An introduction to quadratic equations

Three methods

Zhao Tang

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Table of Content

Table of Content

Table of Content

Introduction

What is a Quadratic Function?

- A quadratic function is:

$$y = ax^2 + bx + c$$

  where $a \neq 0$.
- Common examples:
- $y = x^2 + 2x + 1$
- $y = 3x^2 - 2$

A Quick Test

Which of the following is a quadratic function?

- A. $y = 3x + 2$
- B. $y = -x^2 + 1$
- C. $y = e^{2t} + e^t + 3$.
- D. $y = x^3 - x^2 + 1$

Answer

- A. No. It is a linear one.
- B. Yes.
- C. No. Let $x = e^t$, then $y = x^2 + x + 3$
- D. No. It is a cubic one.

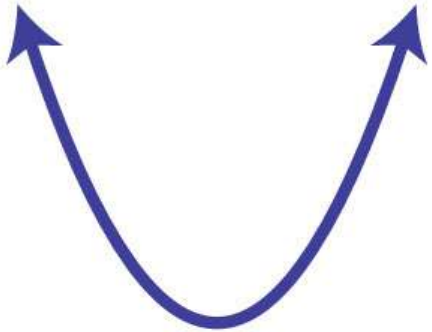
Graph of a Quadratic Function

The graph of a quadratic function is a parabola.

- Line of symmetry: $x = -\frac{b}{2a}$
- If $a > 0$, the parabola opens upward.
- If $a < 0$, the parabola opens downward.

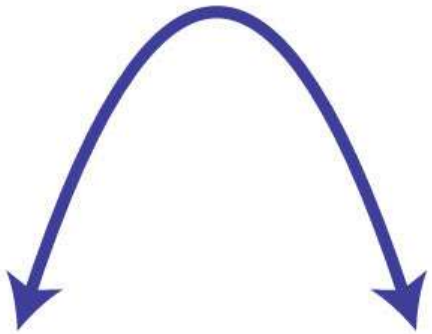
Parabola $y = ax^2 + bx + c$

$a > 0$



opens upward

$a < 0$



opens downward

A Quadratic Equation

- A quadratic equation is:

$$ax^2 + bx + c = 0$$

- Geometrically the intersection of the graph and the x-axis.
- How to solve it ?

Three methods

1. Factorizing

Factorizing

A simple example. Given $x^2 - 5x + 6 = 0$:

- Principle: $6 = 1 * 6 = 2 * 3$
- $x^2 - 5x + 6 = (x - 2)(x - 3) = 0$
- Roots: $x_1 = 2, \quad x_2 = 3$

Factorizing

Another example: Given $2x^2 + x - 6 = 0$:

- $2x^2 + x - 6 = (2x - 3)(x + 2)$
- Roots: $x_1 = \frac{3}{2}, \quad x_2 = -2$

Principle

In order to have $x^2 + bx + c = (x + m)(x + n)$

- $c = m * n$
- $b = m + n$
- A quick calculation of m and n

For general $ax^2 + bx + c = (a_1x + m)(a_2x + n)$

- $a = a_1 * a_2$
- $c = m * n$
- Remember the signs $\pm$.

Exercises

Solve the following equations:

- 1. $x^2 - 8x + 12 = 0$
- 2. $2x^2 - 7x + 6 = 0$
- 3. $6x^2 - x - 1 = 0$

Answer

- 1. $(x-6)(x-2)=0$. $x=6$ or $x=2$
- 2. $(2x-3)(x-2)=0$. $x=\frac{3}{2}$ or $x=2$
- 3. $(3x+1)(2x-1)=0$. $x=-\frac{1}{3}$ or $x=\frac{1}{2}$

2. Completing the square

Completing the square

Example. Solve the equation $x^2 - 4x - 2 = 0$

$$\begin{aligned} x^2 - 4x + 4 - 2 &= 4 \\ x^2 - 4x + 4 &= 4 + 2 \\ x^2 - 4x + 4 &= 6 \\ (x-2)^2 &= 6 \\ x - 2 &= \pm\sqrt{6} \\ x &= 2 \pm \sqrt{6} \end{aligned}$$

The roots are $x_1 = 2 - \sqrt{6}$, $x_2 = 2 + \sqrt{6}$

Completing the square

Another example. Solve the equation $x^2 + 2x - 2 = 0$

$$\begin{aligned} x^2 + 2x + 1 - 2 &= 1 \\ x^2 + 2x + 1 &= 3 \\ (x-1)^2 &= 3 \\ x - 1 &= \pm\sqrt{3} \\ x &= 1 \pm \sqrt{3} \end{aligned}$$

The roots are $x_1 = 1 - \sqrt{3}$, $x_2 = 1 + \sqrt{3}$

Exercises

Solve the following equations:

- 1. $x^2 - 8 = 0$
- 2. $x^2 - 6x + 6 = 0$
- 3. $4x^2 - 16x + 8 = 0$

Answer

- 1. $x^2 = 8$. So $x = \pm 2\sqrt{2}$.
- 2. $x^2 - 6x + 9 = 3$ and $(x-3)^2 = 3$
  So $x = 3 + \sqrt{3}$ or $x = 3 - \sqrt{3}$
- 3. $x^2 - 4x + 2 = 0$ and $x^2 - 4x + 4 = 2$
  Then $(x-2)^2 = 2$
  So $x = 2 + \sqrt{2}$ or $x = 2 - \sqrt{2}$

3. The roots formula

Roots of a Quadratic Equation

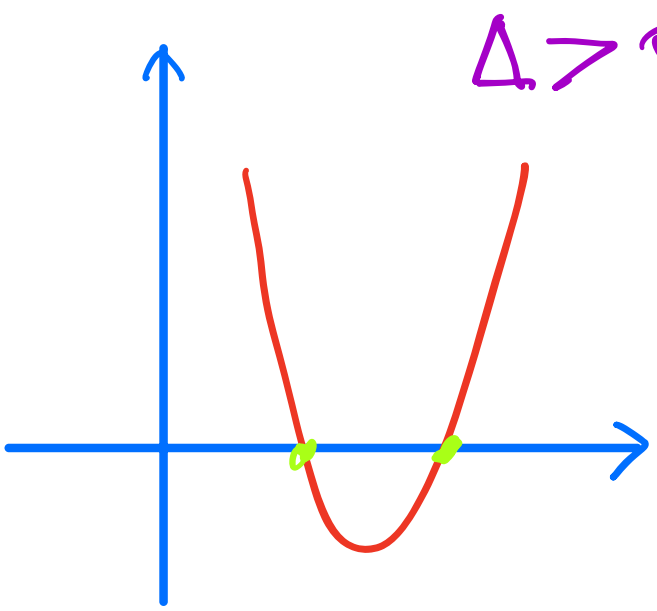
The roots formula of a quadratic equation $ax^2 + bx + c = 0$ is:

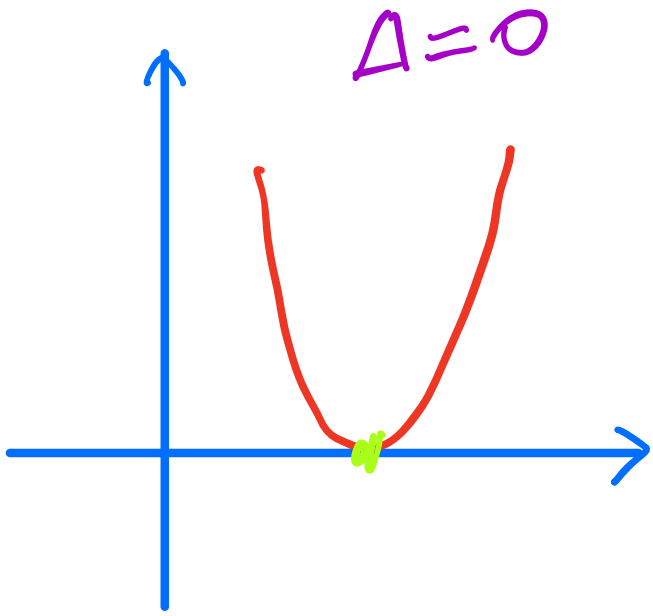
$$x = \frac{-b \pm \sqrt{\Delta}}{2a}$$

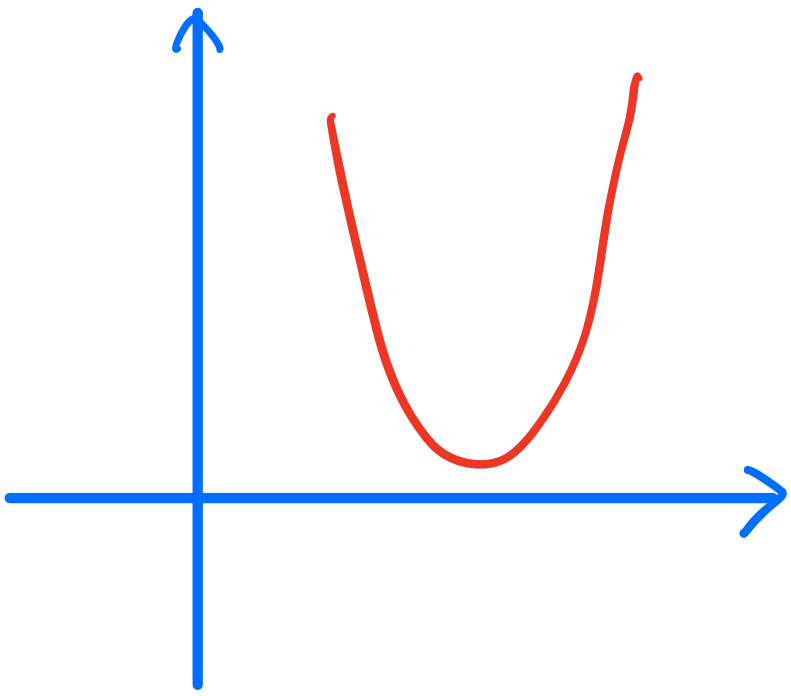
Steps:

1. Find a, b, and c.
2. Compute: $\Delta = b^2 - 4ac$.
3. Δ determines the nature of roots:
   - $\Delta > 0$: 2 distinct real roots.
   - $\Delta = 0$: 1 repeated real root.
   - $\Delta < 0$: 0 real roots.

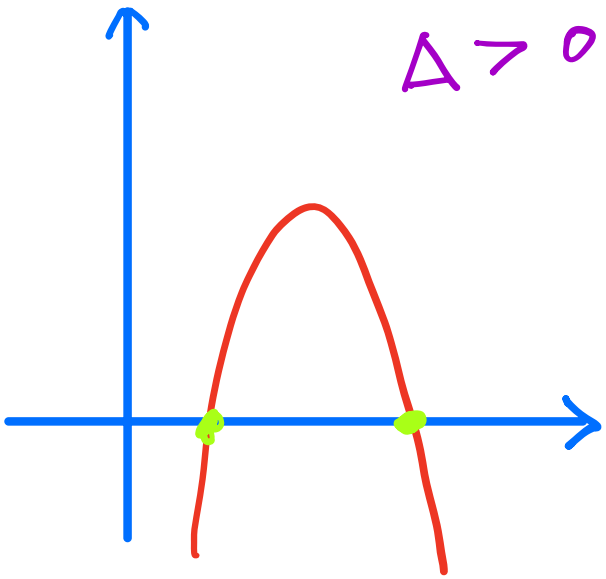
$a > 0$

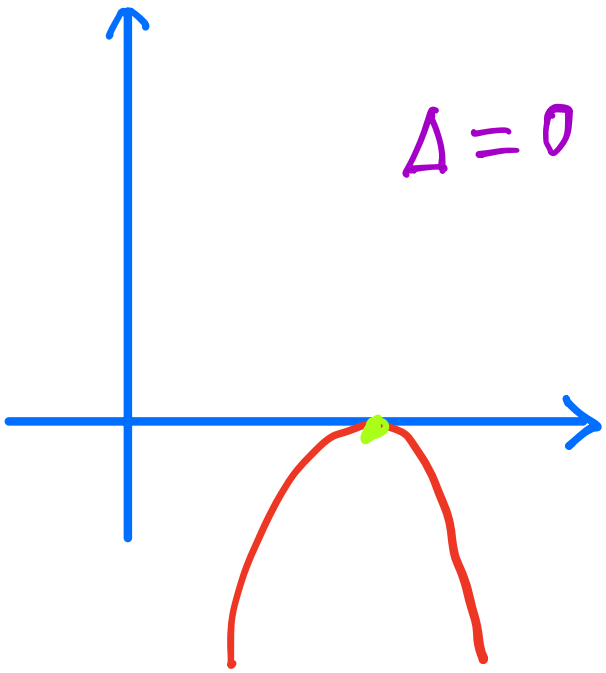


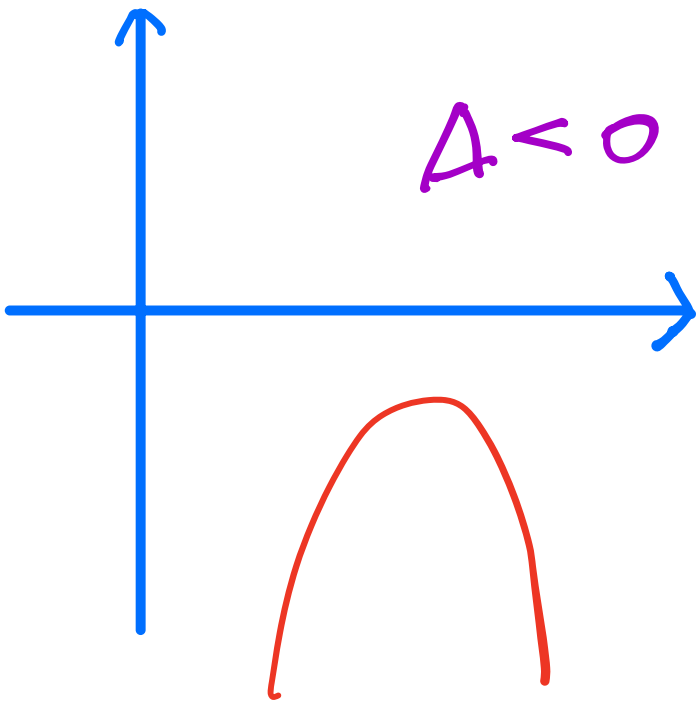




$a < 0$







Example

Given $x^2 - 5x + 6 = 0$:

- Find: $a = 1,\ b = -5,\ c = 6.$
- Compute: $\Delta = 5^2 - 4 * 1 * 6 = 25 - 24 = 1.$
- Formula:
$$x = \frac{-(-5) \pm \sqrt{1}}{2 * 1} = \frac{5 \pm 1}{2}$$
- Roots:
$$x_1 = 3, \quad x_2 = 2$$

Another Example

Given $x^2 - 6x + 9 = 0$:

- Find: $a = 1$, $b = -6$, $c = 9$.
- Compute: $\Delta = (-6)^2 - 4 * 1 * 9 = 36 - 36 = 0$.
- Formula:
$$x = \frac{-(-6) \pm 0}{2 * 1} = \frac{6}{2}$$
- One root:
$$x = 3$$

Exercises

Solve the following equations:

- 1. $2x^2 + 7x - 15 = 0$
- 2. $4x^2 + 5x + 1 = 0$
- 3. $2x^2 - 8x + 8 = 0$

Answer

- 1. $x = -\frac{3}{2}$ or $x = -5$
- 2. $x = -\frac{1}{4}$ or $x = -1$
- 3. One root $x = 2$

Summary

Summary

- A Quadratic Equation $ax^2 + bx + c = 0$, where $a \neq 0$.
- Three Methods:
  - 1. Factorizing
  - 2. Completing the Square
  - 3. The Roots Formula
- Easter Egg

Easter Egg

What if $\Delta < 0$ in the roots formula ?

- Example. $x^2 + x + 1 = 0$
- $\Delta = 1 - 4 = -3 < 0$
- $$x = \frac{-1 \pm \sqrt{-3}}{2}$$

Easter Egg

Let $i = \sqrt{-1}$

$$x = \frac{-1 \pm \sqrt{3} * \sqrt{-1}}{2}$$

$$= \frac{-1 \pm \sqrt{3} * i}{2}$$

The complex roots are:

$$x_1 = \frac{-1 + \sqrt{3}i}{2}$$

$$x_2 = \frac{-1 - \sqrt{3}i}{2}$$

Thank You !