Module 6 Notes (MATH-211)

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General Notes (and Definitions)

- L'Hôpital's Rule

  Indeterminate Form: An expression involving two components where the limit cannot be determined by evaluating the limits of the individual components.

  L'Hôpital's Rule: Suppose f and g are differentiable functions on an open interval I containing the point $x = a$, with $g'(x) \neq 0$ on I when $x \neq a$.

  If $\lim\limits_{x \to a} \frac{f(x)}{g(x)}$ has any of the indeterminate forms: $\frac{0}{0}$, $\frac{\infty}{\infty}$, $-\frac{\infty}{\infty}$, then

  $$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

  provided that one of the following is the case:

  $$\lim_{x \to a} \frac{f'(x)}{g'(x)} \in \mathbb{R}$$

  $$\lim_{x \to a} \frac{f'(x)}{g'(x)} = \infty$$

  $$\lim_{x \to a} \frac{f'(x)}{g'(x)} = -\infty$$

  L'Hôpital's Rule is still valid if $x \to a$ is replaced by any of $x \to a^+$, $x \to a^-$, $x \to \infty$, or $x \to -\infty$. In the last two of these cases, there must be a greatest x-value beyond which both f and g are differentiable at every point.

  Exponential Indeterminate forms: 1^∞, 0^0, ∞^0

  Method for evaluating limits of indeterminate forms 1^∞, 0^0, ∞^0:

  Assume that $L = \lim\limits_{x \to a} f(x)^{g(x)}$ has one of these indeterminate forms.

  1. Use the fact that the natural logarithm and natural exponential functions are inverses to write

     $$L = \lim_{x \to a} e^{\ln\left(f(x)^{g(x)}\right)}$$

  2. Use the power property of logarithm arguments to write

     $$L = \lim_{x \to a} e^{g(x)\ln(f(x))}$$

  3. Use continuity of the exponential function to write

     $$L = e^{\lim\limits_{x \to a} g(x)\ln(f(x))}$$

  4. Rewrite multiplication as division by the reciprocal:

     $$L = e^{\lim\limits_{x \to a}\left(\frac{\ln(f(x))}{\frac{1}{g(x)}}\right)}$$

  5. Use L'Hôpital's Rule to evaluate this limit expression

  Growth Rates: Suppose f and g are functions with $\lim\limits_{x \to \infty} f(x) = \infty$ and $\lim\limits_{x \to \infty} g(x) = \infty$

1. If one of the following are true, f **grows faster than** g, and we use the notation $f \gg g$

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0 \tag{1}$$
$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty \tag{2}$$

2. f **and** g **have comparable growth rates**, if there is some non-zero finite number M such that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = M$$

Ranked Growth Rates as $x \to \infty$
For any base $b > 1$, and for any positive numbers p, q, r, and s

$$\ln^q x \ll x^p \ll x^p \ln^r x \ll x^{p+s} \ll b^x \ll x^x$$

- Antiderivatives
  Antiderivative: A function F is an antiderivative of another function f on an interval I if for all x in I:

  $$F'(x) = f(x)$$

  Family of Antiderivatives: Let $F(x)$ be any antiderivative of $f(x)$ on an interval I. Then all antiderivatives of f on I have the form $F(x) + C$, where C is an arbitrary constant.
  Differential Equations: Any equation involving an unknown function and its derivatives

  – Infinite family of solutions
  – No two solutions from the family pass through the same point
  – Given an initial condition $f(a) = b$, we can identify the particular family member that solves the given problem by solving for C

- Approximating Areas Under Curves

  – If we know the velocity function of a moving object, what can we learn about its position function?
  – Given an object with velocity function $v(t)$, the displacement of the moving object over the interval $[a, b]$ is the area between the velocity curve and the t-axis from $t = a$ to $t = b$.
  – Because objects do not necessarily move at a constant velocity, we can extend this idea to positive velocities that change over an interval of time.
  – The strategy is to divide the time interval into many subintervals, approximate the velocity on each subinterval with a constant velocity, calculate the individual displacements and sum the results.

  Riemann Sums

  – Suppose $f(x)$ is continuous and non-negative on $[a, b]$.
  – Goal is to approximate the area of the region R bounded by the graph of $f(x)$ and the x-axis from $x = a$ to $x = b$.
  – Divide $[a, b]$ into n subintervals $[x_0, x_1], [x_1, x_2], ..., [x_{n-1}, x_n]$ where $a = x_0, b = x_n$.
  – The length of each subinterval is $\Delta x = \frac{b-a}{n}$
  – **Regular Partition**: Suppose $[a, b]$ is a closed interval containing n subintervals

  $$[x_0, x_1], [x_1, x_2], ..., [x_{n-1}, x_n]$$

  of equal length $\Delta x = \frac{b-a}{n}$, with $a = x_0$ and $b = x_n$. The endpoints $x_0, x_1, x_2, ..., x_{n-1}, x_n$ of the subintervals are called **grid points**, and they create a **regular partition** of the interval $[a, b]$. In general the kth grid point is

  $$x_k = a + k\Delta x, \text{ for } k = 0, 1, 2, ..., n$$

  – In the kth subinterval $[x_{k-1}, x_k]$, choose any point x_k^* and build a rectangle whose height is $f(x_k^*)$.
  – The area of the rectangle of the kth subinterval is

  $$\text{height} \cdot \text{base} = f(x_k^*)\Delta x, \text{ where } k = 1, 2, ..., n$$

- Summing the areas of these rectangles, we obtain an approximation to the area of R, which is called a **Riemann sum**:
$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + ... + f(x_n^*)\Delta x$$
- Three notable Riemann sums are the left, right, and midpoint Riemann sums.

Riemann Sum: Suppose f is defined on a closed interval $[a, b]$, which is divided into n subintervals of equal length Δx. If x_k^* is any point in the kth subinterval $[x_{k-1}, x_k]$, for $k = 1, 2, ..., n$, then
$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + ... + f(x_n^*)\Delta x$$
is called a **Riemann sum** for f on $[a, b]$. This sum is called

- a **left Riemann sum** if x_k^* is the left endpoint of $[x_{k-1}, x_k]$
- a **right Riemann sum** if x_k^* is the right endpoint of $[x_{k-1}, x_k]$
- a **midpoint Riemann sum** if x_k^* is the midpoint of $[x_{k-1}, x_k]$

Summation notation (Σ):

- Working with Riemann sums is cumbersome when n is large
- We introduce sigma (summation) notation as a shorthand:
$$1 + 2 + ... + 49 + 50 = \sum_{k=1}^{50} k$$
- The symbol Σ (sigma) stands for sum
- k is the index, and takes on all integer values from $k = 1$ to $k = 50$
- The expression immediately following Σ, the summand, is evaluated for each k, and the resulting values are summed
- The index is a dummy variable, and it does not matter which symbol is chosen for the index:
$$\sum_{k=1}^{99} k = \sum_{n=1}^{99} n = \sum_{p-1}^{99} p$$
- Two Properties of Sums and Sigma Notation
  1. Constant Multiple Rule:
  $$\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$$
  2. Addition Rule:
  $$\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$
- **Theorem**: Sums of Power of Integers
  Let $n \in \mathbb{Z}$ such that $n > 0$ and $c \in \mathbb{R}$
$$\sum_{k=1}^{n} c = cn \tag{1}$$
$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \tag{2}$$
$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \tag{3}$$
$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4} \tag{4}$$

Left, Right, and Midpoint Riemann Sums in Sigma Notation:
Suppose f is defined on a closed interval $[a, b]$, which is divided into subintervals of equal length Δx. If x_k^* is a point in the kth subinterval $[x_{k-1}, x_k]$, for $k = 1, 2, ..., n$, then the **Riemann sum** for f on $[a, b]$ is
$$\sum_{k=1}^{n} f(x_k^*)\Delta x$$
Three cases arise in practice

- $\sum_{k=1}^{n} f(x_k^*)\Delta x$ is a **left Riemann sum** if $x_k^* = a + (k-1)\Delta x$
- $\sum_{k=1}^{n} f(x_k^*)\Delta x$ is a **right Riemann sum** if $x_k^* = a + k\Delta x$
- $\sum_{k=1}^{n} f(x_k^*)\Delta x$ is a **midpoint Riemann sum** if $x_k^* = a + (k - \frac{1}{2})\Delta x$

• Definite Integrals
Net Area: Consider the region R bounded by the graph of a continuous function f and the x-axis between $x = a$ and $x = b$. The **net area** of R is the sum of the area of the parts of R that lie above the x-axis minus the sum of the areas of the parts of R that lie below the x-axis on $[a, b]$.

- Where $f(x) < 0$, Riemann sums approximate the negative of the area of the region bounded by the curve
- On the interval $[a, b]$, we get positive, and negative contributions to the Riemann sum where $f(x)$ is negative
- Riemann sums approximate the area of the regions that lie above the x-axis minus the area of the regions that lie below the x-axis
- The difference is called the **net area**; it can be positive, negative, or zero

$$area_{net} = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

A **general partition** of $[a, b]$ consists of the n subintervals

$$[x_0, x_1], [x_1, x_2], ..., [x_{n-1}, x_n]$$

where $x_0 = a$and $x_n = b$. The length of the kth subinterval is $\Delta x_k = x_k - k_{k-1}$, for $k = 1, 2, ..., n$. We let x_k^* be any point in the subinterval $[x_{k-1}, x_k]$.
General Riemann Sum: Suppose $[x_0, x_1], [x_1, x_2], ..., [x_{n-1}, x_n]$ are subintervals of $[a, b]$ with

$$a = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = b$$

Let $x\Delta x_k$ be the length of the subinterval $[x_{k-1}, x_k]$ and let x_k^* be any point in $[x_{k-1}, x_k]$, for $k = 1, 2, ...n$
If f is defined on $[a, b]$, the sum

$$\sum_{k=1}^{n} f(x_k^*)\Delta x_k = f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + ... + f(x_n^*)\Delta x_n$$

is called a **general Riemann sum** for f on $[a, b]$
Definite Integral: A function f defined on $[a, b]$ is **integrable** on $[a, b]$ if $\lim_{\Delta x\to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$ exists and is unique over all partitions of $[a, b]$ and all choices of x_k^* on a parition. This limit is the **definite integral of f from a to b**, which we rite

$$\int_a^b f(x)dx = \lim_{\Delta x\to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

Integrable Functions: If f is continuous on $[a, b]$ or bounded on $[a, b]$ with a finite number of discontinuities, then f is integrale on $[a, b]$.
Let f and g be integrable function on $[a, b]$, where $b > a$

1. If $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x)\,dx \geq 0$
2. If $f(x) \geq g(x)$ on $[a, b]$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$
3. If $m \leq f(x) \leq M$, then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$

• Fundamental Theorem of Calculus
Area Functions: Let f be a continuous function, for $t \geq a$. The **area function for f with left endpoint** a is

$$A(x) = \int_a^x f(t)\,dt$$

where $x \geq a$. The area function gives the net area of the region bounded by the graph of f and the t-axis on the interval $[a, x]$.
If f is continuous on $[a, b]$, then the area function

$$A(x) = \int_a^x f(t)\,dt, \text{ for } a \leq x \leq b,$$

is continuous on $[a, b]$ and differentiable on (a, b). The area function satisfies $A'(x) = f(x)$. Equivalently,

$$A'(x) = \frac{d}{dx} \int_a^x f(t)\,dt = f(x),$$

which means that the area function of f is an antiderivative of f on $[a, b]$.
If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Antiderivative Rules

- Power Rule
  If $p \neq -1$ and C is an arbitrary constant:

$$\int x^p dx = \frac{x^{p+1}}{p+1} + C$$

- Integral of x^{-1}

$$\int x^{-1} dx = \int \frac{1}{x} dx = \ln|x| + C$$

- Constant Multiple and Sum Rules
  If $c \in \mathbb{R}$:

$$\int cf(x)dx = c \int f(x)dx$$

$$\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$$

- Integral of e^x

$$\int e^x dx = e^x + C$$

- Integral of $\frac{1}{x}$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

Trigonometric (and inverse) Integrals

$$\int \cos(x)dx = \sin x + C \tag{1}$$

$$\int \sin(x)dx = -\cos x + C \tag{2}$$

$$\int \sec^2(x)dx = \tan x + C \tag{3}$$

$$\int \csc^2(x)dx = -\cot x + C \tag{4}$$

$$\int \sec(x)\tan(x)dx = \sec x + C \tag{5}$$

$$\int \csc(x)\cot(x)dx = -\csc x + C \tag{6}$$

$$\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x + C \tag{7}$$

$$\int \frac{1}{1+x^2}dx = \tan^{-1} x + C \tag{8}$$

$$\int \frac{1}{x\sqrt{x^2-1}}dx = \sec^{-1}|x| + C \tag{9}$$

Properties of Definite Integrals

Let f and g be integrable functions on an interval that contains a, b, and p

$$\int_a^a f(x)\,dx = 0 \tag{1}$$

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx \tag{2}$$

$$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx \tag{3}$$

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx, \text{ for any constant } c \tag{4}$$

$$\int_a^b f(x)\,dx = \int_a^p f(x)\,dx + \int_p^b f(x)\,dx \tag{5}$$

The function $|f|$ is integrable on $[a,b]$, and $\int\limits_a^b |f(x)|\,dx$ is the sum of the areas of the regions bounded by the graph of f and the x-axis on $[a,b]$.

Examples

1. Use L'Hôpital's Rule to evaluate a limit with indeterminate form $\frac{0}{0}$

$$\lim_{x\to 0} \frac{e^x - x - 1}{5x^2} = \lim_{x\to 0} \frac{e^x - 1}{10x} \tag{1}$$

$$= \lim_{x\to 0} \frac{e^x}{10} \tag{2}$$

$$= \frac{e^0}{10} \tag{3}$$

$$= \frac{1}{10} \tag{4}$$

2. Use L'Hôpital's Rule to evaluate a limit with indeterminate form $\frac{\infty}{\infty}$

$$\begin{aligned}
\lim_{x\to 0^+} \frac{1-\ln x}{1+\ln x} &= \lim_{x\to 0^+} \frac{-\frac{1}{x}}{\frac{1}{x}} && (1)\\
&= \lim_{x\to 0^+} \frac{-\frac{1}{x}}{\frac{1}{x}} && (2)\\
&= \frac{-1}{1} && (3)\\
&= -1 && (4)
\end{aligned}$$

3. Use L'Hôpital's Rule to evaluate a limit with indeterminate form $0 \cdot \infty$

$$\begin{aligned}
\lim_{x\to 1^-} (1-x)\tan\left(\frac{\pi x}{2}\right) &= \lim_{x\to 1^-} \frac{(1-x)}{\cot\left(\frac{\pi x}{2}\right)} && (1)\\
&= \lim_{x\to 1^-} \frac{-1}{-\frac{\pi}{2}\csc^2\left(\frac{\pi x}{2}\right)} && (2)\\
&= \lim_{x\to 1^-} \frac{2}{\pi}\sin^2\left(\frac{\pi x}{2}\right) && (3)\\
&= \frac{2}{\pi} && (4)
\end{aligned}$$

4. Use L'Hôpital's Rule to evaluate a limit with exponential indeterminate form

$$\begin{aligned}
\lim_{x\to 0^+} x^{\tan x} &= e^{\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\tan x}}} && (1)\\
&= e^{\lim_{x\to 0^+} \frac{\ln x}{\cot x}} && (2)\\
&= e^{\lim_{x\to 0^+} \frac{1}{-x\csc^2 x}} && (3)\\
&= e^{\lim_{x\to 0^+} \frac{-\sin^2 x}{x}} && (4)\\
&= e^{\lim_{x\to 0^+} \frac{-2\sin x\cos x}{1}} && (5)\\
&= e^{\lim_{x\to 0^+} -2\sin x\cos x} && (6)\\
&= e^0 && (7)\\
&= 1 && (8)
\end{aligned}$$

5. Compare the growth rates of functions

$$f(x) = x^2 \ln x$$
$$g(x) = x\ln^2 x$$

$$\begin{aligned}
\lim_{x\to\infty} \frac{f(x)}{g(x)} &= \lim_{x\to\infty} \frac{x^2\ln x}{x\ln^2 x} && (1)\\
&= \lim_{x\to\infty} \frac{x}{\ln x} && (2)\\
&= \lim_{x\to\infty} \frac{1}{\frac{1}{x}} && (3)\\
&= \lim_{x\to\infty} x && (4)\\
&= \infty && (5)
\end{aligned}$$

Since $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$, $f \gg g$

6. Use knowledge of derivatives to find antiderivatives

$$f(x) = -4\cos x - x$$
$$F(x) = -4\sin x - \frac{1}{2}x^2$$
$$F'(x) = -4\cos x - x$$
$$\int (-4\cos x - x)dx = -4\sin x - \frac{1}{2}x^2 + C$$

7. Determine indefinite integrals using antiderivative rules

$$\int \frac{3}{x^4} + 2 - 3x^2 dx = \int 3x^{-4} + 2 - 3x^2 dx = \frac{-1}{x^3} + 2x - x^3 + C$$

8. Rewrite an indefinite integral to find an antiderivative

$$\begin{aligned}
\int \frac{2 + 3\cos y}{\sin^2 y} dy &= \int 2\csc^2 y + 3\cot y \csc y dy && (1)\\
&= 2\int \csc^2 y dy + 3\int \cot y \csc y dy && (2)\\
&= -2\cot y - 3\csc y + C && (3)
\end{aligned}$$

9. Solve an initial value problem

$$f'(u) = 4\cos u - 4\sin u$$

$$f(\pi) = 0$$

$$f(u) = 4\int \cos u\, du - 4\int \sin u\, du = 4\sin u + 4\cos u + C$$

$$\begin{aligned}
4\sin\pi + 4\cos\pi + C &= 0 && (1)\\
0 - 4 + C &= 0 && (2)\\
C &= 4 && (3)
\end{aligned}$$

$$f(u) = 4\sin u + 4\cos u + 4$$

10. Application of differential equations to linear motion

$$a(t) = 2 + 3\sin t$$

$$v(0) = 1$$

$$s(0) = 10$$

$$\begin{aligned}
v(t) &= \int 2 + 3\sin t\, dt && (1)\\
&= 2\int t^0\, dt + 3\int \sin t\, dt && (2)\\
&= 2t - 3\cos t + C && (3)\\
2(0) - 3\cos 0 + C &= 1 && (4)\\
-3 + C &= 1 && (5)\\
C &= 4 && (6)\\
v(t) &= 2t - 3\cos t + 4 && (7)\\
s(t) &= \int 2t - 3\cos t + 4\, dt && (8)\\
&= 2\int t\, dt - 3\int \cos t\, dt + 4\int t^0\, dt && (9)\\
&= 2\frac{t^2}{2} - 3\sin t + 4t + C && (10)\\
&= t^2 - 3\sin t + 4t + C && (11)\\
0^2 - 3\sin 0 + 4(0) + C &= 10 && (12)\\
C &= 10 && (13)\\
s(t) &= t^2 - 3\sin t + 4t + 10 && (14)
\end{aligned}$$

11. Approximating displacement

$$v = \sqrt{10t}$$

$$1 \le t \le 7$$

(a)

$$n = 3$$

$$[1, 3], [3, 5], [5, 7]$$

$$2, 4, 6$$

$$d \approx 2v(2) + 2v(4) + 2v(6) = 2\sqrt{20} + 2\sqrt{40} + 2\sqrt{60} \approx 37.085$$

(b)

$$n = 6$$

$$1.5, 2.5, 3.5, 4.5, 5.5, 6.5$$

$$\begin{aligned} d &\approx v(1.5) + v(2.5) + v(3.5) + v(4.5) + v(5.5) + v(6.5) && (1) \\ &= \sqrt{15} + \sqrt{25} + \sqrt{35} + \sqrt{45} + \sqrt{55} + \sqrt{65} && (2) \\ &\approx 36.976 && (3) \end{aligned}$$

12. Left and right Riemann sums (1)

$$f(x) = x + 1$$

$$[1, 6]$$

$$n = 5$$

(a)

$$\begin{aligned} (1+1) + (2+1) + (3+1) + (4+1) + (5+1) &= 2+3+4+5+6 && (1) \\ &= 20 && (2) \end{aligned}$$

(b)

$$\begin{aligned} (2+1) + (3+1) + (4+1) + (5+1) + (6+1) &= 3+4+5+6+7 && (1) \\ &= 25 && (2) \end{aligned}$$

13. Left and right Riemann sums (2)

$$f(x) = 9 - x$$

$$[3, 8]$$

$$n = 5$$

$$\Delta x = \frac{8-3}{5} = \frac{5}{5} = 1$$

(a) Grid Points:

$$\begin{aligned} x_0 &= 3 && (1) \\ x_1 &= 4 && (2) \\ x_2 &= 5 && (3) \\ x_3 &= 6 && (4) \\ x_4 &= 7 && (5) \\ x_5 &= 8 && (6) \end{aligned}$$

(b) Riemann Sums

$$\begin{aligned} (9-3) + (9-4) + (9-5) + (9-6) + (9-7) &= 6+5+4+3+2 && (1) \\ &= 20 \text{ (Overestimation)} && (2) \end{aligned}$$

$$\begin{aligned} (9-4) + (9-5) + (9-6) + (9-7) + (9-8) &= 5+4+3+2+1 && (1) \\ &= 15 \text{ (Underestimation)} && (2) \end{aligned}$$

14. Midpoint Riemann sum

$$f(x) = 100 - x^2$$

$$[0, 10]$$

$$n = 5$$

$$\Delta x = 2$$

$$\begin{aligned} 2f(1) + 2f(3) + 2f(5) + 2f(7) + 2f(9) &= 2(99) + 2(91) + 2(75) + 2(51) + 2(19) && (1) \\ &= 198 + 182 + 150 + 102 + 38 && (2) \\ &= 670 && (3) \end{aligned}$$

15. Riemann sums from tables

$$[0, 2]$$

$$n = 4$$

$$\Delta x = \frac{1}{2}$$

$$\begin{aligned} \frac{5}{2} + \frac{3}{2} + \frac{2}{2} + \frac{1}{2} &= 5.5 && (1) \\ \frac{3}{2} + \frac{2}{2} + \frac{1}{2} + \frac{1}{2} &= 3.5 && (2) \end{aligned}$$

16. Computing Riemann sums for large values of n

$$f(x) = x^2 + 1$$

$$[-1, 1]$$

$$n = 50$$

$$\begin{aligned} \sum_{k=1}^{50} 0.04 f\left(-1 + 0.04\,(k-1)\right) &= \sum_{k=1}^{50} 0.04\left(\left(-1 + 0.04\,(k-1)\right)^2 + 1\right) && (1) \\ &\approx 2.6672 && (2) \end{aligned}$$

$$\begin{aligned} \sum_{k=1}^{50} 0.04 f\left(-1 + 0.04k\right) &= \sum_{k=1}^{50} 0.04\left(\left(-1 + 0.04k\right)^2 + 1\right) && (3) \\ &\approx 2.6672 && (4) \end{aligned}$$

$$\begin{aligned} \sum_{k-1}^{50} 0.04 f\left(-1 + 0.04\left(k - \frac{1}{2}\right)\right) &= \sum_{k-1}^{50} 0.04\left(\left(-1 + 0.04\left(k - \frac{1}{2}\right)\right)^2 + 1\right) && (5) \\ &\approx 2.6664 && (6) \end{aligned}$$

17. Approximating net area

$$f(x) = 4 - 2x$$

$$[0, 4]$$

$$n = 4$$

$$\Delta x = 1$$

$$\begin{aligned} 4 + 2 + 0 - 2 &= 4 && (1) \\ 2 + 0 - 2 - 4 &= -4 && (2) \\ 3 + 1 - 1 - 3 &= 0 && (3) \end{aligned}$$

18. Identifying definite integrals as limits of sums

$$\lim_{\Delta \to 0} \sum_{k=1}^{n} \left(4 - x_k^{*2}\right) \Delta x_k$$

$$\int_0^2 \left(4 - x^2\right) \, dx$$

19. Using geometry to evaluate definite integrals

$$\int_0^4 (8 - 2x) \, dx = \frac{4 \cdot 8}{2} = 16$$

20. Definite integrals from graphs

$$\begin{aligned}
\int_0^b f(x)\,dx &= 16 - 5 && (1)\\
&= 11 && (2)\\
\int_0^c f(x)\,dx &= 16 - 5 + 11 && (3)\\
&= 22 && (4)\\
\int_a^0 f(x)\,dx &= -\int_0^a f(x)\,dx && (5)\\
&= -16 && (6)\\
\int_0^c |f(x)|\,dx &= 16 + 5 + 11 && (7)\\
&= 32 && (8)
\end{aligned}$$

21. Using properties of integrals

$$\int_1^4 f(x)\,dx = 8$$

$$\int_1^6 f(x)\,dx = 5$$

$$\begin{aligned}
\int_1^4 (-3f(x)) \, dx &= -3\int_1^4 f(x)\,dx && (1)\\
&= -3 \cdot 8 && (2)\\
&= -24 && (3)\\
\int_1^4 3f(x)\,dx &= 3\int_1^4 f(x)\,dx && (4)\\
&= 3 \cdot 8 && (5)\\
&= 24 && (6)\\
\int_6^4 12f(x)\,dx &= -\int_4^6 12f(x)\,dx && (7)\\
&= -12\int_4^6 f(x)\,dx && (8)\\
&= -12\,(5 - 8) && (9)\\
&= 36 && (10)\\
\int_4^6 3f(x)\,dx &= 3\int_4^6 f(x)\,dx && (11)\\
&= 3\,(5 - 8) && (12)\\
&= -9 && (13)
\end{aligned}$$

22. Use definition of definite integral

$$
\begin{aligned}
\int_0^2 (2x+1)\,dx &= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{4k}{n}+1\right)\frac{2}{n} && (1)\\
&= \lim_{n\to\infty} \frac{2}{n} \sum_{k=1}^{n} \left(\frac{4k}{n}+1\right) && (2)\\
&= \lim_{n\to\infty} \frac{2}{n} \left(\sum_{k=1}^{n} \frac{4k}{n} + \sum_{k=1}^{n} 1\right) && (3)\\
&= \lim_{n\to\infty} \frac{2}{n} \left(\frac{4}{n}\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1\right) && (4)\\
&= \lim_{n\to\infty} \frac{2}{n} \left(\frac{4}{n}\left(\frac{n(n+1)}{2}\right) + n\right) && (5)\\
&= \lim_{n\to\infty} \frac{2}{n} \left(\frac{4n(n+1)}{2n} + n\right) && (6)\\
&= \lim_{n\to\infty} \frac{2}{n} \left(2(n+1)+n\right) && (7)\\
&= \lim_{n\to\infty} \frac{2}{n} \left(2n+2+n\right) && (8)\\
&= \lim_{n\to\infty} \frac{2}{n} \left(3n+2\right) && (9)\\
&= \lim_{n\to\infty} \frac{6n+4}{n} && (10)\\
&= \lim_{n\to\infty} 6+\frac{4}{n} && (11)\\
&= 6+0 && (12)\\
&= 6 && (13)
\end{aligned}
$$

23. Area functions (1)

$$
A(x) = \int_0^x f(t)\,dt
$$

$$
F(x) = \int_2^x f(t)\,dt
$$

$$
\begin{aligned}
A(2) &= \int_0^2 f(t)\,dt && (1)\\
&= 8 && (2)\\
F(5) &= \int_2^5 f(t)\,dt && (3)\\
&= -5 && (4)\\
A(0) &= \int_0^0 f(t)\,dt && (5)\\
&= 0 && (6)\\
F(8) &= \int_2^8 f(t)\,dt && (7)\\
&= -16 && (8)\\
A(8) &= \int_0^8 f(t)\,dt && (9)\\
&= -8 && (10)
\end{aligned}
$$

24. Area functions (2)

$$
f(t) = 2t+5
$$

$$
a = 0
$$

$$A(x) = \frac{1}{2}(x-0)(f(0)+f(x)) = \frac{2x^2+10x}{2} = x^2+5x$$
$$A'(x) = 2x+5 = f(x)$$

25. Derivatives of integrals

$$\begin{aligned}
\frac{d}{dx}\int_1^x e^{t^2}\,dt &= e^{x^2} && (1)\\
\frac{d}{dx}\int_x^0 \frac{dp}{p^2+1}\,dt &= -\frac{1}{x^2+1} && (2)\\
\frac{d}{dx}\int_0^{\cos x}\left(t^4+6\right)\,dt &= \left(\cos^4 x+6\right)\cdot\frac{d}{dx}\cos x && (3)\\
&= -\sin x\left(\cos^4 x+6\right) && (4)
\end{aligned}$$

26. Evaluating definite integrals

$$\begin{aligned}
\int\left(3x^2-6x+3\right)\,dx &= 3\int x^2\,dx - 6\int x\,dx + 3\int x^0\,dx && (1)\\
&= 3\int x^2\,dx - 6\int x\,dx + 3\int x^0\,dx && (2)\\
&= 3\frac{x^3}{3} - 6\frac{x^2}{2} + 3x && (3)\\
&= x^3-3x^2+3x && (4)\\
\int_0^3\left(3x^2-6x+3\right)\,dx &= 3^3-3(3)^2+3(3) && (5)\\
&= 27-27+9 && (6)\\
&= 9 && (7)\\
\int\left(8x^{\frac{1}{3}}\right)\,dx &= 8\int x^{\frac{1}{3}}\,dx && (8)\\
&= 8\left(\frac{x^{\frac{4}{3}}}{\frac{4}{3}}\right) && (9)\\
&= 8\left(\frac{3x^{\frac{4}{3}}}{4}\right) && (10)\\
&= 6x^{\frac{4}{3}} && (11)\\
\int_1^8\left(8x^{\frac{1}{3}}\right)\,dx &= 6\left(8^{\frac{4}{3}}\right) - 6\left(1^{\frac{4}{3}}\right) && (12)\\
&= 6(16)-6(1) && (13)\\
&= 90 && (14)\\
\int e^x\,dx &= e^x && (15)\\
\int_0^{\ln 8} e^x\,dx &= e^{\ln 8} - e^0 && (16)\\
&= 8-1 && (17)\\
&= 7 && (18)\\
\int 2\cos x\,dx &= 2\sin x && (19)\\
\int_0^{\frac{\pi}{4}} 2\cos x\,dx &= 2\sin\frac{\pi}{4} - 2\sin 0 && (20)\\
&= \frac{2}{\sqrt{2}} && (21)
\end{aligned}$$

27. Area

$$f(x) = \sqrt{x}$$
$$[1,4]$$

$$A = \int_1^4 \sqrt{x}\,dx$$

$$\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx = \frac{2x^{\frac{3}{2}}}{3}$$

$$\int_1^4 \sqrt{x}\,dx = \frac{16}{3} - \frac{2}{3} = \frac{14}{3}$$

Related Exercises

1. (Section 4.7, Exercise 17)

$$\begin{aligned}
\lim_{x\to 2} \frac{x^2-2x}{x^2-6x+8} &= \lim_{x\to 2} \frac{2x-2}{2x-6} && (1)\\
&= \frac{2(2)-2}{2(2)-6} && (2)\\
&= \frac{4-2}{4-6} && (3)\\
&= \frac{2}{-2} && (4)\\
&= -1 && (5)
\end{aligned}$$

2. (Section 4.7, Exercise 18)

$$\begin{aligned}
\lim_{x\to -1} \frac{x^4+x^3+2x+2}{x+1} &= \lim_{x\to -1} \frac{4x^3+3x^2+2}{1} && (1)\\
&= \lim_{x\to -1} 4x^3+3x^2+2 && (2)\\
&= 4(-1)^3+3(-1)^2+2 && (3)\\
&= -4+3+2 && (4)\\
&= 1 && (5)
\end{aligned}$$

3. (Section 4.7, Exercise 36)

$$\begin{aligned}
\lim_{x\to 0} \frac{e^x-x-1}{5x^2} &= \lim_{x\to 0} \frac{e^x-1}{10x} && (1)\\
&= \lim_{x\to 0} \frac{e^x}{10} && (2)\\
&= \frac{e^0}{10} && (3)\\
&= \frac{1}{10} && (4)
\end{aligned}$$

4. (Section 4.7, Exercise 39)

$$\begin{aligned}
\lim_{x\to 0} \frac{e^x-\sin x-1}{x^4+8x^3+12x^2} &= \lim_{x\to 0} \frac{e^x-\cos x}{4x^3+24x^2+24x} && (1)\\
&= \lim_{x\to 0} \frac{e^x+\sin x}{12x^2+48x+24} && (2)\\
&= \frac{e^0+\sin 0}{12(0)^2+48(0)+24} && (3)\\
&= \frac{1+0}{24} && (4)\\
&= \frac{1}{24} && (5)
\end{aligned}$$

5. (Section 4.7, Exercise 38)

$$\begin{aligned}
\lim_{x\to\infty} \frac{e^{3x}}{3e^{3x}+5} &= \lim_{x\to\infty} \frac{3e^{3x}}{9e^{3x}} && (1)\\
&= \lim_{x\to\infty} \frac{1}{3}\cdot\frac{e^{3x}}{e^{3x}} && (2)\\
&= \lim_{x\to\infty} \frac{1}{3} && (3)\\
&= \frac{1}{3} && (4)
\end{aligned}$$

6. (Section 4.7, Exercise 51)

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^2-\ln\frac{2}{x}}{3x^2+2x} &= \lim_{x\to\infty} \frac{2x+\frac{1}{x}}{6x+2} && (1)\\
&= \lim_{x\to\infty} \frac{2-\frac{1}{x^2}}{6} && (2)\\
&= \frac{2-0}{6} && (3)\\
&= \frac{2}{6} && (4)\\
&= \frac{1}{3} && (5)
\end{aligned}$$

7. (Section 4.7, Exercise 53)

$$\begin{aligned}
\lim_{x\to 0} x\csc x &= \lim_{x\to 0} \frac{x}{\sin x} && (1)\\
&= \lim_{x\to 0} \frac{1}{\cos x} && (2)\\
&= \frac{1}{\cos 0} && (3)\\
&= \frac{1}{1} && (4)\\
&= 1 && (5)
\end{aligned}$$

8. (Section 4.7, Exercise 63)

$$\begin{aligned}
\lim_{x\to\infty}\left(x^2-\sqrt{x^4+16x^2}\right) &= \lim_{x\to\infty}\left(x^2-\sqrt{x^4\left(1+\frac{16}{x^2}\right)}\right) && (1)\\
&= \lim_{x\to\infty}\left(x^2-x^2\sqrt{1+\frac{16}{x^2}}\right) && (2)\\
&= \lim_{x\to\infty} x^2\left(1-\sqrt{1+\frac{16}{x^2}}\right) && (3)\\
&= \lim_{x\to\infty}\frac{1-\sqrt{1+\frac{16}{x^2}}}{\frac{1}{x^2}} && (4)\\
&= \lim_{x\to\infty}\frac{\frac{16}{x^3}}{\frac{-2}{x^3}\sqrt{1+\frac{16}{x^2}}} && (5)\\
&= \lim_{x\to\infty}\frac{\frac{16}{x^3}\cdot\frac{x^3}{-2}}{\sqrt{1+\frac{16}{x^2}}} && (6)\\
&= \lim_{x\to\infty}\frac{\frac{16}{-2}\cdot\frac{x^3}{x^3}}{\sqrt{1+\frac{16}{x^2}}} && (7)\\
&= \lim_{x\to\infty}\frac{-8}{\sqrt{1+\frac{16}{x^2}}} && (8)\\
&= \frac{-8}{\sqrt{1+0}} && (9)\\
&= \frac{-8}{1} && (10)\\
&= -8 && (11)
\end{aligned}$$

9. (Section 4.7, Exercise 64)

$$\begin{aligned}
\lim_{x\to\infty}\left(x-\sqrt{x^2+4x}\right) &= \lim_{x\to\infty}\left(x-\sqrt{x^2\left(1+\frac{4}{x}\right)}\right) && (1)\\
&= \lim_{x\to\infty}\left(x-x\sqrt{1+\frac{4}{x}}\right) && (2)\\
&= \lim_{x\to\infty} x\left(1-\sqrt{1+\frac{4}{x}}\right) && (3)\\
&= \lim_{x\to\infty}\frac{1-\sqrt{1+\frac{4}{x}}}{\frac{1}{x}} && (4)\\
&= \lim_{x\to\infty}\frac{\frac{2}{x^2}}{\frac{-1}{x^2}\sqrt{1+\frac{4}{x}}} && (5)\\
&= \lim_{x\to\infty}\frac{\frac{2}{x^2}\cdot\frac{x^2}{-1}}{\sqrt{1+\frac{4}{x}}} && (6)\\
&= \lim_{x\to\infty}\frac{\frac{2}{-1}\cdot\frac{x^2}{x^2}}{\sqrt{1+\frac{4}{x}}} && (7)\\
&= \lim_{x\to\infty}\frac{-2}{\sqrt{1+\frac{4}{x}}} && (8)\\
&= \frac{-2}{\sqrt{1+0}} && (9)\\
&= -2 && (10)
\end{aligned}$$

10. (Section 4.7, Exercise 75)

$$\begin{aligned}
\lim_{x\to 0^+} x^{2x} &= e^{\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{2x}}} && (1)\\
&= e^{\lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{2x^2}}} && (2)\\
&= e^{\lim_{x\to 0^+} \frac{1}{x}\cdot\frac{2x^2}{-1}} && (3)\\
&= e^{\lim_{x\to 0^+} -\frac{2x^2}{x}} && (4)\\
&= e^{\lim_{x\to 0^+} -2x} && (5)\\
&= e^{-2(0)} && (6)\\
&= e^{0} && (7)\\
&= 1 && (8)
\end{aligned}$$

11. (Section 4.7, Exercise 76)

$$\begin{aligned}
\lim_{x\to 0} (1+4x)^{\frac{3}{x}} &= e^{\lim_{x\to 0^+} \frac{\ln(1+4x)}{\frac{1}{\frac{3}{x}}}} && (1)\\
&= e^{\lim_{x\to 0^+} \frac{\ln(1+4x)}{\frac{x}{3}}} && (2)\\
&= e^{\lim_{x\to 0^+} \frac{\frac{4}{(1+4x)}}{\frac{1}{3}}} && (3)\\
&= e^{\lim_{x\to 0^+} \frac{4}{(1+4x)}\cdot\frac{3}{1}} && (4)\\
&= e^{\lim_{x\to 0^+} \frac{12}{(1+4x)}} && (5)\\
&= e^{\frac{12}{1}} && (6)\\
&= e^{12} && (7)
\end{aligned}$$

12. (Section 4.7, Exercise 96)

$$\begin{aligned}
f(x) &= x^2\ln x && (1)\\
g(x) &= \ln^2 x && (2)
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to\infty}\frac{f(x)}{g(x)} &= \lim_{x\to\infty}\frac{x^2\ln x}{\ln^2 x} && (1)\\
&= \lim_{x\to\infty}\frac{x^2}{\ln x} && (2)\\
&= \lim_{x\to\infty}\frac{2x}{\frac{1}{x}} && (3)\\
&= \lim_{x\to\infty} 2x^2 && (4)\\
&= \infty && (5)
\end{aligned}$$

$$f \gg g$$

13. (Section 4.7, Exercise 100)

$$\begin{aligned}
f(x) &= x^2\ln x && (1)\\
g(x) &= x^3 && (2)
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to\infty}\frac{f(x)}{g(x)} &= \lim_{x\to\infty}\frac{x^2\ln x}{x^3} && (1)\\
&= \lim_{x\to\infty}\frac{\ln x}{x} && (2)\\
&= \lim_{x\to\infty}\frac{\frac{1}{x}}{1} && (3)\\
&= \lim_{x\to\infty}\frac{1}{x} && (4)\\
&= \frac{1}{\infty} \neq \infty && (5)
\end{aligned}$$

$$g \gg f$$

14. (Section 4.7, Exercise 95)

$$\begin{aligned} f(x) &= x^{10} & (1)\\ g(x) &= e^{0.01x} & (2) \end{aligned}$$

$$\begin{aligned} \lim_{x\to\infty} \frac{f(x)}{g(x)} &= \lim_{x\to\infty} \frac{x^{10}}{e^{0.01x}} & (1)\\ &= \lim_{x\to\infty} \frac{10x^9}{0.01e^{0.01x}} & (2)\\ &= \lim_{x\to\infty} \frac{90x^8}{0.01^2e^{0.01x}} & (3)\\ &= \lim_{x\to\infty} \frac{7200x^7}{0.01^3e^{0.01x}} & (4)\\ &= \lim_{x\to\infty} \frac{50400x^6}{0.01^4e^{0.01x}} & (5)\\ &= \lim_{x\to\infty} \frac{302400x^5}{0.01^5e^{0.01x}} & (6)\\ &= \lim_{x\to\infty} \frac{1512000x^4}{0.01^6e^{0.01x}} & (7)\\ &= \lim_{x\to\infty} \frac{6048000x^3}{0.01^7e^{0.01x}} & (8)\\ &= \lim_{x\to\infty} \frac{18144000x^2}{0.01^8e^{0.01x}} & (9)\\ &= \lim_{x\to\infty} \frac{36288000x}{0.01^9e^{0.01x}} & (10)\\ &= \lim_{x\to\infty} \frac{36288000}{0.01^{10}e^{0.01x}} & (11)\\ &= \frac{36288000}{\infty} \neq \infty & (12) \end{aligned}$$

$$g \gg f$$

15. (Section 4.7, Exercise 101)

$$\begin{aligned} f(x) &= x^{20} & (1)\\ g(x) &= 1.00001^x & (2) \end{aligned}$$

$$\begin{aligned} \lim_{x\to\infty} \frac{f(x)}{g(x)} &= \lim_{x\to\infty} \frac{x^{20}}{1.00001^x} & (1)\\ &= \frac{2432902008176640000}{\infty} \neq \infty & (2) \end{aligned}$$

$$g \gg f$$

16. (Section 4.9, Exercise 12)

$$f(x) = 11x^{10}$$

$$\begin{aligned} \int 11x^{10}\,dx &= 11\int x^{10}\,dx & (1)\\ &= 11\frac{x^{11}}{11} + C & (2)\\ &= x^{11} + C & (3) \end{aligned}$$

17. (Section 4.9, Exercise 13)

$$f(x) = 2\sin x + 1$$

$$\begin{aligned} \int 2\sin x + 1\,dx &= 2\int \sin x + \int x^0 & (1)\\ &= -2\cos x + x + C & (2) \end{aligned}$$

18. (Section 4.9, Exercise 24)

$$\begin{aligned}
\int 3u^{-2} - 4u^2 + 1\,du &= 3\int u^{-2}\,du - 4\int u^2\,du + \int u^0\,du && (1)\\
&= 3\frac{u^{-1}}{-1} - 4\frac{u^3}{3} + u + C && (2)\\
&= \frac{-3}{u} - \frac{4u^3}{3} + u + C && (3)
\end{aligned}$$

19. (Section 4.9, Exercise 25)

$$\begin{aligned}
\int 4\sqrt{x} - \frac{4}{\sqrt{x}}\,dx &= 4\int x^{\frac{1}{2}}\,dx - 4\int x^{-\frac{1}{2}}\,dx && (1)\\
&= 4\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - 4\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C && (2)\\
&= \frac{8x^{\frac{3}{2}}}{3} - 8x^{\frac{1}{2}} + C && (3)\\
&= \frac{8x\sqrt{x}}{3} - 8\sqrt{x} + C && (4)
\end{aligned}$$

20. (Section 4.9, Exercise 31)

$$\begin{aligned}
\int (3x+1)(4-x)\,dx &= \int -3x^2 + 11x + 4\,dx && (1)\\
&= -3\int x^2\,dx + 11\int x\,dx + 4\int x^0\,dx && (2)\\
&= -3\frac{x^3}{3} + 11\frac{x^2}{2} + 4x + C && (3)\\
&= -x^3 + \frac{11x^2}{2} + 4x + C && (4)
\end{aligned}$$

21. (Section 4.9, Exercise 35)

$$\begin{aligned}
\int \frac{4x^4 - 6x^2}{x}\,dx &= \int 4x^3 - 6x\,dx && (1)\\
&= 4\int x^3\,dx - 6\int x\,dx && (2)\\
&= 4\frac{x^4}{4} - 6\frac{x^2}{2} + C && (3)\\
&= x^4 - 3x^2 + C && (4)
\end{aligned}$$

22. (Section 4.9, Exercise 40)

$$\begin{aligned}
\int \left(\csc^2\theta + 1\right)d\theta &= \int \csc^2\theta\,d\theta + \int \theta^0\,d\theta && (1)\\
&= -\cot\theta + \theta + C && (2)
\end{aligned}$$

23. (Section 4.9, Exercise 41)

$$\begin{aligned}
\int \frac{2 + 3\cos y}{\sin^2 y}\,dy &= \int \left(\frac{2}{\sin^2 y} + \frac{3\cos y}{\sin^2 y}\right)dy && (1)\\
&= \int \left(2\csc^2 y + 3\cot y\csc y\right)dy && (2)\\
&= 2\int \csc^2 y\,dy + 3\int \cot y\csc y\,dy && (3)\\
&= -2\cot y - 3\csc y + C && (4)
\end{aligned}$$

24. (Section 4.9, Exercise 47)

$$\begin{aligned}
\int \left(3t^2 + 2\csc^2 t\right) dt &= \int 3t^2\,dt + \int 2\csc^2 t\,dt && (1)\\
&= 3\int t^2\,dt + 2\int \csc^2 t\,dt && (2)\\
&= 3\frac{t^3}{3} - 2\cot t + C && (3)\\
&= t^3 - 2\cot t + C && (4)
\end{aligned}$$

25. (Section 4.9, Exercise 45)

$$\begin{aligned}
\int \left(sec^2\theta + \sec\theta\tan\theta\right) d\theta &= \int sec^2\theta\,d\theta + \int \sec\theta\tan\theta\,d\theta && (1)\\
&= \tan\theta + \sec\theta + C && (2)
\end{aligned}$$

26. (Section 4.9, Exercise 50)

$$\begin{aligned}
\int \frac{\csc^3 x + 1}{\csc x}\,dx &= \int \frac{\csc^3 x}{\csc x} + \frac{1}{\csc x}\,dx && (1)\\
&= \int \csc^2 x + \sin x\,dx && (2)\\
&= \int \csc^2 x\,dx + \int \sin x\,dx && (3)\\
&= -\cot x + -\cos x + C && (4)
\end{aligned}$$

27. (Section 4.9, Exercise 51)

$$\begin{aligned}
\int \frac{1}{2y}\,dy &= \frac{1}{2}\int \frac{1}{y}\,dy && (1)\\
&= \frac{1}{2}\ln|y| + C && (2)\\
&= \frac{\ln|y|}{2} + C && (3)
\end{aligned}$$

28. (Section 4.9, Exercise 53)

$$\begin{aligned}
\int \frac{6}{\sqrt{4-4x^2}}\,dx &= \int \frac{6}{\sqrt{4\left(1-x^2\right)}}\,dx && (1)\\
&= \int \frac{6}{2\sqrt{1-x^2}}\,dx && (2)\\
&= 3\int \frac{1}{\sqrt{1-x^2}}\,dx && (3)\\
&= 3\sin^{-1} x + C && (4)
\end{aligned}$$

29. (Section 4.9, Exercise 59)

$$\begin{aligned}
\int \frac{t+1}{t}\,dt &= \int \left(\frac{t}{t} + \frac{1}{t}\right) dt && (1)\\
&= \int \frac{t}{t}\,dt + \int \frac{1}{t}\,dt && (2)\\
&= \int t^0\,dt + \ln|t| + C && (3)\\
&= t + \ln|t| + C && (4)
\end{aligned}$$

30. (Section 4.9, Exercise 61)

$$\begin{aligned}
\int e^{x+2}\,dx &= \int e^2 e^x\,dx && (1)\\
&= e^2\int e^x\,dx && (2)\\
&= e^2 e^x && (3)\\
&= e^{x+2} && (4)
\end{aligned}$$

31. (Section 4.9, Exercise 54)

$$\begin{aligned}
\int \frac{v^3+v+1}{1+v^2}\,dv &= \int \frac{v^3+v}{1+v^2} + \frac{1}{1+v^2}\,dv && (1)\\
&= \int v + \frac{1}{1+v^2}\,dv && (2)\\
&= \int v\,dv + \int \frac{1}{1+v^2}\,dv && (3)\\
&= \frac{v^2}{2} + \tan^{-1} v + C && (4)
\end{aligned}$$

32. (Section 4.9, Exercise 62)

$$\begin{aligned}
\int \frac{10t^5-3}{t}\,dt &= \int \left(\frac{10t^5}{t} - \frac{3}{t}\right) dt && (1)\\
&= \int 10t^4\,dt - \int \frac{3}{t}\,dt && (2)\\
&= 10\int t^4\,dt - 3\int \frac{1}{t}\,dt && (3)\\
&= 10\frac{t^5}{5} - 3\ln|t| + C && (4)\\
&= 2t^5 - 3\ln|t| + C && (5)
\end{aligned}$$

33. (Section 4.9, Exercise 78)

$$\begin{aligned}
g'(x) &= 7x^6 - 4x^3 + 12 && (1)\\
g(1) &= 24 && (2)\\
g(x) &= \int 7x^6 - 4x^3 + 12\,dx && (3)\\
&= 7\int x^6\,dx - 4\int x^3\,dx + 12\int x^0\,dx && (4)\\
&= 7\frac{x^7}{7} - 4\frac{x^4}{4} + 12x + C && (5)\\
&= x^7 - x^4 + 12x + C && (6)\\
(1)^7 - (1)^4 + 12(1) + C &= 24 && (7)\\
1 - 1 + 12 + C &= 24 && (8)\\
12 + C &= 24 && (9)\\
C &= 12 && (10)\\
g(x) &= x^7 - x^4 + 12x + 12 && (11)
\end{aligned}$$

34. (Section 4.9, Exercise 83)

$$\begin{aligned}
y'(t) &= \frac{3}{t} + 6 && (1)\\
y(1) &= 8, t > 0 && (2)\\
y(t) &= \int \left(\frac{3}{t} + 6\right) dt && (3)\\
&= 3\int \frac{1}{t}\,dt + 6\int t^0\,dt && (4)\\
&= 3\ln|t| + 6t + C && (5)\\
3\ln 1 + 6(1) + C &= 8 && (6)\\
6 + C &= 8 && (7)\\
C &= 2 && (8)\\
y(t) &= 3\ln|t| + 6t + 2 && (9)
\end{aligned}$$

35. (Section 4.9, Exercise 105)

$$\begin{aligned}
v(t) &= \sin t && (1)\\
s(0) &= 0 && (2)\\
s(t) &= \int \sin t \, dt && (3)\\
&= -\cos t + C && (4)\\
-\cos 0 + C &= 0 && (5)\\
-1 + C &= 0 && (6)\\
C &= 1 && (7)\\
s(t) &= -\cos t + 1 && (8)
\end{aligned}$$

$$\begin{aligned}
V(t) &= \cos t && (1)\\
S(0) &= 0 && (2)\\
S(t) &= \int \cos t \, dt && (3)\\
&= \sin t + C && (4)\\
\sin 0 + C &= 0 && (5)\\
C &= 0 && (6)\\
S(t) &= \sin t && (7)
\end{aligned}$$

36. (Section 4.9, Exercise 106)

$$\begin{aligned}
v(t) &= e^t && (1)\\
s(0) &= 0 && (2)\\
s(t) &= \int e^t \, dt && (3)\\
&= e^t + C && (4)\\
e^0 + C &= 0 && (5)\\
1 + C &= 0 && (6)\\
C &= -1 && (7)\\
s(t) &= e^t - 1 && (8)
\end{aligned}$$

$$\begin{aligned}
V(t) &= 2 + \cos t && (1)\\
S(0) &= 3 && (2)\\
S(t) &= \int 2 + \cos t \, dt && (3)\\
&= \int 2 \, dt + \int \cos t \, dt && (4)\\
&= 2 \int t^0 \, dt + \sin t + C && (5)\\
&= 2t + \sin t + C && (6)\\
2(0) + \sin 0 + C &= 3 && (7)\\
C &= 3 && (8)
\end{aligned}$$

37. (Section 5.1, Exercise 3)

$$4(40) + 4(70) = 440$$

$$2(30) + 2(50) + 2(80) + 2(40) = 400$$

38. (Section 5.1, Exercise 15)

$$v = 3t^2 + 1$$

$$0 \leq t \leq 4$$

$$f(0.5) + f(1.5) + f(2.5) + f(3.5) = 1.75 + 7.75 + 19.75 + 37.75 \tag{1}$$
$$= 67 \tag{2}$$
$$\frac{f(0.25)}{2} + \frac{f(0.75)}{2} + \frac{f(1.25)}{2} + \frac{f(1.75)}{2} + \frac{f(2.25)}{2} + \frac{f(2.75)}{2} + \frac{f(3.25)}{2} + \frac{f(3.75)}{2} = 67.75 \tag{3}$$

39. (Section 5.1, Exercise 16)

$$v = \sqrt{10t}$$
$$1 \leq t \leq 7$$

(a)

$$n = 3$$
$$[1, 3], [3, 5], [5, 7]$$
$$2, 4, 6$$
$$d \approx 2v(2) + 2v(4) + 2v(6) = 2\sqrt{20} + 2\sqrt{40} + 2\sqrt{60} \approx 37.085$$

(b)

$$n = 6$$
$$1.5, 2.5, 3.5, 4.5, 5.5, 6.5$$

$$d \approx v(1.5) + v(2.5) + v(3.5) + v(4.5) + v(5.5) + v(6.5) \tag{1}$$
$$= \sqrt{15} + \sqrt{25} + \sqrt{35} + \sqrt{45} + \sqrt{55} + \sqrt{65} \tag{2}$$
$$\approx 36.976 \tag{3}$$

40. (Section 5.1, Exercise 23)

$$f(x) = x + 1$$
$$[1, 6]$$
$$n = 5$$

(a)

$$(1 + 1) + (2 + 1) + (3 + 1) + (4 + 1) + (5 + 1) = 2 + 3 + 4 + 5 + 6 \tag{1}$$
$$= 20 \tag{2}$$

(b)

$$(2 + 1) + (3 + 1) + (4 + 1) + (5 + 1) + (6 + 1) = 3 + 4 + 5 + 6 + 7 \tag{1}$$
$$= 25 \tag{2}$$

41. (Section 5.1, Exercise 24)

$$f(x) = \frac{1}{x}$$
$$[1, 5]$$
$$n = 4$$
$$\Delta x = 1$$

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12} \tag{1}$$
$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{77}{60} \tag{2}$$

42. (Section 5.1, Exercise 29)

$$f(x) = x^2 - 1$$
$$[2, 4]$$
$$n = 4$$
$$\Delta x = \frac{1}{2}$$
$$[2, 2.5], [2.5, 3], [3, 3.5], [3.5, 4]$$

$$\frac{3}{2} + \frac{5.25}{2} + \frac{8}{2} + \frac{11.25}{2} = 13.75 \quad (1)$$
$$\frac{5.25}{2} + \frac{8}{2} + \frac{11.25}{2} + \frac{15}{2} = 19.75 \quad (2)$$

43. (Section 5.1, Exercise 33)

$$f(x) = 100 - x^2$$
$$[0, 10]$$
$$n = 5$$
$$\Delta x = 2$$

$$\begin{aligned} 2(99) + 2(91) + 2(75) + 2(51) + 2(19) &= 198 + 182 + 150 + 102 + 38 && (1) \\ &= 670 && (2) \end{aligned}$$

44. (Section 5.1, Exercise 34)

$$f(t) = \cos\frac{t}{2}$$
$$[0, \pi]$$
$$n = 4$$
$$\Delta x = \frac{\pi}{4}$$

$$\begin{aligned} \frac{\pi}{4}\left(\cos\frac{\pi}{16}\right) + \frac{\pi}{4}\left(\cos\frac{3\pi}{16}\right) + \frac{\pi}{4}\left(\cos\frac{5\pi}{16}\right) + \frac{\pi}{4}\left(\cos\frac{7\pi}{16}\right) &= \frac{704}{105} && (1) \\ &\approx 6.704761905 && (2) \end{aligned}$$

45. (Section 5.1, Exercise 39)

$$f(x) = \sqrt{x}$$
$$[1, 3]$$
$$n = 4$$
$$\Delta x = \frac{1}{2}$$

$$\frac{\sqrt{1.25}}{2} + \frac{\sqrt{1.75}}{2} + \frac{\sqrt{2.25}}{2} + \frac{\sqrt{2.75}}{2} \approx 2.853 \quad (1)$$

46. (Section 5.1, Exercise 43)

$$n = 4$$
$$[0, 2]$$
$$\Delta x = \frac{1}{2}$$

$$\frac{5}{2} + \frac{3}{2} + \frac{2}{2} + \frac{1}{2} = 5.5 \quad (1)$$
$$\frac{3}{2} + \frac{2}{2} + \frac{1}{2} + \frac{1}{2} = 3.5 \quad (2)$$

47. (Section 5.1, Exercise 44)

$$n = 8$$

$$[1, 5]$$

$$\Delta x = \frac{1}{2}$$

$$\frac{0}{2} + \frac{2}{2} + \frac{3}{2} + \frac{2}{2} + \frac{2}{2} + \frac{1}{2} + \frac{0}{2} + \frac{2}{2} = 6 \qquad (1)$$

$$\frac{2}{2} + \frac{3}{2} + \frac{2}{2} + \frac{2}{2} + \frac{1}{2} + \frac{0}{2} + \frac{2}{2} + \frac{3}{2} = 7.5 \qquad (2)$$

48. (Section 5.1, Exercise 51)

$$f(x) = 3\sqrt{x}$$

$$[0, 4]$$

$$n = 40$$

$$\Delta x = \frac{1}{10}$$

$$\sum_{k=1}^{40} \frac{3}{10}\sqrt{\frac{k-1}{10}} \approx 15.681 \qquad (1)$$

$$\sum_{k=1}^{40} \frac{3}{10}\sqrt{\frac{k}{10}} \approx 16.281 \qquad (2)$$

$$\sum_{k=1}^{40} \frac{3}{10}\sqrt{\frac{k-\frac{1}{2}}{10}} \approx 16.005 \qquad (3)$$

49. (Section 5.1, Exercise 52)

$$f(x) = x^2 + 1$$

$$[-1, 1]$$

$$n = 50$$

$$\Delta x = \frac{1}{25}$$

$$\sum_{k=1}^{50} 0.04\left((-1 + 0.04\,(k-1))^2 + 1\right) \approx 2.6672 \qquad (1)$$

$$\sum_{k=1}^{50} 0.04\left((-1 + 0.04k)^2 + 1\right) \approx 2.6672 \qquad (2)$$

$$\sum_{k=1}^{50} 0.04\left(\left(-1 + 0.04\left(k - \frac{1}{2}\right)\right)^2 + 1\right) \approx 2.6664 \qquad (3)$$

50. (Section 5.2, Exercise 17)

$$f(x) = -2x - 1$$

$$[0, 4]$$

$$n = 4$$

$$-1 - 3 - 5 - 7 = -16 \qquad (1)$$

$$-3 - 5 - 7 - 9 = -24 \qquad (2)$$

$$0 - 4 - 6 - 8 = -18 \qquad (3)$$

51. (Section 5.2, Exercise 22)

$$f(x) = 8 - 2x^2$$
$$[0, 4]$$
$$n = 4$$

$$\begin{aligned}
8 + 6 + 0 - 10 &= 4 \qquad (1)\\
6 + 0 - 10 - 32 &= -24 \qquad (2)\\
7.5 + 3.5 - 4.5 - 16.5 &= -10 \qquad (3)
\end{aligned}$$

52. (Section 5.2, Exercise 35)

$$\lim_{\Delta \to 0} \sum_{k=1}^{n} \left(x_k^{*2} + 1\right) \Delta x_k$$
$$[0, 2]$$
$$f(x) = x^2 + 1$$
$$\int_0^2 \left(x^2 + 1\right)\, dx$$

53. (Section 5.2, Exercise 36)

$$\lim_{\Delta \to 0} \sum_{k=1}^{n} \left(4 - x_k^{*2}\right) \Delta x_k$$
$$[-2, 2]$$
$$f(x) = 4 - x^2$$
$$\int_{-2}^{2} \left(4 - x^2\right)\, dx$$

54. (Section 5.2, Exercise 59)

$$\int_0^a f(x)\, dx = 16$$

55. (Section 5.2, Exercise 62)

$$\int_0^c f(x)\, dx = 22$$

56. (Section 5.2, Exercise 51)

$$\int_0^4 3x\,(4 - x)\, dx = 32$$

$$\begin{aligned}
\int_4^0 3x\,(4 - x)\, dx &= -\int_0^4 3x\,(4 - x)\, dx \qquad (1)\\
&= -32 \qquad (2)\\
\int_0^4 x\,(x - 4)\, dx &= -\frac{32}{3} \qquad (3)\\
\int_4^0 6x\,(4 - x)\, dx &= -64 \qquad (4)\\
\int_0^8 3x\,(4 - x)\, dx &= \text{Not Possible} \qquad (5)
\end{aligned}$$

57. (Section 5.2, Exercise 52)

$$\int_1^4 f(x)\, dx = 8$$
$$\int_1^6 f(x)\, dx = 5$$

$$\begin{aligned}
\int_1^4 -3f(x)\,dx &= -3\cdot 8 && (1)\\
&= -24 && (2)\\
\int_1^4 3f(x)\,dx &= 3\cdot 8 && (3)\\
&= 24 && (4)\\
\int_6^4 12f(x)\,dx &= -12\int_4^6 f(x)\,dx && (5)\\
&= -12\cdot(5-8) && (6)\\
&= 36 && (7)\\
\int_4^6 3f(x)\,dx &= 3\int_4^6 f(x)\,dx && (8)\\
&= 3\,(5-8) && (9)\\
&= -9 && (10)
\end{aligned}$$

58. (Section 5.2, Exercise 82)

$$\int_0^2 \left(x^2-1\right)\,dx$$

$$\begin{aligned}
\lim_{n\to\infty}\sum_{k=1}^{n}\frac{2}{n}\left(\left(\frac{2k}{n}\right)^2-1\right) &= \lim_{n\to\infty}\frac{2}{n}\sum_{k=1}^{n}\left(\left(\frac{2k}{n}\right)^2-1\right) && (1)\\
&= \lim_{n\to\infty}\frac{2}{n}\left(\sum_{k=1}^{n}\left(\frac{2k}{n}\right)^2-\sum_{k=1}^{n}1\right) && (2)\\
&= \lim_{n\to\infty}\frac{2}{n}\left(\sum_{k=1}^{n}\left(\frac{4k^2}{n^2}\right)-n\right) && (3)\\
&= \lim_{n\to\infty}\frac{2}{n}\left(\frac{4}{n^2}\sum_{k=1}^{n}\left(k^2\right)-n\right) && (4)\\
&= \lim_{n\to\infty}\frac{2}{n}\left(\frac{4}{n^2}\left(\frac{n(n+1)(2n+1)}{6}\right)-n\right) && (5)\\
&= \lim_{n\to\infty}\frac{2}{n}\left(\frac{4n(n+1)(2n+1)}{6n^2}-n\right) && (6)\\
&= \lim_{n\to\infty}\frac{2}{n}\left(\frac{(n+1)(2n+1)-3n^2}{3n}\right) && (7)\\
&= \lim_{n\to\infty}\left(\frac{2\left((n+1)(2n+1)-3n^2\right)}{3n^2}\right) && (8)\\
&= \lim_{n\to\infty}\left(\frac{2\left(2n^2+3n+1\right)-6n^2}{3n^2}\right) && (9)\\
&= \lim_{n\to\infty}\left(\frac{4n^2+6n+2-6n^2}{3n^2}\right) && (10)\\
&= \lim_{n\to\infty}\left(\frac{-2n^2+6n+2}{3n^2}\right) && (11)\\
&= \lim_{n\to\infty}\left(\frac{-4n+6}{6n}\right) && (12)\\
&= \lim_{n\to\infty}\left(\frac{-4}{6}\right) && (13)\\
&= \frac{-4}{6} && (14)
\end{aligned}$$

59. (Section 5.2, Exercise 84)

$$\int_0^2 \left(x^3+x+1\right)\,dx$$

$$\begin{aligned}
\lim_{n\to\infty}\sum_{k=1}^{n}\frac{2}{n}\left(\left(\frac{2k}{n}\right)^3+\frac{2k}{n}+1\right) &= \lim_{n\to\infty}\frac{2}{n}\sum_{k=1}^{n}\left(\left(\frac{2k}{n}\right)^3+\frac{2k}{n}+1\right) && (1)\\
&= \lim_{n\to\infty}\frac{2}{n}\left(\sum_{k=1}^{n}\left(\frac{2k}{n}\right)^3+\sum_{k=1}^{n}\frac{2k}{n}+\sum_{k=1}^{n}1\right) && (2)\\
&= \lim_{n\to\infty}\frac{2}{n}\left(\sum_{k=1}^{n}\frac{8k^3}{n^3}+\frac{2}{n}\sum_{k=1}^{n}k+n\right) && (3)\\
&= \lim_{n\to\infty}\frac{2}{n}\left(\frac{8}{n^3}\sum_{k=1}^{n}k^3+\frac{2}{n}\cdot\frac{n(n+1)}{2}+n\right) && (4)\\
&= \lim_{n\to\infty}\frac{2}{n}\left(\frac{8}{n^3}\cdot\frac{n^2(n+1)^2}{4}+n+1+n\right) && (5)\\
&= \lim_{n\to\infty}\frac{2}{n}\left(\frac{2(n+1)^2}{n}+2n+1\right) && (6)\\
&= \lim_{n\to\infty}\frac{2}{n}\left(\frac{2n^2+4n+2+2n^2+n}{n}\right) && (7)\\
&= \lim_{n\to\infty}\left(\frac{4n^2+8n+4+4n^2+2n}{n^2}\right) && (8)\\
&= \lim_{n\to\infty}\left(\frac{8n^2+10n+4}{n^2}\right) && (9)\\
&= \lim_{n\to\infty}\left(\frac{16n+10}{2n}\right) && (10)\\
&= \lim_{n\to\infty}\left(\frac{16}{2}\right) && (11)\\
&= 8 && (12)
\end{aligned}$$

60. (Section 5.3, Exercise 13)

$$A(x)=\int_{-2}^{x}f(t)\,dt$$

$$F(x)=\int_{4}^{x}f(t)\,dt$$

$$\begin{aligned}
A(-2) &= \int_{-2}^{-2}f(t)\,dt && (1)\\
&= 0 && (2)\\
F(8) &= \int_{4}^{8}f(t)\,dt && (3)\\
&= -9 && (4)\\
A(4) &= \int_{-2}^{4}f(t)\,dt && (5)\\
&= 25 && (6)\\
F(4) &= \int_{4}^{4}f(t)\,dt && (7)\\
&= 0 && (8)\\
A(8) &= \int_{-2}^{8}f(t)\,dt && (9)\\
&= 16 && (10)
\end{aligned}$$

61. (Section 5.3, Exercise 14)

$$A(x)=\int_{0}^{x}f(t)\,dt$$

$$F(x)=\int_{2}^{x}f(t)\,dt$$

$$\begin{aligned}
A(2) &= \int_0^2 f(t)\,dt && (1)\\
&= 8 && (2)\\
F(5) &= \int_2^5 f(t)\,dt && (3)\\
&= -5 && (4)\\
A(0) &= \int_0^0 f(t)\,dt && (5)\\
&= 0 && (6)\\
F(8) &= \int_2^8 f(t)\,dt && (7)\\
&= -16 && (8)\\
A(8) &= \int_0^8 f(t)\,dt && (9)\\
&= -8 && (10)\\
A(5) &= \int_0^5 f(t)\,dt && (11)\\
&= 3 && (12)\\
F(2) &= \int_2^2 f(t)\,dt && (13)\\
&= 0 && (14)
\end{aligned}$$

62. (Section 5.3, Exercise 23)

$$\int_0^1 \left(x^2 - 2x + 3\right)\,dx$$

$$\begin{aligned}
\int x^2 - 2\int x + 3\int x^0 &= \frac{x^3}{3} - 2\frac{x^2}{2} + 3x && (1)\\
&= \frac{x^3}{3} - x^2 + 3x && (2)\\
\int_0^1 \left(x^2 - 2x + 3\right)\,dx &= \frac{1^3}{3} - 1^2 + 3(1) && (3)\\
&= \frac{1}{3} - 1 + 3 && (4)\\
&= \frac{1}{3} + 2 && (5)
\end{aligned}$$

63. (Section 5.3, Exercise 24)

$$\int_{-\frac{\pi}{4}}^{\frac{7\pi}{4}} (\sin x + \cos x)\,dx$$

$$\begin{aligned}
\int (\sin x + \cos x)\,dx &= \int \sin x\,dx + \int \cos x\,dx && (1)\\
&= -\cos x + \sin x && (2)\\
\int_{-\frac{\pi}{4}}^{\frac{7\pi}{4}} (\sin x + \cos x)\,dx &= \left(-\cos\frac{7\pi}{4} + \sin\frac{7\pi}{4}\right) - \left(-\cos -\frac{\pi}{4} + \sin -\frac{\pi}{4}\right) && (3)\\
&= -\frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}} && (4)\\
&= 0 && (5)
\end{aligned}$$

64. (Section 5.3, Exercise 26)

$$\int_0^1 \left(x - \sqrt{x}\right)\,dx$$

$$\begin{aligned}
\int \left(x - \sqrt{x}\right)\, dx &= \int x\, dx - \int x^{\frac{1}{2}} && (1)\\
&= \frac{x^2}{2} - \frac{2^{\frac{3}{2}}}{3} && (2)\\
\int_0^1 \left(x - \sqrt{x}\right)\, dx &= \frac{1^2}{2} - \frac{2\,(1)^{\frac{2}{3}}}{3} && (3)\\
&= \frac{1}{2} - \frac{2}{3} && (4)\\
&= -\frac{1}{6} && (5)
\end{aligned}$$

65. (Section 5.3, Exercise 36)

$$\int_4^9 \frac{2+\sqrt{t}}{\sqrt{t}}\, dt$$

$$\begin{aligned}
\int \frac{2+\sqrt{t}}{\sqrt{t}} &= \int 2\left(t^{-\frac{1}{2}}\right) + 1 && (1)\\
&= 2\int \left(t^{-\frac{1}{2}}\right) + \int x^0 && (2)\\
&= 2\left(2\sqrt{t}\right) + x && (3)\\
&= 4\sqrt{t} + x && (4)\\
\int_4^9 \frac{2+\sqrt{t}}{\sqrt{t}}\, dt &= 4\sqrt{9} + 9 - 4\sqrt{4} - 4 && (5)\\
&= 4(3) + 9 - 4(2) - 4 && (6)\\
&= 12 + 9 - 8 - 4 && (7)\\
&= 9 && (8)
\end{aligned}$$

66. (Section 5.3, Exercise 68)

$$\int_2^4 x^2 - 25$$

$$\begin{aligned}
\int x^2 - 25 &= \int x^2 - 25 \int x^0 && (1)\\
&= \frac{x^3}{3} - 25x && (2)\\
\int_2^4 x^2 - 25 &= \left(\frac{4^3}{3} - 25(4)\right) - \left(\frac{2^3}{3} - 25(2)\right) && (3)\\
&= \frac{64}{3} - 100 - \frac{8}{3} + 50 && (4)\\
&= -\frac{94}{3} && (5)
\end{aligned}$$

67. (Section 5.3, Exercise 70)

$$\int_{-1}^2 x\,(x+1)\,(x-2)$$

$$\begin{aligned}
\int x(x+1)(x-2) &= \int x^3 - x^2 - 2x & (1)\\
&= \int x^3 - \int x^2 - 2\int x & (2)\\
&= \frac{x^4}{4} - \frac{x^3}{3} - 2\frac{x^2}{2} & (3)\\
&= \frac{x^4}{4} - \frac{x^3}{3} - x^2 & (4)\\
\int_{-1}^{2} x(x+1)(x-2) &= \frac{2^4}{4} - \frac{2^3}{3} - 2^2 - \frac{(-1)^4}{4} + \frac{(-1)^3}{3} + (-1)^2 & (5)\\
&= \frac{16}{4} - \frac{8}{3} - 4 - \frac{1}{4} + \frac{-1}{3} + 1 & (6)\\
&= 4 - \frac{8}{3} - 4 - \frac{1}{4} - \frac{1}{3} + 1 & (7)\\
&= -\frac{9}{4} & (8)
\end{aligned}$$

68. (Section 5.3, Exercise 73)

$$\frac{d}{dx}\int_3^x \left(t^2+t+1\right)\,dt$$

$$\frac{d}{dx}\int_3^x \left(t^2+t+1\right)\,dt = x^2+x+1 \qquad (1)$$

69. (Section 5.3, Exercise 75)

$$\frac{d}{dx}\int_x^1 \sqrt{t^4+1}\,dt$$

$$\begin{aligned}
\frac{d}{dx}\int_x^1 \sqrt{t^4+1}\,dt &= \frac{d}{dx} - \int_1^x \sqrt{t^4+1}\,dt & (1)\\
&= -\sqrt{x^4+1} & (2)
\end{aligned}$$

70. (Section 5.3, Exercise 78)

$$\frac{d}{dx}\int_0^{x^2} \frac{dt}{t^2+4}$$

$$\frac{d}{dx}\int_0^{x^2} \frac{dt}{t^2+4} = \frac{2x}{x^4+4} \qquad (1)$$

71. (Section 5.3, Exercise 71)

$$\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin x$$

$$\begin{aligned}
\int \sin x &= -\cos x & (1)\\
\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin x &= -\cos\frac{3\pi}{4} + \cos -\frac{\pi}{4} & (2)\\
&= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} & (3)\\
&= \frac{2}{\sqrt{2}} & (4)\\
&= \sqrt{2} & (5)
\end{aligned}$$

72. (Section 5.3, Exercise 72)

$$\int_{\frac{\pi}{2}}^{\pi} \cos x$$

$$\int \cos x = \sin x \tag{1}$$

$$\int_{\frac{\pi}{2}}^{\pi} \cos x = \sin \pi - \sin \frac{\pi}{2} \tag{2}$$

$$= -1 \tag{3}$$

A copy of my notes (in LaTeX) are available on my GitHub