# TD Analyse :

## Suite Réelle :

### Définition :

$$U : \mathbb{N} \longrightarrow \mathbb{R}$$
$$n \longrightarrow U_n$$

- $U_n = \frac{1}{n}$, $n \in \mathbb{N}^*$, $U_1 = \frac{1}{1} = 1$.

- $U_n = 1 + \frac{n^2}{2}$, $n \in \mathbb{N}$, $U_0 = 1$.

- $U_n$ majorée, $\exists M \in \mathbb{R}$, $U_n \leq M$, $\forall x \in \mathbb{N}$
- $U_n$ minorée, $\exists m \in \mathbb{R}$, $U_n \geq m$, $\forall x \in \mathbb{N}$
- $U_n$ bornée, $\exists M$, $m \in \mathbb{R}$, $U_n \geq m$, $\forall x \in \mathbb{N}$, $U_n = U_{n_0}$.
- $U_n$ périodique $\exists k \in \mathbb{N}$ tq $U_{n+k} = U_n$, $\forall n \in \mathbb{N}$

- $U_n \nearrow$, $U_{n+1} \geq U_n$, $U_{n+1}$, Si $U_n \neq 0$, $\frac{U_{n+1}}{U_n} \geq 1$.

- $U_n \searrow$, $U_{n+1} \leq U_n$, Si $U_n \neq 0$, $\frac{U_{n+1}}{U_n} \leq 1$.
- $U_n$ cte, $U_{n+1} = U_n$,

  Si $U_n \neq 0$, $\frac{U_{n+1}}{U_n} = 1$.

limite :

$$\lim_{+\infty} U_n = \ell \begin{cases} \in \mathbb{R} \\ +\infty \\ -\infty \end{cases}$$

$\ell = +\infty \Rightarrow \forall A > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, U_n \geq A$.

$\ell = -\infty \Rightarrow \forall B > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, U_n \leq B$.

$\ell \in \mathbb{R}$, $\lim_{+\infty} U_n = \ell$, $\Rightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0$
$|U_n - \ell| < \varepsilon$

## Suite Arith

$U_{m+1} = U_m + r$, $U_{m_0} \in \mathbb{R}$.

$U_m = (m - m_0) r + U_{m_0}$

$U_{m_0+1} = U_{m_0} + r$

$U_{m_0+2} = U_{m_0+1} + r = U_{m_0} + 2r$

$U_{m_0+3} = U_{m_0+2} + r = U_{m_0} + 3r$

$U_m = U_{m-1} + r = (m - m_0) r + U_{m_0}$

Sommation :

premier terme dernière terme

$$\sum_{k=m_0}^{m} U_k = \frac{\text{nde}T\,(P^{\text{ère}}T + dT)}{2}$$

$$= (m - m_0 - 1)\left(\frac{U_{m_0} + U_m}{2}\right)$$

## Suite geo

$U_{m+1} = q\, U_m$, $U_{m_0} \in \mathbb{R}$

$U_m = q^{m-m_0}\, U_{m_0}$, $U_{m_0}$ "given"

$U_{m_0+1} = q U_{m_0}$

$U_{m_0+2} = q\, U_{m_0+1} = q^2\, U_{m_0}$

$U_{m_0+3} = q U_{m_0+2} = q^3\, U_{m_0}$

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$\downarrow$

$U_m = q U_{m-1} = q^{m-m_0}\, U_{m_0}$

Sommation :

$$\sum_{k=m_0}^{m} U_k = P^{\text{ère}}T\left(\frac{1 - (\text{raison})^{\text{n.de}T}}{1 - \text{raison}}\right) \text{ Si } q \neq 1$$

$$= U_{m_0}\left(\frac{1 - q^{m-m_0+1}}{1-q}\right)$$

Si $q = 1$ :

$U_m = q^{m-m_0}\, U_{m_0}) = U_{m_0}$,

$$\sum_{k=m_0}^{m} 1\; U_{m_0} = U_{m_0} \cdot \sum_{k=m_0}^{m} 1$$

$$\boxed{a^m = e^{\ln(a^m)} = e^{m\ln(a)}}$$

$$\boxed{(a+b)\cdot(a-b) = a^2 - b^2}$$

Serie 1 :

ex 1 :

Limits :

1) $U_n = \left(1+\frac{1}{n}\right)^n = e^{\ln\left(1+\frac{1}{n}\right)} = e^{0 \cdot \ln\left(\frac{1}{n}\right)} = e^0 = 1.$

2) $U_n = \sqrt{n^2+n+1} - \sqrt{n^2-n+1}$

$(a - b) \times (a + b) = a^2 - b^2$

$$\lim_{+\infty} U_n = \lim_{+\infty} \frac{\left(\sqrt{n^2+n+1} - \sqrt{n^2-n+1}\right)\left(\sqrt{n^2+n+1} + \sqrt{n^2-n+1}\right)}{\left(\sqrt{n^2+n+1} + \sqrt{n^2-n+1}\right)}$$

$$= \lim_{+\infty} \frac{2n}{\left(\sqrt{n^2+n+1} + \sqrt{n^2-n+1}\right)}$$

$$= \lim_{+\infty} \frac{2n}{\left(\sqrt{n^2\left(1+\frac{1}{n}+\frac{1}{n^2}\right)}\right) + \sqrt{n^2\left(1-\frac{1}{n}+\frac{1}{n^2}\right)}}$$

$$= \lim_{+\infty} \frac{2n}{\left(n\underbrace{\sqrt{1}}_{\sqrt{1}}\sqrt{+\underbrace{\frac{1}{n}}_{0}+\underbrace{\frac{1}{n^2}}_{0}}\right) + \left(n\underbrace{\sqrt{1}}_{\sqrt{1}}\sqrt{-\underbrace{\frac{1}{n}}_{0}+\underbrace{\frac{1}{n^2}}_{0}}\right)}$$

$$= \lim_{+\infty} \frac{2n}{2n} = 1$$

3) $U_n = \frac{4^n - (-2)^n}{4^n + (-2)^n}$ ; $\lim_{+\infty} U_n = \frac{(2^{2n}) - (-2)^n}{2^{2n} + (-2)^n}$

$= \lim_{+\infty}$

③ $\lim\limits_{+\infty} U_{n3} = \lim\limits_{+\infty} \dfrac{4^n - (-2)^n}{4^n + (-2)^n} = \lim\limits_{+\infty} \dfrac{(-2)^{2n} - (2)^n}{(-2)^{2n} + (2)^n}$

$= \lim\limits_{+\infty} \dfrac{(-2)^n + 1}{(-2)^n + 1} = 1$ (plus degrée)

④ $\lim\limits_{+\infty} U_{n4} = \lim\limits_{+\infty} 1 + \dfrac{\sqrt{n}}{n+1} = 1 + \lim\limits_{+\infty} \dfrac{\sqrt{n}}{n+1} = 1 + \lim\limits_{+\infty} \dfrac{\sqrt{n}}{\sqrt{n}^2 + 1} = \lim\limits_{+\infty} \dfrac{\overbrace{1}^{0}}{\underbrace{\sqrt{n}}_{+\infty} + \underbrace{\dfrac{1}{\sqrt{n}}}_{0}}$

$= 1.$

⑤ $\lim\limits_{+\infty} U_n = \lim\limits_{+\infty} \dfrac{n^2+1}{n^2+n+1} = \lim\limits_{+\infty} \dfrac{n^2}{n^2} = 1$ (plus degrée)

⑥ $\lim\limits_{+\infty} U_n = \lim\limits_{+\infty} \dfrac{2n+3}{2n+1} = \lim\limits_{+\infty} \dfrac{2n}{2n} = 1$ (plus degrée)

⑦ $\lim\limits_{+\infty} U_{n7} = \lim\limits_{+\infty} \dfrac{n^2 + (-1)^n \sqrt{n}}{n^2 + n + 1} = \lim\limits_{+\infty} \dfrac{n^2\left(1 + \dfrac{(-1)^n \sqrt{n}}{n^2}\right)}{n^2\left(1 + \underbrace{\dfrac{1}{n}}_{0} + \underbrace{\dfrac{1}{n^2}}_{0}\right)}$ Fact $(n^2)$

$= \lim\limits_{+\infty} \dfrac{n^2\left(1 + \dfrac{(-1)^n \cancel{\sqrt{n}}}{n\sqrt{n}\cancel{\sqrt{n}}}\right)}{n^2}$

$= \lim\limits_{+\infty} \dfrac{n^2\left(1 + \overbrace{\dfrac{(-1)^n}{n\sqrt{n}}}^{0}\right)}{n^2} = \lim\limits_{+\infty} \dfrac{n^2}{n^2} = 1.$

⑧ $\lim\limits_{+\infty} U_{n8} = \lim\limits_{+\infty} 1 + \dfrac{\sin(n^2)}{n+1}$ Encadrement.

$\Leftrightarrow \quad -1 \leq \sin(n^2) \leq 1$

$\Leftrightarrow \quad \lim\limits_{+\infty} \quad \dfrac{-1}{n+1} \leq \dfrac{\sin(n^2)}{n+1} \leq \dfrac{1}{n+1}$

$\Leftrightarrow \quad \lim\limits_{+\infty} \quad 0 \leq \dfrac{\sin(n^2)}{n+1} \leq 0$

$\Leftrightarrow \quad \lim\limits_{+\infty} \quad 1 + \overbrace{\dfrac{\sin(n^2)}{n+1}}^{0} = 1$

(9) $\lim_{+\infty} U_{n9} = \lim_{+\infty} \frac{2n+(-1)^n}{2n+1} = \frac{\cancel{n}\left(2+\frac{(-1)^n}{n}\right)}{\cancel{n}\left(2+\frac{1}{n}\right)} = \lim_{+\infty} \frac{2}{2} = 1$.

(10) $\lim_{+\infty} U_{n10} = \lim_{+\infty} \frac{1}{n^2} \cdot \sum_{k=1}^{n} k = \lim_{+\infty} \frac{1}{n^2} \cdot \frac{\cancel{n}(n+1)}{2} = \frac{1}{n} \cdot \frac{(n+1)}{2} = \lim_{+\infty} \frac{n+1}{2n}$

Fact(n)

$= \lim_{+\infty} \frac{\cancel{n}\left(1+\frac{1}{n}\right)}{2\cancel{n}} = \lim_{+\infty} \frac{1}{2} = \frac{1}{2}$

(11) $\lim_{+\infty} U_{n11} = \lim_{+\infty} = \frac{1}{n}\left(\sin\left(\frac{1}{n}\right)\right)^{1/n}$

$\sin\left(\frac{1}{n}\right) \sim \frac{1}{n}$

$\sin\left(\frac{1}{n}\right) \sim \frac{1}{n} \Leftrightarrow \ln\left(\sin\left(\frac{1}{n}\right)\right) \sim \ln\left(\frac{1}{n}\right)$

donc $\frac{1}{n} \ln\left(\sin\left(\frac{1}{n}\right)\right) \sim -\frac{\ln(n)}{n}$

$\Rightarrow \lim_{+\infty} \frac{1}{n} e^0 = \underline{\underline{0}}$.

$\left(\sin\left(\frac{1}{n}\right)\right)^{1/n} = e^{1/n \ln\left(\sin\left(\frac{1}{n}\right)\right)}$

$\rightarrow \frac{\sin(x)}{x}$ $x \to 0$ $\Rightarrow 1$.

$\sin(x) = x + o(x^2)$

DL:

$\sin(n) = n - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + o(x^n)$

(12) $\lim\limits_{+\infty} U_{n12} = \lim\limits_{+\infty} \left(\frac{n-1}{n+1}\right)^n = \lim\limits_{+\infty} e^{n\ln\left(\frac{n-1}{n+1}\right)}$ $\left\{ \ln(1+x) \sim x \right.$

$\ln\left(\frac{n-1}{n+1}\right) = \ln\left(\frac{n+1-2}{n+1}\right) = \ln\left(1-\frac{2}{n+1}\right) \underset{+\infty}{\sim} \frac{-2}{n+1}$.

donc $\lim\limits_{+\infty} n\ln\left(\frac{n-1}{n+2}\right) = \ln\left(\frac{-2n}{n+1}\right) = -2$

d'où $\lim\limits_{+\infty} U_{n12} = \lim\limits_{+\infty} e^{-2} = \frac{1}{e^2} = \left(\frac{1}{e}\right)^2$

(13) $\lim\limits_{+\infty} U_{n13} = \lim\limits_{+\infty} \frac{n-\sqrt{n^2+1}}{n+\sqrt{n^2-1}} = \lim\limits_{+\infty} \frac{n-n\sqrt{1+\frac{1}{n^2}}}{n+n\sqrt{1-\frac{1}{n^2}}} = \lim\limits_{+\infty} \frac{1-\sqrt{1+\frac{1}{n^2}}}{1+\sqrt{1-\frac{1}{n^2}}} = 0$.

(14) $\lim\limits_{+\infty} \frac{\sin(n)}{n} = \frac{1}{n} = 0$ (encadrement)