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A note on algebraic techniques for subgraph detection

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ABSTRACT

The k-path problem asks whether a given graph contains a simple path of length k. Along with other prominent parameterized problems, it reduces to the problem of detecting multilinear terms of degree k (k-MLD), making the latter a fundamental problem in parameterized algorithms.

This has generated significant efforts directed at devising fast deterministic algorithms for k-MLD, and there are now at least two independent approaches that yield the same record bound on the running time. Namely the combinatorial representative-set based approach of Fomin et al. (JACM'16), and the algebraic techniques of Pratt (FOCS'19) and Brand and Pratt (ICALP'21).

In this note, we explore the relationship between the latter results, based on partial differentials, and a previous algebraic approach based on the exterior algebra (Brand, ESA'19; Brand, Dell and Husfeldt, STOC'18). We do so by studying the relevant algebraic objects. These are on the one hand (1) the subalgebras of the tensor square of the exterior algebra generated in degree one. On the other hand, we consider (2) the space of partial derivatives of generic determinants, and closely related, (3) the space of minors of generic matrices.

We prove that (2) arises as a quotient of (1), and that there is an isomorphism between the objects (1) and (3). Hence, the techniques are essentially equivalent, and the quotient relation between (2) and both of (1) and (3) hints at a possible refinement of the techniques.

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1. Introduction

The *k-path problem* asks: Given a graph G, is there a simple path in G of length at least k? Replacing paths of length k with an arbitrary subgraph H on k vertices leads to the more general *subgraph isomorphism problem*, asking whether there is a subgraph of G that is isomorphic to H. This family of problems is part of a larger set of other parameterized problems that efficiently reduce to a single algebraic problem: *k-multilinear detection (k-MLD)*.

Here, we are given an arithmetic circuit C on n variables. We are then asked to decide whether the polynomial computed by C contains a multilinear term of degree k with non-zero coefficient (that is, one where each variable appears with exponent at most one). Readers unfamiliar with arithmetic circuits may think of them as a sort of straight-line programs, where each instruction is either an assignment, an addition or a multiplication, and the available atoms are field constants and the variables over which the polynomials are defined.

Owing to its generality and usefulness, k-MLD has become a fundamental object of study in parameterized algorithms, with the aim of obtaining algorithms that run in $c^k \cdot \mathrm{poly}(n)$ steps and keep the exponential base c as small as possible. Koutis [1] was the first to design efficient *randomized* algorithms for k-MLD, which were subsequently

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improved by Williams [2], leading to the current record bound of $c = 2$ for randomized techniques.

While more classical techniques for e.g. the k-path problem, such as Color-Coding [3] or Divide-and-Color [4,5], have been successfully derandomized, there remains an exponential gap in the parameter k between deterministic and randomized algorithms both for k-MLD and the k-path problem. Significant efforts have been directed at closing this gap for the k-path problem, with the current state-of-the-art being achieved by two independent lines of research: The combinatorial representative-set based techniques of Fomin et al. [6], and algebraic techniques based on partial differentials of polynomials of Brand and Pratt [7]. This was in turn foreshadowed in an earlier work of Pratt [8], employing similar techniques to different ends. Indeed, both the combinatorial and the algebraic approach yield:

Theorem 1.1 *([6,7]). There is an algorithm that, given a directed graph G on n nodes and a number k, decides in time* $\phi^{2k} \cdot poly(n)$ *whether there is a simple path in G on k vertices. Here,* ϕ *is the golden ratio, satisfying* $2.618 < \phi^2 < 2.619$.

In this note, we are concerned with the latter, algebraic approach based on partial differentials and its relation to earlier algebraic approaches of Brand [9] and Brand, Dell and Husfeldt [10], which were based on exterior algebras and yielded a worse base of $c = 4$ in their running times.

To this end, we consider the relevant algebraic objects. These are on the one hand (1) the subalgebras of the tensor square of the exterior algebra generated in degree one, arising in the exterior-algebraic approach [9,10]. On the other hand, we consider (2) the space of partial derivatives of generic determinants (or their so-called apolar algebras), and closely related, (3) the space of minors of generic matrices, which are the protagonists of the partial-differential or apolarity-based approach [8,7].

In [7], computation in the space of partial derivatives is effected by reducing to computation of minors, while [8] relies instead on so-called Waring decompositions of polynomials. We prove an isomorphism between the abovementioned exterior subalgebras and spaces of minors in Theorem 5.1, and show that the apolar algebras of determinants arise as a quotient of exterior subalgebras and minors, who are isomorphic by Theorem 5.1, in Corollary 5.5. This quotient relation is proper in general.

On the algorithmic side, this isomorphism implies that the techniques based on partial differentials via minors [7] and exterior algebraic methods [9,10] are equivalent, and that we may think of the results relying purely on the apolar algebra [8] also in terms of the exterior-algebraic approach, namely as working over a suitable quotient. The fact that (2) arises as a quotient points towards potential improvements in terms of a refinement of the techniques. This avenue for improvement was remarked in the context of partial differentials in [7]. However, it is not at all clear how to make use of this.

We emphasize that reproducing the exact running time of Theorem 1.1 requires some non-trivial new insight into the exterior-algebraic methods over what is given in [10,9] alone, which only yield an exponential base of four instead of ϕ^2 as they are. The present note gives one way to obtain this insight, by identifying the exterior with the apolar approach of [8,7] by means of an isomorphism. This isomorphism may just be used in a black-box manner in all computations in [9,10] to yield the claimed running time. A more satisfactory and explicit way to obtain the faster running time from the exterior approach without resorting to expressing everything in terms of apolar algebras requires some work, and will be published separately.

Establishing similarly explicit and precise connections between the algebraic techniques and the original approach of Fomin et al. [6] remains a challenging and highly interesting open problem. On a more speculative note, this confluence of different techniques and results to the same upper bound may be seen as an indicator that this upper bound actually is the best-possible for some generalization of the k-path problem that is captured both by combinatorial and algebraic techniques. One should note, however, that we know from results of Zehavi [11] and Tsur [12] that the exponent ϕ^2 is not optimal for the plain k-path problem, while the general k-MLD problem is not known be solvable using the combinatorial techniques of Fomin et al. This can be seen as suggesting the existence of some problem of intermediate difficulty for which ϕ^2 is the correct exponential base, and for which both the combinatorial and the algebraic techniques are applicable.

Organization First, we review the ideas that underlie algebraic methods in parameterized algorithms, and in particular the ones treated in this note. This is followed by a brief dramatis personae, as well as a motivating example. We then go on to generalize this example, thereby obtaining our main results, and end by putting them in the context of the algorithmic techniques we consider.

2. Algebraic methods

The general outline for solving a parameterized problem in an algebraic manner is as follows, which we illustrate using the abovementioned problem of finding a path of prescribed length in a graph. First, the entire search space is encoded as some easily computable polynomial, usually some kind of multivariate generating function. For instance, all walks of given length k in the graph G, including those with repeated vertices, are captured by the walk-generating polynomial

$$\mathcal{F}_k = \sum_{u,\ldots,w \text{ } k\text{-walk in } G} x_u \cdots x_w$$

over a polynomial ring in variables x_v for each $v \in V(G)$. This polynomial is easily expressed using the powers of the (symbolic) adjacency matrix $S(G)$ of G, which is obtained from the usual adjacency matrix $A(G)$ via $S(G)_{i,j} = A(G)_{i,j} \cdot x_{v_i}$. The original combinatorial question is then translated to a question about this algebraic object; in this case, G contains a k-path if and only if $\mathcal{F}_k$ contains a multilinear monomial, that is, one without squared terms. The algebraic question is then solved by evaluating $\mathcal{F}_k$ over a suitable algebraic structure of dimension bounded by a function that depends only on k, but not on $|V(G)|$: If the evaluation vanishes, the answer is no, if it doesn't, it is yes. The algebraic structures that have been employed to solve this problem in the past include in particular the exterior and apolar algebras, which we treat in the present note.

For more background, specifically on exterior and apolar algebras and their algorithmic uses, we point to [10,7] and the references therein.

3. Notation and preliminaries

Throughout the paper, we let V be a k-dimensional vector space over a field K of characteristic 0 with standard basis $e_1, \dots, e_k$.[1] The space of all endomorphisms on V, that is, linear maps $V \to V$, is denoted by $\mathrm{End}(V)$. For any vector space U, we let U^* be its dual, i.e., the set of linear functionals, which are the linear maps $U \to K$. Whenever we pick a basis $(u_i)_i$ for U, then $(u_i^*)_i$ will denote the corresponding dual basis, and similarly for any element u of U, u^* is its associated linear map with respect to the choice of basis. The *tensor product* of K-algebras is the tensor product of the underlying vector spaces, endowed with component-wise multiplication.

3.1. Exterior algebras

The exterior algebra of V can be defined in a number of equivalent ways. Very briefly, it is the 2^k-dimensional K-algebra ΛV with linear basis indexed by subsets of $[k]$, say $\{e_S\}_{S \subseteq [k]}$, and a multiplication called the *wedge product*. It is denoted as $\wedge$ and defined on the basis via $e_S \wedge e_T = \sigma(S,T) \cdot e_{S \cup T}$. Hence, it is linearly extended to all of ΛV. Here, $\sigma(S,T)$ is zero if S and T are not disjoint. Otherwise, we enumerate S and T in ascending order as $s_1, \dots, s_{|S|}$ and $t_1, \dots, t_{|T|}$, respectively, and define $\sigma(S,T)$ as the sign of the permutation that brings the concatenated sequence $s_1, \dots, s_{|S|}, t_1, \dots, t_{|T|}$ into ascending order.

One of its most striking properties is that it captures the minors and determinants of matrices in the following sense: Whenever $A_1, \dots, A_t \in V$ are the columns of some matrix A, then the coefficient of e_S in the wedge product $A_1 \wedge \cdots \wedge A_t$ is the minor corresponding to the $t \times t$ submatrix of A with row indices given by S. In particular, if $k = t$, the only possible non-zero coefficient in the corresponding wedge-product is the one of $e_{[k]}$, which is equal to $\det(A)$.

The *degree* of e_S is $|S|$. The set of linear combinations of elements of degree i is the i-th *graded part* of ΛV, and denoted by $\Lambda^i V$. It is a linear space of dimension $\binom{k}{i}$. By definition, $\Lambda^i V \cdot \Lambda^j V \subseteq \Lambda^{i+j} V$.

3.2. Graded endomorphisms of the exterior algebra

We denote the vector space of *graded* linear endomorphisms of degree 0 (that is, elements of degree i are mapped to elements of degree i) on the exterior algebra ΛV over V as $\mathrm{End}(\Lambda V)^\bullet$. As a vector space,

$$\mathrm{End}(\Lambda V)^\bullet = \bigoplus_{i=0}^{k} \mathrm{End}\left(\Lambda^i V\right).$$

For all $0 \le d \le k$, we consequently define

$$\mathrm{End}(\Lambda V)^\bullet_d = \mathrm{End}\left(\Lambda^d V\right).$$

Now, we will first apply the natural isomorphism $\left(\Lambda^i V\right)^* \cong \Lambda^i (V^*)$ and drop the parentheses. We then in turn apply the natural isomorphism

$$\mathrm{End}(\Lambda V)^\bullet \cong \bigoplus_{i=0}^{k} \Lambda^i V \otimes \Lambda^i V^*. \tag{3.1}$$

Note that the expected operation on the left-hand side of this isomorphism is the composition of linear maps, but instead, we shall choose the viewpoint on the right-hand side, for the following reason. Both the exterior algebra and its dual carry the structure of a graded algebra via the wedge product on vectors and linear functionals, respectively. In particular, the tensor product $\Lambda V \otimes \Lambda V^*$ is again an algebra, and the multiplication map of this algebra can be seen to respect the grading, in the sense that it restricts to a map

$$\left(\Lambda^i V \otimes \Lambda^i V^*\right) \otimes \left(\Lambda^j V \otimes \Lambda^j V^*\right) \longrightarrow \Lambda^{i+j} V \otimes \Lambda^{i+j} V^*.$$

That means that the ordinary wedge product turns

$$\mathrm{End}(\Lambda V)^\bullet = \bigoplus_{i=0}^{k} \Lambda^i V \otimes \Lambda^i V^*$$

into a commutative, graded algebra, and we regard $\mathrm{End}(\Lambda V)^\bullet$ as the algebra endowed with the multiplication of $\bigoplus_{i=0}^{k} \Lambda^i V \otimes \Lambda^i V^*$. We define

$$M = \bigoplus_i M_i = \mathrm{End}(\Lambda V)^\bullet.$$

Since $V = E_1$, and $\mathrm{End}(V) = V \otimes V^* = \Lambda^1 V \otimes \Lambda^1 V^* = M_1$, any linear subspace $F \subseteq \mathrm{End}(V)$ of the space of linear endomorphisms of V generates a graded subalgebra $M[F] = \bigoplus_i M[F]_i$ of M. The subalgebras of the form $M[F]$ play the crucial rôle in the approach of [10,9].

3.3. Exterior powers and minors

Let $F \subseteq \mathrm{End}(V)$ again be any linear subspace of the space of endomorphisms of V. Let m be the dimension of F, $R := K[x_1, \dots, x_m]$, and fix a basis for V and F, say $F = \mathrm{span}\{f_1, \dots, f_m\}$, where the f_i are hence considered as matrices. We remark that our result will not depend on the choice of basis. For a linear form $u = \sum_{j=1}^m u_j x_j \in R_1$, we use the notation $\vec{u}$ to denote the vector $\vec{u} = (u_1, \dots, u_m) \in K^m$. Define

$$F(x_1, \dots, x_m) = x_1 f_1 + \cdots + x_m f_m,$$

[1] The results will also hold once the field and its characteristic are large enough, but become generally false over finite fields or in small characteristics. The algorithmic techniques that we consider here are devised in characteristic zero.

such that

$$\det(F(x_1,\dots,x_m)) = \det(x_1 f_1 + \cdots + x_m f_m) \in R\,.$$

Associated to each $\alpha \in \mathrm{End}(V)$ is its t-th exterior power $\wedge^t\alpha \in M_t$, defined by

$$\wedge^t\alpha(v_1\wedge\cdots\wedge v_t) = \alpha(v_1)\wedge\cdots\wedge\alpha(v_t)\,.$$

In coordinates, $\wedge^t$ takes each $k\times k$-matrix to its $\binom{k}{t}\times\binom{k}{t}$-matrix of t-minors. It holds that

$$\wedge^t\alpha = \alpha^t\,,$$

where the power on the right-hand side is taken in M. Therefore, $\mathrm{span}\{\wedge^t(F)\}$ is (the dual of) the space of t-minors of matrices in F, and in particular, $M_t = \mathrm{span}\{\wedge^t(M_1)\}$.

We can of course also consider the minor polynomials of $F(x_1,\dots,x_m)$: We let $X(F)_t$ be the subspace of polynomials in R of t-minors of $x_1 f_1 + \cdots + x_m f_m$.

The space $X(F)_t$ of minor polynomials is algorithmically exploited in [7], where it is used in order to compute in the closely related apolar algebra, which we now introduce.

3.4. Apolarity

For $f, g \in R$, we denote with

$$g\circ f := g(\partial) f \in R$$

the apolar action, which over K of characteristic 0 just means differentiation, of g on f. That is, we associate to each monomial $m = x_1^{i_1}\cdots x_m^{i_m}$ the differential operator

$$m(\partial) := \frac{\partial^{i_1+\cdots+i_m}}{\partial x_1^{i_1}\dots\partial x_m^{i_m}}\,,$$

such that

$$m\circ f = m(\partial) f = \frac{\partial^{i_1+\cdots+i_m} f}{\partial x_1^{i_1}\dots\partial x_m^{i_m}}\,.$$

This is linearly extended to all of R. For a subspace $S\subseteq R$, we denote with $S\circ f = \{g\circ f \mid g\in S\}$ the *apolar orbit* of f under S, which is again a linear space.

For $f, g \in R_t$ homogeneous of equal degree, we can then define a symmetric bilinear form on R_t by

$$\langle f, g\rangle = g\circ f = f\circ g \in K\,.$$

For $f\in R_t$, we denote with $f^\perp \subseteq R_t$ the *linear space* of polynomials *of equal degree* that are orthogonal to f, that is, that annihilate f under the apolar action:

$$f^\perp = \{g\in R_t \mid \langle f, g\rangle = 0\}\,.$$

More generally, we let

$$\mathrm{Ann}(f) := \{g\in R \mid g\circ f = 0\}\,,$$

and

$$\mathrm{Ann}(f)_i := \mathrm{Ann}(f)\cap R_i = \{g\in R_i \mid g\circ f = 0\}\,,$$

such that $\mathrm{Ann}(f)_t = f^\perp$. Note that $R_{t-i}\circ f\subseteq R_i$. For $0\le i\le t$, it is well-known that

$$\mathrm{Ann}(f)_i = (R_{t-i}\circ f)^\perp \subseteq R_i\,.$$

Furthermore, we let $A(f)$ be the quotient $R/\mathrm{Ann}(f)$, which is a graded algebra $A(f) = \bigoplus_{i=0}^t A(f)_i$, where

$$A(f)_i = R_i/\mathrm{Ann}(f)_i = R_i/(R_{t-i}\circ f)^\perp\,.$$

By the dimension formula (the $A(f)_i$ are linear spaces),

$$R_i/\mathrm{Ann}(f)_i \cong R_i\circ f\subseteq R_{t-i}\,. \tag{3.2}$$

The algebra $A(f)$ is known as the *apolar algebra* of f, and is employed in a non-straightforward way in [8], and more directly in [7].

4. A motivating example

An example of the more general phenomenon that is described in this note is provided already in [7], and we reiterate it here to give some context.

To start, we recall the following well-known relation between apolar orbits and minors of matrices with linear forms as entries.

Lemma 4.1. *The space of minors of symbolic matrices is invariant under the apolar action, that is,*

$$R_j\circ X(F)_{i+j}\subseteq X(F)_i\subseteq R_i \tag{4.1}$$

holds for all $i, j\ge 0$. In particular, for $j = k-i$, we see that

$$R_{k-i}\circ\det(F)\subseteq X(F)_i$$

holds.

This fact was used in [7] to indirectly perform computations in the apolar algebra. It can be proven e.g. by Leibniz expansion. Later on, we will give a different proof based on our results.

As in [7], we now consider the case where $F = M$, such that $M(x_{1,1},\dots,x_{k,k})$ is the generic matrix, with generic determinant $\det_k$. In this case, we obtain the following:

Proposition 4.2. *The algebras M and $A(\det_k)$ are isomorphic.*

Proof. Shafiei [13, Lemma 1.3] showed that in the case of generic determinants, the minors actually *are* the apolar orbits. That is,

$$R_{k-i}\circ\det_k = X(F)_i$$

holds. Even more, the algebra $A(\det_k)$ is defined by the relations

$$x_{ij}x_{\ell h} = -x_{ih}x_{\ell j}$$

and

$$x_{ij}x_{ih} = x_{ij}x_{hj} = 0$$

for all $i \neq \ell$, $j \neq h$ [13, Theorem 2.12]. The mapping $R \to M$ that sends $x_{ij} \in R$ to $e_i \otimes e_j^* \in M$ leaves these relations intact, making M a homomorphic image of $A(\det_k)$. Note that this ad hoc argument is in contrast to our later result, which will show that apolar algebras are in general homomorphic images of subalgebras of M.

Finally, comparing dimensions of M and $A(\det_k)$ then yields that this surjection must be an isomorphism. To see this, note first that clearly, $\dim_K M_i = \binom{k}{i}^2$, and hence $\dim_K M = \binom{2k}{k}$. Furthermore, by [13, Eq. (2.2)], $\dim_K A(\det_k) = \binom{2k}{k}$ holds, such that M and $A(\det_k)$ are of equal dimension. □

This algebraic relationship between M and $\det_k$ is not a coincidence, and we will provide a more general form, thereby shedding light on an algorithmic relationship.

5. Results

We now generalize the preceding example and prove the following main result.

Theorem 5.1. *Let $I = \sum_j (X(F)_j)^\perp$. This sum is direct and I is an ideal of R. In particular, $M[F]$ and R/I are isomorphic as algebras.*

In the proof, we will make use of the following fact, see e.g. [14, Eq. (1.1.10)].

Lemma 5.2. *Let $u = \sum_{j=1}^m u_j x_j \in R_1$ and $f \in R_i$, and recall that $\vec{u} = (u_1, \dots, u_m) \in K^m$. Then, the following holds:*

$$u^i \circ f = \langle u^i, f \rangle = i! f(\vec{u}).$$

We start with some technical observations. First, since K is infinite, polynomial interpolation implies that

$$\dim_K \operatorname{span}\{\wedge^t(F)\} = \dim_K X(F)_t \tag{5.1}$$

holds.

Additionally, for any commutative K-algebra A, we let A^d be the set of all d-fold products of elements of A. Then, the following identity, sometimes called *polarization identity*, holds: For $a_1, \dots, a_d \in A$, we have

$$d! \cdot a_1 \cdots a_d = \sum_{I \subseteq [d]} (-1)^{d-|I|} \left(\sum_{i \in I} a_i \right)^d .$$

This implies that $\operatorname{span} A^d = \operatorname{span}\{a^d \mid a \in A\}$ holds. In particular,

$$\dim_K \operatorname{span} A^d = \dim_K \operatorname{span}\{a^d \mid a \in A\} \tag{5.2}$$

holds. Eqs. (5.1) and (5.2) imply the following *linear* isomorphism.

Proposition 5.3. *In each degree i, we have $\dim_K M[F]_i = \dim_K \operatorname{span}\{F^i\} = \dim_K X(F)_i$.*

We then lift this to an *algebraic* isomorphism as follows.

Lemma 5.4. *Define a ring homomorphism*

$$\varphi : R \to M[F],\ x_i \mapsto f_i\,,$$

by the universal property of R. Then,

1. *φ is surjective, and*
2. *$\ker(\varphi)_i \subseteq (X(F)_i)^\perp$.*

Proof. The first claim is clear from the fact that $M[F]$ is by definition generated by $\{f_i\}_i$.

Towards the second claim, suppose that $g \in \ker(\varphi)_i$. Note that, since φ is a graded linear homomorphism of degree 0, it follows that $g \in R_i$. Expand $g = \sum_j u_j^i$ as a sum of powers of linear forms, where $u_j \in R_1$, say

$$u_j = \sum_{k=1}^m u_{j,k} x_k .$$

We want to show that $g \in (X(F)_i)^\perp$. To this end, we write the image of u_j under φ, which is a matrix, as

$$\varphi(u_j) = \sum_j u_{j,k} f_k \in F .$$

Correspondingly, $\varphi(u_j)^i$ is the vector of i-minors of F evaluated at $\vec{u}_j$:

$$\varphi(u_j)^i = (h(\vec{u}_j))_{h \in X(F)_i} .$$

In consequence, for every minor $h \in X(F)_i$, we have

$$\sum_j h(\vec{u}_j) = 0\,, \tag{5.3}$$

because $g \in \ker(\varphi)$ by assumption.

Consider then $g \circ h$ for any such i-minor $h \in X(F)_i$, which is $g \circ h = \sum_j (u_j^i \circ h)$. By Lemma 5.2, it follows that $u_j^i \circ h = i! \cdot h(\vec{u}_j)$, and by Eq. (5.3), $g \circ h = 0$ as desired. □

Now, Theorem 5.1 follows by comparing dimensions: Proposition 5.3 yields that $\ker(\varphi)_i = (X(F)_i)^\perp$, and since φ is of degree 0 and the sum defining the ideal is direct, $\ker(\varphi) = \bigoplus_i \ker(\varphi)_i$, proving the theorem.

Lemma 4.1 then implies the following.

Corollary 5.5. *$A(\det F(x_1, \dots, x_m))$ is a homomorphic image of $M[F]$.*

Remark 5.6. We can also deduce Lemma 4.1 from Theorem 5.1: If $f \in (X(F)_i)^\perp$ and $g \in R_j$, then $fg \in (X(F)_{i+j})^\perp$. Hence, for all $g \in R_j, h \in X(F)_{i+j}$, we have $f \circ g \circ h = 0$, meaning $f \in (R_j \circ X(F)_{i+j})^\perp$, so that $X(F)_i \supseteq R_j \circ X(F)_{i+j}$.

We can of course ask when the homomorphism in Corollary 5.5 is an isomorphism. This is the case when all apolar orbits coincide with the space of minors containing them:

$$R_j \circ X(F)_{i+j} \stackrel{?}{=} X(F)_i . \tag{5.4}$$

We saw in the example of $M = F$ in Section 4 that equality may indeed hold. But, upon inspecting other cases, we find that it is not always satisfied. As we are about to see, the most relevant example for our purposes of establishing a connection between algorithmic techniques, symbolic Hankel determinants, provides us with an instance where this is not the case, and Eq. (5.4) becomes a strict inclusion.

6. Applications and discussion

We now turn to these examples, and try to shed some light on the relation of our result to algorithm design.

6.1. Diagonal and generic symmetric matrices

The arguably simplest example can be obtained as follows. Let D be the space of diagonal matrices, such that $D(x_1, \dots, x_k)$ is the generic diagonal matrix, and

$$\det D(x_1, \dots, x_k) = x_1 x_2 \cdots x_k$$

The apolar ideal of $\det D(x_1, \dots, x_k)$ is generated by $\{x_i^2\}_i$, and the minors of $D(x_1, \dots, x_k)$ do not only contain the apolar orbits, but are actually equal to them, since they are either a single multilinear monomial or zero. Eq. (5.4) holds, and it follows that $A(x_1 \cdots x_k)$ is even isomorphic to $M[D]$, which is the subalgebra of M generated by $e_i \otimes e_i^*$.

For another example, let us consider $S(x_{\{ij\}}) = (x_{\{ij\}})_{ij}$, the generic symmetric matrix, and the corresponding matrix space S. Similar to the generic case, see [15, Eq. (3)], in the generic symmetric case, minors and apolar orbits coincide again as in Eq. (5.4), such that $A(\det(S(x_{\{ij\}}))$ is isomorphic to $M[S]$, which is the subalgebra of M generated by $\{u \otimes u^* \mid u \in V\}$. This is not to be confused with the previous example, since $M[D]$ in particular doesn't contain the elements $e_i \otimes e_j^* + e_j \otimes e_i^* = (e_i + e_j) \otimes (e_i + e_j)^* - e_i \otimes e_i^* - e_j \otimes e_j^* \in M[S]$.

6.2. Hankel matrices

This is the case that most concerns the algorithmic applications [7] and [9,10]. Let $H(x_1, \dots, x_{2k-1}) = (x_{i+j-1})_{ij}$ be the generic $k \times k$ Hankel matrix, H the corresponding matrix space. Then, $M[H]$ is the subalgebra of M generated by $\{u \otimes u^* \mid u = (t, t^2, \cdots, t^k), t \in K\}$. As a shorthand, we write $\kappa_k = \det H(x_1, \dots, x_{2k-1})$.[2]

The space of minors of $H(x_1, \dots, x_{2k-1})$ was used in [7] for algorithmic purposes, namely, to perform computations in $A(\kappa_k)$ indirectly. The exterior subalgebra $M[H]$ was implicitly used in [10], and later explicitly in [9].

Theorem 5.1 and Corollary 5.5 imply that the algorithms given in [7] (which perform computation in $A(\kappa_k)$ by using the minors of H) and [9] (which uses the subalgebra $M[H]$ instead) for e.g. the longest path problem are equivalent.

[2] This polynomial is often referred to as the *catalecticant* of degree k, which should illuminate the choice of κ as a shorthand.

Properness of the quotient This example provides an instance where minors and the apolar algebra do not coincide, that is, where the quotient relation established by Corollary 5.5 is proper. To see this, note the fact that apolar algebras $A(f)$ have the property that their Hilbert functions are symmetric around $d/2$, where d is the degree of the polynomial f. In plain terms, this means that $\dim_K A(f)_i = \dim_K A(f)_{d-i}$ for all $i \le d$ [14, Def. 1.9 and Eq. (1.1.7)].

On the other hand, the minors of H do not have this property, which follows by arguments as made e.g. by Conca [16, Sect. 1], which we quickly sketch. On the monomials in R, we may impose a total order $>_L$, called the degree lexicographic monomial order. It orders the monomials primarily by degree, and lexicographically by their exponent vectors within a single degree. The initial (that is, maximal w.r.t. $>_L$) monomials appearing in the s-minors of $H(x_1, \dots, x_{2k-1})$ are precisely those monomials $x_{i_1} \cdot x_{i_2} \cdots x_{i_s}$ whose index sequences satisfy $i_a + 1 < i_{a+1}$ (which Conca calls a $<_1$-chain). The number of such chains is $\binom{2k-s}{s}$. It is a lower bound on the dimension of the space of s-minors of H, and one can show, as done by Conca, that it is also an upper bound. However, $\binom{2k-s}{s} \neq \binom{k+s}{k-s}$ in general, so the dimensions are not symmetric in the above sense. Therefore, Eq. (5.4) does not hold, and Theorem 5.1 together with Corollary 5.5 shows:

Proposition 6.1. *$A(\kappa_k)$ is a proper quotient of $M[H]$.*

The dimension of $M[H]$ was determined in [9] without having the results of this note available:

Theorem 6.2 *([9]). For all i, $\dim_K M[H]_i = \binom{2k-i}{i}$ holds.*

We can in particular rederive Theorem 6.2 using Theorem 5.1 and the observation of Conca on the dimension of the spaces of s-minors of H. Turning this around, we could also have derived the properness of the quotient relation starting from Theorem 6.2, and using Theorem 5.1.

Potential for improvements As was remarked already in [7], there is computational overhead associated with passing from the actual apolar algebra to the minors using Lemma 4.1, or equivalently, passing to the subalgebra $M[H]$ using Theorem 5.1. Indeed, we have the following.

Proposition 6.3. *The dimension of the kernel of any linear map $M[H] \to A(\kappa_k)$ is exponential in k.*

Proof. We noted above that $\dim_K A(\kappa_k)_i$ is symmetric around $k/2$ and hence bounded by $\min\{\binom{2k-i}{i}, \binom{k+i}{k-i}\}$ from above, by Theorem 6.2. Stirling's approximation then implies that $\dim_K A(\kappa_k) < 2.6^k$, while $\dim_K M[H] > 2.618^k$. $\square$

Therefore, working over the apolar algebra directly instead of over spaces of minors might yield modest exponential speed-ups, with an exponential base of $\xi < 2.6 <$

ϕ^2 in the running time. Unfortunately, we have no explicit description of this algebra available that would enable fast direct computations. Additionally, we have no lower bound on the dimension of $A(\kappa_k)$, which is usually obtained by exploiting a correspondence with related minors (see [13,15]). However, such a correspondence, as shown above, does not hold in the case of κ_k.

This defect between the apolar algebra and the minors appears necessarily from the dimensionality arguments above whenever the dimensions of the minors in some matrix space F are not symmetric around $k/2$. Indeed, whenever $\dim_K M[F]_i > M[F]_{k-i}$, the pairing $M[F]_i \times M[F]_{k-i} \to M[F]_k$, $(x, y) \mapsto xy$ necessarily is not *perfect*. This means that there are elements x of degree i in $M[F]$ that annihilate all elements of degree $k-i$, so that $xy = 0$ holds for all $y \in M[F]_{k-i}$. We call such elements x *degeneracies.*

Let us end with illustrating this by means of $F = H$ in the first interesting case, where we let $K = \mathbb{R}$.

Example. Let $k = 3$, and $i = 2$. Then, the dimensions of $M[H]_i$ are given, by Theorem 6.2, as $(1, 5, 6, 1)$. Since $5 < 6$, the multiplication pairing $M[H]_2 \times M[H]_1 \to M[H]_3$ will *not* be perfect.

For $i, j, h \in \mathbb{R}$, let us write

$$[i\ j] = ((i, i^2, i^3) \wedge (j, j^2, j^3)) \otimes ((i, i^2, i^3) \wedge (j, j^2, j^3))^*$$

and

$$[h] = (h, h^2, h^3) \otimes (h, h^2, h^3)^*$$

as shorthands. A direct computation shows that a linear basis of the degree-two and one parts is given through $\{[1\ 2], [1\ 3], [1\ 4], [2\ 3], [2\ 4], [3\ 4]\}$, and $\{[1], [2], [3], [4], [5]\}$, respectively. A general product of a degree-two element with basis representation $(\lambda_{12}, \ldots, \lambda_{34})$ and a degree-one element with basis representation $(\mu_1, \ldots, \mu_5)$ can then be expressed as

$$\sum_{1 \le i < j \le 6} \sum_{k=1}^{5} \lambda_{ij} \mu_j (i-j)^2 (i-k)^2 (j-k)^2 \cdot e_{[k]} \otimes e^*_{[k]} \,.$$

This corresponds to the product $\lambda \cdot C \cdot \mu^T$ for a suitable matrix $C \in K^{6\times5}$, and computing the left kernel of C then shows that setting

$$x = [1\ 2] - 16 \cdot [1\ 3] + 3 \cdot [1\ 4] + 27 \cdot [2\ 3] - 16 \cdot [2\ 4] + [3\ 4]\,,$$

we have $x \cdot y = 0$ for *all* $y \in M[H]_1$, and x is a degeneracy.

Degeneracies such as x in the preceding example are computational dead weight, for the purposes of algorithms as sketched in Section 2: The output is an element of degree k, and any degeneracy appearing as an intermediate result of the computation will be augmented to an element of degree k before the algorithm terminates, and hence will be sent to zero anyways. Passing to the apolar algebra corresponds to quotienting out such elements (and possibly more), which provides an intuitive explanation of why we pay with additional running time when using the full algebra $M[H]$ instead.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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