

# Module 6 Notes (MATH-211)

Lillie Donato

15 July 2024

## General Notes (and Definitions)

- L'Hôpital's Rule

**Indeterminate Form:** An expression involving two components where the limit cannot be determined by evaluating the limits of the individual components.

**L'Hôpital's Rule:** Suppose  $f$  and  $g$  are differentiable functions on an open interval  $I$  containing the point  $x = a$ , with  $g'(x) \neq 0$  on  $I$  when  $x \neq a$ .

If  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$  has any of the indeterminate forms:  $\frac{0}{0}$ ,  $\frac{\infty}{\infty}$ ,  $-\frac{\infty}{\infty}$ , then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

provided that one of the following is the case:

$$\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} \in \mathbb{R}$$

$$\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = \infty$$

$$\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = -\infty$$

L'Hôpital's Rule is still valid if  $x \rightarrow a$  is replaced by any of  $x \rightarrow a^+$ ,  $x \rightarrow a^-$ ,  $x \rightarrow \infty$ , or  $x \rightarrow -\infty$ . In the last two of these cases, there must be a greatest  $x$ -value beyond which both  $f$  and  $g$  are differentiable at every point.

**Exponential Indeterminate forms:**  $1^\infty$ ,  $0^0$ ,  $\infty^0$

**Method for evaluating limits of indeterminate forms  $1^\infty$ ,  $0^0$ ,  $\infty^0$ :**

Assume that  $L = \lim_{x \rightarrow a} f(x)^{g(x)}$  has one of these indeterminate forms.

1. Use the fact that the natural logarithm and natural exponential functions are inverses to write

$$L = \lim_{x \rightarrow a} e^{\ln(f(x)^{g(x)})}$$

2. Use the power property of logarithm arguments to write

$$L = \lim_{x \rightarrow a} e^{g(x) \ln(f(x))}$$

3. Use continuity of the exponential function to write

$$L = e^{\lim_{x \rightarrow a} g(x) \ln(f(x))}$$

4. Rewrite multiplication as division by the reciprocal:

$$L = e^{\lim_{x \rightarrow a} \left( \frac{\ln(f(x))}{\frac{1}{g(x)}} \right)}$$

5. Use L'Hôpital's Rule to evaluate this limit expression

**Growth Rates:** Suppose  $f$  and  $g$  are functions with  $\lim_{x \rightarrow \infty} f(x) = \infty$  and  $\lim_{x \rightarrow \infty} g(x) = \infty$

1. If one of the following are true,  $f$  **grows faster than**  $g$ , and we use the notation  $f \gg g$

$$\lim_{x \rightarrow \infty} \frac{g(x)}{f(x)} = 0 \quad (1)$$

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \infty \quad (2)$$

2.  $f$  and  $g$  have **comparable growth rates**, if there is some non-zero finite number  $M$  such that

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = M$$

### Ranked Growth Rates as $x \rightarrow \infty$

For any base  $b > 1$ , and for any positive numbers  $p$ ,  $q$ ,  $r$ , and  $s$

$$\ln^q x \ll x^p \ll x^p \ln^r x \ll x^{p+s} \ll b^x \ll x^x$$

### • Antiderivatives

**Antiderivative:** A function  $F$  is an antiderivative of another function  $f$  on an interval  $I$  if for all  $x$  in  $I$ :

$$F'(x) = f(x)$$

**Family of Antiderivatives:** Let  $F(x)$  be any antiderivative of  $f(x)$  on an interval  $I$ . Then all antiderivatives of  $f$  on  $I$  have the form  $F(x) + C$ , where  $C$  is an arbitrary constant.

**Differential Equations:** Any equation involving an unknown function and its derivatives

- Infinite family of solutions
- No two solutions from the family pass through the same point
- Given an initial condition  $f(a) = b$ , we can identify the particular family member that solves the given problem by solving for  $C$

### • Approximating Areas Under Curves

- If we know the velocity function of a moving object, what can we learn about its position function?
- Given an object with velocity function  $v(t)$ , the displacement of the moving object over the interval  $[a, b]$  is the area between the velocity curve and the  $t$ -axis from  $t = a$  to  $t = b$ .
- Because objects do not necessarily move at a constant velocity, we can extend this idea to positive velocities that change over an interval of time.
- The strategy is to divide the time interval into many subintervals, approximate the velocity on each subinterval with a constant velocity, calculate the individual displacements and sum the results.

### Riemann Sums

- Suppose  $f(x)$  is continuous and non-negative on  $[a, b]$ .
- Goal is to approximate the area of the region  $R$  bounded by the graph of  $f(x)$  and the  $x$ -axis from  $x = a$  to  $x = b$ .
- Divide  $[a, b]$  into  $n$  subintervals  $[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]$  where  $a = x_0, b = x_n$ .
- The length of each subinterval is  $\Delta x = \frac{b-a}{n}$
- **Regular Partition:** Suppose  $[a, b]$  is a closed interval containing  $n$  subintervals

$$[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]$$

of equal length  $\Delta x = \frac{b-a}{n}$ , with  $a = x_0$  and  $b = x_n$ . The endpoints  $x_0, x_1, x_2, \dots, x_{n-1}, x_n$  of the subintervals are called **grid points**, and they create a **regular partition** of the interval  $[a, b]$ . In general the  $k$ th grid point is

$$x_k = a + k\Delta x, \text{ for } k = 0, 1, 2, \dots, n$$

- In the  $k$ th subinterval  $[x_{k-1}, x_k]$ , choose any point  $x_k^*$  and build a rectangle whose height is  $f(x_k^*)$ .
- The area of the rectangle of the  $k$ th subinterval is

$$\text{height} \cdot \text{base} = f(x_k^*)\Delta x, \text{ where } k = 1, 2, \dots, n$$

- Summing the areas of these rectangles, we obtain an approximation to the area of  $R$ , which is called a **Riemann sum**:

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \dots + f(x_n^*)\Delta x$$

- Three notable Riemann sums are the left, right, and midpoint Riemann sums.

**Riemann Sum:** Suppose  $f$  is defined on a closed interval  $[a, b]$ , which is divided into  $n$  subintervals of equal length  $\Delta x$ . If  $x_k^*$  is any point in the  $k$ th subinterval  $[x_{k-1}, x_k]$ , for  $k = 1, 2, \dots, n$ , then

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \dots + f(x_n^*)\Delta x$$

is called a **Riemann sum** for  $f$  on  $[a, b]$ . This sum is called

- a **left Riemann sum** if  $x_k^*$  is the left endpoint of  $[x_{k-1}, x_k]$
- a **right Riemann sum** if  $x_k^*$  is the right endpoint of  $[x_{k-1}, x_k]$
- a **midpoint Riemann sum** if  $x_k^*$  is the midpoint of  $[x_{k-1}, x_k]$

**Summation notation ( $\Sigma$ ):**

- Working with Riemann sums is cumbersome when  $n$  is large
- We introduce sigma (summation) notation as a shorthand:

$$1 + 2 + \dots + 49 + 50 = \sum_{k=1}^{50} k$$

- The symbol  $\Sigma$  (sigma) stands for sum
- $k$  is the index, and takes on all integer values from  $k = 1$  to  $k = 50$
- The expression immediately following  $\Sigma$ , the summand, is evaluated for each  $k$ , and the resulting values are summed
- The index is a dummy variable, and it does not matter which symbol is chosen for the index:

$$\sum_{k=1}^{99} k = \sum_{n=1}^{99} n = \sum_{p=1}^{99} p$$

- Two Properties of Sums and Sigma Notation

1. Constant Multiple Rule:

$$\sum_{k=1}^n ca_k = c \sum_{k=1}^n a_k$$

2. Addition Rule:

$$\sum_{k=1}^n (a_k + b_k) = \sum_{k=1}^n a_k + \sum_{k=1}^n b_k$$

- **Theorem:** Sums of Power of Integers

Let  $n \in \mathbb{Z}$  such that  $n > 0$  and  $c \in \mathbb{R}$

$$\sum_{k=1}^n c = cn \tag{1}$$

$$\sum_{k=1}^n k = \frac{n(n+1)}{2} \tag{2}$$

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6} \tag{3}$$

$$\sum_{k=1}^n k^3 = \frac{n^2(n+1)^2}{4} \tag{4}$$

• **Left, Right, and Midpoint Riemann Sums in Sigma Notation:**

Suppose  $f$  is defined on a closed interval  $[a, b]$ , which is divided into subintervals of equal length  $\Delta x$ . If  $x_k^*$  is a point in the  $k$ th subinterval  $[x_{k-1}, x_k]$ , for  $k = 1, 2, \dots, n$ , then the **Riemann sum** for  $f$  on  $[a, b]$  is

$$\sum_{k=1}^n f(x_k^*)\Delta x$$

Three cases arise in practice

- $\sum_{k=1}^n f(x_k^*)\Delta x$  is a **left Riemann sum** if  $x_k^* = a + (k-1)\Delta x$
- $\sum_{k=1}^n f(x_k^*)\Delta x$  is a **right Riemann sum** if  $x_k^* = a + k\Delta x$
- $\sum_{k=1}^n f(x_k^*)\Delta x$  is a **midpoint Riemann sum** if  $x_k^* = a + (k - \frac{1}{2})\Delta x$

## Antiderivative Rules

- Power Rule

If  $p \neq -1$  and  $C$  is an arbitrary constant:

$$\int x^p dx = \frac{x^{p+1}}{p+1} + C$$

- Integral of  $x^{-1}$

$$\int x^{-1} dx = \int \frac{1}{x} dx = \ln|x| + C$$

- Constant Multiple and Sum Rules

If  $c \in \mathbb{R}$ :

$$\int cf(x)dx = c \int f(x)dx$$

$$\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$$

- Integral of  $e^x$

$$\int e^x dx = e^x + C$$

- Integral of  $\frac{1}{x}$

$$\int \frac{1}{x} dx = \ln|x| + C$$

## Trigonometric (and inverse) Integrals

$$\int \cos(x)dx = \sin x + C \tag{1}$$

$$\int \sin(x)dx = -\cos x + C \tag{2}$$

$$\int \sec^2(x)dx = \tan x + C \tag{3}$$

$$\int \csc^2(x)dx = -\cot x + C \tag{4}$$

$$\int \sec(x)\tan(x)dx = \sec x + C \tag{5}$$

$$\int \csc(x)\cot(x)dx = -\csc x + C \tag{6}$$

$$\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}x + C \tag{7}$$

$$\int \frac{1}{1+x^2}dx = \tan^{-1}x + C \tag{8}$$

$$\int \frac{1}{x\sqrt{x^2-1}}dx = \sec^{-1}|x| + C \tag{9}$$

## Examples

1. Use L'Hôpital's Rule to evaluate a limit with indeterminate form  $\frac{0}{0}$

$$\lim_{x \rightarrow 0} \frac{e^x - x - 1}{5x^2} = \lim_{x \rightarrow 0} \frac{e^x - 1}{10x} \quad (1)$$

$$= \lim_{x \rightarrow 0} \frac{e^x}{10} \quad (2)$$

$$= \frac{e^0}{10} \quad (3)$$

$$= \frac{1}{10} \quad (4)$$

2. Use L'Hôpital's Rule to evaluate a limit with indeterminate form  $\frac{\infty}{\infty}$

$$\lim_{x \rightarrow 0^+} \frac{1 - \ln x}{1 + \ln x} = \lim_{x \rightarrow 0^+} \frac{-\frac{1}{x}}{\frac{1}{x}} \quad (1)$$

$$= \lim_{x \rightarrow 0^+} \frac{-1}{1} \quad (2)$$

$$= \frac{-1}{1} \quad (3)$$

$$= -1 \quad (4)$$

3. Use L'Hôpital's Rule to evaluate a limit with indeterminate form  $0 \cdot \infty$

$$\lim_{x \rightarrow 1^-} (1 - x) \tan\left(\frac{\pi x}{2}\right) = \lim_{x \rightarrow 1^-} \frac{(1 - x)}{\cot\left(\frac{\pi x}{2}\right)} \quad (1)$$

$$= \lim_{x \rightarrow 1^-} \frac{-1}{-\frac{\pi}{2} \csc^2\left(\frac{\pi x}{2}\right)} \quad (2)$$

$$= \lim_{x \rightarrow 1^-} \frac{2}{\pi} \sin^2\left(\frac{\pi x}{2}\right) \quad (3)$$

$$= \frac{2}{\pi} \quad (4)$$

4. Use L'Hôpital's Rule to evaluate a limit with exponential indeterminate form

$$\lim_{x \rightarrow 0^+} x^{\tan x} = e^{\lim_{x \rightarrow 0^+} \frac{\ln x}{\tan x}} \quad (1)$$

$$= e^{\lim_{x \rightarrow 0^+} \frac{\ln x}{\cot x}} \quad (2)$$

$$= e^{\lim_{x \rightarrow 0^+} \frac{1}{-x \csc^2 x}} \quad (3)$$

$$= e^{\lim_{x \rightarrow 0^+} \frac{-\sin^2 x}{x}} \quad (4)$$

$$= e^{\lim_{x \rightarrow 0^+} \frac{-2 \sin x \cos x}{1}} \quad (5)$$

$$= e^{\lim_{x \rightarrow 0^+} -2 \sin x \cos x} \quad (6)$$

$$= e^0 \quad (7)$$

$$= 1 \quad (8)$$

5. Compare the growth rates of functions

$$f(x) = x^2 \ln x$$

$$g(x) = x \ln^2 x$$

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{x^2 \ln x}{x \ln^2 x} \quad (1)$$

$$= \lim_{x \rightarrow \infty} \frac{x}{\ln x} \quad (2)$$

$$= \lim_{x \rightarrow \infty} \frac{1}{\frac{1}{x}} \quad (3)$$

$$= \lim_{x \rightarrow \infty} x \quad (4)$$

$$= \infty \quad (5)$$

Since  $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \infty$ ,  $f \gg g$

6. Use knowledge of derivatives to find antiderivatives

$$f(x) = -4 \cos x - x$$

$$F(x) = -4 \sin x - \frac{1}{2}x^2$$

$$F'(x) = -4 \cos x - x$$

$$\int (-4 \cos x - x) dx = -4 \sin x - \frac{1}{2}x^2 + C$$

7. Determine indefinite integrals using antiderivative rules

$$\int \frac{3}{x^4} + 2 - 3x^2 dx = \int 3x^{-4} + 2 - 3x^2 dx = \frac{-1}{x^3} + 2x - x^3 + C$$

8. Rewrite an indefinite integral to find an antiderivative

$$\int \frac{2 + 3 \cos y}{\sin^2 y} dy = \int 2 \csc^2 y + 3 \cot y \csc y dy \quad (1)$$

$$= 2 \int \csc^2 y dy + 3 \int \cot y \csc y dy \quad (2)$$

$$= -2 \cot y - 3 \csc y + C \quad (3)$$

9. Solve an initial value problem

$$f'(u) = 4 \cos u - 4 \sin u$$

$$f(\pi) = 0$$

$$f(u) = 4 \int \cos u du - 4 \int \sin u du = 4 \sin u + 4 \cos u + C$$

$$4 \sin \pi + 4 \cos \pi + C = 0 \quad (1)$$

$$0 - 4 + C = 0 \quad (2)$$

$$C = 4 \quad (3)$$

$$f(u) = 4 \sin u + 4 \cos u + 4$$

10. Application of differential equations to linear motion

$$a(t) = 2 + 3 \sin t$$

$$v(0) = 1$$

$$s(0) = 10$$

$$v(t) = \int 2 + 3 \sin t dt \quad (1)$$

$$= 2 \int t^0 dt + 3 \int \sin t dt \quad (2)$$

$$= 2t - 3 \cos t + C \quad (3)$$

$$2(0) - 3 \cos 0 + C = 1 \quad (4)$$

$$-3 + C = 1 \quad (5)$$

$$C = 4 \quad (6)$$

$$v(t) = 2t - 3 \cos t + 4 \quad (7)$$

$$s(t) = \int 2t - 3 \cos t + 4 dt \quad (8)$$

$$= 2 \int t dt - 3 \int \cos t dt + 4 \int t^0 dt \quad (9)$$

$$= 2 \frac{t^2}{2} - 3 \sin t + 4t + C \quad (10)$$

$$= t^2 - 3 \sin t + 4t + C \quad (11)$$

$$0^2 - 3 \sin 0 + 4(0) + C = 10 \quad (12)$$

$$C = 10 \quad (13)$$

$$s(t) = t^2 - 3 \sin t + 4t + 10 \quad (14)$$

11. Approximating displacement

$$v = \sqrt{10t}$$

$$1 \leq t \leq 7$$

(a)

$$n = 3$$

$$[1, 3], [3, 5], [5, 7]$$

$$2, 4, 6$$

$$d \approx 2v(2) + 2v(4) + 2v(6) = 2\sqrt{20} + 2\sqrt{40} + 2\sqrt{60} \approx 37.085$$

(b)

$$n = 6$$

$$1.5, 2.5, 3.5, 4.5, 5.5, 6.5$$

$$d \approx v(1.5) + v(2.5) + v(3.5) + v(4.5) + v(5.5) + v(6.5) \quad (1)$$

$$= \sqrt{15} + \sqrt{25} + \sqrt{35} + \sqrt{45} + \sqrt{55} + \sqrt{65} \quad (2)$$

$$\approx 36.976 \quad (3)$$

12. Left and right Riemann sums (1)

$$f(x) = x + 1$$

$$[1, 6]$$

$$n = 5$$

(a)

$$(1 + 1) + (2 + 1) + (3 + 1) + (4 + 1) + (5 + 1) = 2 + 3 + 4 + 5 + 6 \quad (1)$$

$$= 20 \quad (2)$$

(b)

$$(2 + 1) + (3 + 1) + (4 + 1) + (5 + 1) + (6 + 1) = 3 + 4 + 5 + 6 + 7 \quad (1)$$

$$= 25 \quad (2)$$

13. Left and right Riemann sums (2)

$$f(x) = 9 - x$$

$$[3, 8]$$

$$n = 5$$

$$\Delta x = \frac{8 - 3}{5} = \frac{5}{5} = 1$$

(a) Grid Points:

$$x_0 = 3 \quad (1)$$

$$x_1 = 4 \quad (2)$$

$$x_2 = 5 \quad (3)$$

$$x_3 = 6 \quad (4)$$

$$x_4 = 7 \quad (5)$$

$$x_5 = 8 \quad (6)$$

(b) Riemann Sums

$$(9 - 3) + (9 - 4) + (9 - 5) + (9 - 6) + (9 - 7) = 6 + 5 + 4 + 3 + 2 \quad (1)$$

$$= 20 \text{ (Overestimation)} \quad (2)$$

$$(9 - 4) + (9 - 5) + (9 - 6) + (9 - 7) + (9 - 8) = 5 + 4 + 3 + 2 + 1 \quad (1)$$

$$= 15 \text{ (Underestimation)} \quad (2)$$

14. Midpoint Riemann sum

$$f(x) = 100 - x^2$$

$$[0, 10]$$

$$n = 5$$

$$\Delta x = 2$$

$$2f(1) + 2f(3) + 2f(5) + 2f(7) + 2f(9) = 2(99) + 2(91) + 2(75) + 2(51) + 2(19) \quad (1)$$

$$= 198 + 182 + 150 + 102 + 38 \quad (2)$$

$$= 670 \quad (3)$$

15. Riemann sums from tables

$$[0, 2]$$

$$n = 4$$

$$\Delta x = \frac{1}{2}$$

$$\frac{5}{2} + \frac{3}{2} + \frac{2}{2} + \frac{1}{2} = 5.5 \quad (1)$$

$$\frac{3}{2} + \frac{2}{2} + \frac{1}{2} + \frac{1}{2} = 3.5 \quad (2)$$

16. Computing Riemann sums for large values of  $n$

$$f(x) = x^2 + 1$$

$$[-1, 1]$$

$$n = 50$$

$$\sum_{k=1}^{50} 0.04 f(-1 + 0.04(k-1)) = \sum_{k=1}^{50} 0.04 \left( (-1 + 0.04(k-1))^2 + 1 \right) \quad (1)$$

$$\approx 2.6672 \quad (2)$$

$$\sum_{k=1}^{50} 0.04 f(-1 + 0.04k) = \sum_{k=1}^{50} 0.04 \left( (-1 + 0.04k)^2 + 1 \right) \quad (3)$$

$$\approx 2.6672 \quad (4)$$

$$\sum_{k=1}^{50} 0.04 f\left(-1 + 0.04\left(k - \frac{1}{2}\right)\right) = \sum_{k=1}^{50} 0.04 \left( \left(-1 + 0.04\left(k - \frac{1}{2}\right)\right)^2 + 1 \right) \quad (5)$$

$$\approx 2.6664 \quad (6)$$

## Related Exercises

1. (Section 4.7, Exercise 17)

$$\lim_{x \rightarrow 2} \frac{x^2 - 2x}{x^2 - 6x + 8} = \lim_{x \rightarrow 2} \frac{2x - 2}{2x - 6} \quad (1)$$

$$= \frac{2(2) - 2}{2(2) - 6} \quad (2)$$

$$= \frac{4 - 2}{4 - 6} \quad (3)$$

$$= \frac{2}{-2} \quad (4)$$

$$= -1 \quad (5)$$



2. (Section 4.7, Exercise 18)

$$\lim_{x \rightarrow -1} \frac{x^4 + x^3 + 2x + 2}{x + 1} = \lim_{x \rightarrow -1} \frac{4x^3 + 3x^2 + 2}{1} \quad (1)$$

$$= \lim_{x \rightarrow -1} 4x^3 + 3x^2 + 2 \quad (2)$$

$$= 4(-1)^3 + 3(-1)^2 + 2 \quad (3)$$

$$= -4 + 3 + 2 \quad (4)$$

$$= 1 \quad (5)$$

3. (Section 4.7, Exercise 36)

$$\lim_{x \rightarrow 0} \frac{e^x - x - 1}{5x^2} = \lim_{x \rightarrow 0} \frac{e^x - 1}{10x} \quad (1)$$

$$= \lim_{x \rightarrow 0} \frac{e^x}{10} \quad (2)$$

$$= \frac{e^0}{10} \quad (3)$$

$$= \frac{1}{10} \quad (4)$$

4. (Section 4.7, Exercise 39)

$$\lim_{x \rightarrow 0} \frac{e^x - \sin x - 1}{x^4 + 8x^3 + 12x^2} = \lim_{x \rightarrow 0} \frac{e^x - \cos x}{4x^3 + 24x^2 + 24x} \quad (1)$$

$$= \lim_{x \rightarrow 0} \frac{e^x + \sin x}{12x^2 + 48x + 24} \quad (2)$$

$$= \frac{e^0 + \sin 0}{12(0)^2 + 48(0) + 24} \quad (3)$$

$$= \frac{1 + 0}{24} \quad (4)$$

$$= \frac{1}{24} \quad (5)$$

5. (Section 4.7, Exercise 38)

$$\lim_{x \rightarrow \infty} \frac{e^{3x}}{3e^{3x} + 5} = \lim_{x \rightarrow \infty} \frac{3e^{3x}}{9e^{3x}} \quad (1)$$

$$= \lim_{x \rightarrow \infty} \frac{1}{3} \cdot \frac{e^{3x}}{e^{3x}} \quad (2)$$

$$= \lim_{x \rightarrow \infty} \frac{1}{3} \quad (3)$$

$$= \frac{1}{3} \quad (4)$$

6. (Section 4.7, Exercise 51)

$$\lim_{x \rightarrow \infty} \frac{x^2 - \ln \frac{2}{x}}{3x^2 + 2x} = \lim_{x \rightarrow \infty} \frac{2x + \frac{1}{x}}{6x + 2} \quad (1)$$

$$= \lim_{x \rightarrow \infty} \frac{2 - \frac{1}{x^2}}{6} \quad (2)$$

$$= \frac{2 - 0}{6} \quad (3)$$

$$= \frac{2}{6} \quad (4)$$

$$= \frac{1}{3} \quad (5)$$

7. (Section 4.7, Exercise 53)

$$\lim_{x \rightarrow 0} x \csc x = \lim_{x \rightarrow 0} \frac{x}{\sin x} \quad (1)$$

$$= \lim_{x \rightarrow 0} \frac{1}{\cos x} \quad (2)$$

$$= \frac{1}{\cos 0} \quad (3)$$

$$= \frac{1}{1} \quad (4)$$

$$= 1 \quad (5)$$

8. (Section 4.7, Exercise 63)

$$\lim_{x \rightarrow \infty} \left( x^2 - \sqrt{x^4 + 16x^2} \right) = \lim_{x \rightarrow \infty} \left( x^2 - \sqrt{x^4 \left( 1 + \frac{16}{x^2} \right)} \right) \quad (1)$$

$$= \lim_{x \rightarrow \infty} \left( x^2 - x^2 \sqrt{1 + \frac{16}{x^2}} \right) \quad (2)$$

$$= \lim_{x \rightarrow \infty} x^2 \left( 1 - \sqrt{1 + \frac{16}{x^2}} \right) \quad (3)$$

$$= \lim_{x \rightarrow \infty} \frac{1 - \sqrt{1 + \frac{16}{x^2}}}{\frac{1}{x^2}} \quad (4)$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{16}{x^3}}{\frac{-2}{x^3} \sqrt{1 + \frac{16}{x^2}}} \quad (5)$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{16}{x^3} \cdot \frac{x^3}{-2}}{\sqrt{1 + \frac{16}{x^2}}} \quad (6)$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{16}{-2} \cdot \frac{x^3}{x^3}}{\sqrt{1 + \frac{16}{x^2}}} \quad (7)$$

$$= \lim_{x \rightarrow \infty} \frac{-8}{\sqrt{1 + \frac{16}{x^2}}} \quad (8)$$

$$= \frac{-8}{\sqrt{1 + 0}} \quad (9)$$

$$= \frac{-8}{1} \quad (10)$$

$$= -8 \quad (11)$$

9. (Section 4.7, Exercise 64)

$$\lim_{x \rightarrow \infty} \left( x - \sqrt{x^2 + 4x} \right) = \lim_{x \rightarrow \infty} \left( x - \sqrt{x^2 \left( 1 + \frac{4}{x} \right)} \right) \quad (1)$$

$$= \lim_{x \rightarrow \infty} \left( x - x \sqrt{1 + \frac{4}{x}} \right) \quad (2)$$

$$= \lim_{x \rightarrow \infty} x \left( 1 - \sqrt{1 + \frac{4}{x}} \right) \quad (3)$$

$$= \lim_{x \rightarrow \infty} \frac{1 - \sqrt{1 + \frac{4}{x}}}{\frac{1}{x}} \quad (4)$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{2}{x^2}}{\frac{-1}{x^2} \sqrt{1 + \frac{4}{x}}} \quad (5)$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{2}{x^2} \cdot \frac{x^2}{-1}}{\sqrt{1 + \frac{4}{x}}} \quad (6)$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{2}{-1} \cdot \frac{x^2}{x^2}}{\sqrt{1 + \frac{4}{x}}} \quad (7)$$

$$= \lim_{x \rightarrow \infty} \frac{-2}{\sqrt{1 + \frac{4}{x}}} \quad (8)$$

$$= \frac{-2}{\sqrt{1 + 0}} \quad (9)$$

$$= -2 \quad (10)$$

10. (Section 4.7, Exercise 75)

$$\lim_{x \rightarrow 0^+} x^{2x} = e^{x \rightarrow 0^+ \frac{\ln x}{2x}} \quad (1)$$

$$= e^{x \rightarrow 0^+ \frac{\frac{1}{x}}{2x^2}} \quad (2)$$

$$= e^{x \rightarrow 0^+ \frac{1}{x} \cdot \frac{2x^2}{-1}} \quad (3)$$

$$= e^{x \rightarrow 0^+ -\frac{2x^2}{x}} \quad (4)$$

$$= e^{x \rightarrow 0^+ -2x} \quad (5)$$

$$= e^{-2(0)} \quad (6)$$

$$= e^0 \quad (7)$$

$$= 1 \quad (8)$$

11. (Section 4.7, Exercise 76)

$$\lim_{x \rightarrow 0} (1 + 4x)^{\frac{3}{x}} = e^{x \rightarrow 0^+ \frac{\ln(1+4x)}{\frac{1}{\frac{3}{x}}}} \quad (1)$$

$$= e^{x \rightarrow 0^+ \frac{\ln(1+4x)}{\frac{3}{x}}} \quad (2)$$

$$= e^{x \rightarrow 0^+ \frac{\frac{4}{(1+4x)}}{\frac{1}{3}}} \quad (3)$$

$$= e^{x \rightarrow 0^+ \frac{4}{(1+4x)} \cdot \frac{3}{1}} \quad (4)$$

$$= e^{x \rightarrow 0^+ \frac{12}{(1+4x)}} \quad (5)$$

$$= e^{\frac{12}{1}} \quad (6)$$

$$= e^{12} \quad (7)$$

12. (Section 4.7, Exercise 96)

$$f(x) = x^2 \ln x \quad (1)$$

$$g(x) = \ln^2 x \quad (2)$$

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{x^2 \ln x}{\ln^2 x} \quad (1)$$

$$= \lim_{x \rightarrow \infty} \frac{x^2}{\ln x} \quad (2)$$

$$= \lim_{x \rightarrow \infty} \frac{2x}{\frac{1}{x}} \quad (3)$$

$$= \lim_{x \rightarrow \infty} 2x^2 \quad (4)$$

$$= \infty \quad (5)$$

$$f \gg g$$

13. (Section 4.7, Exercise 100)

$$f(x) = x^2 \ln x \quad (1)$$

$$g(x) = x^3 \quad (2)$$

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{x^2 \ln x}{x^3} \quad (1)$$

$$= \lim_{x \rightarrow \infty} \frac{\ln x}{x} \quad (2)$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{1} \quad (3)$$

$$= \lim_{x \rightarrow \infty} \frac{1}{x} \quad (4)$$

$$= \frac{1}{\infty} \neq \infty \quad (5)$$

$$g \gg f$$

14. (Section 4.7, Exercise 95)

$$f(x) = x^{10} \quad (1)$$

$$g(x) = e^{0.01x} \quad (2)$$

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{x^{10}}{e^{0.01x}} \quad (1)$$

$$= \lim_{x \rightarrow \infty} \frac{10x^9}{0.01e^{0.01x}} \quad (2)$$

$$= \lim_{x \rightarrow \infty} \frac{90x^8}{0.01^2 e^{0.01x}} \quad (3)$$

$$= \lim_{x \rightarrow \infty} \frac{7200x^7}{0.01^3 e^{0.01x}} \quad (4)$$

$$= \lim_{x \rightarrow \infty} \frac{50400x^6}{0.01^4 e^{0.01x}} \quad (5)$$

$$= \lim_{x \rightarrow \infty} \frac{302400x^5}{0.01^5 e^{0.01x}} \quad (6)$$

$$= \lim_{x \rightarrow \infty} \frac{1512000x^4}{0.01^6 e^{0.01x}} \quad (7)$$

$$= \lim_{x \rightarrow \infty} \frac{6048000x^3}{0.01^7 e^{0.01x}} \quad (8)$$

$$= \lim_{x \rightarrow \infty} \frac{18144000x^2}{0.01^8 e^{0.01x}} \quad (9)$$

$$= \lim_{x \rightarrow \infty} \frac{36288000x}{0.01^9 e^{0.01x}} \quad (10)$$

$$= \lim_{x \rightarrow \infty} \frac{36288000}{0.01^{10} e^{0.01x}} \quad (11)$$

$$= \frac{36288000}{\infty} \neq \infty \quad (12)$$

$$g \gg f$$

15. (Section 4.7, Exercise 101)

$$f(x) = x^{20} \quad (1)$$

$$g(x) = 1.00001^x \quad (2)$$

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{x^{20}}{1.00001^x} \quad (1)$$

$$= \frac{2432902008176640000}{\infty} \neq \infty \quad (2)$$

$$g \gg f$$

16. (Section 4.9, Exercise 12)

$$f(x) = 11x^{10}$$

$$\int 11x^{10} dx = 11 \int x^{10} dx \quad (1)$$

$$= 11 \frac{x^{11}}{11} + C \quad (2)$$

$$= x^{11} + C \quad (3)$$

17. (Section 4.9, Exercise 13)

$$f(x) = 2 \sin x + 1$$

$$\int 2 \sin x + 1 dx = 2 \int \sin x + \int x^0 \quad (1)$$

$$= -2 \cos x + x + C \quad (2)$$

18. (Section 4.9, Exercise 24)

$$\int 3u^{-2} - 4u^2 + 1 du = 3 \int u^{-2} du - 4 \int u^2 du + \int u^0 du \quad (1)$$

$$= 3 \frac{u^{-1}}{-1} - 4 \frac{u^3}{3} + u + C \quad (2)$$

$$= -\frac{3}{u} - \frac{4u^3}{3} + u + C \quad (3)$$

19. (Section 4.9, Exercise 25)

$$\int 4\sqrt{x} - \frac{4}{\sqrt{x}} dx = 4 \int x^{\frac{1}{2}} dx - 4 \int x^{-\frac{1}{2}} dx \quad (1)$$

$$= 4 \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - 4 \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C \quad (2)$$

$$= \frac{8x^{\frac{3}{2}}}{3} - 8x^{\frac{1}{2}} + C \quad (3)$$

$$= \frac{8x\sqrt{x}}{3} - 8\sqrt{x} + C \quad (4)$$

20. (Section 4.9, Exercise 31)

$$\int (3x+1)(4-x) dx = \int -3x^2 + 11x + 4 dx \quad (1)$$

$$= -3 \int x^2 dx + 11 \int x dx + 4 \int x^0 dx \quad (2)$$

$$= -3 \frac{x^3}{3} + 11 \frac{x^2}{2} + 4x + C \quad (3)$$

$$= -x^3 + \frac{11x^2}{2} + 4x + C \quad (4)$$

21. (Section 4.9, Exercise 35)

$$\int \frac{4x^4 - 6x^2}{x} dx = \int 4x^3 - 6x dx \quad (1)$$

$$= 4 \int x^3 dx - 6 \int x dx \quad (2)$$

$$= 4 \frac{x^4}{4} - 6 \frac{x^2}{2} + C \quad (3)$$

$$= x^4 - 3x^2 + C \quad (4)$$

22. (Section 4.9, Exercise 40)

$$\int (\csc^2 \theta + 1) d\theta = \int \csc^2 \theta d\theta + \int \theta^0 d\theta \quad (1)$$

$$= -\cot \theta + \theta + C \quad (2)$$

23. (Section 4.9, Exercise 41)

$$\int \frac{2 + 3 \cos y}{\sin^2 y} dy = \int \left( \frac{2}{\sin^2 y} + \frac{3 \cos y}{\sin^2 y} \right) dy \quad (1)$$

$$= \int (2 \csc^2 y + 3 \cot y \csc y) dy \quad (2)$$

$$= 2 \int \csc^2 y dy + 3 \int \cot y \csc y dy \quad (3)$$

$$= -2 \cot y - 3 \csc y + C \quad (4)$$

24. (Section 4.9, Exercise 47)

$$\int (3t^2 + 2 \csc^2 t) dt = \int 3t^2 dt + \int 2 \csc^2 t dt \quad (1)$$

$$= 3 \int t^2 dt + 2 \int \csc^2 t dt \quad (2)$$

$$= 3 \frac{t^3}{3} - 2 \cot t + C \quad (3)$$

$$= t^3 - 2 \cot t + C \quad (4)$$

25. (Section 4.9, Exercise 45)

$$\int (\sec^2 \theta + \sec \theta \tan \theta) d\theta = \int \sec^2 \theta d\theta + \int \sec \theta \tan \theta d\theta \quad (1)$$

$$= \tan \theta + \sec \theta + C \quad (2)$$

26. (Section 4.9, Exercise 50)

$$\int \frac{\csc^3 x + 1}{\csc x} dx = \int \frac{\csc^3 x}{\csc x} + \frac{1}{\csc x} dx \quad (1)$$

$$= \int \csc^2 x + \sin x dx \quad (2)$$

$$= \int \csc^2 x dx + \int \sin x dx \quad (3)$$

$$= -\cot x + -\cos x + C \quad (4)$$

27. (Section 4.9, Exercise 51)

$$\int \frac{1}{2y} dy = \frac{1}{2} \int \frac{1}{y} dy \quad (1)$$

$$= \frac{1}{2} \ln |y| + C \quad (2)$$

$$= \frac{\ln |y|}{2} + C \quad (3)$$

28. (Section 4.9, Exercise 53)

$$\int \frac{6}{\sqrt{4-4x^2}} dx = \int \frac{6}{\sqrt{4(1-x^2)}} dx \quad (1)$$

$$= \int \frac{6}{2\sqrt{1-x^2}} dx \quad (2)$$

$$= 3 \int \frac{1}{\sqrt{1-x^2}} dx \quad (3)$$

$$= 3 \sin^{-1} x + C \quad (4)$$

29. (Section 4.9, Exercise 59)

$$\int \frac{t+1}{t} dt = \int \left( \frac{t}{t} + \frac{1}{t} \right) dt \quad (1)$$

$$= \int \frac{t}{t} dt + \int \frac{1}{t} dt \quad (2)$$

$$= \int t^0 dt + \ln |t| + C \quad (3)$$

$$= t + \ln |t| + C \quad (4)$$

30. (Section 4.9, Exercise 61)

$$\int e^{x+2} dx = \int e^2 e^x dx \quad (1)$$

$$= e^2 \int e^x dx \quad (2)$$

$$= e^2 e^x \quad (3)$$

$$= e^{x+2} \quad (4)$$

31. (Section 4.9, Exercise 54)

$$\int \frac{v^3 + v + 1}{1 + v^2} dv = \int \frac{v^3 + v}{1 + v^2} + \frac{1}{1 + v^2} dv \quad (1)$$

$$= \int v + \frac{1}{1 + v^2} dv \quad (2)$$

$$= \int v dv + \int \frac{1}{1 + v^2} dv \quad (3)$$

$$= \frac{v^2}{2} + \tan^{-1} v + C \quad (4)$$

32. (Section 4.9, Exercise 62)

$$\int \frac{10t^5 - 3}{t} dt = \int \left( \frac{10t^5}{t} - \frac{3}{t} \right) dt \quad (1)$$

$$= \int 10t^4 dt - \int \frac{3}{t} dt \quad (2)$$

$$= 10 \int t^4 dt - 3 \int \frac{1}{t} dt \quad (3)$$

$$= 10 \frac{t^5}{5} - 3 \ln |t| + C \quad (4)$$

$$= 2t^5 - 3 \ln |t| + C \quad (5)$$

33. (Section 4.9, Exercise 78)

$$g'(x) = 7x^6 - 4x^3 + 12 \quad (1)$$

$$g(1) = 24 \quad (2)$$

$$g(x) = \int 7x^6 - 4x^3 + 12 dx \quad (3)$$

$$= 7 \int x^6 dx - 4 \int x^3 dx + 12 \int x^0 dx \quad (4)$$

$$= 7 \frac{x^7}{7} - 4 \frac{x^4}{4} + 12x + C \quad (5)$$

$$= x^7 - x^4 + 12x + C \quad (6)$$

$$(1)^7 - (1)^4 + 12(1) + C = 24 \quad (7)$$

$$1 - 1 + 12 + C = 24 \quad (8)$$

$$12 + C = 24 \quad (9)$$

$$C = 12 \quad (10)$$

$$g(x) = x^7 - x^4 + 12x + 12 \quad (11)$$

34. (Section 4.9, Exercise 83)

$$y'(t) = \frac{3}{t} + 6 \quad (1)$$

$$y(1) = 8, t > 0 \quad (2)$$

$$y(t) = \int \left( \frac{3}{t} + 6 \right) dt \quad (3)$$

$$= 3 \int \frac{1}{t} dt + 6 \int t^0 dt \quad (4)$$

$$= 3 \ln |t| + 6t + C \quad (5)$$

$$3 \ln 1 + 6(1) + C = 8 \quad (6)$$

$$6 + C = 8 \quad (7)$$

$$C = 2 \quad (8)$$

$$y(t) = 3 \ln |t| + 6t + 2 \quad (9)$$

35. (Section 4.9, Exercise 105)

$$v(t) = \sin t \quad (1)$$

$$s(0) = 0 \quad (2)$$

$$s(t) = \int \sin t dt \quad (3)$$

$$= -\cos t + C \quad (4)$$

$$-\cos 0 + C = 0 \quad (5)$$

$$-1 + C = 0 \quad (6)$$

$$C = 1 \quad (7)$$

$$s(t) = -\cos t + 1 \quad (8)$$



$$V(t) = \cos t \quad (1)$$

$$S(0) = 0 \quad (2)$$

$$S(t) = \int \cos t \, dt \quad (3)$$

$$= \sin t + C \quad (4)$$

$$\sin 0 + C = 0 \quad (5)$$

$$C = 0 \quad (6)$$

$$S(t) = \sin t \quad (7)$$

36. (Section 4.9, Exercise 106)

$$v(t) = e^t \quad (1)$$

$$s(0) = 0 \quad (2)$$

$$s(t) = \int e^t \, dt \quad (3)$$

$$= e^t + C \quad (4)$$

$$e^0 + C = 0 \quad (5)$$

$$1 + C = 0 \quad (6)$$

$$C = -1 \quad (7)$$

$$s(t) = e^t - 1 \quad (8)$$

$$V(t) = 2 + \cos t \quad (1)$$

$$S(0) = 3 \quad (2)$$

$$S(t) = \int 2 + \cos t \, dt \quad (3)$$

$$= \int 2 \, dt + \int \cos t \, dt \quad (4)$$

$$= 2 \int t^0 \, dt + \sin t + C \quad (5)$$

$$= 2t + \sin t + C \quad (6)$$

$$2(0) + \sin 0 + C = 3 \quad (7)$$

$$C = 3 \quad (8)$$

37. (Section 5.1, Exercise 3)

$$4(40) + 4(70) = 440$$

$$2(30) + 2(50) + 2(80) + 2(40) = 400$$

38. (Section 5.1, Exercise 15)

$$v = 3t^2 + 1$$

$$0 \leq t \leq 4$$

$$\begin{aligned} f(0.5) + f(1.5) + f(2.5) + f(3.5) &= 1.75 + 7.75 + 19.75 + 37.75 \\ &= 67 \end{aligned} \quad (2)$$

$$\frac{f(0.25)}{2} + \frac{f(0.75)}{2} + \frac{f(1.25)}{2} + \frac{f(1.75)}{2} + \frac{f(2.25)}{2} + \frac{f(2.75)}{2} + \frac{f(3.25)}{2} + \frac{f(3.75)}{2} = 67.75 \quad (3)$$

39. (Section 5.1, Exercise 16)

$$v = \sqrt{10t}$$

$$1 \leq t \leq 7$$

(a)

$$n = 3$$

$$[1, 3], [3, 5], [5, 7]$$

$$2, 4, 6$$

$$d \approx 2v(2) + 2v(4) + 2v(6) = 2\sqrt{20} + 2\sqrt{40} + 2\sqrt{60} \approx 37.085$$

(b)

$$n = 6$$

$$1.5, 2.5, 3.5, 4.5, 5.5, 6.5$$

$$d \approx v(1.5) + v(2.5) + v(3.5) + v(4.5) + v(5.5) + v(6.5) \quad (1)$$

$$= \sqrt{15} + \sqrt{25} + \sqrt{35} + \sqrt{45} + \sqrt{55} + \sqrt{65} \quad (2)$$

$$\approx 36.976 \quad (3)$$