Module 2 Notes (MATH-211)

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General Notes (and Definitions)

- Derivatives
  - A **derivative** is a new function made up of the slopes of the tangent lines as they change along a curve
  - If a curve represents the trajectory of a moving object, the tangent line at a point indicates the direction of motion at that point
  - As $x \to a$, the slope of the secant lines approaches the slope of the tangent line
  - Alternative definition for Tangent Line(s): Consider the curve $y = f(x)$ and a secant line intersecting the curve at points $P(a, f(a))$ and $Q(a+h, f(a+h))$, with m_{sec} and m_{tan}

$$\text{Interval:}(a, a+h)$$
$$m_{sec} = \frac{f(a+h) - f(a)}{h}$$
$$m_{tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$
$$y - f(a) = m_{tan}(x - a)$$

  - **Definition**: The derivative of f at a, denoted $f'(a)$, is given by either the two following limits, provided the limits exist and a is in the domain of f

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad (1)$$
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad (2)$$
$$(3)$$

  If $f'(a)$ exists, we say that f is **differentiable** at a

Examples

1. Instantaneous Velocity

$$s(t) = -16t^2 + 128t + 192$$
$$t = 2$$

$$\begin{aligned}
\lim_{t \to 2} \frac{s(t) - s(2)}{t - 2} &= \lim_{t \to 2} \frac{(-16t^2 + 128t + 192) - (-16(2^2) + 128(2) + 192)}{t - 2} && (1)\\
&= \lim_{t \to 2} \frac{(-16t^2 + 128t + 192) - (-64 + 256 + 192)}{t - 2} && (2)\\
&= \lim_{t \to 2} \frac{(-16t^2 + 128t + 192) - 384}{t - 2} && (3)\\
&= \lim_{t \to 2} \frac{-16t^2 + 128t - 192}{t - 2} && (4)\\
&= \lim_{t \to 2} \frac{(t-2)(-16t + 96)}{t - 2} && (5)\\
&= \lim_{t \to 2} -16t + 96 && (6)\\
&= -32 + 96 && (7)\\
&= 64 && (8)
\end{aligned}$$

2. Secant Lines

$$y = f(x)$$

Intersection Points: $P(a, f(a))$ and $Q(x, f(x))$

$$\text{Secant Line Slope} = \frac{f(x) - f(a)}{x - a}$$

3. Tangent Lines

$$f(x) = 2x^2 + 4x - 3$$

$$(-1, 5)$$

$$\begin{aligned}
\lim_{x \to -1} \frac{f(x) - f(-1)}{x - (-1)} &= \lim_{x \to -1} \frac{2x^2 + 4x - 3 - (-5)}{x + 1} && (1)\\
&= \lim_{x \to -1} \frac{2x^2 + 4x + 2}{x + 1} && (2)\\
&= \lim_{x \to -1} \frac{(x+1)(2x+2)}{x + 1} && (3)\\
&= \lim_{x \to -1} \frac{(x+1)(2x+2)}{x + 1} && (4)\\
&= \lim_{x \to -1} 2x + 2 && (5)\\
&= 2(-1) + 2 && (6)\\
&= -2 + 2 && (7)\\
&= 0 && (8)
\end{aligned}$$

4. Alternative Tangent Lines

$$f(x) = 5 - x^3$$

$$(2, -3)$$

$$a = 2$$

$$h = -3 - 2 = -5$$

$$\begin{aligned}
\lim_{h \to 0} \frac{f(2+h) - f(2)}{h} &= \lim_{h \to 0} \frac{f(2+h) - (-3)}{h} && (1)\\
&= \lim_{h \to 0} \frac{f(2+h) + 3}{h} && (2)\\
&= \lim_{h \to 0} \frac{5 - (2+h)^3 + 3}{h} && (3)\\
&= \lim_{h \to 0} \frac{8 - (2+h)^3}{h} && (4)\\
&= \lim_{h \to 0} \frac{2^3 - (2+h)^3}{h} && (5)\\
&= \lim_{h \to 0} \frac{(2 - (2+h))(2^2 + 2(2+h) + (2+h)^2)}{h} && (6)\\
&= \lim_{h \to 0} \frac{-h(4 + 4 + 2h + h^2 + 4h + 4)}{h} && (7)\\
&= \lim_{h \to 0} \frac{-h(h^2 + 6h + 12)}{h} && (8)\\
&= \lim_{h \to 0} -(h^2 + 6h + 12) && (9)\\
&= -12 && (10)\\
& && (11)
\end{aligned}$$

$$y + 3 = -12(x - 2) = -12x + 24$$

$$y = -12x + 21$$

5. Derrivative Example

$$f(x) = \sqrt{x-1}$$

$$x = 2$$

$$f(x) = f(2) = \sqrt{2-1} = \sqrt{1} = 1$$

$$(2, 1)$$

$$\begin{aligned}
f'(2) &= \lim_{x \to 2} \frac{f(x) - f(2)}{x-2} && (1)\\
&= \lim_{x \to 2} \frac{\sqrt{x-1} - 1}{x-2} && (2)\\
&= \lim_{x \to 2} \frac{\sqrt{x-1} - 1}{x-2} \cdot \frac{\sqrt{x-1}+1}{\sqrt{x-1}+1} && (3)\\
&= \lim_{x \to 2} \frac{x-2}{(x-2)(\sqrt{x-1}+1)} && (4)\\
&= \lim_{x \to 2} \frac{1}{\sqrt{x-1}+1} && (5)\\
&= \frac{1}{\sqrt{2-1}+1} && (6)\\
&= \frac{1}{\sqrt{1}+1} && (7)\\
&= \frac{1}{1+1} && (8)\\
&= \frac{1}{2} && (9)\\
& && (10)
\end{aligned}$$

$$\begin{aligned}
y - 1 &= \frac{1}{2}(x-2) && (1)\\
y &= \frac{1}{2}(x-2) + 1 && (2)\\
&= \frac{1}{2}x - 1 + 1 && (3)\\
&= \frac{1}{2}x && (4)
\end{aligned}$$

6. Derrivative Application Example

$$V(t) = 3t$$

$$\begin{aligned}
V'(12) &= \lim_{x \to 12} \frac{V(x) - V(12)}{x-12} && (1)\\
&= \lim_{x \to 12} \frac{3x-36}{x-12} && (2)\\
&= \lim_{x \to 12} \frac{3(x-12)}{x-12} && (3)\\
&= \lim_{x \to 12} 3 && (4)\\
&= 3 && (5)
\end{aligned}$$

$$\begin{aligned}
y - 36 &= 3(x-12) && (1)\\
y &= 3x - 36 + 36 && (2)\\
&= 3x && (3)\\
& && (4)
\end{aligned}$$

Related Exercises

1. (Section 3.1, Related Exercise 13)

$$s(t) = -16t^2 + 100t$$

$$a = 1$$

$$\begin{aligned}
\lim_{h\to 0} \frac{s(a+h)-s(a)}{h} &= \lim_{h\to 0} \frac{s(1+h)-84}{h} && (1)\\
&= \lim_{h\to 0} \frac{-16(1+h)^2+100(1+h)-84}{h} && (2)\\
&= \lim_{h\to 0} \frac{-16(h^2+2h+1)+100+100h-84}{h} && (3)\\
&= \lim_{h\to 0} \frac{-16h^2-32h-16+100+100h-84}{h} && (4)\\
&= \lim_{h\to 0} \frac{-16h^2+68h}{h} && (5)\\
&= \lim_{h\to 0} -16h+68 && (6)\\
&= -16(0)+68 && (7)\\
&= 68 && (8)
\end{aligned}$$

2. (Section 3.1, Related Exercise 14)

$$s(t) = -16t^2 + 128t + 192$$

$$a = 2$$

$$\begin{aligned}
\lim_{h\to 0} \frac{s(a+h)-s(a)}{h} &= \lim_{h\to 0} \frac{s(2+h)-384}{h} && (1)\\
&= \lim_{h\to 0} \frac{-16(2+h)^2+128(2+h)+192-384}{h} && (2)\\
&= \lim_{h\to 0} \frac{-16(h^2+4h+4)+128(2+h)+192-384}{h} && (3)\\
&= \lim_{h\to 0} \frac{-16h^2-64h-64+256+128h+192-384}{h} && (4)\\
&= \lim_{h\to 0} \frac{-16h^2+64h}{h} && (5)\\
&= \lim_{h\to 0} -16h+64 && (6)\\
&= -16(0)+64 && (7)\\
&= 64 && (8)
\end{aligned}$$

3. (Section 3.1, Related Exercise 17)

$$f(x) = \frac{1}{x}$$

$$P(-1,-1)$$

$$\begin{aligned}
\lim_{x\to -1}\frac{f(x)-f(-1)}{x-(-1)} &= \lim_{x\to -1}\frac{f(x)-(-1)}{x+1} && (1)\\
&= \lim_{x\to -1}\frac{f(x)+1}{x+1} && (2)\\
&= \lim_{x\to -1}\frac{\frac{1}{x}+1}{x+1} && (3)\\
&= \lim_{x\to -1}\frac{\frac{1+x}{x}}{x+1} && (4)\\
&= \lim_{x\to -1}\frac{\frac{1+x}{x}}{x+1}\cdot\frac{x}{x} && (5)\\
&= \lim_{x\to -1}\frac{1+x}{(x+1)x} && (6)\\
&= \lim_{x\to -1}\frac{1}{x} && (7)\\
&= \frac{1}{-1} && (8)\\
&= -1 && (9)
\end{aligned}$$

$$y-(-1)=-1(x-(-1))$$

$$y=-1(x+1)-1=-x-1-1=-x-2$$

4. (Section 3.1, Related Exercise 18)

$$f(x)=\frac{4}{x^2}$$

$$(-1,4)$$

$$\begin{aligned}
\lim_{x\to -1}\frac{f(x)-4}{x-(-1)} &= \lim_{x\to -1}\frac{f(x)-4}{x+1} && (1)\\
&= \lim_{x\to -1}\frac{\frac{4}{x^2}-4}{x+1} && (2)\\
&= \lim_{x\to -1}\frac{\frac{4-4x^2}{x^2}}{x+1} && (3)\\
&= \lim_{x\to -1}\frac{\frac{4-4x^2}{x^2}}{x+1}\cdot\frac{x^2}{x^2} && (4)\\
&= \lim_{x\to -1}\frac{4-4x^2}{x^2(x+1)} && (5)\\
&= \lim_{x\to -1}\frac{4(1-x^2)}{x^2(x+1)} && (6)\\
&= \lim_{x\to -1}\frac{4(1-x)(1+x)}{x^2(x+1)} && (7)\\
&= \lim_{x\to -1}\frac{4(1-x)}{x^2} && (8)\\
&= \frac{4(1-(-1)^2)}{(-1)^2} && (9)\\
&= \frac{4(1-1)}{1} && (10)\\
&= 0 && (11)
\end{aligned}$$

$$y-4=0$$

$$y=4$$

5. (Section 3.1, Related Exercise 23)

$$f(x)=3x^2-4x$$

$$(1,-1)$$

$$\begin{aligned}
\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} &= \lim_{h\to 0}\frac{f(1+h)-(-1)}{h} && (1)\\
&= \lim_{h\to 0}\frac{3(1+h)^2-4(1+h)+1}{h} && (2)\\
&= \lim_{h\to 0}\frac{3h^2+6h+3-4-4h+1}{h} && (3)\\
&= \lim_{h\to 0}\frac{3h^2+2h+3-4+1}{h} && (4)\\
&= \lim_{h\to 0}\frac{3h^2+2h}{h} && (5)\\
&= \lim_{h\to 0} 3h+2 && (6)\\
&= 3(0)+2 && (7)\\
&= 2 && (8)
\end{aligned}$$

$$y-(-1)=2(x-1)$$

$$y=2(x-1)-1=2x-2-1=2x-3$$

6. (Section 3.1, Related Exercise 27)

$$f(x)=x^3$$

$$(1,1)$$

$$\begin{aligned}
\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} &= \lim_{h\to 0}\frac{f(1+h)-1}{h} && (1)\\
&= \lim_{h\to 0}\frac{(1+h)^3-1}{h} && (2)\\
&= \lim_{h\to 0}\frac{h^3+h^2+2h+2h^2+h+1-1}{h} && (3)\\
&= \lim_{h\to 0}\frac{h^3+2h^2+3h}{h} && (4)\\
&= \lim_{h\to 0} h^2+2h+3 && (5)\\
&= 0^2+2(0)+3 && (6)\\
&= 3 && (7)
\end{aligned}$$

$$y-1=3(x-1)$$

$$y=3x-3+1=3x-2$$

7. (Section 3.1, Related Exercise 39)

$$f(x)=\sqrt{2x+1}$$

$$a=4$$

$$
\begin{aligned}
\lim_{x\to 4}\frac{f(x)-f(a)}{x-a} &= \lim_{x\to 4}\frac{f(x)-3}{x-4} && (1)\\
&= \lim_{x\to 4}\frac{\sqrt{2x+1}-3}{x-4} && (2)\\
&= \lim_{x\to 4}\frac{\sqrt{2x+1}-3}{x-4}\cdot\frac{\sqrt{2x+1}+3}{\sqrt{2x+1}+3} && (3)\\
&= \lim_{x\to 4}\frac{2x+1-9}{(x-4)(\sqrt{2x+1}+3)} && (4)\\
&= \lim_{x\to 4}\frac{2(x-4)}{(x-4)(\sqrt{2x+1}+3)} && (5)\\
&= \lim_{x\to 4}\frac{2}{\sqrt{2x+1}+3} && (6)\\
&= \frac{2}{\sqrt{9}+3} && (7)\\
&= \frac{2}{3+3} && (8)\\
&= \frac{2}{6} && (9)\\
&= \frac{1}{3} && (10)
\end{aligned}
$$

$$y-3=\frac{1}{3}(x-4)$$

$$y=\frac{1}{3}x-\frac{4}{3}+\frac{9}{3}=\frac{1}{3}x+\frac{5}{3}$$

8. (Section 3.1, Related Exercise 40)

$$f(x)=\sqrt{3x}$$

$$a=12$$

$$
\begin{aligned}
\lim_{x\to 12}\frac{f(x)-f(a)}{x-a} &= \lim_{x\to 12}\frac{f(x)-6}{x-12} && (1)\\
&= \lim_{x\to 12}\frac{\sqrt{3x}-6}{x-12} && (2)\\
&= \lim_{x\to 12}\frac{\sqrt{3x}-6}{x-12}\cdot\frac{\sqrt{3x}+6}{\sqrt{3x}+6} && (3)\\
&= \lim_{x\to 12}\frac{3x-36}{(x-12)(\sqrt{3x}+6)} && (4)\\
&= \lim_{x\to 12}\frac{3(x-12)}{(x-12)(\sqrt{3x}+6)} && (5)\\
&= \lim_{x\to 12}\frac{3}{\sqrt{3x}+6} && (6)\\
&= \frac{3}{\sqrt{3(12)}+6} && (7)\\
&= \frac{3}{\sqrt{36}+6} && (8)\\
&= \frac{3}{6+6} && (9)\\
&= \frac{3}{12} && (10)\\
&= \frac{1}{4} && (11)
\end{aligned}
$$

$$y-6=\frac{1}{4}(x-12)$$

$$y=\frac{1}{4}x-3+6=\frac{1}{4}x+3$$

9. (Section 3.1, Related Exercise 49)

$$d(t) = 16t^2$$

$$a = 4$$

$$\begin{aligned}
\lim_{x\to 4} \frac{f(x)-f(a)}{x-a} &= \lim_{h\to 0} \frac{f(x)-256}{x-4} && (1)\\
&= \lim_{x\to 4} \frac{16x^2-256}{x-4} && (2)\\
&= \lim_{x\to 4} \frac{(16x+64)(x-4)}{x-4} && (3)\\
&= \lim_{x\to 4} 16x+64 && (4)\\
&= 16(4)+64 && (5)\\
&= 64+64 && (6)\\
&= 128 && (7)
\end{aligned}$$

10. (Section 3.1, Related Exercise 50)

$$F(x) = \frac{k}{x^2} \text{ where } k \text{ is some constant}$$

$$a = 1$$

$$\begin{aligned}
\lim_{h\to 0} \frac{F(a+h)-F(a)}{h} &= \lim_{h\to 0} \frac{F(1+h)-\frac{k}{1}}{h} && (1)\\
&= \lim_{h\to 0} \frac{F(1+h)-\frac{k}{1}}{h} && (2)\\
&= \lim_{h\to 0} \frac{\frac{k}{(1+h)^2}-\frac{k}{1}}{h} && (3)\\
&= \lim_{h\to 0} \frac{\frac{k}{(1+h)^2}-\frac{k(1+h)^2}{(1+h)^2}}{h} && (4)\\
&= \lim_{h\to 0} \frac{\frac{k-k(1+h)^2}{(1+h)^2}}{h} && (5)\\
&= \lim_{h\to 0} \frac{\frac{k-k(1+h)^2}{(1+h)^2}}{h} && (6)\\
&= \lim_{h\to 0} \frac{\frac{k-(kh^2+2kh+k)}{(1+h)^2}}{h} && (7)\\
&= \lim_{h\to 0} \frac{\frac{k-kh^2-2kh-k}{(1+h)^2}}{h} && (8)\\
&= \lim_{h\to 0} \frac{\frac{-kh^2-2kh}{(1+h)^2}}{h} && (9)\\
&= \lim_{h\to 0} \frac{-kh^2-2kh}{(1+h)^2} \cdot \frac{1}{h} && (10)\\
&= \lim_{h\to 0} \frac{h(-kh-2k)}{h(1+h)^2} && (11)\\
&= \lim_{h\to 0} \frac{-kh-2k}{(1+h)^2} && (12)\\
&= \frac{-kh-2k}{(1+h)^2} && (13)\\
&= \frac{-k(0)-2k}{(1+0)^2} && (14)\\
&= \frac{-2k}{1} && (15)\\
&= -2k && (16)
\end{aligned}$$

11. (Section 3.1, Related Exercise 53)
    Hint: Sketch a Secant Line

$$L'(1.5) \approx 4$$

$$L'(a) \approx 0 \text{ where } a \geq 4$$

12. (Section 3.1, Related Exercise 54)

$$D'(60) \approx 0.6$$

$$D'(170) \approx 0$$

A copy of my notes (in LaTeX) are available on my GitHub