Ontology Merging within Qualitative Conceptual Space

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ABSTRACT

This paper introduces a novel method for merging open-domain ontologies which takes advantage of qualitative conceptual spaces, as formalism used to represent regions in a topological space and to reason about their set-theoretic relationships. To this end, we first propose a faithful translation of ontologies into qualitative conceptual spaces. The merging is performed on these resulting spaces, and the result is then translated back into the underlying language of the input ontologies. Our approach allows us to benefit from the expressivity and the flexibility of conceptual spaces while dealing with conflicting ontological knowledge in a principled way.

CCS CONCEPTS

• Theory of computation → **Constraint and logic programming**; **Description logics**; **• Computing methodologies** → **Description logics**; **Ontology engineering**; **Reasoning about belief and knowledge**;

KEYWORDS

RCC-5, Qualitative Constraint Network, Ontology Merging, Description Logics

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1 INTRODUCTION

Commonsense knowledge is playing an increasingly important role in the development of AI systems. Such knowledge is available, for example, in large open-domain ontologies such as SUMO or Cyc, in knowledge graphs (KGs) such as DBpedia and WikiData, as semantic markup (e.g. RDFa). Ontologies encode structured knowledge about the concepts and properties of a given domain. They play an important role in areas such as Semantic Web [18], Information Retrieval [12], Natural Language Processing [25], and machine learning [17], among others. For instance, Bouraoui and Schockaert (2018) have shown that ontologies, as prior conceptual knowledge, are useful for learning concept representations from few examples. However, the available ontologies (and KGs, as simple ontologies) are inevitably incomplete, where several rules and facts are missing. Several methods have been proposed for automated ontology (KG) completion [4, 20, 24] that exploit statistical regularities in a given ontology to predict plausible missing rules or facts. Unfortunately, meaningful knowledge is difficult to predict, especially since we have few examples of facts or rules. Moreover, as most of the existing approaches are mainly based on statistical regularities, the resulting predictions might be conflicting with others. A repair-based mechanism is then required to maintain the consistency[1], i.e. ensure that there is no conflicting (or contradictory) statements. In the same perspective, to widen the coverage of ontologies to several domains and to deal with incompleteness and conflicting statements, one may combine knowledge from several sources. However, it turns out that merging open-domain ontologies is a particularly challenging task as pointed out for example in [29] reporting the different problems and difficulties encountered when merging Freebase with WikiData. Conflicting information may occur when the statements of several sources are simply gathered together. Ontology merging and alignment has also attracted much attention in the literature [5, 11, 31]. Ontology merging aims to combine two (or more) ontologies having the same terminology while handling conflict, while ontology alignment (or matching) is the process of determining correspondences between terminologies of ontologies.

Let us consider an example taken from crowdsourcing a group of four people to illustrate the merging problem. Assume that a first source says that the concept $Paper$ is $disjoint$ with the concept $Document$, while another source says that every $Paper\ is$ a $Document$. Obviously, these two statements are conflicting. To be faithful to both sources while resolving conflict, a sensible choice would be to assume that $Paper$ and $Document$ are not disjoint concepts, but every $Paper$ is not necessarily a $Document$. This kind of result is clearly consistent and can be seen as a good compromise between both sources. Finding meaningful and relevant compromise between sources during the merging process is difficult to obtain. This is mainly due the fact that ontology languages (Description Logics for example) are not expressive enough to capture salient knowledge that might emerge during the merging process. Consider the example of a first source saying that a $Scientific\ Paper(SP)$ is both a $Paper$ and a $Book$, a second source mentioning that SP is-a $Paper$, and a third source gives that SP is-a $Book$. For this case, $Paper$ and $Book$ should be overlapped to represent information from all three sources. This is clearly a relevant example of piece of knowledge that should be taken into account during the merging process, but that cannot be captured in ontology language. To this end, we take advantage of conceptual spaces, which are geometric representation frameworks, in which the objects are represented as points in a metric space, and concepts are modelled as regions [16]. Motivated by the fact that conceptual knowledge in an ontology can be to some extent modelled as geometric objects and constraints on conceptual spaces [9], this paper proposes a method for ontology merging that takes advantage of qualitative conceptual spaces to find out relevant compromise between sources while solving conflicts. Indeed, qualitative spatial reasoning is a suitable paradigm

[1] A set of statements (axioms) is (logically) **consistent**, iff all the statements can be true together or it involves no contradiction.

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for efficiently reasoning about spatial entities and their relationships. Therein, spatial information is usually represented in terms of basic or non-basic relations in a qualitative calculus, and reasoning tasks are formulated as solving a set of qualitative constraints [8, 23, 27]. In particular, Region Connection Calculus (RCC) is a well-studied formalism for qualitative topological representation and reasoning, including its subsets *RCC-5* [26] and *RCC-8*. Two significant advantages of the RCC framework are (i) its ability to reason efficiently about the relationships between spatial entities, and (ii) its ability to deal with conflicts in qualitative constraint merging as shown in [13, 19, 30].

We take advantage of RCC5 formalism and propose a method for merging open-domain ontologies using qualitative conceptual spaces. We first show how to translate an ontology to a qualitative space while preserving its semantics and properties. Second, we propose a merging operator that produces a single and consistent conceptual space that represents a compromise between sources. Finally, we show how to express the conceptual space in the input ontology language while maintaining the maximum of relevant information.

Proofs are provided in a supplementary material [2].

2 BACKGROUND

We will rely on Description Logics (DLs) encoding of how knowledge is represented to formally provide a mapping from an ontology to conceptual space. We will use RCC and qualitative constraints for performing the merging. This section briefly recalls the technical background required on these two topics.

Description Logics. DLs are one of the standard representations to express ontologies and they underlay the Ontology Web Language (OWL). For simplicity, we consider $\mathcal{EL}$ [1], which is one of the most basic DLs. The main ingredients of DLs are individuals, concepts, and roles, which correspond at the semantic level to objects, sets of objects, and binary relations between objects. More formally, let N_C, N_R, N_I be three pairwise disjoint sets where N_C denotes a set of atomic concepts, N_R denotes a set of atomic relations (roles), and N_I denotes a set of individuals. The $\mathcal{EL}_\perp$ concept expressions are built according to the following grammar:

$$C ::= \top \mid \perp \mid N_C \mid C \sqcap C \mid \exists r.C \text{ where } r \in N_R.$$

Let $C, D \in N_C$, $a, b \in N_I$, and $r \in N_R$. An $\mathcal{EL}$ ontology $\mathcal{O} = \langle \mathcal{T}, \mathcal{A} \rangle$ (a.k.a. knowledge base) comprises two components, the TBox (denoted by $\mathcal{T}$) and ABox (denoted by $\mathcal{A}$). The TBox consists of a set of General Concept Inclusion (GCI) axioms of the form $C \sqsubseteq D$, meaning that C is more specific than D or simply C is subsumed by D, and equivalence axioms of the form $C \equiv D$ which is a shortcut for $C \sqsubseteq D$ and $D \sqsubseteq C$. The ABox is a finite set of assertions on individual objects of the form $C(a)$ or $r(a, b)$.

The semantics is given in terms of interpretations $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$, which consist of a non-empty interpretation domain $\Delta^\mathcal{I}$ and an interpretation function $\cdot^\mathcal{I}$ that maps each individual $a \in N_I$ into an element $a^\mathcal{I} \in \Delta^\mathcal{I}$, each concept $A \in N_C$ into a subset $A^\mathcal{I} \subseteq \Delta^\mathcal{I}$, each role $r \in N_R$ into a subset $r^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$. A summary of the syntax and semantics of $\mathcal{EL}_\perp$ is shown in Table 1

[2] https://anonymous.4open.science/r/OntologyMergingWithinQualitativeConceptualSpace

Syntax	Semantics
$C \sqsubseteq D$	$C^\mathcal{I} \subseteq D^\mathcal{I}$
r	$r^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$
a	$a^\mathcal{I} \in \Delta^\mathcal{I}$
$C \sqcap D$	$C^\mathcal{I} \cap D^\mathcal{I}$
$\top$	$\Delta^\mathcal{I}$
$\perp$	$\emptyset$
$\exists r.C$	$\{x \in \Delta^\mathcal{I} \mid \exists y \in \Delta^\mathcal{I} s.t. (x, y) \in r^\mathcal{I}, y \in C^\mathcal{I}\}$

Table 1: Syntax and semantics of $\mathcal{EL}_\perp$

An interpretation $\mathcal{I}$ is said to be a model of (or satisfies) a GCI axiom, denoted by $\mathcal{I} \models C \sqsubseteq D$, if $C^\mathcal{I} \subseteq D^\mathcal{I}$. Similarly, $\mathcal{I}$ satisfies a concept (resp. role) assertion, denoted $\mathcal{I} \models C(a)$ (resp. $\mathcal{I} \models r(a, b)$), if $a^\mathcal{I} \in C^\mathcal{I}$ (resp. $(a^\mathcal{I}, b^\mathcal{I}) \in r^\mathcal{I}$). An interpretation $\mathcal{I}$ is a model of an ontology $\mathcal{O}$ if it satisfies all the axioms and assertions in $\mathcal{O}$. An ontology is said to be consistent if it has a model. Otherwise, it is inconsistent. An axiom Φ is entailed by an ontology, denoted by $\mathcal{O} \models \Phi$, if Φ is satisfied by every model of $\mathcal{O}$. We say that C is subsumed by D w.r.t. an ontology $\mathcal{O}$ iff $\mathcal{O} \models C \sqsubseteq D$. Similarly, we say that a is an instance of C w.r.t. $\mathcal{O}$ iff $\mathcal{O} \models C(a)$. A fulfilling interpretation is formally defined as follows: An interpretation $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ of an ontology is said to be fulfilling when each concept of the ontology is non-empty in $\mathcal{I}$, i.e., for each concept $C_i \in N_C, \cdot^\mathcal{I}(C_i) \neq \emptyset$. Lastly, given $A, B, A_1, A_2 \in N_C$, an $\mathcal{EL}_\perp$ TBox $\mathcal{T}$ is in the **strict normal form** (a variant of the one introduced by Baader et al. [3]) if it consists of inclusions of the form: $A \sqsubseteq B, A_1 \sqcap A_2 \sqsubseteq \perp, A \sqsubseteq \exists r.B, \exists r.A \sqsubseteq B$,where A, B, A_1, A_2 are atomic concepts. We focus on these forms of inclusions as it is common for most ontologies to use these forms to represent statements in practice. We assume that all ontologies are in the aforementioned normal form. Moreover, we also apply the normalization rules [21] to collect the basic inclusions of the normal form [28].

Region Connection and Qualitative Constraints. The RCC (Region Connection Calculus) formalism allows to represent and reason about the relationships between spatial entities [23]. Among the fragments of the RCC theory, RCC-5 fragment is expressive enough to reason about set-theoretic relations between regions [7, 22]. In RCC-5, regions can simply be interpreted as non-empty subsets of a given set and the focus is given on a set $\mathcal{B} = \{DR, PO, EQ, PP, PPi\}$ of five binary relations between regions called *basic relations*. The set $\mathcal{B}$ forms a jointly exhaustive and pairwise disjoint set of relations, that is, each pair of regions satisfies exactly one relation from $\mathcal{B}$: the relation DR (resp. PO, EQ, PP) holds between two regions whenever the two regions are disjoint (resp. when they partially overlap, are equal, when the first is a strict subset of the second), and PPi is the converse of PP. Based on $\mathcal{B}$, more complex pieces of information about the relative positions of a set of regions can be represented by means of qualitative constraint networks (QCNs). Formally, a QCN is a pair $\mathcal{N} = \langle V, \Psi \rangle$, where $V = \{v_C, v_D, \ldots\}$ is a set of region variables representing the spatial entities and Ψ is a set of binary constraints between these entities. Each constraint $\Psi_{CD} \in \Psi$ is a mapping from $V \times V$ to $2^\mathcal{B}$, and is simply denoted by $\Psi_{CD} = v_C \ \varphi \ v_D$, where $\varphi \subseteq 2^\mathcal{B}$; and Ψ_{CD} is said to be a singleton constraint whenever φ is a singleton. An interpretation of a QCN $\mathcal{N}$ is defined as

Name (Symbol)	Syntax	Semantics
Proper Part of (PP)	$v_C\{PP\}v_D$	$v_C^{\mathcal{S}} \subset v_D^{\mathcal{S}}$
Inverse PP of (PPi)	$v_C\{PPi\}v_D$	$v_D^{\mathcal{S}} \subset v_C^{\mathcal{S}}$
Equals (EQ)	$v_C\{EQ\}v_D$	$v_C^{\mathcal{S}} = v_D^{\mathcal{S}}$
Disjoint From (DR)	$v_C\{DR\}v_D$	$v_C^{\mathcal{S}} \cap v_D^{\mathcal{S}} = \emptyset$
Partially Overlaps (PO)	$v_C\{PO\}v_D$	$v_C^{\mathcal{S}} \cap v_D^{\mathcal{S}} \neq \emptyset$, $v_C^{\mathcal{S}} \not\subseteq v_D^{\mathcal{S}}, v_D^{\mathcal{S}} \not\subseteq v_C^{\mathcal{S}}$

Table 2: Syntax and semantics of *RCC-5*, $v_C, v_D \in V$.

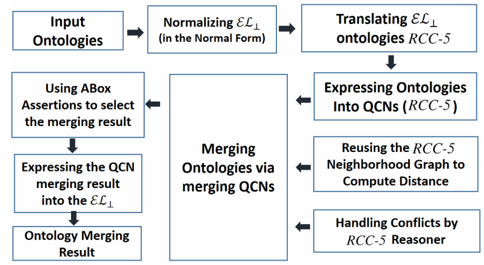

Figure 1: A Framework of Ontology Merging with QCNs

$\mathcal{S} = (\mathcal{D}^{\mathcal{S}}, \cdot^{\mathcal{S}})$, where $\mathcal{D}^{\mathcal{S}}$ is a non-empty set (the domain of the regions), and $\cdot^{\mathcal{S}}$ is an interpretation function which maps each variable v_C to a non-empty subset $v_C^{\mathcal{S}}$ of $\mathcal{D}^{\mathcal{S}}$. Table 2 precises how singleton constraints from Ψ are interpreted in RCC-5, i.e., an interpretation $\mathcal{S}$ of $\mathcal{N}$ satisfies a singleton constraint Ψ_{CD}, denoted by $\mathcal{S} \models \mathcal{N}$, if the relation between $v_C^{\mathcal{S}}$ and $v_D^{\mathcal{S}}$ according to the table is satisfied (e.g., $\mathcal{S} \models v_C\{PP\}v_D\}$ whenever $v_C^{\mathcal{S}} \subset v_D^{\mathcal{S}}$). The satisfaction relation is extended to any (non-singleton) constraint from $2^{\mathcal{B}}$ as follows: for each $\varphi \in 2^{\mathcal{B}}$, $\mathcal{S} \models v_C\ \varphi\ v_D$ iff $\mathcal{S} \models v_C\ \{\varphi_i\}\ v_D$ for some $\varphi_i \in \varphi$ (e.g., $\mathcal{S} \models v_C\{PP, EQ\}v_D$ iff $\mathcal{S} \models v_C\{PP\}v_D$ or $v_C\{EQ\}v_D\}$). An interpretation $\mathcal{S}$ of a QCN $\mathcal{N} = \langle V, \Psi\rangle$ is said to be a solution of $\mathcal{N}$, denoted by $\mathcal{S} \models \mathcal{N}$, iff $\mathcal{S} \models \Psi_{CD}$ for each $\Psi_{CD} \in \Psi$. A QCN is consistent iff it admits a solution. A sub-network of $\mathcal{N}$ is a QCN $\mathcal{N}' = (V, \Psi')$ such that $\Psi' \subseteq \Psi$. A quasi-atomic QCN $\langle V, \Psi\rangle$ is a QCN where for each $v_C, v_D \in V$, there is a unique constraint $\Psi_{CD} \in \Psi$, and where Ψ_{CD} is either a singleton or $\Psi_{CD} \in \{\{PP, EQ\}, \{PPi, EQ\}\}$. A scenario of a QCN is a quasi-atomic sub-network of $\mathcal{N}$. Noteworthy, a QCN is consistent if it admits a consistent scenario.

3 MERGING FRAMEWORK DESCRIPTION

Our aim is to introduce a method for merging open-domain ontologies using qualitative constraints networks. As highlighted in the introduction, qualitative conceptual spaces are more expressive and flexible than ontology languages, which allows in turn to capture relevant information that might emerge during the merging process allowing to select a consistent compromise between sources when expressing the merging result. A key challenge is to find a faithful translation from an ontology to a conceptual space that preserves the semantics and maintains the initial knowledge encoded in the ontology. Another challenging task is to define a merging operator that takes advantage of the QCNs to perform merging while handling conflicts. As constraints of the merged QCN are *sets* of basic relations, this QCN cannot be translated back into the target ontology language in the general case. So we select one of its "best" scenarios which, in contrast, can be expressed as an ontology. In particular, our approach involves the following main steps:

(1) Translating ontologies into QCNs (Section 4). In this step, we present a translation function that ensures the faithfulness of the mapping from ontology to qualitative conceptual space.
(2) Defining a QCN merging operator (Section 5). Exploiting the notion of "distance" between basic relations and constraints, this step associates with the input set of QCNs (the translated ontologies) with a single merged, consistent QCN.
(3) Selecting the "best" consistent scenario of the merged QCN as a representative of the merged result (Section 6). This selection process takes advantage of the notion of distance between scenarios and the ABox assertions from the input ontologies.
(4) Translating the selected consistent scenario back into the underlying language of the sources (Section 7).

4 TRANSLATING ONTOLOGIES INTO QCNS

In this section, we present a translation function from any ontology to a QCN. More precisely, we (1) map concepts names into QCN variables and axioms into constraints, and (2) show that the translation is faithful to the TBox of the initial ontology. Let $\mathcal{C} \stackrel{\text{def}}{=} N_C \cup \{\exists r.A \mid A \in N_C \text{ and } r \in N_R\}$ be the set of *basic concepts*. Clearly, $\mathcal{C}$ is finite and $|\mathcal{C}| = |N_C| + |N_R| \cdot |N_C|$. One can see that $\mathcal{C} \cup \{\perp\}$ contains all possible concepts allowed by the (strict) normal form. Let $\mathcal{V} \stackrel{\text{def}}{=} \{v_C \mid C \in \mathcal{C}\}$ be the set of *basic variable regions* where each variable is tagged with a concept from $\mathcal{C}$. Given $\mathcal{C}$ and $\mathcal{V}$, we can define a (trivial) translation from ontology concepts into regions, which is extended to a translation of GCIs in our (strict) normal form into constraints as follows:

Definition 1 (Forward translation $\tau_\triangleright$). *A **forward translation** is a function $\tau_\triangleright : \mathcal{C} \longrightarrow \mathcal{V}$ s.t. $\tau_\triangleright(C) \stackrel{\text{def}}{=} v_C$. $\tau_\triangleright$ is extended to map GCIs in the (strict) normal form into constraints as follows:*

- $\tau_\triangleright(C \sqsubseteq D) \stackrel{\text{def}}{=} \tau_\triangleright(C)\{PP, EQ\}\tau_\triangleright(D)$, *and*
- $\tau_\triangleright(C \sqcap D \sqsubseteq \perp) \stackrel{\text{def}}{=} \tau_\triangleright(C)\{DR\}\tau_\triangleright(D)$.

Moreover, $\tau_\triangleright$ is extended to translate ontologies in the (strict) normal form into a set of constraints in the expected way: $\tau_\triangleright(\mathcal{O}) \stackrel{\text{def}}{=} \{\tau_\triangleright(\Phi) \mid \Phi \in \mathcal{O}\}$.

To show that the translation is faithful, we provide a "semantic" mapping from $\mathcal{O}$ to $\tau_\triangleright(\mathcal{O})$, and conversely. Let us first show how models of $\mathcal{O}$ correspond to solutions of $\tau_\triangleright(\mathcal{O})$:

Definition 2 (Flattening of an interpretation). *Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be a fulfilling interpretation. With $\mathcal{S}_{\mathcal{I}} \stackrel{\text{def}}{=} (\Delta^{\mathcal{I}}, \cdot^{\mathcal{S}_{\mathcal{I}}})$ we denote the **flattening of** $\mathcal{I}$, where $\cdot^{\mathcal{S}_{\mathcal{I}}} : \mathcal{V} \longrightarrow 2^{\Delta^{\mathcal{I}}}$ is such that $(v_C)^{\mathcal{S}_{\mathcal{I}}} = C^{\mathcal{I}}$.*

Theorem 1. *Let $\mathcal{O}$ be an ontology, and let $\mathcal{I}$ be a fulfilling interpretation of $\mathcal{O}$ such that $\mathcal{I} \models \mathcal{O}$. Then $\mathcal{S}_{\mathcal{I}} \models \tau_\triangleright(\mathcal{O})$.*

The other way around, let us show how solutions of $\tau_\triangleright(\mathcal{O})$ correspond to interpretations satisfying all axioms from $\mathcal{O}$.

Definition 3 (Inflation of a solution). *Let $\mathcal{S} = (\mathcal{D}^{\mathcal{S}}, \cdot^{\mathcal{S}})$ be a semantic solution to a QCN $\mathcal{N}$ over $\mathcal{V}$. With $\mathcal{I}_{\mathcal{S}} \stackrel{\text{def}}{=} (\Delta^{\mathcal{I}_{\mathcal{S}}}, \cdot^{\mathcal{I}_{\mathcal{S}}})$, where $\Delta^{\mathcal{I}_{\mathcal{S}}} = \mathcal{D}^{\mathcal{S}}$ and, for every $A \in N_C$, $A^{\mathcal{I}_{\mathcal{S}}} = (v_A)^{\mathcal{S}}$, we call $\mathcal{I}_{\mathcal{S}}$ an* ***inflation of*** $\mathcal{S}$.

Intuitively, an inflation of $\mathcal{S}$ corresponds to an interpretation blown up from $\mathcal{S}$ by interpreting atomic concept names in the same way their corresponding variable names are "populated" by the solution. Notice that there are as many possible inflations of $\mathcal{S}$ as there are ways of interpreting N_R and N_I over $\Delta^{\mathcal{I}_{\mathcal{S}}}$. An immediate consequence of Definition 3 is that every inflation of a solution $\mathcal{S}$ is fulfilling.

Theorem 2. *Let O be an ontology and let $\mathcal{S}$ be a solution of $\tau_{\triangleright}(O)$. Then there is an inflation $\mathcal{I}_{\mathcal{S}}$ of $\mathcal{S}$ s.t. $\mathcal{I}_{\mathcal{S}} \models \Phi$ for each axiom Φ of O.*

Theorems 1 and 2 establish that our translation is faithful, i.e., that the set of all fulfilling models of an ontology O are captured precisely in its translated QCN $\tau_{\triangleright}(O)$.

5 MERGING THE QCNS

We reduce the merging of a profile of ontologies $\mathcal{P} = \langle O^1, \ldots, O^n \rangle$ to the merging of a profile of QCNs $\mathcal{N} = \langle \mathcal{N}^1, \ldots, \mathcal{N}^n \rangle$, where for each $i \in \{1, \ldots, n\}$, $\mathcal{N}^i = (V, \Psi^i) = \tau_{\triangleright}(O^i)$, based on the faithful translation given the previous section. Inspired by works on syntactical QCN merging [13], this QCN merging process is summarized as follows. We associate with the profile $\mathcal{N}$ a single merged and consistent QCN $\mathcal{N} = (V, \Psi)$ representing $\mathcal{N}$ in a "global" way. This is performed in a constraint-wise fashion: for each pair of variables $v_C, v_D \in V$, we associate each basic relation $b \in \mathcal{B}$ with a value representing its *distance* to the profile of constraints $\mathscr{E}_{CD} = \langle \Psi^1_{CD}, \ldots, \Psi^n_{CD} \rangle$. This distance is the key tool to form the constraint Ψ_{CD} of the merged QCN $\mathcal{N}$. Intuitively, each constraint Ψ_{CD} corresponds to the set of basic relations with the lowest distances to the profile $\mathscr{E}_{CD}$, while ensuring that the resulting QCN is consistent.

Example 1. *Consider the profile of ontologies $\mathcal{P} = \langle O^1, O^2, O^3, O^4 \rangle$ that encodes the following knowledge about the four concepts of Paper, Text, Document and Book, respectively denoted by P, T, D and B.*

- $O^1 = \langle \mathcal{T}^1 = \{P \sqsubseteq T, T \sqcap D \sqsubseteq \bot, P \sqsubseteq B, P \sqcap D \sqsubseteq \bot, B \sqcap D \sqsubseteq \bot\}$, $\mathcal{A}^1 = \{P(p_1), T(p_1), T(t_1), D(d_1), B(p_1), B(b_1)\} \rangle$,
- $O^2 = \langle \mathcal{T}^2 = \{P \sqsubseteq T, T \sqsubseteq B, D \sqsubseteq B, D \sqsubseteq P\}$, $\mathcal{A}^2 = \{P(p_2), P(d_2), T(p_2), T(t_2), D(d_2), B(p_2), B(b_2)\} \rangle$,
- $O^3 = \langle \mathcal{T}^3 = \{B \sqsubseteq D, D \sqsubseteq P, P \sqsubseteq T, D \sqsubseteq T\}$, $\mathcal{A}^3 = \{P(p_3), T(t_3), T(p_3), D(d_3), B(b_3)\} \rangle$,
- $O^4 = \langle \mathcal{T}^4 = \{D \sqcap P \sqsubseteq \bot, P \sqsubseteq T, B \sqsubseteq D, T \sqcap B \sqsubseteq \bot, T \sqcap D \sqsubseteq \bot\}$, $\mathcal{A}^4 = \{P(p_4), T(t_4), T(p_4), D(b_4), D(d_4), B(b_4)\} \rangle$.

Using the forward translation $\tau_{\triangleright}$ presented in the previous section, one associates with $\mathcal{P}$ a profile of QCNs $\langle \tau_{\triangleright}(O^i) \rangle = \langle \mathcal{N}^i \rangle$ ($i \in \{1, \ldots, 4\}$). The four QCNs are depicted in Figure 2 (to alleviate the figures, we do not represent a constraint Ψ_{CD} when $\Psi_{CD} = \mathcal{B}$, i.e., when the QCN does not provide any information between the relationship between v_C and v_D).

Although we do not require it in the general case, note that in this example each ontology O^i is consistent, i.e., the TBox of each input ontology does not contain conflicting information. As a direct consequence of Theorems 1 and 2, each QCN is consistent. However, simply combining these QCNs can easily lead to conflicts. For instance, there is no basic relation shared in the constraints Ψ^1_{BD} and Ψ^3_{BD}, since in $\mathcal{N}^1$ we have that *Book* $\{DR\}$ *Document* whereas in $\mathcal{N}^3$ we have that *Book* $\{PP, EQ\}$ *Document*. This calls for our QCN merging procedure.

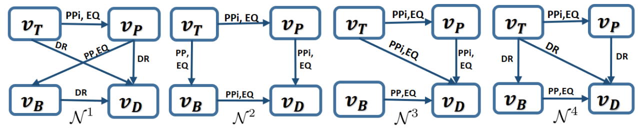

Figure 2: QCN profile of Example 1

Computing a distance between a basic relation and a profile of constraints. We start by considering a distance between basic relations. Firstly introduced in [15] in the context of temporal reasoning, the notion of *conceptual neighborhood* (CN) between relations was later adapted to QCN merging in [14] to define such a distance. Intuitively, two basic relations $b, b' \in \mathcal{B}$ are CNs if a continuous transformation of two regions which satisfy the basic relation $b \in B$ leads them to directly satisfy the basic relation b' without satisfying any other basic relation from $\mathcal{B}$. For instance, PP and EQ are CNs since "shrinking" a first region C initially equal to another region D directly makes it a proper part of D. This results in the neighborhood relation $\{(DR, PO), (PO, DR), (PO, PP), (PP, PO), (PO, PPi), (PPi, PO), (PP, EQ), (EQ, PP), (PPi, EQ), (EQ, PPi)\}$. This neighborhood relation induces a *neighborhood graph* $\mathcal{G}$ whose vertices are the elements of $\mathcal{B}$, and where there is an edge between two basic relations $b, b' \in \mathcal{B}$ whenever b and b' are CNs according the neighborhood relation. The distance $d(b, b')$ between two basic relations b and b' is defined as the length of the shortest path between b and b' in the neighborhood graph. For instance, $d(DR, PO) = 1$ since DR and PO are CNs, and $d(DR, EQ) = 3$ since DR and PO (resp. PO and EQ) are CNs, but DR and EQ are not. This distance is extended to a distance between a basic relation $b \in \mathcal{B}$ and a constraint $\varphi \in 2^{\mathcal{B}}$, defined as $d(b, \varphi) = \min_{b' \in \varphi} d(b, b')$. Lastly, given two variables $v_C, v_D \in V$, the distance between each $b \in \mathcal{B}$ and the profile of constraints $\mathscr{E}_{CD} = \langle \Psi^1_{CD}, \ldots, \Psi^n_{CD} \rangle$ is defined by $d(b, \mathscr{E}_{CD}) = \sum_{i \in \{1, \ldots, n\}} d(b, \Psi^i_{CD})$.

Example 1 (continued). *Let us focus on Text (T) and Document (D). We have $\mathscr{E}_{TD} = \langle \Psi^1_{TD}, \Psi^2_{TD}, \Psi^3_{TD}, \Psi^4_{TD} \rangle = \langle \{DR\}, \mathcal{B}, \{PPi, EQ\}, \{DR\} \rangle$. For the distance between the basic relation PP and $\mathscr{E}_{TD}$, we have that $d(PP, \mathscr{E}_{TD}) = d(b, \Psi^1_{TD}) + d(b, \Psi^2_{TD}) + d(b, \Psi^3_{TD}) + d(b, \Psi^4_{TD})$, where:*

$$\begin{aligned}
d(PP, \Psi^1_{TD}) &= d(PP, \{DR\}) = d(PP, DR) = 2, \\
d(PP, \Psi^2_{TD}) &= d(PP, \mathcal{B}) = d(PP, PP) = 0, \\
d(PP, \Psi^3_{TD}) &= d(PP, \{PPi, EQ\}) = d(PP, EQ) = 1, \\
d(PP, \Psi^4_{TD}) &= d(PP, \{DR\}) = d(PP, DR) = 2.
\end{aligned}$$

We get that $d(PP, \mathscr{E}_{TD}) = 5$. The distances between each basic relation from $\mathcal{B}$ and the profile of constraints $\mathscr{E}_{CD}$ for each pair of variables $v_C, v_D \in V$ is summarized in Table 3.

$\mathscr{E}$ / $\mathcal{B}$	$\mathscr{E}_{TP}$	$\mathscr{E}_{TB}$	$\mathscr{E}_{TD}$	$\mathscr{E}_{PB}$	$\mathscr{E}_{PD}$	$\mathscr{E}_{BD}$
DR	*8*	*2*	*2*	*2*	*4*	*6*
PO	*4*	*2*	*3*	*1*	*4*	*4*
PP	*4*	*2*	*5*	*0*	*6*	*3*
PPi	*0*	*3*	*4*	*1*	*4*	*4*
EQ	*0*	*3*	*6*	*0*	*6*	*3*

Table 3: Distances between relations from $\mathcal{B}$ and the profile of constraints $\mathscr{E}_{CD}$, for each pair of variables $v_C, v_D \in V$.

Using the distance to build a merged, consistent QCN.. We now describe our procedure which associates a profile of QCNs with a merged, consistent QCN. This takes advantage of the distance between basic relations and a profile of constraints $\mathscr{E}_{CD}$. Let us first formally define two intermediate functions $relax_{CD}$ and val_{CD} which are used in our procedure. Given a total preorder[3] $\preceq$ over a finite set E, let us denote by $\min(E, \preceq)$ the set of minimal elements of E w.r.t. $\preceq$, i.e., $\min(E, \preceq) = \{e \in E \mid \forall e' \in E, e \preceq e'\}$. Each pair of variables v_C, v_D is associated with a total preorder $\preceq_{CD}$ over the basic relations from $\mathcal{B}$ defined for all $b, b' \in \mathcal{B}$ as $b \preceq_{CD} b'$ iff $d(b, \mathscr{E}_{CD}) \leq d(b', \mathscr{E}_{CD})$. Then the function $relax_{CD}$ is a mapping $relax_{CD} : 2^{\mathcal{B}} \mapsto 2^{\mathcal{B}}$ defined for each $\varphi \in 2^{\mathcal{B}}$ as $relax_{CD}(\varphi) = \varphi \cup \min(\mathcal{B} \setminus \varphi, \preceq_{CD})$. It corresponds to the *relaxation* of a constraint φ w.r.t.. the total preordering $\preceq_{CD}$. Noteworthy, $relax_{CD}(\emptyset)$ corresponds to the set of basic relations with a minimal distance to the profile of constraints $\mathscr{E}_{CD}$. The function val_{CD} is a mapping $val_{CD} : 2^{\mathcal{B}} \mapsto \mathbb{N}$ defined for each $\varphi \in 2^{\mathcal{B}}$ as $val_{CD}(\varphi) = \max_{b \in \varphi} d(b, \mathscr{E}_{CD})$. For instance, according to Table 3 and focusing on *Book* and *Document* (cf. $\mathscr{E}_{BD}$), we get that $PP, EQ \preceq_{BD} PO, PPi \preceq_{BD} DR$, that $relax_{BD}(\emptyset) = \{PP, EQ\}$, that $relax_{BD}(\{PP, EQ\}) = \{PP, EQ, PO, PPi\}$, and that $val_{BD}(\{PP, EQ\}) = 3$.

We are ready to introduce our main procedure whose outline is given in Algorithm 1 that defines an initial QCN $\mathcal{N}$ by setting each one of its constraints Ψ_{CD} to the set of basic relations from $\mathcal{B}$ having a distance to the profile of constraints $\mathscr{E}_{CD}$ that is minimal (lines 2 to 5). If this QCN is consistent, then it is returned as the merged QCN (line 12). If not, some of the constraints of $\mathcal{N}$ are relaxed in line 7, in the sense that some basic relations are added to these constraints. Such a set of constraints S is selected as follows. In line 7, S is first restricted to those constraints from $\mathcal{N}$ which *can* be relaxed, i.e., those constraints not equal to $\mathcal{B}$. Among those candidate constraints, one selects in line 8 the constraints Ψ_{CD} having a highest value $val_{CD}(\Psi_{CD})$. Indeed, we do not want to relax first the constraints consisting of basic relations which are "close" to the input profile, but rather would one prioritize the relaxation of more "controversial" constraints, i.e., those with a high value according to val_{CD}. For instance, let us look back at Table 3. It can be seen that $d(PPi, \mathscr{E}_{TP}) = d(EQ, \mathscr{E}_{TP}) = 0$, and thus $val_{TP}(\{PPi, EQ\}) = 0$; this low value reflects the consensus between sources on the fact that one of the basic relations PPi, EQ holds between $Text$ and $Paper$, and indeed it can be verified that the axiom $P \sqsubseteq T$ is consistent with each input TBox. On the contrary, one has that $val_{PD}(\{DR, PO, PPi\}) = 4$; this higher

[3] A total preorder over a set E is a total, symmetric and transitive relation.

Algorithm 1: Computing a merged QCN

input: A profile of QCNs $\mathcal{N} = \langle \mathcal{N}_1, \dots, \mathcal{N}_n \rangle$
ouput: A merged, consistent QCN $\mathcal{N}$
1 **begin**
  // Initialization of the output QCN $\mathcal{N}$
2   $\Psi \longleftarrow \{\Psi_{CD} \mid v_C, v_D \in V\}$
3   **foreach** $(v_C, v_D) \in V \times V$ **do**
4     $\Psi_{CD} \longleftarrow relax_{CD}(\emptyset)$
5   $\mathcal{N} \longleftarrow (V, \Psi)$
6   **while** $\mathcal{N}$ *is not consistent* **do**
    // One relaxes some constraints of $\mathcal{N}$
7     $S \longleftarrow \{\Psi_{CD} \mid \Psi_{CD} \in \Psi, \Psi_{CD} \neq \mathcal{B}\}$
8     $S \longleftarrow \arg\max\{val_{CD}(\Psi_{CD}) \mid \Psi_{CD} \in S)\}$
9   **foreach** $\Psi_{CD} \in S$ **do**
10       $\Psi_{CD} \longleftarrow relax_{CD}(\Psi_{CD})$
11     $\mathcal{N} \longleftarrow (V, \Psi)$
12   **return** $\mathcal{N}$

value reflects a disagreement between the input sources about the relationship between the concepts of $Paper$ and $Document$. And in the general case, whenever possible and in order to restore the consistency of the merged QCN, it is a sensible choice to keep unchanged those constraints unanimely accepted by the input sources, and rather weaken first the most disputed constraints. This "relaxation" process is repeated iteratively until the resulting QCN is consistent which, obviously enough, is guaranteed after a finite number of iterations.

Example 1 (continued). *Initially, the merged QCN $\mathcal{N} = \langle V, \Psi \rangle$ is defined by the following set of constraints, which correspond the basic relations highlighted in Table 3 (this QCN is also depicted in Figure 3(a)):*

$$\Psi_{TP} = \{PPi, EQ\} \qquad \Psi_{TB} = \{DR, PO, PP\}$$
$$\Psi_{TD} = \{DR\} \qquad \Psi_{PB} = \{PP, EQ\}$$
$$\Psi_{PD} = \{DR, PO, PPi\} \qquad \Psi_{BD} = \{PP, EQ\}$$

This QCN is inconsistent. One can see that the constraints Ψ_{PB} and Ψ_{BD} imply by transitivity that the relation between P and D must be PP or EQ, yet $\Psi_{PD} \cap \{PP, EQ\} = \emptyset$. Then the constraint Ψ_{PD} is selected (cf. line 8 in the algorithm) as the only candidate for relaxation at this point, since $val_{PD}(\Psi_{PD}) = 4$, which is the highest value among all constraints. And since $relax_{PD}(\Psi_{PD}) = \mathcal{B}$, one updates Ψ_{PD} to $\mathcal{B}$ which results in the QCN depicted in Figure 3(b). This QCN is, again, inconsistent (in this case, explaining its inconsistency is more complex as it involves dependencies between all four variables. We omit the details for space reasons). Then the constraint Ψ_{BD} is selected for relaxation ($val_{BD}(\Psi_{BD}) = 3$) and one updates Ψ_{BD} to $\{PP, EQ, PO, PPi\}$. The resulting QCN (cf. Figure 3(c)) is consistent and returned by the procedure.

6 SELECTING A REPRESENTATIVE SCENARIO OF THE MERGED QCN

Once we have obtained a merged, consistent QCN, our goal is to express it in our initial (target) ontology language. However, not every constraint Ψ_{CD} from the QCN (i.e., a subset of $\mathcal{B}$) can easily

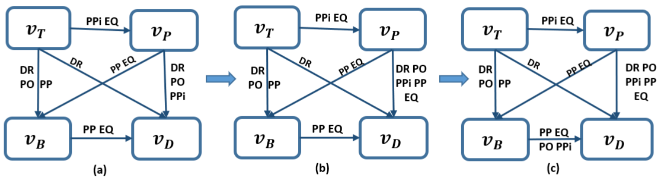

Figure 3: QCNs iteratively generated by our algorithm. Fig. 3(c) corresponds to the final consistent merged QCN.

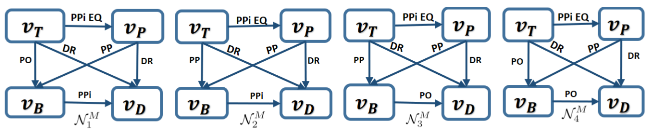

Figure 4: Possible consistent scenarios.

be translated as a set of axioms, since non-singleton constraints express some disjunctive information between two concepts/regions. Moreover, since the merged QCN is consistent, one can remark it necessarily admits at least one consistent scenario, and since a scenario involves singleton constraints (as well as the two constraints $\{PP, EQ\}$, $\{PPi, EQ\}$), it can easily be translated into a single ontology, as will be shown in the next section. Then our aim is to (i) focus on all consistent scenarios of the merged QCN, and (ii) select one representative scenario. This can be done by exploiting information provided by the input ABoxes as we intend to show in the rest of this section.

In our running example, the merged QCN admits four consistent scenarios which are depicted in Figure 4. Let us first discuss why these four scenarios seem to be reasonable candidates to the input ontologies / QCNs provided by the sources. First, note that all input ontologies state that $Paper \sqsubseteq Text$ ($v_T\{PPi, EQ\}v_P$ in the QCN profile). And it can be seen that this consensus is reflected in the four candidate scenarios, which entail that information. Second, while the two sources O_1 and O_4 state that $Text$ and $Document$ are disjoint concepts ($v_T\{DR\}v_D$ in the corresponding input QCNs), only one source says that $Document \sqsubseteq Text$ ($v_T\{PP, EQ\}v_D$). In this case, it make sense to follow the point of view of the majority of the sources. And accordingly, in all four scenarios we have that $v_T\{DR\}v_D$, thus $Text$ and $Document$ are disjoint concepts. Third, the source O_1 states that $Paper \sqsubseteq Book$ and all other sources have no information on these concepts. It is sensible to keep this information in the merged result, and accordingly all four scenarios entail that information. More, one sees that $v_P\{PP\}v_B$ holds in all scenarios, i.e., that $Paper$ is a *strict* part of $Book$, or stated otherwise, that both concepts cannot be equal while keeping the relationships between the remaining concepts consistent. This emergent property is also an interesting feature of the merging process. Last, the reason why there are four, equally reasonable, candidate scenarios is that some strong disagreements hold on the relationships between the concepts $Text$ and $Book$ on the one hand, and the concepts $Book$ and $Document$ on the other hand. Accordingly, the only differences between the four scenarios hold on the constraints between these two pairs of concepts ($v_T\{PO\}v_B$ / $v_T\{PP\}v_B$, and $v_B\{PPi\}v_D$ / $v_B\{PO\}v_D$).

So the remaning step is to select one of these four scenarios. For this purpose, we take advantage of the ABoxes from the input ontologies and see how these ABoxes relate to each scenario. To be as faithful as possible to what each input source says, instead of simply considering each input ABox as such, one considers the "closure" of it according to its corresponding TBox. For instance, if a given source states that $Paper \sqsubseteq Text$ in its TBox and that $Paper(p)$ in its Abox, then it makes sense to also consider that $Text(p)$ also holds in that source's implicit knowledge. Formally, let $O = \langle \mathcal{T}, \mathcal{A} \rangle$ be an ontology. The *deductive closure of* $\mathcal{A}$ w.r.t. $\mathcal{T}$, denoted by $Cl_{\mathcal{T}}(\mathcal{A})$, is defined as $Cl_{\mathcal{T}}(\mathcal{A}) \stackrel{\text{def}}{=} \{B(a) \mid O \models B(a)\} \cup \{r(a, b) \mid O \models r(a, b)\}$ [2, 6]. Accordingly, $\langle \mathcal{T}, \mathcal{A} \rangle$ is logically equivalent to $\langle \mathcal{T}, Cl_{\mathcal{T}}(\mathcal{A}) \rangle$.

Now, to select the representative scenario, one takes advantage of a distance between a scenario and the set of all input (closed) ABox, and then choose the scenario having a minimal distance. Given a scenario, the idea is to count the number of individuals in each input ABox which raise a conflict w.r.t. the constraints of that scenario. This can naturally be done for any scenario constraint $v_C\ \varphi\ v_D$ where $\varphi \neq \{PO\}$. For instance, if $Paper(p)$ and $Document(p)$ hold in the ABox of a given source, and $v_P\{DR\}v_D$ holds in the scenario under consideration, then according to that ABox p is an individual that raises a conflict with that scenario. Another example is if $Paper(p)$ holds but not $Text(p)$, then p is not a member of the concept $Text$ in the ABox (recall that ABoxes are closed w.r.t. their TBox); in that case p raises a conflict with the constraints $v_P\ \varphi\ v_T$ when $\varphi \in \{\{PP\}, \{EQ\}, \{PP, EQ\}\}$. More formally, given an ABox $\mathcal{A}$, a scenario constraint $v_C\ \varphi\ v_D$ where $\varphi \neq \{PO\}$, and an individual p, we say that p *raises a conflict with* φ *w.r.t* $\mathcal{A}$ when:

$$
\begin{array}{ll}
C(p) \in \mathcal{A}, D(p) \notin \mathcal{A} & \text{when } \varphi \subseteq \{PP, EQ\}, \\
D(p) \in \mathcal{A}, C(p) \notin \mathcal{A} & \text{when } \varphi \subseteq \{PPi, EQ\}, \\
C(p), D(p) \in \mathcal{A} & \text{when } \varphi = \{DR\}.
\end{array}
$$

And the number of conflicts raised by an ontology $O = (\mathcal{T}, \mathcal{A})$ w.r.t. a a scenario constraint $v_C\ \varphi\ v_D$ where $\varphi \neq \{PO\}$, is defined as $nbConf(O, \varphi) = |\{p \in N_I \mid p \text{ raises a conflict with } \varphi \text{ w.r.t. } Cl_{\mathcal{T}}(\mathcal{A})\}|$.

The case of PO is more complex. Indeed, it can be easily seen that no individual can raise a conflict with $v_C\{PO\}v_D$. This is because all basic relations $\mathcal{B} \setminus \{PO\}$ express explicit dependencies between concepts / regions, whereas PO is a complementary relation that (explicitely) expresses a notion of *independency* between concepts. For instance, the concepts $Smoker$ and $Researcher$ can naturally be thought of as independent concepts, in the sense that one can easily find in a real-world context individuals that are members of either both concepts, only one of them, and none of them (note that this should not be confused with the case of DR, which expresses an explicit dependency between concepts, e.g., the concepts Dog and Cat.) So to evaluate the number of "conflicts" raised by an ABox w.r.t. a constraint $v_C\{PO\}v_D$, we propose to count how "unbalanced" the number of conflicts are w.r.t. the remaining forms of constraints. Formally, focusing on the scenario constraint between two variables v_C and v_D, $nbConf(O, \varphi) = \max_{\varphi' \neq \{PO\}} nbConf(O, \varphi')) - \min_{\varphi' \neq \{PO\}} nbConf(O, \varphi'))$. For instance, when $nbConf(O, \{PP, EQ\}) = nbConf(O, \{PPi, EQ\}) = nbConf(O, \{DR\})$, then $nbConf(O, \{PO\}) = 0$: since individuals can be found equally (i) in both underlying concepts, and (ii) in one concept but not the other, O raises no conflict w.r.t. $\{PO\}$.

We have now the tools to select the representative scenario from a given set of candidates. The distance between a scenario $\mathcal{N}^M$ and the input profile of ontologies $\mathcal{P} = \langle O^1, \ldots, O^n \rangle$ is simply defined as the overall number of conflicts raised by all input ABoxes w.r.t. all constraints of $\mathcal{N}^M$, i.e., $d(\mathcal{N}^M, \mathcal{P}) = \sum_{i\in\{1,\ldots,n\},\varphi\in\mathcal{N}^M} nbConf(O^i, \varphi)$. And given a set of candidate scenarios, the representative scenario is the one having a minimal distance.

EXAMPLE 1 (CONTINUED). *Let us go back to our running example. We have that* $Cl_{\mathcal{T}^3}(\mathcal{A}^3) = \{P(p_3), P(b_3), P(d_3), T(t_3), T(d_3), T(b_3), T(p_3), D(d_3), D(b_3), B(b_3)\}$. *So focusing on Text and Book (i.e., on the scenario constraints between the variables* v_T *and* v_B*), we have that* $nbConf(O^3, \{PP\}) = |\{t_3, d_3, p_3\}| = 3$, *and one can easily verify that* $nbConf(O^3, \{PO\}) = |\{t_3, d_3, p_3\}| = 3 - 0 = 3$. *Summing up all conflicts, we get that* $d(\mathcal{N}_1^M, \mathcal{P}) = 20$, $d(\mathcal{N}_2^M, \mathcal{P}) = 18$, $d(\mathcal{N}_3^M, \mathcal{P}) = 22$, *and* $d(\mathcal{N}_4^M, \mathcal{P}) = 24$. *Hence, the scenario* $\mathcal{N}_2^M$ *is selected as a representative scenario of the merged QCN.*

7 TRANSLATING THE REPRESENTATIVE SCENARIO INTO AN ONTOLOGY

The last step is to translate the representative scenario selected in the previous step back into an ontology:

DEFINITION 4 (BACKWARD TRANSLATION $\tau_\triangleleft$). *A **backward translation** is a function* $\tau_\triangleleft : \mathcal{V} \longrightarrow C$ *s.t.* $\tau_\triangleleft(v_C) \stackrel{\text{def}}{=} C$. $\tau_\triangleleft$ *is extended to map constraints into an ontology as follows, where* A', C', *and* D' *are new concept names and a, b, and c new individual names:*[4]

- $\tau_\triangleleft(v_C\{EQ\}v_D) \stackrel{\text{def}}{=} \langle\{C \equiv D\}, \emptyset\rangle$;
- $\tau_\triangleleft(v_C\{DR\}v_D) \stackrel{\text{def}}{=} \langle\{C \sqcap D \sqsubseteq \bot\}, \emptyset\rangle$;
- $\tau_\triangleleft(v_C\{PO\}v_D) \stackrel{\text{def}}{=} \langle\{A' \sqsubseteq C \sqcap D, C' \sqsubseteq C, C' \sqcap D \sqsubseteq \bot, D' \sqsubseteq D, D' \sqcap C \sqsubseteq \bot\}, \{A'(a), C(c), C(a), D(d), D(a), C'(c), D'(d)\}\rangle$;
- $\tau_\triangleleft(v_C\{PP, EQ\}v_D) \stackrel{\text{def}}{=} \langle\{C \sqsubseteq D\}, \emptyset\rangle$;
- $\tau_\triangleleft(v_C\{PPi, EQ\}v_D) \stackrel{\text{def}}{=} \langle\{D \sqsubseteq C\}, \emptyset\rangle$;
- $\tau_\triangleleft(v_C\{PP\}v_D) \stackrel{\text{def}}{=} \langle\{C \sqsubseteq D, D' \sqsubseteq D, C \sqcap D' \sqsubseteq \bot\}, \{D'(d), C(c), D(d), D(c)\}\rangle$, *and*
- $\tau_\triangleleft(v_C\{PPi\}v_D)) \stackrel{\text{def}}{=} \langle\{D \sqsubseteq C, C' \sqsubseteq C, D \sqcap C' \sqsubseteq \bot\}, \{C'(c), D(d), C(d), D(c)\}\rangle$.

Moreover, $\tau_\triangleleft$ *is extended to translate a set of constraints into an ontology in the (strict) normal form in the expected way:* $\tau_\triangleleft(\mathcal{N}) \stackrel{\text{def}}{=} \langle\mathcal{T}, \mathcal{A}\rangle$, *where* $\mathcal{T} \stackrel{\text{def}}{=} \bigcup_{\tau_\triangleleft(\Psi)=\langle\mathcal{T}',\mathcal{A}'\rangle,\Psi\in\mathcal{N}} \mathcal{T}'$ *and* $\mathcal{A} \stackrel{\text{def}}{=} \bigcup_{\tau_\triangleleft(\Psi)=\langle\mathcal{T}',\mathcal{A}'\rangle,\Psi\in\mathcal{N}} \mathcal{A}'$.

Accordingly, our back translation is faithful. Using again the notions of inflation and flattening (cf. Definitions 2 and 3), we show that the set of solutions of a scenario $\mathcal{N}$ are captured precsiely in its translated ontology $\tau_\triangleleft(\mathcal{N})$:

THEOREM 3. *Let* $\mathcal{N}$ *be a scenario and* $\mathcal{S}$ *be solution of* $\mathcal{N}$. *Then there is an inflation* $\mathcal{I}_\mathcal{S}$ *of* $\mathcal{S}$ *s.t.* $\mathcal{I}_\mathcal{S}$ *is a model of* $\tau_\triangleleft(\mathcal{N})$.

THEOREM 4. *Let* $\mathcal{N}$ *be a scenario and let* $\mathcal{I}$ *be a fulfilling interpretation of* $\tau_\triangleleft(\mathcal{N})$ *such that* $\mathcal{I}$ *is a model of* $\tau_\triangleleft(\mathcal{N})$. *Then* $\mathcal{S}_\mathcal{I} \models \mathcal{N}$.

EXAMPLE 1 (CONTINUED). *Let us translate the selected scenario* $\mathcal{N}_2^M$ *into an ontology. From Definition 4, we get that:* $\tau_\triangleleft(\mathcal{N}_2^M) = \langle\{(\Psi_{TP})\ P \sqsubseteq T,\ (\Psi_{TD})\ T \sqcap D \sqsubseteq \bot,\ (\Psi_{PD})\ P \sqcap D \sqsubseteq \bot,\ (\Psi_{TB})\ T \sqsubseteq B,\ SubBo1 \sqsubseteq B,\ SubBo1 \sqcap T \sqsubseteq \bot,\ (\Psi_{PB})\ P \sqsubseteq B,\ SubBo2 \sqsubseteq B,\ SubBo2 \sqcap P \sqsubseteq \bot,\ (\Psi_{BD})\ D \sqsubseteq B,\ SubBo3 \sqsubseteq B,\ SubBo3 \sqcap D \sqsubseteq \bot$ $\},\{T(t_1),\ SubBo1(s_1),\ B(t_1),\ B(s_1),\ P(p_1),\ SubBo2(s_2),\ B(p_1),\ B(s_2),\ D(d_1),\ SubBo3(s_3),\ B(d_1),\ B(s_3)\}\rangle$.

[4] Notice that the constraint $\{PO\}$ cannot be translated into a set of GCIs only, whence the use of ABox assertions in the translation.

Discussion. From the merged ontology result, we realize that the method is flexible and solves efficiently the conflicts from both consistent and inconsistent ontologies. The result is useful and consistent with several pieces of evidence as follows:

(1) The merged ontology hierarchy is preserved the essential information from the input sources. i.e., all input ontologies state $Paper \sqsubseteq Text$. Then, the merged result still holds this information.
(2) The merged result respects the nature of the majority vote. i.e., the two sources O_1 and O_4 state that the $Text$ and $Document$ are disjoint while the O_3 says that $Document$ is-a $Text$. Then, the merged result obeys the law of majority vote $Text \sqcap Document \sqsubseteq \bot$.
(3) The merged result does not lose significant pieces of information. i.e., one source O_1 states that the $Paper$ is-a $Book$ and all other sources have no information. Then, the method still holds primary information. However, our method provides more constraints for those statements. The merged result is a more stringent constraint (the $Paper$ is a proper part of the $Book$ without allowing the "equivalence").
(4) If one constraint has many different statements from multi-sources, the method will select the one plausible constraint based on the ability of *RCC-5*'s reasoning. i.e., with $Book$ and the $Documents$ the sources states differently. Finally, the method provides a reasonable constraint depending on the surrounding constraints. In other words, the constraints impact directly each other to give a consistent merged result.

8 CONCLUSION

We introduced an ontology merging procedure based on qualitative conceptual spaces, by providing a two-way translation between ontologies and RCC-5 based qualitative constraints. Accordingly, even if the input ontologies are inconsistent when simply combined together, our approach returns a consistent result. It also remains as close as possible to the input sources, i.e., by preserving as much information as possible. Our next plan is to evaluate our method empirically. Moreover, an approach of representing regions and points in vector space (combining machine learning) will be our next direction for merging ontologies.

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APPENDIX: SYNTAX AND SEMANTICS OF $\mathcal{EL}_\perp$

Syntax	Semantics
$C \sqsubseteq D$	$C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$
r	$r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$
a	$a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$
$C \sqcap D$	$C^{\mathcal{I}} \cap D^{\mathcal{I}}$
$\top$	$\Delta^{\mathcal{I}}$
$\perp$	$\emptyset$
$\exists r.C$	$\{x \in \Delta^{\mathcal{I}} \mid \exists y \in \Delta^{\mathcal{I}} s.t.(x,y) \in r^{\mathcal{I}}, y \in C^{\mathcal{I}}\}$

Table 4: Syntax and semantics of $\mathcal{EL}_\perp$

APPENDIX: PROOFS OF THEOREMS

Theorem 1. Let O be an ontology, and let $\mathcal{I}$ be a fulfilling interpretation of O such that $\mathcal{I} \models O$. Then $\mathcal{S}_{\mathcal{I}} \models \tau_\triangleright(O)$.

PROOF. Let $O = \langle \mathcal{A}, \mathcal{T} \rangle$ be an ontology and $\mathcal{I}$ be a fulfilling interpretation of O such that $\mathcal{I} \models O$. Let us denote by $\tau_\triangleright(O) = \langle V, \Psi \rangle$ the forward translation of O. We need to show that $\mathcal{S}_{\mathcal{I}} \models \tau_\triangleright(O)$. Since $\mathcal{I} \models O$, we know that for each axiom $\Phi \in \mathcal{T}$, we have that $\mathcal{I} \models \Phi$. On the other hand, to show that $\mathcal{S}_{\mathcal{I}} \models \tau_\triangleright(O)$, it is enough to show that $\mathcal{S}_{\mathcal{I}} \models \varphi$ for each constraint $\varphi \in \Psi$. Hence, we need to show that $\mathcal{S}_{\mathcal{I}} \models \tau_\triangleright(\Phi)$ for each axiom $\Phi \in \mathcal{T}$. Let $\Phi \in \mathcal{T}$. We fall into one of the following cases:

- Φ is of the form $C \sqsubseteq D$. Since $\mathcal{I} \models \Phi$, we have that $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ (cf. Table 1), or stated equivalently, that $v_C^{\mathcal{S}_{\mathcal{I}}} \subseteq v_D^{\mathcal{S}_{\mathcal{I}}}$ by definition of $\mathcal{S}_{\mathcal{I}}$ (cf. Definition 2). So $v_C^{\mathcal{S}_{\mathcal{I}}} \subset v_D^{\mathcal{S}_{\mathcal{I}}}$ or $v_C^{\mathcal{S}_{\mathcal{I}}} = v_D^{\mathcal{S}_{\mathcal{I}}}$. Thus from Table 2, we get that $\mathcal{S}_{\mathcal{I}} \models v_C\{PP\}v_D$ or $\mathcal{S}_{\mathcal{I}} \models v_C\{EQ\}v_D$, which can equivalently be written as $\mathcal{S}_{\mathcal{I}} \models v_C\{PP, EQ\}v_D$. Yet we know that $\tau_\triangleright(C \sqsubseteq D) = v_C\{PP, EQ\}v_D$ (cf. Definition 1), so we have that $\mathcal{S}_{\mathcal{I}} \models \tau_\triangleright(C \sqsubseteq D)$. Hence, $\mathcal{S}_{\mathcal{I}} \models \tau_\triangleright(\Phi)$.
- Φ is of the form $C \sqcap D \sqsubseteq \perp$. Since $\mathcal{I} \models \Phi$, we have that $C^{\mathcal{I}} \cap D^{\mathcal{I}} \subseteq \emptyset$ (cf. Table 1), or stated equivalently, that $v_C^{\mathcal{S}_{\mathcal{I}}} \cap v_D^{\mathcal{S}_{\mathcal{I}}} \subseteq \emptyset$ by definition of $\mathcal{S}_{\mathcal{I}}$ (cf. Definition 2), i.e., $v_C^{\mathcal{S}_{\mathcal{I}}} \cap v_D^{\mathcal{S}_{\mathcal{I}}} = \emptyset$. Thus from Table 2, we get that $\mathcal{S}_{\mathcal{I}} \models v_C\{DR\}v_D$. Yet we know that $\tau_\triangleright(C \sqcap D \sqsubseteq \perp) = v_C\{DR\}v_D$ (cf. Definition 1), so we have that $\mathcal{S}_{\mathcal{I}} \models \tau_\triangleright(C \sqcap D \sqsubseteq \perp)$. Hence, $\mathcal{S}_{\mathcal{I}} \models \tau_\triangleright(\Phi)$.

This concludes the proof. □

Theorem 2. Let O be an ontology and let $\mathcal{S}$ be a solution of $\tau_\triangleright(O)$. Then there is an inflation $\mathcal{I}_{\mathcal{S}}$ of $\mathcal{S}$ s.t. $\mathcal{I}_{\mathcal{S}} \models \Phi$ for each axiom Φ of O.

PROOF. Let $O = \langle \mathcal{A}, \mathcal{T} \rangle$ be an ontology. Let us denote by $\tau_\triangleright(O) = \langle V, \Psi \rangle$ the forward translation of O. Let $\mathcal{S} = (\mathcal{D}^{\mathcal{S}}, \cdot^{\mathcal{S}})$ be a solution of $\tau_\triangleright(O)$, e be any element of $\mathcal{D}^{\mathcal{S}}$, and $\mathcal{I}_{\mathcal{S}} \overset{\text{def}}{=} (\Delta^{\mathcal{I}_{\mathcal{S}}}, \cdot^{\mathcal{I}_{\mathcal{S}}})$ be the inflation of $\mathcal{S}$ defined for each $a \in N_I$ as $\cdot^{\mathcal{S}}(a) = e$ and for each $a, b \in N_I$ and each $r \in N_R$ as $\cdot^{\mathcal{S}}(r(a,b)) = (e,e)$. Since $\mathcal{I}_{\mathcal{S}}$ is an inflation of $\mathcal{S}$, we know that $\Delta^{\mathcal{I}_{\mathcal{S}}} = \mathcal{D}^{\mathcal{S}}$ and for every $A \in N_C$, $A^{\mathcal{I}_{\mathcal{S}}} = (v_A)^{\mathcal{S}}$ (cf. Definition 3).

Let Φ be any axiom of $\mathcal{T}$ and let us show that $\mathcal{I}_{\mathcal{S}} \models \Phi$. We fall into one of the following cases:

- Φ is of the form $C \sqsubseteq D$. Since $\mathcal{S}$ is a solution of $\tau_\triangleright(O)$, we have that $\mathcal{S} \models \tau_\triangleright(C \sqsubseteq D)$. Yet we know that $\tau_\triangleright(C \sqsubseteq D) = v_C\{PP, EQ\}v_D$ (cf. Definition 1), so we have that $\mathcal{S} \models v_C\{PP, EQ\}v_D$, which can equivalently be written as $\mathcal{S} \models v_C\{PP\}v_D$ or $\mathcal{S} \models v_C\{EQ\}v_D$. Thus from Table 2, we get that $v_C^{\mathcal{S}} \subset v_D^{\mathcal{S}}$ or $v_C^{\mathcal{S}} = v_D^{\mathcal{S}}$, i.e., $v_C^{\mathcal{S}} \subseteq v_D^{\mathcal{S}}$. So by definition of $\mathcal{I}_{\mathcal{S}}$, we get that $C^{\mathcal{I}_{\mathcal{S}}} \subseteq D^{\mathcal{I}_{\mathcal{S}}}$. From Table 1, this means that $\mathcal{I}_{\mathcal{S}} \models \Phi$.
- Φ is of the form $C \sqcap D \sqsubseteq \perp$. Since $\mathcal{S}$ is a solution of $\tau_\triangleright(O)$, we have that $\mathcal{S} \models \tau_\triangleright(C \sqcap D \sqsubseteq \perp)$. Yet we know that $\tau_\triangleright(C \sqcap D \sqsubseteq \perp) = v_C\{DR\}v_D$ (cf. Definition 1), so we have that $\mathcal{S}_{\mathcal{I}} \models v_C\{DR\}v_D$. Thus from Table 2, we get that $v_C^{\mathcal{S}_{\mathcal{I}}} \cap v_D^{\mathcal{S}_{\mathcal{I}}} \subseteq \emptyset$, i.e., $v_C^{\mathcal{S}_{\mathcal{I}}} \cap v_D^{\mathcal{S}_{\mathcal{I}}} = \emptyset$. So by definition of $\mathcal{I}_{\mathcal{S}}$, we get that $C^{\mathcal{I}} \cap D^{\mathcal{I}} \subseteq \emptyset$. From Table 1, this means that $\mathcal{I}_{\mathcal{S}} \models \Phi$.

This concludes the proof. □

Theorem 3. Let $\mathcal{N}$ be a scenario and $\mathcal{S}$ be solution of $\mathcal{N}$. Then there is an inflation $\mathcal{I}_{\mathcal{S}}$ of $\mathcal{S}$ s.t. $\mathcal{I}_{\mathcal{S}}$ is a model of $\tau_\triangleleft(\mathcal{N})$.

PROOF. Let $\mathcal{N} = \langle V, \Psi \rangle$ be a scenario. Let us denote by $\tau_\triangleleft(\mathcal{N}) = \langle \mathcal{A}, \mathcal{T} \rangle$ the backward translation of $\mathcal{N}$. Let $\mathcal{S} = (\mathcal{D}^{\mathcal{S}}, \cdot^{\mathcal{S}})$ be a solution of $\mathcal{N}$. Note that the set of concept names N_C of $\tau_\triangleleft(\mathcal{N})$ is formed of two parts: the concept names C which are directly associated with a variable v_C from V, and the new concept names introduced in the backward translation (cf. Definition 4). For instance, some new concept names A', C' and D' appear in the translation of a constraint $v_C\{PO\}v_D$. Let us denote by N_C^* the subset of N_C formed by these new concept names. In addition, note that the set of individual names N_I of $\tau_\triangleleft(\mathcal{N})$ is formed only of individual names artificially introduced in the translation; for instance, the individual names a, c and d appear in the translation of a constraint $v_C\{PO\}v_D$, and all these individual names form the set N_I. Now, let $\mathcal{I}_{\mathcal{S}} = (\Delta^{\mathcal{I}_{\mathcal{S}}}, \cdot^{\mathcal{I}_{\mathcal{S}}})$ be an interpretation defined as $\Delta^{\mathcal{I}_{\mathcal{S}}} = \mathcal{D}^{\mathcal{S}}$, and for every $A \in N_C \setminus N_C^*$ as $A^{\mathcal{I}_{\mathcal{S}}} = (v_A)^{\mathcal{S}}$. Note that at this point, $\mathcal{I}_{\mathcal{S}}$ is only partially defined, so one needs to complete its definition. Indeed, one needs to define $(A')^{\mathcal{I}_{\mathcal{S}}}$ for each $A' \in N_C^*$ and to define $\cdot^{\mathcal{I}_{\mathcal{S}}}(a)$ and for each $a \in N_I$. For this purpose, let us consider only the forms of constraints that actually introduce new concept names and new individual names, i.e., the elements of N_C^* and N_I. These constraints are of the form v_C $\{PO\}$ v_D and of the form v_C $\{PP\}$ v_D (the constraints of the form v_C $\{PPi\}$ v_D can be dealt with similarly as to the case of the constraints of the form v_C $\{PP\}$ v_D, since v_C $\{PPi\}$ v_D is equivalent to v_D $\{PP\}$ v_C.) So let φ be a constraint of the form v_C $\{PO\}$ v_D or the form v_C $\{PP\}$ v_D. We consider the two cases separately:

- φ is a constraint of the form v_C $\{PO\}$ v_D. Note that since the concept names C and D are members of $N_C \setminus N_C^*$, $C^{\mathcal{I}_{\mathcal{S}}}$ and $D^{\mathcal{I}_{\mathcal{S}}}$ are already defined as $C^{\mathcal{I}_{\mathcal{S}}} = v_C^{\mathcal{S}}$ and $D^{\mathcal{I}_{\mathcal{S}}} = v_D^{\mathcal{S}}$, where $v_C^{\mathcal{S}}$ and $v_D^{\mathcal{S}}$ are non-empty subsets of $\Delta^{\mathcal{I}_{\mathcal{S}}}$. Additionally, since $\mathcal{S} \models v_C$ $\{PO\}$ v_D, according to Table 2 we have that $v_C^{\mathcal{S}} \cap v_D^{\mathcal{S}} \neq \emptyset$, $v_C^{\mathcal{S}} \not\subseteq v_D^{\mathcal{S}}$ and $v_D^{\mathcal{S}} \not\subseteq v_C^{\mathcal{S}}$. Now, let A', C' and D' be the concept names newly introduced in the translation of v_C $\{PO\}$ v_D according to Definition 4. We have that A', C', $D' \in N_C^*$. Let us define $(A')^{\mathcal{I}_{\mathcal{S}}}$, $(C')^{\mathcal{I}_{\mathcal{S}}}$, and $(D')^{\mathcal{I}_{\mathcal{S}}}$ as follows:

- let $(A')^{\mathcal{I}_\mathcal{S}} = v^\mathcal{S}_{A'}$, where $v^\mathcal{S}_{A'}$ is any non-empty subset of $\Delta^{\mathcal{I}_\mathcal{S}}$ such that $v^\mathcal{S}_{A'} \subseteq v^\mathcal{S}_C \cap v^\mathcal{S}_D$ (this assignment is realizable since $v^\mathcal{S}_C \cap v^\mathcal{S}_D \neq \emptyset$);
- let $(C')^{\mathcal{I}_\mathcal{S}} = (v_{C'})^\mathcal{S}$, where $v^\mathcal{S}_{C'}$ is any non-empty subset of $\Delta^{\mathcal{I}_\mathcal{S}}$ such that $v^\mathcal{S}_{C'} \subseteq v^\mathcal{S}_C$ and $v^\mathcal{S}_{C'} \cap v^\mathcal{S}_D = \emptyset$ (this assignment is realizable since $v^\mathcal{S}_C \not\subseteq v^\mathcal{S}_D$);
- let $(D')^{\mathcal{I}_\mathcal{S}} = v^\mathcal{S}_{D'}$, where $v^\mathcal{S}_{D'}$ is any non-empty subset of $\Delta^{\mathcal{I}_\mathcal{S}}$ such that $v^\mathcal{S}_{D'} \subseteq v^\mathcal{S}_D$ and $v^\mathcal{S}_{D'} \cap v^\mathcal{S}_C = \emptyset$ (this assignment is realizable since $v^\mathcal{S}_D \not\subseteq v^\mathcal{S}_C$).

Similarly, let a, c, d be the individual names newly introduced in that translation. We have that a, c, $d \in N_I$. Let us define $\cdot^{\mathcal{I}_\mathcal{S}}(a)$ (respectively, $\cdot^{\mathcal{I}_\mathcal{S}}(c)$, $\cdot^{\mathcal{I}_\mathcal{S}}(d)$) as any element of $v^\mathcal{S}_{A'}$ (respectively, of $v^\mathcal{S}_{C'}$, of $v^\mathcal{S}_{D'}$).

- φ is a constraint of the form $v_C\ \{PP\}\ v_D$. Note that since the concept names C and D are members of $N_C \setminus N_C^*$, $C^{\mathcal{I}_\mathcal{S}}$ and $D^{\mathcal{I}_\mathcal{S}}$ are already defined as $C^{\mathcal{I}_\mathcal{S}} = v^\mathcal{S}_C$ and $C^{\mathcal{I}_\mathcal{S}} = v^\mathcal{S}_D$, where $v^\mathcal{S}_C$ and $v^\mathcal{S}_D$ are non-empty subsets of $\Delta^{\mathcal{I}_\mathcal{S}}$. Additionally, since $\mathcal{S} \models v_C\ \{PP\}\ v_D$, according to Table 2 we have that $v^\mathcal{S}_C \subset v^\mathcal{S}_D$. Now, let D' be the concept name newly introduced in the translation of $v_C\ \{PP\}\ v_D$ according to Definition 4. We have that $D' \in N_C^*$ and c, $d \in N_I$. Let us define $(D')^{\mathcal{I}_\mathcal{S}}$ as $(D')^{\mathcal{I}_\mathcal{S}} = v^\mathcal{S}_{D'}$, where $v^\mathcal{S}_{D'}$ is any non-empty subset of $\Delta^{\mathcal{I}_\mathcal{S}}$ such that $v^\mathcal{S}_{D'} \subseteq v^\mathcal{S}_D$ and $v^\mathcal{S}_{D'} \cap v^\mathcal{S}_C = \emptyset$ (this assignment is realizable since $v^\mathcal{S}_C \subset v^\mathcal{S}_D$). Similarly, let c, d be the individual names newly introduced in that translation. We have that c, $d \in N_I$. Let us define $\cdot^{\mathcal{I}_\mathcal{S}}(c)$ (respectively, $\cdot^{\mathcal{I}_\mathcal{S}}(d)$) as any element of $v^\mathcal{S}_C$ (respectively, of $v^\mathcal{S}_{D'}$).

At this point, $\mathcal{I}_\mathcal{S}$ is completely defined: we have defined $A^{\mathcal{I}_\mathcal{S}}$ for each $A \in N_C$, and we have defined $\cdot^\mathcal{S}(a)$ for each $a \in N_I$. First, it can easily be verified that $\mathcal{I}_\mathcal{S}$ is an inflation of $\mathcal{S}$, according to Definition 3. Indeed, we have that $\Delta^{\mathcal{I}_\mathcal{S}} = \mathcal{D}^\mathcal{S}$, and for every $A \in N_C$ that $A^{\mathcal{I}_\mathcal{S}} = (v_A)^\mathcal{S}$. We now intend to show that $\mathcal{I}_\mathcal{S}$ is a model of $\tau_\triangleleft(\mathcal{N})$. For this purpose, let us consider each contraint $v_C\ \varphi\ v_D \in \Psi$, and show that $\mathcal{I}_\mathcal{S}$ is such that $\mathcal{I}_\mathcal{S} \models \tau_\triangleleft(v_C\ \varphi\ v_D)$. We fall into one of the following cases:

- φ is a constraint of the form $v_C\ \{EQ\}\ v_D$. Since $\mathcal{S}$ is a solution of $\mathcal{N}$, we have that $\mathcal{S} \models v_C\ \{EQ\}\ v_D$. Thus $v^\mathcal{S}_C = v^\mathcal{S}_D$ (cf. Table 2). So by definition of $\mathcal{I}_\mathcal{S}$, we get that $C^{\mathcal{I}_\mathcal{S}} = D^{\mathcal{I}_\mathcal{S}}$. Thus, from Table 1, we get that $\mathcal{I}_\mathcal{S} \models C \equiv D$. Yet, from Definition 4, we know that $\tau_\triangleleft(v_C\ \{EQ\}\ v_D) = \langle \{C \equiv D\}, \emptyset\rangle$. Hence, $\mathcal{I}_\mathcal{S} \models \tau_\triangleleft(v_C\ \{EQ\}\ v_D)$.
- φ is a constraint of the form $v_C\ \{DR\}\ v_D$. Since $\mathcal{S}$ is a solution of $\mathcal{N}$, we have that $\mathcal{S} \models v_C\ \{DR\}\ v_D$. Thus $v^\mathcal{S}_C \cap v^\mathcal{S}_D = \emptyset$ (cf. Table 2). So by Definition of $\mathcal{I}_\mathcal{S}$, we get that $C^{\mathcal{I}_\mathcal{S}} \cap D^{\mathcal{I}_\mathcal{S}} = \emptyset$, i.e, $C^{\mathcal{I}_\mathcal{S}} \cap D^{\mathcal{I}_\mathcal{S}} \subseteq \emptyset$. Thus, from Table 1, we get that $\mathcal{I}_\mathcal{S} \models C \sqcap D \sqsubseteq \bot$. Yet, from Definition 4, we know that $\tau_\triangleleft(v_C\ \{DR\}\ v_D) = \langle \{C \sqcap D \sqsubseteq \bot\}, \emptyset\rangle$. Hence, $\mathcal{I}_\mathcal{S} \models \tau_\triangleleft(v_C\ \{DR\}\ v_D)$.
- φ is a constraint of the form $v_C\ \{PO\}\ v_D$. Since $\mathcal{S}$ is a solution of $\mathcal{N}$, we have that $\mathcal{S} \models v_C\ \{PO\}\ v_D$. And according to the way we previously defined $(A')^{\mathcal{I}_\mathcal{S}}$, $(C')^{\mathcal{I}_\mathcal{S}}$, and $(D')^{\mathcal{I}_\mathcal{S}}$, we have that $v^\mathcal{S}_{A'} \subseteq v^\mathcal{S}_C \cap v^\mathcal{S}_D$, $v^\mathcal{S}_{C'} \subseteq v^\mathcal{S}_C$, $v^\mathcal{S}_{C'} \cap v^\mathcal{S}_D = \emptyset$, $v^\mathcal{S}_{D'} \subseteq v^\mathcal{S}_D$, and $v^\mathcal{S}_{D'} \cap v^\mathcal{S}_C = \emptyset$. Hence, from Table 1, we get that:
  - $\mathcal{I}_\mathcal{S} \models A' \sqsubseteq C \sqcap D$ (since $v^\mathcal{S}_{A'} \subseteq v^\mathcal{S}_C \cap v^\mathcal{S}_D$);
  - $\mathcal{I}_\mathcal{S} \models C' \sqsubseteq C$ (since $v^\mathcal{S}_{C'} \subseteq v^\mathcal{S}_C$);
  - $\mathcal{I}_\mathcal{S} \models C' \sqcap D \sqsubseteq \bot$ (since $v^\mathcal{S}_{C'} \cap v^\mathcal{S}_D = \emptyset$);
  - $\mathcal{I}_\mathcal{S} \models D' \sqsubseteq D$ (since $v^\mathcal{S}_{D'} \subseteq v^\mathcal{S}_D$);
  - $\mathcal{I}_\mathcal{S} \models D' \sqcap C \sqsubseteq \bot$ (since $v^\mathcal{S}_{D'} \cap v^\mathcal{S}_C = \emptyset$).

  In addition, according to the way we previously defined $\cdot^{\mathcal{I}_\mathcal{S}}(a)$, $\cdot^{\mathcal{I}_\mathcal{S}}(c)$, $\cdot^{\mathcal{I}_\mathcal{S}}(d)$, we know that $\cdot^{\mathcal{I}_\mathcal{S}}(a) \in v^\mathcal{S}_{A'}$, $\cdot^{\mathcal{I}_\mathcal{S}}(c) \in v^\mathcal{S}_{C'}$, and $\cdot^{\mathcal{I}_\mathcal{S}}(d) \in v^\mathcal{S}_{D'}$. Hence, we get that:
  - $\mathcal{I}_\mathcal{S} \models A'(a)$ (since $\cdot^{\mathcal{I}_\mathcal{S}}(a) \in v^\mathcal{S}_{A'}$);
  - $\mathcal{I}_\mathcal{S} \models C(c)$ (since $\cdot^{\mathcal{I}_\mathcal{S}}(c) \in v^\mathcal{S}_{C'}$ and $v^\mathcal{S}_{C'} \subseteq v^\mathcal{S}_C$);
  - $\mathcal{I}_\mathcal{S} \models C(a)$ (since $\cdot^{\mathcal{I}_\mathcal{S}}(a) \in v^\mathcal{S}_{A'}$ and $v^\mathcal{S}_{A'} \subseteq v^\mathcal{S}_C$);
  - $\mathcal{I}_\mathcal{S} \models D(d)$ (since $\cdot^{\mathcal{I}_\mathcal{S}}(d) \in v^\mathcal{S}_{D'}$ and $v^\mathcal{S}_{D'} \subseteq v^\mathcal{S}_D$);
  - $\mathcal{I}_\mathcal{S} \models D(a)$ (since $\cdot^{\mathcal{I}_\mathcal{S}}(a) \in v^\mathcal{S}_{A'}$ and $v^\mathcal{S}_{A'} \subseteq v^\mathcal{S}_D$);
  - $\mathcal{I}_\mathcal{S} \models C'(c)$ (since $\cdot^{\mathcal{I}_\mathcal{S}}(c) \in v^\mathcal{S}_{C'}$);
  - $\mathcal{I}_\mathcal{S} \models D'(d)$ (since $\cdot^{\mathcal{I}_\mathcal{S}}(d) \in v^\mathcal{S}_{D'}$).

  Overall, we got that $\mathcal{I}_\mathcal{S} \models \langle\{A' \sqsubseteq C \sqcap D, C' \sqsubseteq C, C' \sqcap D \sqsubseteq \bot, D' \sqsubseteq D, D' \sqcap C \sqsubseteq \bot\}, \{A'(a), C(c), C(a), D(d), D(a), C'(c), D'(d)\}\rangle$. Hence, $\mathcal{I}_\mathcal{S} \models \tau_\triangleleft(v_C\ \{PO\}\ v_D)$.
- φ is a constraint of the form $v_C\ \{PP, EQ\}\ v_D$. Since $\mathcal{S}$ is a solution of $\mathcal{N}$, we have that $\mathcal{S} \models v_C\ \{PP, EQ\}\ v_D$. Thus $v^\mathcal{S}_C \subseteq v^\mathcal{S}_D$ (cf. Table 2). So by definition of $\mathcal{I}_\mathcal{S}$, we get that $C^{\mathcal{I}_\mathcal{S}} \subseteq D^{\mathcal{I}_\mathcal{S}}$. Thus, from Table 1, we get that $\mathcal{I}_\mathcal{S} \models C \sqsubseteq D$. Yet, from Definition 4, we know that $\tau_\triangleleft(v_C\ \{PP, EQ\}\ v_D) = \langle\{C \sqsubseteq D\}, \emptyset\rangle$. Hence, $\mathcal{I}_\mathcal{S} \models \tau_\triangleleft(v_C\ \{PP, EQ\}\ v_D)$.
- φ is a constraint of the form $v_C\ \{PPi, EQ\}\ v_D$. The proof that $\mathcal{I}_\mathcal{S} \models \tau_\triangleleft(v_C\ \{PPi, EQ\}\ v_D)$ can be reduced equivalently to the proof that $\mathcal{I}_\mathcal{S} \models \tau_\triangleleft(v_D\ \{PP, EQ\}\ v_C)$ similarly to the previous case, since $v_C\ \{PPi, EQ\}\ v_D$ is equivalent to $v_D\ \{PP, EQ\}\ v_C$.
- φ is a constraint of the form $v_C\ \{PP\}\ v_D$. Since $\mathcal{S}$ is a solution of $\mathcal{N}$, we have that $\mathcal{S} \models v_C\ \{PP\}\ v_D$. Thus $v^\mathcal{S}_C \subset v^\mathcal{S}_D$ (cf. Table 2). And according to the way we previously defined $(D')^{\mathcal{I}_\mathcal{S}}$, we have that $v^\mathcal{S}_{D'} \subseteq v^\mathcal{S}_D$ and $v^\mathcal{S}_{D'} \cap v^\mathcal{S}_C = \emptyset$. Hence, from Table 1, we get that:
  - $\mathcal{I}_\mathcal{S} \models C \sqsubseteq D$ (since $v^\mathcal{S}_C \subset v^\mathcal{S}_D$);
  - $\mathcal{I}_\mathcal{S} \models D' \sqsubseteq D$ (since $v^\mathcal{S}_{D'} \subseteq v^\mathcal{S}_D$);
  - $\mathcal{I}_\mathcal{S} \models C \sqcap D' \sqsubseteq \bot$ (since $v^\mathcal{S}_{D'} \cap v^\mathcal{S}_C = \emptyset$).

  In addition, according to the way we previously defined $\cdot^{\mathcal{I}_\mathcal{S}}(c)$ and $\cdot^{\mathcal{I}_\mathcal{S}}(d)$, we know that $\cdot^{\mathcal{I}_\mathcal{S}}(c) \in v^\mathcal{S}_C$, and $\cdot^{\mathcal{I}_\mathcal{S}}(d) \in v^\mathcal{S}_{D'}$. Hence, we get that:
  - $\mathcal{I}_\mathcal{S} \models D'(d)$ (since $\cdot^{\mathcal{I}_\mathcal{S}}(d) \in v^\mathcal{S}_{D'}$);
  - $\mathcal{I}_\mathcal{S} \models C(c)$ (since $\cdot^{\mathcal{I}_\mathcal{S}}(c) \in v^\mathcal{S}_C$);
  - $\mathcal{I}_\mathcal{S} \models D(d)$ (since $\cdot^{\mathcal{I}_\mathcal{S}}(d) \in v^\mathcal{S}_{D'}$ and $v^\mathcal{S}_{D'} \subseteq v^\mathcal{S}_D$);
  - $\mathcal{I}_\mathcal{S} \models D(c)$ (since $\cdot^{\mathcal{I}_\mathcal{S}}(c) \in v^\mathcal{S}_C$ and $v^\mathcal{S}_C \subset v^\mathcal{S}_D$).

  Overall, we got that $\mathcal{I}_\mathcal{S} \models \langle\{C \sqsubseteq D, D' \sqsubseteq D, C \sqcap D' \sqsubseteq \bot\}, \{D'(d), C(c), D(d), D(c)\}\rangle$. Hence, $\mathcal{I}_\mathcal{S} \models \tau_\triangleleft(v_C\ \{PP\}\ v_D)$.
- φ is a constraint of the form $v_C\ \{PPi\}\ v_D$. The proof that $\mathcal{I}_\mathcal{S} \models \tau_\triangleleft(v_C\ \{PPi\}\ v_D)$ can be reduced equivalently to the proof that $\mathcal{I}_\mathcal{S} \models \tau_\triangleleft(v_D\ \{PP\}\ v_C)$ similarly to the previous case, since $v_C\ \{PPi\}\ v_D$ is equivalent to $v_D\ \{PP\}\ v_C$.

We have proved that for each contraint $v_C\ \varphi\ v_D \in \Psi$, we have that $\mathcal{I}_\mathcal{S} \models \tau_\triangleleft(v_C\ \varphi\ v_D)$. Therefore, $\mathcal{I}_\mathcal{S} \models \tau_\triangleleft(\mathcal{N})$. This concludes the proof. □

Theorem 4. Let $\mathcal{N}$ be a scenario and let $\mathcal{I}$ be a fulfilling interpretation of $\tau_\triangleleft(\mathcal{N})$ such that $\mathcal{I}$ is a model of $\tau_\triangleleft(\mathcal{N})$. Then $\mathcal{S}_\mathcal{I} \models \mathcal{N}$.

PROOF. Let $\mathcal{N} = \langle V, \Psi\rangle$ be a scenario and let $\mathcal{I} = \langle \Delta^\mathcal{I}, \cdot^\mathcal{I}\rangle$ be a fulfilling interpretation of $\tau_\triangleleft(\mathcal{N})$ such that $\mathcal{I} \models \tau_\triangleleft(\mathcal{N})$. Let us denote by $\tau_\triangleleft(\mathcal{N}) = \langle \mathcal{A}, \mathcal{T}\rangle$ the backward translation of $\mathcal{N}$. Note that the set of concept names N_C of $\tau_\triangleleft(\mathcal{N})$ is formed of two parts: (1) the concept names C which are directly associated with a variable v_C from V and (2) the new concept names introduced in the backward translation (cf. Definition 4). For instance, some new concepts A', C', and D' appear in the translation of a constraint $v_C\ \{PO\}\ v_D$. Let us denote by N_C^* the subset of N_C formed by these new concept names. In addition, the set of N_I of $\tau_\triangleleft(\mathcal{N})$ is formed only of individual names artificially introduced in the translation; for instance, the individual a, c, and d appear in the translation of a constraint $v_C\ \{PO\}\ v_D$, and all these individual names of N_I.

From Definition 2, we have $\mathcal{S}_\mathcal{I} = \langle \Delta^\mathcal{I}, \cdot^{\mathcal{S}_\mathcal{I}}\rangle$ that is a solution of $\mathcal{N}$, for every $(v_A)^{\mathcal{S}_\mathcal{I}} = A^\mathcal{I}$ where $A \in N_C \setminus N_C^*$ and $v_A \in V$ and $\cdot^{\mathcal{S}_\mathcal{I}}(a)$ for each $a \in N_I$. It can easily be verified that $\mathcal{S}_\mathcal{I}$ is the flattening of $\mathcal{I}$ (cf. Definition 2). Now we need to show that $\mathcal{S}_\mathcal{I} \models \mathcal{N}$. For this purpose, let us consider each constraint $v_C\ \varphi\ v_D$, and show that $\mathcal{I} \models \tau_\triangleleft(v_C\ \varphi\ v_D)$ then $\mathcal{S}_\mathcal{I} \models v_C\ \varphi\ v_D$. We fall into one of the following cases:

- φ is a constraint of the form $v_C\ \{EQ\}\ v_D$. Since $\mathcal{I} \models \tau_\triangleleft(\mathcal{N})$, we have that $\mathcal{I} \models \tau_\triangleleft\{v_C\ \{EQ\}\ v_D\}$. From Definition 4, we get that $\tau_\triangleleft(v_C\ \{EQ\}\ v_D) = \langle C \equiv D, \emptyset\rangle$. Thus, we have $\mathcal{I} \models C \equiv D$, it is equivalent to $C^\mathcal{I} \equiv D^\mathcal{I}$ (cf. Table 1). So by definition of $\mathcal{S}_\mathcal{I}$, we get that $(v_C)^{\mathcal{S}_\mathcal{I}} = (v_D)^{\mathcal{S}_\mathcal{I}}$. Thus, from Table 2, we get that $\mathcal{S}_\mathcal{I} \models v_C\ \{EQ\}\ v_D$.
- φ is a constraint of the form $v_C\ \{DR\}\ v_D$. Since $\mathcal{I} \models \tau_\triangleleft(\mathcal{N})$, we have that $\mathcal{I} \models \tau_\triangleleft\{v_C\ \{DR\}\ v_D\}$. From Definition 4, we get that $\tau_\triangleleft(v_C\ \{DR\}\ v_D) = \langle C \sqcap D \sqsubseteq \bot, \emptyset\rangle$. Thus we have $\mathcal{I} \models C \sqcap D \sqsubseteq \bot$, it is equivalent to $C^\mathcal{I} \cap D^\mathcal{I} \subseteq \emptyset$ (cf. Table 1). So by definition of $\mathcal{S}_\mathcal{I}$, we get that $(v_C)^{\mathcal{S}_\mathcal{I}} \cap (v_D)^{\mathcal{S}_\mathcal{I}} = \emptyset$. Thus, from Table 2, we get that $\mathcal{S}_\mathcal{I} \models v_C\ \{DR\}\ v_D$.
- φ is a constraint of the form $v_C\ \{PO\}\ v_D$. Since $\mathcal{I} \models \tau_\triangleleft(\mathcal{N})$, we have that $\mathcal{I} \models \tau_\triangleleft\{v_C\ \{PO\}\ v_D\}$. From Definition 4, we get that $\tau_\triangleleft(v_C\ \{PO\}\ v_D) = \langle\{A' \sqsubseteq C \sqcap D, C' \sqsubseteq C, C' \sqcap D \sqsubseteq \bot, D' \sqsubseteq D, D' \sqcap C \sqsubseteq \bot\}, \{A'(a), C'(c), D'(d), C(c), C(a), D(d), D(a)\}\rangle$. Thus, we have $\mathcal{I} \models \{A' \sqsubseteq C \sqcap D, C' \sqsubseteq C, C' \sqcap D \sqsubseteq \bot, D' \sqsubseteq D, D' \sqcap C \sqsubseteq \bot\}$ and $\mathcal{I} \models \{A'(a), C'(c), D'(d), C(c), C(a), D(d), D(a)\}$. From Table 1, we get that $(A')^\mathcal{I} \subseteq C^\mathcal{I} \cap D^\mathcal{I}$, $(C')^\mathcal{I} \subseteq C^\mathcal{I}, (C')^\mathcal{I} \cap D^\mathcal{I} \subseteq \emptyset$, and $(D')^\mathcal{I} \subseteq D^\mathcal{I}, (D')^\mathcal{I} \cap C^\mathcal{I} \subseteq \emptyset$. Now, we need to consider the following cases:
  - Since $\mathcal{I} \models \mathcal{A}'(a)$, we get that $a^\mathcal{I} \in (A')^\mathcal{I}$. Moreover, $(A')^\mathcal{I} \subseteq C^\mathcal{I} \cap D^\mathcal{I}$. Thus $a^\mathcal{I} \in C^\mathcal{I} \cap D^\mathcal{I}$. Then we get that $C^\mathcal{I} \cap D^\mathcal{I} \neq \emptyset$.
  - Since $\mathcal{I} \models \{C'(c), C(c)\}$, we get that $c^\mathcal{I} \in (C')^\mathcal{I}$ and $c^\mathcal{I} \in C^\mathcal{I}$. Moreover, $(C')^\mathcal{I} \subseteq C^\mathcal{I}, (C')^\mathcal{I} \cap D^\mathcal{I} \subseteq \emptyset$. Thus $c^\mathcal{I} \in C^\mathcal{I}$ but $c^\mathcal{I} \notin D^\mathcal{I}$. Then, we get that $C^\mathcal{I} \not\subseteq D^\mathcal{I}$.
  - Since $\mathcal{I} \models \{D'(d), D(d)\}$, we get that $d^\mathcal{I} \in (D')^\mathcal{I}$ and $d^\mathcal{I} \in D^\mathcal{I}$. Moreover, $(D')^\mathcal{I} \subseteq D^\mathcal{I}, (D')^\mathcal{I} \cap C^\mathcal{I} \subseteq \emptyset$. Thus $d^\mathcal{I} \in D^\mathcal{I}$ but $d^\mathcal{I} \notin C^\mathcal{I}$. Then, we get that $D^\mathcal{I} \not\subseteq C^\mathcal{I}$.

  From the above cases and by definition of $\mathcal{S}_\mathcal{I}$, we get that $(v_C)^{\mathcal{S}_\mathcal{I}} \cap (v_D)^{\mathcal{S}_\mathcal{I}} \neq \emptyset$, $(v_C)^{\mathcal{S}_\mathcal{I}} \not\subseteq (v_D)^{\mathcal{S}_\mathcal{I}}$, and $(v_D)^{\mathcal{S}_\mathcal{I}} \not\subseteq (v_C)^{\mathcal{S}_\mathcal{I}}$. In other words, we also have $\mathcal{S}_\mathcal{I} \models \{v_{A'} \subseteq v_C \cap v_D, v_C \not\subseteq v_D, v_D \not\subseteq v_C\}$. Thus, from Table 2, we have $\mathcal{S}_\mathcal{I} \models v_C\ \{PO\}\ v_D$.
- φ is a constraint of the form $v_C\ \{PP, EQ\}\ v_D$. Since $\mathcal{I} \models \tau_\triangleleft(\mathcal{N})$, we have that $\mathcal{I} \models \tau_\triangleleft\{v_C\ \{PP, EQ\}\ v_D\}$. From Definition 4, we get that $\tau_\triangleleft(v_C\ \{PP, EQ\}\ v_D) = \langle C \sqsubseteq D, \emptyset\rangle$. Thus, we have $\mathcal{I} \models C \sqsubseteq D$, it is equivalent to $C^\mathcal{I} \subseteq D^\mathcal{I}$ (cf. Table 1). So by definition of $\mathcal{S}_\mathcal{I}$, we get that $(v_C)^{\mathcal{S}_\mathcal{I}} \subseteq (v_D)^{\mathcal{S}_\mathcal{I}}$. Moreover, we also have that $(v_C)^{\mathcal{S}_\mathcal{I}} \subset (v_D)^{\mathcal{S}_\mathcal{I}}$ or $(v_C)^{\mathcal{S}_\mathcal{I}} = (v_D)^{\mathcal{S}_\mathcal{I}}$. Thus, from Table 2, we get that $\mathcal{S}_\mathcal{I} \models \{v_C\ \{PP\}\ v_D$ or $v_C\ \{EQ\}\ v_D\}$. It is also equivalent to $\mathcal{S}_\mathcal{I} \models v_C\ \{PP, EQ\}\ v_D$.
- φ is a constraint of the form $v_C\ \{PPi, EQ\}\ v_D$. The proof that $\mathcal{S}_\mathcal{I} \models v_C\ \{PPi, EQ\}\ v_D$ can be reduced equivalently to the proof that $\mathcal{S}_\mathcal{I} \models v_D\ \{PP, EQ\}\ v_C$ similarly to the previous case, since $v_C\ \{PP, EQ\}\ v_D$ is equivalent to $v_D\ \{PPi, EQ\}\ v_C$.
- φ is a constraint of the form $v_C\{PP\}v_D$. Since $\mathcal{I} \models \tau_\triangleleft(\mathcal{N})$, we have that $\mathcal{I} \models \tau_\triangleleft\{v_C\ \{PP\}\ v_D\}$. From Definition 4, we get that $\tau_\triangleleft(v_C\{PP\}v_D) = \langle\{C \sqsubseteq D, D' \sqsubseteq D, C \sqcap D' \sqsubseteq \bot\}, \{D'(d), C(c), D(d), D(c)\}\rangle$. Thus, we have $\mathcal{I} \models \{C \sqsubseteq D, D' \sqsubseteq D, C \sqcap D' \sqsubseteq \bot\}$ and $\mathcal{I} \models \{D'(d), C(c), D(d), D(c)\}$. From Table 1, we get that $C^\mathcal{I} \subseteq D^\mathcal{I}, (D')^\mathcal{I} \subseteq D^\mathcal{I}, C^\mathcal{I} \cap (D')^\mathcal{I} \subseteq \emptyset$. Moreover, since $\mathcal{I} \models \{D'(d), D(d)\}$, we have $d^\mathcal{I} \in (D')^\mathcal{I}, d^\mathcal{I} \in D^\mathcal{I}$. Then, $d^\mathcal{I} \notin C^\mathcal{I}$ (by $C^\mathcal{I} \cap (D')^\mathcal{I} \subseteq \emptyset$). Hence, $C^\mathcal{I} \not\subset D^\mathcal{I}$. Now, by definition of $\mathcal{S}_\mathcal{I}$, we get that $(v_C)^{\mathcal{S}_\mathcal{I}} \subset (v_D)^{\mathcal{S}_\mathcal{I}}$. In other words, we also have $\mathcal{S}_\mathcal{I} \models v_C \subset v_D$ (shown in the above part). Thus, from Table 2, we have $\mathcal{S}_\mathcal{I} \models v_C\ \{PP\}\ v_D$.
- φ is a constraint of the form $v_C\ \{PPi\}\ v_D$. The proof that $\mathcal{S}_\mathcal{I} \models v_C\ \{PPi\}\ v_D$ can be reduced equivalently to the proof that $\mathcal{S}_\mathcal{I} \models v_D\{PP\}v_C$ similarly to the previous case, since $v_C\ \{PP\}\ v_D$ is equivalent to $v_D\ \{PPi\}\ v_C$.

This concludes the proof. □