Module 1 Notes (MATH-211)

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General Notes (and Definitions)

- Limit Definition(s):
  - Simple: The value that the outputs of a function approach as inputs approach a certain value
  - Preliminary: Suppose a function f is defined for all x near a except possibly at a. If $f(x)$ is arbitrarily close to L all x sufficiently close (but not equal) to a, we write the following.

$$\lim_{x \to a} = L$$

- Secant Line: a line passing through two points $(t_0, s(t_0))$ and $(t_1, s(t_1))$. The slope is given by

$$\frac{s(t_1) - s(t_0)}{t_1 - t_0}$$

- Tangent Line: the line passing through $(t_0, s(t_0))$ with slope

$$\lim_{t \to t_0} \frac{s(t) - s(t_0)}{t - t_0}$$

- One Sided limits:
  - Right-hand (Definition): Suppose a function f is defined for all x near a with $x > a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x > a$ we write

$$\lim_{x \to a^+} f(x) = L$$

  - Left-hand (Definition): Suppose a function f is defined for all x near a with $x < a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x < a$ we write

$$\lim_{x \to a^-} f(x) = L$$

  - In order for their to be a double sided limit, we must have:

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

  - If the limits from sides are not equal, then a the double sided limit, "does not exist"
- Limits can be simplified/solved in an easier way (as compared to numerically/graphically) using Limit Rules/Laws
- Limit Example Types:
  - Tangent lines
  - Velocity
- Velocity
  - Average Veolcity
    - The average velocity over some interval $[t_0, t_1]$ is defined as

$$v_{av} = \frac{s(t_1) - s(t_0)}{t_1 - t_0}$$

- Instantaneous Veolcity
  - The average velocity over some interval $[t_0, t_1]$ is defined as

$$v_{inst} = \lim_{t \to a} v_{av} == \frac{s(t) - s(a)}{t - a}$$

- Solving Techniques
  - Factoring and canceling out
  - Using conjugates
    - When direct substitution is not possible, you may rationalize the numerator
- Infinite Limits: In either case, the limit does not exist (not a real number) if it is infinite
  - Suppose f is defined for all x near a. If $f(x)$ gorws arbitrarily large for all x sufficinetly close (but not equal) to a, we write
$$\lim_{x \to a} f(x) = \infty$$
  - If $f(x)$ is negative and gorws arbitrarily large in magnitude for all x sufficinetly close (but not equal) to a, we write
$$\lim_{x \to a} f(x) = -\infty$$
  - The line $x = a$ is a vertical asymptote for f if any of the following hold

$$\lim_{x \to a} f(x) = \pm\infty$$

$$\lim_{x \to a^+} f(x) = \pm\infty$$

$$\lim_{x \to a^-} f(x) = \pm\infty$$

  - A vertical asymptote exists at $x = a$ if any one sided limit as $x \to a$ is ∞ or $-\infty$
  - If you have a limit of a rational function, where $p(a) = L \neq 0$ and $q(a) = 0$, then the one sided limits for $\frac{p(x)}{q(x)}$ approach $\pm\infty$
$$\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{L}{0}$$

Limit Rules/Laws

Assume $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist.

The following properties hold where c is a real number, and $n > 0$ is an integer.

- **Sum Rule**
$$\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$
- **Difference Rule**
$$\lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$
- **Constant Multiple Rule**
$$\lim_{x \to a} (cf(x)) = c \lim_{x \to a} f(x)$$
- **Product Rule**
$$\lim_{x \to a} (f(x)g(x)) = (\lim_{x \to a} f(x))(\lim_{x \to a} g(x))$$
- **Quotient Rule**
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}, \text{ provided } \lim_{x \to a} g(x) \neq 0$$
- **Power Rule**
$$\lim_{x \to a} f(x)^n = (\lim_{x \to a} f(x))^n$$

- **Root Rule**
$$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}, \text{ provided} f(x) > 0, \text{ for } x \text{ near } a, \text{ if } n \text{ is even}$$

- **Polynomials**
A **Polynomial** is defined as A function of the form $x_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$ where $n \geq 0$ is an integer If $p(x)$ is a polynomial then:
$$\lim_{x\to a} p(x) = p(a)$$
If $p(x)$ and $q(x)$ are polynomials and $q(a) \neq 0$ then (Direct Substitution):
$$\lim_{x\to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$$

- **The Squeeze Theorem**
Assume for some functions f, g and h that satisfy $f(x) \leq g(x) \leq h(x)$ for x near a (except possibly at $x = a$). If
$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$
then
$$\lim_{x\to a} g(x) = L$$
As $x \to a, h(x) \to L$. Therefore, $g(x) \to L$. As x approaches a, if f and h approach the same value, so does g.

Examples

1. (Describing Limits) As x approaches 3, x^2 approaches 9
$$\lim_{x\to 3} x^2 = 9$$

2. (Common Use) Values that are undefined can still have limits, given a graph G where $f(3) =$ undefined ($f(3)$ is a hole), the following limit is valid:
$$\lim_{x\to 3} f(x) = 4$$

3. Calculating Limits Numerically:
$$f(x) = \frac{x^3 - 1}{x - 1}$$

0.9	0.99	0.999	0.9999
2.71	2.9701	2.997001	2.99970001

1.1	1.01	1.001	1.0001
3.31	3.0301	3.003001	3.00030001

As x approaches 1, $f(x)$ approaches 3: $\lim_{x\to 1} \frac{x^3-1}{x-1} = 3$

4. Calculating One-sided limits:
$$g(x) = \frac{x^3 - 4x}{8|x - 2|}$$

1.9	1.09	1.009	1.0009
-0.92625	-0.9925125	-0.999250125	-0.9999250013

2.1	2.01	2.001	2.0001
1.07625	1.0075125	1.000750125	1.000075001

$$\lim_{x\to 2} g(x) = \text{Does not exist}$$
$$\lim_{x\to 2^-} g(x) = -1$$
$$\lim_{x\to 2^+} g(x) = 1$$

5. Calculating piecewise function limits

$$f(x) = \begin{cases} 3 - x \text{ if } x < 2 \\ x - 1 \text{ if } x > 2 \end{cases}$$

$$a = 2$$

1.92	1.99	1.999	1.9999
1.1	1.01	1.001	1.0001

2.1	2.01	2.001	2.0001
1.1	1.01	1.001	1.0001

Explanation: Since $f(2)$ is not defined within the piece wise function, a graph representing this function would have a whole where $x = a$ and have two lines with inverse slopes

$$f(a) = \text{undefined}$$
$$\lim_{x\to a} f(x) = 1$$
$$\lim_{x\to a^-} f(x) = 1$$
$$\lim_{x\to a^+} f(x) = 1$$

6. Limit Rules/Laws:

   (a) Definitions:

$$\lim_{x\to 3} f(x) = 2$$
$$\lim_{x\to 3} g(x) = -1$$
$$\lim_{x\to 3} h(x) = 6$$

   (b) Problems:

      i. Sum, Constant Multiple

$$\begin{aligned} \lim_{x\to 3}(f(x) + 2g(x)) &= \lim_{x\to 3} f(x) + \lim_{x\to 3} 2g(x) && (1) \\ &= \lim_{x\to 3} f(x) + 2(\lim_{x\to 3} g(x)) && (2) \\ &= 2 + 2(-1) && (3) \\ &= 0 && (4) \end{aligned}$$

      ii. Quotient

$$\begin{aligned} \lim_{x\to 3} \frac{h(x)}{g(x)} &= \frac{\lim_{x\to 3} h(x)}{\lim_{x\to 3} g(x)} && (1) \\ &= \frac{6}{-1} && (2) \\ &= -6 && (3) \end{aligned}$$

      iii. Quotient, Root, Difference

$$\begin{aligned} \lim_{x\to 3} \frac{h(x)}{\sqrt{f(x) - g(x)}} &= \frac{\lim_{x\to 3} h(x)}{\lim_{x\to 3} \sqrt{f(x) - g(x)}} && (1) \\ &= \frac{\lim_{x\to 3} h(x)}{\sqrt{\lim_{x\to 3}(f(x) - g(x))}} && (2) \\ &= \frac{\lim_{x\to 3} h(x)}{\sqrt{\lim_{x\to 3} f(x) - \lim_{x\to 3} g(x)}} && (3) \\ &= \frac{6}{\sqrt{2+1}} && (4) \\ &= \frac{6}{\sqrt{3}} && (5) \\ &= 2\sqrt{3} && (6) \end{aligned}$$

7.

$$\begin{aligned}
\lim_{x\to 1}\frac{3x^2-7x+1}{x+2} &= \frac{3(1)^2-7(1)+1}{1+2} && (1)\\
&= \frac{3-7+1}{1+2} && (2)\\
&= \frac{-3}{3} && (3)\\
&= -1 && (4)
\end{aligned}$$

8.

$$\begin{aligned}
\lim_{x\to 4}\frac{\left(\frac{1}{x}-\frac{1}{4}\right)}{x-4} &= \lim_{x\to 4}\frac{\left(\frac{4}{4x}-\frac{x}{4x}\right)}{x-4} && (1)\\
&= \lim_{x\to 4}\frac{\left(\frac{4-x}{4x}\right)}{x-4} && (2)\\
&= \lim_{x\to 4}\frac{\left(\frac{4-x}{4x}\right)}{\left(\frac{x-4}{1}\right)} && (3)\\
&= \lim_{x\to 4}\left(\frac{4-x}{4x}\right)\left(\frac{1}{x-4}\right) && (4)\\
&= \lim_{x\to 4}\frac{4-x}{4x(x-4)} && (5)\\
&= \lim_{x\to 4}\frac{-(-4+x)}{4x(x-4)} && (6)\\
&= \lim_{x\to 4}\frac{-(x-4)}{4x(x-4)} && (7)\\
&= \lim_{x\to 4}\frac{-1}{4x} && (8)\\
&= \lim_{x\to 4}\frac{-1}{4(4)} && (9)\\
&= -\frac{1}{16} && (10)
\end{aligned}$$

9.

$$\begin{aligned}
\lim_{x\to 9}\frac{x-9}{\sqrt{x}-3} &= \lim_{x\to 9}\frac{x-9}{\sqrt{x}-3}\cdot\frac{\sqrt{x}+3}{\sqrt{x}+3} && (1)\\
&= \lim_{x\to 9}\frac{(x-9)(\sqrt{x}+3)}{(\sqrt{x}-3)(\sqrt{x}+3)} && (2)\\
&= \lim_{x\to 9}\frac{(x-9)(\sqrt{x}+3)}{x-9} && (3)\\
&= \lim_{x\to 9}\sqrt{x}+3 && (4)\\
&= \sqrt{9}+3 && (5)\\
&= 3+3 && (6)\\
&= 6 && (7)
\end{aligned}$$

10.

$$1-\frac{x^2}{2}\le\cos x\le 1$$

$$\begin{aligned}
\lim_{x\to 0}\left(1-\frac{x^2}{2}\right) &= 1-\frac{0^2}{2} && (1)\\
&= 1-0 && (2)\\
&= 1 && (3)\\
&= \lim_{x\to 0}1 && (4)\\
\lim_{x\to 0}\cos x &= 1 \qquad \text{(By the Squeeze Theorem)} && (5)
\end{aligned}$$

11.

$$\begin{aligned}
\lim_{x\to 0} \sin x &= 0 && \text{(By the Squeeze Theorem)} \quad (1)\\
\lim_{x\to 0} \cos x &= 1 && \text{(By the Squeeze Theorem)} \quad (2)
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin 2x}{\sin x} &= \lim_{x\to 0} \frac{2\sin x \cos x}{\sin x} && (1)\\
&= \lim_{x\to 0} 2\cos x && (2)\\
&= 2\lim_{x\to 0} \cos x && (3)\\
&= 2\cdot 1 && (4)\\
&= 2 && (5)
\end{aligned}$$

12. Infinite Limits Numerically

$$f(x) = \frac{x}{(x-2)^2}$$

2.1	2.01	2.001	2.0001
210	20100	2001000	200010000

1.9	1.99	1.999	1.9999
190	19900	1999000	199990000

$$\lim_{x\to 2} f(x) = \infty$$

13. Infinite Limits Graphically

$$\lim_{x\to -2^-} h(x) = -\infty$$

$$\lim_{x\to -2^+} h(x) = -\infty$$

$$\lim_{x\to -2} h(x) = -\infty$$

$$\lim_{x\to 3^-} h(x) = \infty$$

$$\lim_{x\to 3^+} h(x) = -\infty$$

$$\lim_{x\to 3} h(x) = \text{Does not exist}$$

14. Infinite Limits Analytically
Hint: Look at the signs of the fractions

$$\frac{x^2-5x+6}{x^4-4x^2} = \frac{(x-3)(x-2)}{x^2(x+2)(x-2)} = \frac{x-3}{x^2(x+2)}$$

$$\lim_{x\to -2^+} \frac{x^2-5x+6}{x^4-4x^2} = \lim_{x\to -2^+} \frac{x-3}{x^2(x+2)} = -\infty$$

$$\lim_{x\to -2^-} \frac{x^2-5x+6}{x^4-4x^2} = \lim_{x\to -2^-} \frac{x-3}{x^2(x+2)} = \infty$$

$$\lim_{x\to -2} \frac{x^2-5x+6}{x^4-4x^2} = \text{Does not exist}$$

15. Infinite Limits Analytically with Square Root

$$\lim_{x\to 1^+} \frac{x+3}{\sqrt{x^2-5x+4}} = \lim_{x\to 1^+} \frac{x+3}{\sqrt{(x-4)(x-1)}} = \text{Does not exist}$$

$$\lim_{x\to 1^-} \frac{x+3}{\sqrt{x^2-5x+4}} = \lim_{x\to 1^-} \frac{x+3}{\sqrt{(x-4)(x-1)}} = \infty$$

$$\lim_{x\to 1} \frac{x+3}{\sqrt{x^2-5x+4}} = \text{Does not exist}$$

16. Infinite Limit with a Trigonometric Function

$$\lim_{\theta\to 0^-} \frac{\sin\theta}{\cos^2\theta - 1} = \lim_{\theta\to 0^-} \frac{\sin\theta}{-\sin^2\theta} = \lim_{\theta\to 0^-} \frac{1}{-\sin^\theta} = \infty$$

17. Locating Veritical Asymptotes

$$f(x) = \frac{x+7}{x^4 - 49x^2} = \frac{x+7}{x^2(x^2-49)} = \frac{x+7}{x^2((x-7)(x+7))} = \frac{1}{x^2(x-7)}$$

Denominator is 0 at $x = 0$, $x = -7$, $x = 7$
$x = -7$ does not fit, as it is connected with $x + 7$, but cancels out
Vertical Asympotetes: $x = 0$, $x = 7$

18. (Section 2.1, Related Exercise 13):
Hint: use the secant line slope formula

$$s(t) = -16t^2 + 128t$$

(a) $[1, 4]$

$$\frac{256 - 112}{4 - 1} = \frac{144}{3} = 48$$

(b) $[1, 3]$

$$\frac{240 - 112}{3 - 1} = \frac{128}{2} = 64$$

(c) $[1, 2]$

$$\frac{192 - 112}{2 - 1} = \frac{80}{1} = 84$$

(d) $[1, 1 + h]$, where $h > 0$ is a real number

$$\frac{112 + -16h^2 + 128h - 112}{1 + h - 1} = \frac{-16h^2 + 128h}{h} = -16h + 128 = 16(-h + 6)$$

19. (Section 2.1, Related Exercise 15): Hint: we use the slope formula for the secant line, and the relationship is referring to the interval

$$s(t) = -16^t + 100t$$

$$\begin{aligned}
\frac{s(t_1) - s(t_0)}{t_1 - t_0} &= \frac{s(2) - s(0.5)}{2 - 0.5} && (1)\\
&= \frac{136 - 46}{1.5} && (2)\\
&= \frac{90}{1.5} && (3)\\
&= 60 && (4)
\end{aligned}$$

The slope of this secant line, through the lens of average velocity could be viewed as the average velocity over the interval $[0.5, 2]$

20. (Section 2.1, Related Exercise 17):

$$s(t) = -16t^2 + 128t$$

$[1, 2]$	$[1, 1.5]$	$[1, 1.1]$	$[1, 1.01]$	$[1, 1.001]$
80	88	94.4	95.84	95.984

$$v_{inst} = \lim_{t\to 1} s(t) = 96$$

21. (Section 2.1, Related Exercise 19):

$$s(t) = -16t^2 + 100t$$

$[2, 3]$	$[2.9, 3]$	$[2.99, 3]$	$[2.999, 3]$	$[2.9999, 3]$
20	5.6	4.16	4.016	4.002

$$v_{inst} = \lim_{t\to 3} s(t) = 4$$

22. (Section 2.2, Related Exercise 3):
    - $h(2) = 5$
    - $\lim_{x \to 2} h(x) = 3$
    - $h(4) =$ Does not exist
    - $\lim_{x \to 4} h(x) = 1$
    - $\lim_{x \to 5} h(x) = 2$
23. (Section 2.2, Related Exercise 4):
    - $g(0) = 0$
    - $\lim_{x \to 0} g(x) = 1$
    - $g(1) = 2$
    - $\lim_{x \to 1} g(x) = 2$
24. (Section 2.2, Related Exercise 7):

$$f(x) = \frac{x^2 - 4}{x - 2}$$

1.9	1.99	1.999	1.9999
3.9	3.99	3.999	3.9999

2.1	2.01	2.001	2.0001
4.1	4.01	4.001	4.0001

$$\lim_{x \to 2} f(x) = 4$$

25. (Section 2.2, Related Exercise 8):

$$f(x) = \frac{x^3 - 1}{x - 1}$$

0.9	0.99	0.999	0.9999
2.71	2.9701	3.997001	3.99970001

1.1	1.01	1.001	1.0001
3.31	3.0301	3.003001	3.00030001

$$\lim_{x \to 1} f(x) = 3$$

26. (Section 2.2, Related Exercise 27):

$$f(x) = \frac{x - 2}{\ln |x - 2|}$$

$$\lim_{x \to 2} f(x) = 2$$

27. (Section 2.2, Related Exercise 28):

$$f(x) = \frac{e^{2x} - 2x - 1}{x^2}$$

$$\lim_{x \to 0} f(x) = 0$$

28. (Section 2.2, Related Exercise 19):

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x \leq -1 \\ 3 & \text{if } x > -1 \end{cases}$$

$$\lim_{x \to -1^-} f(x) = 2$$

$$\lim_{x \to -1^+} f(x) = 3$$

$$\lim_{x \to -1} f(x) = \text{Does not exist}$$

29. (Section 2.2, Related Exercise 20):

$$f(x) = \begin{cases} 3 - x & \text{if } x < 2 \\ x - 1 & \text{if } x > 2 \end{cases}$$

$$\lim_{x \to 2^-} f(x) = 1$$

$$\lim_{x \to 2^+} f(x) = 1$$

$$\lim_{x \to 2} f(x) = 1$$

30. (Section 2.3, Related Exercise 19):

$$\lim_{x \to 4} 3x - 7 = 3(4) - 7 = 12 - 7 = 5$$

31. (Section 2.3, Related Exercise 22):

$$\lim_{x \to 6} 4 = 4$$

32. (Section 2.3, Related Exercise 11): Quotient, Difference

$$\begin{aligned}
\lim_{x \to 1} \frac{f(x)}{g(x) - h(x)} &= \frac{\lim_{x \to 1} f(x)}{\lim_{x \to 1} g(x) - h(x)} && (1) \\
&= \frac{\lim_{x \to 1} f(x)}{\lim_{x \to 1} g(x) - h(x)} && (2) \\
&= \frac{\lim_{x \to 1} f(x)}{\lim_{x \to 1} g(x) - \lim_{x \to 1} h(x)} && (3) \\
&= \frac{8}{3 - 2} && (4) \\
&= \frac{8}{1} && (5) \\
&= 8 && (6)
\end{aligned}$$

33. (Section 2.3, Related Exercise 12): Root, Sum, Product

$$\begin{aligned}
\lim_{x \to 1} \sqrt[3]{f(x)g(x) + 3} &= \sqrt[3]{\lim_{x \to 1} f(x)g(x) + 3} && (1) \\
&= \sqrt[3]{\lim_{x \to 1} f(x)g(x) + 3} && (2) \\
&= \sqrt[3]{\lim_{x \to 1} f(x)g(x) + \lim_{x \to 1} 3} && (3) \\
&= \sqrt[3]{\lim_{x \to 1} f(x) \lim_{x \to 1} g(x) + \lim_{x \to 1} 3} && (4) \\
&= \sqrt[3]{8 \cdot 3 + 3} && (5) \\
&= \sqrt[3]{24 + 3} && (6) \\
&= \sqrt[3]{27} && (7) \\
&= 3 && (8)
\end{aligned}$$

34. (Section 2.3, Related Exercise 25):

$$\begin{aligned}
\lim_{x \to 1} \frac{5x^2 + 6x + 1}{8x - 4} &= \frac{5(1^2) + 6(1) + 1}{8(1) - 4} && (1) \\
&= \frac{5 + 6 + 1}{8 - 4} && (2) \\
&= \frac{12}{4} && (3) \\
&= 3 && (4)
\end{aligned}$$

35. (Section 2.3, Related Exercise 26):

$$\begin{aligned}
\lim_{t\to 3}\sqrt[3]{t^2-10} &= \sqrt[3]{\lim_{t\to 3} t^2-10} && (1)\\
&= \sqrt[3]{3^2-10} && (2)\\
&= \sqrt[3]{9-10} && (3)\\
&= \sqrt[3]{-1} && (4)\\
&= -1 && (5)
\end{aligned}$$

36. (Section 2.3, Related Exercise 27):

$$\begin{aligned}
\lim_{p\to 2}\frac{3p}{\sqrt{4p+1}-1} &= \frac{\lim_{p\to 2} 3p}{\lim_{p\to 2}\sqrt{4p+1}-1} && (1)\\
&= \frac{3(2)}{\sqrt{\lim_{p\to 2} 4p+1}-1} && (2)\\
&= \frac{6}{\sqrt{4(2)+1}-1} && (3)\\
&= \frac{6}{\sqrt{8+1}-1} && (4)\\
&= \frac{6}{\sqrt{9}-1} && (5)\\
&= \frac{6}{3-1} && (6)\\
&= \frac{6}{2} && (7)\\
&= 3 && (8)
\end{aligned}$$

37. (Section 2.3, Related Exercise 72):

$$g(x) = \begin{cases} 5x-15 & \text{if } x<4 \\ \sqrt{6x+1} & \text{if } x \geq 4 \end{cases}$$

$$\lim_{x\to 4^-} g(x) = 5$$

$$\lim_{x\to 4^+} g(x) = 5$$

$$\lim_{x\to 4} g(x) = 5$$

38. (Section 2.3, Related Exercise 73):

$$g(x) = \begin{cases} x^2+1 & \text{if } x<-1 \\ \sqrt{x+1} & \text{if } x \geq -1 \end{cases}$$

$$\lim_{x\to -1^-} g(x) = 2$$

$$\lim_{x\to -1^+} g(x) = 0$$

$$\lim_{x\to -1} g(x) = \text{Does not exist}$$

39. (Section 2.3, Related Exercise 34):

$$\begin{aligned}
\lim_{x\to 3}\frac{x^2-2x-3}{x-3} &= \lim_{x\to 3}\frac{(x-3)(x+1)}{x-3} && (1)\\
&= \lim_{x\to 3} x+1 && (2)\\
&= 3+1 && (3)\\
&= 4 && (4)
\end{aligned}$$

40. (Section 2.3, Related Exercise 41):

$$\begin{aligned}
\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9} &= \lim_{x\to 9}\frac{\sqrt{x}-3}{x-9}\cdot\frac{\sqrt{x}+3}{\sqrt{x}+3} && (1)\\
&= \lim_{x\to 9}\frac{(\sqrt{x}-3)(\sqrt{x}+3)}{(x-9)(\sqrt{x}+3)} && (2)\\
&= \lim_{x\to 9}\frac{x-9}{(x-9)(\sqrt{x}+3)} && (3)\\
&= \lim_{x\to 9}\frac{1}{\sqrt{x}+3} && (4)\\
&= \frac{1}{\sqrt{9}+3} && (5)\\
&= \frac{1}{3+3} && (6)\\
&= \frac{1}{6} && (7)
\end{aligned}$$

41. (Section 2.3, Related Exercise 69):

$$\lim_{x\to 1^+}\frac{x-1}{\sqrt{x^2-1}} = \text{Does not exist}$$

42. (Section 2.3, Related Exercise 70):

$$\begin{aligned}
\lim_{x\to 1^+}\frac{x-1}{\sqrt{x^2-1}} &= \lim_{x\to 1^+}\frac{x-1}{\sqrt{x^2-1}}\cdot\frac{x+1}{x+1} && (1)\\
&= \lim_{x\to 1^+}\frac{x^2-1}{\sqrt{x^2-1}(x+1)} && (2)\\
&= \lim_{x\to 1^+}\frac{x^2-1}{(x^2-1)^{\frac{1}{2}}(x+1)} && (3)\\
&= \lim_{x\to 1^+}\frac{(x^2-1)^{\frac{1}{2}}}{x+1} && (4)\\
&= \lim_{x\to 1^+}\frac{\sqrt{x^2-1}}{x+1} && (5)\\
&= \frac{\sqrt{1-1}}{1+1} && (6)\\
&= \frac{\sqrt{0}}{2} && (7)\\
&= \frac{0}{2} && (8)\\
&= 0 && (9)
\end{aligned}$$

43. (Section 2.3, Related Exercise 95):

$$\frac{2^x-2^0}{x-0} = \frac{2^x-1}{x}$$

-1	-0.1	-0.01	-0.001	-0.0001	-0.00001
0.5	0.6696700846	0.6907504563	0.6929070095	0.6931231585	0.6931447783

$$\lim_{x\to 0^1}\frac{2^x-1}{x} = 0.693$$

44. (Section 2.3, Related Exercise 96):

$$\frac{3^x-3^0}{x-0} = \frac{3^x-1}{x}$$

-0.1	-0.01	-0.001	-0.0001
1.040415402	1.092599583	1.098009035	1.098551943

0.0001	0.001	0.01	0.1
1.098672638	1.099215984	1.104669194	1.161231740

$$\lim_{x\to 0^1} \frac{3^x - 1}{x} = 1.0986$$

45. (Section 2.3, Related Exercise 81):
$-|x| < 0 < |x|$ and $\sin\frac{1}{x} \leq 1$, so $|x|\sin\frac{1}{x} \leq |x|$ and $-|x|\sin\frac{1}{x} \geq -|x|$

$$\lim_{x\to 0} -|x| = -|0| = 0$$

$$\lim_{x\to 0} |x| = |0| = 0$$

$$\lim_{x\to 0} x\sin\frac{1}{x} = 0$$

By the Squeeze Theorem, since $\lim_{x\to 0} -|x| = \lim_{x\to 0} |x|$ and the functions are chronologically greater than the last

46. (Section 2.3, Related Exercise 82):

$$\lim_{x\to 0} 1 - \frac{x^2}{2} = 1 - \frac{0}{2} = 1 - 0 = 1$$

$$\lim_{x\to 0} 1 = 1$$

$$\lim_{x\to 0} \cos x = 1$$

By the Squeeze Theorem, since $\lim_{x\to 0} 1 - \frac{x^2}{2} = \lim_{x\to 0} 1$ and the functions are chronologically greater than the last

47. (Section 2.3, Related Exercise 60):

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin 2x}{\sin x} &= \lim_{x\to 0} \frac{2\sin x\cos x}{\sin x} && (1)\\
&= \lim_{x\to 0} 2\cos x && (2)\\
&= 2\cos 0 && (3)\\
&= 2\cdot 1 && (4)\\
&= 2 && (5)
\end{aligned}$$

48. (Section 2.3, Related Exercise 61):

$$\begin{aligned}
\lim_{x\to 0} \frac{1-\cos x}{\cos^2 x - 3\cos x + 2} &= \lim_{x\to 0} \frac{1}{\cos^2 x - 2\cos x + 2} && (1)\\
&= \lim_{x\to 0} \frac{1}{\cos x\cos x - 2\cos x + 2} && (2)\\
&= \frac{1}{\cos 0\cos 0 - 2\cos 0 + 2} && (3)\\
&= \frac{1}{1\cdot 1 - 2(1) + 2} && (4)\\
&= \frac{1}{1 - 2 + 2} && (5)\\
&= \frac{1}{1} && (6)\\
&= 1 && (7)
\end{aligned}$$

A copy of my notes (in LaTeX) are available on my GitHub