# Ontology Merging within Qualitative Conceptual Space

## (extended version including the proofs of theorems)

PaperID: 1510

## ABSTRACT

This paper introduces a novel method for merging open-domain ontologies which takes advantage of qualitative conceptual spaces, as formalism used to represent regions in a topological space and to reason about their set-theoretic relationships. To this end, we first propose a faithful translation of ontologies into qualitative conceptual spaces. The merging is performed on these resulting spaces, and the result is then translated back into the underlying language of the input ontologies. Our approach allows us to benefit from the expressivity and the flexibility of conceptual spaces while dealing with conflicting ontological knowledge in a principled way.

## CCS CONCEPTS

• **Theory of computation** → **Constraint and logic programming**; **Description logics**; • **Computing methodologies** → **Description logics**; **Ontology engineering**; **Reasoning about belief and knowledge**;

## KEYWORDS

RCC-5, Qualitative Constraint Network, Ontology Merging, Description Logics, Qualitative Conceptual Space.

## 1 INTRODUCTION

Commonsense knowledge is playing an increasingly important role in the development of AI systems. Such knowledge is available, for example, in large open-domain ontologies such as SUMO or Cyc, in knowledge graphs (KGs) such as DBpedia and WikiData, as semantic markup (e.g. RDFa). Ontologies encode structured knowledge about the concepts and properties of a given domain. They play an important role in areas such as Semantic Web [22], Information Retrieval [14], Natural Language Processing [31], and machine learning [21], among others. For instance, Bouraoui and Schockaert (2018) have shown that ontologies, as prior conceptual knowledge, are useful for learning concept representations from few examples. However, the available ontologies (and KGs, as simple ontologies) are inevitably incomplete, where several rules and facts are missing.

Several methods have been proposed for automated ontology (KG) completion [4, 26, 30] that exploit statistical regularities in a given ontology to predict plausible missing rules or facts. Unfortunately, meaningful knowledge is difficult to predict, especially since we have few examples of facts or rules. Moreover, as most of the existing approaches are mainly based on statistical regularities, the resulting predictions might be conflicting with others. A repair-based mechanism is then required to maintain the consistency[1], i.e., ensure that there are no conflicting (or contradictory) statements. In the same perspective, to widen the coverage of ontologies to several domains and to deal with incompleteness and conflicting statements, one may combine knowledge from several sources. However, it turns out that merging open-domain ontologies is a particularly challenging task as pointed out, for example, in [35] reporting the different problems and difficulties encountered when merging Freebase with WikiData. Conflicting information may occur when the statements of several sources are simply gathered together. Ontology merging and alignment / matching have also attracted much attention in the literature [6, 13, 25, 37, 37, 39]. Their approach studies an equivalence matching between a source and target taxonomy. In our work we assume that all source ontologies are already aligned and mapped to each other. Namely, we suppose that they share the same terminology.. Ontology merging aims to combine two (or more) ontologies having the same terminology while handling conflict, while ontology alignment (or matching) is the process of determining correspondences between terminologies of ontologies.

Let us consider an example taken from crowdsourcing a group of four people to illustrate the merging problem. Assume that a first source says that the concept *Paper* is *disjoint* with the concept *Document*, while another source says that every *Paper is a Document*. Obviously, these two statements are conflicting. To be faithful to both sources while resolving conflict, a sensible choice would be to assume that *Paper* and *Document* are not disjoint concepts, but every *Paper* is not necessarily a *Document*. This kind of result is clearly consistent and can be seen as a good compromise between both sources. Finding meaningful and relevant compromise between sources during the merging process is difficult to obtain. This is mainly due the fact that ontology languages (Description Logics for example) are not expressive enough to capture salient knowledge that might emerge during the merging process. Consider the example of a first source saying that a *Scientific Paper* (*SP*) is both a *Paper* and a *Book*, a second source mentioning that *SP* is-a *Paper*, a third source gives that *SP* is-a *Book*, and a fourth source tells that *Paper* is not a *Book*. For this case, *Paper* and *Book* should be overlapped to represent information from all three sources. This is clearly a relevant example of piece of knowledge that should be taken into account during the merging process, but that cannot be captured in ontology language. The problem of ontology (or DL)

---

merging is close to the problem of belief merging in a propositional setting [5, 6, 24, 38]. For instance, Benferhat et al. [6] studied merging assertional bases in *DL-Lite* fragment. They have determined the minimal subsets of assertions to resolve conflicts based on the inconsistency minimization principle. As another example, Bouraoui et al. [11] proposed a model-based merging operator for merging $\mathcal{EL}$ ontologies. They solve the semantic conflicts that arise during the merging process. Intuitively, the DL rules (or constraints) are strict and hard for merging sources, while the region constraints, i.e., qualitative constraint networks (QCN), are more flexible since it allows for expressing disjunctions in the constraints. To handle the merging problem, we take advantage of conceptual spaces, which are geometric representation frameworks, in which the objects are represented as points in a metric space, and concepts are modelled as regions [19, 20]. Motivated by the fact that conceptual knowledge in an ontology can be to some extent modelled as geometric objects and constraints on conceptual spaces [10], this paper proposes a method for ontology merging that takes advantage of qualitative conceptual spaces to find out relevant compromise between sources while solving conflicts.

Indeed, qualitative spatial reasoning is a suitable paradigm for efficiently reasoning about spatial entities and their relationships. Therein, spatial information is usually represented in terms of basic or non-basic relations in a qualitative calculus, and reasoning tasks are formulated as solving a set of qualitative constraints [9, 29, 33]. In particular, Region Connection Calculus (RCC) is a well-studied formalism for qualitative topological representation and reasoning, including its subsets *RCC-5* [32] and *RCC-8*. Two significant advantages of the RCC framework are (i) its ability to reason efficiently about the relationships between spatial entities, and (ii) its ability to deal with conflicts in qualitative constraint merging as shown in [16, 36].

Several QCN merging operators have been introduced in [15, 16, 23, 36]. For instance, Condotta and colleagues [16] introduced a QCN merging framework with two merging operators. Roughly speaking, these operators compute a distance between QCN scenarios and the input QCNs. Then the scenarios with a minimal distance are selected as the best candidates for the merged result. Similar to the paper [16], our method is syntactical, i.e., the merged result relies on explicit statements from the input TBoxes rather than their deductive closure. As explained in Condotta et al.'s paper, the benefits of this approach are two-fold. First, this method considers as relevant all statements explicitly stated in the TBoxes of the ontology sources, which allows one, in particular, to take advantage of inconsistent pieces of information provided by each source. Second, it is been shown [16] that this kind of approach is interesting from the computational viewpoint. The approach we develop in this paper departs from the work of Condotta et al. [16] in that our merged result is not only consistent but also can be expressed in a succinct way, in a size polynomial to the size of the input sources (this is not the case in Condotta et al.'s framework[16]).

We take advantage of RCC-5 formalism and propose a method for merging open-domain ontologies using qualitative conceptual spaces. We first show how to translate an ontology into a qualitative space while preserving its semantics and properties. Our translation focuses crucially on the popular / practical constraints in the ontology. Second, we propose a merging operator that produces a single and

| Syntax | Semantics |
|--------|-----------|
| $C \sqsubseteq D$ | $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ |
| $r$ | $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ |
| $a$ | $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$ |
| $C \sqcap D$ | $C^{\mathcal{I}} \cap D^{\mathcal{I}}$ |
| $\top$ | $\Delta^{\mathcal{I}}$ |
| $\bot$ | $\emptyset$ |
| $\exists r.C$ | $\{x \in \Delta^{\mathcal{I}} \mid \exists y \in \Delta^{\mathcal{I}} s.t. (x,y) \in r^{\mathcal{I}}, y \in C^{\mathcal{I}}\}$ |

**Table 1: Syntax and semantics of $\mathcal{EL}_{\bot}$**

consistent conceptual space representing a compromise between sources. Finally, we show how to express the conceptual space in the input ontology language while maintaining the maximum relevant information. [2]

## 2 BACKGROUND

We will rely on Description Logics (DLs) encoding of how knowledge is represented to formally provide a mapping from an ontology to conceptual space. We will use RCC and qualitative constraints for performing the merging. This section briefly recalls the technical background required on these two topics.

*Description Logics.* DLs are one of the standard representations to express ontologies and they underlay the Ontology Web Language (OWL). For simplicity, we consider $\mathcal{EL}$ [1], which is one of the most basic DLs. The main ingredients of DLs are individuals, concepts, and roles, which correspond at the semantic level to objects, sets of objects, and binary relations between objects. More formally, let $N_C, N_R, N_I$ be three pairwise disjoint sets where $N_C$ denotes a set of atomic concepts, $N_R$ denotes a set of atomic relations (roles), and $N_I$ denotes a set of individuals. The $\mathcal{EL}_{\bot}$ concept expressions are built according to the following grammar:

$$C ::= \top \mid \bot \mid N_C \mid C \sqcap C \mid \exists r.C \text{ where } r \in N_R.$$

Let $C, D \in N_C$, $a, b \in N_I$, and $r \in N_R$. An $\mathcal{EL}$ ontology $O = \langle \mathcal{T}, \mathcal{A} \rangle$ (a.k.a. knowledge base) comprises two components, the TBox (denoted by $\mathcal{T}$) and ABox (denoted by $\mathcal{A}$). The TBox consists of a set of General Concept Inclusion (GCI) axioms of the form $C \sqsubseteq D$, meaning that $C$ is more specific than $D$ or simply $C$ is subsumed by $D$, and equivalence axioms of the form $C \equiv D$ which is a shortcut for $C \sqsubseteq D$ and $D \sqsubseteq C$. The ABox is a finite set of assertions on individual objects of the form $C(a)$ or $r(a, b)$.

The semantics is given in terms of interpretations $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, which consist of a non-empty interpretation domain $\Delta^{\mathcal{I}}$ and an interpretation function $\cdot^{\mathcal{I}}$ that maps each individual $a \in N_I$ into an element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$, each concept $A \in N_C$ into a subset $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$, each role $r \in N_R$ into a subset $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. A summary of the syntax and semantics of $\mathcal{EL}_{\bot}$ is shown in Table 1.

An interpretation $\mathcal{I}$ is said to be a model of (or satisfies) a GCI axiom, denoted by $\mathcal{I} \models C \sqsubseteq D$, if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$. Similarly, $\mathcal{I}$ satisfies a concept (resp. role) assertion, denoted $\mathcal{I} \models C(a)$ (resp. $\mathcal{I} \models r(a, b)$), if $a^{\mathcal{I}} \in C^{\mathcal{I}}$ (resp. $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}$). An interpretation $\mathcal{I}$ is a model of an ontology $O$ if it satisfies all the axioms and assertions in $O$. An

---

[2]An implementation to illustrate our method is made available at the following link https://anonymous.4open.science/r/OntologyMergingWithinQualitativeConceptualSpace.

ontology is said to be consistent if it has a model. Otherwise, it is inconsistent. An axiom $\Phi$ is entailed by an ontology, denoted by $O \models \Phi$, if $\Phi$ is satisfied by every model of $O$. We say that $C$ is subsumed by $D$ w.r.t. an ontology $O$ iff $O \models C \sqsubseteq D$. Similarly, we say that $a$ is an instance of $C$ w.r.t. $O$ iff $O \models C(a)$. A fulfilling interpretation is formally defined as follows: An interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ of an ontology is said to be fulfilling when each concept of the ontology is non-empty in $\mathcal{I}$, i.e., for each concept $C_i \in N_C$, $\cdot^{\mathcal{I}}(C_i) \neq \emptyset$.

Lastly, given $A, B, A_1, A_2 \in N_C$, an $\mathcal{EL}_\perp$ TBox $\mathcal{T}$ is in the **strict normal form** (a variant of the one introduced by Baader et al. [3]) if it consists of inclusions of the form: $A \sqsubseteq B, A_1 \sqcap A_2 \sqsubseteq \perp, A \sqsubseteq \exists r.B, \exists r.A \sqsubseteq B$, where $A, B, A_1, A_2$ are atomic concepts. We focus on these forms of inclusions as it is common for most ontologies to use these forms to represent statements in practice. We assume that all ontologies are in the aforementioned normal form. Moreover, we also apply the normalization rules [27] to collect the basic inclusions of the normal form [34].

*Region Connection and Qualitative Constraints.* The RCC (Region Connection Calculus) formalism allows to represent and reason about the relationships between spatial entities [29]. Among the fragments of the RCC theory, RCC-5 fragment is expressive enough to reason about set-theoretic relations between regions [8, 28]. In RCC-5, regions can simply be interpreted as non-empty subsets of a given set and the focus is given on a set $\mathcal{B} = \{DR, PO, EQ, PP, PPi\}$ of five binary relations between regions called *basic relations*. The set $\mathcal{B}$ forms a jointly exhaustive and pairwise disjoint set of relations, that is, each pair of regions satisfies exactly one relation from $\mathcal{B}$: the relation $DR$ (resp. $PO$, $EQ$, $PP$) holds between two regions whenever the two regions are disjoint (resp. when they partially overlap, are equal, when the first is a strict subset of the second), and $PPi$ is the converse of $PP$. Based on $\mathcal{B}$, more complex pieces of information about the relative positions of a set of regions can be represented by means of qualitative constraint networks (QCNs). Formally, a QCN is a pair $\mathcal{N} = \langle V, \Psi \rangle$, where $V = \{v_C, v_D, \ldots\}$ is a set of region variables representing the spatial entities and $\Psi$ is a set of binary constraints between these entities. Each constraint $\Psi_{CD} \in \Psi$ is a mapping from $V \times V$ to $2^{\mathcal{B}}$, and is simply denoted by $\Psi_{CD} = v_C \varphi v_D$, where $\varphi \subseteq 2^{\mathcal{B}}$; and $\Psi_{CD}$ is said to be a singleton constraint whenever $\varphi$ is a singleton.

An interpretation of a QCN $\mathcal{N}$ is defined as $\mathcal{S} = (\mathcal{D}^{\mathcal{S}}, \cdot^{\mathcal{S}})$, where $\mathcal{D}^{\mathcal{S}}$ is a non-empty set (the domain of the regions), and $\cdot^{\mathcal{S}}$ is an interpretation function which maps each variable $v_C$ to a non-empty subset $v_C^{\mathcal{S}}$ of $\mathcal{D}^{\mathcal{S}}$. Table 2 precises how singleton constraints from $\Psi$ are interpreted in RCC-5, i.e., an interpretation $\mathcal{S}$ of $\mathcal{N}$ satisfies a singleton constraint $\Psi_{CD}$, denoted by $\mathcal{S} \models \mathcal{N}$, if the relation between $v_C^{\mathcal{S}}$ and $v_D^{\mathcal{S}}$ according to the table is satisfied (e.g., $\mathcal{S} \models v_C\{PP\}v_D$ whenever $v_C^{\mathcal{S}} \subset v_D^{\mathcal{S}}$). The satisfaction relation is extended to any (non-singleton) constraint from $2^{\mathcal{B}}$ as follows: for each $\varphi \in 2^{\mathcal{B}}$, $\mathcal{S} \models v_C \varphi v_D$ iff $\mathcal{S} \models v_C \{\varphi_i\} v_D$ for some $\varphi_i \in \varphi$ (e.g., $\mathcal{S} \models v_C\{PP, EQ\}v_D$ iff $\mathcal{S} \models v_C\{PP\}v_D$ or $v_C\{EQ\}v_D$). An interpretation $\mathcal{S}$ of a QCN $\mathcal{N} = \langle V, \Psi \rangle$ is said to be a solution of $\mathcal{N}$, denoted by $\mathcal{S} \models \mathcal{N}$, iff $\mathcal{S} \models \Psi_{CD}$ for each $\Psi_{CD} \in \Psi$. A QCN is consistent iff it admits a solution. A sub-network of $\mathcal{N}$ is a QCN $\mathcal{N}' = (V, \Psi')$ such that $\Psi' \subseteq \Psi$. A quasi-atomic QCN $\langle V, \Psi \rangle$ is a QCN where for each $v_C, v_D \in V$, there is a unique constraint $\Psi_{CD} \in \Psi$, and

| Name (Symbol) | Syntax | Semantics |
|---|---|---|
| Proper Part of (PP ) | $v_C\{PP\}v_D$ | $v_C^{\mathcal{S}} \subset v_D^{\mathcal{S}}$ |
| Inverse PP of (PPi) | $v_C\{PPi\}v_D$ | $v_D^{\mathcal{S}} \subset v_C^{\mathcal{S}}$ |
| Equals (EQ) | $v_C\{EQ\}v_D$ | $v_C^{\mathcal{S}} = v_D^{\mathcal{S}}$ |
| Disjoint From (DR) | $v_C\{DR\}v_D$ | $v_C^{\mathcal{S}} \cap v_D^{\mathcal{S}} = \emptyset$ |
| Partially Overlaps (PO) | $v_C\{PO\}v_D$ | $v_C^{\mathcal{S}} \cap v_D^{\mathcal{S}} \neq \emptyset$ $v_C^{\mathcal{S}} \not\subseteq v_D^{\mathcal{S}}, v_D^{\mathcal{S}} \not\subseteq v_C^{\mathcal{S}}$ |

**Table 2: Syntax and semantics of *RCC-5*,** $v_C, v_D \in V$**.**

where $\Psi_{CD}$ is either a singleton or $\Psi_{CD} \in \{\{PP, EQ\}, \{PPi, EQ\}\}$. A scenario of a QCN is a quasi-atomic sub-network of $\mathcal{N}$. Noteworthy, a QCN is consistent if it admits a consistent scenario.

## 3 MERGING FRAMEWORK DESCRIPTION

Our aim is to introduce a method for merging open-domain ontologies using qualitative constraint networks. As highlighted in the introduction, qualitative conceptual spaces are more expressive and flexible than ontology languages, which allows in turn to capture relevant information that might emerge during the merging process allowing to select a consistent compromise between sources when expressing the merging result.

A key challenge is to find a faithful translation from an ontology to a conceptual space that preserves the semantics and maintains the initial knowledge encoded in the ontology. Another challenging task is to define a merging operator that takes advantage of the QCNs to perform merging while handling conflicts. As constraints of the merged QCN are *sets* of basic relations, this QCN cannot be translated back into the target ontology language in the general case. So we select one of its "best" scenarios which, in contrast, can be expressed as an ontology. The merging framework is depicted in Figure 1. In particular, our approach involves the following main steps:

(1) Translating ontologies into QCNs (Section 4). In this step, we present a translation function that ensures the faithfulness of the mapping from ontology to qualitative conceptual space.
(2) Defining a QCN merging operator (Section 5). Exploiting the notion of "distance" between basic relations and constraints, this step associates with the input set of QCNs (the translated ontologies) with a single merged, consistent QCN.
(3) Selecting the "best" consistent scenario of the merged QCN as a representative of the merged result (Section 6). This selection process takes advantage of the notion of distance between scenarios and the ABox assertions from the input ontologies.
(4) Translating the selected consistent scenario back into the underlying language of the sources (Section 7).

## 4 TRANSLATING ONTOLOGIES INTO QCNS

In this section, we present a translation function from any ontology to a QCN. More precisely, we (1) map concepts names into QCN variables and axioms into constraints, and (2) show that the translation is faithful to the TBox of the initial ontology. Let $C \stackrel{\text{def}}{=} N_C \cup \{\exists r.A \mid A \in N_C \text{ and } r \in N_R\}$ be the set of *basic concepts*. Clearly, $C$ is
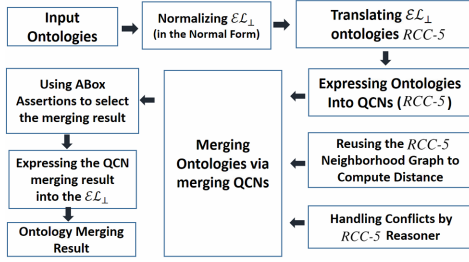
**Figure 1: A Framework of Ontology Merging with QCNs**

finite and $|C| = |N_C| + |N_R| \cdot |N_C|$. One can see that $C \cup \{\bot\}$ contains all possible concepts allowed by the (strict) normal form. Let $\mathcal{V} \overset{\text{def}}{=} \{v_C \mid C \in C\}$ be the set of *basic variable regions* where each variable is tagged with a concept from $C$. Given $C$ and $\mathcal{V}$, we can define a (trivial) translation from ontology concepts into regions, which is extended to a translation of GCIs in our (strict) normal form into constraints as follows:

DEFINITION 1 (FORWARD TRANSLATION $\tau_\triangleright$). *A **forward translation** is a function $\tau_\triangleright : C \longrightarrow \mathcal{V}$ s.t. $\tau_\triangleright(C) \overset{\text{def}}{=} v_C$. $\tau_\triangleright$ is extended to map GCIs in the (strict) normal form into constraints as follows:*

- $\tau_\triangleright(C \sqsubseteq D) \overset{\text{def}}{=} \tau_\triangleright(C)\{PP, EQ\}\tau_\triangleright(D)$, *and*
- $\tau_\triangleright(C \sqcap D \sqsubseteq \bot) \overset{\text{def}}{=} \tau_\triangleright(C)\{DR\}\tau_\triangleright(D)$.

*Moreover, $\tau_\triangleright$ is extended to translate ontologies in the (strict) normal form into a set of constraints in the expected way: $\tau_\triangleright(O) \overset{\text{def}}{=} \{\tau_\triangleright(\Phi) \mid \Phi \in O\}$.*

To show that the translation is faithful, we provide a "semantic" mapping from $O$ to $\tau_\triangleright(O)$, and conversely. Let us first show how models of $O$ correspond to solutions of $\tau_\triangleright(O)$:

DEFINITION 2 (FLATTENING OF AN INTERPRETATION). *Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be a fulfilling interpretation. With $\mathcal{S}_{\mathcal{I}} \overset{\text{def}}{=} (\Delta^{\mathcal{I}}, \cdot^{\mathcal{S}_{\mathcal{I}}})$ we denote the **flattening** of $\mathcal{I}$, where $\cdot^{\mathcal{S}_{\mathcal{I}}} : \mathcal{V} \longrightarrow 2^{\Delta^{\mathcal{I}}}$ is such that $(v_C)^{\mathcal{S}_{\mathcal{I}}} = C^{\mathcal{I}}$.*

THEOREM 1. *Let $O$ be an ontology, and let $\mathcal{I}$ be a fulfilling interpretation of $O$ such that $\mathcal{I} \models O$. Then $\mathcal{S}_{\mathcal{I}} \models \tau_\triangleright(O)$.*

The other way around, let us show how solutions of $\tau_\triangleright(O)$ correspond to interpretations satisfying all axioms from $O$.

DEFINITION 3 (INFLATION OF A SOLUTION). *Let $\mathcal{S} = (\mathcal{D}^{\mathcal{S}}, \cdot^{\mathcal{S}})$ be a semantic solution to a QCN $\mathcal{N}$ over $\mathcal{V}$. With $\mathcal{I}_{\mathcal{S}} \overset{\text{def}}{=} (\Delta^{\mathcal{I}_{\mathcal{S}}}, \cdot^{\mathcal{I}_{\mathcal{S}}})$, where $\Delta^{\mathcal{I}_{\mathcal{S}}} = \mathcal{D}^{\mathcal{S}}$ and, for every $A \in N_C$, $A^{\mathcal{I}_{\mathcal{S}}} = (v_A)^{\mathcal{S}}$, we call $\mathcal{I}_{\mathcal{S}}$ an **inflation** of $\mathcal{S}$.*

Intuitively, the inflation of $\mathcal{S}$ corresponds to an interpretation blown up from $\mathcal{S}$ by interpreting atomic concept names in the same way their corresponding variable names are "populated" by the solution. Notice that there are as many possible inflations of $\mathcal{S}$ as there are ways of interpreting $N_R$ and $N_I$ over $\Delta^{\mathcal{I}_{\mathcal{S}}}$. An immediate consequence of Definition 3 is that every inflation of a solution $\mathcal{S}$ is fulfilling.

THEOREM 2. *Let $O$ be an ontology and let $\mathcal{S}$ be a solution of $\tau_\triangleright(O)$. Then there is an inflation $\mathcal{I}_{\mathcal{S}}$ of $\mathcal{S}$ s.t. $\mathcal{I}_{\mathcal{S}} \models \Phi$ for each axiom $\Phi$ of $O$.*
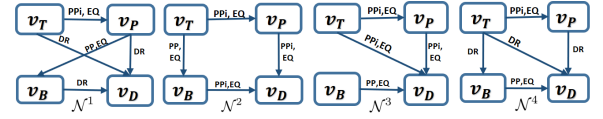


**Figure 2: QCN profile of Example 1**

Theorems 1 and 2 establish that our translation is faithful, i.e., that the set of all fulfilling models of an ontology $O$ are captured precisely in its translated QCN $\tau_\triangleright(O)$.

## 5 MERGING THE QCNS

We reduce the merging of a profile of ontologies $\mathcal{P} = \langle O^1, \dots, O^n \rangle$ to the merging of a profile of QCNs $\mathcal{N} = \langle \mathcal{N}^1, \dots, \mathcal{N}^n \rangle$, where for each $i \in \{1, \dots, n\}$, $\mathcal{N}^i = (V, \Psi^i) = \tau_\triangleright(O^i)$, based on the faithful translation given the previous section. Inspired by works on syntactical QCN merging [16], this QCN merging process is summarized as follows. We associate with the profile $\mathcal{N}$ a single merged and consistent QCN $\mathcal{N} = (V, \Psi)$ representing $\mathcal{N}$ in a "global" way. This is performed in a constraint-wise fashion: for each pair of variables $v_C, v_D \in V$, we associate each basic relation $b \in \mathcal{B}$ with a value representing its *distance* to the profile of constraints $\mathscr{E}_{CD} = \langle \Psi^1_{CD}, \dots, \Psi^n_{CD} \rangle$. This distance is the key tool to form the constraint $\Psi_{CD}$ of the merged QCN $\mathcal{N}$. Intuitively, each constraint $\Psi_{CD}$ corresponds to the set of basic relations with the lowest distances to the profile $\mathscr{E}_{CD}$, while ensuring that the resulting QCN is consistent.

EXAMPLE 1. *Consider the profile of ontologies $\mathcal{P} = \langle O^1, O^2, O^3, O^4 \rangle$ that encodes the following knowledge about the four concepts of Paper, Text, Document and Book, respectively denoted by P, T, D and B.*

- *$O^1 = \langle \mathcal{T}^1 = \{P \sqsubseteq T, T \sqcap D \sqsubseteq \bot, P \sqsubseteq B, P \sqcap D \sqsubseteq \bot, B \sqcap D \sqsubseteq \bot\}$, $\mathcal{A}^1 = \{P(p_1), T(p_1), T(t_1), D(d_1), B(p_1), B(b_1)\} \rangle$,*
- *$O^2 = \langle \mathcal{T}^2 = \{P \sqsubseteq T, T \sqsubseteq B, D \sqsubseteq B, D \sqsubseteq P\}, \mathcal{A}^2 = \{P(p_2), P(d_2), T(p_2), T(t_2), D(d_2), B(p_2), B(b_2)\} \rangle$,*
- *$O^3 = \langle \mathcal{T}^3 = \{B \sqsubseteq D, D \sqsubseteq P, P \sqsubseteq T, D \sqsubseteq T\}, \mathcal{A}^3 = \{P(p_3), T(t_3), T(p_3), D(d_3), B(b_3)\} \rangle$,*
- *$O^4 = \langle \mathcal{T}^4 = \{D \sqcap P \sqsubseteq \bot, P \sqsubseteq T, B \sqsubseteq D, T \sqcap B \sqsubseteq \bot, T \sqcap D \sqsubseteq \bot\}, \mathcal{A}^4 = \{P(p_4), T(t_4), T(p_4), D(b_4), D(d_4), B(b_4)\} \rangle$.*

*Using the forward translation $\tau_\triangleright$ presented in the previous section, one associates with $\mathcal{P}$ a profile of QCNs $\langle \tau_\triangleright(O^i) \rangle = \langle \mathcal{N}^i \rangle$ ($i \in \{1, \dots, 4\}$). The four QCNs are depicted in Figure 2 (to alleviate the figures, we do not represent a constraint $\Psi_{CD}$ when $\Psi_{CD} = \mathcal{B}$, i.e., when the QCN does not provide any information between the relationship between $v_C$ and $v_D$).*

Although we do not require it in the general case, note that in this example each ontology $O^i$ is consistent, i.e., the TBox of each input ontology does not contain conflicting information. As a direct consequence of Theorems 1 and 2, each QCN is consistent. However, simply combining these QCNs can easily lead to conflicts. For instance, there is no basic relation shared in the constraints $\Psi^1_{BD}$ and $\Psi^3_{BD}$, since in $\mathcal{N}^1$ we have that *Book* $\{DR\}$ *Document* whereas in $\mathcal{N}^3$ we have that *Book* $\{PP, EQ\}$ *Document*. This calls for our QCN merging procedure.

*Computing a distance between a basic relation and a profile of constraints.* We start by considering a distance between basic relations. Firstly introduced in [18] in the context of temporal reasoning, the notion of *conceptual neighborhood* (CN) between relations was later adapted to QCN merging in [17] to define such a distance. Intuitively, two basic relations $b, b' \in \mathcal{B}$ are CNs if a continuous transformation of two regions which satisfy the basic relation $b \in B$ leads them to directly satisfy the basic relation $b'$ without satisfying any other basic relation from $\mathcal{B}$. For instance, $PP$ and $EQ$ are CNs since "shrinking" a first region $C$ initially equal to another region $D$ directly makes it a proper part of $D$. This results in the neighborhood relation $\{(DR, PO), (PO, DR), (PO, PP), (PP, PO), (PO, PPi), (PPi, PO), (PP, EQ), (EQ, PP), (PPi, EQ), (EQ, PPi)\}$. This neighborhood relation induces a *neighborhood graph* $\mathcal{G}$ whose vertices are the elements of $\mathcal{B}$, and where there is an edge between two basic relations $b, b' \in \mathcal{B}$ whenever $b$ and $b'$ are CNs according the neighborhood relation. The distance $d(b, b')$ between two basic relations $b$ and $b'$ is defined as the length of the shortest path between $b$ and $b'$ in the neighborhood graph. For instance, $d(DR, PO) = 1$ since $DR$ and $PO$ are CNs, and $d(DR, EQ) = 3$ since $DR$ and $PO$ (resp. $(PO, PP)$ and $(PP, EQ)$) are CNs, but $DR$ and $EQ$ are not. This distance is extended to a distance between a basic relation $b \in \mathcal{B}$ and a constraint $\varphi \in 2^{\mathcal{B}}$, defined as $d(b, \varphi) = \min_{b' \in \varphi} d(b, b')$. Lastly, given two variables $v_C, v_D \in V$, the distance between each $b \in \mathcal{B}$ and the profile of constraints $\mathscr{E}_{CD} = \langle \Psi_{CD}^1, \ldots, \Psi_{CD}^n \rangle$ is defined by $d(b, \mathscr{E}_{CD}) = \sum_{i \in \{1, \ldots, n\}} d(b, \Psi_{CD}^i)$.

EXAMPLE 1 (CONTINUED). *Let us focus on Text (T) and Document (D). We have* $\mathscr{E}_{TD} = \langle \Psi_{TD}^1, \Psi_{TD}^2, \Psi_{TD}^3, \Psi_{TD}^4 \rangle = \langle \{DR\}, \mathcal{B}, \{PPi, EQ\}, \{DR\} \rangle$. *For the distance between the basic relation PP and* $\mathscr{E}_{TD}$, *we have that* $d(PP, \mathscr{E}_{TD}) = d(b, \Psi_{TD}^1) + d(b, \Psi_{TD}^2) + d(b, \Psi_{TD}^3) + d(b, \Psi_{TD}^4)$, *where:*

$$d(PP, \Psi_{TD}^1) = d(PP, \{DR\}) = d(PP, DR) = 2,$$
$$d(PP, \Psi_{TD}^2) = d(PP, \mathcal{B}) = d(PP, PP) = 0,$$
$$d(PP, \Psi_{TD}^3) = d(PP, \{PPi, EQ\}) = d(PP, EQ) = 1,$$
$$d(PP, \Psi_{TD}^4) = d(PP, \{DR\}) = d(PP, DR) = 2.$$

*We get that* $d(PP, \mathscr{E}_{TD}) = 5$. *The distances between each basic relation from* $\mathcal{B}$ *and the profile of constraints* $\mathscr{E}_{CD}$ *for each pair of variables* $v_C, v_D \in V$ *is summarized in Table 3.*

| $\mathscr{E}$ <br> $\mathcal{B}$ | $\mathscr{E}_{TP}$ | $\mathscr{E}_{TB}$ | $\mathscr{E}_{TD}$ | $\mathscr{E}_{PB}$ | $\mathscr{E}_{PD}$ | $\mathscr{E}_{BD}$ |
|---|---|---|---|---|---|---|
| $DR$ | 8 | 2 | 2 | 2 | 4 | 6 |
| $PO$ | 4 | 2 | 3 | 1 | 4 | 4 |
| $PP$ | 4 | 2 | 5 | 0 | 6 | 3 |
| $PPi$ | 0 | 3 | 4 | 1 | 4 | 4 |
| $EQ$ | 0 | 3 | 6 | 0 | 6 | 3 |

**Table 3: Distances between relations from $\mathcal{B}$ and the profile of constraints $\mathscr{E}_{CD}$, for each pair of variables $v_C, v_D \in V$.**

*Using the distance to build a merged, consistent QCN.* We now describe our procedure which associates a profile of QCNs with a merged, consistent QCN. This takes advantage of the distance

between basic relations and a profile of constraints $\mathscr{E}_{CD}$. Let us first formally define two intermediate functions $relax_{CD}$ and $val_{CD}$ which are used in our procedure. Given a total preorder[3] $\preceq$ over a finite set $E$, let us denote by $\min(E, \preceq)$ the set of minimal elements of $E$ w.r.t. $\preceq$, i.e., $\min(E, \preceq) = \{e \in E \mid \forall e' \in E, e \preceq e'\}$. Each pair of variables $v_C, v_D$ is associated with a total preorder $\preceq_{CD}$ over the basic relations from $\mathcal{B}$ defined for all $b, b' \in \mathcal{B}$ as $b \preceq_{CD} b'$ iff $d(b, \mathscr{E}_{CD}) \leq d(b', \mathscr{E}_{CD})$. Then the function $relax_{CD}$ is a mapping $relax_{CD} : 2^{\mathcal{B}} \mapsto 2^{\mathcal{B}}$ defined for each $\varphi \in 2^{\mathcal{B}}$ as $relax_{CD}(\varphi) = \varphi \cup \min(\mathcal{B} \setminus \varphi, \preceq_{CD})$. It corresponds to the *relaxation* of a constraint $\varphi$ w.r.t.. the total preordering $\preceq_{CD}$. Noteworthy, $relax_{CD}(\emptyset)$ corresponds to the set of basic relations with a minimal distance to the profile of constraints $\mathscr{E}_{CD}$. The function $val_{CD}$ is a mapping $val_{CD} : 2^{\mathcal{B}} \mapsto \mathbb{N}$ defined for each $\varphi \in 2^{\mathcal{B}}$ as $val_{CD}(\varphi) = \max_{b \in \varphi} d(b, \mathscr{E}_{CD})$. For instance, according to Table 3 and focusing on *Book* and *Document* (cf. $\mathscr{E}_{BD}$), we get that $PP, EQ \preceq_{BD} PO, PPi \preceq_{BD} DR$, that $relax_{BD}(\emptyset) = \{PP, EQ\}$, that $relax_{BD}(\{PP, EQ\}) = \{PP, EQ, PO, PPi\}$, and that $val_{BD}(\{PP, EQ\}) = 3$.

We are ready to introduce our main procedure, whose outline is given in Algorithm 1 that defines an initial QCN $\mathcal{N}$ by setting each one of its constraints $\Psi_{CD}$ to the set of basic relations from $\mathcal{B}$ having a distance to the profile of constraints $\mathscr{E}_{CD}$ that is minimal (lines 2 to 5). If this QCN is consistent, then it is returned as the merged QCN (line 12). If not, some of the constraints of $\mathcal{N}$ are relaxed in line 7, in the sense that some basic relations are added to these constraints. Such a set of constraints $S$ is selected as follows. In line 7, $S$ is first restricted to those constraints from $\mathcal{N}$ which *can* be relaxed, i.e., those constraints not equal to $\mathcal{B}$. Among those candidate constraints, one selects in line 8 the constraints $\Psi_{CD}$ having a highest value $val_{CD}(\Psi_{CD})$. Indeed, we do not want to relax first the constraints consisting of basic relations which are "close" to the input profile, but rather would one prioritize the relaxation of more "controversial" constraints, i.e., those with a high value according to $val_{CD}$. For instance, let us look back at Table 3. It can be seen that $d(PPi, \mathscr{E}_{TP}) = d(EQ, \mathscr{E}_{TP}) = 0$, and thus $val_{TP}(\{PPi, EQ\}) = 0$; this low value reflects the consensus between sources on the fact that one of the basic relations $PPi, EQ$ holds between *Text* and *Paper*, and indeed it can be verified that the axiom $P \sqsubseteq T$ is consistent with each input TBox. On the contrary, one has that $val_{PD}(\{DR, PO, PPi\}) = 4$; this higher value reflects a disagreement between the input sources about the relationship between the concepts of *Paper* and *Document*. And in the general case, whenever possible and in order to restore the consistency of the merged QCN, it is a sensible choice to keep unchanged those constraints unanimely accepted by the input sources, and rather weaken first the most disputed constraints. This "relaxation" process is repeated iteratively until the resulting QCN is consistent which, obviously enough, is guaranteed after a finite number of iterations.

EXAMPLE 1 (CONTINUED). *Initially, the merged QCN* $\mathcal{N} = \langle V, \Psi \rangle$ *is defined by the following set of constraints, which correspond the basic relations highlighted in Table 3 (this QCN is also*

---

[3] A total preorder over a set $E$ is a total, symmetric and transitive relation.

**Algorithm 1:** Computing a merged QCN

**input:** A profile of QCNs $\mathcal{N} = \langle \mathcal{N}_1, \ldots, \mathcal{N}_n \rangle$
**ouput:** A merged, consistent QCN $\mathcal{N}$

1  **begin**
   // Initialization of the output QCN $\mathcal{N}$
2     $\Psi \longleftarrow \{\Psi_{CD} \mid v_C, v_D \in V\}$
3     **foreach** $(v_C, v_D) \in V \times V$ **do**
4        $\Psi_{CD} \longleftarrow relax_{CD}(\emptyset)$
5     $\mathcal{N} \longleftarrow (V, \Psi)$
6     **while** $\mathcal{N}$ *is not consistent* **do**
         // One relaxes some constraints of $\mathcal{N}$
7        $S \longleftarrow \{\Psi_{CD} \mid \Psi_{CD} \in \Psi, \Psi_{CD} \neq \mathcal{B}\}$
8        $S \longleftarrow \arg\max\{val_{CD}(\Psi_{CD}) \mid \Psi_{CD} \in S)\}$
9        **foreach** $\Psi_{CD} \in S$ **do**
10          $\Psi_{CD} \longleftarrow relax_{CD}(\Psi_{CD})$
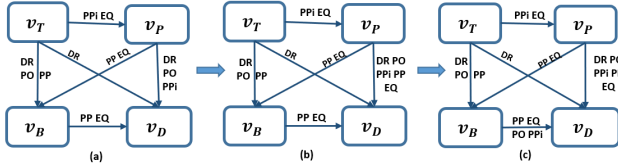11       $\mathcal{N} \longleftarrow (V, \Psi)$
12    **return** $\mathcal{N}$



**Figure 3: QCNs iteratively generated by our algorithm. Fig. 3(c) corresponds to the final consistent merged QCN.**

*depicted in Figure 3(a)):*

$$\Psi_{TP} = \{PPi, EQ\} \qquad \Psi_{TB} = \{DR, PO, PP\}$$
$$\Psi_{TD} = \{DR\} \qquad \Psi_{PB} = \{PP, EQ\}$$
$$\Psi_{PD} = \{DR, PO, PPi\} \qquad \Psi_{BD} = \{PP, EQ\}$$

*This QCN is inconsistent. One can see that the constraints $\Psi_{PB}$ and $\Psi_{BD}$ imply by transitivity that the relation between P and D must be PP or EQ, yet $\Psi_{PD} \cap \{PP, EQ\} = \emptyset$. Then the constraint $\Psi_{PD}$ is selected (cf. line 8 in the algorithm) as the only candidate for relaxation at this point, since $val_{PD}(\Psi_{PD}) = 4$, which is the highest value among all constraints. And since $relax_{PD}(\Psi_{PD}) = \mathcal{B}$, one updates $\Psi_{PD}$ to $\mathcal{B}$ which results in the QCN depicted in Figure 3(b). This QCN is, again, inconsistent (in this case, explaining its inconsistency is more complex as it involves dependencies between all four variables. We omit the details for space reasons). Then the constraint $\Psi_{BD}$ is selected for relaxation ($val_{BD}(\Psi_{BD}) = 3$) and one updates $\Psi_{BD}$ to $\{PP, EQ, PO, PPi\}$. The resulting QCN (cf. Figure 3(c)) is consistent and returned by the procedure.*

## 6 SELECTING A REPRESENTATIVE SCENARIO OF THE MERGED QCN

Once we have obtained a merged, consistent QCN, our goal is to express it in our initial (target) ontology language. However, not every constraint $\Psi_{CD}$ from the QCN (i.e., a subset of $\mathcal{B}$) can easily be translated as a set of axioms, since non-singleton constraints express some disjunctive information between two concepts/regions.
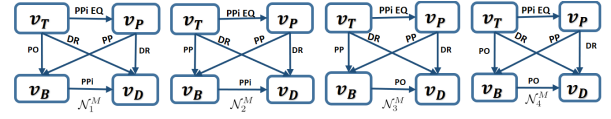


**Figure 4: Possible consistent scenarios.**

Moreover, since the merged QCN is consistent, one can remark it necessarily admits at least one consistent scenario, and since a scenario involves singleton constraints (as well as the two constraints $\{PP, EQ\}$, $\{PPi, EQ\}$), it can easily be translated into a single ontology, as will be shown in the next section. Then our aim is to (i) focus on all consistent scenarios of the merged QCN, and (ii) select one representative scenario. This can be done by exploiting information provided by the input ABoxes as we intend to show in the rest of this section.

In our running example, the merged QCN admits four consistent scenarios which are depicted in Figure 4. Let us first discuss why these four scenarios seem to be reasonable candidates to the input ontologies / QCNs provided by the sources. First, note that all input ontologies state that $Paper \sqsubseteq Text$ ($v_T\{PPi, EQ\}v_P$ in the QCN profile). And it can be seen that this consensus is reflected in the four candidate scenarios, which entail that information. Second, while the two sources $O_1$ and $O_4$ state that $Text$ and $Document$ are disjoint concepts ($v_T\{DR\}v_D$ in the corresponding input QCNs), only one source says that $Document \sqsubseteq Text$ ($v_T\{PP, EQ\}v_D$). In this case, it make sense to follow the point of view of the majority of the sources. And accordingly, in all four scenarios we have that $v_T\{DR\}v_D$, thus $Text$ and $Document$ are disjoint concepts. Third, the source $O_1$ states that $Paper \sqsubseteq Book$ and all other sources have no information on these concepts. It is sensible to keep this information in the merged result, and accordingly all four scenarios entail that information. More, one sees that $v_P\{PP\}v_B$ holds in all scenarios, i.e., that $Paper$ is a *strict* part of $Book$, or stated otherwise, that both concepts cannot be equal while keeping the relationships between the remaining concepts consistent. This emergent property is also an interesting feature of the merging process. Last, the reason why there are four, equally reasonable, candidate scenarios is that some strong disagreements hold on the relationships between the concepts $Text$ and $Book$ on the one hand, and the concepts $Book$ and $Document$ on the other hand. Accordingly, the only differences between the four scenarios hold on the constraints between these two pairs of concepts ($v_T\{PO\}v_B$ / $v_T\{PP\}v_B$, and $v_B\{PPi\}v_D$ / $v_B\{PO\}v_D$).

So the remaning step is to select one of these four scenarios. For this purpose, we take advantage of the ABoxes from the input ontologies and see how these ABoxes relate to each scenario. To be as faithful as possible to what each input source says, instead of simply considering each input ABox as such, one considers the "closure" of it according to its corresponding TBox. For instance, if a given source states that $Paper \sqsubseteq Text$ in its TBox and that $Paper(p)$ in its Abox, then it makes sense to also consider that $Text(p)$ also holds in that source's implicit knowledge. Formally, let $O = \langle \mathcal{T}, \mathcal{A} \rangle$ be an ontology. The *deductive closure of* $\mathcal{A}$ w.r.t. $\mathcal{T}$, denoted by $Cl_{\mathcal{T}}(\mathcal{A})$, is defined as $Cl_{\mathcal{T}}(\mathcal{A}) \stackrel{\text{def}}{=} \{B(a) \mid O \models B(a)\} \cup \{r(a, b) \mid O \models r(a, b)\}$ [2, 7]. Accordingly, $\langle \mathcal{T}, \mathcal{A} \rangle$ is logically equivalent to $\langle \mathcal{T}, Cl_{\mathcal{T}}(\mathcal{A}) \rangle$.

Now, to select the representative scenario, one takes advantage of a distance between a scenario and the set of all input (closed) ABox, and then choose the scenario having a minimal distance. Given a scenario, the idea is to count the number of individuals in each input ABox which raise a conflict w.r.t. the constraints of that scenario. This can naturally be done for any scenario constraint $v_C \varphi v_D$ where $\varphi \neq \{PO\}$. For instance, if $Paper(p)$ and $Document(p)$ hold in the ABox of a given source, and $v_P\{DR\}v_D$ holds in the scenario under consideration, then according to that ABox $p$ is an individual that raises a conflict with that scenario. Another example is if $Paper(p)$ holds but not $Text(p)$, then $p$ is not a member of the concept $Text$ in the ABox (recall that ABoxes are closed w.r.t. their TBox); in that case $p$ raises a conflict with the constraints $v_P \varphi v_T$ when $\varphi \in \{\{PP\}, \{EQ\}, \{PP, EQ\}\}$. More formally, given an ABox $\mathcal{A}$, a scenario constraint $v_C \varphi v_D$ where $\varphi \neq \{PO\}$, and an individual $p$, we say that $p$ raises a conflict with $\varphi$ w.r.t $\mathcal{A}$ when:

$$C(p) \in \mathcal{A}, D(p) \notin \mathcal{A} \quad \text{when } \varphi \subseteq \{PP, EQ\},$$
$$D(p) \in \mathcal{A}, C(p) \notin \mathcal{A} \quad \text{when } \varphi \subseteq \{PPi, EQ\},$$
$$C(p), D(p) \in \mathcal{A} \quad \text{when } \varphi = \{DR\}.$$

And the number of conflicts raised by an ontology $O = \langle \mathcal{T}, \mathcal{A} \rangle$ w.r.t. a a scenario constraint $v_C \varphi v_D$ where $\varphi \neq \{PO\}$, is defined as $nbConf(O, \varphi) = |\{p \in N_I \mid p \text{ raises a conflict with } \varphi \text{ w.r.t. } Cl_\mathcal{T}(\mathcal{A})\}|$.

The case of $PO$ is more complex. Indeed, it can be easily seen that no individual can raise a conflict with $v_C\{PO\}v_D$. This is because all basic relations $\mathcal{B} \setminus \{PO\}$ express explicit dependencies between concepts / regions, whereas $PO$ is a complementary relation that (explicitly) expresses a notion of *independency* between concepts. For instance, the concepts *Smoker* and *Researcher* can naturally be thought of as independent concepts, in the sense that one can easily find in a real-world context individuals that are members of either both concepts, only one of them, and none of them (note that this should not be confused with the case of $DR$, which expresses an explicit dependency between concepts, e.g., the concepts *Dog* and *Cat*.) So to evaluate the number of "conflicts" raised by an ABox w.r.t. a constraint $v_C\{PO\}v_D$, we propose to count how "unbalanced" the number of conflicts are w.r.t. the remaining forms of constraints. Formally, focusing on the scenario constraint between two variables $v_C$ and $v_D$, $nbConf(O, \varphi) = \max_{\varphi' \neq \{PO\}} nbConf(O, \varphi')) - \min_{\varphi' \neq \{PO\}} nbConf(O, \varphi'))$. For instance, when $nbConf(O, \{PP, EQ\}) = nbConf(O, \{PPi, EQ\}) = nbConf(O, \{DR\})$, then $nbConf(O, \{PO\}) = 0$: since individuals can be found equally (i) in both underlying concepts, and (ii) in one concept but not the other, $O$ raises no conflict w.r.t. $\{PO\}$.

We have now the tools to select the representative scenario from a given set of candidates. The distance between a scenario $\mathcal{N}^M$ and the input profile of ontologies $\mathcal{P} = \langle O^1, \dots, O^n \rangle$ is simply defined as the overall number of conflicts raised by all input ABoxes w.r.t. all constraints of $\mathcal{N}^M$, i.e., $d(\mathcal{N}^M, \mathcal{P}) = \sum_{i \in \{1, \dots, n\}, \varphi \in \mathcal{N}^M} nbConf(O^i, \varphi)$. And given a set of candidate scenarios, the representative scenario is the one having a minimal distance.

EXAMPLE 1 (CONTINUED). *Let us go back to our running example. We have that $Cl_{\mathcal{T}^3}(\mathcal{A}^3) = \{P(p_3), P(b_3), P(d_3), T(t_3), T(d_3), T(b_3), T(p_3), D(d_3), D(b_3), B(b_3)\}$. So focusing on Text and Book (i.e., on the scenario constraints between the variables $v_T$ and $v_B$), we have that $nbConf(O^3, \{PP\}) = |\{t_3, d_3, p_3\}| = 3$, and one can*

*easily verify that $nbConf(O^3, \{PO\}) = |\{t_3, d_3, p_3\}| = 3 - 0 = 3$. Summing up all conflicts, we get that $d(\mathcal{N}^M_1, \mathcal{P}) = 20$, $d(\mathcal{N}^M_2, \mathcal{P}) = 18$, $d(\mathcal{N}^M_3, \mathcal{P}) = 22$, and $d(\mathcal{N}^M_4, \mathcal{P}) = 24$. Hence, the scenario $\mathcal{N}^M_2$ is selected as a representative scenario of the merged QCN.*

## 7 TRANSLATING THE REPRESENTATIVE SCENARIO INTO AN ONTOLOGY

The last step is to translate the representative scenario selected in the previous step back into an ontology:

DEFINITION 4 (BACKWARD TRANSLATION $\tau_\lhd$). *A **backward translation** is a function $\tau_\lhd : \mathcal{V} \longrightarrow C$ s.t. $\tau_\lhd(v_C) \stackrel{\text{def}}{=} C$. $\tau_\lhd$ is extended to map constraints into an ontology as follows, where $A'$, $C'$, and $D'$ are new concept names and a, b, and c new individual names:*[4]

- $\tau_\lhd(v_C\{EQ\}v_D) \stackrel{\text{def}}{=} \langle \{C \equiv D\}, \emptyset \rangle$;
- $\tau_\lhd(v_C\{DR\}v_D) \stackrel{\text{def}}{=} \langle \{C \sqcap D \sqsubseteq \bot\}, \emptyset \rangle$;
- $\tau_\lhd(v_C\{PO\}v_D) \stackrel{\text{def}}{=} \langle \{A' \sqsubseteq C \sqcap D, C' \sqsubseteq C, C' \sqcap D \sqsubseteq \bot, D' \sqsubseteq D, D' \sqcap C \sqsubseteq \bot\}, \{A'(a), C(c), C(a), D(d), D(a), C'(c), D'(d)\} \rangle$;
- $\tau_\lhd(v_C\{PP, EQ\}v_D) \stackrel{\text{def}}{=} \langle \{C \sqsubseteq D\}, \emptyset \rangle$;
- $\tau_\lhd(v_C\{PPi, EQ\}v_D) \stackrel{\text{def}}{=} \langle \{D \sqsubseteq C\}, \emptyset \rangle$;
- $\tau_\lhd(v_C\{PP\}v_D) \stackrel{\text{def}}{=} \langle \{C \sqsubseteq D, D' \sqsubseteq D, C \sqcap D' \sqsubseteq \bot\}, \{D'(d), C(c), D(d), D(c)\} \rangle$, *and*
- $\tau_\lhd(v_C\{PPi\}v_D)) \stackrel{\text{def}}{=} \langle \{D \sqsubseteq C, C' \sqsubseteq C, D \sqcap C' \sqsubseteq \bot\}, \{C'(c), D(d), C(d), D(c)\} \rangle$.

*Moreover, $\tau_\lhd$ is extended to translate a set of constraints into an ontology in the (strict) normal form in the expected way: $\tau_\lhd(\mathcal{N}) \stackrel{\text{def}}{=} \langle \mathcal{T}, \mathcal{A} \rangle$, where $\mathcal{T} \stackrel{\text{def}}{=} \bigcup_{\tau_\lhd(\Psi) = \langle \mathcal{T}', \mathcal{A}' \rangle, \Psi \in \mathcal{N}} \mathcal{T}'$ and $\mathcal{A} \stackrel{\text{def}}{=} \bigcup_{\tau_\lhd(\Psi) = \langle \mathcal{T}', \mathcal{A}' \rangle, \Psi \in \mathcal{N}} \mathcal{A}'$.*

Accordingly, our back translation is faithful. Using again the notions of inflation and flattening (cf. Definitions 2 and 3), we show that the set of solutions of a scenario $\mathcal{N}$ are captured precisely in its translated ontology $\tau_\lhd(\mathcal{N})$:

THEOREM 3. *Let $\mathcal{N}$ be a scenario and $\mathcal{S}$ be solution of $\mathcal{N}$. Then there is an inflation $\mathcal{I}_\mathcal{S}$ of $\mathcal{S}$ s.t. $\mathcal{I}_\mathcal{S}$ is a model of $\tau_\lhd(\mathcal{N})$.*

THEOREM 4. *Let $\mathcal{N}$ be a scenario and let $\mathcal{I}$ be a fulfilling interpretation of $\tau_\lhd(\mathcal{N})$ such that $\mathcal{I}$ is a model of $\tau_\lhd(\mathcal{N})$. Then $\mathcal{S}_\mathcal{I} \models \mathcal{N}$.*

EXAMPLE 1 (CONTINUED). *Let us translate the selected scenario $\mathcal{N}^M_2$ into an ontology. From Definition 4, we get that: $\tau_\lhd(\mathcal{N}^M_2) = \langle \{(\Psi_{TP}) \ P \sqsubseteq T, (\Psi_{TD}) \ T \sqcap D \sqsubseteq \bot, (\Psi_{PD}) \ P \sqcap D \sqsubseteq \bot, (\Psi_{TB}) \ T \sqsubseteq B, SubBo1 \sqsubseteq B, SubBo1 \sqcap T \sqsubseteq \bot, (\Psi_{PB}) \ P \sqsubseteq B, SubBo2 \sqsubseteq B, SubBo2 \sqcap P \sqsubseteq \bot, (\Psi_{BD}) \ D \sqsubseteq B, SubBo3 \sqsubseteq B, SubBo3 \sqcap D \sqsubseteq \bot \}, \{T(t_1), SubBo1(s_1), B(t_1), B(s_1), P(p_1), SubBo2(s_2), B(p_1), B(s_2), D(d_1), SubBo3(s_3), B(d_1), B(s_3)\} \rangle$.*

## 8 CONCLUSION

We introduced an ontology merging procedure based on qualitative conceptual spaces, by providing a two-way translation between ontologies and RCC-5 based qualitative constraints. Accordingly, even if the input ontologies are inconsistent when simply combined together, our approach returns a consistent result. It also remains as close as possible to the input sources, i.e., by preserving as much information as possible. Our next plan is to evaluate our method empirically. Moreover, an approach of representing regions and

---

[4]Notice that the constraint $\{PO\}$ cannot be translated into a set of GCIs only, whence the use of ABox assertions in the translation.

points in vector space (combining machine learning) will be our next direction for merging ontologies.

## REFERENCES

[1] Franz Baader, Sebastian Brandt, and Carsten Lutz. 2005. Pushing the EL Envelope. In *the proceeding of International Joint Conferences on Artificial Intelligence (IJCAI'05)*. Morgan Kaufmann Publishers Inc., San Francisco, CA, USA, 364–369.

[2] Franz Baader, Diego Calvanese, Deborah Mcguinness, Daniele Nardi, and Peter Patel-Schneider. 2007. *The Description Logic Handbook: Theory, Implementation, and Applications*. Cambridge University Press, USA.

[3] Franz Baader, Rafael Peñaloza, and Boontawee Suntisrivaraporn. 2007. Pinpointing in the Description Logic $\mathcal{EL}$. In *Proceedings of the 30th German Conference on Artificial Intelligence (KI'2007)*. Springer Berlin Heidelberg, Germany, 52–67.

[4] Islam Beltagy, Cuong Chau, Gemma Boleda, Dan Garrette, Katrin Erk, and Raymond Mooney. 2013. Montague Meets Markov: Deep Semantics with Probabilistic Logical Form. In *Second Joint Conference on Lexical and Computational Semantics (*SEM)*. Association for Computational Linguistics, USA, 11–21.

[5] Salem Benferhat, Zied Bouraoui, Sylvain Lagrue, and Julien Rossit. 2014. Min-based Assertional Merging Approach for Prioritized DL-Lite Knowledge Bases. In *Scalable Uncertainty Management - 13th International Conference*, Umberto Straccia and Andrea Calì (Eds.). Springer, Cham, 8–21.

[6] Salem Benferhat, Zied Bouraoui, Odile Papini, and Eric Würbel. 2019. Assertional Removed Sets Merging of DL-Lite Knowledge Bases. In *Scalable Uncertainty Management - 13th International Conference, SUM 2019, Compiègne, France, December 16-18, 2019, Proceedings (Lecture Notes in Computer Science)*, Nahla Ben Amor, Benjamin Quost, and Martin Theobald (Eds.), Vol. 11940. Springer, Compiègne, France., 207–220. https://doi.org/10.1007/978-3-030-35514-2_16

[7] Salem Benferhat, Zied Bouraoui, and Karim Tabia. 2015. How to Select One Preferred Assertional-Based Repair from Inconsistent and Prioritized DL-Lite Knowledge Bases?. In *Proceedings of the 24th International Conference on Artificial Intelligence (IJCAI'15)*. AAAI Press, Buenos Aires, Argentina, 1450–1456.

[8] Brandon Bennett. 1994. Spatial Reasoning with Propositional Logics. In *Principles of Knowledge Representation and Reasoning (KR'94) (The Morgan Kaufmann Series in Representation and Reasoning)*, Jon Doyle, Erik Sandewall, and Pietro Torasso (Eds.). Morgan Kaufmann, Germany, 51–62.

[9] Mehul Bhatt, Hans Guesgen, Stefan Wölfl, and Shyamanta Hazarika. 2011. Qualitative Spatial and Temporal Reasoning: Emerging Applications, Trends, and Directions. *Spatial Cognition & Computation* 11 (02 2011), 1–14.

[10] Zied Bouraoui, José Camacho-Collados, Luis Espinosa Anke, and S. Schockaert. 2020. Modelling Semantic Categories using Conceptual Neighborhood. In *Proceedings of the AAAI Conference on Artificial Intelligence (AAAI'20)*. International Joint Conferences on Artificial Intelligence Organization, New York, New York, United States, 7448–7455.

[11] Z. Bouraoui, S. Konieczny, T. T. Ma, and I. Varzinczak. 2020. Model-based Merging of Open-Domain Ontologies. In *the proceeding of the IEEE International Conference on Tools with Artificial Intelligence (ICTAI'20)*. IEEE, Baltimore, MD, 29–34. https://doi.org/10.1109/ICTAI50040.2020.00015

[12] Zied Bouraoui and S. Schockaert. 2018. Learning Conceptual Space Representations of Interrelated Concepts. In *International Joint Conference on Artificial Intelligence (IJCAI'18)*. ijcai.org, Stockholm, Sweden, 1760–1766.

[13] Fei Chang, Guo wei Chen, and Songmao Zhang. 2019. FCAMap-KG Results for OAEI 2019. In *the proceeding of International Semantic Web Conference (OM@ISWC'19)*. CEUR-WS.org, Auckland, New Zealand, 138–145.

[14] Zheng Chen, Sun Yu, Wan Shengxian, and Yu Dianhai. 2019. RLTM: An Efficient Neural IR Framework for Long Documents. In *the proceeding of International Joint Conference on Artificial Intelligence (IJCAI'19)*. ijcai.org, Macao, China, 5457–5463. https://doi.org/10.24963/ijcai.2019/758

[15] Jean-Francois Condotta, Souhila Kaci, Pierre Marquis, and Nicolas Schwind. 2009. Merging Qualitative Constraint Networks in a Piecewise Fashion. In *the proceeding of the IEEE International Conference on Tools with Artificial Intelligence (ICTAI'09)*. IEEE, NJ, USA, 605–608.

[16] Jean-Francois Condotta, Souhila Kaci, Pierre Marquis, and Nicolas Schwind. 2010. A Syntactical Approach to Qualitative Constraint Networks Merging. In *the proceeding of the International Conference on Logic Programming and Automated Reasoning (LPAR'10)*. Springer, Dakar, Senegal, 233–247.

[17] Jean-Francois Condotta, Souhila Kaci, and Nicolas Schwind. 2008. A Framework for Merging Qualitative Constraints Networks. In *the proceeding of the International Florida Artificial Intelligence Research Society Conference (FLAIRS'08)*. AAAI Press, Florida, USA, 586–591.

[18] Christian Freksa. 1992. Temporal reasoning based on semi-intervals. *Artificial Intelligence* 54, 1 (1992), 199–227.

[19] Peter Gärdenfors. 2000. *Conceptual Spaces: The Geometry of Thought*. MIT Press, US.

[20] V. Haarslev and R. Möller. 1997. SBox: A Qualitative Spatial Reasoner. In *Proceedings of the 11th International Workshop on Qualitative Reasoning*. IWQR, Italy, 105–113.

[21] Patrick Hohenecker and Thomas Lukasiewicz. 2020. Ontology Reasoning with Deep Neural Networks. In *the proceeding of International Joint Conference on Artificial Intelligence (IJCAI-20)*. ijcai.org, Yokohama, Japan, 5060–5064.

[22] Timo Homburg, Steffen Staab, and Daniel Janke. 2020. GeoSPARQL+: Syntax, Semantics and System for Integrated Querying of Graph, Raster and Vector Data. In *the proceeding of the International Semantic Web Conference (ISWC'2020)*. CEUR-WS.org, Athens, Greece, 258–275.

[23] Hué Julien and Matthias Westphal. 2012. Revising Qualitative Constraint Networks: Definition and Implementation. *the proceeding of the IEEE International Conference on Tools with Artificial Intelligence (ICTAI'12)* 1 (2012), 548–555.

[24] Sri Krishna Kumar and Jennifer A Harding. 2016. Description logic–based knowledge merging for concrete- and fuzzy-domain ontologies. *Journal of Engineering Manufacture* 230, 5 (2016), 954–971. https://doi.org/10.1177/0954405414564404

[25] Amir Laadhar, Faïza Ghozzi, Imen Megdiche, Franck Ravat, Olivier Teste, and Faïez Gargouri. 2017. POMap results for OAEI 2017. In *the International Semantic Web Conference (ISWC'17)*. CEUR-WS.org, Vienna, Austria, pp. 1–7.

[26] Na Li, Zied Bouraoui, and Steven Schockaert. 2019. Ontology Completion Using Graph Convolutional Networks. In *the proceeding of the International Semantic Web Conference (ISWC'19)*. Springer, Auckland, New Zealand, 435–452.

[27] Despoina Magka, Yevgeny Kazakov, and Ian Horrocks. 2011. Tractable Extensions of the Description Logic $\mathcal{EL}$ with Numerical Datatypes. *Journal of Automated Reasoning* 47, 4 (Dec. 2011), 427–450.

[28] Jonsson Peter and Drakengren Thomas. 1997. A Complete Classification of Tractability in RCC-5. *J. of Artificial Intelligence Research* 6 (1997), 211–221.

[29] David A. Randell, Zhan Cui, and Anthony G. Cohn. 1992. A Spatial Logic Based on Regions and Connection. In *Proceedings of the Third International Conference on Principles of Knowledge Representation and Reasoning (KR'92)*. Morgan Kaufmann Publishers Inc., San Francisco, CA, USA, 165–175.

[30] Tim Rocktäschel and Sebastian Riedel. 2017. End-to-end Differentiable Proving. In *Proceedings of International Conference on Neural Information Processing Systems (NIPS'17)*. Curran Associates Inc., Red Hook, NY, USA, 3791–3803.

[31] Marco Rospocher and Francesco Corcoglioniti. 2018. Joint Posterior Revision of NLP Annotations via Ontological Knowledge. In *the proceeding of International Joint Conference on Artificial Intelligence (IJCAI-18)*. ijcai.org, Stockholm, Sweden, 4316–4322. https://doi.org/10.24963/ijcai.2018/600

[32] Steven Schockaert and Sanjiang Li. 2013. Combining RCC5 Relations with Betweenness Information. In *IJCAI 2013, Proceedings of the 23rd International Joint Conference on Artificial Intelligence, Beijing, China, August 3-9, 2013*, Francesca Rossi (Ed.). IJCAI/AAAI, Beijing, China, 1083–1089.

[33] Michael Sioutis, Zhiguo Long, and Tomi Janhunen. 2020. On Robustness in Qualitative Constraint Networks. In *the proceeding of International Joint Conference on Artificial Intelligence (IJCAI'20)*, Christian Bessiere (Ed.). ijcai.org, Yokohama, Japan, 1813–1819.

[34] Boontawee Suntisrivaraporn. 2009. *Polynomial-Time Reasoning Support for Design and Maintenance of Large-Scale Biomedical Ontologies*. Ph.D. thesis. Technical University of Dresden, Germany.

[35] Thomas Pellissier Tanon, Denny Vrandecic, Sebastian Schaffert, Thomas Steiner, and Lydia Pintscher. 2016. From Freebase to Wikidata: The Great Migration. In *International World Wide Web Conference (WWW'2016)*. World Wide Web Conference, Switzerland, 1419–1428.

[36] David Thau, Shawn Bowers, and Bertram Ludäscher. 2009. Merging Taxonomies under RCC-5 Algebraic Articulations. *Journal of Computing Science and Engineering* 3 (06 2009), 109–126. https://doi.org/10.5626/JCSE.2009.3.2.109

[37] Élodie Thiéblin, Ollivier Haemmerlé, and Cássia Trojahn. 2018. CANARD complex matching system. In *the proceeding of the International Semantic Web Conference (OM@ISWC'18)*. Springer, Monterey, CA (USA), 138–143.

[38] Z. Wang, K. Wang, Y. Jin, and G. Qi. 2012. Ontomerge: A system for merging DL-Lite ontologies. *CEUR Workshop Proceedings* 969 (01 2012), 16–27.

[39] Mengyi Zhao and Songmao Zhang. 2016. FCA-Map results for OAEI 2016. In *the International Semantic Web Conference (OM@ISWC'16)*. CEUR-WS.org, Kobe, Japan, 1–7.

# APPENDIX: PROOFS OF THEOREMS

**Theorem 1.** Let $O$ be an ontology, and let $\mathcal{I}$ be a fulfilling interpretation of $O$ such that $\mathcal{I} \models O$. Then $\mathcal{S}_{\mathcal{I}} \models \tau_{\triangleright}(O)$.

PROOF. Let $O = \langle \mathcal{A}, \mathcal{T} \rangle$ be an ontology and $\mathcal{I}$ be a fulfilling interpretation of $O$ such that $\mathcal{I} \models O$. Let us denote by $\tau_{\triangleright}(O) = \langle V, \Psi \rangle$ the forward translation of $O$. We need to show that $\mathcal{S}_{\mathcal{I}} \models \tau_{\triangleright}(O)$. Since $\mathcal{I} \models O$, we know that for each axiom $\Phi \in \mathcal{T}$, we have that $\mathcal{I} \models \Phi$. On the other hand, to show that $\mathcal{S}_{\mathcal{I}} \models \tau_{\triangleright}(O)$, it is enough to show that $\mathcal{S}_{\mathcal{I}} \models \varphi$ for each constraint $\varphi \in \Psi$. Hence, we need to show that $\mathcal{S}_{\mathcal{I}} \models \tau_{\triangleright}(\Phi)$ for each axiom $\Phi \in \mathcal{T}$. Let $\Phi \in \mathcal{T}$. We fall into one of the following cases:

- $\Phi$ is of the form $C \sqsubseteq D$. Since $\mathcal{I} \models \Phi$, we have that $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ (cf. Table 1), or stated equivalently, that $v_C^{\mathcal{S}_{\mathcal{I}}} \subseteq v_D^{\mathcal{S}_{\mathcal{I}}}$ by definition of $\mathcal{S}_{\mathcal{I}}$ (cf. Definition 2). So $v_C^{\mathcal{S}_{\mathcal{I}}} \subset v_D^{\mathcal{S}_{\mathcal{I}}}$ or $v_C^{\mathcal{S}_{\mathcal{I}}} = v_D^{\mathcal{S}_{\mathcal{I}}}$. Thus from Table 2, we get that $\mathcal{S}_{\mathcal{I}} \models v_C\{PP\}v_D$ or $\mathcal{S}_{\mathcal{I}} \models v_C\{EQ\}v_D$, which can equivalently be written as $\mathcal{S}_{\mathcal{I}} \models v_C\{PP, EQ\}v_D$. Yet we know that $\tau_{\triangleright}(C \sqsubseteq D) = v_C\{PP, EQ\}v_D$ (cf. Definition 1), so we have that $\mathcal{S}_{\mathcal{I}} \models \tau_{\triangleright}(C \sqsubseteq D)$. Hence, $\mathcal{S}_{\mathcal{I}} \models \tau_{\triangleright}(\Phi)$.
- $\Phi$ is of the form $C \sqcap D \sqsubseteq \bot$. Since $\mathcal{I} \models \Phi$, we have that $C^{\mathcal{I}} \cap D^{\mathcal{I}} \subseteq \emptyset$ (cf. Table 1), or stated equivalently, that $v_C^{\mathcal{S}_{\mathcal{I}}} \cap v_D^{\mathcal{S}_{\mathcal{I}}} \subseteq \emptyset$ by definition of $\mathcal{S}_{\mathcal{I}}$ (cf. Definition 2), i.e., $v_C^{\mathcal{S}_{\mathcal{I}}} \cap v_D^{\mathcal{S}_{\mathcal{I}}} = \emptyset$. Thus from Table 2, we get that $\mathcal{S}_{\mathcal{I}} \models v_C\{DR\}v_D$. Yet we know that $\tau_{\triangleright}(C \sqcap D \sqsubseteq \bot) = v_C\{DR\}v_D$ (cf. Definition 1), so we have that $\mathcal{S}_{\mathcal{I}} \models \tau_{\triangleright}(C \sqcap D \sqsubseteq \bot)$. Hence, $\mathcal{S}_{\mathcal{I}} \models \tau_{\triangleright}(\Phi)$.

This concludes the proof. □

**Theorem 2.** Let $O$ be an ontology and let $\mathcal{S}$ be a solution of $\tau_{\triangleright}(O)$. Then there is an inflation $\mathcal{I}_{\mathcal{S}}$ of $\mathcal{S}$ s.t. $\mathcal{I}_{\mathcal{S}} \models \Phi$ for each axiom $\Phi$ of $O$.

PROOF. Let $O = \langle \mathcal{A}, \mathcal{T} \rangle$ be an ontology. Let us denote by $\tau_{\triangleright}(O) = \langle \mathcal{A}, \mathcal{T} \rangle$ the forward translation of $O$. Let $\mathcal{S} = (\mathcal{D}^{\mathcal{S}}, \cdot^{\mathcal{S}})$ be a solution of $\tau_{\triangleright}(O)$, $e$ be any element of $\mathcal{D}^{\mathcal{S}}$, and $\mathcal{I}_{\mathcal{S}} \stackrel{\text{def}}{=} (\Delta^{\mathcal{I}_{\mathcal{S}}}, \cdot^{\mathcal{I}_{\mathcal{S}}})$ be the inflation of $\mathcal{S}$ defined for each $a \in N_I$ as $\cdot^{\mathcal{S}}(a) = e$ and for each $a, b \in N_I$ and each $r \in N_R$ as $\cdot^{\mathcal{S}}(r(a, b)) = (e, e)$. Since $\mathcal{I}_{\mathcal{S}}$ is an inflation of $\mathcal{S}$, we know that $\Delta^{\mathcal{I}_{\mathcal{S}}} = \mathcal{D}^{\mathcal{S}}$ and for every $A \in N_C$, $A^{\mathcal{I}_{\mathcal{S}}} = (v_A)^{\mathcal{S}}$ (cf. Definition 3).

Let $\Phi$ be any axiom of $\mathcal{T}$ and let us show that $\mathcal{I}_{\mathcal{S}} \models \Phi$. We fall into one of the following cases:

- $\Phi$ is of the form $C \sqsubseteq D$. Since $\mathcal{S}$ is a solution of $\tau_{\triangleright}(O)$, we have that $\mathcal{S} \models \tau_{\triangleright}(C \sqsubseteq D)$. Yet we know that $\tau_{\triangleright}(C \sqsubseteq D) = v_C\{PP, EQ\}v_D$ (cf. Definition 1), so we have that $\mathcal{S} \models v_C\{PP, EQ\}v_D$, which can equivalently be written as $\mathcal{S} \models v_C\{PP\}v_D$ or $\mathcal{S} \models v_C\{EQ\}v_D$. Thus from Table 2, we get that $v_C^{\mathcal{S}} \subset v_D^{\mathcal{S}}$ or $v_C^{\mathcal{S}} = v_D^{\mathcal{S}}$, i.e., $v_C^{\mathcal{S}} \subseteq v_D^{\mathcal{S}}$. So by definition of $\mathcal{I}_{\mathcal{S}}$, we get that $C^{\mathcal{I}_{\mathcal{S}}} \subseteq D^{\mathcal{I}_{\mathcal{S}}}$. From Table 1, this means that $\mathcal{I}_{\mathcal{S}} \models \Phi$.
- $\Phi$ is of the form $C \sqcap D \sqsubseteq \bot$. Since $\mathcal{S}$ is a solution of $\tau_{\triangleright}(O)$, we have that $\mathcal{S} \models \tau_{\triangleright}(C \sqcap D \sqsubseteq \bot)$. Yet we know that $\tau_{\triangleright}(C \sqcap D \sqsubseteq \bot) = v_C\{DR\}v_D$ (cf. Definition 1), so we have that $\mathcal{S}_{\mathcal{I}} \models v_C\{DR\}v_D$. Thus from Table 2, we get that $v_C^{\mathcal{S}_{\mathcal{I}}} \cap v_D^{\mathcal{S}_{\mathcal{I}}} \subseteq \emptyset$,

i.e., $v_C^{\mathcal{S}_{\mathcal{I}}} \cap v_D^{\mathcal{S}_{\mathcal{I}}} = \emptyset$. So by definition of $\mathcal{I}_{\mathcal{S}}$, we get that $C^{\mathcal{I}} \cap D^{\mathcal{I}} \subseteq \emptyset$. From Table 1, this means that $\mathcal{I}_{\mathcal{S}} \models \Phi$.

This concludes the proof. □

**Theorem 3.** Let $\mathcal{N}$ be a scenario and $\mathcal{S}$ be solution of $\mathcal{N}$. Then there is an inflation $\mathcal{I}_{\mathcal{S}}$ of $\mathcal{S}$ s.t. $\mathcal{I}_{\mathcal{S}}$ is a model of $\tau_{\triangleleft}(\mathcal{N})$.

PROOF. Let $\mathcal{N} = \langle V, \Psi \rangle$ be a scenario. Let us denote by $\tau_{\triangleleft}(\mathcal{N}) = \langle \mathcal{A}, \mathcal{T} \rangle$ the backward translation of $\mathcal{N}$. Let $\mathcal{S} = (\mathcal{D}^{\mathcal{S}}, \cdot^{\mathcal{S}})$ be a solution of $\mathcal{N}$. Note that the set of concept names $N_C$ of $\tau_{\triangleleft}(\mathcal{N})$ is formed of two parts: the concept names $C$ which are directly associated with a variable $v_C$ from $V$, and the new concept names introduced in the backward translation (cf. Definition 4). For instance, some new concept names $A'$, $C'$ and $D'$ appear in the translation of a constraint $v_C\{PO\}v_D$. Let us denote by $N_C^*$ the subset of $N_C$ formed by these new concept names. In addition, note that the set of individual names $N_I$ of $\tau_{\triangleleft}(\mathcal{N})$ is formed only of individual names artificially introduced in the translation; for instance, the individual names $a$, $c$ and $d$ appear in the translation of a constraint $v_C\{PO\}v_D$, and all these individual names form the set $N_I$. Now, let $\mathcal{I}_{\mathcal{S}} = (\Delta^{\mathcal{I}_{\mathcal{S}}}, \cdot^{\mathcal{I}_{\mathcal{S}}})$ be an interpretation defined as $\Delta^{\mathcal{I}_{\mathcal{S}}} = \mathcal{D}^{\mathcal{S}}$, and for every $A \in N_C \setminus N_C^*$ as $A^{\mathcal{I}_{\mathcal{S}}} = (v_A)^{\mathcal{S}}$. Note that at this point, $\mathcal{I}_{\mathcal{S}}$ is only partially defined, so one needs to complete its definition. Indeed, one needs to define $(A')^{\mathcal{I}_{\mathcal{S}}}$ for each $A' \in N_C^*$ and to define $\cdot^{\mathcal{I}_{\mathcal{S}}}(a)$ and for each $a \in N_I$. For this purpose, let us consider only the forms of constraints that actually introduce new concept names and new individual names, i.e., the elements of $N_C^*$ and $N_I$. These constraints are of the form $v_C \{PO\} v_D$ and of the form $v_C \{PP\} v_D$ (the constraints of the form $v_C \{PPi\} v_D$ can be dealt with similarly as to the case of the constraints of the form $v_C \{PP\} v_D$, since $v_C \{PPi\} v_D$ is equivalent to $v_D \{PP\} v_C$.) So let $\varphi$ be a constraint of the form $v_C \{PO\} v_D$ or the form $v_C \{PP\} v_D$. We consider the two cases separately:

- $\varphi$ is a constraint of the form $v_C \{PO\} v_D$. Note that since the concept names $C$ and $D$ are members of $N_C \setminus N_C^*$, $C^{\mathcal{I}_{\mathcal{S}}}$ and $D^{\mathcal{I}_{\mathcal{S}}}$ are already defined as $C^{\mathcal{I}_{\mathcal{S}}} = v_C^{\mathcal{S}}$ and $D^{\mathcal{I}_{\mathcal{S}}} = v_D^{\mathcal{S}}$, where $v_C^{\mathcal{S}}$ and $v_D^{\mathcal{S}}$ are non-empty subsets of $\Delta^{\mathcal{I}_{\mathcal{S}}}$. Additionally, since $\mathcal{S} \models v_C \{PO\} v_D$, according to Table 2 we have that $v_C^{\mathcal{S}} \cap v_D^{\mathcal{S}} \neq \emptyset$, $v_C^{\mathcal{S}} \not\subseteq v_D^{\mathcal{S}}$ and $v_D^{\mathcal{S}} \not\subseteq v_C^{\mathcal{S}}$. Now, let $A'$, $C'$ and $D'$ be the concept names newly introduced in the translation of $v_C \{PO\} v_D$ according to Definition 4. We have that $A'$, $C'$, $D' \in N_C^*$. Let us define $(A')^{\mathcal{I}_{\mathcal{S}}}$, $(C')^{\mathcal{I}_{\mathcal{S}}}$, and $(D')^{\mathcal{I}_{\mathcal{S}}}$ as follows:
  - let $(A')^{\mathcal{I}_{\mathcal{S}}} = v_{A'}^{\mathcal{S}}$, where $v_{A'}^{\mathcal{S}}$ is any non-empty subset of $\Delta^{\mathcal{I}_{\mathcal{S}}}$ such that $v_{A'}^{\mathcal{S}} \subseteq v_C^{\mathcal{S}} \cap v_D^{\mathcal{S}}$ (this assignment is realizable since $v_C^{\mathcal{S}} \cap v_D^{\mathcal{S}} \neq \emptyset$);
  - let $(C')^{\mathcal{I}_{\mathcal{S}}} = (v_{C'})^{\mathcal{S}}$, where $v_{C'}^{\mathcal{S}}$ is any non-empty subset of $\Delta^{\mathcal{I}_{\mathcal{S}}}$ such that $v_{C'}^{\mathcal{S}} \subseteq v_C^{\mathcal{S}}$ and $v_{C'}^{\mathcal{S}} \cap v_D^{\mathcal{S}} = \emptyset$ (this assignment is realizable since $v_C^{\mathcal{S}} \not\subseteq v_D^{\mathcal{S}}$);
  - let $(D')^{\mathcal{I}_{\mathcal{S}}} = v_{D'}^{\mathcal{S}}$, where $v_{D'}^{\mathcal{S}}$ is any non-empty subset of $\Delta^{\mathcal{I}_{\mathcal{S}}}$ such that $v_{D'}^{\mathcal{S}} \subseteq v_D^{\mathcal{S}}$ and $v_{D'}^{\mathcal{S}} \cap v_C^{\mathcal{S}} = \emptyset$ (this assignment is realizable since $v_D^{\mathcal{S}} \not\subseteq v_C^{\mathcal{S}}$).

  Similarly, let $a$, $c$, $d$ be the individual names newly introduced in that translation. We have that $a$, $c$, $d \in N_I$. Let us define

$\cdot^{\mathcal{I}_S}(a)$ (respectively, $\cdot^{\mathcal{I}_S}(c)$, $\cdot^{\mathcal{I}_S}(d)$) as any element of $v_{A'}^S$ (respectively, of $v_{C'}^S$, of $v_{D'}^S$).

- $\varphi$ is a constraint of the form $v_C \{PP\} v_D$. Note that since the concept names $C$ and $D$ are members of $N_C \setminus N_C^*$, $C^{\mathcal{I}_S}$ and $D^{\mathcal{I}_S}$ are already defined as $C^{\mathcal{I}_S} = v_C^S$ and $C^{\mathcal{I}_S} = v_D^S$, where $v_C^S$ and $v_D^S$ are non-empty subsets of $\Delta^{\mathcal{I}_S}$. Additionally, since $\mathcal{S} \models v_C \{PP\} v_D$, according to Table 2 we have that $v_C^S \subset v_D^S$. Now, let $D'$ be the concept name newly introduced in the translation of $v_C \{PP\} v_D$ according to Definition 4. We have that $D' \in N_C^*$ and $c, d \in N_I$. Let us define $(D')^{\mathcal{I}_S}$ as $(D')^{\mathcal{I}_S} = v_{D'}^S$, where $v_{D'}^S$ is any non-empty subset of $\Delta^{\mathcal{I}_S}$ such that $v_{D'}^S \subseteq v_D^S$ and $v_{D'}^S \cap v_C^S = \emptyset$ (this assignment is realizable since $v_C^S \subset v_D^S$). Similarly, let $c, d$ be the individual names newly introduced in that translation. We have that $c, d \in N_I$. Let us define $\cdot^{\mathcal{I}_S}(c)$ (respectively, $\cdot^{\mathcal{I}_S}(d)$) as any element of $v_C^S$ (respectively, of $v_{D'}^S$).

At this point, $\mathcal{I}_S$ is completely defined: we have defined $A^{\mathcal{I}_S}$ for each $A \in N_C$, and we have defined $\cdot^S(a)$ for each $a \in N_I$. First, it can easily be verified that $\mathcal{I}_S$ is an inflation of $\mathcal{S}$, according to Definition 3. Indeed, we have that $\Delta^{\mathcal{I}_S} = \mathcal{D}^S$, and for every $A \in N_C$ that $A^{\mathcal{I}_S} = (v_A)^S$. We now intend to show that $\mathcal{I}_S$ is a model of $\tau_\triangleleft(\mathcal{N})$. For this purpose, let us consider each contraint $v_C \varphi v_D \in \Psi$, and show that $\mathcal{I}_S$ is such that $\mathcal{I}_S \models \tau_\triangleleft(v_C \varphi v_D)$. We fall into one of the following cases:

- $\varphi$ is a constraint of the form $v_C \{EQ\} v_D$. Since $\mathcal{S}$ is a solution of $\mathcal{N}$, we have that $\mathcal{S} \models v_C \{EQ\} v_D$. Thus $v_C^S = v_D^S$ (cf. Table 2). So by definition of $\mathcal{I}_S$, we get that $C^{\mathcal{I}_S} = D^{\mathcal{I}_S}$. Thus, from Table 1, we get that $\mathcal{I}_S \models C \equiv D$. Yet, from Definition 4, we know that $\tau_\triangleleft(v_C \{EQ\} v_D) = \langle\{C \equiv D\}, \emptyset\rangle$. Hence, $\mathcal{I}_S \models \tau_\triangleleft(v_C \{EQ\} v_D)$.

- $\varphi$ is a constraint of the form $v_C \{DR\} v_D$. Since $\mathcal{S}$ is a solution of $\mathcal{N}$, we have that $\mathcal{S} \models v_C \{DR\} v_D$. Thus $v_C^S \cap v_D^S = \emptyset$ (cf. Table 2). So by Definition of $\mathcal{I}_S$, we get that $C^{\mathcal{I}_S} \cap D^{\mathcal{I}_S} = \emptyset$, i.e, $C^{\mathcal{I}_S} \cap D^{\mathcal{I}_S} \subseteq \emptyset$. Thus, from Table 1, we get that $\mathcal{I}_S \models C \sqcap D \sqsubseteq \bot$. Yet, from Definition 4, we know that $\tau_\triangleleft(v_C \{DR\} v_D) = \langle\{C \sqcap D \sqsubseteq \bot\}, \emptyset\rangle$. Hence, $\mathcal{I}_S \models \tau_\triangleleft(v_C \{DR\} v_D)$.

- $\varphi$ is a constraint of the form $v_C \{PO\} v_D$. Since $\mathcal{S}$ is a solution of $\mathcal{N}$, we have that $\mathcal{S} \models v_C \{PO\} v_D$. And according to the way we previously defined $(A')^{\mathcal{I}_S}$, $(C')^{\mathcal{I}_S}$, and $(D')^{\mathcal{I}_S}$, we have that $v_{A'}^S \subseteq v_C^S \cap v_D^S$, $v_{C'}^S \subseteq v_C^S$, $v_{C'}^S \cap v_D^S = \emptyset$, $v_{D'}^S \subseteq v_D^S$, and $v_{D'}^S \cap v_C^S = \emptyset$. Hence, from Table 1, we get that:
  - $\mathcal{I}_S \models A' \sqsubseteq C \sqcap D$ (since $v_{A'}^S \subseteq v_C^S \cap v_D^S$);
  - $\mathcal{I}_S \models C' \sqsubseteq C$ (since $v_{C'}^S \subseteq v_C^S$);
  - $\mathcal{I}_S \models C' \sqcap D \sqsubseteq \bot$ (since $v_{C'}^S \cap v_D^S = \emptyset$);
  - $\mathcal{I}_S \models D' \sqsubseteq D$ (since $v_{D'}^S \subseteq v_D^S$);
  - $\mathcal{I}_S \models D' \sqcap C \sqsubseteq \bot$ (since $v_{D'}^S \cap v_C^S = \emptyset$).

  In addition, according to the way we previously defined $\cdot^{\mathcal{I}_S}(a)$, $\cdot^{\mathcal{I}_S}(c)$, $\cdot^{\mathcal{I}_S}(d)$, we know that $\cdot^{\mathcal{I}_S}(a) \in v_{A'}^S$, $\cdot^{\mathcal{I}_S}(c) \in v_{C'}^S$, and $\cdot^{\mathcal{I}_S}(d) \in v_{D'}^S$. Hence, we get that:
  - $\mathcal{I}_S \models A'(a)$ (since $\cdot^{\mathcal{I}_S}(a) \in v_{A'}^S$);
  - $\mathcal{I}_S \models C(c)$ (since $\cdot^{\mathcal{I}_S}(c) \in v_{C'}^S$ and $v_{C'}^S \subseteq v_C^S$);
  - $\mathcal{I}_S \models C(a)$ (since $\cdot^{\mathcal{I}_S}(a) \in v_{A'}^S$ and $v_{A'}^S \subseteq v_C^S$);
  - $\mathcal{I}_S \models D(d)$ (since $\cdot^{\mathcal{I}_S}(d) \in v_{D'}^S$ and $v_{D'}^S \subseteq v_D^S$);
  - $\mathcal{I}_S \models D(a)$ (since $\cdot^{\mathcal{I}_S}(a) \in v_{A'}^S$ and $v_{A'}^S \subseteq v_D^S$);
  - $\mathcal{I}_S \models C'(c)$ (since $\cdot^{\mathcal{I}_S}(c) \in v_{C'}^S$);
  - $\mathcal{I}_S \models D'(d)$ (since $\cdot^{\mathcal{I}_S}(d) \in v_{D'}^S$).

  Overall, we got that $\mathcal{I}_S \models \langle\{A' \sqsubseteq C \sqcap D, C' \sqsubseteq C, C' \sqcap D \sqsubseteq \bot, D' \sqsubseteq D, D' \sqcap C \sqsubseteq \bot\}, \{A'(a), C(c), C(a), D(d), D(a), C'(c), D'(d)\}\rangle$. Hence, $\mathcal{I}_S \models \tau_\triangleleft(v_C \{PO\} v_D)$.

- $\varphi$ is a constraint of the form $v_C \{PP, EQ\} v_D$. Since $\mathcal{S}$ is a solution of $\mathcal{N}$, we have that $\mathcal{S} \models v_C \{PP, EQ\} v_D$. Thus $v_C^S \subseteq v_D^S$ (cf. Table 2). So by definition of $\mathcal{I}_S$, we get that $C^{\mathcal{I}_S} \subseteq D^{\mathcal{I}_S}$. Thus, from Table 1, we get that $\mathcal{I}_S \models C \sqsubseteq D$. Yet, from Definition 4, we know that $\tau_\triangleleft(v_C \{PP, EQ\} v_D) = \langle\{C \sqsubseteq D\}, \emptyset\rangle$. Hence, $\mathcal{I}_S \models \tau_\triangleleft(v_C \{PP, EQ\} v_D)$.

- $\varphi$ is a constraint of the form $v_C \{PPi, EQ\} v_D$. The proof that $\mathcal{I}_S \models \tau_\triangleleft(v_C \{PPi, EQ\} v_D)$ can be reduced equivalently to the proof that $\mathcal{I}_S \models \tau_\triangleleft(v_D \{PP, EQ\} v_C)$ similarly to the previous case, since $v_C \{PPi, EQ\} v_D$ is equivalent to $v_D \{PP, EQ\} v_C$.

- $\varphi$ is a constraint of the form $v_C \{PP\} v_D$. Since $\mathcal{S}$ is a solution of $\mathcal{N}$, we have that $\mathcal{S} \models v_C \{PP\} v_D$. Thus $v_C^S \subset v_D^S$ (cf. Table 2). And according to the way we previously defined $(D')^{\mathcal{I}_S}$, we have that $v_{D'}^S \subseteq v_D^S$ and $v_{D'}^S \cap v_C^S = \emptyset$. Hence, from Table 1, we get that:
  - $\mathcal{I}_S \models C \sqsubseteq D$ (since $v_C^S \subset v_D^S$);
  - $\mathcal{I}_S \models D' \sqsubseteq D$ (since $v_{D'}^S \subseteq v_D^S$);
  - $\mathcal{I}_S \models C \sqcap D' \sqsubseteq \bot$ (since $v_{D'}^S \cap v_C^S = \emptyset$).

  In addition, according to the way we previously defined $\cdot^{\mathcal{I}_S}(c)$ and $\cdot^{\mathcal{I}_S}(d)$, we know that $\cdot^{\mathcal{I}_S}(c) \in v_C^S$, and $\cdot^{\mathcal{I}_S}(d) \in v_{D'}^S$. Hence, we get that:
  - $\mathcal{I}_S \models D'(d)$ (since $\cdot^{\mathcal{I}_S}(d) \in v_{D'}^S$);
  - $\mathcal{I}_S \models C(c)$ (since $\cdot^{\mathcal{I}_S}(c) \in v_C^S$);
  - $\mathcal{I}_S \models D(d)$ (since $\cdot^{\mathcal{I}_S}(d) \in v_{D'}^S$ and $v_{D'}^S \subseteq v_D^S$);
  - $\mathcal{I}_S \models D(c)$ (since $\cdot^{\mathcal{I}_S}(c) \in v_C^S$ and $v_C^S \subset v_D^S$).

  Overall, we got that $\mathcal{I}_S \models \langle\{C \sqsubseteq D, D' \sqsubseteq D, C \sqcap D' \sqsubseteq \bot\}, \{D'(d), C(c), D(d), D(c)\}\rangle$. Hence, $\mathcal{I}_S \models \tau_\triangleleft(v_C \{PP\} v_D)$.

- $\varphi$ is a constraint of the form $v_C \{PPi\} v_D$. The proof that $\mathcal{I}_S \models \tau_\triangleleft(v_C \{PPi\} v_D)$ can be reduced equivalently to the proof that $\mathcal{I}_S \models \tau_\triangleleft(v_D \{PP\} v_C)$ similarly to the previous case, since $v_C \{PPi\} v_D$ is equivalent to $v_D \{PP\} v_C$.

We have proved that for each contraint $v_C \varphi v_D \in \Psi$, we have that $\mathcal{I}_S \models \tau_\triangleleft(v_C \varphi v_D)$. Therefore, $\mathcal{I}_S \models \tau_\triangleleft(\mathcal{N})$. This concludes the proof. □

**Theorem 4.** Let $\mathcal{N}$ be a scenario and let $\mathcal{I}$ be a fulfilling interpretation of $\tau_\triangleleft(\mathcal{N})$ such that $\mathcal{I}$ is a model of $\tau_\triangleleft(\mathcal{N})$. Then $\mathcal{S}_\mathcal{I} \models \mathcal{N}$.

PROOF. Let $\mathcal{N} = \langle V, \Psi\rangle$ be a scenario and let $\mathcal{I} = \langle\Delta^\mathcal{I}, \cdot^\mathcal{I}\rangle$ be a fulfilling interpretation of $\tau_\triangleleft(\mathcal{N})$ such that $\mathcal{I} \models \tau_\triangleleft(\mathcal{N})$. Let us denote by $\tau_\triangleleft(\mathcal{N}) = \langle\mathcal{A}, \mathcal{T}\rangle$ the backward translation of $\mathcal{N}$. Note that the set of concept names $N_C$ of $\tau_\triangleleft(\mathcal{N})$ is formed of two parts: (1) the concept names $C$ which are directly associated with a variable $v_C$ from $V$ and (2) the new concept names introduced in the backward translation (cf. Definition 4). For instance, some new concepts $A'$, $C'$, and $D'$ appear in the translation of a constraint $v_C \{PO\} v_D$. Let

us denote by $N_C^*$ the subset of $N_C$ formed by these new concept names. In addition, the set of $N_I$ of $\tau_\lhd(\mathcal{N})$ is formed only of individual names artificially introduced in the translation; for instance, the individual $a$, $c$, and $d$ appear in the translation of a constraint $v_C \{PO\} v_D$, and all these individual names of $N_I$.

From Definition 2, we have $\mathcal{S}_I = \langle \Delta^I, \cdot^{\mathcal{S}_I} \rangle$ that is a solution of $\mathcal{N}$, for every $(v_A)^{\mathcal{S}_I} = A^I$ where $A \in N_C \setminus N_C^*$ and $v_A \in V$ and $\cdot^{\mathcal{S}_I}(a)$ for each $a \in N_I$. It can easily be verified that $\mathcal{S}_I$ is the flattening of $\mathcal{I}$ (cf. Definition 2). Now we need to show that $\mathcal{S}_I \models \mathcal{N}$. For this purpose, let us consider each constraint $v_C \varphi v_D$, and show that $\mathcal{I} \models \tau_\lhd(v_C \varphi v_D)$ then $\mathcal{S}_I \models v_C \varphi v_D$. We fall into one of the following cases:

- $\varphi$ is a constraint of the form $v_C \{EQ\} v_D$. Since $\mathcal{I} \models \tau_\lhd(\mathcal{N})$, we have that $\mathcal{I} \models \tau_\lhd\{v_C \{EQ\} v_D\}$. From Definition 4, we get that $\tau_\lhd(v_C \{EQ\} v_D) = \langle C \equiv D, \emptyset \rangle$. Thus, we have $\mathcal{I} \models C \equiv D$, it is equivalent to $C^I \equiv D^I$ (cf. Table 1). So by definition of $\mathcal{S}_I$, we get that $(v_C)^{\mathcal{S}_I} = (v_D)^{\mathcal{S}_I}$. Thus, from Table 2, we get that $\mathcal{S}_I \models v_C \{EQ\} v_D$.

- $\varphi$ is a constraint of the form $v_C \{DR\} v_D$. Since $\mathcal{I} \models \tau_\lhd(\mathcal{N})$, we have that $\mathcal{I} \models \tau_\lhd\{v_C \{DR\} v_D\}$. From Definition 4, we get that $\tau_\lhd(v_C \{DR\} v_D) = \langle C \sqcap D \sqsubseteq \bot, \emptyset \rangle$. Thus we have $\mathcal{I} \models C \sqcap D \sqsubseteq \bot$, it is equivalent to $C^I \cap D^I \subseteq \emptyset$ (cf. Table 1). So by definition of $\mathcal{S}_I$, we get that $(v_C)^{\mathcal{S}_I} \cap (v_D)^{\mathcal{S}_I} = \emptyset$. Thus, from Table 2, we get that $\mathcal{S}_I \models v_C \{DR\} v_D$.

- $\varphi$ is a constraint of the form $v_C \{PO\} v_D$. Since $\mathcal{I} \models \tau_\lhd(\mathcal{N})$, we have that $\mathcal{I} \models \tau_\lhd\{v_C \{PO\} v_D\}$. From Definition 4, we get that $\tau_\lhd(v_C \{PO\} v_D) = \langle \{A' \sqsubseteq C \sqcap D, C' \sqsubseteq C, C' \sqcap D \sqsubseteq \bot, D' \sqsubseteq D, D' \sqcap C \sqsubseteq \bot\}, \{A'(a), C'(c), D'(d), C(c), C(a), D(d), D(a)\} \rangle$. Thus, we have $\mathcal{I} \models \{A' \sqsubseteq C \sqcap D, C' \sqsubseteq C, C' \sqcap D \sqsubseteq \bot, D' \sqsubseteq D, D' \sqcap C \sqsubseteq \bot\}$ and $\mathcal{I} \models \{A'(a), C'(c), D'(d), C(c), C(a), D(d), D(a)\}$. From Table 1, we get that $(A')^I \subseteq C^I \cap D^I$, $(C')^I \subseteq C^I, (C')^I \cap D^I \subseteq \emptyset$, and $(D')^I \subseteq D^I, (D')^I \cap C^I \subseteq \emptyset$. Now, we need to consider the following cases:
  - Since $\mathcal{I} \models \mathcal{A}'(a)$, we get that $a^I \in (A')^I$. Moreover, $(A')^I \subseteq C^I \cap D^I$. Thus $a^I \in C^I \cap D^I$. Then we get that $C^I \cap D^I \neq \emptyset$.
  - Since $\mathcal{I} \models \{C'(c), C(c)\}$, we get that $c^I \in (C')^I$ and $c^I \in C^I$. Moreover, $(C')^I \subseteq C^I, (C')^I \cap D^I \subseteq \emptyset$. Thus $c^I \in C^I$ but $c^I \notin D^I$. Then, we get that $C^I \nsubseteq D^I$.
  - Since $\mathcal{I} \models \{D'(d), D(d)\}$, we get that $d^I \in (D')^I$ and $d^I \in D^I$. Moreover, $(D')^I \subseteq D^I, (D')^I \cap C^I \subseteq \emptyset$. Thus $d^I \in D^I$ but $d^I \notin C^I$. Then, we get that $D^I \nsubseteq C^I$.

  From the above cases and by definition of $\mathcal{S}_I$, we get that $(v_C)^{\mathcal{S}_I} \cap (v_D)^{\mathcal{S}_I} \neq \emptyset$, $(v_C)^{\mathcal{S}_I} \nsubseteq (v_D)^{\mathcal{S}_I}$, and $(v_D)^{\mathcal{S}_I} \nsubseteq (v_C)^{\mathcal{S}_I}$. In other words, we also have $\mathcal{S}_I \models \{v_{A'} \subseteq v_C \cap v_D, v_C \nsubseteq v_D, v_D \nsubseteq v_C\}$. Thus, from Table 2, we have $\mathcal{S}_I \models v_C \{PO\} v_D$.

- $\varphi$ is a constraint of the form $v_C \{PP, EQ\} v_D$. Since $\mathcal{I} \models \tau_\lhd(\mathcal{N})$, we have that $\mathcal{I} \models \tau_\lhd\{v_C \{PP, EQ\} v_D\}$. From Definition 4, we get that $\tau_\lhd(v_C \{PP, EQ\} v_D) = \langle C \sqsubseteq D, \emptyset \rangle$. Thus, we have $\mathcal{I} \models C \sqsubseteq D$, it is equivalent to $C^I \subseteq D^I$ (cf. Table 1). So by definition of $\mathcal{S}_I$, we get that $(v_C)^{\mathcal{S}_I} \subseteq (v_D)^{\mathcal{S}_I}$. Moreover, we also have that $(v_C)^{\mathcal{S}_I} \subset (v_D)^{\mathcal{S}_I}$ or $(v_C)^{\mathcal{S}_I} = (v_D)^{\mathcal{S}_I}$. Thus, from Table 2, we get that $\mathcal{S}_I \models$ $\{v_C \{PP\} v_D$ or $v_C \{EQ\} v_D\}$. It is also equivalent to $\mathcal{S}_I \models$ $v_C \{PP, EQ\} v_D$.

- $\varphi$ is a constraint of the form $v_C \{PPi, EQ\} v_D$. The proof that $\mathcal{S}_I \models v_C \{PPi, EQ\} v_D$ can be reduced equivalently to the proof that $\mathcal{S}_I \models v_D \{PP, EQ\} v_C$ similarly to the previous case, since $v_C \{PP, EQ\} v_D$ is equivalent to $v_D \{PPi, EQ\} v_C$.

- $\varphi$ is a constraint of the form $v_C\{PP\}v_D$. Since $\mathcal{I} \models \tau_\lhd(\mathcal{N})$, we have that $\mathcal{I} \models \tau_\lhd\{v_C \{PP\} v_D\}$. From Definition 4, we get that $\tau_\lhd(v_C\{PP\}v_D) = \langle \{C \sqsubseteq D, D' \sqsubseteq D, C \sqcap D' \sqsubseteq \bot\}, \{D'(d), C(c), D(d), D(c)\} \rangle$. Thus, we have $\mathcal{I} \models \{C \sqsubseteq D, D' \sqsubseteq D, C \sqcap D' \sqsubseteq \bot\}$ and $\mathcal{I} \models \{D'(d), C(c), D(d), D(c)\}$. From Table 1, we get that $C^I \subseteq D^I, (D')^I \subseteq D^I, C^I \cap (D')^I \subseteq \emptyset$. Moreover, since $\mathcal{I} \models \{D'(d), D(d)\}$, we have $d^I \in (D')^I, d^I \in D^I$. Then, $d^I \notin C^I$ (by $C^I \cap (D')^I \subseteq \emptyset$). Hence, $C^I \nsubseteq D^I$. Now, by definition of $\mathcal{S}_I$, we get that $(v_C)^{\mathcal{S}_I} \subset (v_D)^{\mathcal{S}_I}$. In other words, we also have $\mathcal{S}_I \models v_C \subset v_D$ (shown in the above part). Thus, from Table 2, we have $\mathcal{S}_I \models v_C \{PP\} v_D$.

- $\varphi$ is a constraint of the form $v_C \{PPi\} v_D$. The proof that $\mathcal{S}_I \models v_C \{PPi\} v_D$ can be reduced equivalently to the proof that $\mathcal{S}_I \models v_D\{PP\}v_C$ similarly to the previous case, since $v_C \{PP\} v_D$ is equivalent to $v_D \{PPi\} v_C$.

This concludes the proof.                                          □