

Module 7 Notes (MATH-211)

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General Notes (and Definitions)

- Working with Integrals

A function $f(x)$ is **even** if $f(-x) = f(x)$.

A function $f(x)$ is **odd** if $f(-x) = -f(x)$.

Let $a \in \mathbb{R}$ such that $a > 0$ and let f be an integrable function on the interval $[-a, a]$.

$$\text{If } f \text{ is even, } \int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$$

$$\text{If } f \text{ is odd, } \int_{-a}^a f(x) dx = 0$$

The average value of an integrable function f on the interval $[a, b]$ is

$$\bar{f} = \frac{1}{b-a} \int_a^b f(x) dx$$

Let f be continuous on the interval $[a, b]$. There exists a point c in (a, b) such that (Mean Value Theorem)

$$f(c) = \bar{f} = \frac{1}{b-a} \int_a^b f(t) dx$$

- Substitution Rule

Let $u = g(x)$, where g is differentiable on an interval, and let f be continuous on the corresponding range of g . On that interval,

$$\int f(g(x))g'(x) dx = \int f(u) du$$

1. Given an indefinite integral involving a composite function $f(g(x))$, identify an inner function $u = g(x)$ such that a constant multiple of $g'(x)$ appears in the integrand.
2. Substitute $u = g(x)$ and $du = g'(x) dx$ in the integral.
3. Evaluate the new indefinite integral with respect to u .
4. Write the result in terms of x using $u = g(x)$.

Let $u = g(x)$, where g' is continuous on $[a, b]$, and let f be continuous on the range of g . Then

$$\int_a^b f(g(x))g'(x) dx = \int_{g(a)}^{g(b)} f(u) du$$

- Velocity and Net Change

Position, Velocity, Displacement, and Distance:

1. The **position** of an object moving along a line at time t , denoted $s(t)$, is the location of the object relative to the origin.
2. The **velocity** of an object at time t is $v(t) = s'(t)$.
3. The **displacement** of the object between $t = a$ and $t = b > a$ is

$$s(b) - s(a) = \int_a^b v(t) dt$$

4. The **distance traveled** by the object between $t = a$ and $t = b > a$ is

$$\int_a^b |v(t)| \, dt$$

where $|v(t)|$ is the **speed** of the object at time t .

Theorem: Position from Velocity

Given the velocity $v(t)$ of an object moving along a line and its initial position $s(0)$, the position function of the object for future times $t \geq 0$ is

$$s(t) = s(0) + \int_0^t v(x) \, dx$$

Theorem: Velocity from Acceleration

Given the acceleration $a(t)$ of an object moving along a line and its initial velocity $v(0)$, the velocity of the object for future times $t \geq 0$ is

$$v(t) = v(0) + \int_0^t a(x) \, dx$$

Theorem: Net Change and Future Value

Suppose a quantity Q changes over time at a known rate Q' . Then the **net change** in Q between $t = a$ and $t = b > a$ is

$$Q(b) - Q(a) = \int_a^b Q'(t) \, dt$$

Given the initial value $Q(0)$, the **future value** of Q at time $t \geq 0$ is

$$Q(t) = Q(0) + \int_0^t Q'(x) \, dx$$

• Area Between Curves

Area of a Region Between Two Curves:

Suppose that f and g are continuous functions with $f(x) \geq g(x)$ on the interval $[a, b]$. The area of the region bounded by the graphs of f and g on $[a, b]$ is

$$A = \int_a^b (f(x) - g(x)) \, dx$$

Area of a Region Between Two Curves with Respect to y :

Suppose that f and g are continuous functions with $f(y) \geq g(y)$ on the interval $[c, d]$. The area of the region bounded by the graphs $x = f(y)$ and $x = g(y)$ on $[c, d]$ is

$$A = \int_c^d (f(y) - g(y)) \, dy$$

General formulas for indefinite integrals

$$\int \cos ax \, dx = \frac{1}{a} \sin ax + C \quad (1)$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax + C \quad (2)$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax + C \quad (3)$$

$$\int \csc^2 ax \, dx = -\frac{1}{a} \cot ax + C \quad (4)$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax + C \quad (5)$$

$$\int \csc ax \cot ax \, dx = -\frac{1}{a} \csc ax + C \quad (6)$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax} + C \quad (7)$$

$$\int b^x \, dx = \frac{1}{\ln b} b^x + C, b > 0, b \neq 1 \quad (8)$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C \quad (9)$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + C, a > 0 \quad (10)$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1} \left| \frac{x}{a} \right| + C, a > 0 \quad (11)$$

Examples

1. Use symmetry to evaluate integrals

$$\int_{-200}^{200} 2x^5 \, dx = 0$$

$$\int_{-2}^2 (x^2 + x^3) \, dx = \int_{-2}^2 x^2 \, dx + \int_{-2}^2 x^3 \, dx \quad (1)$$

$$= 2 \int_0^2 x^2 \, dx + 0 \quad (2)$$

$$= 2 \frac{x^3}{3} \quad (3)$$

$$= \frac{16}{3} \quad (4)$$

2. A derivative calculation

$$s(t) = -16t^2 + 64t$$

$$t = 4$$

$$[0, 4]$$

$$v(t) = s'(t) \quad (1)$$

$$\bar{v} = \frac{1}{4} \int_0^4 v(t) \, dt \quad (2)$$

$$= \frac{1}{4} \int_0^4 s'(t) \, dt \quad (3)$$

$$= \frac{1}{4} s(t) \quad (4)$$

$$= \frac{1}{4} (s(4) - s(0)) \quad (5)$$

$$= 0 \quad (6)$$

3. Applying MVT for integrals

$$\begin{aligned} f(x) &= e^x \\ [0, 2] \end{aligned}$$

$$\bar{f} = \frac{1}{2} \left(\int_0^2 e^x dx \right) \quad (1)$$

$$= \frac{e^x}{2} \quad (2)$$

$$= \frac{e^2}{2} - \frac{e^0}{2} \quad (3)$$

$$= \frac{e^2 - 1}{2} \quad (4)$$

$$e^x = \frac{e^2 - 1}{2} \quad (5)$$

$$\ln e^x = \ln \frac{e^2 - 1}{2} \quad (6)$$

4. Perfect substitutions in indefinite integrals

$$u = 4x^3 - 8 \quad (1)$$

$$du = 12x^2 dx \quad (2)$$

$$\int 12x^2 (4x^3 - 8)^5 dx = \int 12x^2 u^5 dx \quad (3)$$

$$= \frac{u^6}{6} + C \quad (4)$$

$$= \frac{(4x^3 - 8)^6}{6} + C \quad (5)$$

$$u = \sin t \quad (1)$$

$$du = \cos t dt \quad (2)$$

$$\int (\cos t) e^{\sin t} dt = \int e^u du \quad (3)$$

$$= e^u + C \quad (4)$$

$$= e^{\sin t} + C \quad (5)$$

5. Introducing constants when integrating by substitution

$$u = 6x + 4 \quad (1)$$

$$du = 6 dx \quad (2)$$

$$dx = \frac{du}{6} \quad (3)$$

$$\int (6x + 4)^9 dx = \int \frac{1}{6} \cdot u^9 du \quad (4)$$

$$= \frac{1}{6} \int u^9 du \quad (5)$$

$$= \frac{1}{6} \cdot \frac{u^9}{9} + C \quad (6)$$

$$= \frac{(6x + 4)^9}{54} + C \quad (7)$$

$$u = \cot x \quad (1)$$

$$du = -\csc^2 x dx \quad (2)$$

$$\int \cot^2 x \csc^2 x dx = \int -u^2 du \quad (3)$$

$$= -\frac{u^3}{3} + C \quad (4)$$

$$= -\frac{\csc^3 x}{3} + C \quad (5)$$

6. Variations on the substitution method

$$u = x - 1 \quad (1)$$

$$du = dx \quad (2)$$

$$x = u + 1 \quad (3)$$

$$\int x\sqrt{x-1} \, dx = \int (u+1)\sqrt{u} \, du \quad (4)$$

$$= \int u\sqrt{u} + \sqrt{u} \, du \quad (5)$$

$$= \int u^{\frac{3}{2}} + u^{\frac{1}{2}} \, du \quad (6)$$

$$= \frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}} + C \quad (7)$$

$$= \frac{2}{5}(x-1)^{\frac{5}{2}} + \frac{2}{3}(x-1)^{\frac{3}{2}} + C \quad (8)$$

7. Use known formulas to evaluate indefinite integrals

$$\int 2e^{-4x} \, dx = 2 \int e^{-4x} \, dx \quad (1)$$

$$= \frac{2}{-4}e^{-4x} + C \quad (2)$$

$$= -\frac{1}{2}e^{-4x} + C \quad (3)$$

$$(4)$$

$$\int \frac{dx}{\sqrt{36-x^2}} = \int \frac{dx}{\sqrt{6^2-x^2}} \quad (1)$$

$$= \sin^{-1} \frac{x}{6} + C \quad (2)$$

8. Evaluating definite integrals using substitution

$$u = 2^x + 4 \quad (1)$$

$$du = 2^x \ln 2 \, dx \quad (2)$$

$$\frac{1}{\ln 2} du = 2^x \, dx \quad (3)$$

$$\int_1^3 \frac{2^x}{2^x + 4} \, dx = \int_1^3 \frac{1}{u \ln 2} \, du \quad (4)$$

$$\int_{g(1)}^{g(3)} \frac{1}{u \ln 2} \, du = \int_6^{12} \frac{1}{u \ln 2} \, du \quad (5)$$

$$= \frac{1}{\ln 2} \int_6^{12} \frac{du}{u} \quad (6)$$

$$= \frac{1}{\ln 2} \cdot (\ln 12 - \ln 6) \quad (7)$$

$$= \frac{\ln 2}{\ln 2} \quad (8)$$

$$= 1 \quad (9)$$

$$u = \ln p \quad (1)$$

$$du = \frac{1}{p} \, dx \quad (2)$$

$$\int_1^{e^2} \frac{\ln p}{p} \, dx = \int_0^2 u \, du \quad (3)$$

$$= \frac{2^2}{2} - \frac{0^2}{2} \quad (4)$$

$$= \frac{4}{2} \quad (5)$$

$$= 2 \quad (6)$$

9. Integrals involving $\cos^2 x$ and $\sin^2 x$

$$u = 2x \quad (1)$$

$$du = 2 dx \quad (2)$$

$$dx = \frac{1}{2} du \quad (3)$$

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad (4)$$

$$\int_0^\pi \sin^2 x dx = \int_0^\pi \frac{1 - \cos 2x}{2} dx \quad (5)$$

$$= \frac{1}{2} \int_0^\pi 1 - \cos 2x dx \quad (6)$$

$$= \frac{1}{2} \left(\int_0^\pi 1 dx - \int_0^\pi \cos 2x dx \right) \quad (7)$$

$$= \frac{1}{2} \left((\pi - 0) - \frac{1}{2} \int_0^{2\pi} \cos u du \right) \quad (8)$$

$$= \frac{1}{2} \left(\pi - \frac{1}{2} (\sin 2\pi - \sin 0) \right) \quad (9)$$

$$= \frac{1}{2} (\pi - 0) \quad (10)$$

$$= \frac{\pi}{2} \quad (11)$$

10. Displacement and distance from velocity

$$v(t) = 4t^3 - 24t^2 + 20t$$

(a)

$$v(t) = 0 \quad (1)$$

$$4t^3 - 24t^2 + 20t = 0 \quad (2)$$

$$4t(t^2 - 6t + 5) = 0 \quad (3)$$

$$4t(t - 1)(t - 5) = 0 \quad (4)$$

$$t = 0 \quad (5)$$

$$t = 1 \quad (6)$$

$$t = 5 \quad (7)$$

$$0 < t < 1 = \text{Positive} \quad (8)$$

$$1 < t < 5 = \text{Negative} \quad (9)$$

$$t > 5 = \text{Positive} \quad (10)$$

(b)

$$\int_0^5 4t^3 - 24t^2 + 20t dt = 4 \int_0^5 t^3 dt - 24 \int_0^5 t^2 dt + 20 \int_0^5 t dt \quad (1)$$

$$= 4 \left(\frac{5^4}{4} - \frac{0^4}{4} \right) - 24 \left(\frac{5^3}{3} - \frac{0^3}{3} \right) + 20 \left(\frac{5^2}{2} - \frac{0^2}{2} \right) \quad (2)$$

$$= 4 \left(\frac{625}{4} \right) - 24 \left(\frac{125}{3} \right) + 20 \left(\frac{25}{2} \right) \quad (3)$$

$$= -125 \quad (4)$$

(c)

$$\int_0^5 |4t^3 - 24t^2 + 20t| dt = \int_0^1 4t^3 - 24t^2 + 20t dt + \int_1^5 -4t^3 + 24t^2 - 20t dt \quad (1)$$

$$= 3 + 128 \quad (2)$$

$$= 131 \quad (3)$$

11. Position and velocity from acceleration

$$a(t) = \frac{20}{(t+2)^2}$$

$$v(0) = 20$$

$$s(0) = 10$$

$$v(t) = v(0) + \int_0^t a(t) dt \quad (1)$$

$$= 20 + \int_0^t \frac{20}{(t+2)^2} dt \quad (2)$$

$$= 20 - \frac{20}{t+2} + 10 \quad (3)$$

$$= 30 - \frac{20}{t+2} \quad (4)$$

$$s(t) = s(0) + \int_0^t v(t) dt \quad (5)$$

$$= 10 + \int_0^t \left(30 - \frac{20}{t+2} \right) dt \quad (6)$$

$$= 10 + 30t - 20 \ln |t+2| + 20 \ln 2 \quad (7)$$

12. Acceleration application

$$a(t) = -15$$

$$v(0) = 60$$

$$s(0) = 0$$

$$v(t) = 60 + \int_0^t -15 dt \quad (1)$$

$$= 60 - 15 \int_0^t t^0 dt \quad (2)$$

$$= -15t + 60 \quad (3)$$

$$s(t) = 0 + \int_0^t -15t + 60 dt \quad (4)$$

$$= -15 \int_0^t t dt + 60t \quad (5)$$

$$= -\frac{15}{2}t^2 + 60t \quad (6)$$

$$v(t) = 0 \quad (7)$$

$$60 - 15t = 0 \quad (8)$$

$$15t = 60 \quad (9)$$

$$t = \frac{60}{15} \quad (10)$$

$$= 4 \quad (11)$$

$$s(4) - s(0) = 60(4) - \frac{15}{2}(4)^2 - 0 \quad (12)$$

$$= 120 \quad (13)$$

13. Application of net change

$$V'(t) = 70(1 + \sin 2\pi t)$$

$$[0, t]$$

$$V(0) = 0$$

$$V(t) = V(0) + \int_0^t V'(x) dx \quad (1)$$

$$= 0 + \int_0^t 70(1 + \sin(2\pi x)) dx \quad (2)$$

$$= 70 \left(t - \frac{\cos 2\pi t}{2\pi} + \frac{1}{2\pi} \right) \quad (3)$$

$$V(60) = 70 \left(60 - \frac{\cos 120\pi}{2\pi} + \frac{1}{2\pi} \right) \quad (4)$$

$$= 70 \cdot 60 \quad (5)$$

$$= 4200 \quad (6)$$

14. Area between curves (one integral)

$$y = x$$

$$y = x^2 - 2$$

$$x = x^2 - 2 \quad (1)$$

$$0 = x^2 - x - 2 \quad (2)$$

$$= (x - 2)(x + 1) \quad (3)$$

$$x = -1 \quad (4)$$

$$x = 2 \quad (5)$$

$$\int_{-1}^2 x - (x^2 - 2) dx = \int_{-1}^2 -x^2 + x + 2 dx \quad (6)$$

$$= \left(\frac{-2^3}{3} + \frac{2^2}{2} + 2(2) \right) - \left(\frac{-(-1)^3}{3} + \frac{(-1)^2}{2} + 2(-1) \right) \quad (7)$$

$$= \left(\frac{-8}{3} + 6 \right) - \left(\frac{1}{3} + \frac{1}{2} - 2 \right) \quad (8)$$

$$= \frac{-9}{3} + 8 - \frac{1}{2} \quad (9)$$

$$= 5 - \frac{1}{2} \quad (10)$$

$$= 4.5 \quad (11)$$

15. Area between curves (multiple integrals)

$$y = x$$

$$y = x^3$$

$$x^3 = x \quad (1)$$

$$0 = x^3 - x \quad (2)$$

$$= x(x^2 - 1) \quad (3)$$

$$x = -1 \quad (4)$$

$$x = 0 \quad (5)$$

$$x = 1 \quad (6)$$

$$\int_{-1}^0 x^3 - x dx + \int_0^1 x - x^3 dx = - \left(\frac{(-1)^4}{4} - \frac{(-1)^2}{2} \right) + \left(\frac{1^2}{2} - \frac{(-1)^4}{4} \right) \quad (7)$$

$$= - \left(\frac{1}{4} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{4} \right) \quad (8)$$

$$= - \left(-\frac{1}{4} \right) + \frac{1}{4} \quad (9)$$

$$= \frac{1}{4} + \frac{1}{4} \quad (10)$$

$$= \frac{1}{2} \quad (11)$$

16. Area between curves (multiple integrals 2)

$$y = 4x - x^2$$

$$y = 4x - 4$$

$$[0, 2]$$

$$\int_0^1 4x - x^2 dx + \int_1^2 4x - x^2 - 4x + 4 dx = \left(2 + \frac{1}{3}\right) + \left(\left(-\frac{2^3}{3} + 4(2)\right) - \left(-\frac{1^3}{3} + 4(1)\right)\right) \quad (1)$$

$$= \left(2 + \frac{1}{3}\right) + \left(\left(-\frac{8}{3} + 8\right) - \left(-\frac{1}{3} + 4\right)\right) \quad (2)$$

$$= \frac{7}{3} + \left(\frac{16}{3} - \frac{11}{3}\right) \quad (3)$$

$$= \frac{7}{3} + \left(\frac{16}{3} - \frac{11}{3}\right) \quad (4)$$

$$= 4 \quad (5)$$

17. Area between curves (integrating dy)

$$x = \sqrt{y}$$

$$x \frac{y}{4}$$

$$\int_0^4 \left(\sqrt{y} - \frac{y}{4}\right) dy = \left(\frac{2}{3} 4^{\frac{3}{2}} - \frac{4^2}{8}\right) \quad (1)$$

$$= \left(\frac{2}{3} \cdot 8 - \frac{16}{8}\right) \quad (2)$$

$$= \left(\frac{16}{3} - 2\right) \quad (3)$$

$$= \frac{10}{3} \quad (4)$$

18. Area between curves (choosing a method)

$$y = \sqrt{\frac{x}{2} + 1}$$

$$y = \sqrt{1 - x}$$

$$x = 2y^2 - 2 \quad (1)$$

$$x = 1 - y^2 \quad (2)$$

$$2y^2 - 2 = 1 - y^2 \quad (3)$$

$$3y^2 - 3 = 0 \quad (4)$$

$$3(y^2 - 1) = 0 \quad (5)$$

$$y = -1 \quad (6)$$

$$y = 1 \quad (7)$$

$$\int_0^1 (1 - y^2 - 2y^2 + 2) dy = \int_0^1 (3 - 3y^2) dy \quad (8)$$

$$= (3(1) - 1^3) \quad (9)$$

$$= 3 - 1 \quad (10)$$

$$= 2 \quad (11)$$

19. Using geometry to find area between curves

$$x = 2y$$

$$x = y + 1$$

$$(2, 1)$$

$$A_1 = 1 \quad (1)$$

$$A_2 = \frac{1}{2} \quad (2)$$

$$A = 1 - \frac{1}{2} \quad (3)$$

$$= \frac{1}{2} \quad (4)$$

Related Exercises

1. (Section 5.4, Exercise 15)

$$\int_{-2}^2 (x^2 + x^3) dx = \int_{-2}^2 x^2 dx + \int_{-2}^2 x^3 dx \quad (1)$$

$$= 2 \int_0^2 x^2 dx + 0 \quad (2)$$

$$= 2 \frac{x^3}{3} \quad (3)$$

$$= 2 \frac{2^3}{3} - 2 \frac{0^3}{3} \quad (4)$$

$$= 2 \frac{8}{3} \quad (5)$$

$$= \frac{16}{3} \quad (6)$$

2. (Section 5.4, Exercise 16)

$$\int_{-\pi}^{\pi} t^2 \sin t dx = 0$$

3. (Section 5.4, Exercise 26)

$$f(x) = x^2 + 1$$

$$[-2, 2]$$

$$\bar{f} = \frac{1}{2 - (-2)} \int_{-2}^2 x^2 + 1 dx \quad (1)$$

$$= \frac{1}{4} \left(\int_{-2}^2 x^2 dx + 1 \int_{-2}^2 x^0 dx \right) \quad (2)$$

$$= \frac{1}{4} \left(\frac{x^3}{3} + x \right) \quad (3)$$

$$= \frac{1}{4} \left(\int_{-2}^2 x^2 dx + \int_{-2}^2 1 dx \right) \quad (4)$$

$$= \frac{1}{4} \left(\frac{2^3}{3} - \frac{(-2)^3}{3} + 2 - (-2) \right) \quad (5)$$

$$= \frac{1}{4} \left(\frac{8}{3} - \frac{-8}{3} + 4 \right) \quad (6)$$

$$= \frac{1}{4} \left(\frac{16}{3} + 4 \right) \quad (7)$$

$$= \frac{1}{4} \left(\frac{28}{3} \right) \quad (8)$$

$$= \frac{7}{3} \quad (9)$$

4. (Section 5.4, Exercise 34)

$$f(x) = x^3 - 5x^2 + 30$$

$$[0, 4]$$

$$\bar{f} = \frac{1}{4} \left(\int_0^4 (x^3 - 5x^2 + 30) dx \right) \quad (1)$$

$$= \frac{1}{4} \left(\int_0^4 x^3 - 5 \int_0^4 x^2 + 30 \int_0^4 x^0 \right) \quad (2)$$

$$= \frac{1}{4} \left(\frac{x^4}{4} - 5 \frac{x^3}{3} + 30x \right) \quad (3)$$

$$= \frac{1}{4} \left(\left(\frac{4^4}{4} - \frac{0^4}{4} \right) - \left(5 \frac{4^3}{3} - 5 \frac{0^3}{3} \right) + (30(4) - 30(0)) \right) \quad (4)$$

$$= \frac{1}{4} \left(64 - \frac{320}{3} + 120 \right) \quad (5)$$

$$= \frac{1}{4} \left(\frac{232}{3} \right) \quad (6)$$

$$= \frac{58}{3} \quad (7)$$

5. (Section 5.4, Exercise 41)

$$f(x) = 1 - \frac{x^2}{a^2}$$

$$[0, a]$$

$$\bar{f} = \frac{1}{a} \left(\int_0^a 1 - \frac{x^2}{a^2} dx \right) \quad (1)$$

$$= \frac{1}{a} \left(\int_0^a 1 dx - \int_0^a \frac{x^2}{a^2} dx \right) \quad (2)$$

$$= \frac{1}{a} \left(x - \frac{1}{a^2} \int_0^a x^2 dx \right) \quad (3)$$

$$= \frac{1}{a} \left(x - \frac{1}{a^2} \frac{x^3}{3} \right) \quad (4)$$

$$= \frac{1}{a} \left(x - \frac{x^3}{3a^2} \right) \quad (5)$$

$$= \frac{1}{a} \left((a - 0) - \frac{1}{a^2} \left(\frac{a^3}{3} - \frac{0^3}{3} \right) \right) \quad (6)$$

$$= \frac{1}{a} \left(a - \frac{a^3}{3a^2} \right) \quad (7)$$

$$= \frac{1}{a} \left(a - \frac{a}{3} \right) \quad (8)$$

$$= \frac{1}{a} \left(\frac{2a}{3} \right) \quad (9)$$

$$= \frac{2}{3} \quad (10)$$

$$1 - \frac{c^2}{a^2} = \frac{2}{3} \quad (11)$$

$$\frac{c^2}{a^2} = \frac{1}{3} \quad (12)$$

$$c^2 = \frac{a^2}{3} \quad (13)$$

$$c = \sqrt{\frac{a^2}{3}} \quad (14)$$

$$= \frac{a}{\sqrt{3}} \quad (15)$$

6. (Section 5.4, Exercise 42)

$$f(x) = \frac{\pi}{4} \sin x$$

$$[0, \pi]$$

$$\bar{f} = \frac{1}{\pi} \int_0^{\pi} \frac{\pi}{4} \sin x \quad (1)$$

$$= \frac{1}{\pi} \frac{\pi}{4} \int_0^{\pi} \sin x \quad (2)$$

$$= \frac{1}{\pi} \frac{\pi}{4} (-\cos x) \quad (3)$$

$$= \frac{1}{\pi} \frac{\pi}{4} (-\cos \pi + \cos 0) \quad (4)$$

$$= \frac{1}{\pi} \frac{\pi}{4} (1 + 1) \quad (5)$$

$$= \frac{1}{\pi} \frac{\pi}{2} \quad (6)$$

$$= \frac{1}{2} \quad (7)$$

$$\frac{\pi}{4} \sin x = \frac{1}{2} \quad (8)$$

$$\sin x = \frac{2}{\pi} \quad (9)$$

$$\sin^{-1} \sin x = \sin^{-1} \frac{2}{\pi} \quad (10)$$

$$x = \sin^{-1} \frac{2}{\pi} \quad (11)$$

7. (Section 5.5, Exercise 17)

$$u = x^2 - 1 \quad (1)$$

$$du = 2x \, dx \quad (2)$$

$$\int 2x (x^2 - 1)^{99} \, dx = \int u^{99} \, du \quad (3)$$

$$= \frac{u^{100}}{100} + C \quad (4)$$

$$= \frac{(x^2 - 1)^{100}}{100} + C \quad (5)$$

8. (Section 5.5, Exercise 20)

$$u = \sqrt{x} + 1 \quad (1)$$

$$du = \frac{1}{2\sqrt{x}} \, dx \quad (2)$$

$$\int \frac{(\sqrt{x} + 1)^4}{2\sqrt{x}} \, dx = \int u^4 \, du \quad (3)$$

$$= \frac{u^5}{5} + C \quad (4)$$

$$= \frac{(\sqrt{x} + 1)^5}{5} + C \quad (5)$$

9. (Section 5.5, Exercise 21)

$$u = x^2 + x \quad (1)$$

$$du = 2x + 1 \, dx \quad (2)$$

$$\int (x^2 + x)^{10} (2x + 1) \, dx = \int u^{10} \, du \quad (3)$$

$$= \frac{u^{11}}{11} + C \quad (4)$$

$$= \frac{(x^2 + x)^{11}}{11} + C \quad (5)$$

10. (Section 5.5, Exercise 23)

$$u = x^4 + 16 \quad (1)$$

$$du = 4x^3 dx \quad (2)$$

$$x^3 dx = \frac{1}{4} du \quad (3)$$

$$\int x^3 (x^4 + 16)^6 dx = \int \frac{1}{4} u^6 du \quad (4)$$

$$= \frac{1}{4} \int u^6 du \quad (5)$$

$$= \frac{1}{4} \cdot \frac{u^7}{7} + C \quad (6)$$

$$= \frac{(x^4 + 16)^7}{28} + C \quad (7)$$

11. (Section 5.5, Exercise 24)

$$u = \sin \theta \quad (1)$$

$$du = \cos \theta d\theta \quad (2)$$

$$\int \sin^{10} \theta \cos \theta d\theta = \int u^{10} du \quad (3)$$

$$= \frac{u^{11}}{11} + C \quad (4)$$

$$= \frac{\sin^{11} \theta}{11} + C \quad (5)$$

12. (Section 5.5, Exercise 78)

$$u = x - 2 \quad (1)$$

$$x = u + 2 \quad (2)$$

$$du = dx \quad (3)$$

$$\int \frac{x}{x-2} dx = \int \frac{u+2}{u} du \quad (4)$$

$$= \int \frac{u}{u} du + \int \frac{2}{u} du \quad (5)$$

$$= u + 2 \int \frac{1}{u} du + C \quad (6)$$

$$= u + 2 \ln |u| + C \quad (7)$$

$$= x - 2 + 2 \ln |u| + C \quad (8)$$

13. (Section 5.5, Exercise 79)

$$u = \sqrt{x-4} \quad (1)$$

$$u^2 = x - 4 \quad (2)$$

$$x = u^2 + 4 \quad (3)$$

$$dx = 2u du \quad (4)$$

$$\int \frac{x}{\sqrt{x-4}} dx = \int 2u \frac{u^2+4}{u} du \quad (5)$$

$$= \int \frac{2u^3+8u}{u} du \quad (6)$$

$$= 2 \left(\int u^2 du + \int 4 du \right) \quad (7)$$

$$= 2 \left(\frac{\sqrt{x-4}^3}{3} + 4\sqrt{x-4} \right) + C \quad (8)$$

$$= \frac{2}{3} \sqrt{x-4}^3 + 8\sqrt{x-4} + C \quad (9)$$

14. (Section 5.5, Exercise 15)

$$\int e^{10x} dx = \frac{1}{10}e^{10x} + C \quad (1)$$

$$\int \sec 5x \tan 5x dx = \frac{1}{5} \sec 5x + C \quad (2)$$

$$\int \sin 7x dx = -\frac{1}{7} \cos 7x + C \quad (3)$$

$$\int \cos \frac{x}{7} dx = 7 \sin \frac{x}{7} + C \quad (4)$$

$$\int \frac{dx}{81 + 9x^2} = \int \frac{dx}{9^2 + 9x^2} \quad (5)$$

$$= \int \frac{dx}{9(9 + x^2)} \quad (6)$$

$$= \frac{1}{9} \int \frac{dx}{3^2 + x^2} \quad (7)$$

$$= \frac{1}{27} \tan^{-1} \frac{x}{3} + C \quad (8)$$

$$\int \frac{dx}{\sqrt{36 - x^2}} = \int \frac{dx}{\sqrt{6^2 - x^2}} \quad (9)$$

$$= \sin^{-1} \frac{x}{6} + C \quad (10)$$

15. (Section 5.5, Exercise 16)

$$\int_0^1 10^x dx = \frac{1}{\ln 10} 10 - \frac{1}{\ln 10} \quad (1)$$

$$= \frac{9}{\ln 10} \quad (2)$$

$$\int_0^{\frac{\pi}{40}} \cos 20x dx = \cos \frac{20\pi}{40} - \cos 20(0) \quad (3)$$

$$= 0 - 1 \quad (4)$$

$$= -1 \quad (5)$$

$$\int_{3\sqrt{2}}^6 \frac{dx}{x\sqrt{x^2 - 9}} = \int_{3\sqrt{2}}^6 \frac{dx}{x\sqrt{x^2 - 3^2}} \quad (6)$$

$$= \frac{1}{3} \sec^{-1} \left| \frac{6}{3} \right| - \frac{1}{3} \sec^{-1} \left| \frac{3\sqrt{2}}{3} \right| \quad (7)$$

$$= \frac{1}{3} \sec^{-1} \frac{\pi}{3} - \frac{1}{3} \sec^{-1} \frac{\pi}{4} \quad (8)$$

$$\int_0^{\frac{\pi}{16}} \sec^2 4x dx = \frac{1}{4} \tan \frac{4\pi}{16} - \frac{1}{4} \tan 0 \quad (9)$$

$$= \frac{1}{4} - 0 \quad (10)$$

$$= \frac{1}{4} \quad (11)$$

16. (Section 5.5, Exercise 49)

$$u = 2^x + 4 \quad (1)$$

$$du = 2^x \ln 2 \, dx \quad (2)$$

$$2^x \, dx = \frac{1}{\ln 2} \, du \quad (3)$$

$$\int_1^3 \frac{2^x}{2^x + 4} \, dx = \frac{1}{\ln 2} \int_6^{12} \frac{1}{u} \, du \quad (4)$$

$$= \frac{1}{\ln 2} (\ln 12 - \ln 6) \quad (5)$$

$$= \frac{1}{\ln 2} \cdot \ln 2 \quad (6)$$

$$= 1 \quad (7)$$

17. (Section 5.5, Exercise 51)

$$u = \sin \theta \quad (1)$$

$$du = \cos \theta \, d\theta \quad (2)$$

$$\int_0^{\frac{\pi}{2}} \sin^2 \theta \cos \theta \, d\theta = \int_0^{\frac{\pi}{2}} u^2 \, du \quad (3)$$

$$= \int_0^1 u^2 \, du \quad (4)$$

$$= \frac{1^3}{3} - \frac{0^3}{3} \quad (5)$$

$$= \frac{1}{3} \quad (6)$$

18. (Section 5.5, Exercise 64)

$$u = 3 + 2e^x \quad (1)$$

$$du = 2e^x \, dx \quad (2)$$

$$e^x \, dx = \frac{1}{2} \, du \quad (3)$$

$$\int_0^{\ln 4} \frac{e^x}{3 + 2e^x} \, dx = \frac{1}{2} \int_{u=5}^{u=11} \frac{1}{u} \, du \quad (4)$$

$$= \frac{\ln 11 - \ln 5}{2} \quad (5)$$

19. (Section 5.5, Exercise 87)

$$u = 2x \quad (1)$$

$$du = 2 dx \quad (2)$$

$$dx = \frac{1}{2} du \quad (3)$$

$$\cos^2 x = \frac{1 + \cos 2x}{2} \quad (4)$$

$$\int_{-\pi}^{\pi} \cos^2 x dx = \int_{-\pi}^{\pi} \frac{1}{2} + \frac{\cos 2x}{2} dx \quad (5)$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} 1 + \cos 2x dx \quad (6)$$

$$= \frac{1}{2} \left(\int_{-\pi}^{\pi} 1 dx + \frac{1}{2} \int_{-\pi}^{\pi} \cos u du \right) \quad (7)$$

$$= \frac{1}{2} \left((\pi - (-\pi)) + \frac{1}{2} \int_{u=-2\pi}^{u=2\pi} \cos u du \right) \quad (8)$$

$$= \frac{1}{2} \left(2\pi + \frac{1}{2} (\sin 2\pi - \sin(-2\pi)) \right) \quad (9)$$

$$= \frac{1}{2} \left(2\pi + \frac{1}{2} (0) \right) \quad (10)$$

$$= \frac{2\pi}{2} \quad (11)$$

$$= \pi \quad (12)$$

20. (Section 5.5, Exercise 91)

$$u = 4\theta \quad (1)$$

$$du = 4 d\theta \quad (2)$$

$$d\theta = \frac{1}{4} du \quad (3)$$

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad (4)$$

$$\sin^2 2\theta = \frac{1 - \cos 4\theta}{2} \quad (5)$$

$$\int_{-\pi}^{\pi} \sin^2 2\theta d\theta = \int_{-\pi}^{\pi} \frac{1 - \cos 4\theta}{2} d\theta \quad (6)$$

$$= \int_{-\pi}^{\pi} \frac{1}{2} - \frac{\cos u}{2} d\theta \quad (7)$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} 1 - \cos u d\theta \quad (8)$$

$$= \frac{1}{2} \left(\int_{-\pi}^{\pi} 1 d\theta - \int_{-\pi}^{\pi} \cos u d\theta \right) \quad (9)$$

$$= \frac{1}{2} \left(2\pi - \frac{1}{4} \int_{u=-4\pi}^{u=4\pi} \cos u du \right) \quad (10)$$

$$= \frac{1}{2} \left(2\pi - \frac{1}{4} (\sin 4\pi - \sin(-4\pi)) \right) \quad (11)$$

$$= \frac{2\pi}{2} \quad (12)$$

$$= \pi \quad (13)$$

21. (Section 6.1, Exercise 7)

(a) $0 < t < 1$ and $3 < t < 5$

(b) -4

(c) 26

(d) 6

22. (Section 6.1, Exercise 8)

(a) $0 < t < 2$ and $4 < t < 6$

(b) 4

(c) 44

(d) -10

23. (Section 6.1, Exercise 17)

$$v(t) = \sin t$$

$$s(0) = 1$$

$$s(t) = 1 + \int_0^t \sin t \, dt \quad (1)$$

$$= 1 - \cos t + \cos 0 \quad (2)$$

$$= 2 - \cos t \quad (3)$$

24. (Section 6.1, Exercise 20)

$$v(t) = 3 \sin \pi t$$

$$s(0) = 1$$

$$u = \pi t \quad (1)$$

$$du = \pi \, dt \quad (2)$$

$$s(t) = 1 + \int_0^t 3 \sin \pi t \, dt \quad (3)$$

$$= 1 + \frac{3}{\pi} \int_0^t \sin u \, du \quad (4)$$

$$= 1 + \frac{3}{\pi} (-\cos \pi t + \cos 0) \quad (5)$$

$$= 1 + \frac{3}{\pi} - \frac{3}{\pi} \cos \pi t \quad (6)$$

25. (Section 6.1, Exercise 27)

$$\int_0^{20} 3t \, dt + \int_{20}^{30} 60 \, dt = \left(\frac{3(20)^2}{2} - \frac{3(0)^2}{2} \right) + (60(30) - 60(20)) \quad (1)$$

$$= \left(\frac{1200}{2} \right) + (1800 - 1200) \quad (2)$$

$$= 600 + 600 \quad (3)$$

$$= 1200 \quad (4)$$

$$\int_0^{20} 3t \, dt + \int_{20}^{45} 60 \, dt + \int_{45}^{60} 240 - 4t \, dt = 600 + 2700 - 1200 + 14400 - 7200 - 10800 - 4050 \quad (5)$$

$$= 2100 + ((14400 - 7200) - (10800 - 4050)) \quad (6)$$

$$= 2100 + (7200 - 6750) \quad (7)$$

$$= 2550 \quad (8)$$

$$s(t) = \int_0^{20} 3t \, dt + \int_{20}^{45} 60 \, dt + \int_{45}^t 240 - 4t \, dt \quad (9)$$

$$= 2100 + (240t - 2t^2 - 6750) \quad (10)$$

$$= 240t - 2t^2 - 4650 \quad (11)$$

$$s(75) = 2100 \quad (12)$$

26. (Section 6.1, Exercise 28)

$$\int_0^{10} 9.8t \, dt + \int_{10}^{30} 10 \, dt = (4.9(10)^2 - 4.9(0^2)) + (10(30) - 10(10)) \quad (1)$$

$$= (4.9(100)) + (300 - 100) \quad (2)$$

$$= 490 + 200 \quad (3)$$

$$= 690 \quad (4)$$

27. (Section 6.1, Exercise 30)

$$a(t) = -32$$

$$v(0) = 50$$

$$s(0) = 0$$

$$v(t) = v(0) + \int_0^t -32 \, dt \quad (1)$$

$$= 50 - 32t \quad (2)$$

$$s(t) = s(0) + \int_0^t 50 - 32t \, dt \quad (3)$$

$$= 50t - 16t^2 \quad (4)$$

28. (Section 6.1, Exercise 31)

$$a(t) = -9.8$$

$$v(0) = 20$$

$$s(0) = 0$$

$$v(t) = v(0) + \int_0^t -9.8 \, dt \quad (1)$$

$$= 20 - 9.8t \quad (2)$$

$$s(t) = s(0) + \int_0^t 20 - 9.8t \, dt \quad (3)$$

$$= 20t - 4.9t^2 \quad (4)$$

29. (Section 6.1, Exercise 43)

$$N'(t) = 100e^{-0.25t}$$

$$u = -0.25t \quad (1)$$

$$du = -0.25 \, dt \quad (2)$$

$$N(t) = N(0) + \int_0^t 100e^{-0.25t} \, dt \quad (3)$$

$$= 1900 + 100 \int_0^t \frac{1}{-0.25} e^u \, du \quad (4)$$

$$= 1900 - 400e^{-0.25t} \quad (5)$$

$$N(20) = 1900 - 400e^{-5} \quad (6)$$

$$\approx 1897.305 \quad (7)$$

$$N(40) = 1900 - 400e^{-10} \quad (8)$$

$$\approx 1899.98 \quad (9)$$

30. (Section 6.1, Exercise 44)

$$r(t) = 0.0025e^{0.25t} - 0.1485e^{-0.15t}$$

$$t_0 = \frac{\ln 0.0025 - \ln 0.1485}{-0.4} \quad (1)$$

$$\int_0^{t_0} 0.0025e^{0.25t} - 0.1485e^{-0.15t} \, dt \approx 271 \quad (2)$$

31. (Section 6.1, Exercise 55)

$$C'(x) = 2000 - 0.5x$$

$$\int_{100}^{150} 2000 - 0.5x \, dx = (2000(150) - 0.25(150)^2) - (2000(100) - 0.25(100)^2) \quad (1)$$

$$= (300000 - 5625) - (200000 - 2500) \quad (2)$$

$$= 294375 - 197500 \quad (3)$$

$$= 96875 \quad (4)$$

$$\int_{500}^{550} 2000 - 0.5x \, dx = (1024375) - (937500) \quad (5)$$

$$= 86875 \quad (6)$$

32. (Section 6.1, Exercise 56)

$$C'(x) = 200 - 0.05x$$

$$\int_{100}^{150} 200 - 0.05x \, dx = (200(150) - 0.025(150)^2) - (200(100) - 0.025(100)^2) \quad (1)$$

$$= (29437.5) - (19750) \quad (2)$$

$$= 9687.5 \quad (3)$$

$$\int_{500}^{550} 200 - 0.05x \, dx = (200(550) - 0.025(550)^2) - (200(500) - 0.025(500)^2) \quad (4)$$

$$= 102437.5 - 93750 \quad (5)$$

$$= 8687.5 \quad (6)$$