Module 7 Notes (MATH-211)

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General Notes (and Definitions)

- Working with Integrals
  A function $f(x)$ is **even** if $f(-x) = f(x)$.
  A function $f(x)$ is **odd** if $f(-x) = -f(x)$.
  Let $a \in \mathbb{R}$ such that $a > 0$ and let f be an integrable function on the interval $[-a, a]$.

  $$\text{If } f \text{ is even, } \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

  $$\text{If } f \text{ is odd, } \int_{-a}^{a} f(x)\,dx = 0$$

  The average value of an integrable function f on the interval $[a, b]$ is

  $$\overline{f} = \frac{1}{b-a}\int_{a}^{b} f(x)\,dx$$

  Let f be continuous on the interval $[a, b]$. There exists a point c in (a, b) such that (Mean Value Theorem)

  $$f(c) = \overline{f} = \frac{1}{b-a}\int_{a}^{b} f(t)\,dx$$

- Substitution Rule
  Let $u = g(x)$, where g is differentiable on an interval, and let f be continuous on the corresponding range of g. On that interval,

  $$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

  1. Given an indefinite integral involving a commposite function $f(g(x))$, identify an inner function $u = g(x)$ such that a constant multiple of $g'(x)$ appears in the integrand.
  2. Substitute $u = g(x)$ and $du = g'(x)\,dx$ in the integral.
  3. Evaluate the new indefinite integral with respect to u.
  4. Write the result in terms of x using $u = g(x)$.

  Let $u = g(x)$, where g' is continuous on $[a, b]$, and let f be continuous on the range of g. Then

  $$\int_{a}^{b} f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

- Velocity and Net Change
  Position, Velocity, Displacement, and Distance:

  1. The **position** of an object moving along a line at time t, denoted $s(t)$, is the location of the object relative to the origin.
  2. The **velocity** of an object at time t is $v(t) = s'(t)$.
  3. The **displacement** of the object between $t = a$ and $t = b > a$ is

  $$s(b) - s(a) = \int_{a}^{b} v(t)\,dt$$

4. The **distance traveled** by the object between $t = a$ and $t = b > a$ is

$$\int_a^b |v(t)| \, dt$$

where $|v(t)|$ is the **speed** of the object at time t.

Theorem: Position from Velocity
Given the velocity $v(t)$ of an object moving along a line and its initial position $s(0)$, the position function of the object for future times $t \geq 0$ is

$$s(t) = s(0) + \int_0^t v(x) \, dx$$

Theorem: Velocity from Acceleration
Given the acceleration $a(t)$ of an object moving along a line and its initial velocity $v(0)$, the velocity of the object for future times $t \geq 0$ is

$$v(t) = v(0) + \int_0^t a(x) \, dx$$

Theorem: Net Change and Future Value
Suppose a quantity Q changes over time at a known rate Q'. Then the **net change** in Q between $t = a$ and $t = b > a$ is

$$Q(b) - Q(a) = \int_a^b Q'(t) \, dt$$

Given the initial value $Q(0)$, the **future value** of Q at time $t \geq 0$ is

$$Q(t) = Q(0) + \int_0^t Q'(x) \, dx$$

- Area Between Curves
  Area of a Region Between Two Curves:
  Suppose that f and g are continuous functions with $f(x) \geq g(x)$ on the interval $[a, b]$. The area of the region bounded by the graphs of f and g on $[a, b]$ is

  $$A = \int_a^b (f(x) - g(x)) \, dx$$

  Area of a Region Between Two Curves with Respect to y:
  Suppose that f and g are continuous functions with $f(y) \geq g(y)$ on the interval $[c, d]$. The area of the region bounded by the graphs $x = f(y)$ and $x = g(y)$ on $[c, d]$ is

  $$A = \int_c^d (f(y) - g(y)) \, dy$$

- Volume by Slicing
  General Slicing Method:
  Suppose a solid object extends from $x = a$ to $x = b$ and the cross section of the solid perpendicular to the x-axis has an area given by a function A that is integrable on $[a, b]$. The volume of the solid is

  $$V = \int_a^b A(x) \, dx$$

  Disk Method about the x-Axis:
  Let f be continuous with $f(x) \geq 0$ on the interval $[a, b]$. If the region R bounded by the graph of f, the x-axis, and the lines $x = a$ and $x = b$ is revolved about the x-axis, the volume of the resulting solid of revolution is

  $$V = \int_a^b \pi \, f(x)^2 \, dx$$

  Washer Method about the x-Axis:
  Let f and g be continuous functions with $f(x) \geq g(x) \geq 0$ on $[a, b]$. Let R be the region bounded by

$y = f(x)$, $y = g(x)$, and the lines $x = a$ and $x = b$. When R is revolved about the x-axis, the volume of the resulting solid of revolution is

$$V = \int_a^b \left(f(x)^2 - g(x)^2\right)\, dx$$

Disk and Washer Methods about the y-Axis:
Let p and q be continuous functions with $p(y) \geq q(y) \geq 0$ on $[c, d]$. Let R be the region bounded by $x = p(y)$, $x = q(y)$, and the lines $y = c$ and $y = d$. When R is revolved about the y-axis, the volume of the resulting solid of revolution is given by

$$V = \int_c^d \pi \left(p(y)^2 - q(y)^2\right)\, dy$$

If $q(y) = 0$, the disk method results:

$$V = \int_c^d \pi\, p(y)^2\, dy$$

General formulas for indefinite integrals

$$\int \cos ax\, dx = \frac{1}{a}\sin ax + C \tag{1}$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax + C \tag{2}$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax + C \tag{3}$$

$$\int \csc^2 ax\, dx = -\frac{1}{a}\cot ax + C \tag{4}$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax + C \tag{5}$$

$$\int \csc ax \cot ax\, dx = -\frac{1}{a}\csc ax + C \tag{6}$$

$$\int e^{ax}\, dx = \frac{1}{a}e^{ax} + C \tag{7}$$

$$\int b^x\, dx = \frac{1}{\ln b}b^x + C, b > 0, b \neq 1 \tag{8}$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C \tag{9}$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C, a > 0 \tag{10}$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C, a > 0 \tag{11}$$

Examples

1. Use symmetry to evaluate integrals

$$\int_{-200}^{200} 2x^5\, dx = 0$$

$$\begin{aligned}
\int_{-2}^{2} \left(x^2 + x^3\right)\, dx &= \int_{-2}^{2} x^2\, dx + \int_{-2}^{2} x^3\, dx && (1)\\
&= 2\int_0^2 x^2\, dx + 0 && (2)\\
&= 2\frac{x^3}{3} && (3)\\
&= \frac{16}{3} && (4)
\end{aligned}$$

2. A derivative calculation

$$s(t) = -16t^2 + 64t$$

$$t = 4$$

$$[0, 4]$$

$$\begin{aligned}
v(t) &= s'(t) && (1)\\
\overline{v} &= \frac{1}{4}\int_0^4 v(t)\,dx && (2)\\
&= \frac{1}{4}\int_0^4 s'(t)\,dx && (3)\\
&= \frac{1}{4}s(t) && (4)\\
&= \frac{1}{4}\left(s(4) - s(0)\right) && (5)\\
&= 0 && (6)
\end{aligned}$$

3. Applying MVT for integrals

$$f(x) = e^x$$

$$[0, 2]$$

$$\begin{aligned}
\overline{f} &= \frac{1}{2}\left(\int_0^2 e^x\,dx\right) && (1)\\
&= \frac{e^x}{2} && (2)\\
&= \frac{e^2}{2} - \frac{e^0}{2} && (3)\\
&= \frac{e^2 - 1}{2} && (4)\\
e^x &= \frac{e^2 - 1}{2} && (5)\\
\ln e^x &= \ln\frac{e^2 - 1}{2} && (6)
\end{aligned}$$

4. Perfect substitutions in indefinite integrals

$$\begin{aligned}
u &= 4x^3 - 8 && (1)\\
du &= 12x^2\,dx && (2)\\
\int 12x^2\left(4x^3 - 8\right)^5\,dx &= \int 12x^2u^5\,dx && (3)\\
&= \frac{u^6}{6} + C && (4)\\
&= \frac{\left(4x^3 - 8\right)^6}{6} + C && (5)
\end{aligned}$$

$$\begin{aligned}
u &= \sin t && (1)\\
du &= \cos t\,dt && (2)\\
\int (\cos t)\,e^{\sin t}\,dt &= \int e^u\,du && (3)\\
&= e^u + C && (4)\\
&= e^{\sin t} + C && (5)
\end{aligned}$$

5. Introducting constants when integrating by substitution

$$\begin{aligned}
u &= 6x+4 && (1)\\
du &= 6\,dx && (2)\\
dx &= \frac{du}{6} && (3)\\
\int (6x+4)^9 \; dx &= \int \frac{1}{6} \cdot u^9 \, du && (4)\\
&= \frac{1}{6} \int u^8 \, du && (5)\\
&= \frac{1}{6} \cdot \frac{u^9}{9} + C && (6)\\
&= \frac{(6x+4)^9}{54} + C && (7)
\end{aligned}$$

$$\begin{aligned}
u &= \cot x && (1)\\
du &= -\csc^2 x \, dx && (2)\\
\int \cot^2 x \csc^2 x \, dx &= \int -u^2 \, du && (3)\\
&= -\frac{u^3}{3} + C && (4)\\
&= -\frac{\csc^3 x}{3} + C && (5)
\end{aligned}$$

6. Variations on the substitution method

$$\begin{aligned}
u &= x-1 && (1)\\
du &= dx && (2)\\
x &= u+1 && (3)\\
\int x\sqrt{x-1}\, dx &= \int (u+1)\sqrt{u}\, du && (4)\\
&= \int u\sqrt{u} + \sqrt{u}\, du && (5)\\
&= \int u^{\frac{3}{2}} + u^{\frac{1}{2}} \; du && (6)\\
&= \frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}} + C && (7)\\
&= \frac{2}{5}(x-1)^{\frac{5}{2}} + \frac{2}{3}(x-1)^{\frac{3}{2}} + C && (8)
\end{aligned}$$

7. Use known formulas to evaluate indefinite integrals

$$\begin{aligned}
\int 2e^{-4x}\, dx &= 2\int e^{-4x}\, dx && (1)\\
&= \frac{2}{-4}e^{-4x} + C && (2)\\
&= -\frac{1}{2}e^{-4x} + C && (3)\\
& && (4)
\end{aligned}$$

$$\begin{aligned}
\int \frac{dx}{\sqrt{36-x^2}} &= \int \frac{dx}{\sqrt{6^2-x^2}} && (1)\\
&= \sin^{-1}\frac{x}{6} + C && (2)
\end{aligned}$$

8. Evaluating definite integrals using substitution

$$\begin{aligned}
u &= 2^x + 4 && (1)\\
du &= 2^x \ln 2 \, dx && (2)\\
\frac{1}{\ln 2} du &= 2^x \, dx && (3)\\
\int_1^3 \frac{2^x}{2^x + 4} \, dx &= \int_1^3 \frac{1}{u \ln 2} \, du && (4)\\
\int_{g(1)}^{g(3)} \frac{1}{u \ln 2} \, du &= \int_6^{12} \frac{1}{u \ln 2} \, du && (5)\\
&= \frac{1}{\ln 2} \int_6^{12} \frac{du}{u} && (6)\\
&= \frac{1}{\ln 2} \cdot (\ln 12 - \ln 6) && (7)\\
&= \frac{\ln 2}{\ln 2} && (8)\\
&= 1 && (9)
\end{aligned}$$

$$\begin{aligned}
u &= \ln p && (1)\\
du &= \frac{1}{p} \, dx && (2)\\
\int_1^{e^2} \frac{\ln p}{p} &= \int_0^2 u \, du && (3)\\
&= \frac{2^2}{2} - \frac{0^2}{2} && (4)\\
&= \frac{4}{2} && (5)\\
&= 2 && (6)
\end{aligned}$$

9. Integrals involving $\cos^2 x$ and $\sin^2 x$

$$\begin{aligned}
u &= 2x && (1)\\
du &= 2 \, dx && (2)\\
dx &= \frac{1}{2} du && (3)\\
\sin^2 x &= \frac{1 - \cos 2x}{2} && (4)\\
\int_0^{\pi} \sin^2 x \, dx &= \int_0^{\pi} \frac{1 - \cos 2x}{2} \, dx && (5)\\
&= \frac{1}{2} \int_0^{\pi} 1 - \cos 2x \, dx && (6)\\
&= \frac{1}{2} \left(\int_0^{\pi} 1 \, dx - \int_0^{\pi} \cos 2x \, dx \right) && (7)\\
&= \frac{1}{2} \left((\pi - 0) - \frac{1}{2} \int_0^{2\pi} \cos u \, du \right) && (8)\\
&= \frac{1}{2} \left(\pi - \frac{1}{2} (\sin 2\pi - \sin 0) \right) && (9)\\
&= \frac{1}{2} (\pi - 0) && (10)\\
&= \frac{\pi}{2} && (11)
\end{aligned}$$

10. Displacement and distance from velocity

$$v(t) = 4t^3 - 24t^2 + 20t$$

(a)

$$\begin{aligned}
v(t) &= 0 && (1)\\
4t^3 - 24t^2 + 20t &= 0 && (2)\\
4t(t^2 - 6t + 5) &= 0 && (3)\\
4t(t-1)(t-5) &= 5 && (4)\\
t &= 0 && (5)\\
t &= 1 && (6)\\
t &= 5 && (7)\\
0 < t < 1 &= \text{Positive} && (8)\\
1 < t < 5 &= \text{Negative} && (9)\\
t > 5 &= \text{Positive} && (10)
\end{aligned}$$

(b)

$$\begin{aligned}
\int_0^5 4t^3 - 24t^2 + 20t\,dt &= 4\int_0^5 t^3\,dt - 24\int_0^5 t^2\,dt + 20\int_0^5 t\,dt && (1)\\
&= 4\left(\frac{5^4}{4} - \frac{0^4}{4}\right) - 24\left(\frac{5^3}{3} - \frac{0^3}{3}\right) + 20\left(\frac{5^2}{2} - \frac{0^2}{2}\right) && (2)\\
&= 4\left(\frac{625}{4}\right) - 24\left(\frac{125}{3}\right) + 20\left(\frac{25}{2}\right) && (3)\\
&= -125 && (4)
\end{aligned}$$

(c)

$$\begin{aligned}
\int_0^5 |4t^3 - 24t^2 + 20t|\,dt &= \int_0^1 4t^3 - 24t^2 + 20t\,dt + \int_1^5 -4t^3 + 24t^2 - 20t\,dt && (1)\\
&= 3 + 128 && (2)\\
&= 131 && (3)
\end{aligned}$$

11. Position and velocity from acceleration

$$a(t) = \frac{20}{(t+2)^2}$$

$$v(0) = 20$$

$$s(0) = 10$$

$$\begin{aligned}
v(t) &= v(0) + \int_0^t a(t)\,d5 && (1)\\
&= 20 + \int_0^t \frac{20}{(t+2)^2}\,d5 && (2)\\
&= 20 - \frac{20}{t+2} + 10 && (3)\\
&= 30 - \frac{20}{t+2} && (4)\\
s(t) &= s(0) + \int_0^t v(t)\,dt && (5)\\
&= 10 + \int_0^t \left(30 - \frac{20}{t+2}\right)\,dt && (6)\\
&= 10 + 30t - 20\ln|t+2| + 20\ln 2 && (7)
\end{aligned}$$

12. Acceleration application

$$a(t) = -15$$

$$v(0) = 60$$

$$s(0) = 0$$

$$\begin{aligned}
v(t) &= 60 + \int_0^t -15\,dt && (1)\\
&= 60 + -15\int_0^t t^0\,dt && (2)\\
&= -15t + 60 && (3)\\
s(t) &= 0 + \int_0^t -15t + 60\,dt && (4)\\
&= -15\int_0^t t\,dt + 60t && (5)\\
&= -\frac{15}{2}t^2 + 60t && (6)\\
v(t) &= 0 && (7)\\
60 - 15t &= 0 && (8)\\
15t &= 60 && (9)\\
t &= \frac{60}{15} && (10)\\
&= 4 && (11)\\
s(4) - s(0) &= 60(4) - \frac{15}{2}(4)^2 - 0 && (12)\\
&= 120 && (13)
\end{aligned}$$

13. Application of net change

$$V'(t) = 70(1 + \sin 2\pi t)$$

$$[0, t]$$

$$V(0) = 0$$

$$\begin{aligned}
V(t) &= V(0) + \int_0^t V'(x)\,dx && (1)\\
&= 0 + \int_0^t 70(1 + \sin(2\pi x)\,dx && (2)\\
&= 70\left(t - \frac{\cos 2\pi t}{2\pi} + \frac{1}{2\pi}\right) && (3)\\
V(60) &= 70\left(60 - \frac{\cos 120\pi}{2\pi} + \frac{1}{2\pi}\right) && (4)\\
&= 70 \cdot 60 && (5)\\
&= 4200 && (6)
\end{aligned}$$

14. Area between curves (one integral)

$$y = x$$

$$y = x^2 - 2$$

$$\begin{aligned}
x &= x^2 - 2 && (1)\\
0 &= x^2 - x - 2 && (2)\\
&= (x-2)(x+1) && (3)\\
x &= -1 && (4)\\
x &= 2 && (5)\\
\int_{-1}^{2} x - \left(x^2 - 2\right)\, dx &= \int_{-1}^{2} -x^2 + x + 2\, dx && (6)\\
&= \left(\frac{-2^3}{3} + \frac{2^2}{2} + 2(2)\right) - \left(\frac{-(-1)^3}{3} + \frac{(-1)^2}{2} + 2(-1)\right) && (7)\\
&= \left(\frac{-8}{3} + 6\right) - \left(\frac{1}{3} + \frac{1}{2} - 2\right) && (8)\\
&= \frac{-9}{3} + 8 - \frac{1}{2} && (9)\\
&= 5 - \frac{1}{2} && (10)\\
&= 4.5 && (11)
\end{aligned}$$

15. Area between curves (multiple integrals)

$$y = x$$
$$y = x^3$$

$$\begin{aligned}
x^3 &= x && (1)\\
0 &= x^3 - x && (2)\\
&= x\left(x^2 - 1\right) && (3)\\
x &= -1 && (4)\\
x &= 0 && (5)\\
x &= 1 && (6)\\
\int_{-1}^{0} x^3 - x\, dx + \int_{0}^{1} x - x^3\, dx &= -\left(\frac{(-1)^4}{4} - \frac{(-1)^2}{2}\right) + \left(\frac{1^2}{2} - \frac{(-1)^4}{4}\right) && (7)\\
&= -\left(\frac{1}{4} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) && (8)\\
&= -\left(-\frac{1}{4}\right) + \frac{1}{4} && (9)\\
&= \frac{1}{4} + \frac{1}{4} && (10)\\
&= \frac{1}{2} && (11)
\end{aligned}$$

16. Area between curves (multiple integrals 2)

$$y = 4x - x^2$$
$$y = 4x - 4$$
$$[0, 2]$$

$$\begin{aligned}
\int_{0}^{1} 4x - x^2\, dx + \int_{1}^{2} 4x - x^2 - 4x + 4\, dx &= \left(2 + \frac{1}{3}\right) + \left(\left(-\frac{2^3}{3} + 4(2)\right) - \left(-\frac{1^3}{3} + 4(1)\right)\right) && (1)\\
&= \left(2 + \frac{1}{3}\right) + \left(\left(-\frac{8}{3} + 8\right) - \left(-\frac{1}{3} + 4\right)\right) && (2)\\
&= \frac{7}{3} + \left(\frac{16}{3} - \frac{11}{3}\right) && (3)\\
&= \frac{7}{3} + \left(\frac{16}{3} - \frac{11}{3}\right) && (4)\\
&= 4 && (5)
\end{aligned}$$

17. Area between curves (integrating dy)

$$x = \sqrt{y}$$

$$x\frac{y}{4}$$

$$\begin{aligned}
\int_0^4 \left(\sqrt{y} - \frac{y}{4}\right)\, dy &= \left(\frac{2}{3}4^{\frac{3}{2}} - \frac{4^2}{8}\right) && (1)\\
&= \left(\frac{2}{3}\cdot 8 - \frac{16}{8}\right) && (2)\\
&= \left(\frac{16}{3} - 2\right) && (3)\\
&= \frac{10}{3} && (4)
\end{aligned}$$

18. Area between curves (choosing a method)

$$y = \sqrt{\frac{x}{2} + 1}$$

$$y = \sqrt{1-x}$$

$$\begin{aligned}
x &= 2y^2 - 2 && (1)\\
x &= 1 - y^2 && (2)\\
2y^2 - 2 &= 1 - y^2 && (3)\\
3y^2 - 3 &= 0 && (4)\\
3\left(y^2 - 1\right) &= 0 && (5)\\
y &= -1 && (6)\\
y &= 1 && (7)\\
\int_0^1 \left(1 - y^2 - 2y^2 + 2\right)\, dy &= \int_0^1 \left(3 - 3y^2\right)\, dy && (8)\\
&= \left(3(1) - 1^3\right) && (9)\\
&= 3 - 1 && (10)\\
&= 2 && (11)
\end{aligned}$$

19. Using geometry to find area between curves

$$x = 2y$$

$$x = y + 1$$

$$(2, 1)$$

$$\begin{aligned}
A_1 &= 1 && (1)\\
A_2 &= \frac{1}{2} && (2)\\
A &= 1 - \frac{1}{2} && (3)\\
&= \frac{1}{2} && (4)
\end{aligned}$$

20. Applying the general slicing method

$$y = \sqrt{1 - x^2}$$

$$
\begin{aligned}
S &= \sqrt{1-x^2} && (1)\\
A(x) &= \sqrt{1-x^2}^2 && (2)\\
&= 1-x^2 && (3)\\
x &= -1, 1 && (4)\\
V &= \int_{-1}^{1} 1-x^2\,dx && (5)\\
&= \left(1-\frac{1^3}{3}\right)-\left(-1-\frac{(-1)^3}{3}\right) && (6)\\
&= 1-\frac{1}{3}+1-\frac{1}{3} && (7)\\
&= 2-\frac{2}{3} && (8)\\
&= \frac{4}{3} && (9)
\end{aligned}
$$

21. Graphing and applying the general slicing method

$$y = x^2 \qquad \text{(Base Region)}$$

$$y = 1 \qquad \text{(Base Line)}$$

$$
\begin{aligned}
S &= 1-x^2 && (1)\\
A(x) &= \left(1-x^2\right)^2 && (2)\\
&= 1-2x^2+x^4 && (3)\\
1 &= x^2 && (4)\\
x &= -1, 1 && (5)\\
\int_{-1}^{1} 1-2x^2+x^4\,dx &= \left(1-2\frac{1^3}{3}+\frac{1^5}{5}\right)-\left(-1-2\frac{(-1)^3}{3}+\frac{(-1)^5}{5}\right) && (6)\\
&= 1-\frac{2}{3}+\frac{1}{5}+1-\frac{2}{3}+\frac{1}{5} && (7)\\
&= \frac{16}{15} && (8)
\end{aligned}
$$

22. Applying the disk method (x-axis)

$$y = e^{-x}$$

$$y = 0$$

$$x = 0$$

$$x = \ln 4$$

$$
\begin{aligned}
V &= \int_{0}^{\ln 4} \pi\left(e^{-x}\right)^2\,dx && (1)\\
&= \int_{0}^{\ln 4} \left(\pi\, e^{-2x}\right)\,dx && (2)\\
&= \frac{15}{32}\pi && (3)
\end{aligned}
$$

23. Applying the washer method (x-axis)

$$y = x$$

$$y = \sqrt[4]{x}$$

$$\begin{aligned}
r_o &= \sqrt[4]{x} && (1)\\
r_i &= x && (2)\\
x &= \sqrt[4]{x} && (3)\\
x^4 &= x && (4)\\
0 &= x^4 - x && (5)\\
&= x\left(x^3 - 1\right) && (6)\\
x &= 0, 1 && (7)\\
V &= \int_0^1 \pi\left(\left(\sqrt[4]{x}\right)^2 - x^2\right)\, dx && (8)\\
&= \int_0^1 \pi\left(x^{\frac{1}{2}} - x^2\right)\, dx && (9)\\
&= \frac{1}{3}\pi && (10)
\end{aligned}$$

24. Applying the washer method (y-axis)

$$y = x^3$$
$$y = 0$$
$$x = 1$$

$$\begin{aligned}
r_o &= 1 && (1)\\
x &= y^{\frac{1}{3}} && (2)\\
r_i &= \sqrt[3]{y} && (3)\\
V &= \int_0^1 \pi\left(1^2 - \sqrt[3]{y}^2\right)\, dy && (4)\\
&= \int_0^1 \pi\left(1 - y^{\frac{2}{3}}\right)\, dy && (5)\\
&= \frac{2}{5}\pi && (6)
\end{aligned}$$

25. Which solid of rotation is bigger

$$y = x^2$$
$$y = \sqrt{8x}$$

$$
\begin{aligned}
x^2 &= \sqrt{8x} && (1)\\
x^4 &= 8x && (2)\\
0 &= x^4 - 8x && (3)\\
&= x\left(x^3 - 8\right) && (4)\\
x &= 0, 2 && (5)\\
y &= 0, 4 && (6)\\
r_o &= \sqrt{8x} && (7)\\
r_i &= x^2 && (8)\\
V &= \int_0^2 \pi \left(\sqrt{8x}^2 - \left(x^2\right)^2\right)\, dx && (9)\\
&= \int_0^2 \pi \left(8x - x^4\right)\, dx && (10)\\
&= \frac{48}{5}\pi && (11)\\
r_o &= \sqrt{y} && (12)\\
r_i &= \frac{y^2}{8} && (13)\\
V &= \int_0^4 \pi \left(\sqrt{y}^2 - \left(\frac{y^2}{8}\right)^2\right)\, dy && (14)\\
&= \int_0^4 \pi \left(y - \frac{y^4}{64}\right)\, dy && (15)\\
&= \frac{24}{5}\pi && (16)
\end{aligned}
$$

26. Volume and rotation about a parallel axis

$$
\begin{gathered}
x = 0\\
y = \sqrt{x}\\
y = 2\\
x = 4 \quad \text{(about)}
\end{gathered}
$$

$$
\begin{aligned}
x &= y^2 && (1)\\
r_o &= 4 && (2)\\
r_i &= 4 - y^2 && (3)\\
V &= \int_0^2 \pi \left(4^2 - \left(4 - y^2\right)^2\right)\, dy && (4)\\
&= \int_0^2 \pi \left(8y^2 - y^4\right)\, dy && (5)\\
&= \frac{224}{15}\pi && (6)
\end{aligned}
$$

Related Exercises

1. (Section 5.4, Exercise 15)

$$\begin{aligned}
\int_{-2}^{2}\left(x^2+x^3\right)\,dx &= \int_{-2}^{2} x^2\,dx + \int_{-2}^{2} x^3\,dx && (1)\\
&= 2\int_{0}^{2} x^2\,dx + 0 && (2)\\
&= 2\frac{x^3}{3} && (3)\\
&= 2\frac{2^3}{3} - 2\frac{0^3}{3} && (4)\\
&= 2\frac{8}{3} && (5)\\
&= \frac{16}{3} && (6)
\end{aligned}$$

2. (Section 5.4, Exercise 16)

$$\int_{-\pi}^{\pi} t^2 \sin t\,dx = 0$$

3. (Section 5.4, Exercise 26)

$$f(x) = x^2+1$$
$$[-2,2]$$

$$\begin{aligned}
\overline{f} &= \frac{1}{2-(-2)}\int_{-2}^{2} x^2+1\,dx && (1)\\
&= \frac{1}{4}\left(\int_{-2}^{2} x^2\,dx + 1\int_{-2}^{2} x^0\,dx\right) && (2)\\
&= \frac{1}{4}\left(\frac{x^3}{3}+x\right) && (3)\\
&= \frac{1}{4}\left(\int_{-2}^{2} x^2\,dx + \int_{-2}^{2} 1\,dx\right) && (4)\\
&= \frac{1}{4}\left(\frac{2^3}{3} - \frac{(-2)^3}{3} + 2 - (-2)\right) && (5)\\
&= \frac{1}{4}\left(\frac{8}{3} - \frac{-8}{3} + 4\right) && (6)\\
&= \frac{1}{4}\left(\frac{16}{3} + 4\right) && (7)\\
&= \frac{1}{4}\left(\frac{28}{3}\right) && (8)\\
&= \frac{7}{3} && (9)
\end{aligned}$$

4. (Section 5.4, Exercise 34)

$$f(x) = x^3 - 5x^2 + 30$$
$$[0,4]$$

$$\begin{aligned}
\overline{f} &= \frac{1}{4}\left(\int_0^4 \left(x^3 - 5x^2 + 30\right)\, dx\right) && (1)\\
&= \frac{1}{4}\left(\int_0^4 x^3 - 5\int_0^4 x^2 + 30\int_0^4 x^0\right) && (2)\\
&= \frac{1}{4}\left(\frac{x^4}{4} - 5\frac{x^3}{3} + 30x\right) && (3)\\
&= \frac{1}{4}\left(\left(\frac{4^4}{4} - \frac{0^4}{4}\right) - \left(5\frac{4^3}{3} - 5\frac{0^3}{3}\right) + (30(4) - 30(0))\right) && (4)\\
&= \frac{1}{4}\left(64 - \frac{320}{3} + 120\right) && (5)\\
&= \frac{1}{4}\left(\frac{232}{3}\right) && (6)\\
&= \frac{58}{3} && (7)
\end{aligned}$$

5. (Section 5.4, Exercise 41)

$$f(x) = 1 - \frac{x^2}{a^2}$$

$$[0, a]$$

$$\begin{aligned}
\overline{f} &= \frac{1}{a}\left(\int_0^a 1 - \frac{x^2}{a^2}\, dx\right) && (1)\\
&= \frac{1}{a}\left(\int_0^a 1\, dx - \int_0^a \frac{x^2}{a^2}\, dx\right) && (2)\\
&= \frac{1}{a}\left(x - \frac{1}{a^2}\int_0^a x^2\, dx\right) && (3)\\
&= \frac{1}{a}\left(x - \frac{1}{a^2}\frac{x^3}{3}\right) && (4)\\
&= \frac{1}{a}\left(x - \frac{x^3}{3a^2}\right) && (5)\\
&= \frac{1}{a}\left((a-0) - \frac{1}{a^2}\left(\frac{a^3}{3} - \frac{0^3}{3}\right)\right) && (6)\\
&= \frac{1}{a}\left(a - \frac{a^3}{3a^2}\right) && (7)\\
&= \frac{1}{a}\left(a - \frac{a}{3}\right) && (8)\\
&= \frac{1}{a}\left(\frac{2a}{3}\right) && (9)\\
&= \frac{2}{3} && (10)\\
1 - \frac{c^2}{a^2} &= \frac{2}{3} && (11)\\
\frac{c^2}{a^2} &= \frac{1}{3} && (12)\\
c^2 &= \frac{a^2}{3} && (13)\\
c &= \sqrt{\frac{a^2}{3}} && (14)\\
&= \frac{a}{\sqrt{3}} && (15)
\end{aligned}$$

6. (Section 5.4, Exercise 42)

$$f(x) = \frac{\pi}{4}\sin x$$

$$[0, \pi]$$

$$\begin{aligned}
\overline{f} &= \frac{1}{\pi}\int_0^{\pi}\frac{\pi}{4}\sin x && (1)\\
&= \frac{1}{\pi}\frac{\pi}{4}\int_0^{\pi}\sin x && (2)\\
&= \frac{1}{\pi}\frac{\pi}{4}(-\cos x) && (3)\\
&= \frac{1}{\pi}\frac{\pi}{4}(-\cos\pi+\cos 0) && (4)\\
&= \frac{1}{\pi}\frac{\pi}{4}(1+1) && (5)\\
&= \frac{1}{\pi}\frac{\pi}{2} && (6)\\
&= \frac{1}{2} && (7)\\
\frac{\pi}{4}\sin x &= \frac{1}{2} && (8)\\
\sin x &= \frac{2}{\pi} && (9)\\
\sin^{-1}\sin x &= \sin^{-1}\frac{2}{\pi} && (10)\\
x &= \sin^{-1}\frac{2}{\pi} && (11)
\end{aligned}$$

7. (Section 5.5, Exercise 17)

$$\begin{aligned}
u &= x^2-1 && (1)\\
du &= 2x\,dx && (2)\\
\int 2x\left(x^2-1\right)^{99}\,dx &= \int u^{99}\,du && (3)\\
&= \frac{u^{100}}{100}+C && (4)\\
&= \frac{\left(x^2-1\right)^{100}}{100}+C && (5)
\end{aligned}$$

8. (Section 5.5, Exercise 20)

$$\begin{aligned}
u &= \sqrt{x}+1 && (1)\\
du &= \frac{1}{2\sqrt{x}}\,dx && (2)\\
\int\frac{\left(\sqrt{x}+1\right)^4}{2\sqrt{x}}\,dx &= \int u^4\,du && (3)\\
&= \frac{u^5}{5}+C && (4)\\
&= \frac{\left(\sqrt{x}+1\right)^5}{5}+C && (5)
\end{aligned}$$

9. (Section 5.5, Exercise 21)

$$\begin{aligned}
u &= x^2+x && (1)\\
du &= 2x+1\,dx && (2)\\
\int\left(x^2+x\right)^{10}(2x+1)\,dx &= \int u^{10}\,du && (3)\\
&= \frac{u^{11}}{11}+C && (4)\\
&= \frac{\left(x^2+x\right)^{11}}{11}+C && (5)
\end{aligned}$$

10. (Section 5.5, Exercise 23)

$$\begin{aligned}
u &= x^4+16 && (1)\\
du &= 4x^3\,dx && (2)\\
x^3\,dx &= \frac{1}{4}du && (3)\\
\int x^3\left(x^4+16\right)^6\,dx &= \int \frac{1}{4}u^6\,du && (4)\\
&= \frac{1}{4}\int u^6\,du && (5)\\
&= \frac{1}{4}\cdot\frac{u^7}{7}+C && (6)\\
&= \frac{\left(x^4+16\right)^7}{28}+C && (7)
\end{aligned}$$

11. (Section 5.5, Exercise 24)

$$\begin{aligned}
u &= \sin\theta && (1)\\
du &= \cos\theta\,d\theta && (2)\\
\int \sin^{10}\theta\cos\theta\,d\theta &= \int u^{10}\,du && (3)\\
&= \frac{u^{11}}{11}+C && (4)\\
&= \frac{\sin^{11}\theta}{11}+C && (5)
\end{aligned}$$

12. (Section 5.5, Exercise 78)

$$\begin{aligned}
u &= x-2 && (1)\\
x &= u+2 && (2)\\
du &= dx && (3)\\
\int \frac{x}{x-2}\,dx &= \int \frac{u+2}{u}\,du && (4)\\
&= \int \frac{u}{u}\,du + \int \frac{2}{u}\,du && (5)\\
&= u+2\int \frac{1}{u}\,du+C && (6)\\
&= u+2\ln|u|+C && (7)\\
&= x-2+2\ln|u|+C && (8)
\end{aligned}$$

13. (Section 5.5, Exercise 79)

$$\begin{aligned}
u &= \sqrt{x-4} && (1)\\
u^2 &= x-4 && (2)\\
x &= u^2+4 && (3)\\
dx &= 2u\,du && (4)\\
\int \frac{x}{\sqrt{x-4}}\,dx &= \int 2u\frac{u^2+4}{u}\,du && (5)\\
&= \int \frac{2u^3+8u}{u}\,du && (6)\\
&= 2\left(\int u^2\,du+\int 4\,du\right) && (7)\\
&= 2\left(\frac{\sqrt{x-4}^3}{3}+4\sqrt{x-4}\right)+C && (8)\\
&= \frac{2}{3}\sqrt{x-4}+8\sqrt{x-4}+C && (9)
\end{aligned}$$

14. (Section 5.5, Exercise 15)

$$\begin{aligned}
\int e^{10x}\,dx &= \frac{1}{10}e^{10x}+C && (1)\\
\int \sec 5x\tan 5x\,dx &= \frac{1}{5}\sec 5x+C && (2)\\
\int \sin 7x\,dx &= -\frac{1}{7}\cos 7x+C && (3)\\
\int \cos\frac{x}{7}\,dx &= 7\sin\frac{x}{7}+C && (4)\\
\int \frac{dx}{81+9x^2} &= \int \frac{dx}{9^2+9x^2} && (5)\\
&= \int \frac{dx}{9(9+x^2)} && (6)\\
&= \frac{1}{9}\int \frac{dx}{3^2+x^2} && (7)\\
&= \frac{1}{27}\tan^{-1}\frac{x}{3}+C && (8)\\
\int \frac{dx}{\sqrt{36-x^2}} &= \int \frac{dx}{\sqrt{6^2-x^2}} && (9)\\
&= \sin^{-1}\frac{x}{6}+C && (10)
\end{aligned}$$

15. (Section 5.5, Exercise 16)

$$\begin{aligned}
\int_0^1 10^x\,dx &= \frac{1}{\ln 10}10-\frac{1}{\ln 10} && (1)\\
&= \frac{9}{\ln 10} && (2)\\
\int_0^{\frac{\pi}{40}} \cos 20x\,dx &= \cos\frac{20\pi}{40}-\cos 20(0) && (3)\\
&= 0-1 && (4)\\
&= -1 && (5)\\
\int_{3\sqrt{2}}^6 \frac{dx}{x\sqrt{x^2-9}} &= \int_{3\sqrt{2}}^6 \frac{dx}{x\sqrt{x^2-3^2}} && (6)\\
&= \frac{1}{3}\sec^{-1}\left|\frac{6}{3}\right|-\frac{1}{3}\sec^{-1}\left|\frac{3\sqrt{2}}{3}\right| && (7)\\
&= \frac{1}{3}\sec^{-1}\frac{\pi}{3}-\frac{1}{3}\sec^{-1}\frac{\pi}{4} && (8)\\
\int_0^{\frac{\pi}{16}} \sec^2 4x\,dx &= \frac{1}{4}\tan\frac{4\pi}{16}-\frac{1}{4}\tan 0 && (9)\\
&= \frac{1}{4}-0 && (10)\\
&= \frac{1}{4} && (11)
\end{aligned}$$

16. (Section 5.5, Exercise 49)

$$\begin{aligned}
u &= 2^x + 4 && (1)\\
du &= 2^x \ln 2 \, dx && (2)\\
2^x \, dx &= \frac{1}{\ln 2} \, du && (3)\\
\int_1^3 \frac{2^x}{2^x+4} \, dx &= \frac{1}{\ln 2} \int_6^{12} \frac{1}{u} \, du && (4)\\
&= \frac{1}{\ln 2} (\ln 12 - \ln 6) && (5)\\
&= \frac{1}{\ln 2} \cdot \ln 2 && (6)\\
&= 1 && (7)
\end{aligned}$$

17. (Section 5.5, Exercise 51)

$$\begin{aligned}
u &= \sin\theta && (1)\\
du &= \cos\theta \, d\theta && (2)\\
\int_0^{\frac{\pi}{2}} \sin^2\theta \cos\theta \, d\theta &= \int_0^{\frac{\pi}{2}} u^2 \, du && (3)\\
&= \int_0^1 u^2 \, du && (4)\\
&= \frac{1^3}{3} - \frac{0^3}{3} && (5)\\
&= \frac{1}{3} && (6)
\end{aligned}$$

18. (Section 5.5, Exercise 64)

$$\begin{aligned}
u &= 3 + 2e^x && (1)\\
du &= 2e^x \, dx && (2)\\
e^x \, dx &= \frac{1}{2} \, du && (3)\\
\int_0^{\ln 4} \frac{e^x}{3+2e^x} \, dx &= \frac{1}{2} \int_{u=5}^{u=11} \frac{1}{u} \, du && (4)\\
&= \frac{\ln 11 - \ln 5}{2} && (5)
\end{aligned}$$

19. (Section 5.5, Exercise 87)

$$\begin{aligned}
u &= 2x && (1)\\
du &= 2\,dx && (2)\\
dx &= \frac{1}{2}\,du && (3)\\
\cos^2 x &= \frac{1+\cos 2x}{2} && (4)\\
\int_{-\pi}^{\pi} \cos^2 x\,dx &= \int_{-\pi}^{\pi} \frac{1}{2} + \frac{\cos 2x}{2}\,dx && (5)\\
&= \frac{1}{2}\int_{-\pi}^{\pi} 1 + \cos 2x\,dx && (6)\\
&= \frac{1}{2}\left(\int_{-\pi}^{\pi} 1\,dx + \frac{1}{2}\int_{-\pi}^{\pi} \cos u\,du\right) && (7)\\
&= \frac{1}{2}\left((\pi - (-\pi)) + \frac{1}{2}\int_{u=-2\pi}^{u=2\pi} \cos u\,du\right) && (8)\\
&= \frac{1}{2}\left(2\pi + \frac{1}{2}\left(\sin 2\pi - \sin(-2\pi)\right)\right) && (9)\\
&= \frac{1}{2}\left(2\pi + \frac{1}{2}(0)\right) && (10)\\
&= \frac{2\pi}{2} && (11)\\
&= \pi && (12)
\end{aligned}$$

20. (Section 5.5, Exercise 91)

$$\begin{aligned}
u &= 4\theta && (1)\\
du &= 4\,d\theta && (2)\\
d\theta &= \frac{1}{4}\,du && (3)\\
\sin^2 x &= \frac{1-\cos 2x}{2} && (4)\\
\sin^2 2\theta &= \frac{1-\cos 4\theta}{2} && (5)\\
\int_{-\pi}^{\pi} \sin^2 2\theta\,d\theta &= \int_{-\pi}^{\pi} \frac{1-\cos 4\theta}{2}\,d\theta && (6)\\
&= \int_{-\pi}^{\pi} \frac{1}{2} - \frac{\cos u}{2}\,d\theta && (7)\\
&= \frac{1}{2}\int_{-\pi}^{\pi} 1 - \cos u\,d\theta && (8)\\
&= \frac{1}{2}\left(\int_{-\pi}^{\pi} 1\,d\theta - \int_{-\pi}^{\pi} \cos u\,d\theta\right) && (9)\\
&= \frac{1}{2}\left(2\pi - \frac{1}{4}\int_{u=-4\pi}^{u=4\pi} \cos u\,du\right) && (10)\\
&= \frac{1}{2}\left(2\pi - \frac{1}{4}\left(\sin 4\pi - \sin(-4\pi)\right)\right) && (11)\\
&= \frac{2\pi}{2} && (12)\\
&= \pi && (13)
\end{aligned}$$

21. (Section 6.1, Exercise 7)

(a) $0 < t < 1$ and $3 < t < 5$

(b) -4

(c) 26

(d) 6

22. (Section 6.1, Exercise 8)

(a) $0 < t < 2$ and $4 < t < 6$

(b) 4

(c) 44

(d) -10

23. (Section 6.1, Exercise 17)

$$v(t) = \sin t$$
$$s(0) = 1$$

$$\begin{aligned} s(t) &= 1 + \int_0^t \sin t \, dt && (1)\\ &= 1 - \cos t + \cos 0 && (2)\\ &= 2 - \cos t && (3) \end{aligned}$$

24. (Section 6.1, Exercise 20)

$$v(t) = 3\sin \pi t$$
$$s(0) = 1$$

$$\begin{aligned} u &= \pi t && (1)\\ du &= \pi \, dt && (2)\\ s(t) &= 1 + \int_0^t 3\sin \pi t \, dt && (3)\\ &= 1 + \frac{3}{\pi}\int_0^t \sin u \, du && (4)\\ &= 1 + \frac{3}{\pi}(-\cos \pi t + \cos 0) && (5)\\ &= 1 + \frac{3}{\pi} - \frac{3}{\pi}\cos \pi t && (6) \end{aligned}$$

25. (Section 6.1, Exercise 27)

$$\begin{aligned} \int_0^{20} 3t\,dt + \int_{20}^{30} 60 &= \left(\frac{3(20)^2}{2} - \frac{3(0)^2}{2}\right) + (60(30) - 60(20)) && (1)\\ &= \left(\frac{1200}{2}\right) + (1800 - 1200) && (2)\\ &= 600 + 600 && (3)\\ &= 1200 && (4)\\ \int_0^{20} 3t\,dt + \int_{20}^{45} 60 + \int_{45}^{60} 240 - 4t\,dt &= 600 + 2700 - 1200 + 14400 - 7200 - 10800 - 4050 && (5)\\ &= 2100 + ((14400 - 7200) - (10800 - 4050)) && (6)\\ &= 2100 + (7200 - 6750) && (7)\\ &= 2550 && (8)\\ s(t) &= \int_0^{20} 3t\,dt + \int_{20}^{45} 60 + \int_{45}^{t} 240 - 4t\,dt && (9)\\ &= 2100 + \left(240t - 2t^2 - 6750\right) && (10)\\ &= 240t - 2t^2 - 4650 && (11)\\ s(75) &= 2100 && (12) \end{aligned}$$

26. (Section 6.1, Exercise 28)

$$\begin{aligned}
\int_0^{10} 9.8t\,dt + \int_{10}^{30} 10 &= \left(4.9(10)^2 - 4.9(0^2)\right) + (10(30) - 10(10)) && (1)\\
&= (4.9(100)) + (300 - 100) && (2)\\
&= 490 + 200 && (3)\\
&= 690 && (4)
\end{aligned}$$

27. (Section 6.1, Exercise 30)

$$a(t) = -32$$
$$v(0) = 50$$
$$s(0) = 0$$

$$\begin{aligned}
v(t) &= v(0) + \int_0^t -32 && (1)\\
&= 50 - 32t && (2)\\
s(t) &= s(0) + \int_0^t 50 - 32t && (3)\\
&= 50t - 16t^2 && (4)
\end{aligned}$$

28. (Section 6.1, Exercise 31)

$$a(t) = -9.8$$
$$v(0) = 20$$
$$s(0) = 0$$

$$\begin{aligned}
v(t) &= v(0) + \int_0^t -9.8 && (1)\\
&= 20 - 9.8t && (2)\\
s(t) &= s(0) + \int_0^t 20 - 9.8t && (3)\\
&= 20t - 4.9t^2 && (4)
\end{aligned}$$

29. (Section 6.1, Exercise 43)

$$N'(t) = 100e^{-0.25t}$$

$$\begin{aligned}
u &= -0.25t && (1)\\
du &= -0.25\,dt && (2)\\
N(t) &= N(0) + \int_0^t 10e^{-0.25t}\,dt && (3)\\
&= 1900 + 100\int_0^t \frac{1}{-0.25}e^u\,du && (4)\\
&= 1900 - 400e^{-0.25t} && (5)\\
N(20) &= 1900 - 400e^{-5} && (6)\\
&\approx 1897.305 && (7)\\
N(40) &= 1900 - 400e^{-10} && (8)\\
&\approx 1899.98 && (9)
\end{aligned}$$

30. (Section 6.1, Exercise 44)

$$r(t) = 0.0025e^{0.25t} - 0.1485e^{-0.15t}$$

$$\begin{aligned}
t_0 &= \frac{\ln 0.0025 - \ln 0.1485}{-0.4} && (1)\\
\int_0^{t_0} 0.0025e^{0.25t} - 0.1485e^{-0.15t} &\approx 271 && (2)
\end{aligned}$$

31. (Section 6.1, Exercise 55)

$$C'(x) = 2000 - 0.5x$$

$$\begin{aligned}
\int_{100}^{150} 2000 - 0.5x\,dx &= \big(2000(150) - 0.25(150)^2\big) - \big(2000(100) - 0.25(100)^2\big) && (1)\\
&= (300000 - 5625) - (200000 - 2500) && (2)\\
&= 294375 - 197500 && (3)\\
&= 96875 && (4)\\
\int_{500}^{550} 2000 - 0.5x\,dx &= (1024375) - (937500) && (5)\\
&= 86875 && (6)
\end{aligned}$$

32. (Section 6.1, Exercise 56)

$$C'(x) = 200 - 0.05x$$

$$\begin{aligned}
\int_{100}^{150} 200 - 0.05x\,dx &= \big(200(150) - 0.025(150)^2\big) - \big(200(100) - 0.025(100)^2\big) && (1)\\
&= (29437.5) - (19750) && (2)\\
&= 9687.5 && (3)\\
\int_{500}^{500} 200 - 0.05x\,dx &= \big(200(550) - 0.025(550)^2\big) - \big(200(500) - 0.025(500)^2\big) && (4)\\
&= 102437.5 - 93750 && (5)\\
&= 8687.5 && (6)
\end{aligned}$$

33. (Section 6.2, Exercise 9)

$$y = x$$

$$y = x^2 - 2$$

$$\begin{aligned}
x &= x^2 - 2 && (1)\\
0 &= x^2 - x - 2 && (2)\\
&= (x+1)(x-2) && (3)\\
x &= -1, 2 && (4)\\
\int_{-2}^{1} \left(-x^2 + x + 2\right)\,dx &= \left(-\frac{2^3}{3} + \frac{2^2}{2} + 2(2)\right) - \left(-\frac{(-1)^3}{3} + \frac{(-1)^2}{2} + 2(-1)\right) && (5)\\
&= \left(-\frac{8}{3} + \frac{4}{2} + 4\right) - \left(\frac{1}{3} + \frac{1}{2} - 2\right) && (6)\\
&= -\frac{8}{3} + \frac{4}{2} + 4 - \frac{1}{3} - \frac{1}{2} + 2 && (7)\\
&= -\frac{9}{3} + \frac{3}{2} + 6 && (8)\\
&= -3 + \frac{3}{2} + 6 && (9)\\
&= \frac{3}{2} + 3 && (10)\\
&= 4.5 && (11)
\end{aligned}$$

34. (Section 6.2, Exercise 10)

$$y = -x^2 + 4x$$

$$y = x^2 - 2x$$

$$\begin{aligned}
-x^2+4x &= x^2-2x && (1)\\
0 &= 2x^2-6x && (2)\\
&= 2x\left(x-6\right) && (3)\\
x &= 0,6 && (4)\\
\int_0^6 \left(-x^2+4x-x^2+2x\right)\,dx &= \left(\frac{-2(6)^3}{3}+3(6)^2\right) && (5)\\
&= \left(\frac{-2(216)}{3}+3(36)\right) && (6)\\
&= \left(\frac{-432}{3}+108\right) && (7)\\
&= -144+108 && (8)\\
&= -36 && (9)
\end{aligned}$$

35. (Section 6.2, Exercise 15)

$$y=\sin x$$
$$y=\cos x$$

$$\begin{aligned}
\sin x &= \cos x && (1)\\
0 &= \sin x-\cos x && (2)\\
x &= \frac{\pi}{4} && (3)\\
\int_0^{\frac{\pi}{4}} \sin x\,dx+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x\,dx &= \left(-\cos\frac{\pi}{4}+\cos 0\right)+\left(\sin\frac{\pi}{2}-\sin\frac{\pi}{4}\right) && (4)\\
&= \left(-\frac{1}{\sqrt{2}}+1\right)+\left(1-\frac{1}{\sqrt{2}}\right) && (5)\\
&= 2-\sqrt{2} && (6)
\end{aligned}$$

36. (Section 6.2, Exercise 16)

$$y=x^3$$
$$y=x$$

$$\begin{aligned}
x^3 &= x && (1)\\
0 &= x^3-x && (2)\\
&= x\left(x^2-1\right) && (3)\\
x &= -1,0,1 && (4)\\
\int_{-1}^0 \left(x^3-x\right)\,dx+\int_0^1 \left(x-x^3\right)\,dx &= -\left(\frac{(-1)^4}{4}-\frac{(-1)^2}{2}\right)+\left(\frac{1^2}{2}-\frac{1^4}{4}\right) && (5)\\
&= -\left(\frac{1}{4}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{4}\right) && (6)\\
&= \frac{1}{4}+\frac{1}{4} && (7)\\
&= \frac{1}{2} && (8)
\end{aligned}$$

37. (Section 6.2, Exercise 19)

$$x=y^2-3$$
$$x=2y$$

$$
\begin{aligned}
2y &= y^2 - 3 && (1)\\
0 &= y^2 - 2y - 3 && (2)\\
&= (y-3)(y+1) && (3)\\
y &= -1, 3 && (4)\\
\int_{-1}^{3} \left(2y - y^2 + 3\right)\, dy &= \left(3^2 - \frac{3^3}{3} + 3(3)\right) - \left((-1)^2 - \frac{(-1)^3}{3} + 3(-1)\right) && (5)\\
&= \left(9 - \frac{27}{3} + 9\right) - \left(1 - \frac{-1}{3} - 3\right) && (6)\\
&= (18-9) - \left(-2 + \frac{1}{3}\right) && (7)\\
&= 9 + 2 - \frac{1}{3} && (8)\\
&= 11 - \frac{1}{3} && (9)\\
&= \frac{32}{3} && (10)
\end{aligned}
$$

38. (Section 6.2, Exercise 20)

$$x = \frac{y}{4}$$
$$x = \sqrt{y}$$

$$
\begin{aligned}
\int_0^4 \left(\sqrt{y} - \frac{y}{4}\right) &= \left(\frac{2}{3}4^{\frac{3}{2}} - \frac{4^2}{8}\right) && (1)\\
&= \left(\frac{2}{3}\cdot 8 - \frac{16}{8}\right) && (2)\\
&= \frac{16}{3} - 2 && (3)\\
&= \frac{10}{3} && (4)
\end{aligned}
$$

39. (Section 6.2, Exercise 34)

$$y = 2x^2$$
$$y = 3 - x$$

$$
\begin{aligned}
2x^2 &= 3 - x && (1)\\
0 &= 2x^2 + x - 3 && (2)\\
&= (2x+3)(x-1) && (3)\\
x &= 0, 1 && (4)\\
\int_0^1 3 - x - 2x^2\, dx &= \left(3 - \frac{1}{2} - 2\frac{1}{3}\right) && (5)\\
&= \left(3 - \frac{1}{2} - \frac{2}{3}\right) && (6)\\
&= \frac{11}{6} && (7)\\
A_1 &= \frac{11}{6} && (8)\\
A &= A_1 + A_2 && (9)\\
&= \frac{9}{2} && (10)\\
A_2 &= A - A_1 && (11)\\
&= \frac{9}{2} - \frac{11}{6} && (12)\\
&= \frac{8}{3} && (13)
\end{aligned}
$$

A copy of my notes (in LaTeX) are available on my GitHub