Module 3 Notes (MATH-211)

Lillie Donato

24 June 2024

General Notes (and Definitions)

- The Chain Rule
  Suppose $y = f(u)$ is differentiable at $u = g(x)$ and $u = g(x)$ is differentiable at x. The composite function $y = f(g(x))$ is differentiable at x, and its derivative can be expressed in two equivalent ways.

$$\frac{dy}{dx} = \frac{dy}{dy} \cdot \frac{du}{dx} \tag{1}$$

$$\frac{d}{dx}\left(f\left(g\left(x\right)\right)\right) = f'\left(g\left(x\right)\right) \cdot g'\left(x\right) \tag{2}$$

  Application of the Chain Rule (Assume the differentiable function $y = f(g(x))$ is given):

  1. Identify an outer function f and an inner function g, and let $u = g(x)$.
  2. Replace $g(x)$ with u to express y in terms of u:
  $$y = f(g(x)) = f(u)$$
  3. Calculate the product
  $$\frac{dy}{du} \cdot \frac{du}{dx}$$
  4. Replace u with $g(x)$ in $\frac{dy}{du}$ to obtain $\frac{dy}{dx}$

  If g is differentiable for all x in its domain and $p \in \mathbb{R}$,

$$\frac{d}{dx}\left(\left(g\left(x\right)\right)^{p}\right) = p\left(g\left(x\right)\right)^{p-1} g'\left(x\right)$$

- Implicit Differentiation
  When we are unable to solve for y explicitly, we treat y as a function of x ($y = y(x)$) and apply the Chain Rule:

$$y' = \frac{dy}{dx}$$

$$\frac{d}{dx}y^{n} = ny^{n-1}\frac{dy}{dx}$$

- Derivatives of Logarithmic and Exponential Functions

$$\frac{d}{dx}\left(\ln x\right) = \frac{1}{x}, \text{ for } x > 0$$

$$\frac{d}{dx}\left(\ln |x|\right) = \frac{1}{x}, \text{ for } x \neq 0$$

  If u is differentiable at x and $u(x) \neq 0$, then

$$\frac{d}{dx}\left(\ln |u(x)|\right) = \frac{u'(x)}{u(x)}$$

  If $b > 0$ and $b \neq 1$, then for all x,

$$\frac{d}{dx}\left(b^{x}\right) = b^{x} \ln b$$

  General Power Rule:

$$\text{For } p \in \mathbb{R} \text{ and for } x > 0,\ \frac{d}{dx}\left(x^{p}\right) = px^{p-1}$$

Furthermore, if u is a positive differentiable function on its domain, then

$$\frac{d}{dx}\left(u\left(x\right)^{p}\right) = p\left(u\left(x\right)\right)^{p-1} \cdot u'\left(x\right)$$

Functions of the form $f(x) = (g(x))^{h(x)}$, where both g and h are nonconstant functions, are neither exponential function nor power functions (they are sometimes called **tower functions**). To compute their derivatives, we use the identity $b^x = e^{x \ln b}$ to rewrite f with base e:

$$f(x) = (g(x))^{h(x)} = e^{h(x) \ln g(x)}$$

If $b > 0$ and $b \neq 1$, then

$$\frac{d}{dx}\left(\log_b x\right) = \frac{1}{x \ln b}, \text{ for } x > 0$$
$$\frac{d}{dx}\left(\log_b |x|\right) = \frac{1}{x \ln b}, \text{ for } x \neq 0$$

Examples

1. The Chain Rule

$$\begin{aligned}
y &= \left(5x^2 + 11x\right)^{\frac{4}{3}} && (1)\\
u &= 5x^2 + 11x && (2)\\
f &= u^{\frac{4}{3}} && (3)\\
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} && (4)\\
&= \frac{4}{3} u^{\frac{1}{3}} \cdot 10x + 11 && (5)\\
&= \frac{4}{3}\left(5x^2 + 11x\right)^{\frac{1}{3}} \cdot 10x + 11 && (6)\\
& && (7)
\end{aligned}$$

$$\begin{aligned}
y &= e^{4x^2+1} && (1)\\
u &= 4x^2 + 1 && (2)\\
y &= e^{u} && (3)\\
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} && (4)\\
&= e^{u} \cdot 8x && (5)\\
&= e^{4x^2+1} \cdot 8x && (6)\\
&= 8xe^{4x^2+1} && (7)
\end{aligned}$$

2. The Chain Rule (with Tables)

$$h(x) = f(g(x))$$

$$\begin{aligned}
y &= f(g(x)) && (1)\\
u &= g(x) && (2)\\
y &= f(u) && (3)\\
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} && (4)\\
&= f(u) \cdot g'(x) && (5)\\
&= f'(g(x)) \cdot g'(x) && (6)
\end{aligned}$$

$$\begin{aligned}
h'(1) &= f'(g(1)) \cdot g'(1) && (7)\\
&= f'(4) \cdot 9 && (8)\\
&= 7 \cdot 9 && (9)\\
&= 63 && (10)
\end{aligned}$$

$$\begin{aligned}
h'(2) &= f'(g(2)) \cdot g'(2) && (11)\\
&= f'(1) \cdot 7 && (12)\\
&= -6 \cdot 7 && (13)\\
&= -42 && (14)
\end{aligned}$$

$$\begin{aligned}
h'(3) &= f'(g(3)) \cdot g'(3) && (15)\\
&= f'(5) \cdot 3 && (16)\\
&= 2 \cdot 3 && (17)\\
&= 6 && (18)
\end{aligned}$$

$$k(x) = g(g(x))$$

$$\begin{aligned}
y &= g(g(x)) && (1)\\
u &= g(x) && (2)\\
y &= g(u) && (3)\\
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} && (4)\\
&= g'(u) \cdot g'(x) && (5)\\
&= g'(g(x)) \cdot g'(x) && (6)
\end{aligned}$$

$$\begin{aligned}
k'(3) &= g'(g(3)) \cdot g'(3) && (7)\\
&= g'(5) \cdot 3 && (8)\\
&= -5 \cdot 3 && (9)\\
&= -15 && (10)
\end{aligned}$$

$$\begin{aligned}
k'(1) &= g'(g(1)) \cdot g'(1) && (11)\\
&= g'(4) \cdot 9 && (12)\\
&= -1 \cdot 9 && (13)\\
&= -9 && (14)
\end{aligned}$$

$$\begin{aligned}
k'(5) &= g'(g(5)) \cdot g'(5) && (15)\\
&= g'(3) \cdot -5 && (16)\\
&= 3 \cdot -5 && (17)\\
&= -15 && (18)
\end{aligned}$$

3. The Chain Rule (All Forms)

$$\begin{aligned}
y &= \sqrt[3]{2x^2 - x - 5} && (1)\\
u &= 2x^2 - x - 5 && (2)\\
y &= \sqrt[3]{u} && (3)\\
y' &= u^{-\frac{2}{3}} \cdot 4x - 1 && (4)\\
&= \frac{1}{3}\left(2x^2 - x - 5\right)^{-\frac{2}{3}} \cdot 4x - 1 && (5)
\end{aligned}$$

$$\begin{aligned}
y &= \csc(\tan t) && (1)\\
u &= \tan t && (2)\\
y &= \csc u && (3)\\
y' &= -\csc u \cot u \cdot \sec^2 t && (4)\\
&= -\csc(\tan t)\cot(\tan t) \cdot \sec^2 t && (5)
\end{aligned}$$

4. The Chain Rule (Nested)

$$\begin{aligned}
y &= \tan(\sin e^x) && (1)\\
u_2 &= e^x && (2)\\
u_1 &= \sin u_2 && (3)\\
y &= \tan u_1 && (4)\\
y' &= \sec^2(\sin e^x) \cdot \cos e^x \cdot e^x && (5)
\end{aligned}$$

5. The Chain Rule (Combination of Rules)

$$\begin{aligned}
y &= \left(\frac{e^x}{x+1}\right)^8 && (1)\\
y' &= 8\left(\frac{e^x}{x+1}\right)^7 \cdot \frac{xe^x}{(x+1)^2} && (2)\\
&= 8\frac{e^{7x}}{(x+1)^7} \cdot \frac{xe^x}{(x+1)^2} && (3)\\
&= \frac{8xe^{8x}}{(x+1)^9} && (4)
\end{aligned}$$

6. Implicit Differentiation

$$\begin{aligned}
x^4 + y^4 &= 2 && (1)\\
4x^3 + 4y^3\frac{dy}{dx} &= 0 && (2)\\
4y^3\frac{dy}{dx} &= -4x^3 && (3)\\
\frac{dy}{dx} &= \frac{-4x^3}{4y^3} && (4)\\
&= \frac{-x^3}{y^3} && (5)
\end{aligned}$$

$$\left.\frac{dy}{dx}\right|_{(1,-1)} = \frac{-(1)^3}{(-1)^3} = \frac{-1}{-1} = 1$$

7. Implicit Differentiation (Finding y)

$$\begin{aligned}
y &= y = xe^y && (1)\\
y' &= e^y + xe^y y' && (2)\\
y' - y'xe^y &= e^y && (3)\\
y'(1 - xe^y) &= e^y && (4)\\
y' &= \frac{e^y}{1 - xe^y} && (5)
\end{aligned}$$

8. Implicit Differentiation (Tangent Line)

$$\begin{aligned}
\cos(x-y)+\sin y &= \sqrt{2} && (1)\\
(-\sin(x-y))(1-1y')+\cos y\,(y') &= 0 && (2)\\
-\sin(x-y)+y'\sin(x-y)+y'\cos y &= 0 && (3)\\
y'(\sin(x-y)+\cos y) &= \sin(x-y) && (4)\\
y' &= \frac{\sin(x-y)}{\sin(x-y)+\cos y} && (5)\\
y'\Big|_{\left(\frac{\pi}{2},\frac{\pi}{4}\right)} &= \frac{\sin\left(\frac{\pi}{2}-\frac{\pi}{2}\right)}{\sin\left(\frac{\pi}{2}-\frac{\pi}{2}\right)+\cos\frac{\pi}{4}} && (6)\\
&= \frac{1}{2} && (7)\\
y &= \frac{1}{2}x && (8)
\end{aligned}$$

9. Implicit Differentiation (Higher Order)

$$\begin{aligned}
x^4+y^4 &= 64 && (1)\\
4x^3+4y^3\frac{dy}{dx} &= 0 && (2)\\
4y^3\frac{dy}{dx} &= -4x^3 && (3)\\
\frac{dy}{dx} &= \frac{-4x^3}{4y^3} && (4)\\
&= \frac{-x^3}{y^3} && (5)\\
\frac{d^2y}{dx^2} &= \frac{-3x^2y^3-\left(-x^3 3y^2\frac{dy}{dx}\right)}{(y^3)^2} && (6)\\
&= \frac{-3x^2y^3+3x^3y^2\frac{dy}{dx}}{y^6} && (7)\\
&= \frac{-3x^2y^3+3x^3y^2\frac{-x^3}{y^3}}{y^6} && (8)\\
&= \frac{-3x^2y^3+\frac{-3x^6y^2}{y^3}}{y^6} && (9)\\
&= \frac{-3x^2y^3-\frac{3x^6}{y}}{y^6} && (10)\\
&= \frac{\frac{-3x^2y^4-3x^6}{y}}{y^6} && (11)\\
&= \frac{-3x^2y^4-3x^6}{y^7} && (12)
\end{aligned}$$

10. Derivatives with $\ln x$

$$\begin{aligned}
y &= \ln 2x^8 && (1)\\
\frac{dy}{dx} &= \frac{1}{2x^8}\cdot 16x^7 && (2)\\
&= \frac{16x^7}{2x^8} && (3)\\
&= \frac{8}{x} && (4)
\end{aligned}$$

$$\begin{aligned} y &= x^2\left(1-\ln x^2\right) && (1)\\ \frac{dy}{dx} &= 2x\left(1-\ln x^2\right)+x^2\left(-\frac{2x}{x^2}\right) && (2)\\ &= 2x-2x\ln x^2-2x && (3)\\ &= -2x\ln x^2 && (4) \end{aligned}$$

11. Derivatives with b^x

$$\begin{aligned} y &= 2^{2x} && (1)\\ \frac{dy}{dx} &= 2^{2x}\ln 2\cdot 2 && (2)\\ &= 2^{2x+1}\ln 2 && (3) \end{aligned}$$

$$\begin{aligned} f(x) &= 7^{-x}\cos x && (1)\\ \frac{dy}{dx} &= -7^{-x}\ln 7\cos x-7^{-x}\sin x && (2)\\ &= -7^{-x}\left(\ln 7\cos x+\sin x\right) && (3) \end{aligned}$$

12. Derivatives with the General Power Rule

$$\begin{aligned} y &= x^e && (1)\\ \frac{dy}{dx} &= ex^{e-1} && (2) \end{aligned}$$

$$\begin{aligned} f(x) &= \left(x^3+3^x\right)^{\pi} && (1)\\ \frac{dy}{dx} &= \pi\left(x^3+3^x\right)^{\pi-1}\cdot\left(3x^2+3^x\ln 3\right) && (2) \end{aligned}$$

13. Derivatives with Tower Functions

$$\begin{aligned} g(x) &= x^{\ln x} && (1)\\ &= e^{\ln x\ln x} && (2)\\ &= e^{(\ln x)^2} && (3)\\ g'(x) &= e^{(\ln x)^2}\cdot\frac{2\ln x}{x} && (4)\\ g'(e) &= e^{(\ln e)^2}\cdot\frac{2\ln e}{e} && (5)\\ &= \frac{2e}{e} && (6)\\ &= 2 && (7)\\ y &= 2x-2e+e && (8)\\ &= 2x-e && (9) \end{aligned}$$

14. Derivatives of Logarithmic Functions

$$\begin{aligned} y &= \log_7 5x && (1)\\ \frac{dy}{dx} &= \frac{5}{5x\ln 7} && (2)\\ &= \frac{1}{x\ln 7} && (3) \end{aligned}$$

$$\begin{aligned} y &= \log\left(\log x\right) && (1)\\ \frac{dy}{dx} &= \frac{1}{\log x\ln 10}\cdot\frac{1}{x\ln 10} && (2)\\ &= \frac{1}{x\log x\ln\left(10\right)^2} && (3) \end{aligned}$$

15. Logarithmic Differentiation

$$\begin{aligned}
f(x) &= (\cos x)^{\sec x} && (1)\\
\ln f(x) &= \ln\left((\cos x)^{\sec x}\right) && (2)\\
&= \sec x \cdot \ln(\cos x) && (3)\\
\frac{f'(x)}{f(x)} &= \sec x \cdot \tan x \cdot \ln(\cos x) + \sec x \cdot \frac{-\sin x}{\cos x} && (4)\\
&= \sec x \cdot \tan x \cdot \ln(\cos x) + \sec x \cdot (-\tan x) && (5)\\
&= \tan x \sec x\left(\ln(\cos x) - 1\right) && (6)\\
f'(x) &= f(x)\tan x \sec x\left(\ln(\cos x) - 1\right) && (7)\\
&= (\cos x)^{\sec x}\tan x \sec x\left(\ln(\cos x) - 1\right) && (8)\\
& && (9)
\end{aligned}$$

Related Exercises

1. (Section 3.7 Exercise 15)

$$\begin{aligned}
y &= (3x+7)^{10} && (1)\\
u &= 3x+7 && (2)\\
f(u) &= u^{10} && (3)\\
y' &= 10u^9 \cdot 3 && (4)\\
&= 10(3x+7)^9 \cdot 3 && (5)\\
&= 30(3x+7)^9 && (6)
\end{aligned}$$

2. (Section 3.7 Exercise 17)

$$\begin{aligned}
y &= \sin^5 x && (1)\\
u &= \sin x && (2)\\
f(u) &= u^5 && (3)\\
y' &= 5u^4 \cdot \cos x && (4)\\
&= 5\sin^4 x \cos x && (5)
\end{aligned}$$

3. (Section 3.7 Exercise 28)

$$\begin{aligned}
y &= \left(x^2+2x+7\right)^8 && (1)\\
u &= x^2+2x+7 && (2)\\
f(u) &= u^8 && (3)\\
y' &= 8u^7 \cdot (2x+2) && (4)\\
&= 8\left(x^2+2x+7\right)^7 \cdot (2x+2) && (5)\\
&= (16x+16)\left(x^2+2x+7\right)^7 && (6)
\end{aligned}$$

4. (Section 3.7 Exercise 29)

$$\begin{aligned}
y &= \sqrt{10x+1} && (1)\\
u &= 10x+1 && (2)\\
f(u) &= \sqrt{u} && (3)\\
y' &= \frac{1}{2\sqrt{u}} \cdot 10 && (4)\\
&= \frac{1}{2\sqrt{10x+1}} \cdot 10 && (5)\\
&= \frac{10}{2\sqrt{10x+1}} && (6)\\
&= \frac{5}{\sqrt{10x+1}} && (7)
\end{aligned}$$

5. (Section 3.7 Exercise 41)

$$\begin{aligned}
y &= \sqrt[4]{\frac{2x}{4x-3}} && (1)\\
u &= \frac{2x}{4x-3} && (2)\\
f(u) &= \sqrt[4]{u} && (3)\\
y' &= \frac{1}{4}u^{-\frac{3}{4}} \cdot -\frac{6}{(4x-3)^2} && (4)\\
&= \frac{1}{4}\left(\frac{2x}{4x-3}\right)^{-\frac{3}{4}} \cdot -\frac{6}{(4x-3)^2} && (5)\\
&= -\frac{6}{4\,(4x-3)^2}\left(\frac{2x}{4x-3}\right)^{-\frac{3}{4}} && (6)
\end{aligned}$$

6. (Section 3.7 Exercise 23)

$$\begin{aligned}
y &= \tan 5x^2 && (1)\\
u &= 5x^2 && (2)\\
f(u) &= \tan u && (3)\\
y' &= \sec^2 u \cdot 10x && (4)\\
&= \sec^2 5x^2 \cdot 10x && (5)\\
&= 10x \sec^2 5x^2 && (6)\\
& && (7)
\end{aligned}$$

7. (Section 3.7 Exercise 24)

$$\begin{aligned}
y &= \sin \frac{x}{4} && (1)\\
u &= \frac{x}{4} && (2)\\
f(u) &= \sin u && (3)\\
y' &= \cos u \cdot \frac{4}{16} && (4)\\
&= \cos \frac{x}{4} \cdot \frac{1}{4} && (5)\\
&= \frac{1}{4} \cos \frac{x}{4} && (6)
\end{aligned}$$

8. (Section 3.7 Exercise 45)

$$\begin{aligned}
y &= (2x^6 - 3x^3 + 3)^{25} && (1)\\
u &= 2x^6 - 3x^3 + 3 && (2)\\
f(u) &= u^{25} && (3)\\
y' &= 25\,(u)^{24} \cdot 12x^5 - 9x^2 && (4)\\
&= 25\left(2x^6 - 3x^3 + 3\right)^{24} \cdot 12x^5 - 9x^2 && (5)\\
&= 25\left(12x^5 - 9x^2\right)\left(2x^6 - 3x^3 + 3\right)^{24} && (6)
\end{aligned}$$

9. (Section 3.7 Exercise 46)

$$\begin{aligned}
y &= (\cos x + 2\sin x)^8 && (1)\\
u &= \cos x + 2\sin x && (2)\\
f(u) &= u^8 && (3)\\
y' &= 8u^7 \cdot (-\sin x + 2\cos x) && (4)\\
&= 8\,(\cos x + 2\sin x)^7 \cdot (-\sin x + 2\cos x) && (5)\\
&= 8(-\sin x + 2\cos x)\,(\cos x + 2\sin x)^7 && (6)\\
& && (7)
\end{aligned}$$

10. (Section 3.7 Exercise 53)

$$\begin{aligned} y &= \sin(\sin(e^x)) && (1)\\ y' &= \cos(\sin e^x)\cos e^x e^x && (2) \end{aligned}$$

11. (Section 3.7 Exercise 54)

$$\begin{aligned} y &= \sin^2 e^{3x+1} && (1)\\ y' &= 6\sin e^{3x+1}\cos e^{3x+1} && (2) \end{aligned}$$

12. (Section 3.7 Exercise 68)

$$\begin{aligned} y &= \left(\frac{3x}{4x+2}\right)^5 && (1)\\ y' &= 5u^4 \cdot \frac{12x+6-12x}{(4x+2)^2} && (2)\\ &= 5\left(\frac{3x}{4x+2}\right)^4 \cdot \frac{6}{(4x+2)^2} && (3)\\ &= \frac{30}{(4x+2)^2}\left(\frac{3x}{4x+2}\right)^4 && (4) \end{aligned}$$

13. (Section 3.7 Exercise 69)

$$\begin{aligned} y &= \left((x+2)\left(x^2+1\right)\right)^4 && (1)\\ y' &= 4u^3 \cdot x^2+1+2x^2+4x && (2)\\ &= 4\left(3x^2+4x+1\right)\left((x+2)\left(x^2+1\right)\right)^3 && (3)\\ &= 4(3x+1)(x+1)\left((x+2)\left(x^2+1\right)\right)^3 && (4) \end{aligned}$$

14. (Section 3.8 Exercise 13)

$$\begin{aligned} x^4+y^4 &= 2 && (1)\\ (1,-1) & && (2)\\ 4x^3+4y^3\frac{dy}{dx} &= 0 && (3)\\ 4y^3\frac{dy}{dx} &= -4x^3 && (4)\\ \frac{dy}{dx} &= \frac{-4x^3}{4y^3} && (5)\\ &= \frac{-x^3}{y^3} && (6)\\ \left.\frac{dy}{dx}\right|_{(1,-1)} &= \frac{-(1^3)}{(-1)^3} && (7)\\ &= \frac{-1}{-1} && (8)\\ &= 1 && (9) \end{aligned}$$

15. (Section 3.8 Exercise 15)

$$\begin{aligned} y^2 &= 4x && (1)\\ (1,2) & && (2)\\ 2y\frac{dy}{dx} &= 4 && (3)\\ \frac{dy}{dx} &= \frac{4}{2y} && (4)\\ &= \frac{2}{y} && (5)\\ \left.\frac{dy}{dx}\right|_{(1,2)} &= \frac{2}{2} && (6)\\ &= 1 && (7) \end{aligned}$$

16. (Section 3.8 Exercise 31)

$$\begin{aligned}
\sin xy &= x+y && (1)\\
\cos xy \cdot \left(y + x\frac{dy}{dx}\right) &= 1+\frac{dy}{dx} && (2)\\
y\cos xy + \frac{dy}{dx}x\cos xy &= 1+\frac{dy}{dx} && (3)\\
\frac{dy}{dx}x\cos xy - \frac{dy}{dx} &= 1-y\cos xy && (4)\\
\frac{dy}{dx}(x\cos xy - 1) &= 1-y\cos xy && (5)\\
\frac{dy}{dx} &= \frac{1-y\cos xy}{x\cos xy-1} && (6)
\end{aligned}$$

17. (Section 3.8 Exercise 33)

$$\begin{aligned}
\cos y^2 + x &= e^y && (1)\\
-\frac{dy}{dx}2y\sin y^2 + 1 &= \frac{dy}{dx}e^y && (2)\\
\frac{dy}{dx}2y\sin y^2 + \frac{dy}{dx}e^y &= 1 && (3)\\
\frac{dy}{dx}\left(2y\sin y^2 + e^y\right) &= 1 && (4)\\
\frac{dy}{dx} &= \frac{1}{2y\sin y^2 + e^y} && (5)
\end{aligned}$$

18. (Section 3.8 Exercise 47)

$$\begin{aligned}
x^2+xy+y^2 &= 7 && (1)\\
(2,1) & && (2)\\
2x+y+\frac{dx}{dy}x+\frac{dx}{dy}2y &= 0 && (3)\\
\frac{dx}{dy}(x+2y) &= -2x-y && (4)\\
\frac{dx}{dy} &= \frac{-2x-y}{x+2y} && (5)\\
\left.\frac{dy}{dx}\right|_{(2,1)} &= \frac{-2(2)-1}{2+2(1)} && (6)\\
&= \frac{-4-1}{2+2} && (7)\\
&= \frac{-5}{4} && (8)\\
y &= \frac{-5}{4}x - 2\frac{-5}{4}+1 && (9)\\
&= \frac{-5}{4}x + \frac{5}{2} + \frac{2}{2} && (10)\\
&= \frac{-5}{4}x + \frac{7}{2} && (11)
\end{aligned}$$

19. (Section 3.8 Exercise 48)

$$
\begin{aligned}
x^4 - x^2 y + y^4 &= 1 && (1)\\
(-1,1) & && (2)\\
4x^3 - 2xy - \frac{dy}{dx}x^2 + \frac{dy}{dx}4y^3 &= 0 && (3)\\
\frac{dy}{dx}\left(-x^2 + 4y^3\right) &= 2xy - 4x^3 && (4)\\
\frac{dy}{dx} &= \frac{2xy - 4x^3}{4y^3 - x^2} && (5)\\
\left.\frac{dy}{dx}\right|_{(-1,1)} &= \frac{2(-1)(1) - 4(-1)^3}{4(1)^3 - (-1)^2} && (6)\\
&= \frac{-2+4}{4+1} && (7)\\
&= \frac{2}{5} && (8)\\
y &= \frac{2}{5}x + \frac{2}{5} + 1 && (9)\\
&= \frac{2}{5}x + \frac{7}{5} && (10)
\end{aligned}
$$

20. (Section 3.8 Exercise 25)

$$
\begin{aligned}
x\sqrt[3]{y} + y &= 10 && (1)\\
(1,8) & && (2)\\
\sqrt[3]{y} + \frac{dy}{dx}\left(\frac{x}{3y^{\frac{2}{3}}}\right) + \frac{dy}{dx} &= 0 && (3)\\
\frac{dy}{dx}\left(\frac{x}{3y^{\frac{2}{3}}} + 1\right) &= -\sqrt[3]{y} && (4)\\
\frac{dy}{dx} &= \frac{-3y^{\frac{2}{3}}\sqrt[3]{y}}{x + 3y^{\frac{2}{3}}} && (5)\\
&= \frac{-3y}{3y^{\frac{2}{3}} + x} && (6)\\
\left.\frac{dy}{dx}\right|_{(1,8)} &= \frac{-3(8)}{3(8)^{\frac{2}{3}} + 1} && (7)\\
&= \frac{-24}{3(4)+1} && (8)\\
&= \frac{-24}{13} && (9)
\end{aligned}
$$

21. (Section 3.8 Exercise 26)

$$\begin{aligned}
(x+y)^{\frac{2}{3}} &= y && (1)\\
(4,4) & && (2)\\
\frac{2}{3}(x+y)^{-\frac{1}{3}}\cdot\left(1+\frac{dy}{dx}\right) &= \frac{dy}{dx} && (3)\\
\frac{2}{3}(x+y)^{-\frac{1}{3}}+\frac{dy}{dx}\frac{2}{3}(x+y)^{-\frac{1}{3}} &= \frac{dy}{dx} && (4)\\
\frac{2}{3}(x+y)^{-\frac{1}{3}} &= \frac{dy}{dx}-\frac{dy}{dx}\frac{2}{3}(x+y)^{-\frac{1}{3}} && (5)\\
\frac{dy}{dx}\left(1-\frac{2}{3}(x+y)^{-\frac{1}{3}}\right) &= \frac{2}{3}(x+y)^{-\frac{1}{3}} && (6)\\
\frac{dy}{dx} &= \frac{\frac{2}{3}(x+y)^{-\frac{1}{3}}}{1-\frac{2}{3}(x+y)^{-\frac{1}{3}}} && (7)\\
\left.\frac{dy}{dx}\right|_{(4,4)} &= \frac{\frac{2}{3}(4+4)^{-\frac{1}{3}}}{1-\frac{2}{3}(4+4)^{-\frac{1}{3}}} && (8)\\
&= \frac{\frac{2}{3}\frac{1}{2}}{1-\frac{2}{3}\frac{1}{2}} && (9)\\
&= \frac{\frac{1}{2}}{-\frac{1}{2}} && (10)\\
&= -1 && (11)
\end{aligned}$$

22. (Section 3.8 Exercise 51)

$$\begin{aligned}
x+y^2 &= 1 && (1)\\
1+\frac{dy}{dx}2y &= 0 && (2)\\
\frac{dy}{dx}2y &= -1 && (3)\\
\frac{dy}{dx} &= \frac{-1}{2y} && (4)\\
\frac{d^2y}{dx^2} &= \frac{-1}{2y} && (5)\\
&= \frac{-1}{2}\frac{1}{y} && (6)\\
&= \frac{-1}{2}\frac{dy}{dx}\frac{-1}{y^2} && (7)\\
&= \frac{-1}{2}\frac{-1}{2y}\frac{-1}{y^2} && (8)\\
&= \frac{-1}{4y^3} && (9)
\end{aligned}$$

23. (Section 3.8 Exercise 52)

$$2x^2 + y^2 = 4 \tag{1}$$

$$4x + 2y\frac{dy}{dx} = 0 \tag{2}$$

$$2y\frac{dy}{dx} = -4x \tag{3}$$

$$\frac{dy}{dx} = \frac{-4x}{2y} \tag{4}$$

$$= \frac{-2x}{y} \tag{5}$$

$$\frac{d^2y}{dx^2} = \frac{-2x}{y} \tag{6}$$

$$= -2x\frac{1}{y} \tag{7}$$

$$= -2x\frac{dy}{dx}\frac{-1}{y^2} \tag{8}$$

$$= -2x\frac{-2x}{y}\frac{-1}{y^2} \tag{9}$$

$$= \frac{-4x^2}{y^3} \tag{10}$$

A copy of my notes (in LaTeX) are available on my GitHub