

Module 3 Notes (MATH-211)

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24 June 2024

General Notes (and Definitions)

- The Chain Rule

Suppose $y = f(u)$ is differentiable at $u = g(x)$ and $u = g(x)$ is differentiable at x . The composite function $y = f(g(x))$ is differentiable at x , and its derivative can be expressed in two equivalent ways.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad (1)$$

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x) \quad (2)$$

Application of the Chain Rule (Assume the differentiable function $y = f(g(x))$ is given):

1. Identify an outer function f and an inner function g , and let $u = g(x)$.
2. Replace $g(x)$ with u to express y in terms of u :

$$y = f(g(x)) = f(u)$$

3. Calculate the product

$$\frac{dy}{du} \cdot \frac{du}{dx}$$

4. Replace u with $g(x)$ in $\frac{dy}{du}$ to obtain $\frac{dy}{dx}$

If g is differentiable for all x in its domain and $p \in \mathbb{R}$,

$$\frac{d}{dx}((g(x))^p) = p(g(x))^{p-1} g'(x)$$

- Implicit Differentiation

When we are unable to solve for y explicitly, we treat y as a function of x ($y = y(x)$) and apply the Chain Rule:

$$y' = \frac{dy}{dx}$$

$$\frac{d}{dx}y^n = ny^{n-1}\frac{dy}{dx}$$

Examples

1. The Chain Rule

$$y = (5x^2 + 11x)^{\frac{4}{3}} \quad (1)$$

$$u = 5x^2 + 11x \quad (2)$$

$$f = u^{\frac{4}{3}} \quad (3)$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad (4)$$

$$= \frac{4}{3}u^{\frac{1}{3}} \cdot 10x + 11 \quad (5)$$

$$= \frac{4}{3}(5x^2 + 11x)^{\frac{1}{3}} \cdot 10x + 11 \quad (6)$$

$$(7)$$

$$y = e^{4x^2+1} \quad (1)$$

$$u = 4x^2 + 1 \quad (2)$$

$$y = e^u \quad (3)$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad (4)$$

$$= e^u \cdot 8x \quad (5)$$

$$= e^{4x^2+1} \cdot 8x \quad (6)$$

$$= 8xe^{4x^2+1} \quad (7)$$

2. The Chain Rule (with Tables)

$$h(x) = f(g(x))$$

$$y = f(g(x)) \quad (1)$$

$$u = g(x) \quad (2)$$

$$y = f(u) \quad (3)$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad (4)$$

$$= f(u) \cdot g'(x) \quad (5)$$

$$= f'(g(x)) \cdot g'(x) \quad (6)$$

$$h'(1) = f'(g(1)) \cdot g'(1) \quad (7)$$

$$= f'(4) \cdot 9 \quad (8)$$

$$= 7 \cdot 9 \quad (9)$$

$$= 63 \quad (10)$$

$$h'(2) = f'(g(2)) \cdot g'(2) \quad (11)$$

$$= f'(1) \cdot 7 \quad (12)$$

$$= -6 \cdot 7 \quad (13)$$

$$= -42 \quad (14)$$

$$h'(3) = f'(g(3)) \cdot g'(3) \quad (15)$$

$$= f'(5) \cdot 3 \quad (16)$$

$$= 2 \cdot 3 \quad (17)$$

$$= 6 \quad (18)$$

$$k(x) = g(g(x))$$

$$y = g(g(x)) \quad (1)$$

$$u = g(x) \quad (2)$$

$$y = g(u) \quad (3)$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad (4)$$

$$= g'(u) \cdot g'(x) \quad (5)$$

$$= g'(g(x)) \cdot g'(x) \quad (6)$$

$$k'(3) = g'(g(3)) \cdot g'(3) \quad (7)$$

$$= g'(5) \cdot 3 \quad (8)$$

$$= -5 \cdot 3 \quad (9)$$

$$= -15 \quad (10)$$

$$k'(1) = g'(g(1)) \cdot g'(1) \quad (11)$$

$$= g'(4) \cdot 9 \quad (12)$$

$$= -1 \cdot 9 \quad (13)$$

$$= -9 \quad (14)$$

$$k'(5) = g'(g(5)) \cdot g'(5) \quad (15)$$

$$= g'(3) \cdot -5 \quad (16)$$

$$= 3 \cdot -5 \quad (17)$$

$$= -15 \quad (18)$$

3. The Chain Rule (All Forms)

$$y = \sqrt[3]{2x^2 - x - 5} \quad (1)$$

$$u = 2x^2 - x - 5 \quad (2)$$

$$y = \sqrt[3]{u} \quad (3)$$

$$y' = u^{-\frac{2}{3}} \cdot 4x - 1 \quad (4)$$

$$= \frac{1}{3} (2x^2 - x - 5)^{-\frac{2}{3}} \cdot 4x - 1 \quad (5)$$

$$y = \csc(\tan t) \quad (1)$$

$$u = \tan t \quad (2)$$

$$y = \csc u \quad (3)$$

$$y' = -\csc u \cot u \cdot \sec^2 t \quad (4)$$

$$= -\csc(\tan t) \cot(\tan t) \cdot \sec^2 t \quad (5)$$

4. The Chain Rule (Nested)

$$y = \tan(\sin e^x) \quad (1)$$

$$u_2 = e^x \quad (2)$$

$$u_1 = \sin u_2 \quad (3)$$

$$y = \tan u_1 \quad (4)$$

$$y' = \sec^2(\sin e^x) \cdot \cos e^x \cdot e^x \quad (5)$$

5. The Chain Rule (Combination of Rules)

$$y = \left(\frac{e^x}{x+1} \right)^8 \quad (1)$$

$$y' = 8 \left(\frac{e^x}{x+1} \right)^7 \cdot \frac{xe^x}{(x+1)^2} \quad (2)$$

$$= 8 \frac{e^{7x}}{(x+1)^7} \cdot \frac{xe^x}{(x+1)^2} \quad (3)$$

$$= \frac{8xe^{8x}}{(x+1)^9} \quad (4)$$

6. Implicit Differentiation

$$x^4 + y^4 = 2 \quad (1)$$

$$4x^3 + 4y^3 \frac{dy}{dx} = 0 \quad (2)$$

$$4y^3 \frac{dy}{dx} = -4x^3 \quad (3)$$

$$\frac{dy}{dx} = \frac{-4x^3}{4y^3} \quad (4)$$

$$= \frac{-x^3}{y^3} \quad (5)$$

$$\left. \frac{dy}{dx} \right|_{(1,-1)} = \frac{-(1)^3}{(-1)^3} = \frac{-1}{-1} = 1$$

7. Implicit Differentiation (Finding y)

$$y = ye^y \quad (1)$$

$$y' = e^y + ye^y y' \quad (2)$$

$$y' - y'xe^y = e^y \quad (3)$$

$$y'(1 - xe^y) = e^y \quad (4)$$

$$y' = \frac{e^y}{1 - xe^y} \quad (5)$$

8. Implicit Differentiation (Tangent Line)

$$\cos(x-y) + \sin y = \sqrt{2} \quad (1)$$

$$(-\sin(x-y))(1-y') + \cos y (y') = 0 \quad (2)$$

$$-\sin(x-y) + y' \sin(x-y) + y' \cos y = 0 \quad (3)$$

$$y'(\sin(x-y) + \cos y) = \sin(x-y) \quad (4)$$

$$y' = \frac{\sin(x-y)}{\sin(x-y) + \cos y} \quad (5)$$

$$\left. y' \right|_{\left(\frac{\pi}{2}, \frac{\pi}{4}\right)} = \frac{\sin\left(\frac{\pi}{2} - \frac{\pi}{2}\right)}{\sin\left(\frac{\pi}{2} - \frac{\pi}{2}\right) + \cos \frac{\pi}{4}} \quad (6)$$

$$= \frac{1}{2} \quad (7)$$

$$y = \frac{1}{2}x \quad (8)$$

9. Implicit Differentiation (Higher Order)

$$x^4 + y^4 = 64 \quad (1)$$

$$4x^3 + 4y^3 \frac{dy}{dx} = 0 \quad (2)$$

$$4y^3 \frac{dy}{dx} = -4x^3 \quad (3)$$

$$\frac{dy}{dx} = \frac{-4x^3}{4y^3} \quad (4)$$

$$= \frac{-x^3}{y^3} \quad (5)$$

$$\frac{d^2y}{dx^2} = \frac{-3x^2y^3 - \left(-x^3 3y^2 \frac{dy}{dx}\right)}{(y^3)^2} \quad (6)$$

$$= \frac{-3x^2y^3 + 3x^3y^2 \frac{dy}{dx}}{y^6} \quad (7)$$

$$= \frac{-3x^2y^3 + 3x^3y^2 \frac{-x^3}{y^3}}{y^6} \quad (8)$$

$$= \frac{-3x^2y^3 + \frac{-3x^6y^2}{y^3}}{y^6} \quad (9)$$

$$= \frac{-3x^2y^3 - \frac{3x^6}{y}}{y^6} \quad (10)$$

$$= \frac{\frac{-3x^2y^4 - 3x^6}{y}}{y^6} \quad (11)$$

$$= \frac{-3x^2y^4 - 3x^6}{y^7} \quad (12)$$

Related Exercises

1. (Section 3.7 Exercise 15)

$$y = (3x + 7)^{10} \quad (1)$$

$$u = 3x + 7 \quad (2)$$

$$f(u) = u^{10} \quad (3)$$

$$y' = 10u^9 \cdot 3 \quad (4)$$

$$= 10(3x + 7)^9 \cdot 3 \quad (5)$$

$$= 30(3x + 7)^9 \quad (6)$$

2. (Section 3.7 Exercise 17)

$$y = \sin^5 x \quad (1)$$

$$u = \sin x \quad (2)$$

$$f(u) = u^5 \quad (3)$$

$$y' = 5u^4 \cdot \cos x \quad (4)$$

$$= 5 \sin^4 x \cos x \quad (5)$$

3. (Section 3.7 Exercise 28)

$$y = (x^2 + 2x + 7)^8 \quad (1)$$

$$u = x^2 + 2x + 7 \quad (2)$$

$$f(u) = u^8 \quad (3)$$

$$y' = 8u^7 \cdot (2x + 2) \quad (4)$$

$$= 8(x^2 + 2x + 7)^7 \cdot (2x + 2) \quad (5)$$

$$= (16x + 16)(x^2 + 2x + 7)^7 \quad (6)$$

4. (Section 3.7 Exercise 29)

$$y = \sqrt{10x+1} \quad (1)$$

$$u = 10x+1 \quad (2)$$

$$f(u) = \sqrt{u} \quad (3)$$

$$y' = \frac{1}{2\sqrt{u}} \cdot 10 \quad (4)$$

$$= \frac{1}{2\sqrt{10x+1}} \cdot 10 \quad (5)$$

$$= \frac{10}{2\sqrt{10x+1}} \quad (6)$$

$$= \frac{5}{\sqrt{10x+1}} \quad (7)$$

5. (Section 3.7 Exercise 41)

$$y = \sqrt[4]{\frac{2x}{4x-3}} \quad (1)$$

$$u = \frac{2x}{4x-3} \quad (2)$$

$$f(u) = \sqrt[4]{u} \quad (3)$$

$$y' = \frac{1}{4}u^{-\frac{3}{4}} \cdot -\frac{6}{(4x-3)^2} \quad (4)$$

$$= \frac{1}{4} \left(\frac{2x}{4x-3} \right)^{-\frac{3}{4}} \cdot -\frac{6}{(4x-3)^2} \quad (5)$$

$$= -\frac{6}{4(4x-3)^2} \left(\frac{2x}{4x-3} \right)^{-\frac{3}{4}} \quad (6)$$

6. (Section 3.7 Exercise 23)

$$y = \tan 5x^2 \quad (1)$$

$$u = 5x^2 \quad (2)$$

$$f(u) = \tan u \quad (3)$$

$$y' = \sec^2 u \cdot 10x \quad (4)$$

$$= \sec^2 5x^2 \cdot 10x \quad (5)$$

$$= 10x \sec^2 5x^2 \quad (6)$$

$$(7)$$

7. (Section 3.7 Exercise 24)

$$y = \sin \frac{x}{4} \quad (1)$$

$$u = \frac{x}{4} \quad (2)$$

$$f(u) = \sin u \quad (3)$$

$$y' = \cos u \cdot \frac{1}{4} \quad (4)$$

$$= \cos \frac{x}{4} \cdot \frac{1}{4} \quad (5)$$

$$= \frac{1}{4} \cos \frac{x}{4} \quad (6)$$

8. (Section 3.7 Exercise 45)

$$y = (2x^6 - 3x^3 + 3)^{25} \quad (1)$$

$$u = 2x^6 - 3x^3 + 3 \quad (2)$$

$$f(u) = u^{25} \quad (3)$$

$$y' = 25(u)^{24} \cdot 12x^5 - 9x^2 \quad (4)$$

$$= 25(2x^6 - 3x^3 + 3)^{24} \cdot 12x^5 - 9x^2 \quad (5)$$

$$= 25(12x^5 - 9x^2)(2x^6 - 3x^3 + 3)^{24} \quad (6)$$

9. (Section 3.7 Exercise 46)

$$y = (\cos x + 2 \sin x)^8 \quad (1)$$

$$u = \cos x + 2 \sin x \quad (2)$$

$$f(u) = u^8 \quad (3)$$

$$y' = 8u^7 \cdot (-\sin x + 2 \cos x) \quad (4)$$

$$= 8(\cos x + 2 \sin x)^7 \cdot (-\sin x + 2 \cos x) \quad (5)$$

$$= 8(-\sin x + 2 \cos x)(\cos x + 2 \sin x)^7 \quad (6)$$

$$(7)$$

10. (Section 3.7 Exercise 53)

$$y = \sin(\sin(e^x)) \quad (1)$$

$$y' = \cos(\sin e^x) \cos e^x e^x \quad (2)$$

11. (Section 3.7 Exercise 54)

$$y = \sin^2 e^{3x+1} \quad (1)$$

$$y' = 6 \sin e^{3x+1} \cos e^{3x+1} \quad (2)$$

12. (Section 3.7 Exercise 68)

$$y = \left(\frac{3x}{4x+2} \right)^5 \quad (1)$$

$$y' = 5u^4 \cdot \frac{12x+6-12x}{(4x+2)^2} \quad (2)$$

$$= 5 \left(\frac{3x}{4x+2} \right)^4 \cdot \frac{6}{(4x+2)^2} \quad (3)$$

$$= \frac{30}{(4x+2)^2} \left(\frac{3x}{4x+2} \right)^4 \quad (4)$$

13. (Section 3.7 Exercise 69)

$$y = ((x+2)(x^2+1))^4 \quad (1)$$

$$y' = 4u^3 \cdot x^2 + 1 + 2x^2 + 4x \quad (2)$$

$$= 4(3x^2 + 4x + 1)((x+2)(x^2+1))^3 \quad (3)$$

$$= 4(3x+1)(x+1)((x+2)(x^2+1))^3 \quad (4)$$

14. (Section 3.8 Exercise 13)

$$x^4 + y^4 = 2 \quad (1)$$

$$(1, -1) \quad (2)$$

$$4x^3 + 4y^3 \frac{dy}{dx} = 0 \quad (3)$$

$$4y^3 \frac{dy}{dx} = -4x^3 \quad (4)$$

$$\frac{dy}{dx} = \frac{-4x^3}{4y^3} \quad (5)$$

$$= \frac{-x^3}{y^3} \quad (6)$$

$$\left. \frac{dy}{dx} \right|_{(1,-1)} = \frac{-(1^3)}{(-1)^3} \quad (7)$$

$$= \frac{-1}{-1} \quad (8)$$

$$= 1 \quad (9)$$

15. (Section 3.8 Exercise 15)

$$y^2 = 4x \quad (1)$$

$$(1, 2) \quad (2)$$

$$2y \frac{dy}{dx} = 4 \quad (3)$$

$$\frac{dy}{dx} = \frac{4}{2y} \quad (4)$$

$$= \frac{2}{y} \quad (5)$$

$$\left. \frac{dy}{dx} \right|_{(1,2)} = \frac{2}{2} \quad (6)$$

$$= 1 \quad (7)$$

16. (Section 3.8 Exercise 31)

$$\sin xy = x + y \quad (1)$$

$$\cos xy \cdot \left(y + x \frac{dy}{dx} \right) = 1 + \frac{dy}{dx} \quad (2)$$

$$y \cos xy + \frac{dy}{dx} x \cos xy = 1 + \frac{dy}{dx} \quad (3)$$

$$\frac{dy}{dx} x \cos xy - \frac{dy}{dx} = 1 - y \cos xy \quad (4)$$

$$\frac{dy}{dx} (x \cos xy - 1) = 1 - y \cos xy \quad (5)$$

$$\frac{dy}{dx} = \frac{1 - y \cos xy}{x \cos xy - 1} \quad (6)$$

17. (Section 3.8 Exercise 33)

$$\cos y^2 + x = e^y \quad (1)$$

$$-\frac{dy}{dx} 2y \sin y^2 + 1 = \frac{dy}{dx} e^y \quad (2)$$

$$\frac{dy}{dx} 2y \sin y^2 + \frac{dy}{dx} e^y = 1 \quad (3)$$

$$\frac{dy}{dx} (2y \sin y^2 + e^y) = 1 \quad (4)$$

$$\frac{dy}{dx} = \frac{1}{2y \sin y^2 + e^y} \quad (5)$$

18. (Section 3.8 Exercise 47)

$$x^2 + xy + y^2 = 7 \quad (1)$$

$$(2, 1) \quad (2)$$

$$2x + y + \frac{dx}{dy}x + \frac{dx}{dy}2y = 0 \quad (3)$$

$$\frac{dx}{dy}(x + 2y) = -2x - y \quad (4)$$

$$\frac{dx}{dy} = \frac{-2x - y}{x + 2y} \quad (5)$$

$$\left. \frac{dy}{dx} \right|_{(2,1)} = \frac{-2(2) - 1}{2 + 2(1)} \quad (6)$$

$$= \frac{-4 - 1}{2 + 2} \quad (7)$$

$$= \frac{-5}{4} \quad (8)$$

$$y = \frac{-5}{4}x - 2\frac{-5}{4} + 1 \quad (9)$$

$$= \frac{-5}{4}x + \frac{5}{2} + \frac{2}{2} \quad (10)$$

$$= \frac{-5}{4}x + \frac{7}{2} \quad (11)$$

19. (Section 3.8 Exercise 48)

$$x^4 - x^2y + y^4 = 1 \quad (1)$$

$$(-1, 1) \quad (2)$$

$$4x^3 - 2xy - \frac{dy}{dx}x^2 + \frac{dy}{dx}4y^3 = 0 \quad (3)$$

$$\frac{dy}{dx}(-x^2 + 4y^3) = 2xy - 4x^3 \quad (4)$$

$$\frac{dy}{dx} = \frac{2xy - 4x^3}{4y^3 - x^2} \quad (5)$$

$$\left. \frac{dy}{dx} \right|_{(-1,1)} = \frac{2(-1)(1) - 4(-1)^3}{4(1)^3 - (-1)^2} \quad (6)$$

$$= \frac{-2 + 4}{4 + 1} \quad (7)$$

$$= \frac{2}{5} \quad (8)$$

$$y = \frac{2}{5}x + \frac{2}{5} + 1 \quad (9)$$

$$= \frac{2}{5}x + \frac{7}{5} \quad (10)$$

20. (Section 3.8 Exercise 25)

$$x\sqrt[3]{y} + y = 10 \quad (1)$$

$$(1, 8) \quad (2)$$

$$\sqrt[3]{y} + \frac{dy}{dx} \left(\frac{x}{3y^{\frac{2}{3}}} \right) + \frac{dy}{dx} = 0 \quad (3)$$

$$\frac{dy}{dx} \left(\frac{x}{3y^{\frac{2}{3}}} + 1 \right) = -\sqrt[3]{y} \quad (4)$$

$$\frac{dy}{dx} = \frac{-3y^{\frac{2}{3}} \sqrt[3]{y}}{x + 3y^{\frac{2}{3}}} \quad (5)$$

$$= \frac{-3y}{3y^{\frac{2}{3}} + x} \quad (6)$$

$$\left. \frac{dy}{dx} \right|_{(1,8)} = \frac{-3(8)}{3(8)^{\frac{2}{3}} + 1} \quad (7)$$

$$= \frac{-24}{3(4) + 1} \quad (8)$$

$$= \frac{-24}{13} \quad (9)$$

21. (Section 3.8 Exercise 26)

$$(x + y)^{\frac{2}{3}} = y \quad (1)$$

$$(4, 4) \quad (2)$$

$$\frac{2}{3} (x + y)^{-\frac{1}{3}} \cdot \left(1 + \frac{dy}{dx} \right) = \frac{dy}{dx} \quad (3)$$

$$\frac{2}{3} (x + y)^{-\frac{1}{3}} + \frac{dy}{dx} \frac{2}{3} (x + y)^{-\frac{1}{3}} = \frac{dy}{dx} \quad (4)$$

$$\frac{2}{3} (x + y)^{-\frac{1}{3}} = \frac{dy}{dx} - \frac{dy}{dx} \frac{2}{3} (x + y)^{-\frac{1}{3}} \quad (5)$$

$$\frac{dy}{dx} \left(1 - \frac{2}{3} (x + y)^{-\frac{1}{3}} \right) = \frac{2}{3} (x + y)^{-\frac{1}{3}} \quad (6)$$

$$\frac{dy}{dx} = \frac{\frac{2}{3} (x + y)^{-\frac{1}{3}}}{1 - \frac{2}{3} (x + y)^{-\frac{1}{3}}} \quad (7)$$

$$\left. \frac{dy}{dx} \right|_{(4,4)} = \frac{\frac{2}{3} (4 + 4)^{-\frac{1}{3}}}{1 - \frac{2}{3} (4 + 4)^{-\frac{1}{3}}} \quad (8)$$

$$= \frac{\frac{2}{3} \frac{1}{2}}{1 - \frac{2}{3} \frac{1}{2}} \quad (9)$$

$$= \frac{\frac{1}{2}}{-\frac{1}{2}} \quad (10)$$

$$= -1 \quad (11)$$

22. (Section 3.8 Exercise 51)

$$x + y^2 = 1 \quad (1)$$

$$1 + \frac{dy}{dx} 2y = 0 \quad (2)$$

$$\frac{dy}{dx} 2y = -1 \quad (3)$$

$$\frac{dy}{dx} = \frac{-1}{2y} \quad (4)$$

$$\frac{d^2y}{dx^2} = \frac{-1}{2y} \quad (5)$$

$$= \frac{-1}{2} \frac{1}{y} \quad (6)$$

$$= \frac{-1}{2} \frac{dy}{dx} \frac{-1}{y^2} \quad (7)$$

$$= \frac{-1}{2} \frac{-1}{2y} \frac{-1}{y^2} \quad (8)$$

$$= \frac{-1}{4y^3} \quad (9)$$

23. (Section 3.8 Exercise 52)

$$2x^2 + y^2 = 4 \quad (1)$$

$$4x + 2y \frac{dy}{dx} = 0 \quad (2)$$

$$2y \frac{dy}{dx} = -4x \quad (3)$$

$$\frac{dy}{dx} = \frac{-4x}{2y} \quad (4)$$

$$= \frac{-2x}{y} \quad (5)$$

$$\frac{d^2y}{dx^2} = \frac{-2x}{y} \quad (6)$$

$$= -2x \frac{1}{y} \quad (7)$$

$$= -2x \frac{dy}{dx} \frac{-1}{y^2} \quad (8)$$

$$= -2x \frac{-2x}{y} \frac{-1}{y^2} \quad (9)$$

$$= \frac{-4x^2}{y^3} \quad (10)$$