Module 7 Notes (MATH-211)

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General Notes (and Definitions)

- Working with Integrals
  A function $f(x)$ is **even** if $f(-x) = f(x)$.
  A function $f(x)$ is **odd** if $f(-x) = -f(x)$.
  Let $a \in \mathbb{R}$ such that $a > 0$ and let f be an integrable function on the interval $[-a, a]$.

  $$\text{If } f \text{ is even, } \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

  $$\text{If } f \text{ is odd, } \int_{-a}^{a} f(x)\,dx = 0$$

  The average value of an integrable function f on the interval $[a, b]$ is

  $$\overline{f} = \frac{1}{b-a}\int_{a}^{b} f(x)\,dx$$

  Let f be continuous on the interval $[a, b]$. There exists a point c in (a, b) such that (Mean Value Theorem)

  $$f(c) = \overline{f} = \frac{1}{b-a}\int_{a}^{b} f(t)\,dx$$

- Substitution Rule
  Let $u = g(x)$, where g is differentiable on an interval, and let f be continuous on the corresponding range of g. On that interval,

  $$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

  1. Given an indefinite integral involving a commposite function $f(g(x))$, identify an inner function $u = g(x)$ such that a constant multiple of $g'(x)$ appears in the integrand.
  2. Substitute $u = g(x)$ and $du = g'(x)\,dx$ in the integral.
  3. Evaluate the new indefinite integral with respect to u.
  4. Write the result in terms of x using $u = g(x)$.

  Let $u = g(x)$, where g' is continuous on $[a, b]$, and let f be continuous on the range of g. Then

  $$\int_{a}^{b} f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

- Velocity and Net Change
  Position, Velocity, Displacement, and Distance:

  1. The **position** of an object moving along a line at time t, denoted $s(t)$, is the location of the object relative to the origin.
  2. The **velocity** of an object at time t is $v(t) = s'(t)$.
  3. The **displacement** of the object between $t = a$ and $t = b > a$ is

     $$s(b) - s(a) = \int_{a}^{b} v(t)\,dt$$

4. The **distance traveled** by the object between $t = a$ and $t = b > a$ is

$$\int_a^b |v(t)|\, dt$$

where $|v(t)|$ is the **speed** of the object at time t.

Theorem: Position from Velocity
Given the velocity $v(t)$ of an object moving along a line and its initial position $s(0)$, the position function of the object for future times $t \geq 0$ is

$$s(t) = s(0) + \int_0^t v(x)\, dx$$

Theorem: Velocity from Acceleration
Given the acceleration $a(t)$ of an object moving along a line and its initial velocity $v(0)$, the velocity of the object for future times $t \geq 0$ is

$$v(t) = v(0) + \int_0^t a(x)\, dx$$

Theorem: Net Change and Future Value
Suppose a quantity Q changes over time at a known rate Q'. Then the **net change** in Q between $t = a$ and $t = b > a$ is

$$Q(b) - Q(a) = \int_a^b Q'(t)\, dt$$

Given the initial value $Q(0)$, the **future value** of Q at time $t \geq 0$ is

$$Q(t) = Q(0) + \int_0^t Q'(x)\, dx$$

General formulas for indefinite integrals

$$\int \cos ax\, dx = \frac{1}{a}\sin ax + C \qquad (1)$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax + C \qquad (2)$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax + C \qquad (3)$$

$$\int \csc^2 ax\, dx = -\frac{1}{a}\cot ax + C \qquad (4)$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax + C \qquad (5)$$

$$\int \csc ax \cot ax\, dx = -\frac{1}{a}\csc ax + C \qquad (6)$$

$$\int e^{ax}\, dx = \frac{1}{a}e^{ax} + C \qquad (7)$$

$$\int b^x\, dx = \frac{1}{\ln b}b^x + C, b > 0, b \neq 1 \qquad (8)$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C \qquad (9)$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C, a > 0 \qquad (10)$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C, a > 0 \qquad (11)$$

Examples

1. Use symmetry to evaluate integrals

$$\int_{-200}^{200} 2x^5\, dx = 0$$

$$
\begin{aligned}
\int_{-2}^{2} \left(x^2 + x^3\right)\, dx &= \int_{-2}^{2} x^2\, dx + \int_{-2}^{2} x^3\, dx && (1)\\
&= 2\int_{0}^{2} x^2\, dx + 0 && (2)\\
&= 2\frac{x^3}{3} && (3)\\
&= \frac{16}{3} && (4)
\end{aligned}
$$

2. A derivative calculation

$$s(t) = -16t^2 + 64t$$

$$t = 4$$

$$[0, 4]$$

$$
\begin{aligned}
v(t) &= s'(t) && (1)\\
\overline{v} &= \frac{1}{4}\int_{0}^{4} v(t)\, dx && (2)\\
&= \frac{1}{4}\int_{0}^{4} s'(t)\, dx && (3)\\
&= \frac{1}{4}s(t) && (4)\\
&= \frac{1}{4}\left(s(4) - s(0)\right) && (5)\\
&= 0 && (6)
\end{aligned}
$$

3. Applying MVT for integrals

$$f(x) = e^x$$

$$[0, 2]$$

$$
\begin{aligned}
\overline{f} &= \frac{1}{2}\left(\int_{0}^{2} e^x\, dx\right) && (1)\\
&= \frac{e^x}{2} && (2)\\
&= \frac{e^2}{2} - \frac{e^0}{2} && (3)\\
&= \frac{e^2 - 1}{2} && (4)\\
e^x &= \frac{e^2 - 1}{2} && (5)\\
\ln e^x &= \ln\frac{e^2 - 1}{2} && (6)
\end{aligned}
$$

4. Perfect substitutions in indefinite integrals

$$
\begin{aligned}
u &= 4x^3 - 8 && (1)\\
du &= 12x^2\, dx && (2)\\
\int 12x^2 \left(4x^3 - 8\right)^5\, dx &= \int 12x^2 u^5\, dx && (3)\\
&= \frac{u^6}{6} + C && (4)\\
&= \frac{\left(4x^3 - 8\right)^6}{6} + C && (5)
\end{aligned}
$$

$$\begin{aligned} u &= \sin t && (1)\\ du &= \cos t\, dt && (2)\\ \int (\cos t)\, e^{\sin t}\, dt &= \int e^u\, du && (3)\\ &= e^u + C && (4)\\ &= e^{\sin t} + C && (5) \end{aligned}$$

5. Introducting constants when integrating by substitution

$$\begin{aligned} u &= 6x+4 && (1)\\ du &= 6\, dx && (2)\\ dx &= \frac{du}{6} && (3)\\ \int (6x+4)^9\; dx &= \int \frac{1}{6}\cdot u^9\, du && (4)\\ &= \frac{1}{6}\int u^8\, du && (5)\\ &= \frac{1}{6}\cdot\frac{u^9}{9} + C && (6)\\ &= \frac{(6x+4)^9}{54} + C && (7) \end{aligned}$$

$$\begin{aligned} u &= \cot x && (1)\\ du &= -\csc^2 x\, dx && (2)\\ \int \cot^2 x \csc^2 x\, dx &= \int -u^2\, du && (3)\\ &= -\frac{u^3}{3} + C && (4)\\ &= -\frac{\csc^3 x}{3} + C && (5) \end{aligned}$$

6. Variations on the substitution method

$$\begin{aligned} u &= x-1 && (1)\\ du &= dx && (2)\\ x &= u+1 && (3)\\ \int x\sqrt{x-1}\, dx &= \int (u+1)\sqrt{u}\, du && (4)\\ &= \int u\sqrt{u} + \sqrt{u}\, du && (5)\\ &= \int u^{\frac{3}{2}} + u^{\frac{1}{2}}\; du && (6)\\ &= \frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}} + C && (7)\\ &= \frac{2}{5}(x-1)^{\frac{5}{2}} + \frac{2}{3}(x-1)^{\frac{3}{2}} + C && (8) \end{aligned}$$

7. Use known formulas to evaluate indefinite integrals

$$\begin{aligned} \int 2e^{-4x}\, dx &= 2\int e^{-4x}\, dx && (1)\\ &= \frac{2}{-4}e^{-4x} + C && (2)\\ &= -\frac{1}{2}e^{-4x} + C && (3)\\ & && (4) \end{aligned}$$

$$\begin{aligned}
\int \frac{dx}{\sqrt{36-x^2}} &= \int \frac{dx}{\sqrt{6^2-x^2}} && (1)\\
&= \sin^{-1}\frac{x}{6} + C && (2)
\end{aligned}$$

8. Evaluating definite integrals using substitution

$$\begin{aligned}
u &= 2^x + 4 && (1)\\
du &= 2^x \ln 2\, dx && (2)\\
\frac{1}{\ln 2}\, du &= 2^x\, dx && (3)\\
\int_1^3 \frac{2^x}{2^x+4}\, dx &= \int_1^3 \frac{1}{u \ln 2}\, du && (4)\\
\int_{g(1)}^{g(3)} \frac{1}{u\ln 2}\, du &= \int_6^{12} \frac{1}{u\ln 2}\, du && (5)\\
&= \frac{1}{\ln 2}\int_6^{12} \frac{du}{u} && (6)\\
&= \frac{1}{\ln 2}\cdot(\ln 12 - \ln 6) && (7)\\
&= \frac{\ln 2}{\ln 2} && (8)\\
&= 1 && (9)
\end{aligned}$$

$$\begin{aligned}
u &= \ln p && (1)\\
du &= \frac{1}{p}\, dx && (2)\\
\int_1^{e^2} \frac{\ln p}{p} &= \int_0^2 u\, du && (3)\\
&= \frac{2^2}{2} - \frac{0^2}{2} && (4)\\
&= \frac{4}{2} && (5)\\
&= 2 && (6)
\end{aligned}$$

9. Integrals involving $\cos^2 x$ and $\sin^2 x$

$$\begin{aligned}
u &= 2x && (1)\\
du &= 2\, dx && (2)\\
dx &= \frac{1}{2} du && (3)\\
\sin^2 x &= \frac{1-\cos 2x}{2} && (4)\\
\int_0^\pi \sin^2 x\, dx &= \int_0^\pi \frac{1-\cos 2x}{2}\, dx && (5)\\
&= \frac{1}{2}\int_0^\pi 1 - \cos 2x\, dx && (6)\\
&= \frac{1}{2}\left(\int_0^\pi 1\, dx - \int_0^\pi \cos 2x\, dx\right) && (7)\\
&= \frac{1}{2}\left((\pi - 0) - \frac{1}{2}\int_0^{2\pi} \cos u\, du\right) && (8)\\
&= \frac{1}{2}\left(\pi - \frac{1}{2}(\sin 2\pi - \sin 0)\right) && (9)\\
&= \frac{1}{2}(\pi - 0) && (10)\\
&= \frac{\pi}{2} && (11)
\end{aligned}$$

10. Displacement and distance from velocity

$$v(t) = 4t^3 - 24t^2 + 20t$$

(a)

$$\begin{aligned}
v(t) &= 0 && (1)\\
4t^3 - 24t^2 + 20t &= 0 && (2)\\
4t(t^2 - 6t + 5) &= 0 && (3)\\
4t(t-1)(t-5) &= 5 && (4)\\
t &= 0 && (5)\\
t &= 1 && (6)\\
t &= 5 && (7)\\
0 < t < 1 &= \text{Positive} && (8)\\
1 < t < 5 &= \text{Negative} && (9)\\
t > 5 &= \text{Positive} && (10)
\end{aligned}$$

(b)

$$\begin{aligned}
\int_0^5 4t^3 - 24t^2 + 20t\,dt &= 4\int_0^5 t^3\,dt - 24\int_0^5 t^2\,dt + 20\int_0^5 t\,dt && (1)\\
&= 4\left(\frac{5^4}{4} - \frac{0^4}{4}\right) - 24\left(\frac{5^3}{3} - \frac{0^3}{3}\right) + 20\left(\frac{5^2}{2} - \frac{0^2}{2}\right) && (2)\\
&= 4\left(\frac{625}{4}\right) - 24\left(\frac{125}{3}\right) + 20\left(\frac{25}{2}\right) && (3)\\
&= -125 && (4)
\end{aligned}$$

(c)

$$\begin{aligned}
\int_0^5 |4t^3 - 24t^2 + 20t|\,dt &= \int_0^1 4t^3 - 24t^2 + 20t\,dt + \int_1^5 -4t^3 + 24t^2 - 20t\,dt && (1)\\
&= 3 + 128 && (2)\\
&= 131 && (3)
\end{aligned}$$

11. Position and velocity from acceleration

$$a(t) = \frac{20}{(t+2)^2}$$

$$v(0) = 20$$

$$s(0) = 10$$

$$\begin{aligned}
v(t) &= v(0) + \int_0^t a(t)\,d5 && (1)\\
&= 20 + \int_0^t \frac{20}{(t+2)^2}\,d5 && (2)\\
&= 20 - \frac{20}{t+2} + 10 && (3)\\
&= 30 - \frac{20}{t+2} && (4)\\
s(t) &= s(0) + \int_0^t v(t)\,dt && (5)\\
&= 10 + \int_0^t \left(30 - \frac{20}{t+2}\right)\,dt && (6)\\
&= 10 + 30t - 20\ln|t+2| + 20\ln 2 && (7)
\end{aligned}$$

12. Acceleration application

$$a(t) = -15$$
$$v(0) = 60$$
$$s(0) = 0$$

$$\begin{aligned}
v(t) &= 60 + \int_0^t -15\,dt & (1)\\
&= 60 + -15\int_0^t t^0\,dt & (2)\\
&= -15t + 60 & (3)\\
s(t) &= 0 + \int_0^t -15t + 60\,dt & (4)\\
&= -15\int_0^t t\,dt + 60t & (5)\\
&= -\frac{15}{2}t^2 + 60t & (6)\\
v(t) &= 0 & (7)\\
60 - 15t &= 0 & (8)\\
15t &= 60 & (9)\\
t &= \frac{60}{15} & (10)\\
&= 4 & (11)\\
s(4) - s(0) &= 60(4) - \frac{15}{2}(4)^2 - 0 & (12)\\
&= 120 & (13)
\end{aligned}$$

13. Application of net change

$$V'(t) = 70(1 + \sin 2\pi t)$$
$$[0, t]$$
$$V(0) = 0$$

$$\begin{aligned}
V(t) &= V(0) + \int_0^t V'(x)\,dx & (1)\\
&= 0 + \int_0^t 70(1 + \sin(2\pi x)\,dx & (2)\\
&= 70\left(t - \frac{\cos 2\pi t}{2\pi} + \frac{1}{2\pi}\right) & (3)\\
V(60) &= 70\left(60 - \frac{\cos 120\pi}{2\pi} + \frac{1}{2\pi}\right) & (4)\\
&= 70 \cdot 60 & (5)\\
&= 4200 & (6)
\end{aligned}$$

Related Exercises

1. (Section 5.4, Exercise 15)

$$\begin{aligned}
\int_{-2}^{2} \left(x^2 + x^3\right)\, dx &= \int_{-2}^{2} x^2\, dx + \int_{-2}^{2} x^3\, dx && (1)\\
&= 2\int_{0}^{2} x^2\, dx + 0 && (2)\\
&= 2\frac{x^3}{3} && (3)\\
&= 2\frac{2^3}{3} - 2\frac{0^3}{3} && (4)\\
&= 2\frac{8}{3} && (5)\\
&= \frac{16}{3} && (6)
\end{aligned}$$

2. (Section 5.4, Exercise 16)

$$\int_{-\pi}^{\pi} t^2 \sin t\, dx = 0$$

3. (Section 5.4, Exercise 26)

$$f(x) = x^2 + 1$$

$$[-2, 2]$$

$$\begin{aligned}
\overline{f} &= \frac{1}{2-(-2)} \int_{-2}^{2} x^2 + 1\, dx && (1)\\
&= \frac{1}{4}\left(\int_{-2}^{2} x^2\, dx + 1\int_{-2}^{2} x^0\, dx\right) && (2)\\
&= \frac{1}{4}\left(\frac{x^3}{3} + x\right) && (3)\\
&= \frac{1}{4}\left(\int_{-2}^{2} x^2\, dx + \int_{-2}^{2} 1\, dx\right) && (4)\\
&= \frac{1}{4}\left(\frac{2^3}{3} - \frac{(-2)^3}{3} + 2 - (-2)\right) && (5)\\
&= \frac{1}{4}\left(\frac{8}{3} - \frac{-8}{3} + 4\right) && (6)\\
&= \frac{1}{4}\left(\frac{16}{3} + 4\right) && (7)\\
&= \frac{1}{4}\left(\frac{28}{3}\right) && (8)\\
&= \frac{7}{3} && (9)
\end{aligned}$$

4. (Section 5.4, Exercise 34)

$$f(x) = x^3 - 5x^2 + 30$$

$$[0, 4]$$

$$
\begin{aligned}
\overline{f} &= \frac{1}{4}\left(\int_0^4 \left(x^3-5x^2+30\right)\,dx\right) && (1)\\
&= \frac{1}{4}\left(\int_0^4 x^3 - 5\int_0^4 x^2 + 30\int_0^4 x^0\right) && (2)\\
&= \frac{1}{4}\left(\frac{x^4}{4} - 5\frac{x^3}{3} + 30x\right) && (3)\\
&= \frac{1}{4}\left(\left(\frac{4^4}{4}-\frac{0^4}{4}\right) - \left(5\frac{4^3}{3}-5\frac{0^3}{3}\right) + (30(4)-30(0))\right) && (4)\\
&= \frac{1}{4}\left(64 - \frac{320}{3} + 120\right) && (5)\\
&= \frac{1}{4}\left(\frac{232}{3}\right) && (6)\\
&= \frac{58}{3} && (7)
\end{aligned}
$$

5. (Section 5.4, Exercise 41)

$$
f(x) = 1 - \frac{x^2}{a^2}
$$

$$
[0, a]
$$

$$
\begin{aligned}
\overline{f} &= \frac{1}{a}\left(\int_0^a 1 - \frac{x^2}{a^2}\,dx\right) && (1)\\
&= \frac{1}{a}\left(\int_0^a 1\,dx - \int_0^a \frac{x^2}{a^2}\,dx\right) && (2)\\
&= \frac{1}{a}\left(x - \frac{1}{a^2}\int_0^a x^2\,dx\right) && (3)\\
&= \frac{1}{a}\left(x - \frac{1}{a^2}\frac{x^3}{3}\right) && (4)\\
&= \frac{1}{a}\left(x - \frac{x^3}{3a^2}\right) && (5)\\
&= \frac{1}{a}\left((a-0) - \frac{1}{a^2}\left(\frac{a^3}{3}-\frac{0^3}{3}\right)\right) && (6)\\
&= \frac{1}{a}\left(a - \frac{a^3}{3a^2}\right) && (7)\\
&= \frac{1}{a}\left(a - \frac{a}{3}\right) && (8)\\
&= \frac{1}{a}\left(\frac{2a}{3}\right) && (9)\\
&= \frac{2}{3} && (10)\\
1 - \frac{c^2}{a^2} &= \frac{2}{3} && (11)\\
\frac{c^2}{a^2} &= \frac{1}{3} && (12)\\
c^2 &= \frac{a^2}{3} && (13)\\
c &= \sqrt{\frac{a^2}{3}} && (14)\\
&= \frac{a}{\sqrt{3}} && (15)
\end{aligned}
$$

6. (Section 5.4, Exercise 42)

$$
f(x) = \frac{\pi}{4}\sin x
$$

$$
[0, \pi]
$$

$$\begin{aligned}
\overline{f} &= \frac{1}{\pi}\int_0^{\pi}\frac{\pi}{4}\sin x & (1)\\
&= \frac{1}{\pi}\frac{\pi}{4}\int_0^{\pi}\sin x & (2)\\
&= \frac{1}{\pi}\frac{\pi}{4}(-\cos x) & (3)\\
&= \frac{1}{\pi}\frac{\pi}{4}(-\cos\pi+\cos 0) & (4)\\
&= \frac{1}{\pi}\frac{\pi}{4}(1+1) & (5)\\
&= \frac{1}{\pi}\frac{\pi}{2} & (6)\\
&= \frac{1}{2} & (7)\\
\frac{\pi}{4}\sin x &= \frac{1}{2} & (8)\\
\sin x &= \frac{2}{\pi} & (9)\\
\sin^{-1}\sin x &= \sin^{-1}\frac{2}{\pi} & (10)\\
x &= \sin^{-1}\frac{2}{\pi} & (11)
\end{aligned}$$

7. (Section 5.5, Exercise 17)

$$\begin{aligned}
u &= x^2-1 & (1)\\
du &= 2x\,dx & (2)\\
\int 2x\left(x^2-1\right)^{99}dx &= \int u^{99}\,du & (3)\\
&= \frac{u^{100}}{100}+C & (4)\\
&= \frac{\left(x^2-1\right)^{100}}{100}+C & (5)
\end{aligned}$$

8. (Section 5.5, Exercise 20)

$$\begin{aligned}
u &= \sqrt{x}+1 & (1)\\
du &= \frac{1}{2\sqrt{x}}\,dx & (2)\\
\int\frac{\left(\sqrt{x}+1\right)^4}{2\sqrt{x}}\,dx &= \int u^4\,du & (3)\\
&= \frac{u^5}{5}+C & (4)\\
&= \frac{\left(\sqrt{x}+1\right)^5}{5}+C & (5)
\end{aligned}$$

9. (Section 5.5, Exercise 21)

$$\begin{aligned}
u &= x^2+x & (1)\\
du &= 2x+1\,dx & (2)\\
\int\left(x^2+x\right)^{10}(2x+1)\,dx &= \int u^{10}\,du & (3)\\
&= \frac{u^{11}}{11}+C & (4)\\
&= \frac{\left(x^2+x\right)^{11}}{11}+C & (5)
\end{aligned}$$

10. (Section 5.5, Exercise 23)

$$\begin{aligned}
u &= x^4 + 16 && (1)\\
du &= 4x^3\,dx && (2)\\
x^3\,dx &= \frac{1}{4}du && (3)\\
\int x^3\left(x^4+16\right)^6\,dx &= \int \frac{1}{4}u^6\,du && (4)\\
&= \frac{1}{4}\int u^6\,du && (5)\\
&= \frac{1}{4}\cdot\frac{u^7}{7} + C && (6)\\
&= \frac{\left(x^4+16\right)^7}{28} + C && (7)
\end{aligned}$$

11. (Section 5.5, Exercise 24)

$$\begin{aligned}
u &= \sin\theta && (1)\\
du &= \cos\theta\,d\theta && (2)\\
\int \sin^{10}\theta\cos\theta\,d\theta &= \int u^{10}\,du && (3)\\
&= \frac{u^{11}}{11} + C && (4)\\
&= \frac{\sin^{11}\theta}{11} + C && (5)
\end{aligned}$$

12. (Section 5.5, Exercise 78)

$$\begin{aligned}
u &= x - 2 && (1)\\
x &= u + 2 && (2)\\
du &= dx && (3)\\
\int \frac{x}{x-2}\,dx &= \int \frac{u+2}{u}\,du && (4)\\
&= \int \frac{u}{u}\,du + \int \frac{2}{u}\,du && (5)\\
&= u + 2\int \frac{1}{u}\,du + C && (6)\\
&= u + 2\ln|u| + C && (7)\\
&= x - 2 + 2\ln|u| + C && (8)
\end{aligned}$$

13. (Section 5.5, Exercise 79)

$$\begin{aligned}
u &= \sqrt{x-4} && (1)\\
u^2 &= x - 4 && (2)\\
x &= u^2 + 4 && (3)\\
dx &= 2u\,du && (4)\\
\int \frac{x}{\sqrt{x-4}}\,dx &= \int 2u\frac{u^2+4}{u}\,du && (5)\\
&= \int \frac{2u^3+8u}{u}\,du && (6)\\
&= 2\left(\int u^2\,du + \int 4\,du\right) && (7)\\
&= 2\left(\frac{\sqrt{x-4}^3}{3} + 4\sqrt{x-4}\right) + C && (8)\\
&= \frac{2}{3}\sqrt{x-4} + 8\sqrt{x-4} + C && (9)
\end{aligned}$$

14. (Section 5.5, Exercise 15)

$$\begin{aligned}
\int e^{10x}\,dx &= \frac{1}{10}e^{10x}+C && (1)\\
\int \sec 5x \tan 5x\,dx &= \frac{1}{5}\sec 5x + C && (2)\\
\int \sin 7x\,dx &= -\frac{1}{7}\cos 7x + C && (3)\\
\int \cos\frac{x}{7}\,dx &= 7\sin\frac{x}{7}+C && (4)\\
\int \frac{dx}{81+9x^2} &= \int \frac{dx}{9^2+9x^2} && (5)\\
&= \int \frac{dx}{9(9+x^2)} && (6)\\
&= \frac{1}{9}\int \frac{dx}{3^2+x^2} && (7)\\
&= \frac{1}{27}\tan^{-1}\frac{x}{3}+C && (8)\\
\int \frac{dx}{\sqrt{36-x^2}} &= \int \frac{dx}{\sqrt{6^2-x^2}} && (9)\\
&= \sin^{-1}\frac{x}{6}+C && (10)
\end{aligned}$$

15. (Section 5.5, Exercise 16)

$$\begin{aligned}
\int_0^1 10^x\,dx &= \frac{1}{\ln 10}10-\frac{1}{\ln 10} && (1)\\
&= \frac{9}{\ln 10} && (2)\\
\int_0^{\frac{\pi}{40}} \cos 20x\,dx &= \cos\frac{20\pi}{40}-\cos 20(0) && (3)\\
&= 0-1 && (4)\\
&= -1 && (5)\\
\int_{3\sqrt{2}}^{6} \frac{dx}{x\sqrt{x^2-9}} &= \int_{3\sqrt{2}}^{6} \frac{dx}{x\sqrt{x^2-3^2}} && (6)\\
&= \frac{1}{3}\sec^{-1}\left|\frac{6}{3}\right| - \frac{1}{3}\sec^{-1}\left|\frac{3\sqrt{2}}{3}\right| && (7)\\
&= \frac{1}{3}\sec^{-1}\frac{\pi}{3}-\frac{1}{3}\sec^{-1}\frac{\pi}{4} && (8)\\
\int_0^{\frac{\pi}{16}} \sec^2 4x\,dx &= \frac{1}{4}\tan\frac{4\pi}{16}-\frac{1}{4}\tan 0 && (9)\\
&= \frac{1}{4}-0 && (10)\\
&= \frac{1}{4} && (11)
\end{aligned}$$

16. (Section 5.5, Exercise 49)

$$\begin{aligned}
u &= 2^x + 4 && (1)\\
du &= 2^x \ln 2 \, dx && (2)\\
2^x \, dx &= \frac{1}{\ln 2} \, du && (3)\\
\int_1^3 \frac{2^x}{2^x + 4} \, dx &= \frac{1}{\ln 2} \int_6^{12} \frac{1}{u} \, du && (4)\\
&= \frac{1}{\ln 2} (\ln 12 - \ln 6) && (5)\\
&= \frac{1}{\ln 2} \cdot \ln 2 && (6)\\
&= 1 && (7)
\end{aligned}$$

17. (Section 5.5, Exercise 51)

$$\begin{aligned}
u &= \sin\theta && (1)\\
du &= \cos\theta \, d\theta && (2)\\
\int_0^{\frac{\pi}{2}} \sin^2\theta \cos\theta \, d\theta &= \int_0^{\frac{\pi}{2}} u^2 \, du && (3)\\
&= \int_0^1 u^2 \, du && (4)\\
&= \frac{1^3}{3} - \frac{0^3}{3} && (5)\\
&= \frac{1}{3} && (6)
\end{aligned}$$

18. (Section 5.5, Exercise 64)

$$\begin{aligned}
u &= 3 + 2e^x && (1)\\
du &= 2e^x \, dx && (2)\\
e^x \, dx &= \frac{1}{2} \, du && (3)\\
\int_0^{\ln 4} \frac{e^x}{3 + 2e^x} \, dx &= \frac{1}{2} \int_{u=5}^{u=11} \frac{1}{u} \, du && (4)\\
&= \frac{\ln 11 - \ln 5}{2} && (5)
\end{aligned}$$

19. (Section 5.5, Exercise 87)

$$\begin{aligned}
u &= 2x && (1)\\
du &= 2\,dx && (2)\\
dx &= \frac{1}{2}\,du && (3)\\
\cos^2 x &= \frac{1+\cos 2x}{2} && (4)\\
\int_{-\pi}^{\pi} \cos^2 x\,dx &= \int_{-\pi}^{\pi} \frac{1}{2} + \frac{\cos 2x}{2}\,dx && (5)\\
&= \frac{1}{2}\int_{-\pi}^{\pi} 1 + \cos 2x\,dx && (6)\\
&= \frac{1}{2}\left(\int_{-\pi}^{\pi} 1\,dx + \frac{1}{2}\int_{-\pi}^{\pi} \cos u\,du\right) && (7)\\
&= \frac{1}{2}\left((\pi - (-\pi)) + \frac{1}{2}\int_{u=-2\pi}^{u=2\pi} \cos u\,du\right) && (8)\\
&= \frac{1}{2}\left(2\pi + \frac{1}{2}\left(\sin 2\pi - \sin\left(-2\pi\right)\right)\right) && (9)\\
&= \frac{1}{2}\left(2\pi + \frac{1}{2}\left(0\right)\right) && (10)\\
&= \frac{2\pi}{2} && (11)\\
&= \pi && (12)
\end{aligned}$$

20. (Section 5.5, Exercise 91)

$$\begin{aligned}
u &= 4\theta && (1)\\
du &= 4\,d\theta && (2)\\
d\theta &= \frac{1}{4}\,du && (3)\\
\sin^2 x &= \frac{1-\cos 2x}{2} && (4)\\
\sin^2 2\theta &= \frac{1-\cos 4\theta}{2} && (5)\\
\int_{-\pi}^{\pi} \sin^2 2\theta\,d\theta &= \int_{-\pi}^{\pi} \frac{1-\cos 4\theta}{2}\,d\theta && (6)\\
&= \int_{-\pi}^{\pi} \frac{1}{2} - \frac{\cos u}{2}\,d\theta && (7)\\
&= \frac{1}{2}\int_{-\pi}^{\pi} 1 - \cos u\,d\theta && (8)\\
&= \frac{1}{2}\left(\int_{-\pi}^{\pi} 1\,d\theta - \int_{-\pi}^{\pi} \cos u\,d\theta\right) && (9)\\
&= \frac{1}{2}\left(2\pi - \frac{1}{4}\int_{u=-4\pi}^{u=4\pi} \cos u\,du\right) && (10)\\
&= \frac{1}{2}\left(2\pi - \frac{1}{4}\left(\sin 4\pi - \sin\left(-4\pi\right)\right)\right) && (11)\\
&= \frac{2\pi}{2} && (12)\\
&= \pi && (13)
\end{aligned}$$

A copy of my notes (in LaTeX) are available on my GitHub