Midterm Review (MATH-211)

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Limits (M01)

- Limit Definition(s):
  - Simple: The value that the outputs of a function approach as inputs approach a certain value
  - Preliminary: Suppose a function f is defined for all x near a except possibly at a. If $f(x)$ is arbitrarily close to L all x sufficiently close (but not equal) to a, we write the following.

$$\lim_{x \to a} = L$$

- Secant Line: a line passing through two points $(t_0, s(t_0))$ and $(t_1, s(t_1))$. The slope is given by

$$\frac{s(t_1) - s(t_0)}{t_1 - t_0}$$

- Tangent Line: the line passing through $(t_0, s(t_0))$ with slope

$$\lim_{t \to t_0} \frac{s(t) - s(t_0)}{t - t_0}$$

- One Sided limits:
  - Right-hand (Definition): Suppose a function f is defined for all x near a with $x > a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x > a$ we write

$$\lim_{x \to a^+} f(x) = L$$

  - Left-hand (Definition): Suppose a function f is defined for all x near a with $x < a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x < a$ we write

$$\lim_{x \to a^-} f(x) = L$$

  - In order for their to be a double sided limit, we must have:

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

  - If the limits from sides are not equal, then a the double sided limit, "does not exist"
- Velocity
  - Average Veolcity
    * The average velocity over some interval $[t_0, t_1]$ is defined as

$$v_{av} = \frac{s(t_1) - s(t_0)}{t_1 - t_0}$$

  - Instantaneous Veolcity
    * The average velocity over some interval $[t_0, t_1]$ is defined as

$$v_{inst} = \lim_{t \to a} v_{av} == \frac{s(t) - s(a)}{t - a}$$

- Solving Techniques
  - Factoring and canceling out
  - Using conjugates
    * When direct substitution is not possible, you may rationalize the numerator
- Infinite Limits: In either case, the limit does not exist (not a real number) if it is infinite
  - Suppose f is defined for all x near a. If $f(x)$ gorws arbitrarily large for all x sufficiently close (but not equal) to a, we write
    $$\lim_{x \to a} f(x) = \infty$$
  - If $f(x)$ is negative and gorws arbitrarily large in magnitude for all x sufficiently close (but not equal) to a, we write
    $$\lim_{x \to a} f(x) = -\infty$$
  - The line $x = a$ is a vertical asymptote for f if any of the following hold
    $$\lim_{x \to a} f(x) = \pm\infty$$
    $$\lim_{x \to a^+} f(x) = \pm\infty$$
    $$\lim_{x \to a^-} f(x) = \pm\infty$$
  - A vertical asymptote exists at $x = a$ if any one sided limit as $x \to a$ is ∞ or $-\infty$
  - If you have a limit of a rational function, where $p(a) = L \neq 0$ and $q(a) = 0$, then the one sided limits for $\frac{p(x)}{q(x)}$ approach $\pm\infty$
    $$\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{L}{0}$$
- Limits as Infinity
  - **Definition**: If $f(x)$ becoomes arbitrarily close to a finite number L for all sufficiently large and positive x, the we write
    $$\lim_{x \to \infty} f(x) = L$$
    The definition for
    $$\lim_{x \to -\infty} f(x) = M$$
    is analogous.
  - If $\lim_{x \to \infty} f(x) = L$ we say that the function $f(x)$ has a horizontal asymptote at $y = L$
  - If $\lim_{x \to -\infty} f(x) = M$ we say that the function $f(x)$ has a horizontal asymptote at $y = M$
  - **Principle**: If $n > 0$ is an integer then
    $$\lim_{x \to \pm\infty} \frac{1}{x^n} = 0$$
  - Suppose $f(x) = \frac{p(x)}{q(x)}$ is a rational function where
    $$p(x) = a_m x^m + a_{m-1} x^{x-1} + ... + a_1 x + a_0$$
    $$q(x) = b_n x^n + b_{n-1} x^{x-1} + ... + b_1 x + b_0$$
    If the degree of $p(x)$ is less than the degree of $q(x)$ then
    $$\lim_{x \to \pm\infty} f(x) = 0$$
    If the degree of $p(x)$ equals the degree of $q(x)$ then
    $$\lim_{x \to \pm\infty} f(x) = \frac{a_m}{b_n}$$
    If the degree of $p(x)$ is greater than the degree of $q(x)$ then
    $$\lim_{x \to \pm\infty} f(x) = -\infty \text{ or } \infty$$
    If the graph of a function f approaches a line (with finite and nonzero slope) as $x \to \pm\infty$, then that line is a slant asymptote/oblique asymptote of f

- End behaviour for transcendental functions

$$\lim_{x \to \pm\infty} \sin x = \text{Does not exist}$$

$$\lim_{x \to \infty} e^x = \infty$$

$$\lim_{x \to \infty} e^{-x} = 0$$

$$\lim_{x \to -\infty} e^x = 0$$

$$\lim_{x \to -\infty} e^{-x} = \infty$$

$$\lim_{x \to \infty} \ln x = \infty$$

$$\lim_{x \to 0^+} \ln x = -\infty$$

- Continuity

  - **Definition**: A function f is continuous at a if

  $$\lim_{x \to a} f(x) = f(a)$$

  - A function f is continuous at a if
    1. $f(a)$ is defined (Removable Discontinuity)
    2. $\lim_{x \to a} f(x)$ exists (Jump Discontinuity)
    3. $\lim x \to af(x) = f(a)$ (Removable Discontinuity)
  - A function f has an **Infinite Discontinuity** at a if the function has a Vertical Asymptote at a

  - Suppose f is a function defined on an interval I. We say that f is continuous on interval I if f is continuous at every point on the interior of I and the following hold:

    1. If a is the the left-hand endpoint of I and a is contained in I then

    $$\lim_{x \to a^+} f(x) = f(a) \text{ (} f \text{ is continuous from the right)}$$

    2. If b is the the righ-hand endpoint of I and b is contained in I then

    $$\lim_{x \to b^-} f(x) = f(b) \text{ (} f \text{ is continuous from the left)}$$

  - **Theorem**: All of the following functions are continuous on the intervals where they are defined.
    1. Polynomials (continuous everywhere)
    2. Rational Functions (continuous except where denominator is zero)
    3. Exponential functions
    4. Logarithmic functions
    5. Trigonometric functions
    6. Inverse trigonometric functions
  - **Theorem**: If f and g are continuous at a, then the following functions are also continuous at a. Assume c is a constant and $n > 0$ is an integer.
    1. $f + g$
    2. $f - g$
    3. cf
    4. fg
    5. $\frac{f}{g}$ provided $g(a) \neq 0$
    6. $(f(x))^n$
  - **Theorem**:
    1. A polynomial function is continuous for all x

2. A rational function (a function of the form $\frac{p}{q}$, where p and q are polynomials) is continuous for all x for which $q(x) \neq 0$

– **Theorem**: If g is continuous at a and f is continuous at $g(a)$ then the composite function $f \circ g$ is continuous at a.

– **Theorem**: Assume n is a positive integer. If n is odd then $(f(x))^{1/n}$ is continuous at all points at which f is continuous. If n is even then $(f(x))^{1/n}$ is continuous at all points a at which f is continuous and $f(a) > 0$

– **Intermediate Value Theorem**: Suppose f is continuous on the interval $[a, b]$ and L is a number strictly between $f(a)$ and $f(b)$. Then there exists at least one number c in (a, b) satisfying $f(c) = L$.

Derivatives (M02)

- Derivatives
  - A **derivative** is a new function made up of the slopes of the tangent lines as they change along a curve
  - If a curve represents the trajectory of a moving object, the tangent line at a point indicates the direction of motion at that point
  - As $x \to a$, the slope of the secant lines approaches the slope of the tangent line
  - Alternative definition for Tangent Line(s): Consider the curve $y = f(x)$ and a secant line intersecting the curve at points $P(a, f(a))$ and $Q(a + h, f(a + h))$, with m_{sec} and m_{tan}

$$\text{Interval: } (a, a + h)$$
$$m_{\text{sec}} = \frac{f(a + h) - f(a)}{h}$$
$$m_{\text{tan}} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$
$$y - f(a) = m_{\text{tan}}(x - a)$$

  - **Definition**: The derivative of f at a, denoted $f'(a)$, is given by either the two following limits, provided the limits exist and a is in the domain of f

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{1}$$
$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \tag{2}$$

    If $f'(a)$ exists, we say that f is **differentiable** at a

- Derivatives as Functions
  - The slope of the tangent line of somefunction f is a function called the derivative of f

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

  - If $f'(x)$ exists, we say that f is **differentiable** at x
  - If f is differentiable at every point in some open interval I, we say that f is differentiable on I
  - For some function f we can denote the derivative of f like such:

$$f'(x) \tag{1}$$
$$\frac{dy}{dx} \tag{2}$$
$$\frac{df}{dx} \tag{3}$$
$$\frac{d}{dx}(f(x)) \tag{4}$$
$$D_x(f(x)) \tag{5}$$
$$y'(x) \tag{6}$$

- When evaluating some derivative f at a, we can use the following:

$$f'(a) \tag{1}$$
$$y'(a) \tag{2}$$
$$\left.\frac{df}{dx}\right|_{x=a} \tag{3}$$
$$\left.\frac{dy}{dx}\right|_{x=a} \tag{4}$$

- If f is differentiable at a, then f is continuous at a
- If f is not continuous at a, then f is not differentiable at a

- Derivatives as Rate of Change
  - Secant Lines give average velocities

$$v_{av} = \frac{f(a+\Delta t) - f(a)}{\Delta t}$$

  - Tangent Line gives instantaneous velocity

$$v(a) = \lim_{\Delta t \to 0} \frac{f(a+\Delta t) - f(a)}{\Delta t} = f'(a)$$

  - Velocity, speed, and acceleration
    Suppose and object moves along a line with position $s = f(t)$

$$\text{the } \textbf{velocity} \text{ at time } t \text{ is } v = \frac{ds}{dt} = f'(t)$$

$$\text{the } \textbf{speed} \text{ at time } t \text{ is } |v| = |f'(t)|$$

$$\text{the } \textbf{acceleration} \text{ at time } t \text{ is } a = \frac{dv}{dt} = \frac{d^2 s}{dt^2} = f''(t)$$

  - Average and marginal cost

    The **cost function** $C(x)$ gives the cost to produce the first x items in a manufacturing process

$$\text{The } \textbf{average cost} \text{ to produce } x \text{ items is } \overline{C}(x) = \frac{C(x)}{x}$$

    The **marginal cost** $C'(x)$ is the approximate cost to produce one additional item after producing x items

  - Elasticity

$$E(p) = \frac{dD}{dp} \frac{p}{D} \text{ where } D = f(p)$$

Rules of Differentiation

- Constant Rule

$$\text{If } c \in \mathbb{R}, \text{ then } \frac{d}{dx}(c) = 0$$

- Power Rule

$$\text{If } n \in \mathbb{Z} \text{ and } n > 0, \text{ then } \frac{d}{dx}(x^n) = nx^{n-1}$$

- Derivative of a Root

$$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

- Constant Multiple Rule

$$\text{If } f \text{ is differentiable at } x \text{ and } c \text{ is a constant, then } \frac{d}{dx}(cf(x)) = cf'(x)$$

- Sum Rule

$$\text{If } f \text{ and } g \text{ are differentiable at } x, \text{ then } \frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$$

- Generalized Sum Rule

$$\frac{d}{dx}\left(f_1(x) + f_2(x) + ... + f_x(x)\right) = f_1'(x) + f_2'(x) + ... + f_n'(x)$$

- Difference Rule

$$\frac{d}{dx}\left(f(x) - g(x)\right) = f'(x) - g'(x)$$

- Euler's Number

$$\text{The function } f(x) = e^x \text{ is differentiable for all } x \in \mathbb{R}, \text{ and } \frac{d}{dx}\left(e^x\right) = e^x$$

- Higher-order Derivatives
  Assuming $y = f(x)$ can be differentiated as often as necessary, the **second derivative** of f is

$$f''(x) = \frac{d}{dx}\left(f'(x)\right)$$

  For $n \in \mathbb{Z}$ where $n \geq 1$, the **nth derivative** of f is

$$f^{(n)}(x) = \frac{d}{dx}\left(f^{(n-1)}(x)\right)$$

- Product Rule

$$\text{If } f \text{ and } g \text{ are differentiable at } x, \text{ then } \frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x)$$

- Quotient Rule

$$\text{If } f \text{ and } g \text{ are differentiable at } x \text{ and } g(x) \neq 0, \text{ then the derivative of } \frac{f}{g} \text{ at } x \text{ exists and}$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

Trigonometric (and inverse) Derivatives

$$\begin{aligned}
\lim_{x\to 0}\frac{\sin x}{x} &= 1 && (1)\\
\lim_{x\to 0}\frac{\cos x - 1}{x} &= 0 && (2)\\
\frac{d}{dx}(\sin x) &= \cos x && (3)\\
\frac{d}{dx}(\cos x) &= -\sin x && (4)\\
\frac{d}{dx}(\tan x) &= \sec^2 x && (5)\\
\frac{d}{dx}(\cot x) &= -\csc^2 x && (6)\\
\frac{d}{dx}(\sec x) &= \sec x \tan x && (7)\\
\frac{d}{dx}(\csc x) &= -\csc x \cot x && (8)\\
\frac{d}{dx}\left(\sin^{-1} x\right) &= \frac{1}{\sqrt{1-x^2}},\ \text{for } -1 < x < 1 && (9)\\
\frac{d}{dx}\left(\cos^{-1} x\right) &= -\frac{1}{\sqrt{1-x^2}},\ \text{for } -1 < x < 1 && (10)\\
\frac{d}{dx}\left(\tan^{-1} x\right) &= \frac{1}{1+x^2},\ \text{for } -\infty < x < \infty && (11)\\
\frac{d}{dx}\left(\cot^{-1} x\right) &= -\frac{1}{1+x^2},\ \text{for } -\infty < x < \infty && (12)\\
\frac{d}{dx}\left(\sec^{-1} x\right) &= \frac{1}{|x|\sqrt{x^2-1}},\ \text{for } |x| > 1 && (13)\\
\frac{d}{dx}\left(\csc^{-1} x\right) &= -\frac{1}{|x|\sqrt{x^2-1}},\ \text{for } |x| > 1 && (14)\\
& && (15)
\end{aligned}$$

More Derivatives (M03)

- The Chain Rule
  Suppose $y = f(u)$ is differentiable at $u = g(x)$ and $u = g(x)$ is differentiable at x. The composite function $y = f(g(x))$ is differentiable at x, and its derivative can be expressed in two equivalent ways.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{dy}\cdot\frac{du}{dx} && (1)\\
\frac{d}{dx}\left(f\left(g\left(x\right)\right)\right) &= f'\left(g\left(x\right)\right)\cdot g'\left(x\right) && (2)
\end{aligned}$$

  Application of the Chain Rule (Assume the differentiable function $y = f(g(x))$ is given):

  1. Identify an outer function f and an inner function g, and let $u = g(x)$.
  2. Replace $g(x)$ with u to express y in terms of u:
  $$y = f(g(x)) = f(u)$$
  3. Calculate the product
  $$\frac{dy}{du}\cdot\frac{du}{dx}$$
  4. Replace u with $g(x)$ in $\frac{dy}{du}$ to obtain $\frac{dy}{dx}$

  If g is differentiable for all x in its domain and $p \in \mathbb{R}$,

$$\frac{d}{dx}\left(\left(g\left(x\right)\right)^p\right) = p\left(g\left(x\right)\right)^{p-1} g'\left(x\right)$$

- Implicit Differentiation
  When we are unable to solve for y explicitly, we treat y as a function of x ($y = y(x)$) and apply the Chain Rule:
$$y' = \frac{dy}{dx}$$
$$\frac{d}{dx}y^n = ny^{n-1}\frac{dy}{dx}$$
- Derivatives of Logarithmic and Exponential Functions
$$\frac{d}{dx}(\ln x) = \frac{1}{x}, \text{ for } x > 0$$
$$\frac{d}{dx}(\ln |x|) = \frac{1}{x}, \text{ for } x \neq 0$$
  If u is differentiable at x and $u(x) \neq 0$, then
$$\frac{d}{dx}(\ln |u(x)|) = \frac{u'(x)}{u(x)}$$
  If $b > 0$ and $b \neq 1$, then for all x,
$$\frac{d}{dx}(b^x) = b^x \ln b$$
  General Power Rule:
$$\text{For } p \in \mathbb{R} \text{ and for } x > 0,\ \frac{d}{dx}(x^p) = px^{p-1}$$
  Furthermore, if u is a positive differentiable function on its domain, then
$$\frac{d}{dx}\left(u(x)^p\right) = p\left(u(x)\right)^{p-1} \cdot u'(x)$$
  Functions of the form $f(x) = (g(x))^{h(x)}$, where both g and h are nonconstant functions, are neither exponential function nor power functions (they are sometimes called **tower functions**). To compute their derivatives, we use the identity $b^x = e^{x \ln b}$ to rewrite f with base e:
$$f(x) = (g(x))^{h(x)} = e^{h(x) \ln g(x)}$$
  If $b > 0$ and $b \neq 1$, then
$$\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}, \text{ for } x > 0$$
$$\frac{d}{dx}(\log_b |x|) = \frac{1}{x \ln b}, \text{ for } x \neq 0$$
  Useful Properties of Logarithms
$$\ln xy = \ln x + \ln y \quad (1)$$
$$\ln\left(\frac{x}{y}\right) = \ln x - \ln y \quad (2)$$
$$\ln x^z = z \ln x \quad (3)$$
  Let f be differentiable and have an inverse on an interval I. If x_0 is a point of I at which $f'(x_0) \neq 0$, then f^{-1} is differentiable at $y_0 = f(x_0)$ and
$$\left(f^{-1}\right)'(y_0) = \frac{1}{f'(x_0)}, \text{ where } y_0 = f(x_0)$$
- Related Rates
  Procedure
  1. Read the problem carefully, making a sketch to organize the given information. Identify the rates that are given and the rate that is to be determined.
  2. Write one or more equations that express the basic relationships among the variables.
  3. Introduce rates of change by differentiating the appropriate equation(s) with respect to time t.
  4. Substitute known values and solve for the desired quantity.
  5. Check that units are consistent and the answer is reasonable. (For example, does it have the correct sign?)

A copy of my notes (in LaTeX) are available on my GitHub