The Hang Seng University of Hong Kong
Department of Mathematics, Statistics and Insurance
AMS1302 Probability and Statistical Theory
2022-2023 Semester 2 Assignment 2

Answers

1	For the exponential distribution we have: $$\mu = E(X_i) = \frac{1}{5},$$ $$\sigma^2 = V(X_i) = \frac{1}{5^2} = \frac{1}{25}.$$ Hence, $$E(\Sigma X_i) = 2500\left(\frac{1}{5}\right) = 500,$$ $$V(\Sigma X_i) = 2500\left(\frac{1}{25}\right) = 100.$$ Using the Central Limit Theorem, $\Sigma X_i \sim N(500,100)$. We have $$P(\Sigma X_i \leq 550) = P\left(\frac{\Sigma X_i - 500}{\sqrt{100}} \leq \frac{550-500}{\sqrt{100}}\right) = P(Z \leq 5) = 1$$ **(7 marks)**
2(a)	$$f(x) = cxe^{-0.0016x^2}, \quad x > 0$$ Let $u = 0.0016x^2$. $\frac{du}{dx} = 0.0032x \to \frac{1}{0.0032}du = xdx$ When $x = 0, u = 0$. When $x = \infty, u = \infty$. The total probability =1 $$\int_0^\infty cxe^{-0.0016x^2}dx = 1$$ $$c\int_0^\infty e^{-0.0016x^2}(xdx) = 1$$ $$\int_0^\infty e^{-u}\frac{1}{0.0032}du = \frac{1}{c} \quad (change\ of\ variable)$$ $$\int_0^\infty e^{-u}du = \frac{0.0032}{c}$$ $$lim_{u\to\infty}(-e^{-u}) + e^{-0} = \frac{0.0032}{c}$$ $$0 + 1 = \frac{0.0032}{c} \quad (by\ (3.19)\ in\ ch3)$$

$$c = 0.0032$$

(7 marks)

2(b)

Method 1:

$$f(x) = \begin{cases} 0.0032xe^{-0.0016x^2}, & \text{for } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

By using change of variables,

let $t = 0.0016u^2$. $\frac{dt}{du} = 0.0032u \rightarrow \frac{1}{0.0032} dt = udu$

When $u = 0$, then $t = 0.0016(0^2) = 0$;

When $u = x$, then $t = 0.0016(x^2) = 0.0016x^2$

$$F(x) = \int_0^x 0.0032ue^{-0.0016u^2}\, du = \int_0^x 0.0032e^{-(0.0016u^2)}\,(udu)$$

$$= \int_0^{0.0016x^2} 0.0032e^{-t}\left(\frac{1}{0.0032}\, dt\right) = \int_0^{0.0016x^2} e^{-t}\, dt$$

$$= -e^{-t}\big|_0^{0.0016x^2} = -e^{-0.0016x^2} + 1, \qquad if\ x > 0$$

$$P(X > 25) = 1 - F(25) = 1 - \left(-e^{-0.0016(25^2)} + 1\right) = e^{-0.0016(25^2)} = e^{-1}$$

$$= 0.3679$$

Method 2:

Let $u = 0.0016x^2$. $\frac{du}{dx} = 0.0032x \rightarrow \frac{1}{0.0032} du = xdx$

When $x = 25, u = 0.0016(25^2) = 1$. When $x = \infty, u = \infty$.

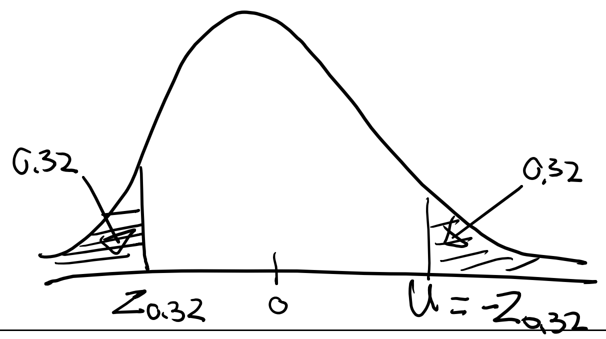
$$P(X > 25) = \int_{25}^{\infty} cxe^{-0.0016x^2} dx = 0.0032 \int_{25}^{\infty} e^{-0.0016x^2} xdx$$

$$= 0.0032 \int_1^{\infty} e^{-u}\left(\frac{1}{0.0032}\right) du = \int_1^{\infty} e^{-u}\, du$$

$$= lim_{u\to\infty}(-e^{-u}) - (-e^{-1}) = 0 + e^{-1} = 0.3679$$

(7 marks)

3(a)



	Define $u = -z_{0.32}$. By symmetric result and the figure above, $P(Z \le z_{0.32}) = P(Z \ge -z_{0.32}) = P(Z \ge u)$ =0.32 $$P(Z \ge u) = 0.32$$ $$1 - P(Z < u) = 0.32$$ $$P(Z < u) = 0.68$$ By the z-table, $u = 0.47$. Hence, $z_{0.32} = -0.47$ **(5 marks)**
3(b)	Let X be the random variable following normal distribution with mean μ and variance σ^2. Let $Z = \frac{X-\mu}{\sigma}$ be the standard normal variable. Since 32th percentile of X equal to 10σ, $$x_{0.32} = 10\sigma.$$ By the transformation rule for the percentile of X and percentile of Z (3.40), $$x_{0.32} = \mu + z_{0.32}\sigma$$ $$z_{0.32} = -0.47(by\ (a))$$ $$10\sigma = \mu + (-0.47)\sigma$$ $$\mu = 10.47\sigma$$ **(5 marks)**
4	$$M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$ $$M_X'(t) = \frac{d}{dt}\left(e^{\mu t + \frac{\sigma^2 t^2}{2}}\right) = \frac{d}{dt}\left(\mu t + \frac{\sigma^2 t^2}{2}\right) e^{\mu t + \frac{\sigma^2 t^2}{2}} = (\mu + \sigma^2 t) e^{\mu t + \frac{\sigma^2 t^2}{2}}$$ $$E(X) = M_X'(0) = \mu$$ $$M_X''(t) = (\mu + \sigma^2 t)\frac{d}{dt} e^{\mu t + \frac{\sigma^2 t^2}{2}} + e^{\mu t + \frac{\sigma^2 t^2}{2}} \frac{d}{dt}(\mu + \sigma^2 t)$$ $$= (\mu + \sigma^2 t)^2 e^{\mu t + \frac{\sigma^2 t^2}{2}} + e^{\mu t + \frac{\sigma^2 t^2}{2}}(\sigma^2)$$ $$= e^{\mu t + \frac{\sigma^2 t^2}{2}}[(\mu + \sigma^2 t)^2 + \sigma^2]$$ $$E(X^2) = M_X''(0) = \mu^2 + \sigma^2$$ $$V(X) = E(X^2) - E(X)^2 = \sigma^2$$ **(10 marks)**

5	$$\int_0^1 1.2x - 2.5e^{-3x}\,dx = \left[\frac{1.2x^2}{2} - 2.5\left(\frac{1}{-3}\right)e^{-3x}\right]_0^1$$ $$= \left[\frac{1.2(1)^2}{2} + \frac{2.5}{3}e^{-3(1)}\right] - \left[\frac{1.2(0)^2}{2} + \frac{2.5}{3}e^{-3(0)}\right]$$ $$= 0.6 + \frac{5}{6}e^{-3} - \frac{5}{6} = -\frac{7}{30} + \frac{5}{6}e^{-3} = -0.1918$$ **(4 marks)**
6(a)	a=0.1 b=0.4 c=0.2 d=0.4 e=0.5 f=0.5 **(3 marks)**
(b)	$$P(X < Y) = P(1,3) + P(1,5) + P(4,5) = 0.1 + 0.1 + 0.3 = 0.5$$ **(2 marks)**