Module 3 Notes (MATH-211)

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General Notes (and Definitions)

- The Chain Rule
  Suppose $y = f(u)$ is differentiable at $u = g(x)$ and $u = g(x)$ is differentiable at x. The composite function $y = f(g(x))$ is differentiable at x, and its derivative can be expressed in two equivalent ways.

$$\frac{dy}{dx} = \frac{dy}{dy} \cdot \frac{du}{dx} \tag{1}$$

$$\frac{d}{dx}\left(f\left(g\left(x\right)\right)\right) = f'\left(g\left(x\right)\right) \cdot g'\left(x\right) \tag{2}$$

  Application of the Chain Rule (Assume the differentiable function $y = f(g(x))$ is given):

  1. Identify an outer function f and an inner function g, and let $u = g(x)$.
  2. Replace $g(x)$ with u to express y in terms of u:
  $$y = f(g(x)) = f(u)$$
  3. Calculate the product
  $$\frac{dy}{du} \cdot \frac{du}{dx}$$
  4. Replace u with $g(x)$ in $\frac{dy}{du}$ to obtain $\frac{dy}{dx}$

  If g is differentiable for all x in its domain and $p \in \mathbb{R}$,
  $$\frac{d}{dx}\left(\left(g\left(x\right)\right)^{p}\right) = p\left(g\left(x\right)\right)^{p-1} g'\left(x\right)$$

- Implicit Differentiation
  When we are unable to solve for y explicitly, we treat y as a function of x ($y = y(x)$) and apply the Chain Rule:
  $$y' = \frac{dy}{dx}$$
  $$\frac{d}{dx}y^n = ny^{n-1}\frac{dy}{dx}$$

Examples

1. The Chain Rule

$$
\begin{aligned}
y &= \left(5x^2 + 11x\right)^{\frac{4}{3}} && (1)\\
u &= 5x^2 + 11x && (2)\\
f &= u^{\frac{4}{3}} && (3)\\
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} && (4)\\
&= \frac{4}{3}u^{\frac{1}{3}} \cdot 10x + 11 && (5)\\
&= \frac{4}{3}\left(5x^2 + 11x\right)^{\frac{1}{3}} \cdot 10x + 11 && (6)\\
& && (7)
\end{aligned}
$$

$$\begin{aligned}
y &= e^{4x^2+1} && (1)\\
u &= 4x^2+1 && (2)\\
y &= e^u && (3)\\
\frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx} && (4)\\
&= e^u \cdot 8x && (5)\\
&= e^{4x^2+1}\cdot 8x && (6)\\
&= 8xe^{4x^2+1} && (7)
\end{aligned}$$

2. The Chain Rule (with Tables)

$$h(x) = f(g(x))$$

$$\begin{aligned}
y &= f(g(x)) && (1)\\
u &= g(x) && (2)\\
y &= f(u) && (3)\\
\frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx} && (4)\\
&= f(u)\cdot g'(x) && (5)\\
&= f'(g(x))\cdot g'(x) && (6)
\end{aligned}$$

$$\begin{aligned}
h'(1) &= f'(g(1))\cdot g'(1) && (7)\\
&= f'(4)\cdot 9 && (8)\\
&= 7\cdot 9 && (9)\\
&= 63 && (10)
\end{aligned}$$

$$\begin{aligned}
h'(2) &= f'(g(2))\cdot g'(2) && (11)\\
&= f'(1)\cdot 7 && (12)\\
&= -6\cdot 7 && (13)\\
&= -42 && (14)
\end{aligned}$$

$$\begin{aligned}
h'(3) &= f'(g(3))\cdot g'(3) && (15)\\
&= f'(5)\cdot 3 && (16)\\
&= 2\cdot 3 && (17)\\
&= 6 && (18)
\end{aligned}$$

$$k(x) = g(g(x))$$

$$\begin{aligned}
y &= g(g(x)) && (1)\\
u &= g(x) && (2)\\
y &= g(u) && (3)\\
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} && (4)\\
&= g'(u) \cdot g'(x) && (5)\\
&= g'(g(x)) \cdot g'(x) && (6)
\end{aligned}$$

$$\begin{aligned}
k'(3) &= g'(g(3)) \cdot g'(3) && (7)\\
&= g'(5) \cdot 3 && (8)\\
&= -5 \cdot 3 && (9)\\
&= -15 && (10)
\end{aligned}$$

$$\begin{aligned}
k'(1) &= g'(g(1)) \cdot g'(1) && (11)\\
&= g'(4) \cdot 9 && (12)\\
&= -1 \cdot 9 && (13)\\
&= -9 && (14)
\end{aligned}$$

$$\begin{aligned}
k'(5) &= g'(g(5)) \cdot g'(5) && (15)\\
&= g'(3) \cdot -5 && (16)\\
&= 3 \cdot -5 && (17)\\
&= -15 && (18)
\end{aligned}$$

3. The Chain Rule (All Forms)

$$\begin{aligned}
y &= \sqrt[3]{2x^2 - x - 5} && (1)\\
u &= 2x^2 - x - 5 && (2)\\
y &= \sqrt[3]{u} && (3)\\
y' &= u^{-\frac{2}{3}} \cdot 4x - 1 && (4)\\
&= \frac{1}{3}\left(2x^2 - x - 5\right)^{-\frac{2}{3}} \cdot 4x - 1 && (5)
\end{aligned}$$

$$\begin{aligned}
y &= \csc(\tan t) && (1)\\
u &= \tan t && (2)\\
y &= \csc u && (3)\\
y' &= -\csc u \cot u \cdot \sec^2 t && (4)\\
&= -\csc(\tan t)\cot(\tan t) \cdot \sec^2 t && (5)
\end{aligned}$$

4. The Chain Rule (Nested)

$$\begin{aligned}
y &= \tan(\sin e^x) && (1)\\
u_2 &= e^x && (2)\\
u_1 &= \sin u_2 && (3)\\
y &= \tan u_1 && (4)\\
y' &= \sec^2(\sin e^x) \cdot \cos e^x \cdot e^x && (5)
\end{aligned}$$

5. The Chain Rule (Combination of Rules)

$$
\begin{aligned}
y &= \left(\frac{e^x}{x+1}\right)^8 && (1)\\
y' &= 8\left(\frac{e^x}{x+1}\right)^7 \cdot \frac{xe^x}{(x+1)^2} && (2)\\
&= 8\frac{e^{7x}}{(x+1)^7} \cdot \frac{xe^x}{(x+1)^2} && (3)\\
&= \frac{8xe^{8x}}{(x+1)^9} && (4)
\end{aligned}
$$

6. Implicit Differentiation

$$
\begin{aligned}
x^4 + y^4 &= 2 && (1)\\
4x^3 + 4y^3\frac{dy}{dx} &= 0 && (2)\\
4y^3\frac{dy}{dx} &= -4x^3 && (3)\\
\frac{dy}{dx} &= \frac{-4x^3}{4y^3} && (4)\\
&= \frac{-x^3}{y^3} && (5)
\end{aligned}
$$

$$\left.\frac{dy}{dx}\right|_{(1,-1)} = \frac{-(1)^3}{(-1)^3} = \frac{-1}{-1} = 1$$

7. Implicit Differentiation (Finding y)

$$
\begin{aligned}
y &= y = xe^y && (1)\\
y' &= e^y + xe^y y' && (2)\\
y' - y'xe^y &= e^y && (3)\\
y'(1 - xe^y) &= e^y && (4)\\
y' &= \frac{e^y}{1 - xe^y} && (5)
\end{aligned}
$$

8. Implicit Differentiation (Tangent Line)

$$
\begin{aligned}
\cos(x-y) + \sin y &= \sqrt{2} && (1)\\
(-\sin(x-y))(1 - 1y') + \cos y\,(y') &= 0 && (2)\\
-\sin(x-y) + y'\sin(x-y) + y'\cos y &= 0 && (3)\\
y'(\sin(x-y) + \cos y) &= \sin(x-y) && (4)\\
y' &= \frac{\sin(x-y)}{\sin(x-y) + \cos y} && (5)\\
\left.y'\right|_{\left(\frac{\pi}{2},\frac{\pi}{4}\right)} &= \frac{\sin\left(\frac{\pi}{2} - \frac{\pi}{2}\right)}{\sin\left(\frac{\pi}{2} - \frac{\pi}{2}\right) + \cos\frac{\pi}{4}} && (6)\\
&= \frac{1}{2} && (7)\\
y &= \frac{1}{2}x && (8)
\end{aligned}
$$

9. Implicit Differentiation (Higher Order)

$$\begin{aligned}
x^4+y^4 &= 64 && (1)\\
4x^3+4y^3\frac{dy}{dx} &= 0 && (2)\\
4y^3\frac{dy}{dx} &= -4x^3 && (3)\\
\frac{dy}{dx} &= \frac{-4x^3}{4y^3} && (4)\\
&= \frac{-x^3}{y^3} && (5)\\
\frac{d^2y}{dx^2} &= \frac{-3x^2y^3-\left(-x^3 3y^2\frac{dy}{dx}\right)}{\left(y^3\right)^2} && (6)\\
&= \frac{-3x^2y^3+3x^3y^2\frac{dy}{dx}}{y^6} && (7)\\
&= \frac{-3x^2y^3+3x^3y^2\frac{-x^3}{y^3}}{y^6} && (8)\\
&= \frac{-3x^2y^3+\frac{-3x^6y^2}{y^3}}{y^6} && (9)\\
&= \frac{-3x^2y^3-\frac{3x^6}{y}}{y^6} && (10)\\
&= \frac{\frac{-3x^2y^4-3x^6}{y}}{y^6} && (11)\\
&= \frac{-3x^2y^4-3x^6}{y^7} && (12)
\end{aligned}$$

Related Exercises

1. (Section 3.7 Exercise 15)

$$\begin{aligned}
y &= (3x+7)^{10} && (1)\\
u &= 3x+7 && (2)\\
f(u) &= u^{10} && (3)\\
y' &= 10u^9\cdot 3 && (4)\\
&= 10\,(3x+7)^9\cdot 3 && (5)\\
&= 30\,(3x+7)^9 && (6)
\end{aligned}$$

2. (Section 3.7 Exercise 17)

$$\begin{aligned}
y &= \sin^5 x && (1)\\
u &= \sin x && (2)\\
f(u) &= u^5 && (3)\\
y' &= 5u^4\cdot\cos x && (4)\\
&= 5\sin^4 x\cos x && (5)
\end{aligned}$$

3. (Section 3.7 Exercise 28)

$$\begin{aligned}
y &= \left(x^2+2x+7\right)^8 && (1)\\
u &= x^2+2x+7 && (2)\\
f(u) &= u^8 && (3)\\
y' &= 8u^7\cdot(2x+2) && (4)\\
&= 8\left(x^2+2x+7\right)^7\cdot(2x+2) && (5)\\
&= (16x+16)\left(x^2+2x+7\right)^7 && (6)
\end{aligned}$$

4. (Section 3.7 Exercise 29)

$$\begin{aligned}
y &= \sqrt{10x+1} && (1)\\
u &= 10x+1 && (2)\\
f(u) &= \sqrt{u} && (3)\\
y' &= \frac{1}{2\sqrt{u}} \cdot 10 && (4)\\
&= \frac{1}{2\sqrt{10x+1}} \cdot 10 && (5)\\
&= \frac{10}{2\sqrt{10x+1}} && (6)\\
&= \frac{5}{\sqrt{10x+1}} && (7)
\end{aligned}$$

5. (Section 3.7 Exercise 41)

$$\begin{aligned}
y &= \sqrt[4]{\frac{2x}{4x-3}} && (1)\\
u &= \frac{2x}{4x-3} && (2)\\
f(u) &= \sqrt[4]{u} && (3)\\
y' &= \frac{1}{4}u^{-\frac{3}{4}} \cdot -\frac{6}{(4x-3)^2} && (4)\\
&= \frac{1}{4}\left(\frac{2x}{4x-3}\right)^{-\frac{3}{4}} \cdot -\frac{6}{(4x-3)^2} && (5)\\
&= -\frac{6}{4\,(4x-3)^2}\left(\frac{2x}{4x-3}\right)^{-\frac{3}{4}} && (6)
\end{aligned}$$

6. (Section 3.7 Exercise 23)

$$\begin{aligned}
y &= \tan 5x^2 && (1)\\
u &= 5x^2 && (2)\\
f(u) &= \tan u && (3)\\
y' &= \sec^2 u \cdot 10x && (4)\\
&= \sec^2 5x^2 \cdot 10x && (5)\\
&= 10x \sec^2 5x^2 && (6)\\
& && (7)
\end{aligned}$$

7. (Section 3.7 Exercise 24)

$$\begin{aligned}
y &= \sin\frac{x}{4} && (1)\\
u &= \frac{x}{4} && (2)\\
f(u) &= \sin u && (3)\\
y' &= \cos u \cdot \frac{4}{16} && (4)\\
&= \cos\frac{x}{4} \cdot \frac{1}{4} && (5)\\
&= \frac{1}{4}\cos\frac{x}{4} && (6)
\end{aligned}$$

8. (Section 3.7 Exercise 45)

$$
\begin{aligned}
y &= (2x^6 - 3x^3 + 3)^{25} && (1)\\
u &= 2x^6 - 3x^3 + 3 && (2)\\
f(u) &= u^{25} && (3)\\
y' &= 25\,(u)^{24} \cdot 12x^5 - 9x^2 && (4)\\
&= 25\left(2x^6 - 3x^3 + 3\right)^{24} \cdot 12x^5 - 9x^2 && (5)\\
&= 25\left(12x^5 - 9x^2\right)\left(2x^6 - 3x^3 + 3\right)^{24} && (6)
\end{aligned}
$$

9. (Section 3.7 Exercise 46)

$$
\begin{aligned}
y &= (\cos x + 2\sin x)^8 && (1)\\
u &= \cos x + 2\sin x && (2)\\
f(u) &= u^8 && (3)\\
y' &= 8u^7 \cdot (-\sin x + 2\cos x) && (4)\\
&= 8\,(\cos x + 2\sin x)^7 \cdot (-\sin x + 2\cos x) && (5)\\
&= 8(-\sin x + 2\cos x)\,(\cos x + 2\sin x)^7 && (6)\\
& && (7)
\end{aligned}
$$

10. (Section 3.7 Exercise 53)

$$
\begin{aligned}
y &= \sin\left(\sin\left(e^x\right)\right) && (1)\\
y' &= \cos\left(\sin e^x\right)\cos e^x e^x && (2)
\end{aligned}
$$

11. (Section 3.7 Exercise 54)

$$
\begin{aligned}
y &= \sin^2 e^{3x+1} && (1)\\
y' &= 6\sin e^{3x+1}\cos e^{3x+1} && (2)
\end{aligned}
$$

12. (Section 3.7 Exercise 68)

$$
\begin{aligned}
y &= \left(\frac{3x}{4x+2}\right)^5 && (1)\\
y' &= 5u^4 \cdot \frac{12x + 6 - 12x}{(4x+2)^2} && (2)\\
&= 5\left(\frac{3x}{4x+2}\right)^4 \cdot \frac{6}{(4x+2)^2} && (3)\\
&= \frac{30}{(4x+2)^2}\left(\frac{3x}{4x+2}\right)^4 && (4)
\end{aligned}
$$

13. (Section 3.7 Exercise 69)

$$
\begin{aligned}
y &= \left((x+2)\left(x^2+1\right)\right)^4 && (1)\\
y' &= 4u^3 \cdot x^2 + 1 + 2x^2 + 4x && (2)\\
&= 4\left(3x^2 + 4x + 1\right)\left((x+2)\left(x^2+1\right)\right)^3 && (3)\\
&= 4\,(3x+1)\,(x+1)\left((x+2)\left(x^2+1\right)\right)^3 && (4)
\end{aligned}
$$

A copy of my notes (in LaTeX) are available on my GitHub