

Module 5 Notes (MATH-211)

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General Notes (and Definitions)

- Maxima and Minima

Absolute Maximum: Assume a function f is defined on a set D , and $x = c$ is a point in D . Then, $y = f(c)$ is an **absolute maximum value** of f on D if $f(c) \geq f(x)$ for every x in D . Changing the set on which f is defined may change the absolute maximum value.

Absolute Minimum: Assume a function f is defined on a set D , and $x = c$ is a point in D . Then, $y = f(c)$ is an **absolute minimum value** of f on D if $f(c) \leq f(x)$ for every x in D . Changing the set on which f is defined may change the absolute minimum value.

Extreme Value Theorem: A function that is continuous on a closed interval is guaranteed to have both an absolute maximum value and an absolute minimum value.

A discontinuous function, or a function defined on an interval that is not closed, may still have absolute extrema.

Local Maximum and Minimum Values: Assume $x = c$ is an interior point (not an endpoint) of some interval I in the domain of f . Then, $y = f(c)$ is a **local maximum value** of f if $f(c) \geq f(x)$ for every x in I , and $y = f(c)$ is a **local minimum value** of f if $f(c) \leq f(x)$ for every x in I .

Critical Points: An interior point $x = c$ of the domain of f is called a **critical point** of f if either $f'(c) = 0$ or $f'(c)$ does not exist.

Local Extreme Value Theorem: If a function f has a local maximum or a local minimum at a point $x = c$, then either $f'(c) = 0$ or $f'(c)$ does not exist.

If f has a local extreme, it must occur at a critical point.

Not every critical point is the location of a local extreme value.

For a continuous function f on a closed interval $[a, b]$, absolute extremes are guaranteed to exist, and they must occur either at the endpoints of interval or at critical points of f within the interval.

- Mean Value Theorem

Rolle's Theorem: Let f be a continuous function on a closed interval $[a, b]$ that is differentiable on (a, b) , with $f(a) = f(b)$. Then, there is at least one point $x = c$ in (a, b) where $f'(c) = 0$.

Mean Value Theorem: If f is a continuous function on a closed interval $[a, b]$ that is differentiable on (a, b) , then there is at least one point $x = c$ in (a, b) where

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Zero Derivative Implies Constant Function: If f is differentiable on an open interval I , and $f'(x) = 0$ for all x in I , then f is a constant function on I .

Function with Equal Derivative Differ by a Constant: If $f'(x) = g'(x)$ for all x in an open interval I , then $f(x) = g(x) + C$ for some constant C .

- What Derivatives Tell Us

Increasing and Decreasing Functions: Suppose a function f is defined on an interval I . We say f is **increasing** on I if $f(x_2) > f(x_1)$ whenever x_1 and x_2 are in I and $x_2 > x_1$, and we say f is **decreasing** on I if $f(x_2) < f(x_1)$ whenever x_1 and x_2 are in I and $x_2 > x_1$.

Test for Intervals of Increase and Decrease: Suppose a function f is defined on an interval I , and differentiable inside I . If $f'(x) > 0$ at all interior points of I , then f is increasing on I ; If $f'(x) < 0$ at all interior points of I , then f is decreasing on I .

First Derivative Test: Assume f is continuous on an interval containing a critical point c , and that f is differentiable on an interval containing c (except possibly at c itself). Under these conditions:

- If f' changes sign from positive to negative as x increases through c , then f has a local maximum at c .

- If f' changes sign from negative to positive as x increases through c , then f has a local minimum at c .
- If f' is positive on both sides of c , or negative on both sides of c , then f has no local extreme value at c .

One local extremum implies absolute extremum: Suppose f is continuous on an interval I that contains exactly one local extremum $x = c$.

- If f has a local max at c , then $f(c)$ is the absolute max of f on I .
- If f has a local min at c , then $f(c)$ is the absolute min of f on I .

Concavity: Suppose a function f is twice differentiable on an open interval I .

- If f' is increasing on I , then f is **concave up** on I , and $f'' > 0$ on I .
- If f' is decreasing on I , then f is **concave down** on I , and $f'' < 0$ on I .

Inflection Point: Suppose a function f is twice differentiable on an open interval I . If f is continuous at a point c in I and f changes concavity at c , then f has an **inflection point** at c .

Second Derivative Test: Assume f'' is continuous on an open interval containing $x = c$, with $f'(c) = 0$. Under these conditions:

- If $f''(c) > 0$, then f has a local minimum at c .
- If $f''(c) < 0$, then f has a local maximum at c .
- If $f''(c) = 0$, then the test is inconclusive; f may have a local minimum, a local maximum, or neither of these at $x = c$.

Examples

1. Locate absolute maxima and minima from a graph
Absolute Maximum: $f(c)$ and occurs at $x = c$
Absolute Minimum: None, as the $f(b)$ does not exist

2. Locate local maxima and minima from a graph

Absolute Min at $(a, f(a))$
Absolute Max at $(p, f(p))$
Local Max at $(p, f(p))$
Local Max at $(r, f(r))$
Local Min at $(q, f(q))$
Local Min at $(s, f(s))$

3. Find critical points of a function

$$f(t) = t^2 - 2 \ln(t^2 + 1)$$

$$f'(t) = \frac{2t(t+1)(t-1)}{t^2 + 1}$$

Critical Point at $x = -1$

Critical Point at $x = 0$

Critical Point at $x = 1$

4. Find absolute extremes of a continuous function on a closed interval

$$f(x) = \frac{x}{(x^2 + 9)^5}$$

$$f'(x) = \frac{-9x^2 + 9}{(x^2 + 9)^6}$$

$$[-2, 2]$$

$$f(-2) \approx -0.000005$$

$$f(2) \approx 0.000005$$

$$f(-1) = -0.00001$$

$$f(1) = 0.00001$$

Absolute Min at $(-1, f(-1))$

Absolute Max at $(1, f(1))$

5. Application of finding absolute extreme values

$$P(x) = 2x + \frac{128}{x}$$

$$P'(x) = 2 + \frac{-128}{x^2}$$

$$(0, \infty)$$

$$f(8) = 18$$

Absolute min at $(8, 32)$ or a perimeter of 32 units

6. Verifying Rolle's Theorem

$$f(x) = x^3 - 2x^2 - 8x$$

$$f'(x) = 3x^2 - 4x - 8$$

$$[-2, 4]$$

$$f(-2) = 0 = f(4)$$

$$x = \frac{2 + 2\sqrt{7}}{3} \approx 2.43$$

$$x = \frac{2 - 2\sqrt{7}}{3} \approx -1.097$$

$$x = \frac{2 \pm 2\sqrt{7}}{3}$$

7. Verifying the Mean Value Theorem

$$f(x) = x^3 - 2x^2$$

$$f'(x) = 3x^2 - 4x$$

$$[0, 1]$$

$$f'(c) = -1$$

$$(3x - 1)(x - 1) = 0$$

$$x = \frac{1}{3}$$

$$f\left(\frac{1}{3}\right) = -1$$

8. Application of the Mean Value Theorem

$$\frac{30}{27} = \frac{30}{0.45} \approx 66.667$$

$$66.667 > 60$$

9. Find the intervals of increase and decrease of a function

$$f(x) = \frac{x^3}{3} - \frac{5x^2}{2} + 4x$$

$$f'(x) = (x - 4)(x - 1)$$

Critical Points: $x = 1$ and $x = 4$

For some $x \in (-\infty, 1)$, $f(x) > 0$

For some $x \in (1, 4)$, $f(x) < 0$

For some $x \in (4, \infty)$, $f(x) > 0$

f is increasing at the following intervals: $(-\infty, 1)$ and $(4, \infty)$

f is decreasing at the following intervals: $(1, 4)$

10. Use the First Derivative Test to find local extrema

$$f(x) = -x^3 + 9x$$

$$f'(x) = -3x^2 + 9$$

There are critical points at $x = \pm\sqrt{3}$

There is a local minimum at $x = -\sqrt{3}$ and $f(-\sqrt{3}) \approx -10.39230485$

There is a local maximum at $x = \sqrt{3}$ and $f(\sqrt{3}) \approx 10.39230485$

There is an absolute minimum at $x = -\sqrt{3}$ and $f(-\sqrt{3}) \approx -10.39230485$

There is an absolute maximum at $x = 4$ and $f(4) = 28$

11. Find absolute extrema on unclosed intervals of functions with one critical point

$$f(x) = 4x + \frac{1}{\sqrt{x}}$$

$$f'(x) = 4 - \frac{1}{2x\sqrt{x^3}}$$

There is a critical point at $x = 0$ and $x = \frac{1}{4}$

There is a local minimum at $x = \frac{1}{4}$ and $f\left(\frac{1}{4}\right) = 3$ There is a absolute minimum at $x = \frac{1}{4}$ and $f\left(\frac{1}{4}\right) = 3$

12. Find intervals where a function is concave up or concave down and identify inflection points

$$h(t) = 2 + \cos 2t$$

$$h'(t) = -2 \sin 2t$$

$$h''(t) = -5 \cos 2t$$

$$[0, 2\pi]$$

h is concave up at the following intervals: $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right), \left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)$

h is concave up at the following intervals: $\left(0, \frac{\pi}{4}\right), \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right), \left(\frac{7\pi}{4}, 2\pi\right)$

The inflection points for h occur when $t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

13. Use the Second Derivative Test to find local extrema

$$f(x) = \frac{e^x}{x+1}$$

$$f'(x) = \frac{xe^x}{(x+1)^2}$$

$$f''(x) = \frac{e^x(x^2+1)}{(x+1)^3}$$

There is a critical point at $x = 0$

$$f''(0) = 1$$

There is a local minimum at $x = 0$

Related Exercises

- (Section 4.1, Exercise 11)
Absolute Min at $x = c_2$
Absolute Max at $x = b$
- (Section 4.1, Exercise 14)
Absolute Min at $x = c$
Absolute Max at $x = b$
- (Section 4.1, Exercise 15)
Absolute Max at $x = b$
Absolute Min at $x = a$
Local Max at $x = p$
Local Max at $x = r$
Local Min at $x = q$
Local Min at $x = s$

4. (Section 4.1, Exercise 18) Absolute Max at $x = p$
 Absolute Min at $x = u$
 Local Max at $x = p$
 Local Max at $x = r$
 Local Max at $x = t$
 Local Min at $x = q$
 Local Min at $x = s$
 Local Min at $x = u$

5. (Section 4.1, Exercise 35)

$$f(x) = \frac{1}{x} + \ln x$$

$$f'(x) = \frac{x-1}{x^2}$$

Critical Points at $x = 1$

6. (Section 4.1, Exercise 36)

$$f(t) = t^2 - 2 \ln(t^2 + 1)$$

$$f'(t) = \frac{2t(t+1)(t-1)}{t^2 + 1}$$

Critical Points at $t = -1$, $t = 0$ and $t = 1$

7. (Section 4.1, Exercise 46)

$$f(x) = x^4 - 4x^3 + 4x^2$$

$$f'(x) = 4x^3 - 12x^2 + 8x$$

$$[-1, 3]$$

$$f(-1) = 9$$

$$f(0) = 0$$

$$f(1) = 1$$

$$f(2) = 0$$

$$f(3) = 9$$

Absolute Max at $(-1, 9)$ and $(3, 9)$

Absolute Min at $(0, 0)$ and $(2, 0)$

8. (Section 4.1, Exercise 52)

$$f(x) = 3x^{\frac{2}{3}}$$

$$f'(x) = \frac{2}{x^{\frac{1}{3}}}$$

$$[0, 27]$$

$$f(0) = 0$$

$$f(27) = 27$$

Absolute Min at $(0, 0)$

Absolute Min at $(27, 27)$

9. (Section 4.1, Exercise 73)

$$s(t) = -16t^2 + 64t + 192$$

$$s'(t) = -32t + 64$$

$$0 \leq t \leq 6$$

$$s(0) = 192$$

$$s(2) = 256$$

$$s(6) = 0$$

The stone will reach its maximum height at 2 seconds

10. (Section 4.2, Exercise 11)

$$\begin{aligned}f(x) &= x(x-1)^2 \\f'(x) &= (x-1)^2 + 2x(x-1) \\&[0, 1] \\f(0) &= 0 \\f(1) &= 0 \\f'\left(\frac{1}{3}\right) &= 0\end{aligned}$$

11. (Section 4.2, Exercise 16)

$$\begin{aligned}f(x) &= x^3 - 2x^2 - 8x \\f'(x) &= 3x^2 - 4x - 8 \\&[-2, 4] \\f(-2) &= 0 \\f(4) &= 0 \\x &\approx -1.097 \\x &\approx 2.431\end{aligned}$$

12. (Section 4.2, Exercise 19)

$$\begin{aligned}f(6.1) &= -10.3 \\f(3.2) &= 8.0 \\ \frac{-10.3 - 8.0}{6.1 - 3.2} &= \frac{-18.3}{2.9} & (1) \\ &\approx -6.3 & (2)\end{aligned}$$

Because the average lapse rate is approximately -6.3 , we are unable to conclude that it exceeds 7.

13. (Section 4.2, Exercise 42)

- (a) Formations of a weak layer are likely as the following temperature gradient is greater than 10 degrees celsius.

$$\frac{14}{1.1} \approx 12.72$$

- (b) Formations of a weak layer are not likely as the following temperature gradient is less than 10 degrees celsius.

$$\frac{11}{1.4} \approx 7.86$$

- (c) A weak layer is more likely to form when there is less of a difference in the deepness of the snowpack, as there is a higher chance of a greater temperature gradient.
- (d) A weak layer most likely will not form in isothermal snow because if the temperatures are the same, then we know the value of the temperature gradient would be 0.

14. (Section 4.2, Exercise 21)

$$\begin{aligned}f(x) &= 7 - x^2 \\f'(x) &= -2x \\&[-1, 2] \\f(-1) &= 6 \\f(2) &= 3 \\f(c) &= \frac{-3}{3} = -1 \\c &= \frac{1}{2}\end{aligned}$$

15. (Section 4.2, Exercise 22)

$$f(x) = x^3 - 2x^2$$

$$f'(x) = 3x^2 - 4x$$

$$[0, 1]$$

$$f(0) = 0$$

$$f(1) = -1$$

$$f(c) = \frac{-1}{1} = -1$$

$$c = \frac{1}{3}$$

16. (Section 4.3, Exercise 22)

$$f(x) = x^3 + 4x$$

$$f'(x) = 3x^2 + 4$$

f is increasing on the interval(s): $(-\infty, \infty)$

17. (Section 4.3, Exercise 27)

$$f(x) = -\frac{x^4}{4} + x^3 - x^2$$

$$f'(x) = -x^3 + 3x^2 - 2x$$

There are Critical Points at $x = 0$, $x = 1$, and $x = 2$

f is increasing on the interval(s): $(-\infty, 0)$ and $(1, 2)$

f is decreasing on the interval(s): $(0, 1)$ and $(2, \infty)$

18. (Section 4.3, Exercise 29)

$$f(x) = x^2 \ln x^2 + 1$$

$$f'(x) = 2x (\ln x^2 + 1)$$

There are Critical Points at $x = 0$, $x = \frac{1}{\sqrt{e}}$ and $x = -\frac{1}{\sqrt{e}}$

f is increasing on the interval(s): $(-\frac{1}{\sqrt{e}}, 0)$ and $(\frac{1}{\sqrt{e}}, \infty)$

f is decreasing on the interval(s): $(-\infty, -\frac{1}{\sqrt{e}})$ and $(0, \frac{1}{\sqrt{e}})$

19. (Section 4.3, Exercise 49)

$$f(x) = -x^3 + 9x$$

$$f'(x) = -3x^2 + 9$$

$$[-4, 3]$$

There are Critical Points at $x = \pm\sqrt{3}$

There is a local minimum at $x = -\sqrt{3}$

There is a local maximum at $x = \sqrt{3}$

There is an absolute minimum at $x = -6\sqrt{3}$

There is an absolute maximum at $x = -4$

20. (Section 4.3, Exercise 50)

$$f(x) = 2x^5 - 5x^4 - 10x^3 + 4$$

$$f'(x) = 10x^4 - 20x^3 - 30x^2$$

$$[-2, 4]$$

There are Critical Points at $x = -1$, $x = 0$ and $x = 3$

There is a local maximum at $x = -1$

There is a local minimum at $x = 3$

There is an absolute maximum at $x = 4$

There is an absolute minimum at $x = 3$

21. (Section 4.3, Exercise 51)

$$f(x) = x^{\frac{2}{3}}(x - 5)$$
$$f'(x) = \frac{5x - 10}{3x^{\frac{1}{3}}}$$
$$[-5, 5]$$

There are Critical Points at $x = 0$ and $x = 2$

There is a local minimum at $x = 2$

There is a local maximum at $x = 0$

There is an absolute minimum at $x = -5$

There is an absolute maximum at $x = 0$ and $x = 5$

22. (Section 4.3, Exercise 53)

$$f(x) = \sqrt{x} \ln x$$
$$f'(x) = \frac{\ln x + 2}{2\sqrt{x}}$$
$$(0, \infty)$$

There are Critical Points at $x = e^{-2}$

There is a local minimum at $x = e^{-2}$

There is an absolute minimum at $x = e^{-2}$

23. (Section 4.3, Exercise 55)

$$f(x) = xe^{-x}$$
$$f'(x) = \frac{1 - x}{e^x}$$

There are Critical Points at $x = 1$

There is a local maximum at $x = 1$

There is an absolute maximum at $x = 1$

24. (Section 4.3, Exercise 56)

$$f(x) = 4x + \frac{1}{\sqrt{x}}$$
$$f'(x) = 4 - \frac{1}{2x\sqrt{x}}$$

There are Critical Points at $x = 0$ and $x = \frac{1}{4}$

There is a local minimum at $x = 0.25$

There is an absolute minimum at $x = 0.25$

25. (Section 4.3, Exercise 64)

$$f(x) = -x^4 - 2x^3 + 12x^2$$
$$f'(x) = -4x^3 - 6x^2 + 24x$$
$$f''(x) = -12x^2 - 12x + 24$$

The inflection points for f are at $x = -2$ and $x = 1$

f is concave up at the following intervals: $(-2, 1)$

f is concave down at the following intervals: $(-\infty, -2), (1, \infty)$

26. (Section 4.3, Exercise 67)

$$f(x) = e^x(x - 3)$$
$$f'(x) = xe^x - 2e^x$$
$$f''(x) = xe^x - e^x$$

The inflection points for f are at $x = 1$

f is concave up at the following intervals: $(1, \infty)$

f is concave down at the following intervals: $(-\infty, 1)$

27. (Section 4.3, Exercise 78)

$$f(x) = 6x^2 - x^3$$

$$f'(x) = 12x - 3x^2$$

$$f''(x) = 12 - 6x$$

There are Critical Points at $x = 0$ and $x = 4$

There is a local minimum at $x = 0$

There is a local maximum at $x = 4$

28. (Section 4.3, Exercise 80)

$$f(x) = x^3 - \frac{3}{2}x^2 - 36x$$

$$f'(x) = 3x^2 - 3x - 36$$

$$f''(x) = 6x - 3$$

There are Critical Points at $x = -3$ and $x = 4$

There is a local maximum at $x = -3$

There is a local minimum at $x = 4$