Related Exercise 49)

$$d(t)=16t^2$$

$$a=4$$

$$\begin{aligned}
\lim_{x\to 4}\frac{f(x)-f(a)}{x-a} &= \lim_{h\to 0}\frac{f(x)-256}{x-4} && (1)\\
&= \lim_{x\to 4}\frac{16x^2-256}{x-4} && (2)\\
&= \lim_{x\to 4}\frac{(16x+64)(x-4)}{x-4} && (3)\\
&= \lim_{x\to 4} 16x+64 && (4)\\
&= 16(4)+64 && (5)\\
&= 64+64 && (6)\\
&= 128 && (7)
\end{aligned}$$

10. (Section 3.1, Related Exercise 50)

$$F(x)=\frac{k}{x^2}\text{ where }k\text{ is some constant}$$

$$a=1$$

$$
\begin{aligned}
\lim_{h\to 0}\frac{F(a+h)-F(a)}{h} &= \lim_{h\to 0}\frac{F(1+h)-\frac{k}{1}}{h} && (1)\\
&= \lim_{h\to 0}\frac{F(1+h)-\frac{k}{1}}{h} && (2)\\
&= \lim_{h\to 0}\frac{\frac{k}{(1+h)^2}-\frac{k}{1}}{h} && (3)\\
&= \lim_{h\to 0}\frac{\frac{k}{(1+h)^2}-\frac{k(1+h)^2}{(1+h)^2}}{h} && (4)\\
&= \lim_{h\to 0}\frac{\frac{k-k(1+h)^2}{(1+h)^2}}{h} && (5)\\
&= \lim_{h\to 0}\frac{\frac{k-k(1+h)^2}{(1+h)^2}}{h} && (6)\\
&= \lim_{h\to 0}\frac{\frac{k-(kh^2+2kh+k)}{(1+h)^2}}{h} && (7)\\
&= \lim_{h\to 0}\frac{\frac{k-kh^2-2kh-k}{(1+h)^2}}{h} && (8)\\
&= \lim_{h\to 0}\frac{\frac{-kh^2-2kh}{(1+h)^2}}{h} && (9)\\
&= \lim_{h\to 0}\frac{-kh^2-2kh}{(1+h)^2}\cdot\frac{1}{h} && (10)\\
&= \lim_{h\to 0}\frac{h(-kh-2k)}{h(1+h)^2} && (11)\\
&= \lim_{h\to 0}\frac{-kh-2k}{(1+h)^2} && (12)\\
&= \frac{-kh-2k}{(1+h)^2} && (13)\\
&= \frac{-k(0)-2k}{(1+0)^2} && (14)\\
&= \frac{-2k}{1} && (15)\\
&= -2k && (16)
\end{aligned}
$$

11. (Section 3.1, Related Exercise 53)
    Hint: Sketch a Secant Line

$$L'(1.5) \approx 4$$

$$L'(a) \approx 0 \text{ where } a \geq 4$$

12. (Section 3.1, Related Exercise 54)

$$D'(60) \approx 0.6$$

$$D'(170) \approx 0$$

13. (Section 3.2, Related Exercise 23)

$$f(x) = 4x^2 + 1$$

$$a = 2, 4$$

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} && (1)\\
&= \lim_{h\to 0}\frac{4(x+h)^2+1-(4x^2+1)}{h} && (2)\\
&= \lim_{h\to 0}\frac{4(x^2+2xh+h^2)+1-4x^2-1}{h} && (3)\\
&= \lim_{h\to 0}\frac{4x^2+8xh+4h^2-4x^2}{h} && (4)\\
&= \lim_{h\to 0}\frac{8xh+4h^2}{h} && (5)\\
&= \lim_{h\to 0} 8x+4h && (6)\\
&= 8x+4(0) && (7)\\
&= 8x && (8)
\end{aligned}
$$

$$f'(2) = 8(2) = 16$$

$$f'(4) = 8(4) = 32$$

14. (Section 3.2, Related Exercise 24)

$$f(x) = x^2 + 3x$$

$$a = -1, 4$$

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} && (1)\\
&= \lim_{h\to 0}\frac{(x+h)^2+3(x+h)-(x^2+3x)}{h} && (2)\\
&= \lim_{h\to 0}\frac{x^2+h^2+xh+3x+3h-x^2-3x}{h} && (3)\\
&= \lim_{h\to 0}\frac{h^2+xh+3h}{h} && (4)\\
&= \lim_{h\to 0} h+x+3 && (5)\\
&= 0+x+3 && (6)\\
&= x+3 && (7)
\end{aligned}
$$

$$f'(-1) = -1 + 3 = 2$$

$$f'(4) = 4 + 3 = 7$$

15. (Section 3.2, Related Exercise 37)

$$f(x) = \sqrt{3x+1}$$

$$a = 8$$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} && (1)\\
&= \lim_{h\to 0}\frac{\sqrt{3(x+h)+1}-\sqrt{3x+1}}{h} && (2)\\
&= \lim_{h\to 0}\frac{\sqrt{3(x+h)+1}-\sqrt{3x+1}}{h}\cdot\frac{\sqrt{3(x+h)+1}+\sqrt{3x+1}}{\sqrt{3(x+h)+1}+\sqrt{3x+1}} && (3)\\
&= \lim_{h\to 0}\frac{3(x+h)+1-(3x+1)}{h(\sqrt{3(x+h)+1}+\sqrt{3x+1})} && (4)\\
&= \lim_{h\to 0}\frac{3x+3h+1-3x-1}{h(\sqrt{3(x+h)+1}+\sqrt{3x+1})} && (5)\\
&= \lim_{h\to 0}\frac{3h}{h(\sqrt{3(x+h)+1}+\sqrt{3x+1})} && (6)\\
&= \lim_{h\to 0}\frac{3}{\sqrt{3(x+h)+1}+\sqrt{3x+1}} && (7)\\
&= \frac{3}{\sqrt{3(x+0)+1}+\sqrt{3x+1}} && (8)\\
&= \frac{3}{\sqrt{3x+1}+\sqrt{3x+1}} && (9)\\
&= \frac{3}{2\sqrt{3x+1}} && (10)
\end{aligned}$$

$$f'(8)=\frac{3}{2\sqrt{3(8)+1}}=\frac{3}{2\sqrt{24+1}}=\frac{3}{2\sqrt{25}}=\frac{3}{2(5)}=\frac{3}{10}$$

$$y-f(8)=f'(8)(x-8)$$

$$y=\frac{3}{10}(x-8)+5=\frac{3}{10}x-\frac{12}{5}+5=\frac{3}{10}x+\frac{13}{5}$$

16. (Section 3.2, Related Exercise 38)

$$f(x)=\sqrt{x+2}$$

$$a=7$$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} && (1)\\
&= \lim_{h\to 0}\frac{\sqrt{x+h+2}-\sqrt{x+2}}{h} && (2)\\
&= \lim_{h\to 0}\frac{\sqrt{x+h+2}-\sqrt{x+2}}{h}\cdot\frac{\sqrt{x+h+2}+\sqrt{x+2}}{\sqrt{x+h+2}+\sqrt{x+2}} && (3)\\
&= \lim_{h\to 0}\frac{x+h+2-(x+2)}{h(\sqrt{x+h+2}+\sqrt{x+2})} && (4)\\
&= \lim_{h\to 0}\frac{x+h+2-x-2}{h(\sqrt{x+h+2}+\sqrt{x+2})} && (5)\\
&= \lim_{h\to 0}\frac{h}{h(\sqrt{x+h+2}+\sqrt{x+2})} && (6)\\
&= \lim_{h\to 0}\frac{1}{\sqrt{x+h+2}+\sqrt{x+2}} && (7)\\
&= \frac{1}{\sqrt{x+0+2}+\sqrt{x+2}} && (8)\\
&= \frac{1}{\sqrt{x+2}+\sqrt{x+2}} && (9)\\
&= \frac{1}{2\sqrt{x+2}} && (10)
\end{aligned}$$

$$f'(7)=\frac{1}{2\sqrt{7+2}}=\frac{1}{2\sqrt{9}}=\frac{1}{2\cdot 3}=\frac{1}{6}$$

$$y - f(7) = f'(7)(x-7)$$

$$y = \frac{1}{6}(x-7) + 3 = \frac{1}{6}x - \frac{7}{6} + 3 = \frac{1}{6}x + \frac{11}{6}$$

17. (Section 3.2, Related Exercise 25)

$$f(x) = \frac{1}{x+1}$$

$$a = -\frac{1}{2}, 5$$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} && (1)\\
&= \lim_{h\to 0} \frac{\frac{1}{x+h+1} - \frac{1}{x+1}}{h} && (2)\\
&= \lim_{h\to 0} \frac{\frac{1}{x+h+1} - \frac{1}{x+1}}{h} && (3)\\
&= \lim_{h\to 0} \frac{\frac{x+1-(x+h+1)}{(x+h+1)(x+1)}}{h} && (4)\\
&= \lim_{h\to 0} \frac{x+1-x-h-1}{h(x+h+1)(x+1)} && (5)\\
&= \lim_{h\to 0} \frac{-h}{h(x+h+1)(x+1)} && (6)\\
&= \lim_{h\to 0} \frac{-1}{(x+h+1)(x+1)} && (7)\\
&= \frac{-1}{(x+0+1)(x+1)} && (8)\\
&= \frac{-1}{(x+1)(x+1)} && (9)\\
&= \frac{-1}{(x+1)^2} && (10)
\end{aligned}$$

$$f'(-\frac{1}{2}) = \frac{-1}{(-\frac{1}{2}+1)^2} = \frac{-1}{(\frac{1}{2})^2} = \frac{-1}{\frac{1}{4}} = -1 \cdot 4 = -4$$

$$f'(5) = \frac{-1}{(5+1)^2} = \frac{-1}{(6)^2} = \frac{-1}{36}$$

18. (Section 3.2, Related Exercise 27)

$$f(t) = \frac{1}{\sqrt{t}}$$

$$a = 9, \frac{1}{4}$$

$$
\begin{aligned}
f'(t) &= \lim_{h\to 0} \frac{f(t+h)-f(t)}{h} && (1)\\
&= \lim_{h\to 0} \frac{\frac{1}{\sqrt{t+h}} - \frac{1}{\sqrt{t}}}{h} && (2)\\
&= \lim_{h\to 0} \frac{\frac{\sqrt{t}-\sqrt{t+h}}{\sqrt{t+h}\sqrt{t}}}{h} && (3)\\
&= \lim_{h\to 0} \frac{\frac{\sqrt{t}-\sqrt{t+h}}{\sqrt{t+h}\sqrt{t}} \cdot \frac{\sqrt{t}+\sqrt{t+h}}{\sqrt{t}+\sqrt{t+h}}}{h} && (4)\\
&= \lim_{h\to 0} \frac{t-(t+h)}{h(t\sqrt{t+h}+(t+h)\sqrt{t})} && (5)\\
&= \lim_{h\to 0} \frac{-h}{h(t\sqrt{t+h}+(t+h)\sqrt{t})} && (6)\\
&= \lim_{h\to 0} \frac{-1}{t\sqrt{t+h}+(t+h)\sqrt{t}} && (7)\\
&= \frac{-1}{t\sqrt{t+0}+(t+0)\sqrt{t}} && (8)\\
&= \frac{-1}{t\sqrt{t}+t\sqrt{t}} && (9)\\
&= \frac{-1}{2(t\sqrt{t})} && (10)
\end{aligned}
$$

$$
f'(9) = \frac{-1}{2\left(9\sqrt{9}\right)} = \frac{-1}{2\,(9\cdot 3)} = \frac{-1}{2\,(27)} = \frac{-1}{54}
$$

$$
f'\left(\frac{1}{4}\right) = \frac{-1}{2\left(\frac{1}{4}\sqrt{\frac{1}{4}}\right)} = \frac{-1}{2\left(\frac{1}{4}\cdot\frac{1}{2}\right)} = \frac{-1}{2\left(\frac{1}{8}\right)} = \frac{-1}{\frac{2}{8}} = -1\,(4) = -4
$$

19. (Section 3.2, Related Exercise 53)
 f is not continuous at $x = 1$
 f is not differentiable at $x = 1, 2$

20. (Section 3.2, Related Exercise 54)
 f is not continuous at $x = 1$
 f is not differentiable at $x = 1, 2$

21. (Section 3.3, Related Exercise 19)
$$\frac{d}{dx}\left(x^5\right) = 5x^4$$

22. (Section 3.3, Related Exercise 22)
$$\frac{d}{dx}\left(e^3\right) = 0$$

23. (Section 3.3, Related Exercise 23)
$$\frac{d}{dx}\left(5x^3\right) = 15x^2$$

24. (Section 3.3, Related Exercise 24)
$$\frac{d}{dx}\left(\frac{5}{6}w^{12}\right) = 10w^{11}$$

25. (Section 3.3, Related Exercise 28)
$$\frac{d}{dx}\left(6\sqrt{t}\right) = \frac{3}{\sqrt{t}}$$

26. (Section 3.3, Related Exercise 31)
$$\frac{d}{dx}\left(3x^4 + 7x\right) = 12x^3 + 7$$

27. (Section 3.3, Related Exercise 33)
$$\frac{d}{dx}\left(10x^4 - 32x + e^2\right) = 40x^3 - 32$$

28. (Section 3.3, Related Exercise 60)

$$f(x) = x^3 - 4x^2 + 2x - 1$$
$$a = 2$$
$$f'(x) = 3x^2 - 8x + 2$$
$$y = -2x - 1$$

29. (Section 3.3, Related Exercise 61)

$$f(x) = e^x$$
$$a = \ln 3$$
$$f'(x) = e^x$$
$$y = 3x - 3(\ln 3) + 3$$

30. (Section 3.3, Related Exercise 63)

$$f(x) = x^2 - 6x + 5$$
$$f'(x) = 2x - 6$$
$$y = f(x) \text{ is } 0, \text{ when } x = 3$$
$$y = f(x) \text{ is } 2, \text{ when } x = 4$$

31. (Section 3.3, Related Exercise 64)

$$f(t) = t^3 - 27t + 5$$
$$f'(t) = 3t^2 - 27$$
$$y = f(t) \text{ is } 0, \text{ when } x = 3$$
$$y = f(t) \text{ is } 21, \text{ when } x = 4$$

32. (Section 3.3, Related Exercise 69)

$$f(x) = 5x^4 + 10x^3 + 3x + 6$$
$$f'(x) = 20x^3 + 30x^2 + 3$$
$$f''(x) = 60x^2 + 60x$$
$$f'''(x) = 120x + 60$$

33. (Section 3.3, Related Exercise 70)

$$f(x) = 3x^2 + 5e^x$$
$$f'(x) = 6x + 5e^x$$
$$f''(x) = 6 + 5e^x$$
$$f'''(x) = 5e^x$$

34. (Section 3.4, Related Exercise 19)

$$\frac{d}{dx}\left(3x^4(2x^2 - 1)\right) = 12x^3(2x^2 - 1) + 3x^4(4x) = 36x^5 - 12x^3$$

35. (Section 3.4, Related Exercise 20)

$$\frac{d}{dx}\left(6x - 2xe^x\right) = 6 - 2(e^x + 2xe^x) = 6 - 2e^x - 2xe^x$$

36. (Section 3.4, Related Exercise 22)

$$\begin{aligned}
\frac{d}{dx}\left(\frac{x^3 - 4x^2 + x}{x - 2}\right) &= \frac{(3x^2 - 8x + 1)(x - 2) - (x^3 - 4x^2 + x)}{(x - 2)^2} && (1)\\
&= \frac{3x^3 - 6x^2 - 8x^2 + 16x + x - 2 - x^3 + 4x^2 - x}{(x - 2)^2} && (2)\\
&= \frac{2x^3 + -10x^2 + 16x - 2}{(x - 2)^2} && (3)
\end{aligned}$$

37. (Section 3.4, Related Exercise 27)

$$\begin{aligned}
\frac{d}{dx}\left(xe^{-x}\right) &= \frac{d}{dx}\left(\frac{x}{e^x}\right) && (1)\\
&= \frac{e^x - xe^x}{\left(e^x\right)^2} && (2)\\
&= \frac{e^x(1-x)}{\left(e^x\right)^2} && (3)\\
&= \frac{1-x}{e^x} && (4)\\
&= e^{-x}\left(1-x\right) && (5)
\end{aligned}$$

38. (Section 3.4, Related Exercise 61)

$$f(x) = \frac{x+5}{x-1}$$
$$a = 3$$

$$\begin{aligned}
\frac{d}{dx}\left(\frac{x+5}{x-1}\right) &= \frac{(x-1)-(x+5)}{(x-1)^2} && (1)\\
&= \frac{x-1-x-5}{(x-1)^2} && (2)\\
&= \frac{-6}{(x-1)^2} && (3)
\end{aligned}$$

$$y = -\frac{3}{2}x + \frac{17}{2}$$

39. (Section 3.4, Related Exercise 62)

$$f(x) = \frac{2x^2}{3x-1}$$
$$a = 1$$

$$\begin{aligned}
\frac{d}{dx}\left(\frac{2x^2}{3x-1}\right) &= \frac{4x(3x-1)-3(2x^2)}{(3x-1)^2} && (1)\\
&= \frac{12x^2-4x-6x^2}{(3x-1)^2} && (2)\\
&= \frac{6x^2-4x}{(3x-1)^2} && (3)
\end{aligned}$$

$$y = \frac{1}{2}x + \frac{1}{2}$$

40. (Section 3.4, Related Exercise 28)

$$\frac{d}{dx}\left(e^x\sqrt[3]{x}\right) = e^x\sqrt[3]{x} + \frac{e^x}{3x^{\frac{2}{3}}} \quad (1)$$

41. (Section 3.4, Related Exercise 39)

$$\begin{aligned}
\frac{d}{dx}\left(3x^{-9}\right) &= -27x^{-10} && (1)\\
&= -\frac{27}{x^{10}} && (2)
\end{aligned}$$

42. (Section 3.4, Related Exercise 43)

$$\begin{aligned}
\frac{d}{dt}\left(\frac{t^3+3t^2+t}{t^3}\right) &= \frac{t^3(3t^2+6t+1)-3t^2(t^3+3t^2+t)}{(t^3)^2} && (1)\\
&= \frac{3t^5+6t^4+t^3-3t^5-9t^4-3t^3}{(t^3)^2} && (2)\\
&= \frac{-3t^4-2t^3}{t^6} && (3)\\
&= \frac{-3t^4}{t^6} - \frac{2t^3}{t^6} && (4)\\
&= \frac{-3}{t^2} - \frac{2}{t^3} && (5)
\end{aligned}$$

43. (Section 3.4, Related Exercise 51)

$$\begin{aligned}
\frac{d}{dw}\left(\frac{w^{\frac{5}{3}}}{w^{\frac{5}{3}}+1}\right) &= \frac{\frac{5}{3}w^{\frac{2}{3}}(w^{\frac{5}{3}}+1)-\frac{5}{3}w^{\frac{2}{3}}w^{\frac{5}{3}}}{\left(w^{\frac{5}{3}}+1\right)^2} && (1)\\
&= \frac{\frac{5}{3}w^{\frac{7}{3}}+\frac{5}{3}w^{\frac{2}{3}}-\frac{5}{3}w^{\frac{7}{3}}}{\left(w^{\frac{5}{3}}+1\right)^2} && (2)\\
&= \frac{\frac{5}{3}w^{\frac{2}{3}}}{\left(w^{\frac{5}{3}}+1\right)^2} && (3)\\
&= \frac{5w^{\frac{2}{3}}}{3\left(w^{\frac{5}{3}}+1\right)^2} && (4)
\end{aligned}$$

44. (Section 3.4, Related Exercise 45)

$$\begin{aligned}
\frac{d}{dx}\left(\frac{(x+1)e^x}{x-2}\right) &= \frac{(x-2)(2e^x+xe^x)-((x+1)e^x)}{(x-2)^2} && (1)\\
&= \frac{2xe^x+x^2e^x-4e^x-2xe^x-(xe^x+e^x)}{(x-2)^2} && (2)\\
&= \frac{2xe^x+x^2e^x-4e^x-2xe^x-xe^x-e^x}{(x-2)^2} && (3)\\
&= \frac{x^2e^x-xe^x-5e^x}{(x-2)^2} && (4)\\
&= \frac{e^x(x^2-x-5)}{(x-2)^2} && (5)
\end{aligned}$$

45. (Section 3.4, Related Exercise 46)

$$\begin{aligned}
\frac{d}{dx}\left(\frac{(x-1)(2x^2-1)}{(x^3-1)}\right) &= \frac{(6x^2-4x-1)(x^3-1)-3x^2(2x^3-2x^2-x+1)}{(x^3-1)^2} && (1)\\
&= \frac{6x^5-6x^2-4x^4+4x-x^3+1-6x^5+6x^4+3x^3-3x^2}{(x^3-1)^2} && (2)\\
&= \frac{2x^4+2x^3-9x^2+5}{(x^3-1)^2} && (3)\\
& && (4)
\end{aligned}$$

46. (Section 3.5, Related Exercise 12)

$$\begin{aligned}
\lim_{x\to 0}\frac{\sin 5x}{3x} &= \lim_{x\to 0}\frac{\sin 5x}{3x}\cdot\frac{5}{5} && (1)\\
&= \lim_{x\to 0}\frac{5\sin 5x}{15x} && (2)\\
&= \lim_{x\to 0}\frac{5}{3}\cdot\frac{\sin 5x}{5x} && (3)\\
&= \frac{5}{3}\cdot\lim_{x\to 0}\frac{\sin 5x}{5x} && (4)\\
&= \frac{5}{3}\cdot 1 && (5)\\
&= \frac{5}{3} && (6)
\end{aligned}$$

47. (Section 3.5, Related Exercise 13)

$$\begin{aligned}
\lim_{x\to 0}\frac{\sin 7x}{\sin 3x} &= \lim_{x\to 0}\frac{\left(\frac{\sin 7x}{x}\right)}{\left(\frac{\sin 3x}{x}\right)} && (1)\\
&= \lim_{x\to 0}\frac{\left(\frac{\sin 7x}{x}\right)}{\left(\frac{\sin 3x}{x}\right)} && (2)\\
&= \frac{\lim_{x\to 0}\left(\frac{\sin 7x}{x}\right)}{\lim_{x\to 0}\left(\frac{\sin 3x}{x}\right)} && (3)\\
&= \frac{7}{3} && (4)
\end{aligned}$$

48. (Section 3.5, Related Exercise 25)

$$\begin{aligned}
\frac{d}{dx}\left(e^{-x}\sin x\right) &= \frac{d}{dx}\left(\frac{\sin x}{e^x}\right) && (1)\\
&= \frac{\cos x e^x - \sin x e^x}{\left(e^x\right)^2} && (2)\\
&= \frac{e^x\left(\cos x - \sin x\right)}{e^{2x}} && (3)\\
&= \frac{\cos x - \sin x}{e^x} && (4)\\
&= e^{-x}\left(\cos x - \sin x\right) && (5)
\end{aligned}$$

49. (Section 3.5, Related Exercise 27)

$$\frac{d}{dx}\left(x\sin x\right) = \sin x + x\cos x \qquad (1)$$

50. (Section 3.5, Related Exercise 29)

$$\begin{aligned}
\frac{d}{dx}\left(\frac{\cos x}{\sin x + 1}\right) &= \frac{-\sin x(\sin x + 1) - \cos x\cos x}{\left(\sin x + 1\right)^2} && (1)\\
&= \frac{-\sin^2 x - \sin x - \cos^2 x}{\left(\sin x + 1\right)^2} && (2)\\
&= \frac{-(\sin^2 x + \cos^2 x) - \sin x}{\left(\sin x + 1\right)^2} && (3)\\
&= \frac{-1 - \sin x}{\left(\sin x + 1\right)^2} && (4)\\
&= \frac{-\left(\sin x + 1\right)}{\left(\sin x + 1\right)^2} && (5)\\
&= \frac{-1}{\sin x + 1} && (6)\\
&= -\frac{1}{\sin x + 1} && (7)
\end{aligned}$$

51. (Section 3.5, Related Exercise 52)

$$
\begin{aligned}
\frac{d}{dx}(\cot x) &= \frac{d}{dx}\left(\frac{1}{\tan x}\right) && (1)\\
&= \frac{d}{dx}\left(\frac{1}{\left(\frac{\sin x}{\cos x}\right)}\right) && (2)\\
&= \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) && (3)\\
&= \frac{-\sin x \sin x - \cos x \cos x}{\sin^2 x} && (4)\\
&= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} && (5)\\
&= -\frac{\sin^2 x}{\sin^2 x} - \frac{\cos^2 x}{\sin^2 x} && (6)\\
&= -1 - \frac{1-\sin^2 x}{\sin^2 x} && (7)\\
&= -1 - \left(\frac{1}{\sin^2 x} - \frac{\sin^2 x}{\sin^2 x}\right) && (8)\\
&= -1 - \frac{1}{\sin^2 x} + 1 && (9)\\
&= -\frac{1}{\sin^2 x} && (10)\\
&= -\csc^2 x && (11)
\end{aligned}
$$

52. (Section 3.5, Related Exercise 54)

$$
\begin{aligned}
\frac{d}{dx}(\csc x) &= \frac{d}{dx}\left(\frac{1}{\sin x}\right) && (1)\\
&= \frac{-\cos x}{\sin^2 x} && (2)\\
&= \frac{-\cot x}{\sin x} && (3)\\
&= -\cot x \csc x && (4)
\end{aligned}
$$

53. (Section 3.5, Related Exercise 43)

$$
\frac{d}{dx}(\sec x + \csc x) = \sec x \tan x - \csc x \cot x \qquad (1)
$$

54. (Section 3.5, Related Exercise 44)

$$
\begin{aligned}
\frac{d}{dx}(\sec x \tan x) &= \sec x \tan x \tan x + \sec^2 x \sec x && (1)\\
&= \sec x \tan^2 x + \sec^3 x && (2)
\end{aligned}
$$

55. (Section 3.5, Related Exercise 61)

$$
\begin{aligned}
\frac{d}{dx}\left(\frac{d}{dx}(\cot x)\right) &= \frac{d}{dx}\left(-\csc^2 x\right) && (1)\\
&= -2\left(-\csc x \cot x\right) && (2)\\
&= 2\csc x \cot x && (3)
\end{aligned}
$$

56. (Section 3.5, Related Exercise 62)

$$
\begin{aligned}
\frac{d}{dx}\left(\frac{d}{dx}(\tan x)\right) &= \frac{d}{dx}\left(\sec^2 x\right) && (1)\\
&= 2\sec x \tan x && (2)\\
& && (3)
\end{aligned}
$$

57. (Section 3.6, Related Exercise 11)

$$\frac{30}{0.75} = 40$$

$$\frac{30-10}{0.75-0.25} = \frac{20}{0.5} = 40$$

$$\frac{-15-15}{2.25-1.75} = \frac{-30}{0.5} = -60$$

Away, Back, Away (Other Way), Back

58. (Section 3.6, Related Exercise 12)

$$\frac{600}{1.5} = 400$$

$$\frac{-300}{8.5-7.5} = -300$$

$$2.75 \le t \le 5.25$$

$$f'(6) = -400$$

59. (Section 3.6, Related Exercise 15)

$$f(t) = t^2 - 4t$$

$$0 \le t \le 5$$

$$f'(t) = 2t - 4$$

$$f''(t) = 2$$

$$f'(1) = -2$$

$$f''(1) = 2$$

$$(2, 5]$$

60. (Section 3.6, Related Exercise 16)

$$f(t) = -t^2 + 4t - 3$$

$$0 \le t \le 5$$

$$f'(t) = -2t + 4$$

$$f''(t) = -2$$

$$f'(1) = 2$$

$$f''(1) = -2$$

$$[0, 5)$$

61. (Section 3.6, Related Exercise 24)

$$s(t) = -4.9t^2 + 19.6t + 24.5$$

$$v(t) = s'(t) = -9.8t + 19.6$$

$$t = 1$$

$$s(1) = -4.9 + 19.6 + 24.5 = 39.2$$

$$t = 5$$

$$v(5) = -49 + 19.6 = -29.4$$

62. (Section 3.6, Related Exercise 25)

$$s(t) = -16t^2 + 64t + 32$$

$$v(t) = s'(t) = 32t + 64$$

$$t = 2$$

$$s(2) = -16(4) + 64(2) + 32 = -64 + 128 + 32 = 96$$

$$t = 2 + \sqrt{6}$$

$$s(2+\sqrt{6}) = -16(2+\sqrt{6})^2 + 64(2+\sqrt{6}) + 32 = -32\sqrt{6}$$

63. (Section 3.6, Related Exercise 28)

$$p(t) = 0.0078t^2 + 0.028t + 6.73$$

$$\frac{7.1788 - 6.73}{6 - 0} = \frac{0.4488}{6} = 0.0748$$

$$p'(t) = 0.0156t + 0.028$$

$$p'(1) = 0.0156 + 0.028 = 0.0436$$

$$p'(5) = 0.0156(5) + 0.028 = 0.078 + 0.028 = 0.106$$

64. (Section 3.6, Related Exercise 29)

$$C(x) = 1000 + 0.1x$$

$$0 \leq x \leq 5000$$

$$a = 2000$$

$$\overline{C}(x) = \frac{1000}{x} + 0.1$$

$$C'(x) = 0.1$$

$$\overline{C}(2000) = \frac{1000}{2000} + 0.1 = \frac{1}{2} + 0.1 = 0.5 + 0.1 = 0.6$$

$$C'(2000) = 0.1$$

65. (Section 3.6, Related Exercise 30)

$$C(x) = 500 + 0.02x$$

$$0 \leq x \leq 2000$$

$$a = 1000$$

$$\overline{C}(x) = \frac{500}{x} + 0.02$$

$$C'(x) = 0.02$$

$$\overline{C}(x) = \frac{500}{1000} + 0.02 = 0.5 + 0.02 = 0.52$$

$$C'(x) = 0.02$$

66. (Section 3.6, Related Exercise 33)

$$D(p) = 40 - 2p$$

$$D(10) = 40 - 20 = 20$$

$$\$20$$

$$\begin{aligned}
\frac{d}{dp}(40 - 2p)\frac{p}{40 - 2p} &= -2\frac{p}{40 - 2p} && (1)\\
&= -2\frac{p}{40 - 2p} && (2)\\
&= -2\frac{p}{2(20 - p)} && (3)\\
&= -2\frac{p}{2}\frac{1}{20 - p} && (4)\\
&= -\frac{p}{20 - p} && (5)
\end{aligned}$$

Hint: For elasticity, when $E < -1$, it is elastic and vice versa
Elastic when $p > 10$ and inelastic when $0 < p < 10$
Hint: Use $\frac{\Delta p}{p}$

$$\frac{10.25 - 10.00}{10.00} = \frac{0.25}{10} = 0.025 = 2.5\%$$

Hint: Use $\frac{\Delta D}{D}$

$$E(10) \cdot \frac{D(10.25) - D(10.00)}{D(10.00)} = -1 \cdot \frac{19.5 - 20}{20} = -\frac{-0.5}{20} = 0.025 = 2.5\%$$

67. (Section 3.6, Related Exercise 34)

$$D(p) = \frac{1800}{p-40}$$

$$D(60) = \frac{1800}{60-40} = \frac{1800}{20} = 90$$

$$(40, \infty)$$

$$\begin{aligned}
\frac{d}{dp}\left(\frac{1800}{p-40}\right)\frac{p(p-40)}{1800} &= \frac{1800}{(p-40)^2}\frac{p(p-40)}{1800} && (1)\\
&= \frac{1800p\,(p-40)}{1800\,(p-40)^2} && (2)\\
&= \frac{p\,(p-40)}{(p-40)^2} && (3)\\
&= \frac{p\,(p-40)}{(p-40)^2} && (4)\\
&= \frac{p}{p-40} && (5)
\end{aligned}$$

$$20 < p < 40$$

$$E(60) \cdot \frac{D(62) - D(60)}{D(60)} = 3\frac{\frac{900}{11} - 90}{90} \approx -27\%$$

A copy of my notes (in LaTeX) are available on my GitHub