# Module 6 Notes (MATH-211)

### Lillie Donato

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## General Notes (and Definitions)

- L'Hôpital's Rule

  **Indeterminate Form**: An expression involving two components where the limit cannot be determined by evaluating the limits of the individual components.

  **L'Hôpital's Rule**: Suppose $f$ and $g$ are differentiable functions on an open interval $I$ containing the point $x = a$, with $g'(x) \neq 0$ on $I$ when $x \neq a$.

  If $\lim\limits_{x \to a} \frac{f(x)}{g(x)}$ has any of the indeterminate forms: $\frac{0}{0}$, $\frac{\infty}{\infty}$, $-\frac{\infty}{\infty}$, then

  $$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

  provided that one of the following is the case:

  $$\lim_{x \to a} \frac{f'(x)}{g'(x)} \in \mathbb{R}$$

  $$\lim_{x \to a} \frac{f'(x)}{g'(x)} = \infty$$

  $$\lim_{x \to a} \frac{f'(x)}{g'(x)} = -\infty$$

  L'Hôpital's Rule is still valid if $x \to a$ is replaced by any of $x \to a^+$, $x \to a^-$, $x \to \infty$, or $x \to -\infty$. In the last two of these cases, there must be a greatest $x$-value beyond which both $f$ and $g$ are differentiable at every point.

  **Exponential Indeterminate forms**: $1^\infty$, $0^0$, $\infty^0$

  **Method for evaluating limits of indeterminate forms $1^\infty$, $0^0$, $\infty^0$**:

  Assume that $L = \lim\limits_{x \to a} f(x)^{g(x)}$ has one of these indeterminate forms.

  1. Use the fact that the natural logarithm and natural exponential functions are inverses to write

     $$L = \lim_{x \to a} e^{\ln\left(f(x)^{g(x)}\right)}$$

  2. Use the power property of logarithm arguments to write

     $$L = \lim_{x \to a} e^{g(x)\ln(f(x))}$$

  3. Use continuity of the exponential function to write

     $$L = e^{\lim\limits_{x \to a} g(x)\ln(f(x))}$$

  4. Rewrite multiplication as division by the reciprocal:

     $$L = e^{\lim\limits_{x \to a} \left(\frac{\ln(f(x))}{\frac{1}{g(x)}}\right)}$$

  5. Use L'Hôpital's Rule to evaluate this limit expression

  **Growth Rates**: Suppose $f$ and $g$ are functions with $\lim\limits_{x \to \infty} f(x) = \infty$ and $\lim\limits_{x \to \infty} g(x) = \infty$

1. If one of the following are true, $f$ **grows faster than** $g$, and we use the notation $f \gg g$

$$\lim_{x \to \infty} \frac{g(x)}{f(x)} = 0 \qquad (1)$$

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \infty \qquad (2)$$

2. $f$ and $g$ **have comparable growth rates**, if there is some non-zero finite number $M$ such that

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = M$$

**Ranked Growth Rates as $x \to \infty$**
For any base $b > 1$, and for any positive numbers $p$, $q$, $r$, and $s$

$$\ln^q x \ll x^p \ll x^p \ln^r x \ll x^{p+s} \ll b^x \ll x^x$$

- Antiderivatives
  **Antiderivative**: A function $F$ is an antiderivative of another function $f$ on an interval $I$ if for all $x$ in $I$:

  $$F'(x) = f(x)$$

  **Family of Antiderivatives**: Let $F(x)$ be any antiderivative of $f(x)$ on an interval $I$. Then all antiderivatives of $f$ on $I$ have the form $F(x) + C$, where $C$ is an arbitrary constant.
  **Differential Equations**: Any equation involving an unknown function and its derivatives

  - Infinite family of solutions
  - No two solutions from the family pass through the same point
  - Given an initial condition $f(a) = b$, we can identify the particular family member that solves the given problem by solving for $C$

- Approximating Areas Under Curves

  - If we know the velocity function of a moving object, what can we learn about its position function?
  - Given an object with velocity function $v(t)$, the displacement of the moving object over the interval $[a, b]$ is the area between the velocity curve and the $t$-axis from $t = a$ to $t = b$.
  - Because objects do not necessarily move at a constant velocity, we can extend this idea to positive velocities that change over an interval of time.
  - The strategy is to divide the time interval into many subintervals, approximate the velocity on each subinterval with a constant velocity, calculate the individual displacements and sum the results.

**Riemann Sums**

  - Suppose $f(x)$ is continuous and non-negative on $[a, b]$.
  - Goal is to approximate the area of the region $R$ bounded by the graph of $f(x)$ and the $x$-axis from $x = a$ to $x = b$.
  - Divide $[a, b]$ into $n$ subintervals $[x_0, x_1], [x_1, x_2], ..., [x_{n-1}, x_n]$ where $a = x_0, b = x_n$.
  - The length of each subinterval is $\Delta x = \frac{b-a}{n}$
  - **Regular Partition**: Suppose $[a, b]$ is a closed interval containing $n$ subintervals

  $$[x_0, x_1], [x_1, x_2], ..., [x_{n-1}, x_n]$$

  of equal length $\Delta x = \frac{b-a}{n}$, with $a = x_0$ and $b = x_n$. The endpoints $x_0, x_1, x_2, ..., x_{n-1}, x_n$ of the subintervals are called **grid points**, and they create a **regular partition** of the interval $[a, b]$. In general the $k$th grid point is
  $$x_k = a + k\Delta x, \text{ for } k = 0, 1, 2, ..., n$$

  - In the $k$th subinterval $[x_{k-1}, x_k]$, choose any point $x_k^*$ and build a rectangle whose height is $f(x_k^*)$.
  - The area of the rectangle of the $k$th subinterval is

  $$\text{height} \cdot \text{base} = f(x_k^*)\Delta x, \text{ where } k = 1, 2, ..., n$$

- Summing the areas of these rectangles, we obtain an approximation to the area of $R$, which is called a **Riemann sum**:
$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + ... + f(x_n^*)\Delta x$$
- Three notable Riemann sums are the left, right, and midpoint Riemann sums.

**Riemann Sum**: Suppose $f$ is defined on a closed interval $[a, b]$, which is divided into $n$ subintervals of equal length $\Delta x$. If $x_k^*$ is any point in the $k$th subinterval $[x_{k-1}, x_k]$, for $k = 1, 2, ..., n$, then
$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + ... + f(x_n^*)\Delta x$$
is called a **Riemann sum** for $f$ on $[a, b]$. This sum is called

- a **left Riemann sum** if $x_k^*$ is the left endpoint of $[x_{k-1}, x_k]$
- a **right Riemann sum** if $x_k^*$ is the right endpoint of $[x_{k-1}, x_k]$
- a **midpoint Riemann sum** if $x_k^*$ is the midpoint of $[x_{k-1}, x_k]$

**Summation notation ($\Sigma$)**:

- Working with Riemann sums is cumbersome when $n$ is large
- We introduce sigma (summation) notation as a shorthand:
$$1 + 2 + ... + 49 + 50 = \sum_{k=1}^{50} k$$

- The symbol $\Sigma$ (sigma) stands for sum
- $k$ is the index, and takes on all integer values from $k = 1$ to $k = 50$
- The expression immediately following $\Sigma$, the summand, is evaluated for each $k$, and the resulting values are summed
- The index is a dummy variable, and it does not matter which symbol is chosen for the index:
$$\sum_{k=1}^{99} k = \sum_{n=1}^{99} n = \sum_{p-1}^{99} p$$

- Two Properties of Sums and Sigma Notation
  1. Constant Multiple Rule:
  $$\sum_{k=1}^{n} ca_k = c \sum_{k=1}^{n} a_k$$

  2. Addition Rule:
  $$\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

- **Theorem**: Sums of Power of Integers
  Let $n \in \mathbb{Z}$ such that $n > 0$ and $c \in \mathbb{R}$

$$\sum_{k=1}^{n} c = cn \tag{1}$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \tag{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \tag{3}$$

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4} \tag{4}$$

- **Left, Right, and Midpoint Riemann Sums in Sigma Notation**:
  Suppose $f$ is defined on a closed interval $[a, b]$, which is divided into subintervals of equal length $\Delta x$. If $x_k^*$ is a point in the $k$th subinterval $[x_{k-1}, x_k]$, for $k = 1, 2, ..., n$, then the **Riemann sum** for $f$ on $[a, b]$ is

$$\sum_{k=1}^{n} f(x_k^*)\Delta x$$

Three cases arise in practice

- $\displaystyle\sum_{k=1}^{n} f(x_k^*)\Delta x$ is a **left Riemann sum** if $x_k^* = a + (k-1)\Delta x$

- $\displaystyle\sum_{k=1}^{n} f(x_k^*)\Delta x$ is a **right Riemann sum** if $x_k^* = a + k\Delta x$

- $\displaystyle\sum_{k=1}^{n} f(x_k^*)\Delta x$ is a **midpoint Riemann sum** if $x_k^* = a + (k - \frac{1}{2})\Delta x$

# Antiderivative Rules

- Power Rule
  If $p \neq -1$ and $C$ is an arbitrary constant:

$$\int x^p dx = \frac{x^{p+1}}{p+1} + C$$

- Integral of $x^{-1}$

$$\int x^{-1}dx = \int \frac{1}{x}dx = \ln|x| + C$$

- Constant Multiple and Sum Rules
  If $c \in \mathbb{R}$:

$$\int cf(x)dx = c \int f(x)dx$$

$$\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$$

- Integral of $e^x$

$$\int e^x dx = e^x + C$$

- Integral of $\frac{1}{x}$

$$\int \frac{1}{x} dx = \ln|x| + C$$

# Trigonometric (and inverse) Integrals

$$\int \cos(x)dx = \sin x + C \tag{1}$$

$$\int \sin(x)dx = -\cos x + C \tag{2}$$

$$\int \sec^2(x)dx = \tan x + C \tag{3}$$

$$\int \csc^2(x)dx = -\cot x + C \tag{4}$$

$$\int \sec(x)\tan(x)dx = \sec x + C \tag{5}$$

$$\int \csc(x)\cot(x)dx = -\csc x + C \tag{6}$$

$$\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x + C \tag{7}$$

$$\int \frac{1}{1+x^2}dx = \tan^{-1} x + C \tag{8}$$

$$\int \frac{1}{x\sqrt{x^2-1}}dx = \sec^{-1}|x| + C \tag{9}$$

# Examples

1. Use L'Hôpital's Rule to evaluate a limit with indeterminate form $\frac{0}{0}$

$$\lim_{x \to 0} \frac{e^x - x - 1}{5x^2} \quad = \quad \lim_{x \to 0} \frac{e^x - 1}{10x} \tag{1}$$

$$= \quad \lim_{x \to 0} \frac{e^x}{10} \tag{2}$$

$$= \quad \frac{e^0}{10} \tag{3}$$

$$= \quad \frac{1}{10} \tag{4}$$

2. Use L'Hôpital's Rule to evaluate a limit with indeterminate form $\frac{\infty}{\infty}$

$$\lim_{x \to 0^+} \frac{1 - \ln x}{1 + \ln x} \quad = \quad \lim_{x \to 0^+} \frac{-\frac{1}{x}}{\frac{1}{x}} \tag{1}$$

$$= \quad \lim_{x \to 0^+} \frac{-\frac{1}{x}}{\frac{1}{x}} \tag{2}$$

$$= \quad \frac{-1}{1} \tag{3}$$

$$= \quad -1 \tag{4}$$

3. Use L'Hôpital's Rule to evaluate a limit with indeterminate form $0 \cdot \infty$

$$\lim_{x \to 1^-} (1 - x) \tan\left(\frac{\pi x}{2}\right) \quad = \quad \lim_{x \to 1^-} \frac{(1 - x)}{\cot\left(\frac{\pi x}{2}\right)} \tag{1}$$

$$= \quad \lim_{x \to 1^-} \frac{-1}{-\frac{\pi}{2} \csc^2\left(\frac{\pi x}{2}\right)} \tag{2}$$

$$= \quad \lim_{x \to 1^-} \frac{2}{\pi} \sin^2\left(\frac{\pi x}{2}\right) \tag{3}$$

$$= \quad \frac{2}{\pi} \tag{4}$$

4. Use L'Hôpital's Rule to evaluate a limit with exponential indeterminate form

$$\lim_{x \to 0^+} x^{\tan x} \quad = \quad e^{\lim_{x \to 0^+} \frac{\ln x}{\frac{1}{\tan x}}} \tag{1}$$

$$= \quad e^{\lim_{x \to 0^+} \frac{\ln x}{\cot x}} \tag{2}$$

$$= \quad e^{\lim_{x \to 0^+} \frac{\frac{1}{x}}{-x \csc^2 x}} \tag{3}$$

$$= \quad e^{\lim_{x \to 0^+} \frac{-\sin^2 x}{x}} \tag{4}$$

$$= \quad e^{\lim_{x \to 0^+} \frac{-2 \sin x \cos x}{1}} \tag{5}$$

$$= \quad e^{\lim_{x \to 0^+} -2 \sin x \cos x} \tag{6}$$

$$= \quad e^0 \tag{7}$$

$$= \quad 1 \tag{8}$$

5. Compare the growth rates of functions

$$f(x) = x^2 \ln x$$
$$g(x) = x \ln^2 x$$

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} \quad = \quad \lim_{x \to \infty} \frac{x^2 \ln x}{x \ln^2 x} \tag{1}$$

$$= \quad \lim_{x \to \infty} \frac{x}{\ln x} \tag{2}$$

$$= \quad \lim_{x \to \infty} \frac{1}{\frac{1}{x}} \tag{3}$$

$$= \quad \lim_{x \to \infty} x \tag{4}$$

$$= \quad \infty \tag{5}$$

Since $\lim\limits_{x \to \infty} \frac{f(x)}{g(x)} = \infty$, $f \gg g$

6. Use knowledge of derivatives to find antiderivatives
$$f(x) = -4\cos x - x$$
$$F(x) = -4\sin x - \frac{1}{2}x^2$$
$$F'(x) = -4\cos x - x$$
$$\int (-4\cos x - x)dx = -4\sin x - \frac{1}{2}x^2 + C$$

7. Determine indefinite integrals using antiderivative rules
$$\int \frac{3}{x^4} + 2 - 3x^2 dx = \int 3x^{-4} + 2 - 3x^2 dx = \frac{-1}{x^3} + 2x - x^3 + C$$

8. Rewrite an indefinite integral to find an antiderivative
$$\int \frac{2 + 3\cos y}{\sin^2 y} dy \quad = \quad \int 2\csc^2 y + 3\cot y \csc y \, dy \tag{1}$$
$$= \quad 2\int \csc^2 y \, dy + 3\int \cot y \csc y \, dy \tag{2}$$
$$= \quad -2\cot y - 3\csc y + C \tag{3}$$

9. Solve an initial value problem
$$f'(u) = 4\cos u - 4\sin u$$
$$f(\pi) = 0$$
$$f(u) = 4\int \cos u \, du - 4\int \sin u \, du = 4\sin u + 4\cos u + C$$

$$4\sin \pi + 4\cos \pi + C \quad = \quad 0 \tag{1}$$
$$0 - 4 + C \quad = \quad 0 \tag{2}$$
$$C \quad = \quad 4 \tag{3}$$
$$f(u) = 4\sin u + 4\cos u + 4$$

10. Application of differential equations to linear motion
$$a(t) = 2 + 3\sin t$$
$$v(0) = 1$$
$$s(0) = 10$$

$$v(t) \quad = \quad \int 2 + 3\sin t \, dt \tag{1}$$
$$= \quad 2\int t^0 \, dt + 3\int \sin t \, dt \tag{2}$$
$$= \quad 2t - 3\cos t + C \tag{3}$$
$$2(0) - 3\cos 0 + C \quad = \quad 1 \tag{4}$$
$$-3 + C \quad = \quad 1 \tag{5}$$
$$C \quad = \quad 4 \tag{6}$$
$$v(t) \quad = \quad 2t - 3\cos t + 4 \tag{7}$$
$$s(t) \quad = \quad \int 2t - 3\cos t + 4 \, dt \tag{8}$$
$$= \quad 2\int t \, dt - 3\int \cos t \, dt + 4\int t^0 \, dt \tag{9}$$
$$= \quad 2\frac{t^2}{2} - 3\sin t + 4t + C \tag{10}$$
$$= \quad t^2 - 3\sin t + 4t + C \tag{11}$$
$$0^2 - 3\sin 0 + 4(0) + C \quad = \quad 10 \tag{12}$$
$$C \quad = \quad 10 \tag{13}$$
$$s(t) \quad = \quad t^2 - 3\sin t + 4t + 10 \tag{14}$$

11. Approximating displacement

$$v = \sqrt{10t}$$

$$1 \le t \le 7$$

(a)

$$n = 3$$

$$[1, 3], [3, 5], [5, 7]$$

$$2, 4, 6$$

$$d \approx 2v(2) + 2v(4) + 2v(6) = 2\sqrt{20} + 2\sqrt{40} + 2\sqrt{60} \approx 37.085$$

(b)

$$n = 6$$

$$1.5, 2.5, 3.5, 4.5, 5.5, 6.5$$

$$
\begin{aligned}
d &\approx v(1.5) + v(2.5) + v(3.5) + v(4.5) + v(5.5) + v(6.5) & (1)\\
&= \sqrt{15} + \sqrt{25} + \sqrt{35} + \sqrt{45} + \sqrt{55} + \sqrt{65} & (2)\\
&\approx 36.976 & (3)
\end{aligned}
$$

12. Left and right Riemann sums (1)

$$f(x) = x + 1$$

$$[1, 6]$$

$$n = 5$$

(a)

$$
\begin{aligned}
(1+1) + (2+1) + (3+1) + (4+1) + (5+1) &= 2 + 3 + 4 + 5 + 6 & (1)\\
&= 20 & (2)
\end{aligned}
$$

(b)

$$
\begin{aligned}
(2+1) + (3+1) + (4+1) + (5+1) + (6+1) &= 3 + 4 + 5 + 6 + 7 & (1)\\
&= 25 & (2)
\end{aligned}
$$

13. Left and right Riemann sums (2)

$$f(x) = 9 - x$$

$$[3, 8]$$

$$n = 5$$

$$\Delta x = \frac{8 - 3}{5} = \frac{5}{5} = 1$$

(a) Grid Points:

$$
\begin{aligned}
x_0 &= 3 & (1)\\
x_1 &= 4 & (2)\\
x_2 &= 5 & (3)\\
x_3 &= 6 & (4)\\
x_4 &= 7 & (5)\\
x_5 &= 8 & (6)
\end{aligned}
$$

(b) Riemann Sums

$$
\begin{aligned}
(9-3) + (9-4) + (9-5) + (9-6) + (9-7) &= 6 + 5 + 4 + 3 + 2 & (1)\\
&= 20 \ (\text{Overestimation}) & (2)
\end{aligned}
$$

$$
\begin{aligned}
(9-4) + (9-5) + (9-6) + (9-7) + (9-8) &= 5 + 4 + 3 + 2 + 1 & (1)\\
&= 15 \ (\text{Underestimation}) & (2)
\end{aligned}
$$

14. Midpoint Riemann sum

$$f(x) = 100 - x^2$$

$$[0, 10]$$

$$n = 5$$

$$\Delta x = 2$$

$$
\begin{align}
2f(1) + 2f(3) + 2f(5) + 2f(7) + 2f(9) &= 2(99) + 2(91) + 2(75) + 2(51) + 2(19) \tag{1} \\
&= 198 + 182 + 150 + 102 + 38 \tag{2} \\
&= 670 \tag{3}
\end{align}
$$

15. Riemann sums from tables

$$[0, 2]$$

$$n = 4$$

$$\Delta x = \frac{1}{2}$$

$$
\begin{align}
\frac{5}{2} + \frac{3}{2} + \frac{2}{2} + \frac{1}{2} &= 5.5 \tag{1} \\
\frac{3}{2} + \frac{2}{2} + \frac{1}{2} + \frac{1}{2} &= 3.5 \tag{2}
\end{align}
$$

16. Computing Riemann sums for large values of $n$

$$f(x) = x^2 + 1$$

$$[-1, 1]$$

$$n = 50$$

$$
\begin{align}
\sum_{k=1}^{50} 0.04 f\left(-1 + 0.04\,(k-1)\right) &= \sum_{k=1}^{50} 0.04 \left(\left(-1 + 0.04\,(k-1)\right)^2 + 1\right) \tag{1} \\
&\approx 2.6672 \tag{2}
\end{align}
$$

$$
\begin{align}
\sum_{k=1}^{50} 0.04 f\left(-1 + 0.04k\right) &= \sum_{k=1}^{50} 0.04 \left(\left(-1 + 0.04k\right)^2 + 1\right) \tag{3} \\
&\approx 2.6672 \tag{4}
\end{align}
$$

$$
\begin{align}
\sum_{k-1}^{50} 0.04 f\left(-1 + 0.04\left(k - \frac{1}{2}\right)\right) &= \sum_{k-1}^{50} 0.04 \left(\left(-1 + 0.04\left(k - \frac{1}{2}\right)\right)^2 + 1\right) \tag{5} \\
&\approx 2.6664 \tag{6}
\end{align}
$$

# Related Exercises

1. (Section 4.7, Exercise 17)

$$
\begin{align}
\lim_{x \to 2} \frac{x^2 - 2x}{x^2 - 6x + 8} &= \lim_{x \to 2} \frac{2x - 2}{2x - 6} \tag{1} \\
&= \frac{2(2) - 2}{2(2) - 6} \tag{2} \\
&= \frac{4 - 2}{4 - 6} \tag{3} \\
&= \frac{2}{-2} \tag{4} \\
&= -1 \tag{5}
\end{align}
$$

2. (Section 4.7, Exercise 18)

$$\lim_{x \to -1} \frac{x^4 + x^3 + 2x + 2}{x + 1} = \lim_{x \to -1} \frac{4x^3 + 3x^2 + 2}{1} \tag{1}$$

$$= \lim_{x \to -1} 4x^3 + 3x^2 + 2 \tag{2}$$

$$= 4(-1)^3 + 3(-1)^2 + 2 \tag{3}$$

$$= -4 + 3 + 2 \tag{4}$$

$$= 1 \tag{5}$$

3. (Section 4.7, Exercise 36)

$$\lim_{x \to 0} \frac{e^x - x - 1}{5x^2} = \lim_{x \to 0} \frac{e^x - 1}{10x} \tag{1}$$

$$= \lim_{x \to 0} \frac{e^x}{10} \tag{2}$$

$$= \frac{e^0}{10} \tag{3}$$

$$= \frac{1}{10} \tag{4}$$

4. (Section 4.7, Exercise 39)

$$\lim_{x \to 0} \frac{e^x - \sin x - 1}{x^4 + 8x^3 + 12x^2} = \lim_{x \to 0} \frac{e^x - \cos x}{4x^3 + 24x^2 + 24x} \tag{1}$$

$$= \lim_{x \to 0} \frac{e^x + \sin x}{12x^2 + 48x + 24} \tag{2}$$

$$= \frac{e^0 + \sin 0}{12(0)^2 + 48(0) + 24} \tag{3}$$

$$= \frac{1 + 0}{24} \tag{4}$$

$$= \frac{1}{24} \tag{5}$$

5. (Section 4.7, Exercise 38)

$$\lim_{x \to \infty} \frac{e^{3x}}{3e^{3x} + 5} = \lim_{x \to \infty} \frac{3e^{3x}}{9e^{3x}} \tag{1}$$

$$= \lim_{x \to \infty} \frac{1}{3} \cdot \frac{e^{3x}}{e^{3x}} \tag{2}$$

$$= \lim_{x \to \infty} \frac{1}{3} \tag{3}$$

$$= \frac{1}{3} \tag{4}$$

6. (Section 4.7, Exercise 51)

$$\lim_{x \to \infty} \frac{x^2 - \ln \frac{2}{x}}{3x^2 + 2x} = \lim_{x \to \infty} \frac{2x + \frac{1}{x}}{6x + 2} \tag{1}$$

$$= \lim_{x \to \infty} \frac{2 - \frac{1}{x^2}}{6} \tag{2}$$

$$= \frac{2 - 0}{6} \tag{3}$$

$$= \frac{2}{6} \tag{4}$$

$$= \frac{1}{3} \tag{5}$$

7. (Section 4.7, Exercise 53)

$$\lim_{x \to 0} x \csc x \quad = \quad \lim_{x \to 0} \frac{x}{\sin x} \tag{1}$$

$$= \quad \lim_{x \to 0} \frac{1}{\cos x} \tag{2}$$

$$= \quad \frac{1}{\cos 0} \tag{3}$$

$$= \quad \frac{1}{1} \tag{4}$$

$$= \quad 1 \tag{5}$$

8. (Section 4.7, Exercise 63)

$$\lim_{x \to \infty} \left( x^2 - \sqrt{x^4 + 16x^2} \right) \quad = \quad \lim_{x \to \infty} \left( x^2 - \sqrt{x^4 \left( 1 + \frac{16}{x^2} \right)} \right) \tag{1}$$

$$= \quad \lim_{x \to \infty} \left( x^2 - x^2 \sqrt{1 + \frac{16}{x^2}} \right) \tag{2}$$

$$= \quad \lim_{x \to \infty} x^2 \left( 1 - \sqrt{1 + \frac{16}{x^2}} \right) \tag{3}$$

$$= \quad \lim_{x \to \infty} \frac{1 - \sqrt{1 + \frac{16}{x^2}}}{\frac{1}{x^2}} \tag{4}$$

$$= \quad \lim_{x \to \infty} \frac{\frac{16}{x^3}}{\frac{-2}{x^3} \sqrt{1 + \frac{16}{x^2}}} \tag{5}$$

$$= \quad \lim_{x \to \infty} \frac{\frac{16}{x^3} \cdot \frac{x^3}{-2}}{\sqrt{1 + \frac{16}{x^2}}} \tag{6}$$

$$= \quad \lim_{x \to \infty} \frac{\frac{16}{-2} \cdot \frac{x^3}{x^3}}{\sqrt{1 + \frac{16}{x^2}}} \tag{7}$$

$$= \quad \lim_{x \to \infty} \frac{-8}{\sqrt{1 + \frac{16}{x^2}}} \tag{8}$$

$$= \quad \frac{-8}{\sqrt{1 + 0}} \tag{9}$$

$$= \quad \frac{-8}{1} \tag{10}$$

$$= \quad -8 \tag{11}$$

9. (Section 4.7, Exercise 64)

$$\lim_{x\to\infty} \left(x - \sqrt{x^2+4x}\right) = \lim_{x\to\infty} \left(x - \sqrt{x^2\left(1+\frac{4}{x}\right)}\right) \tag{1}$$

$$= \lim_{x\to\infty} \left(x - x\sqrt{1+\frac{4}{x}}\right) \tag{2}$$

$$= \lim_{x\to\infty} x\left(1 - \sqrt{1+\frac{4}{x}}\right) \tag{3}$$

$$= \lim_{x\to\infty} \frac{1 - \sqrt{1+\frac{4}{x}}}{\frac{1}{x}} \tag{4}$$

$$= \lim_{x\to\infty} \frac{\frac{2}{x^2}}{\frac{-1}{x^2}\sqrt{1+\frac{4}{x}}} \tag{5}$$

$$= \lim_{x\to\infty} \frac{\frac{2}{x^2}\cdot\frac{x^2}{-1}}{\sqrt{1+\frac{4}{x}}} \tag{6}$$

$$= \lim_{x\to\infty} \frac{\frac{2}{-1}\cdot\frac{x^2}{x^2}}{\sqrt{1+\frac{4}{x}}} \tag{7}$$

$$= \lim_{x\to\infty} \frac{-2}{\sqrt{1+\frac{4}{x}}} \tag{8}$$

$$= \frac{-2}{\sqrt{1+0}} \tag{9}$$

$$= -2 \tag{10}$$

10. (Section 4.7, Exercise 75)

$$\lim_{x\to 0^+} x^{2x} = e^{\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{2x}}} \tag{1}$$

$$= e^{\lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{2x^2}}} \tag{2}$$

$$= e^{\lim_{x\to 0^+} \frac{1}{x}\cdot\frac{2x^2}{-1}} \tag{3}$$

$$= e^{\lim_{x\to 0^+} -\frac{2x^2}{x}} \tag{4}$$

$$= e^{\lim_{x\to 0^+} -2x} \tag{5}$$

$$= e^{-2(0)} \tag{6}$$

$$= e^0 \tag{7}$$

$$= 1 \tag{8}$$

11. (Section 4.7, Exercise 76)

$$\lim_{x\to 0} (1+4x)^{\frac{3}{x}} = e^{\lim_{x\to 0^+} \frac{\ln(1+4x)}{\frac{1}{\frac{3}{x}}}} \tag{1}$$

$$= e^{\lim_{x\to 0^+} \frac{\ln(1+4x)}{\frac{x}{3}}} \tag{2}$$

$$= e^{\lim_{x\to 0^+} \frac{\frac{4}{(1+4x)}}{\frac{1}{3}}} \tag{3}$$

$$= e^{\lim_{x\to 0^+} \frac{4}{(1+4x)}\cdot\frac{3}{1}} \tag{4}$$

$$= e^{\lim_{x\to 0^+} \frac{12}{(1+4x)}} \tag{5}$$

$$= e^{\frac{12}{1}} \tag{6}$$

$$= e^{12} \tag{7}$$

12. (Section 4.7, Exercise 96)

$$f(x) = x^2 \ln x \tag{1}$$
$$g(x) = \ln^2 x \tag{2}$$

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{x^2 \ln x}{\ln^2 x} \tag{1}$$
$$= \lim_{x \to \infty} \frac{x^2}{\ln x} \tag{2}$$
$$= \lim_{x \to \infty} \frac{2x}{\frac{1}{x}} \tag{3}$$
$$= \lim_{x \to \infty} 2x^2 \tag{4}$$
$$= \infty \tag{5}$$

$$f \gg g$$

13. (Section 4.7, Exercise 100)

$$f(x) = x^2 \ln x \tag{1}$$
$$g(x) = x^3 \tag{2}$$

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{x^2 \ln x}{x^3} \tag{1}$$
$$= \lim_{x \to \infty} \frac{\ln x}{x} \tag{2}$$
$$= \lim_{x \to \infty} \frac{\frac{1}{x}}{1} \tag{3}$$
$$= \lim_{x \to \infty} \frac{1}{x} \tag{4}$$
$$= \frac{1}{\infty} \neq \infty \tag{5}$$

$$g \gg f$$

14. (Section 4.7, Exercise 95)

$$f(x) = x^{10} \tag{1}$$
$$g(x) = e^{0.01x} \tag{2}$$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^{10}}{e^{0.01x}} \tag{1}$$

$$= \lim_{x\to\infty} \frac{10x^9}{0.01e^{0.01x}} \tag{2}$$

$$= \lim_{x\to\infty} \frac{90x^8}{0.01^2 e^{0.01x}} \tag{3}$$

$$= \lim_{x\to\infty} \frac{7200x^7}{0.01^3 e^{0.01x}} \tag{4}$$

$$= \lim_{x\to\infty} \frac{50400x^6}{0.01^4 e^{0.01x}} \tag{5}$$

$$= \lim_{x\to\infty} \frac{302400x^5}{0.01^5 e^{0.01x}} \tag{6}$$

$$= \lim_{x\to\infty} \frac{1512000x^4}{0.01^6 e^{0.01x}} \tag{7}$$

$$= \lim_{x\to\infty} \frac{6048000x^3}{0.01^7 e^{0.01x}} \tag{8}$$

$$= \lim_{x\to\infty} \frac{18144000x^2}{0.01^8 e^{0.01x}} \tag{9}$$

$$= \lim_{x\to\infty} \frac{36288000x}{0.01^9 e^{0.01x}} \tag{10}$$

$$= \lim_{x\to\infty} \frac{36288000}{0.01^{10} e^{0.01x}} \tag{11}$$

$$= \frac{36288000}{\infty} \neq \infty \tag{12}$$

$$g \gg f$$

15. (Section 4.7, Exercise 101)

$$f(x) = x^{20} \tag{1}$$
$$g(x) = 1.00001^x \tag{2}$$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^{20}}{1.00001^x} \tag{1}$$

$$= \frac{2432902008176640000}{\infty} \neq \infty \tag{2}$$

$$g \gg f$$

16. (Section 4.9, Exercise 12)

$$f(x) = 11x^{10}$$

$$\int 11x^{10}\, dx = 11 \int x^{10}\, dx \tag{1}$$

$$= 11\frac{x^{11}}{11} + C \tag{2}$$

$$= x^{11} + C \tag{3}$$

17. (Section 4.9, Exercise 13)

$$f(x) = 2\sin x + 1$$

$$\int 2\sin x + 1\, dx = 2 \int \sin x + \int x^0 \tag{1}$$

$$= -2\cos x + x + C \tag{2}$$

18. (Section 4.9, Exercise 24)

$$\int 3u^{-2} - 4u^2 + 1\, du = 3 \int u^{-2}\, du - 4 \int u^2\, du + \int u^0\, du \tag{1}$$

$$= 3\frac{u^{-1}}{-1} - 4\frac{u^3}{3} + u + C \tag{2}$$

$$= \frac{-3}{u} - \frac{4u^3}{3} + u + C \tag{3}$$

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19. (Section 4.9, Exercise 25)

$$\int 4\sqrt{x} - \frac{4}{\sqrt{x}}\,dx \;=\; 4\int x^{\frac{1}{2}}\,dx - 4\int x^{-\frac{1}{2}}\,dx \tag{1}$$

$$= \; 4\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - 4\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C \tag{2}$$

$$= \; \frac{8x^{\frac{3}{2}}}{3} - 8x^{\frac{1}{2}} + C \tag{3}$$

$$= \; \frac{8x\sqrt{x}}{3} - 8\sqrt{x} + C \tag{4}$$

20. (Section 4.9, Exercise 31)

$$\int (3x+1)(4-x)\,dx \;=\; \int -3x^2 + 11x + 4\,dx \tag{1}$$

$$= \; -3\int x^2\,dx + 11\int x\,dx + 4\int x^0\,dx \tag{2}$$

$$= \; -3\frac{x^3}{3} + 11\frac{x^2}{2} + 4x + C \tag{3}$$

$$= \; -x^3 + \frac{11x^2}{2} + 4x + C \tag{4}$$

21. (Section 4.9, Exercise 35)

$$\int \frac{4x^4 - 6x^2}{x}\,dx \;=\; \int 4x^3 - 6x\,dx \tag{1}$$

$$= \; 4\int x^3\,dx - 6\int x\,dx \tag{2}$$

$$= \; 4\frac{x^4}{4} - 6\frac{x^2}{2} + C \tag{3}$$

$$= \; x^4 - 3x^2 + C \tag{4}$$

22. (Section 4.9, Exercise 40)

$$\int \left(\csc^2\theta + 1\right)d\theta \;=\; \int \csc^2\theta\,d\theta + \int \theta^0\,d\theta \tag{1}$$

$$= \; -\cot\theta + \theta + C \tag{2}$$

23. (Section 4.9, Exercise 41)

$$\int \frac{2 + 3\cos y}{\sin^2 y}\,dy \;=\; \int \left(\frac{2}{\sin^2 y} + \frac{3\cos y}{\sin^2 y}\right)dy \tag{1}$$

$$= \; \int \left(2\csc^2 y + 3\cot y\csc y\right)dy \tag{2}$$

$$= \; 2\int \csc^2 y\,dy + 3\int \cot y\csc y\,dy \tag{3}$$

$$= \; -2\cot y - 3\csc y + C \tag{4}$$

24. (Section 4.9, Exercise 47)

$$\int \left(3t^2 + 2\csc^2 t\right)dt \;=\; \int 3t^2\,dt + \int 2\csc^2 t\,dt \tag{1}$$

$$= \; 3\int t^2\,dt + 2\int \csc^2 t\,dt \tag{2}$$

$$= \; 3\frac{t^3}{3} - 2\cot t + C \tag{3}$$

$$= \; t^3 - 2\cot t + C \tag{4}$$

25. (Section 4.9, Exercise 45)

$$\int \left( sec^2\theta + \sec\theta\tan\theta \right) d\theta \ = \ \int sec^2\theta\, d\theta + \int \sec\theta\tan\theta\, d\theta \tag{1}$$

$$= \ \tan\theta + \sec\theta + C \tag{2}$$

26. (Section 4.9, Exercise 50)

$$\int \frac{\csc^3 x + 1}{\csc x}\, dx \ = \ \int \frac{\csc^3 x}{\csc x} + \frac{1}{\csc x}\, dx \tag{1}$$

$$= \ \int \csc^2 x + \sin x\, dx \tag{2}$$

$$= \ \int \csc^2 x\, dx + \int \sin x\, dx \tag{3}$$

$$= \ -\cot x + -\cos x + C \tag{4}$$

27. (Section 4.9, Exercise 51)

$$\int \frac{1}{2y}\, dy \ = \ \frac{1}{2}\int \frac{1}{y}\, dy \tag{1}$$

$$= \ \frac{1}{2}\ln|y| + C \tag{2}$$

$$= \ \frac{\ln|y|}{2} + C \tag{3}$$

28. (Section 4.9, Exercise 53)

$$\int \frac{6}{\sqrt{4 - 4x^2}}\, dx \ = \ \int \frac{6}{\sqrt{4\left(1 - x^2\right)}}\, dx \tag{1}$$

$$= \ \int \frac{6}{2\sqrt{1 - x^2}}\, dx \tag{2}$$

$$= \ 3\int \frac{1}{\sqrt{1 - x^2}}\, dx \tag{3}$$

$$= \ 3\sin^{-1} x + C \tag{4}$$

29. (Section 4.9, Exercise 59)

$$\int \frac{t + 1}{t}\, dt \ = \ \int \left( \frac{t}{t} + \frac{1}{t} \right) dt \tag{1}$$

$$= \ \int \frac{t}{t}\, dt + \int \frac{1}{t}\, dt \tag{2}$$

$$= \ \int t^0\, dt + \ln|t| + C \tag{3}$$

$$= \ t + \ln|t| + C \tag{4}$$

30. (Section 4.9, Exercise 61)

$$\int e^{x+2}\, dx \ = \ \int e^2 e^x\, dx \tag{1}$$

$$= \ e^2 \int e^x\, dx \tag{2}$$

$$= \ e^2 e^x \tag{3}$$

$$= \ e^{x+2} \tag{4}$$

31. (Section 4.9, Exercise 54)

$$\int \frac{v^3 + v + 1}{1 + v^2}\, dv \ = \ \int \frac{v^3 + v}{1 + v^2} + \frac{1}{1 + v^2}\, dv \tag{1}$$

$$= \ \int v + \frac{1}{1 + v^2}\, dv \tag{2}$$

$$= \ \int v\, dv + \int \frac{1}{1 + v^2}\, dv \tag{3}$$

$$= \ \frac{v^2}{2} + \tan^{-1} v + C \tag{4}$$

32. (Section 4.9, Exercise 62)

$$\int \frac{10t^5 - 3}{t}\, dt \;=\; \int \left( \frac{10t^5}{t} - \frac{3}{t} \right) dt \tag{1}$$

$$=\; \int 10t^4\, dt - \int \frac{3}{t}\, dt \tag{2}$$

$$=\; 10 \int t^4\, dt - 3 \int \frac{1}{t}\, dt \tag{3}$$

$$=\; 10\frac{t^5}{5} - 3\ln|t| + C \tag{4}$$

$$=\; 2t^5 - 3\ln|t| + C \tag{5}$$

33. (Section 4.9, Exercise 78)

$$g'(x) \;=\; 7x^6 - 4x^3 + 12 \tag{1}$$

$$g(1) \;=\; 24 \tag{2}$$

$$g(x) \;=\; \int 7x^6 - 4x^3 + 12\, dx \tag{3}$$

$$=\; 7 \int x^6\, dx - 4 \int x^3\, dx + 12 \int x^0\, dx \tag{4}$$

$$=\; 7\frac{x^7}{7} - 4\frac{x^4}{4} + 12x + C \tag{5}$$

$$=\; x^7 - x^4 + 12x + C \tag{6}$$

$$(1)^7 - (1)^4 + 12(1) + C \;=\; 24 \tag{7}$$

$$1 - 1 + 12 + C \;=\; 24 \tag{8}$$

$$12 + C \;=\; 24 \tag{9}$$

$$C \;=\; 12 \tag{10}$$

$$g(x) \;=\; x^7 - x^4 + 12x + 12 \tag{11}$$

34. (Section 4.9, Exercise 83)

$$y'(t) \;=\; \frac{3}{t} + 6 \tag{1}$$

$$y(1) \;=\; 8, t > 0 \tag{2}$$

$$y(t) \;=\; \int \left( \frac{3}{t} + 6 \right) dt \tag{3}$$

$$=\; 3 \int \frac{1}{t}\, dt + 6 \int t^0\, dt \tag{4}$$

$$=\; 3\ln|t| + 6t + C \tag{5}$$

$$3\ln 1 + 6(1) + C \;=\; 8 \tag{6}$$

$$6 + C \;=\; 8 \tag{7}$$

$$C \;=\; 2 \tag{8}$$

$$y(t) \;=\; 3\ln|t| + 6t + 2 \tag{9}$$

35. (Section 4.9, Exercise 105)

$$v(t) \;=\; \sin t \tag{1}$$

$$s(0) \;=\; 0 \tag{2}$$

$$s(t) \;=\; \int \sin t\, dt \tag{3}$$

$$=\; -\cos t + C \tag{4}$$

$$-\cos 0 + C \;=\; 0 \tag{5}$$

$$-1 + C \;=\; 0 \tag{6}$$

$$C \;=\; 1 \tag{7}$$

$$s(t) \;=\; -\cos t + 1 \tag{8}$$

$$
\begin{align}
V(t) &= \cos t \tag{1}\\
S(0) &= 0 \tag{2}\\
S(t) &= \int \cos t \, dt \tag{3}\\
&= \sin t + C \tag{4}\\
\sin 0 + C &= 0 \tag{5}\\
C &= 0 \tag{6}\\
S(t) &= \sin t \tag{7}
\end{align}
$$

36. (Section 4.9, Exercise 106)

$$
\begin{align}
v(t) &= e^t \tag{1}\\
s(0) &= 0 \tag{2}\\
s(t) &= \int e^t \, dt \tag{3}\\
&= e^t + C \tag{4}\\
e^0 + C &= 0 \tag{5}\\
1 + C &= 0 \tag{6}\\
C &= -1 \tag{7}\\
s(t) &= e^t - 1 \tag{8}
\end{align}
$$

$$
\begin{align}
V(t) &= 2 + \cos t \tag{1}\\
S(0) &= 3 \tag{2}\\
S(t) &= \int 2 + \cos t \, dt \tag{3}\\
&= \int 2 \, dt + \int \cos t \, dt \tag{4}\\
&= 2 \int t^0 \, dt + \sin t + C \tag{5}\\
&= 2t + \sin t + C \tag{6}\\
2(0) + \sin 0 + C &= 3 \tag{7}\\
C &= 3 \tag{8}
\end{align}
$$