

# Module 1 Notes (MATH-211)

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## General Notes (and Definitions)

- Limit Definition(s):

- Simple: The value that the outputs of a function approach as inputs approach a certain value
- Preliminary: Suppose a function  $f$  is defined for all  $x$  near  $a$  except possibly at  $a$ . If  $f(x)$  is arbitrarily close to  $L$  all  $x$  sufficiently close (but not equal) to  $a$ , we write the following.

$$\lim_{x \rightarrow a} = L$$

- Secant Line: a line passing through two points  $(t_0, s(t_0))$  and  $(t_1, s(t_1))$ . The slope is given by

$$\frac{s(t_1) - s(t_0)}{t_1 - t_0}$$

- Tangent Line: the line passing through  $(t_0, s(t_0))$  with slope

$$\lim_{t \rightarrow t_0} \frac{s(t) - s(t_0)}{t - t_0}$$

- One Sided limits:

- Right-hand (Definition): Suppose a function  $f$  is defined for all  $x$  near  $a$  with  $x > a$ . If  $f(x)$  is arbitrarily close to  $L$  for all  $x$  sufficiently close to  $a$  with  $x > a$  we write

$$\lim_{x \rightarrow a^+} f(x) = L$$

- Left-hand (Definition): Suppose a function  $f$  is defined for all  $x$  near  $a$  with  $x < a$ . If  $f(x)$  is arbitrarily close to  $L$  for all  $x$  sufficiently close to  $a$  with  $x < a$  we write

$$\lim_{x \rightarrow a^-} f(x) = L$$

- In order for there to be a double sided limit, we must have:

$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x)$$

- If the limits from sides are not equal, then a the double sided limit, "does not exist"

- Limits can be simplified/solved in an easier way (as compared to numerically/graphically) using Limit Rules/Laws

- Limit Example Types:

- Tangent lines
- Velocity

- Velocity

- Average Velocity

\* The average velocity over some interval  $[t_0, t_1]$  is defined as

$$v_{av} = \frac{s(t_1) - s(t_0)}{t_1 - t_0}$$

- Instantaneous Velocity

\* The average velocity over some interval  $[t_0, t_1]$  is defined as

$$v_{inst} = \lim_{t \rightarrow a} v_{av} = \frac{s(t) - s(a)}{t - a}$$

- Solving Techniques

- Factoring and canceling out

- Using conjugates

\* When direct substitution is not possible, you may rationalize the numerator

- Infinite Limits: In either case, the limit does not exist (not a real number) if it is infinite

- Suppose  $f$  is defined for all  $x$  near  $a$ . If  $f(x)$  grows arbitrarily large for all  $x$  sufficiently close (but not equal) to  $a$ , we write

$$\lim_{x \rightarrow a} f(x) = \infty$$

- If  $f(x)$  is negative and grows arbitrarily large in magnitude for all  $x$  sufficiently close (but not equal) to  $a$ , we write

$$\lim_{x \rightarrow a} f(x) = -\infty$$

- The line  $x = a$  is a vertical asymptote for  $f$  if any of the following hold

$$\lim_{x \rightarrow a} f(x) = \pm\infty$$

$$\lim_{x \rightarrow a^+} f(x) = \pm\infty$$

$$\lim_{x \rightarrow a^-} f(x) = \pm\infty$$

- A vertical asymptote exists at  $x = a$  if any one sided limit as  $x \rightarrow a$  is  $\infty$  or  $-\infty$

- If you have a limit of a rational function, where  $p(a) = L \neq 0$  and  $q(a) = 0$ , then the one sided limits for  $\frac{p(x)}{q(x)}$  approach  $\pm\infty$

$$\lim_{x \rightarrow a} \frac{p(x)}{q(x)} = \frac{L}{0}$$

- Limits as Infinity

- **Definition:** If  $f(x)$  becomes arbitrarily close to a finite number  $L$  for all sufficiently large and positive  $x$ , then we write

$$\lim_{x \rightarrow \infty} f(x) = L$$

The definition for

$$\lim_{x \rightarrow -\infty} f(x) = M$$

is analogous.

- If  $\lim_{x \rightarrow \infty} f(x) = L$  we say that the function  $f(x)$  has a horizontal asymptote at  $y = L$

- If  $\lim_{x \rightarrow -\infty} f(x) = M$  we say that the function  $f(x)$  has a horizontal asymptote at  $y = M$

- **Principle:** If  $n > 0$  is an integer then

$$\lim_{x \rightarrow \pm\infty} \frac{1}{x^n} = 0$$

- Suppose  $f(x) = \frac{p(x)}{q(x)}$  is a rational function where

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

$$q(x) = b_n x^n + b_{n-1} x^{n-1} + \dots + b_1 x + b_0$$

If the degree of  $p(x)$  is less than the degree of  $q(x)$  then

$$\lim_{x \rightarrow \pm\infty} f(x) = 0$$

If the degree of  $p(x)$  equals the degree of  $q(x)$  then

$$\lim_{x \rightarrow \pm\infty} f(x) = \frac{a_m}{b_n}$$

If the degree of  $p(x)$  is greater than the degree of  $q(x)$  then

$$\lim_{x \rightarrow \pm\infty} f(x) = -\infty \text{ or } \infty$$

If the graph of a function  $f$  approaches a line (with finite and nonzero slope) as  $x \rightarrow \pm\infty$ , then that line is a slant asymptote/oblique asymptote of  $f$

– End behaviour for transcendental functions

$$\lim_{x \rightarrow \pm\infty} \sin x = \text{Does not exist}$$

$$\lim_{x \rightarrow \infty} e^x = \infty$$

$$\lim_{x \rightarrow \infty} e^{-x} = \lim_{x \rightarrow \infty} \frac{1}{e^x} = 0$$

$$\lim_{x \rightarrow -\infty} e^x = 0$$

$$\lim_{x \rightarrow -\infty} e^{-x} = \infty$$

$$\lim_{x \rightarrow \infty} \ln x = \infty$$

$$\lim_{x \rightarrow 0^+} \ln x = -\infty$$

## Limit Rules/Laws

Assume  $\lim_{x \rightarrow a} f(x)$  and  $\lim_{x \rightarrow a} g(x)$  exist.

The following properties hold where  $c$  is a real number, and  $n > 0$  is an integer.

- **Sum Rule**

$$\lim_{x \rightarrow a} (f(x) + g(x)) = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$$

- **Difference Rule**

$$\lim_{x \rightarrow a} (f(x) - g(x)) = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$$

- **Constant Multiple Rule**

$$\lim_{x \rightarrow a} (cf(x)) = c \lim_{x \rightarrow a} f(x)$$

- **Product Rule**

$$\lim_{x \rightarrow a} (f(x)g(x)) = (\lim_{x \rightarrow a} f(x))(\lim_{x \rightarrow a} g(x))$$

- **Quotient Rule**

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}, \text{ provided } \lim_{x \rightarrow a} g(x) \neq 0$$

- **Power Rule**

$$\lim_{x \rightarrow a} f(x)^n = (\lim_{x \rightarrow a} f(x))^n$$

- **Root Rule**

$$\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)}, \text{ provided } f(x) > 0, \text{ for } x \text{ near } a, \text{ if } n \text{ is even}$$

- **Polynomials**

A **Polynomial** is defined as A function of the form  $x_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$  where  $n \geq 0$  is an integer If  $p(x)$  is a polynomial then:

$$\lim_{x \rightarrow a} p(x) = p(a)$$

If  $p(x)$  and  $q(x)$  are polynomials and  $q(a) \neq 0$  then (Direct Substitution):

$$\lim_{x \rightarrow a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$$

- **The Squeeze Theorem**

Assume for some functions  $f$ ,  $g$  and  $h$  that satisfy  $f(x) \leq g(x) \leq h(x)$  for  $x$  near  $a$  (except possibly at  $x = a$ ). If

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$$

then

$$\lim_{x \rightarrow a} g(x) = L$$

As  $x \rightarrow a$ ,  $h(x) \rightarrow L$ . Therefore,  $g(x) \rightarrow L$ . As  $x$  approaches  $a$ , if  $f$  and  $h$  approach the same value, so does  $g$ .

## Examples

1. (Describing Limits) As  $x$  approaches 3,  $x^2$  approaches 9

$$\lim_{x \rightarrow 3} x^2 = 9$$

2. (Common Use) Values that are undefined can still have limits, given a graph  $G$  where  $f(3) = \text{undefined}$  ( $f(3)$  is a hole), the following limit is valid:

$$\lim_{x \rightarrow 3} f(x) = 4$$

3. Calculating Limits Numerically:

$$f(x) = \frac{x^3 - 1}{x - 1}$$

0.9	0.99	0.999	0.9999
2.71	2.9701	2.997001	2.99970001

1.1	1.01	1.001	1.0001
3.31	3.0301	3.003001	3.00030001

As  $x$  approaches 1,  $f(x)$  approaches 3:  $\lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1} = 3$

4. Calculating One-sided limits:

$$g(x) = \frac{x^3 - 4x}{8|x - 2|}$$

1.9	1.09	1.009	1.0009
-0.92625	-0.9925125	-0.999250125	-0.9999250013

2.1	2.01	2.001	2.0001
1.07625	1.0075125	1.000750125	1.000075001

$$\lim_{x \rightarrow 2} g(x) = \text{Does not exist}$$

$$\lim_{x \rightarrow 2^-} g(x) = -1$$

$$\lim_{x \rightarrow 2^+} g(x) = 1$$

5. Calculating piecewise function limits

$$f(x) = \begin{cases} 3 - x & \text{if } x < 2 \\ x - 1 & \text{if } x > 2 \end{cases}$$

$$a = 2$$

1.92	1.99	1.999	1.9999
1.1	1.01	1.001	1.0001

2.1	2.01	2.001	2.0001
1.1	1.01	1.001	1.0001

**Explanation:** Since  $f(2)$  is not defined within the piece wise function, a graph representing this function would have a whole where  $x = a$  and have two lines with inverse slopes

$$f(a) = \text{undefined}$$

$$\lim_{x \rightarrow a} f(x) = 1$$

$$\lim_{x \rightarrow a^-} f(x) = 1$$

$$\lim_{x \rightarrow a^+} f(x) = 1$$

## 6. Limit Rules/Laws:

(a) Definitions:

$$\lim_{x \rightarrow 3} f(x) = 2$$

$$\lim_{x \rightarrow 3} g(x) = -1$$

$$\lim_{x \rightarrow 3} h(x) = 6$$

(b) Problems:

i. Sum, Constant Multiple

$$\lim_{x \rightarrow 3} (f(x) + 2g(x)) = \lim_{x \rightarrow 3} f(x) + \lim_{x \rightarrow 3} 2g(x) \quad (1)$$

$$= \lim_{x \rightarrow 3} f(x) + 2(\lim_{x \rightarrow 3} g(x)) \quad (2)$$

$$= 2 + 2(-1) \quad (3)$$

$$= 0 \quad (4)$$

ii. Quotient

$$\lim_{x \rightarrow 3} \frac{h(x)}{g(x)} = \frac{\lim_{x \rightarrow 3} h(x)}{\lim_{x \rightarrow 3} g(x)} \quad (1)$$

$$= \frac{6}{-1} \quad (2)$$

$$= -6 \quad (3)$$

iii. Quotient, Root, Difference

$$\lim_{x \rightarrow 3} \frac{h(x)}{\sqrt{f(x) - g(x)}} = \frac{\lim_{x \rightarrow 3} h(x)}{\lim_{x \rightarrow 3} \sqrt{f(x) - g(x)}} \quad (1)$$

$$= \frac{\lim_{x \rightarrow 3} h(x)}{\sqrt{\lim_{x \rightarrow 3} (f(x) - g(x))}} \quad (2)$$

$$= \frac{\lim_{x \rightarrow 3} h(x)}{\sqrt{\lim_{x \rightarrow 3} f(x) - \lim_{x \rightarrow 3} g(x)}} \quad (3)$$

$$= \frac{6}{\sqrt{2 + 1}} \quad (4)$$

$$= \frac{6}{\sqrt{3}} \quad (5)$$

$$= 2\sqrt{3} \quad (6)$$

7.

$$\lim_{x \rightarrow 1} \frac{3x^2 - 7x + 1}{x + 2} = \frac{3(1)^2 - 7(1) + 1}{1 + 2} \quad (1)$$

$$= \frac{3 - 7 + 1}{1 + 2} \quad (2)$$

$$= \frac{-3}{3} \quad (3)$$

$$= -1 \quad (4)$$

8.

$$\lim_{x \rightarrow 4} \frac{\left(\frac{1}{x} - \frac{1}{4}\right)}{x - 4} = \lim_{x \rightarrow 4} \frac{\left(\frac{4}{4x} - \frac{x}{4x}\right)}{x - 4} \quad (1)$$

$$= \lim_{x \rightarrow 4} \frac{\left(\frac{4-x}{4x}\right)}{x - 4} \quad (2)$$

$$= \lim_{x \rightarrow 4} \frac{\left(\frac{4-x}{4x}\right)}{\left(\frac{x-4}{1}\right)} \quad (3)$$

$$= \lim_{x \rightarrow 4} \left(\frac{4-x}{4x}\right) \left(\frac{1}{x-4}\right) \quad (4)$$

$$= \lim_{x \rightarrow 4} \frac{4-x}{4x(x-4)} \quad (5)$$

$$= \lim_{x \rightarrow 4} \frac{-(-4+x)}{4x(x-4)} \quad (6)$$

$$= \lim_{x \rightarrow 4} \frac{-(x-4)}{4x(x-4)} \quad (7)$$

$$= \lim_{x \rightarrow 4} \frac{-1}{4x} \quad (8)$$

$$= \lim_{x \rightarrow 4} \frac{-1}{4(4)} \quad (9)$$

$$= -\frac{1}{16} \quad (10)$$

9.

$$\lim_{x \rightarrow 9} \frac{x-9}{\sqrt{x}-3} = \lim_{x \rightarrow 9} \frac{x-9}{\sqrt{x}-3} \cdot \frac{\sqrt{x}+3}{\sqrt{x}+3} \quad (1)$$

$$= \lim_{x \rightarrow 9} \frac{(x-9)(\sqrt{x}+3)}{(\sqrt{x}-3)(\sqrt{x}+3)} \quad (2)$$

$$= \lim_{x \rightarrow 9} \frac{(x-9)(\sqrt{x}+3)}{x-9} \quad (3)$$

$$= \lim_{x \rightarrow 9} \sqrt{x}+3 \quad (4)$$

$$= \sqrt{9}+3 \quad (5)$$

$$= 3+3 \quad (6)$$

$$= 6 \quad (7)$$

10.

$$1 - \frac{x^2}{2} \leq \cos x \leq 1$$

$$\lim_{x \rightarrow 0} \left(1 - \frac{x^2}{2}\right) = 1 - \frac{0^2}{2} \quad (1)$$

$$= 1 - 0 \quad (2)$$

$$= 1 \quad (3)$$

$$= \lim_{x \rightarrow 0} 1 \quad (4)$$

$$\lim_{x \rightarrow 0} \cos x = 1 \quad (\text{By the Squeeze Theorem}) \quad (5)$$

11.

$$\lim_{x \rightarrow 0} \sin x = 0 \quad (\text{By the Squeeze Theorem}) \quad (1)$$

$$\lim_{x \rightarrow 0} \cos x = 1 \quad (\text{By the Squeeze Theorem}) \quad (2)$$

$$\lim_{x \rightarrow 0} \frac{\sin 2x}{\sin x} = \lim_{x \rightarrow 0} \frac{2 \sin x \cos x}{\sin x} \quad (1)$$

$$= \lim_{x \rightarrow 0} 2 \cos x \quad (2)$$

$$= 2 \lim_{x \rightarrow 0} \cos x \quad (3)$$

$$= 2 \cdot 1 \quad (4)$$

$$= 2 \quad (5)$$

## 12. Infinite Limits Numerically

$$f(x) = \frac{x}{(x-2)^2}$$

2.1	2.01	2.001	2.0001
210	20100	2001000	200010000

1.9	1.99	1.999	1.9999
190	19900	1999000	199990000

$$\lim_{x \rightarrow 2} f(x) = \infty$$

## 13. Infinite Limits Graphically

$$\lim_{x \rightarrow -2^-} h(x) = -\infty$$

$$\lim_{x \rightarrow -2^+} h(x) = -\infty$$

$$\lim_{x \rightarrow -2} h(x) = -\infty$$

$$\lim_{x \rightarrow 3^-} h(x) = \infty$$

$$\lim_{x \rightarrow 3^+} h(x) = -\infty$$

$$\lim_{x \rightarrow 3} h(x) = \text{Does not exist}$$

## 14. Infinite Limits Analytically

Hint: Look at the signs of the fractions

$$\frac{x^2 - 5x + 6}{x^4 - 4x^2} = \frac{(x-3)(x-2)}{x^2(x+2)(x-2)} = \frac{x-3}{x^2(x+2)}$$

$$\lim_{x \rightarrow -2^+} \frac{x^2 - 5x + 6}{x^4 - 4x^2} = \lim_{x \rightarrow -2^+} \frac{x-3}{x^2(x+2)} = -\infty$$

$$\lim_{x \rightarrow -2^-} \frac{x^2 - 5x + 6}{x^4 - 4x^2} = \lim_{x \rightarrow -2^-} \frac{x-3}{x^2(x+2)} = \infty$$

$$\lim_{x \rightarrow -2} \frac{x^2 - 5x + 6}{x^4 - 4x^2} = \text{Does not exist}$$

## 15. Infinite Limits Analytically with Square Root

$$\lim_{x \rightarrow 1^+} \frac{x+3}{\sqrt{x^2 - 5x + 4}} = \lim_{x \rightarrow 1^+} \frac{x+3}{\sqrt{(x-4)(x-1)}} = \text{Does not exist}$$

$$\lim_{x \rightarrow 1^-} \frac{x+3}{\sqrt{x^2 - 5x + 4}} = \lim_{x \rightarrow 1^-} \frac{x+3}{\sqrt{(x-4)(x-1)}} = \infty$$

$$\lim_{x \rightarrow 1} \frac{x+3}{\sqrt{x^2 - 5x + 4}} = \text{Does not exist}$$

## 16. Infinite Limit with a Trigonometric Function

$$\lim_{\theta \rightarrow 0^-} \frac{\sin \theta}{\cos^2 \theta - 1} = \lim_{\theta \rightarrow 0^-} \frac{\sin \theta}{-\sin^2 \theta} = \lim_{\theta \rightarrow 0^-} \frac{1}{-\sin^2 \theta} = \infty$$

17. Locating Vertical Asymptotes

$$f(x) = \frac{x+7}{x^4-49x^2} = \frac{x+7}{x^2(x^2-49)} = \frac{x+7}{x^2((x-7)(x+7))} = \frac{1}{x^2(x-7)}$$

Denominator is 0 at  $x = 0$ ,  $x = -7$ ,  $x = 7$

$x = -7$  does not fit, as it is connected with  $x + 7$ , but cancels out

Vertical Asymptotes:  $x = 0$ ,  $x = 7$

18. Limits at Infinity

$$\lim_{x \rightarrow \infty} 5 + \frac{1}{x} + \frac{10}{x^2} = 5 + 0 + 0 = 5$$

$$\lim_{x \rightarrow \infty} 5 = 5$$

$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

$$\lim_{x \rightarrow \infty} \frac{10}{x^2} = 0$$

19. End behaviour for rational functions (different degrees)

Hint: the degree of the numerator is less than the denominator

$$\lim_{x \rightarrow \infty} \frac{6x+1}{2x^2-5x+2} = \lim_{x \rightarrow \infty} \frac{\frac{6}{x} + \frac{1}{x^2}}{2 - \frac{5}{x} + \frac{2}{x^2}} = \lim_{x \rightarrow \infty} \frac{0+0}{2-0+0} = \frac{0}{2} = 0$$

20. End behaviour for rational functions (equal degrees)

Hint: the degree of the numerator is the same as the denominator

$$\lim_{x \rightarrow \infty} \frac{6x^2+1}{2x^2-5x+2} = \lim_{x \rightarrow \infty} \frac{6 + \frac{1}{x^2}}{2 - \frac{5}{x} + \frac{2}{x^2}} = \lim_{x \rightarrow \infty} \frac{6+0}{2-0+0} = \frac{6}{2} = 3$$

21. End behaviour for rational functions (different degrees)

Hint: the degrees of the numerator is greater than the degree of the denominator

$$\lim_{x \rightarrow \infty} \frac{6x^4+1}{2x^2-5x+2} = \lim_{x \rightarrow \infty} \frac{6x^2 + \frac{1}{x^2}}{2 - \frac{5}{x} + \frac{2}{x^2}} = \lim_{x \rightarrow \infty} \frac{6x^2+0}{2-0+0} = \frac{\infty}{2} = \infty$$

22. End behaviours for rational functions

Hint: If there is a negative exponent like  $2x^{-2}$ , we can rewrite that as  $\frac{2}{x^2}$

Hint: Keep in mind the direction at which  $x$  is changing (increasing or decreasing)

$$\lim_{x \rightarrow -\infty} 2x^{-8} + 4x^3 = \lim_{x \rightarrow -\infty} \frac{2}{x^8} + 4x^3 = 0 - \infty = -\infty$$

$$\lim_{x \rightarrow \infty} \frac{14x^3 + 3x^2 - 2x}{21x^3 + x^2 + 2x + 1} = \lim_{x \rightarrow \infty} \frac{14 + \frac{3}{x} - \frac{2}{x^2}}{21 + \frac{1}{x} + \frac{2}{x^2} + \frac{1}{x^3}} = \frac{14}{21} = \frac{2}{3}$$

$$\lim_{x \rightarrow \infty} \frac{9x^3 + x^2 - 5}{3x^4 + 4x^2} = \lim_{x \rightarrow \infty} \frac{\frac{9}{x} + \frac{1}{x^2} - \frac{5}{x^4}}{3 + \frac{4}{x^2}} = \frac{0}{3} = 0$$

23. Asymptotes for a rational function

$$f(x) = \frac{3x^2 - 7}{x^2 + 5x}$$

Horizontal Asymptote(s):  $y = 3$

**Explanation:** The end behaviour for this function approaches 3 (on both ends), so there is a single horizontal asymptote



24. End behaviour for algebraic function

$$\lim_{x \rightarrow -\infty} \frac{\sqrt{16x^2 + x}}{x} = \lim_{x \rightarrow -\infty} \frac{\frac{\sqrt{16x^2 + x}}{x}}{\frac{x}{x}} \quad (1)$$

$$= \lim_{x \rightarrow -\infty} \frac{\frac{1}{x}\sqrt{16x^2 + x}}{1} \quad (2)$$

$$= \lim_{x \rightarrow -\infty} \frac{1}{-\sqrt{x^2}} \sqrt{16x^2 + x} \quad (3)$$

$$= \lim_{x \rightarrow -\infty} -\sqrt{\frac{16x^2}{x^2} + \frac{x}{x^2}} \quad (4)$$

$$= \lim_{x \rightarrow -\infty} -\sqrt{16 + \frac{1}{x}} \quad (5)$$

$$= \lim_{x \rightarrow -\infty} -\sqrt{16} \quad (6)$$

$$= -4 \quad (7)$$

$$(8)$$

25. End behaviour for transcendental function

$$\lim_{x \rightarrow \infty} \frac{\sin x}{e^x + \ln x} = \lim_{x \rightarrow \infty} \frac{\sin x}{\infty + \infty} = 0$$

**Explanation:** Since  $\sin x$  is bounded between  $-1$  and  $1$ , and the denominator is a very large number, we know as  $x$  increases, the function will approach zero

26. (Section 2.1, Related Exercise 13):

Hint: use the secant line slope formula

$$s(t) = -16t^2 + 128t$$

(a)  $[1, 4]$

$$\frac{256 - 112}{4 - 1} = \frac{144}{3} = 48$$

(b)  $[1, 3]$

$$\frac{240 - 112}{3 - 1} = \frac{128}{2} = 64$$

(c)  $[1, 2]$

$$\frac{192 - 112}{2 - 1} = \frac{80}{1} = 84$$

(d)  $[1, 1 + h]$ , where  $h > 0$  is a real number

$$\frac{112 + -16h^2 + 128h - 112}{1 + h - 1} = \frac{-16h^2 + 128h}{h} = -16h + 128 = 16(-h + 6)$$

27. (Section 2.1, Related Exercise 15): Hint: we use the slope formula for the secant line, and the relationship is referring to the interval

$$s(t) = -16t^2 + 100t$$

$$\frac{s(t_1) - s(t_0)}{t_1 - t_0} = \frac{s(2) - s(0.5)}{2 - 0.5} \quad (1)$$

$$= \frac{136 - 46}{1.5} \quad (2)$$

$$= \frac{90}{1.5} \quad (3)$$

$$= 60 \quad (4)$$

The slope of this secant line, through the lens of average velocity could be viewed as the average velocity over the interval  $[0.5, 2]$

28. (Section 2.1, Related Exercise 17):

$$s(t) = -16t^2 + 128t$$

[1, 2]	[1, 1.5]	[1, 1.1]	[1, 1.01]	[1, 1.001]
80	88	94.4	95.84	95.984

$$v_{inst} = \lim_{t \rightarrow 1} s(t) = 96$$

29. (Section 2.1, Related Exercise 19):

$$s(t) = -16t^2 + 100t$$

[2, 3]	[2.9, 3]	[2.99, 3]	[2.999, 3]	[2.9999, 3]
20	5.6	4.16	4.016	4.002

$$v_{inst} = \lim_{t \rightarrow 3} s(t) = 4$$

30. (Section 2.2, Related Exercise 3):

- $h(2) = 5$
- $\lim_{x \rightarrow 2} h(x) = 3$
- $h(4) = \text{Does not exist}$
- $\lim_{x \rightarrow 4} h(x) = 1$
- $\lim_{x \rightarrow 5} h(x) = 2$

31. (Section 2.2, Related Exercise 4):

- $g(0) = 0$
- $\lim_{x \rightarrow 0} g(x) = 1$
- $g(1) = 2$
- $\lim_{x \rightarrow 1} g(x) = 2$

32. (Section 2.2, Related Exercise 7):

$$f(x) = \frac{x^2 - 4}{x - 2}$$

1.9	1.99	1.999	1.9999
3.9	3.99	3.999	3.9999

2.1	2.01	2.001	2.0001
4.1	4.01	4.001	4.0001

$$\lim_{x \rightarrow 2} f(x) = 4$$

33. (Section 2.2, Related Exercise 8):

$$f(x) = \frac{x^3 - 1}{x - 1}$$

0.9	0.99	0.999	0.9999
2.71	2.9701	3.997001	3.99970001

1.1	1.01	1.001	1.0001
3.31	3.0301	3.003001	3.00030001

$$\lim_{x \rightarrow 1} f(x) = 3$$

34. (Section 2.2, Related Exercise 27):

$$f(x) = \frac{x-2}{\ln|x-2|}$$

$$\lim_{x \rightarrow 2} f(x) = 2$$

35. (Section 2.2, Related Exercise 28):

$$f(x) = \frac{e^{2x} - 2x - 1}{x^2}$$

$$\lim_{x \rightarrow 0} f(x) = 0$$

36. (Section 2.2, Related Exercise 19):

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x \leq -1 \\ 3 & \text{if } x > -1 \end{cases}$$

$$\lim_{x \rightarrow -1^-} f(x) = 2$$

$$\lim_{x \rightarrow -1^+} f(x) = 3$$

$$\lim_{x \rightarrow -1} f(x) = \text{Does not exist}$$

37. (Section 2.2, Related Exercise 20):

$$f(x) = \begin{cases} 3 - x & \text{if } x < 2 \\ x - 1 & \text{if } x > 2 \end{cases}$$

$$\lim_{x \rightarrow 2^-} f(x) = 1$$

$$\lim_{x \rightarrow 2^+} f(x) = 1$$

$$\lim_{x \rightarrow 2} f(x) = 1$$

38. (Section 2.3, Related Exercise 19):

$$\lim_{x \rightarrow 4} 3x - 7 = 3(4) - 7 = 12 - 7 = 5$$

39. (Section 2.3, Related Exercise 22):

$$\lim_{x \rightarrow 6} 4 = 4$$

40. (Section 2.3, Related Exercise 11): Quotient, Difference

$$\lim_{x \rightarrow 1} \frac{f(x)}{g(x) - h(x)} = \frac{\lim_{x \rightarrow 1} f(x)}{\lim_{x \rightarrow 1} g(x) - h(x)} \quad (1)$$

$$= \frac{\lim_{x \rightarrow 1} f(x)}{\lim_{x \rightarrow 1} g(x) - h(x)} \quad (2)$$

$$= \frac{\lim_{x \rightarrow 1} f(x)}{\lim_{x \rightarrow 1} g(x) - \lim_{x \rightarrow 1} h(x)} \quad (3)$$

$$= \frac{8}{3 - 2} \quad (4)$$

$$= \frac{8}{1} \quad (5)$$

$$= 8 \quad (6)$$

41. (Section 2.3, Related Exercise 12): Root, Sum, Product

$$\lim_{x \rightarrow 1} \sqrt[3]{f(x)g(x) + 3} = \sqrt[3]{\lim_{x \rightarrow 1} f(x)g(x) + 3} \quad (1)$$

$$= \sqrt[3]{\lim_{x \rightarrow 1} f(x)g(x) + 3} \quad (2)$$

$$= \sqrt[3]{\lim_{x \rightarrow 1} f(x)g(x) + \lim_{x \rightarrow 1} 3} \quad (3)$$

$$= \sqrt[3]{\lim_{x \rightarrow 1} f(x) \lim_{x \rightarrow 1} g(x) + \lim_{x \rightarrow 1} 3} \quad (4)$$

$$= \sqrt[3]{8 \cdot 3 + 3} \quad (5)$$

$$= \sqrt[3]{24 + 3} \quad (6)$$

$$= \sqrt[3]{27} \quad (7)$$

$$= 3 \quad (8)$$

42. (Section 2.3, Related Exercise 25):

$$\lim_{x \rightarrow 1} \frac{5x^2 + 6x + 1}{8x - 4} = \frac{5(1^2) + 6(1) + 1}{8(1) - 4} \quad (1)$$

$$= \frac{5 + 6 + 1}{8 - 4} \quad (2)$$

$$= \frac{12}{4} \quad (3)$$

$$= 3 \quad (4)$$

43. (Section 2.3, Related Exercise 26):

$$\lim_{t \rightarrow 3} \sqrt[3]{t^2 - 10} = \sqrt[3]{\lim_{t \rightarrow 3} t^2 - 10} \quad (1)$$

$$= \sqrt[3]{3^2 - 10} \quad (2)$$

$$= \sqrt[3]{9 - 10} \quad (3)$$

$$= \sqrt[3]{-1} \quad (4)$$

$$= -1 \quad (5)$$

44. (Section 2.3, Related Exercise 27):

$$\lim_{p \rightarrow 2} \frac{3p}{\sqrt{4p + 1} - 1} = \frac{\lim_{p \rightarrow 2} 3p}{\lim_{p \rightarrow 2} \sqrt{4p + 1} - 1} \quad (1)$$

$$= \frac{3(2)}{\sqrt{\lim_{p \rightarrow 2} 4p + 1} - 1} \quad (2)$$

$$= \frac{6}{\sqrt{4(2) + 1} - 1} \quad (3)$$

$$= \frac{6}{\sqrt{8 + 1} - 1} \quad (4)$$

$$= \frac{6}{\sqrt{9} - 1} \quad (5)$$

$$= \frac{6}{3 - 1} \quad (6)$$

$$= \frac{6}{2} \quad (7)$$

$$= 3 \quad (8)$$

45. (Section 2.3, Related Exercise 72):

$$g(x) = \begin{cases} 5x - 15 & \text{if } x < 4 \\ \sqrt{6x + 1} & \text{if } x \geq 4 \end{cases}$$

$$\lim_{x \rightarrow 4^-} g(x) = 5$$

$$\lim_{x \rightarrow 4^+} g(x) = 5$$

$$\lim_{x \rightarrow 4} g(x) = 5$$

46. (Section 2.3, Related Exercise 73):

$$g(x) = \begin{cases} x^2 + 1 & \text{if } x < -1 \\ \sqrt{x+1} & \text{if } x \geq -1 \end{cases}$$

$$\lim_{x \rightarrow -1^-} g(x) = 2$$

$$\lim_{x \rightarrow -1^+} g(x) = 0$$

$$\lim_{x \rightarrow -1} g(x) = \text{Does not exist}$$

47. (Section 2.3, Related Exercise 34):

$$\lim_{x \rightarrow 3} \frac{x^2 - 2x - 3}{x - 3} = \lim_{x \rightarrow 3} \frac{(x-3)(x+1)}{x-3} \quad (1)$$

$$= \lim_{x \rightarrow 3} x + 1 \quad (2)$$

$$= 3 + 1 \quad (3)$$

$$= 4 \quad (4)$$

48. (Section 2.3, Related Exercise 41):

$$\lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x - 9} = \lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x - 9} \cdot \frac{\sqrt{x} + 3}{\sqrt{x} + 3} \quad (1)$$

$$= \lim_{x \rightarrow 9} \frac{(\sqrt{x} - 3)(\sqrt{x} + 3)}{(x - 9)(\sqrt{x} + 3)} \quad (2)$$

$$= \lim_{x \rightarrow 9} \frac{x - 9}{(x - 9)(\sqrt{x} + 3)} \quad (3)$$

$$= \lim_{x \rightarrow 9} \frac{1}{\sqrt{x} + 3} \quad (4)$$

$$= \frac{1}{\sqrt{9} + 3} \quad (5)$$

$$= \frac{1}{3 + 3} \quad (6)$$

$$= \frac{1}{6} \quad (7)$$

49. (Section 2.3, Related Exercise 69):

$$\lim_{x \rightarrow 1^+} \frac{x - 1}{\sqrt{x^2 - 1}} = \text{Does not exist}$$

50. (Section 2.3, Related Exercise 70):

$$\lim_{x \rightarrow 1^+} \frac{x-1}{\sqrt{x^2-1}} = \lim_{x \rightarrow 1^+} \frac{x-1}{\sqrt{x^2-1}} \cdot \frac{x+1}{x+1} \quad (1)$$

$$= \lim_{x \rightarrow 1^+} \frac{x^2-1}{\sqrt{x^2-1}(x+1)} \quad (2)$$

$$= \lim_{x \rightarrow 1^+} \frac{x^2-1}{(x^2-1)^{\frac{1}{2}}(x+1)} \quad (3)$$

$$= \lim_{x \rightarrow 1^+} \frac{(x^2-1)^{\frac{1}{2}}}{x+1} \quad (4)$$

$$= \lim_{x \rightarrow 1^+} \frac{\sqrt{x^2-1}}{x+1} \quad (5)$$

$$= \frac{\sqrt{1-1}}{1+1} \quad (6)$$

$$= \frac{\sqrt{0}}{2} \quad (7)$$

$$= \frac{0}{2} \quad (8)$$

$$= 0 \quad (9)$$

51. (Section 2.3, Related Exercise 95):

$$\frac{2^x - 2^0}{x - 0} = \frac{2^x - 1}{x}$$

-1	-0.1	-0.01	-0.001	-0.0001	-0.00001
0.5	0.6696700846	0.6907504563	0.6929070095	0.6931231585	0.6931447783

$$\lim_{x \rightarrow 0^1} \frac{2^x - 1}{x} = 0.693$$

52. (Section 2.3, Related Exercise 96):

$$\frac{3^x - 3^0}{x - 0} = \frac{3^x - 1}{x}$$

-0.1	-0.01	-0.001	-0.0001
1.040415402	1.092599583	1.098009035	1.098551943

  

0.0001	0.001	0.01	0.1
1.098672638	1.099215984	1.104669194	1.161231740

$$\lim_{x \rightarrow 0^1} \frac{3^x - 1}{x} = 1.0986$$

53. (Section 2.3, Related Exercise 81):

$-|x| < 0 < |x|$  and  $\sin \frac{1}{x} \leq 1$ , so  $|x| \sin \frac{1}{x} \leq |x|$  and  $-|x| \sin \frac{1}{x} \geq -|x|$

$$\lim_{x \rightarrow 0} -|x| = -|0| = 0$$

$$\lim_{x \rightarrow 0} |x| = |0| = 0$$

$$\lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0$$

By the Squeeze Theorem, since  $\lim_{x \rightarrow 0} -|x| = \lim_{x \rightarrow 0} |x|$  and the functions are chronologically greater than the last

54. (Section 2.3, Related Exercise 82):

$$\lim_{x \rightarrow 0} 1 - \frac{x^2}{2} = 1 - \frac{0}{2} = 1 - 0 = 1$$

$$\lim_{x \rightarrow 0} 1 = 1$$

$$\lim_{x \rightarrow 0} \cos x = 1$$

By the Squeeze Theorem, since  $\lim_{x \rightarrow 0} 1 - \frac{x^2}{2} = \lim_{x \rightarrow 0} 1$  and the functions are chronologically greater than the last

55. (Section 2.3, Related Exercise 60):

$$\lim_{x \rightarrow 0} \frac{\sin 2x}{\sin x} = \lim_{x \rightarrow 0} \frac{2 \sin x \cos x}{\sin x} \quad (1)$$

$$= \lim_{x \rightarrow 0} 2 \cos x \quad (2)$$

$$= 2 \cos 0 \quad (3)$$

$$= 2 \cdot 1 \quad (4)$$

$$= 2 \quad (5)$$

56. (Section 2.3, Related Exercise 61):

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{\cos^2 x - 3 \cos x + 2} = \lim_{x \rightarrow 0} \frac{1}{\cos^2 x - 2 \cos x + 2} \quad (1)$$

$$= \lim_{x \rightarrow 0} \frac{1}{\cos x \cos x - 2 \cos x + 2} \quad (2)$$

$$= \frac{1}{\cos 0 \cos 0 - 2 \cos 0 + 2} \quad (3)$$

$$= \frac{1}{1 \cdot 1 - 2(1) + 2} \quad (4)$$

$$= \frac{1}{1 - 2 + 2} \quad (5)$$

$$= \frac{1}{1} \quad (6)$$

$$= 1 \quad (7)$$

57. (Section 2.4, Related Exercise 6):

$$f(x) = \frac{x}{(x^2 - 2x - 3)^2}$$

$$\lim_{x \rightarrow -1} f(x) = -\infty$$

$$\lim_{x \rightarrow 3} f(x) = \infty$$

58. (Section 2.4, Related Exercise 7):

$$\lim_{x \rightarrow 1^-} f(x) = \infty$$

$$\lim_{x \rightarrow 1^+} f(x) = \infty$$

$$\lim_{x \rightarrow 1} f(x) = \infty$$

$$\lim_{x \rightarrow 2^-} f(x) = \infty$$

$$\lim_{x \rightarrow 2^+} f(x) = -\infty$$

$$\lim_{x \rightarrow 2} f(x) = \text{Does not exist}$$

59. (Section 2.4, Related Exercise 8):

$$\lim_{x \rightarrow 2^-} g(x) = \infty$$

$$\lim_{x \rightarrow 2^+} g(x) = -\infty$$

$$\lim_{x \rightarrow 2} g(x) = \text{Does not exist}$$

$$\lim_{x \rightarrow 4^-} g(x) = -\infty$$

$$\lim_{x \rightarrow 4^+} g(x) = -\infty$$

$$\lim_{x \rightarrow 4} g(x) = -\infty$$

60. (Section 2.4, Related Exercise 21):

$$\begin{aligned}\lim_{x \rightarrow 2^+} \frac{1}{x-2} &= \infty \\ \lim_{x \rightarrow 2^-} \frac{1}{x-2} &= -\infty \\ \lim_{x \rightarrow 2} \frac{1}{x-2} &= \text{Does not exist}\end{aligned}$$

61. (Section 2.4, Related Exercise 22):

$$\begin{aligned}\lim_{x \rightarrow 3^+} \frac{2}{(x-3)^3} &= \infty \\ \lim_{x \rightarrow 3^-} \frac{2}{(x-3)^3} &= -\infty \\ \lim_{x \rightarrow 3} \frac{2}{(x-3)^3} &= \text{Does not exist}\end{aligned}$$

62. (Section 2.4, Related Exercise 28):

$$\begin{aligned}\lim_{t \rightarrow -2^+} \frac{t^3 - 5t^2 + 6t}{t^4 - 4t^2} &= \lim_{t \rightarrow -2^+} \frac{t(t-2)(t-3)}{t^2(t^2-4)} = \lim_{t \rightarrow -2^+} \frac{t(t-2)(t-3)}{t^2(t-2)(t+2)} = \lim_{t \rightarrow -2^+} \frac{t(t-3)}{t^2(t+2)} = \lim_{t \rightarrow -2^+} \frac{t^2-3t}{t^3+2t^2} = -\infty \\ \lim_{t \rightarrow -2^-} \frac{t^3 - 5t^2 + 6t}{t^4 - 4t^2} &= \lim_{t \rightarrow -2^-} \frac{t(t-2)(t-3)}{t^2(t^2-4)} = \lim_{t \rightarrow -2^-} \frac{t(t-2)(t-3)}{t^2(t-2)(t+2)} = \lim_{t \rightarrow -2^-} \frac{t(t-3)}{t^2(t+2)} = \lim_{t \rightarrow -2^-} \frac{t^2-3t}{t^3+2t^2} = -\infty \\ \lim_{t \rightarrow -2} \frac{t^3 - 5t^2 + 6t}{t^4 - 4t^2} &= \lim_{t \rightarrow -2} \frac{t(t-2)(t-3)}{t^2(t^2-4)} = \lim_{t \rightarrow -2} \frac{t(t-2)(t-3)}{t^2(t-2)(t+2)} = \lim_{t \rightarrow -2} \frac{t(t-3)}{t^2(t+2)} = \lim_{t \rightarrow -2} \frac{t^2-3t}{t^3+2t^2} = -\infty \\ \lim_{t \rightarrow 2} \frac{t^3 - 5t^2 + 6t}{t^4 - 4t^2} &= \lim_{t \rightarrow 2} \frac{t(t-2)(t-3)}{t^2(t^2-4)} = -\frac{1}{8}\end{aligned}$$

63. (Section 2.4, Related Exercise 31): Remember, if you are able to solve by direct substitution after canceling terms (where the denominator does not equal zero), that's your answer

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{x-3}{x^4-9x^2} &= \lim_{x \rightarrow 0} \frac{x-3}{x^2(x-3)(x+3)} = \lim_{x \rightarrow 0} \frac{1}{x^2(x+3)} = \lim_{x \rightarrow 0} \frac{1}{x^3+3x^2} = \infty \\ \lim_{x \rightarrow 3} \frac{x-3}{x^4-9x^2} &= \lim_{x \rightarrow 3} \frac{x-3}{x^2(x-3)(x+3)} = \lim_{x \rightarrow 3} \frac{1}{x^2(x+3)} = \lim_{x \rightarrow 3} \frac{1}{x^3+3x^2} = \frac{1}{54} \\ \lim_{x \rightarrow -3} \frac{x-3}{x^4-9x^2} &= \lim_{x \rightarrow -3} \frac{x-3}{x^2(x-3)(x+3)} = \lim_{x \rightarrow -3} \frac{1}{x^2(x+3)} = \lim_{x \rightarrow -3} \frac{1}{x^3+3x^2} = \text{Does not exist}\end{aligned}$$

64. (Section 2.4, Related Exercise 45):

$$f(x) = \frac{x-5}{x^2-25} = \frac{x-5}{(x-5)(x+5)} = \frac{1}{x+5}$$

Vertical Asymptotes:  $x = -5$

$$\begin{aligned}\lim_{x \rightarrow 5} f(x) &= \lim_{x \rightarrow 5} \frac{1}{x+5} = \frac{1}{5+5} = \frac{1}{10} \\ \lim_{x \rightarrow -5^-} f(x) &= \lim_{x \rightarrow -5^-} \frac{1}{x+5} = -\infty \\ \lim_{x \rightarrow -5^+} f(x) &= \lim_{x \rightarrow -5^+} \frac{1}{x+5} = \infty\end{aligned}$$

65. (Section 2.4, Related Exercise 46):

$$f(x) = \frac{x+7}{x^4-49x^2} = \frac{x+7}{x^2(x^2-49)} = \frac{x+7}{x^2(x+7)(x-7)} = \frac{1}{x^2(x-7)} = \frac{1}{x^3-7x^2}$$

Vertical Asymptotes:  $x = 0$ ,  $x = 7$ ,  $x = -7$

$$\lim_{x \rightarrow 7^-} f(x) = \lim_{x \rightarrow 7^-} \frac{1}{x^3-7x^2} = -\infty$$



$$\lim_{x \rightarrow 7^+} f(x) = \lim_{x \rightarrow 7^+} \frac{1}{x^3 - 6x^2} = \infty$$

$$\lim_{x \rightarrow -7} f(x) = \lim_{x \rightarrow -7} \frac{1}{x^3 - 7x^2} = \text{Does not exist}$$

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{1}{x^3 - 7x^2} = -\infty$$

66. (Section 2.4, Related Exercise 39):

$$\lim_{\theta \rightarrow 0^+} \csc \theta = \infty$$

67. (Section 2.4, Related Exercise 40):

$$\lim_{x \rightarrow 0^-} \csc x = -\infty$$

68. (Section 2.5, Related Exercise 10)

$$\lim_{x \rightarrow \infty} 5 + \frac{1}{x} + \frac{10}{x^2} = 5 + 0 + 0 = 5$$

69. (Section 2.5, Related Exercise 19)

$$\lim_{x \rightarrow \infty} \frac{\cos x^5}{\sqrt{x}} = 0$$

70. (Section 2.5, Related Exercise 21)

$$\lim_{x \rightarrow \infty} 3x^1 2 - 9x^7 = \infty$$

71. (Section 2.5, Related Exercise 23)

$$\lim_{x \rightarrow -\infty} -3x^1 6 + 2 = -\infty$$

72. (Section 2.5, Related Exercise 38)

$$f(x) = \frac{3x^2 - 7}{x^2 + 5x}$$

Horizontal Asymptote:  $y = 3$

73. (Section 2.5, Related Exercise 41)

$$f(x) = \frac{3x^3 - 7}{x^4 + 5x^2}$$

Horizontal Asymptote:  $y = 0$

74. (Section 2.5, Related Exercise 43)

$$f(x) = \frac{40x^5 + x^2}{16x^4 - 2x}$$

Horizontal Asymptote: None

**Explanation:** Since the limit is infinity, there is no horizontal asymptote.

75. (Section 2.5, Related Exercise 51)

$$f(x) = \frac{x^2 - 3}{x + 6}$$

Slant Asymptote:  $y = x - 6$

Vertical Asymptote:  $x = -6$

76. (Section 2.5, Related Exercise 52)

$$f(x) = \frac{x^2 - 1}{x + 2}$$

Slant Asymptote:  $y = x - 2$

Vertical Asymptote:  $x = -2$

77. (Section 2.5, Related Exercise 46)

$$f(x) = \frac{\sqrt{x^2 + 1}}{2x + 1}$$

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 1}}{2x + 1} \quad (1)$$

$$= \lim_{x \rightarrow \infty} \frac{\sqrt{\frac{x^2}{x^2} + \frac{1}{x^2}}}{\frac{2x}{x} + \frac{1}{x}} \quad (2)$$

$$= \lim_{x \rightarrow \infty} \frac{\sqrt{1 + \frac{1}{x^2}}}{2 + \frac{1}{x}} \quad (3)$$

$$= \frac{\sqrt{1 + 0}}{2 + 0} \quad (4)$$

$$= \frac{\sqrt{1}}{2} \quad (5)$$

$$= \frac{1}{2} \quad (6)$$

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} \frac{-\sqrt{x^2 + 1}}{2x + 1} \quad (1)$$

$$= \lim_{x \rightarrow -\infty} \frac{-\sqrt{\frac{x^2}{x^2} + \frac{1}{x^2}}}{\frac{2x}{x} + \frac{1}{x}} \quad (2)$$

$$= \frac{-\sqrt{1 + 0}}{2 + 0} \quad (3)$$

$$= \frac{-1}{2} \quad (4)$$

$$= -\frac{1}{2} \quad (5)$$

Horizontal Asymptotes:  $y = \frac{1}{2}$ ,  $y = -\frac{1}{2}$

78. (Section 2.5, Related Exercise 47)

$$f(x) = \frac{4x^3 + 1}{2x^3 + \sqrt{16x^6 + 1}}$$

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{4x^3 + 1}{2x^3 + \sqrt{16x^6 + 1}} \quad (1)$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{4x^3}{x^3} + \frac{1}{x^3}}{\frac{2x^3}{x^3} + \sqrt{\frac{16x^6}{x^6} + \frac{1}{x^6}}} \quad (2)$$

$$= \frac{4 + 0}{2 + \sqrt{16 + 0}} \quad (3)$$

$$= \frac{4}{2 + \sqrt{16}} \quad (4)$$

$$= \frac{4}{2 + 4} \quad (5)$$

$$= \frac{4}{6} \quad (6)$$

$$= \frac{2}{3} \quad (7)$$

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} \frac{4x^3 + 1}{2x^3 - \sqrt{16x^6 + 1}} \quad (1)$$

$$= \lim_{x \rightarrow -\infty} \frac{\frac{4x^3}{x^3} + \frac{1}{x^3}}{\frac{2x^3}{x^3} - \sqrt{\frac{16x^6}{x^6} + \frac{1}{x^6}}} \quad (2)$$

$$= \frac{4 + 0}{2 - \sqrt{16 + 0}} \quad (3)$$

$$= \frac{4}{2 - \sqrt{16}} \quad (4)$$

$$= \frac{4}{2 - 4} \quad (5)$$

$$= \frac{4}{-2} \quad (6)$$

$$= -2 \quad (7)$$

Horizontal Asymptotes:  $y = \frac{2}{3}$ ,  $y = -2$

79. (Section 2.5, Related Exercise 57)

$$f(x) = -3e^{-x}$$

$$\lim_{x \rightarrow \infty} -3e^{-x} = 3(0) = 0$$

$$\lim_{x \rightarrow -\infty} -3e^{-x} = 3(\infty) = \infty$$

Horizontal Asymptotes:  $y = 0$

80. (Section 2.5, Related Exercise 59)

$$f(x) = 1 - \ln x$$

$$\lim_{x \rightarrow \infty} (1 - \ln x) = 1 - \infty = -\infty$$

$$\lim_{x \rightarrow 0^+} (1 - \ln x) = 1 + \infty = \infty$$

81. (Section 2.5, Related Exercise 62)

$$f(x) = \frac{50}{e^{2x}}$$

$$\lim_{x \rightarrow \infty} \frac{50}{e^{2x}} = \frac{50}{\infty} = 0$$

$$\lim_{x \rightarrow -\infty} \frac{50}{e^{2x}} = \text{Does not exist}$$