Module 2 Notes (MATH-211)

Lillie Donato

17 June 2024

General Notes (and Definitions)

- Derivatives
  - A **derivative** is a new function made up of the slopes of the tangent lines as they change along a curve
  - If a curve represents the trajectory of a moving object, the tangent line at a point indicates the direction of motion at that point
  - As $x \to a$, the slope of the secant lines approaches the slope of the tangent line
  - Alternative definition for Tangent Line(s): Consider the curve $y = f(x)$ and a secant line intersecting the curve at points $P(a, f(a))$ and $Q(a+h, f(a+h))$, with m_{sec} and m_{tan}

$$\text{Interval:}(a, a+h)$$
$$m_{sec} = \frac{f(a+h) - f(a)}{h}$$
$$m_{tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$
$$y - f(a) = m_{tan}(x - a)$$

  - **Definition**: The derivative of f at a, denoted $f'(a)$, is given by either the two following limits, provided the limits exist and a is in the domain of f

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad (1)$$
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad (2)$$
$$(3)$$

If $f'(a)$ exists, we say that f is **differentiable** at a

Examples

1. Instantaneous Velocity

$$s(t) = -16t^2 + 128t + 192$$
$$t = 2$$

$$\begin{aligned}
\lim_{t \to 2} \frac{s(t) - s(2)}{t - 2} &= \lim_{t \to 2} \frac{(-16t^2 + 128t + 192) - (-16(2^2) + 128(2) + 192)}{t - 2} && (1)\\
&= \lim_{t \to 2} \frac{(-16t^2 + 128t + 192) - (-64 + 256 + 192)}{t - 2} && (2)\\
&= \lim_{t \to 2} \frac{(-16t^2 + 128t + 192) - 384}{t - 2} && (3)\\
&= \lim_{t \to 2} \frac{-16t^2 + 128t - 192}{t - 2} && (4)\\
&= \lim_{t \to 2} \frac{(t - 2)(-16t + 96)}{t - 2} && (5)\\
&= \lim_{t \to 2} -16t + 96 && (6)\\
&= -32 + 96 && (7)\\
&= 64 && (8)
\end{aligned}$$

2. Secant Lines

$$y = f(x)$$

Intersection Points: $P(a, f(a))$ and $Q(x, f(x))$

$$\text{Secant Line Slope} = \frac{f(x) - f(a)}{x - a}$$

3. Tangent Lines

$$f(x) = 2x^2 + 4x - 3$$

$$(-1, 5)$$

$$\begin{aligned}
\lim_{x \to -1} \frac{f(x) - f(-1)}{x - (-1)} &= \lim_{x \to -1} \frac{2x^2 + 4x - 3 - (-5)}{x + 1} && (1)\\
&= \lim_{x \to -1} \frac{2x^2 + 4x + 2}{x + 1} && (2)\\
&= \lim_{x \to -1} \frac{(x + 1)(2x + 2)}{x + 1} && (3)\\
&= \lim_{x \to -1} \frac{(x + 1)(2x + 2)}{x + 1} && (4)\\
&= \lim_{x \to -1} 2x + 2 && (5)\\
&= 2(-1) + 2 && (6)\\
&= -2 + 2 && (7)\\
&= 0 && (8)
\end{aligned}$$

4. Alternative Tangent Lines

$$f(x) = 5 - x^3$$

$$(2, -3)$$

$$a = 2$$

$$h = -3 - 2 = -5$$

$$\begin{aligned}
\lim_{h \to 0} \frac{f(2 + h) - f(2)}{h} &= \lim_{h \to 0} \frac{f(2 + h) - (-3)}{h} && (1)\\
&= \lim_{h \to 0} \frac{f(2 + h) + 3}{h} && (2)\\
&= \lim_{h \to 0} \frac{5 - (2 + h)^3 + 3}{h} && (3)\\
&= \lim_{h \to 0} \frac{8 - (2 + h)^3}{h} && (4)\\
&= \lim_{h \to 0} \frac{2^3 - (2 + h)^3}{h} && (5)\\
&= \lim_{h \to 0} \frac{(2 - (2 + h))(2^2 + 2(2 + h) + (2 + h)^2)}{h} && (6)\\
&= \lim_{h \to 0} \frac{-h(4 + 4 + 2h + h^2 + 4h + 4)}{h} && (7)\\
&= \lim_{h \to 0} \frac{-h(h^2 + 6h + 12)}{h} && (8)\\
&= \lim_{h \to 0} -(h^2 + 6h + 12) && (9)\\
&= -12 && (10)\\
& && (11)
\end{aligned}$$

$$y + 3 = -12(x - 2) = -12x + 24$$

$$y = -12x + 21$$

5. Derrivative Example

$$f(x) = \sqrt{x-1}$$

$$x = 2$$

$$f(x) = f(2) = \sqrt{2-1} = \sqrt{1} = 1$$

$$(2, 1)$$

$$\begin{aligned}
f'(2) &= \lim_{x\to 2} \frac{f(x) - f(2)}{x-2} && (1)\\
&= \lim_{x\to 2} \frac{\sqrt{x-1}-1}{x-2} && (2)\\
&= \lim_{x\to 2} \frac{\sqrt{x-1}-1}{x-2} \cdot \frac{\sqrt{x-1}+1}{\sqrt{x-1}+1} && (3)\\
&= \lim_{x\to 2} \frac{x-2}{(x-2)(\sqrt{x-1}+1)} && (4)\\
&= \lim_{x\to 2} \frac{1}{\sqrt{x-1}+1} && (5)\\
&= \frac{1}{\sqrt{2-1}+1} && (6)\\
&= \frac{1}{\sqrt{1}+1} && (7)\\
&= \frac{1}{1+1} && (8)\\
&= \frac{1}{2} && (9)\\
& && (10)
\end{aligned}$$

$$\begin{aligned}
y - 1 &= \frac{1}{2}(x-2) && (1)\\
y &= \frac{1}{2}(x-2) + 1 && (2)\\
&= \frac{1}{2}x - 1 + 1 && (3)\\
&= \frac{1}{2}x && (4)
\end{aligned}$$

6. Derrivative Application Example

$$V(t) = 3t$$

$$\begin{aligned}
V'(12) &= \lim_{x\to 12} \frac{V(x) - V(12)}{x-12} && (1)\\
&= \lim_{x\to 12} \frac{3x-36}{x-12} && (2)\\
&= \lim_{x\to 12} \frac{3(x-12)}{x-12} && (3)\\
&= \lim_{x\to 12} 3 && (4)\\
&= 3 && (5)
\end{aligned}$$

$$\begin{aligned}
y - 36 &= 3(x-12) && (1)\\
y &= 3x - 36 + 36 && (2)\\
&= 3x && (3)\\
& && (4)
\end{aligned}$$

Related Exercises

1. Example

A copy of my notes (in LaTeX) are available on my GitHub