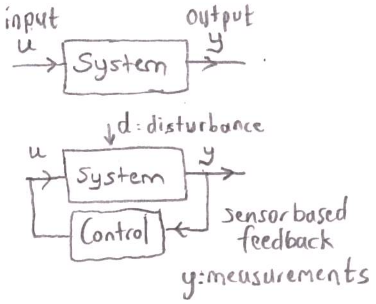
Control Bootcamp: Overview

Optimal & Modern Control

- Passive Control
- Active Control
  - Open Loop
  - Closed Loop (feedback)
    - • sensors



$$\dot{x} = Ax$$
$$x(t) = e^{At}x(0)$$

$$\dot{x} = Ax + B\overset{\text{actuator}}{u}$$
$$y = Cx$$

$$u = -Kx \quad \rightarrow \text{Assume } y = x$$

$$\dot{x} = Ax - BKx$$
$$\dot{x} = (A - BK)x$$

* Why feedback?

$\boxed{\begin{array}{l} 1.\ \text{Uncertainty} \\ 2.\ \text{Instability} \\ 3.\ \text{Disturbances} \\ 4.\ \text{Efficient} \end{array}}$

Linear Systems

$$\dot{x} = Ax + Bu$$

$$\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x} \qquad x \in \mathbb{R}^n$$
$$x(t) = e^{At}x(0)$$

$$\underbrace{e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots}_{\text{• not easy to compute}}$$

Coordinate transformation
Eigenvalues & Eigenvectors

$$\boxed{AT = TD}$$

$$\underline{\underline{A}}\,\underline{\xi} = \lambda\,\underline{\xi} \qquad \xi\text{: Eigenvectors}$$

$$T = \begin{bmatrix} \xi_1 & \xi_2 & \xi_3 & \dots & \xi_n \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix}$$

$\dot{x} = T\dot{z}$ ↳ (coordinate system in the eigenvector directions)

$$\dot{x} = T\dot{z} = Ax$$
$$T\dot{z} = ATz$$
$$\dot{z} = \underbrace{T^{-1}AT}_{D}z$$

(eigenvector coordinate) $\boxed{\dot{z} = Dz}$ • Components of z is uncoupled from each other

$$AT = TD$$
$$A = TDT^{-1}$$
$$e^{At} = I + At + \frac{A^2t^2}{2!} + \dots$$

$$e^{TDT^{-1}t} = \underset{\downarrow\ TT^{-1}}{I} + TDT^{-1}t + TD^2T^{-1}\frac{t^2}{2!} + \dots$$

$$e^{TDT^{-1}t} = T\Big[\underbrace{I + Dt + \frac{D^2t^2}{2!} + \frac{D^3t^3}{3!} + \dots}_{D'\dots}\Big]T^{-1} \Rightarrow \boxed{\underbrace{e^{At}}_{\text{not easy}} = T\underbrace{e^{Dt}}_{\text{easy to}}T^{-1}}$$

$$\underbrace{\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots\ \lambda_n \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}}_{\text{• easy to compute.}}$$

$$z(t) = e^{Dt}z(0) = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & & \ddots\ e^{\lambda_n t} \end{bmatrix} z(0)$$

$$x(t) = Te^{Dt}\underbrace{T^{-1}x(0)}_{z(0)} \Rightarrow \boxed{x(t) = Tz(t)}$$

Stability and Eigenvalues

$\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x} \qquad x \in \mathbb{R}^n$

$[T, D] = \mathrm{eig}(A);$

$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$

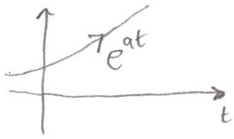
$x(t) = T e^{Dt} T^{-1} x(0)$

$e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}$

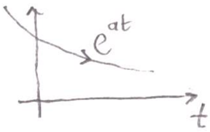
$\lambda = a \pm ib$ ← euler's formula

$e^{\pm\lambda t} = e^{at}\left[\cos(bt) \pm i \sin(bt)\right]$

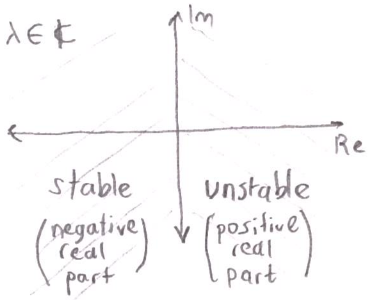
If $a > 0$



If $a < 0$

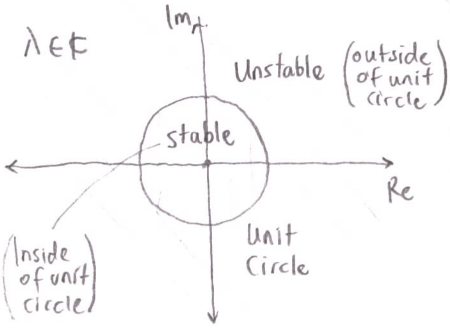


$\lambda \in \mathbb{C}$



Continuous Time

$\lambda \in \mathbb{C}$



Discrete Time

$x_{k+1} = \tilde{A} x_k \quad , \quad x_k = x(k\Delta t) \qquad \tilde{A} = e^{A\Delta t}$

$x_1 = \tilde{A} x_0 \qquad \lambda$

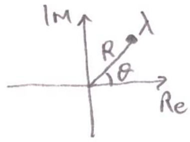
$x_2 = \tilde{A} x_1 = \tilde{A}^2 x_0 \qquad \lambda^2$

$\vdots$

$x_N = \tilde{A}^N x_0 \qquad \lambda^N$

$\tilde{A}\tilde{T} = \tilde{T}\tilde{D}$

$\tilde{A} = \tilde{T}\tilde{D}\tilde{T}^{-1}$



$\lambda = a \pm ib$

$\lambda = R e^{i\theta}$

$\lambda^N = R^N e^{iN\theta}$

If $R > 0$

as $N \to \infty$ unstable.

If $R < 0$

as $N \to \infty$ stable.

Linearizing Around a Fixed Point

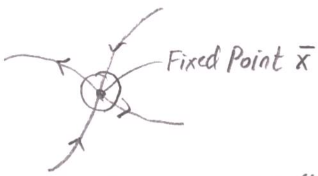
$\dot{x} = f(x) \Rightarrow \dot{x} = Ax$

1. Find fixed points

   $\bar{x}$ such that $f(\bar{x}) = 0$

2. Linearize about $\bar{x}$

   $\left.\frac{Df}{Dx}\right|_{\bar{x}} = \left[\frac{\partial f_i}{\partial x_j}\right]$



From "Engineering Math" Part 22

$$\frac{d}{dt}(\Delta x) = \frac{Df}{Dx}(\bar{x})\,\Delta x$$

$$\Delta\dot{x} = A\,\Delta x$$

- If x is really close to $\bar{x}$ fixed points then the dynamics look linear.
- Good control can keep the system close to linear region.

$\dot{x}_1 = f_1(x_1, x_2)$

$\dot{x}_2 = f_2(x_1, x_2)$

$\left.\frac{Df}{Dx}\right|_{\bar{x}} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\right|_{\bar{x}}$

Jacobian

Ex: $f_1 = x_1 x_2$

$f_2 = x_1^2 + x_2^2$

$\frac{Df}{Dx} = \begin{bmatrix} x_2 & x_1 \\ 2x_1 & 2x_2 \end{bmatrix}$

Hartman-Grobman theorem ☆

The linearization does not always characterize the dynamics around a non-linear system.

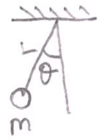
- If the eigenvalues of the linearization around the fixed point is hyperbolic meaning that all of the eigenvalues have a non-zero real part, then this linear system really does describe the neighbourhood well.

$\dot{x} = f(x)$ Taylor Series Expansion

$$\dot{x} = f(\bar{x}) + \left.\frac{Df}{Dx}\right|_{\bar{x}}(x - \bar{x}) + \underbrace{\left.\frac{D^2 f}{Dx^2}\right|_{\bar{x}}(x - \bar{x})^2 + \cdots}_{\text{truncated terms}}$$

- If there are purely imaginary eigenvalues, non linear terms in the dynamics that we truncated, could break the behaviour

Ex: Pendulum



$$\ddot{\theta} = \frac{-g}{L}\sin(\theta) \underbrace{-\delta\dot{\theta}}_{\text{damping}}$$

Assume that, $\frac{g}{L} = 1$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \qquad \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\sin(x_1) - \delta x_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

1. Fixed points

$$\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ \bar{x}\begin{bmatrix} \pi \\ 0 \end{bmatrix} \qquad f(\bar{x}) = 0 \qquad \begin{array}{l} x_2 = 0 \\ -\sin x = 0 \\ x = 0, \pi \end{array}$$

2. Linearize at fixed points

$$\frac{Df}{Dx} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\cos(x_1) & -\delta \end{bmatrix}$$

For $\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\frac{Df}{Dx} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & -\delta \end{bmatrix}}_{A_{down}}$$

For $\bar{x} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$

$$\frac{Df}{Dx} = \underbrace{\begin{bmatrix} 0 & 1 \\ +1 & -\delta \end{bmatrix}}_{A_{up}}$$

$$A_{down} = \begin{bmatrix} 0 & 1 \\ -1 & -\delta \end{bmatrix} \qquad A_{up} = \begin{bmatrix} 0 & 1 \\ 1 & -\delta \end{bmatrix}$$

Let $\delta = 0.1$

$$A_d = \begin{bmatrix} 0 & 1 \\ -1 & -0.1 \end{bmatrix} \qquad A_u = \begin{bmatrix} 0 & 1 \\ 1 & -0.1 \end{bmatrix}$$

Eigenvalues of A_d and A_u

$$eig(A_d) \quad \left.\begin{array}{l} -0.05 + 0.9987i \\ -0.05 - 0.9987i \end{array}\right\} \text{stable}$$

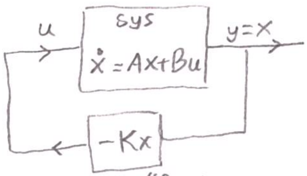
$$eig(A_u) \quad \left.\begin{array}{l} -1.0512 \\ 0.9512 \end{array}\right\} \begin{array}{l} \text{saddle point} \\ \text{unstable} \end{array}$$

Controllability

$$\dot{x} = Ax + Bu \qquad y = Cx$$

$x \in \mathbb{R}^n$ $\quad A \in \mathbb{R}^{n\times n}$ $\quad y \in \mathbb{R}^p$

$u \in \mathbb{R}^q$ $\quad B \in \mathbb{R}^{n\times q}$



"Optimal" for lin. sys.

$u = -Kx$

$$\dot{x} = (A - BK)x$$

with the right choice of K, the dynamics can be completely changed. (If the system is controllable)

ctrb(A, B)

Ex:

$$(*) \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u$$

The system is uncontrollable. (uncoupled)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

The system is controllable. (multiple inputs)

★ Note: If the linearizated system is uncontrollable, the non-linec system may or may not be controllable.

$$(**) \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u$$

The system is controllable. (coupled)

$$\boxed{\mathcal{C} = \begin{bmatrix} B & AB & A^2B & A^3B \ldots & A^{n-1}B \end{bmatrix}}$$

Controllability matrix.

★ If the $\mathcal{C}$ has n linearly independent columns, then the system is controllable.

Column rank of n.

$\text{rank}(\mathcal{C}) = n \rightarrow$ full rank

$$(*) \quad \mathcal{C} = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \quad \begin{array}{l} \text{rank} = 1 \\ \text{uncontrollable.} \end{array}$$

$$\boxed{\text{iff rank}(\mathcal{C}) = n, \text{ then the system is controllable.}}$$

$$(**) \quad \mathcal{C} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \quad \begin{array}{l} \text{rank} = 2 \\ \text{Controllable} \end{array}$$

(It does not tell how controllable the system is.) } $\mathcal{C} \rightarrow$ SVD $\rightarrow$

★ left singular vectors ar ordered from most controllable states to

Controllability, Reachability, and Eigenvalue Placement

(6)

1. System is controllable
2. Arbitrary eigenvalue (pole) placement

$$u = -Kx \Rightarrow \dot{x} = \underbrace{(A-BK)}_{\text{arbitrary eigens}}x$$

Eigenvalues can be placed anywhere.

Matlab code

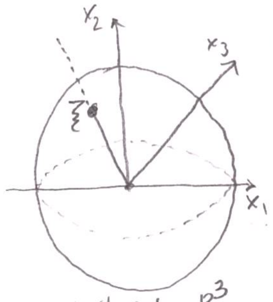
`K=place(A,B,eigs)`

3. Reachability (full in $\mathbb{R}^n$)

Reachable set $R_t = \left\{ \xi \in \mathbb{R}^n \;\middle|\; \text{there is an input } u(t) \text{ so that } x(t) = \xi \right\}$

states

$R_t = \mathbb{R}^n$ the states can be drived anywhere in R^n with some $u(t)$



unit circle: R^3

- If $u(t)$ is ten times bigger ξ (states) will be ten times bigger. (linear system)
- If it is reachable in all of the directions of R^n ξ can be arbitrarily far just making input $u(t)$ as big as we want.

Controllability and the Discrete-Time Impulse Response

(7)

$$x_{k+1} = Ax_k + Bu_k$$

Discrete time matrices.

Impulse response

$u_0 = 1 \quad x_0 = 0 \rightarrow$ initial condition

$u_1 = 0 \quad x_1 = B$

$u_2 = 0 \quad x_2 = AB$

$u_3 = 0 \quad x_3 = A^2B$

$\vdots \qquad \vdots$

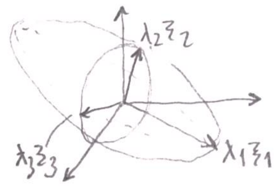
$u_m = 0 \quad x_m = A^{m-1}B$

Degrees of Controllability and Gramians

SVD : singular value decomposition

SVD of $\mathcal{C}$: Controllability Gramian. $\longrightarrow$ $W_t = \int_0^t e^{A\tau} B B^T e^{A^T \tau} d\tau \in \mathbb{R}^{n \times n}$

W_t: time t gramian



$\dot{x} = Ax + Bu$

$$x(t) = e^{At} x(0) + \underbrace{\int_0^t e^{A(t-\tau)} B u(\tau) d\tau}_{\text{convolution integral.}}$$

(e^{At} is a kernel
we are sliding input u
across the kernel
and convolving.)

$W_t \xi = \lambda \xi$ eigen decomposition of W_t

λ, ξ: eigenvalue / eigenvectors } of the controllability grammian.

★ ordering from biggest eigenvalue down to a smallest one. and the eigenvectors corresponding to the biggest eigenvalues are the most controllable directions in the state-space.

★ the eigenvectors that have the biggest eigenvalue are the most controllable directions in the state-space meaning I can go farther away in those directions on the same amount of input energy. I can go farther in the directions with bigger lambda than I can in the directions with smaller lambda.

★★ Stabilizability : Stab. iff all unstable and lightly damped eigen vectors of A are in controllable subspace. ★

Note: A set of vectors are linearly independent if and only if the Gram determinant is non-zero.

Linear Independence: A set of vector is said to be linearly independent if there exists no nontrivial linear combination of the vectors that equal to zero-vector.

$W_t \simeq \mathcal{C}\mathcal{C}^T$ for discrete-time — eigenvalues & eigenvectors of this matrix are the singular vectors & singular values of $\mathcal{C}$

Matlab code

```
[U, Σ, V] = svd(C, 'econ')
```

columns of U : ξ
$\underbrace{\text{singular values squared}}_{\sigma}$ λ : Diagonal matrix Σ

Controllability and the PBH Test

Popov-Belevitch-Hautus (PBH) Test.

(A,B) is controllable iff

$\text{rank}\begin{bmatrix}(A-\lambda I) & B\end{bmatrix} = n$

$\forall \lambda \in \mathbb{C}$: for all λ in the complex plane.

1. $\text{rank}(A-\lambda I) = n$ except for eigenvalues λ.
2. B needs to have some component in each eigenvector direction.
3. If B is a random vector $B = \text{rand}(n,1)$ the (A,B) will be controllable with high probability.

Note: If a matrix is rank deficient It means determinant of the matrix is zero.

★ $\text{rank}\begin{bmatrix}\underbrace{(A-\lambda I)} & B\end{bmatrix} = n$

⇒ This is only rank deficient in the eigenvector directions corresponding those eigenvalues.

⇒ So, for this $\text{rank}[(A-\lambda I \;\; B] = n$ is true for all those eigenvalues λ, then B has to have some components in those eigenvector directions that missing from $(A-\lambda I)$.

single or multiple input (Matrix or Vector)

$\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\,\underline{u}$

★ If all eigenvalues are distinct, with only a single input column B we can control all of the directions.

★ Grey areas: If there are nearly identical eigenvalues (almost rank deficient), maybe we need two input B

★ If we have repeated eigenvalues let's say it is repeated three times, that tells us that we need three independent columns of B to fill in that rank. (3 control knobs)

Cayley-Hamilton Theorem

Every square matrix A satisfies its own characteristic (eigenvalue) equation.

$$\det(A-\lambda I)=0$$

$$\lambda^n+a_{n-1}\lambda^{n-1}+\dots+a_2\lambda^2+a_1\lambda+a_0=0$$

$$A^n+a_{n-1}A^{n-1}+\dots+a_2A^2+a_1A+a_0=0$$

Note: Almost every matrix A. There are exceptions.

$$A^n=-a_0I-a_1A-a_2A^2-\dots-a_{n-1}A^{n-1}$$

$$A^{\geq n}=\sum_{J=0}^{n-1}\alpha_J A^J$$

$$\dot{x}=Ax+Bu$$

$$e^{At}=I+At+\frac{A^2t^2}{2!}+\dots$$

$$\downarrow$$

$$e^{At}=\alpha_0(t)I+\alpha_1(t)A+\alpha_2(t)A^2+\dots+\alpha_{n-1}(t)A^{n-1}$$

$$\mathcal{C}=[B\ AB\ A^2B\dots A^{n-1}B]$$

Reachability and Controllability with Cayley-Hamilton

$$\dot{x}=Ax+Bu$$
$$x\in\mathbb{R}^n$$
$$u\in\mathbb{R}^q$$

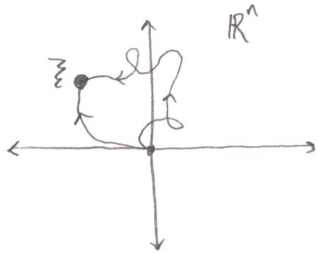
$$e^{At}=\phi_0(t)I+\phi_1(t)A+\phi_2(t)A^2+\dots+\phi_{n-1}(t)A^{n-1}$$

If $\underset{\text{(state)}}{\xi}\in\mathbb{R}^n$ is reachable then $\xi=\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$ for some $u(t)$

$$\xi=\int_0^t\left(\phi_0(t-\tau)u(\tau)IB+\phi_1(t-\tau)u(\tau)AB+\dots+\phi_{n-1}(t-\tau)u(\tau)A^{n-1}B\right)d\tau$$

$$\xi=B\int_0^t\phi_0(t-\tau)u(\tau)d\tau+AB\int_0^t\phi_1(t-\tau)u(\tau)d\tau+\dots+A^{n-1}B\int_0^t\phi_{n-1}(t-\tau)u(\tau)d\tau$$

$$\xi=\underbrace{\begin{bmatrix}B & AB & \dots & A^{n-1}B\end{bmatrix}}_{\substack{\text{If this is rank } n\\ \xi \text{ can be anywhere}\\ \text{in the state-space}}}\left.\begin{bmatrix}\int_0^t\phi_0(t-\tau)u(\tau)d\tau\\ \int_0^t\phi_1(t-\tau)u(\tau)d\tau\\ \vdots\\ \int_0^t\phi_{n-1}(t-\tau)u(\tau)d\tau\end{bmatrix}\right\}n\cdot q$$

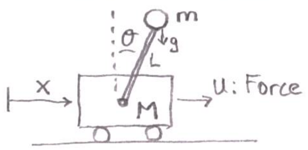


* There are infinitely many $u(t)$'s to reach ξ if the system is controllable.

*Controllability is equivalent to reachability.

$$\mathcal{C}=[B\ AB\ A^2B\dots A^{n-1}B]$$

Inverted Pendulum on a Cart



m: mass of the pendulum
M: mass of the cart
L: length
g: gravity
d: damping on the cart
u: force

$$\underline{x} = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} \qquad \frac{d}{dt}\underline{x} = \underline{f}(\underline{x})$$

non-linear

Fixed Points

$\theta = 0, \pi$
$\dot{\theta} = 0$
$\dot{x} = 0$
x = free variable.

$\Downarrow$ Linearization

$$\left.\frac{Df}{Dx}\right|_{\bar{x}} \Rightarrow \dot{\underline{x}} = A\underline{x} + Bu$$

Jacobian Matrix

$m = 1$, $M = 5$, $L = 2$, $g = -10$, $d = 1$ } 2 Fixed Points: $\theta = 0$ down. $\theta = \pi$ up. $\dot{\theta} = 0, \dot{x} = 0$, x = free } $\xrightarrow{\text{Linearization}}$ Pendulum up ($\theta = \pi$) $\Rightarrow$ A matrix

$$\mathrm{eig}(A) = \begin{bmatrix} 0 \\ -2.4674 \\ -0.1645 \\ 2.4339 \end{bmatrix} \rightarrow \text{Unstable.}$$

} rank($\mathcal{C}$) = 4, controllable. }

$u = -Kx$
$\dot{x} = (A - BK)x$
poles can be anywhere.

If controllable, we can use $-K$ full state feedback to make the eigenvalues be what we want.

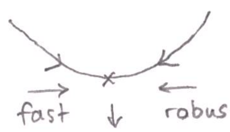
Pole Placement for the Inverted Pendulum on a Cart.

$K = \text{place}(A, B, \text{eigs})$
↑ specify what we want.

$\mathrm{eig}(A - BK)$ ⇄

- If poles are far away on the left half plane, aggressive behaviour but less robust.

* Non-linear system may not act like linear system (if the controlled system is super aggressive.

★ 

↓ Finding the sweet spot (LQR) (Performance – Control Effort)

Linear Quadratic Regulator (LQR) Control for the Inverted Pendulum on a Cart.

Where are the best eigenvalues?

Cost function
↳ States
↳ Inputs

$$J = \int_0^\infty \big(\underbrace{x^T Q x}_{\text{positive semi-definite}} + u^T R u\big)\, dt \qquad \underbrace{Q, R}_{\text{penalties}}$$

min → $\boxed{u = -Kx}$ Riccati Equation $\boxed{\dot{x} = (A - BK)x}$

LQR → $O(n^3)$
↳ Expensive for high dimensional systems.

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 100 \end{bmatrix} \begin{array}{l} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{array}$$

x ⇒ Means: I penalize it with 1 if it is not where I tell it to be.

θ ⇒ More penalized if it is not where I tell it to be.

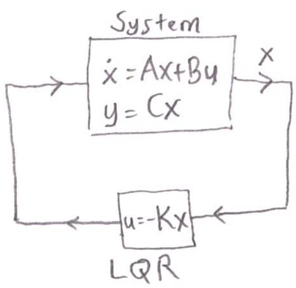
$\dot{\theta}$ ⇒ larger penalty

$R = 0.001$ [illegible]

Motivation for Full-State Estimation

$$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n$$
$$y = Cx \qquad u \in \mathbb{R}^q \qquad p \leq n$$
$$y = \in \mathbb{R}^p$$

In reality, I may not have access to measure all states. Only have access to some limited measurements y.



controllability — Can I steer the system anywhere given some $u(t)$

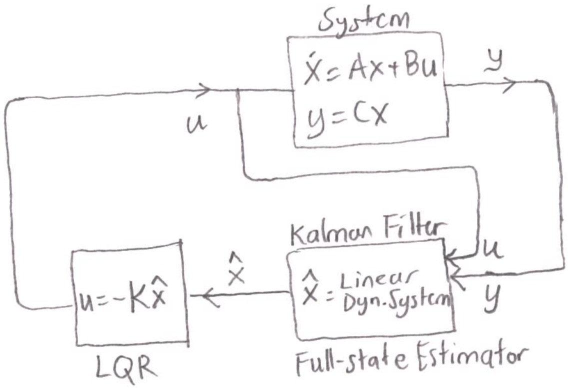
ctrb(A,B)

observability — Can I estimate any high dimensional state x from measurements $y(t)$

obsv(A,C)

$$\dot{x} = Ax - BKx$$
$$\dot{x} = (A - BK)x$$

$K_r \Rightarrow$ Regulator



sensor-based feedback.

(Optimal full-state controller) + (Optimal full-state estimator)

Observability

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} \qquad \mathcal{C} = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$$

1. Observable if
   $\text{rank}(\mathcal{O}) = n \Rightarrow \det(\text{obsv}(A,C)) \neq 0$
2. Can estimate x from y

* Eigenvalues of Kalman Filter are gonna tell how fast $\hat{x}$ converges to x.
* If we make Kalman Filter too aggressive, it is gonna over-emphasize noise and disturbances.

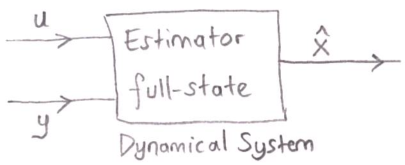
→ There is some balance just like LQR.

$[U, \Sigma, V] = \text{svd}(\mathcal{O}) \Rightarrow$ how observable

$\Rightarrow$ columns of V or rows of V^T in order, the most observable in the state-space

* most observable direction = higher signal to noise

Full-State Estimation



$$\frac{d}{dt}\hat{x} = A\hat{x} + B\hat{u} + \underbrace{K_f(y-\hat{y})}_{\text{Update based on new data } y.}$$

$$\hat{y} = C\hat{x}$$

$$\frac{d}{dt}\hat{x} = A\hat{x} + Bu + K_f y - K_f C\hat{x}$$

$$\frac{d}{dt}\hat{x} = (A - K_f C)\hat{x} + \begin{bmatrix} B & K_f \end{bmatrix}\begin{bmatrix} u \\ y \end{bmatrix}$$

It has a model A, B and C

⇒ rely on · y and u

Error $\varepsilon = x - \hat{x}$

$$\frac{d}{dt}\varepsilon = \frac{d}{dt}x - \frac{d}{dt}\hat{x} = Ax + \cancel{Bu} - A\hat{x} + K_f C\hat{x} - K_f y - \cancel{Bu}$$

$$\frac{d}{dt}\varepsilon = A(x-\hat{x}) + K_f C(\hat{x} - x)$$

$$\frac{d}{dt}\varepsilon = A\varepsilon - K_f C\varepsilon$$

$$\boxed{\frac{d}{dt}\varepsilon = (A - K_f C)\varepsilon}$$

If observable then place eigenvalues by choosing K_f

$$\dot{x} = Ax + Bu + w_d \quad \text{↳ disturbance}$$

$$y = Cx + w_n \quad \text{↳ measurement noise}$$

* What is the optimal filter gain, given some knowledge about magnitude of disturbances and magnitudes of sensor noise?

Kalman Filter

Optimal state estimator

w_d : Gaussian (disturbance)

V_d : Variance

w_n : Gaussian (noise)

V_n : Variance

$$\varepsilon = x - \hat{x}$$

$$\dot{\varepsilon} = \underbrace{(A - K_f C)}_{\text{there is a sweet spot for the eigenvalues.}}\varepsilon$$

Kalman filter model rely on y and u

If I have bigger disturbances than noise, then I should trust on measurements.
If I have bigger noise than disturbances, then I should trust my model.

Cost function

$$\dot{\varepsilon} = (A - K_f C)\varepsilon \longrightarrow O(n^3) \quad \text{Algebraic Riccati Equation.}$$

$$J = \mathbb{E}\left((x-\hat{x})^T (x-\hat{x})\right)$$

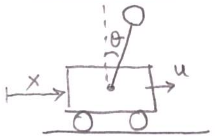
$\mathbb{E}$ ↓ Expectation value.

$$K_f = lqe(A, C, V_d, V_n \ldots) \leftarrow \text{Matlab command.}$$

lqe ↓ linear quadratic estimator

Observability Example (Part 1)

(19)



$\underline{x} = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix}$ $\quad \dot{x} = \underset{4\times4}{A}x + \underset{4\times1}{B}u$ $\quad$ obsv(A,C)

$y = \underset{4\times4}{C}x$

$C = \begin{array}{cccc} x & \dot{x} & \theta & \dot{\theta} \end{array}$
$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$

only measure x position

Ex1 ⇒ $\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix}$ $\quad \underbrace{\det(\mathcal{O}) \neq 0}_{\text{observable}}$

$C = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$

only measure θ angle

$\underbrace{\det(\mathcal{O}) = 0}_{\text{not observable.}}$

Ex2 ⇒ $\mathcal{O} = \begin{bmatrix} 0 & \cdots \\ 0 & \cdots \\ 0 & \cdots \\ 0 & \cdots \end{bmatrix}$

↳ x is not observable.

Observability Example (Part 2)

(20)

What if we don't care about the cart position x, but I only care about stabilizing inverted pendulum.

$\underline{x} = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix}$ → don't care (x); only estimate these ($\dot{x}, \theta, \dot{\theta}$).

A matrix: 3x3 $\quad$ B matrix: 3x1

⇒ A : 3x3, B : 3x1 } obsv(A,C)

$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ ($\dot{x}$) ⇒ $\det(\mathcal{O}) \neq 0$ observable

$C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$ (θ) ⇒ $\det(\mathcal{O}) \neq 0$ observable

$C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ ($\dot{\theta}$) ⇒ $\det(\mathcal{O}) \neq 0$ observable

$\det(\mathcal{O}) \neq 0$
↳ binary → observable / → not observable

sys = ss(A,B,C,D) ⇐

det(gram(sys,'o'))

↖ only for
★ stable dynamics (pendulum down)

observability gramian

y	$\det(W_o)$
$\dot{x}$	50.0
θ	0.03
$\dot{\theta}$	0.03

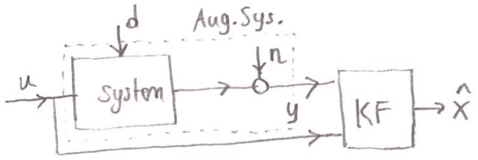
} Volume of observability ellipsoid.

If measure $\dot{x}$, It is easier to observe full-state. (But maybe eigenvectors are only in one direction)

Kalman Filter Example

$$\underline{x} = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} \qquad \begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

$C = [1\ 0\ 0\ 0]$ only measure x



$V_d = \begin{bmatrix} 0.1 & & & 0 \\ & 0.1 & & \\ & & 0.1 & \\ 0 & & & 0.1 \end{bmatrix}$ disturbance covariance

$V_n = 1$ noise covariance

$BF = [B\ \ V_d\ \ 0{*}B]$ augmented inputs with disturbances and noise

$$\dot{x} = Ax + \boxed{Bu + V_d d + 0n}$$
$$y = Cx + \boxed{Du + 0d + V_n n}$$

$D = [0\ \ 0\ \ V_n]\begin{bmatrix} u \\ d \\ n \end{bmatrix}$ 4 seperate disturbance input for each state.

$$Aug.Sys = ss(A, BF, C, [\underbrace{0}_{u}\ \underbrace{0\ 0\ 0\ 0}_{d}\ \underbrace{V_n}_{n}])$$

$$\dot{x} = Ax + Bu + V_d d + 0n$$

$[4\times4]$, $[4\times1]$, $[1\times1]$, $[4\times1]$, $[1\times1]$, $[4\times1]$, $[4\times4]$

$\star$ $K_f = (lqr(\underbrace{A^T}_{A}, \underbrace{C^T}_{B}, \underbrace{V_d}_{Q}, \underbrace{V_n}_{R}))^T$ ← Just like LQR

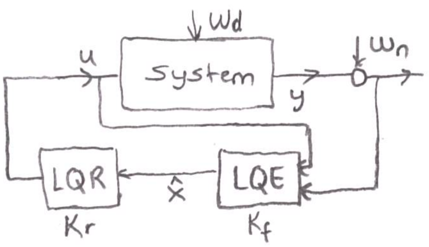
Kalman filter gain matrix

$\star$ system $KF = ss(A - K_f C, [B\ K_f], \underbrace{eye(4)}_{\text{full state } \hat{x}}, 0{*}[B\ K_f])$

Linear Quadratic Gaussian (LQG)

$$\dot{x} = Ax + Bu + w_d$$
$$y = Cx + w_n$$

$\underbrace{w_d, w_n}$ Gaussian white noise disturbances and measurement noise.



LQG control. → can suffer from robussness issue.

(Combination of Kalman Filter and LQR)

$$\varepsilon = x - \hat{x}$$
$$\dot{x} = Ax - BK_r\hat{x} + w_d$$
$$\hat{x} = x(x - \hat{x})$$
$$\boxed{\dot{x} = Ax - BK_r x + BK_r(x - \hat{x}) + w_d} \star$$
$$\boxed{\dot{\varepsilon} = (A - K_f C)\varepsilon + w_d - K_f w_n} \star$$

Seperation Principle

$$\frac{d}{dt}\begin{bmatrix} x \\ \varepsilon \end{bmatrix} = \begin{bmatrix} (A - BK_r) & BK_r \\ 0 & (A - CK_f) \end{bmatrix}\begin{bmatrix} x \\ \varepsilon \end{bmatrix} + \begin{bmatrix} I & 0 \\ I & -K_f \end{bmatrix}\begin{bmatrix} w_d \\ w_n \end{bmatrix}$$

states

error

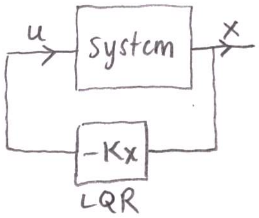
LQR

Kalman Filter LQE

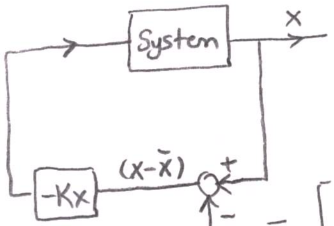
Seperate eigenvalues!

$\star$ We can design LQR and LQE seperately, the combined system retains the desirable properties of each other.

LQG Example



$$x = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 0 \\ \pi \\ 0 \end{bmatrix} \rightarrow \text{up position.}$$

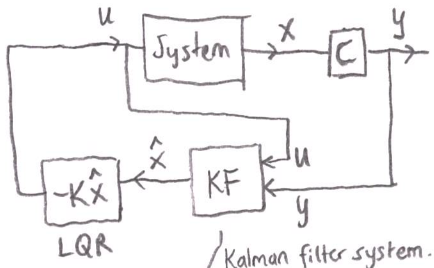


$$\bar{x} = \begin{bmatrix} 0 \\ 0 \\ \pi \\ 0 \end{bmatrix}$$ equilibrium point. ★

LQR

The system linearized around the $\bar{x}$.

LQG



Sensor based feedback of unstable system.

Kalman filter system.

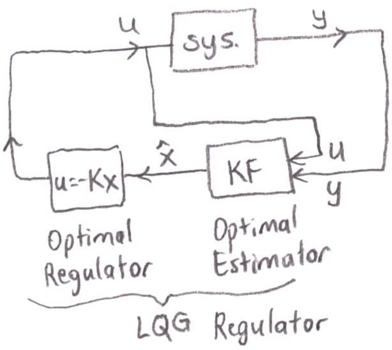
$$K_f = \left(lqr(A^T, C^T, Vd, Vn)\right)^T$$

$$sysKF = ss\left(A - CK_f, [B\ K_f], eye(4), 0^*[B\ K_f]\right)$$

LQR+LQE : combined system is more sensitive to disturbances and noise. This is a general issue of LQG. Non-robust.

Introduction to Robust Control

Optimal Control



Doyle in 1978

Guaranteed stability margins for LQG regulators

Abstract: There are none.

★ Robustness and Performance

Three Equivalent Representations of Linear Systems.

$u \rightarrow \boxed{\text{sys}} \rightarrow y$

G : Transfer Function

Linear

$$\boxed{\begin{array}{c} u_1 \rightarrow y_1 \\ u_2 \rightarrow y_2 \\ (u_1+u_2) \rightarrow (y_1+y_2) \end{array}}$$

★ 1. State-Space

$$\boxed{\begin{array}{l} \dot{x} = Ax + Bu \\ y = Cx \end{array}}$$

★ 2. Frequency Domain

$$\boxed{G(s) = C(sI-A)^{-1}B}$$

Transfer function

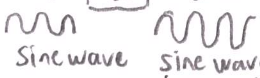
★ $\left.\begin{array}{l}\text{To assess the} \\ \text{Robustness}\end{array}\right\} G(s)$

★ 3. Time Domain

$$\boxed{y(t) = \int_0^t h(t-\tau)u(\tau)d\tau}$$

Impulse response

$u \rightarrow \boxed{\text{sys}} \rightarrow y$



Sine wave

Sine wave
- different amplitude
- different phase
- the same frequency

$$\boxed{\sin(\omega t) \longrightarrow A\sin(\omega t + \phi)}$$

Amplitude   phase lead or lag

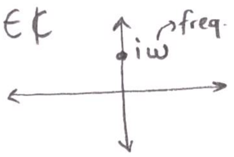
$$|G(i\omega)| = A$$

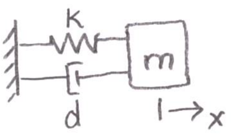
$$\angle G(i\omega) = \phi$$

Example Frequency Response (Bode Plot) for Spring-Mass Damper

$G(s)$ $s \in \mathbb{C}$

$G(i\omega)$





$$m\ddot{x} + d\dot{x} + kx = u$$

$$\mathcal{L}\left\{\frac{d}{dt}(x)\right\} = s\bar{x}(s) - x(0)$$

For $x(0) = 0$

$$= s\bar{x}(s)$$

$$\mathcal{L}\left\{\frac{d^2}{dt^2}(x)\right\} = s^2\bar{x}(s)$$

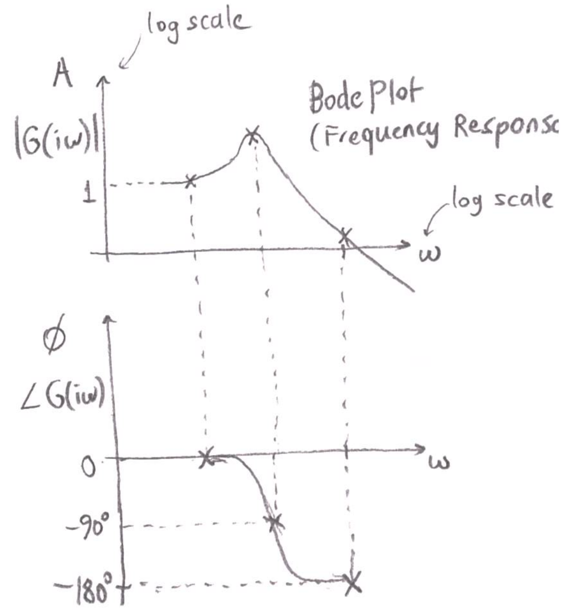
$$m\ddot{x} + d\dot{x} + kx = u$$

$$m = d = k = 1$$

$$\ddot{x} + \dot{x} + x = u$$

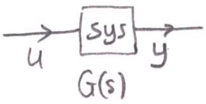
$$s^2\bar{x}(s) + s\bar{x}(s) + \bar{x}(s) = \bar{u}$$

$$\boxed{\frac{\bar{x}}{\bar{u}} = \frac{1}{s^2+s+1}} = G(s)$$



Bode Plot.
- → Robustness
- → Sensitivity
- → Disturbance
- → Noise etc.

Laplace Transform and the Transfer Function



$$\left.\begin{aligned}\dot{x} &= Ax+Bu\\ y &= Cx\end{aligned}\right\}\text{Laplace} \Rightarrow \boxed{G(s) = C(sI-A)^{-1}B}$$

↳ If there is $Du \Rightarrow \boxed{G(s) = C(sI-A)^{-1}B + D}$

Laplace transform

$s \in \mathbb{C}$ $(a+bi)$

$$\mathcal{L}\{x(t)\} = \int_{0^-}^{\infty} x(t)\, e^{-st}\, dt = \bar{X}(s)$$

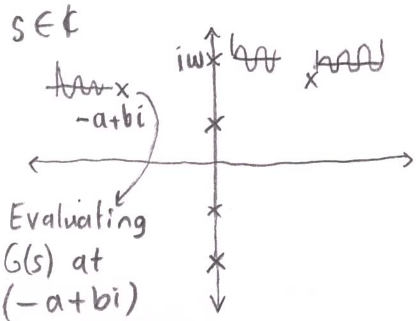
↓ Generalized Fourier transform

Laplace looks like a one-sided fourier transform.

$$\mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, e^{-iwx}\, dx = \bar{f}(w)$$

$$\bar{Y}(s) = \bar{G}(s)\bar{U}(s)$$

$s \in \mathbb{C}$



Evaluating $G(s)$ at $(-a+bi)$

★ Fourier Transform only be able to handle pure sines and cosines, but it is possible to represent the transfer function with the exponentially decaying and growing modes also.

★ Impulse response $\boxed{\mathcal{L}^{-1}(G(s))}$
$y(t)$ given $u = \delta(t)$

$$\mathcal{L}\left\{\frac{d}{dt}x(t)\right\} = \int_{0^-}^{\infty} \underbrace{\frac{d}{dt}x(t)}_{dv}\, \underbrace{e^{-st}}_{u}\, dt$$

$$= [uv]_{0^-}^{\infty} - \int_{0^-}^{\infty} v\, du\, dt$$

$$= \left[e^{-st}x\right]_{0^-}^{\infty} + s\underbrace{\int_{0^-}^{\infty} x\, e^{-st}\, dt}_{\text{Laplace transform of } x}$$

$$= 0 - x(0) + s\,\mathcal{L}\{x\}$$

$$\boxed{\mathcal{L}\{\dot{x}\} = s\underbrace{\mathcal{L}\{x\}}_{\bar{x}(s)} - x(0)}$$

$$s\bar{x}(s) - x(0) = A\bar{x}(s) + B\bar{u}(s)$$

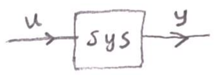
$$(sI-A)\bar{x}(s) = B\bar{u} + \cancel{x(0)}$$

$$\bar{x}(s) = (sI-A)^{-1}B\bar{u} + \underbrace{\cancel{(sI-A)^{-1}x(0)}}_{\text{steady-state behaviour}}$$

★ $\boxed{\bar{y}(s) = \underbrace{C(sI-A)^{-1}B}_{G(s)}\,\bar{u}(s)}$

Benefits of Feedback on Cruise Control Example (Part 1/2)

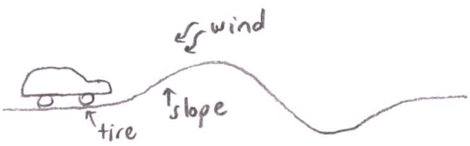
- stability
- uncertainty
- disturbance

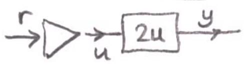


Model $\boxed{y = 2u}$

$u_{\text{open loop}} = \dfrac{r}{2}$

★ If the actual system differs from the model the open loop control does not correct.

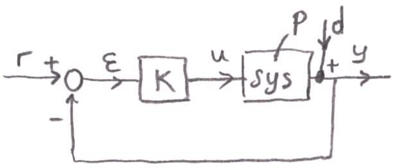




Let's say, true car ⇒ $\boxed{y = u + d}$ $\Big\}$ $y_{\text{open loop}} = \dfrac{r}{2} + d$

← reference velocity off by %50 because the model was bad

↖ disturbance isn't reduced at all.



Proportional Feedback Control.

$u_{\text{closed loop}} \Rightarrow \left.\begin{aligned}u_{CL} &= K\varepsilon = K(r-y)\\ y_{CL} &= Pu + d\end{aligned}\right\}$ $y_{CL} = PKr - PKy_{CL} + d$

$$(1+PK)\,y_{CL} = PKr + d$$

$$\boxed{y_{CL} = \underbrace{\frac{PK}{1+PK}}_{\text{how good } y_{CL} \text{ matches with } r} r + \underbrace{\frac{1}{1+PK}}_{\text{how much disturbances get reduced by effective control.}} d}$$

★ Let's say, model $P=2$ but true car $P=1$

$\left.\dfrac{PK}{1+PK}\right\}$ want this 1. $\left.\dfrac{1}{1+PK}\right\}$ want this 0.

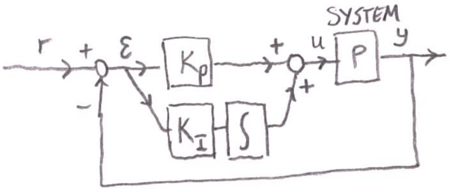
⇒ Big K works!

For $K=100$, around %1 disturbance effect. → $\left(\dfrac{1}{1+P.100}\right) = \left(\dfrac{1}{101}\right)$

around %99 reference tracking → $\left(\dfrac{100P}{1+P100}\right) = \left(\dfrac{100}{101}\right)$

↓

Cruise Control Example with Proportional-Integral (PI) Control.



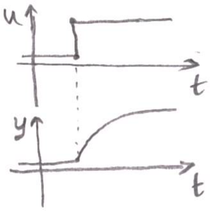
PI Control.

* Integral term reduces steady state error → 0.

Model dynamics

$$\left.\begin{aligned} \dot{x} &= -x + u \\ y &= x \end{aligned}\right\}$$

$$\frac{\bar{y}(s)}{\bar{u}(s)} = \boxed{P(s) = \frac{1}{s+1}}$$



$$y = \underbrace{\frac{PK}{1+PK}}\, r$$ ⇐ Proportional Feedback Control.

This can never equal to one.

There will be a steady-state error.

$y \neq r$

$$u = K_p(r-y) + K_I \overbrace{\int (r-y)\,dt}^{z = \int \varepsilon}$$

$$\dot{z} = r - x$$

$$u = K_p(r-x) + K_I z$$

$$\boxed{\begin{aligned} \dot{x} &= -x + K_p r - K_p x + K_I z \\ \dot{z} &= r - x \end{aligned}}$$

two state system. (x, z)

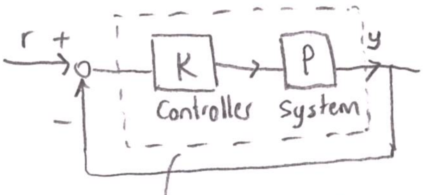
* r is acting like an input.

$$\boxed{\begin{aligned} \frac{d}{dt}\begin{bmatrix} x \\ z \end{bmatrix} &= \begin{bmatrix} -1-K_p & K_I \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} +K_p \\ 1 \end{bmatrix} r \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix} \end{aligned}}$$ *

⇒ $r \rightarrow \boxed{sys} \rightarrow y$

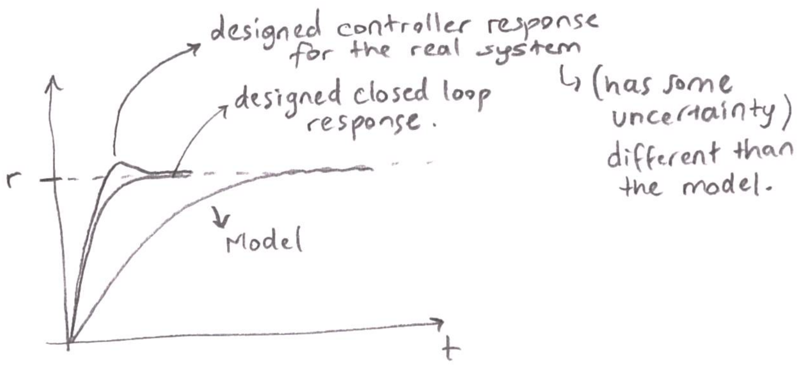
Closed Loop System.

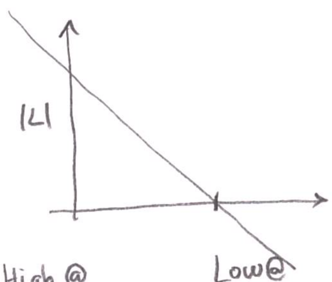
Loop Shaping Example For Cruise Control



Loop transfer function

$L = PK$





High @ low freq.

Low @ high freq.

- track reference
- reject disturbances.

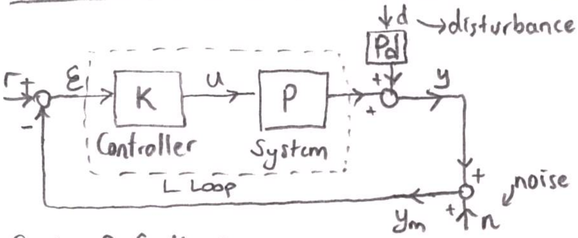
- attenuate noise.

* If you have a model for the system.
  - Develop K instead of LQR,

I want my loop transfer function looks like an integrator.

(Loop shaping Controller Design)

Sensitivity and Complementary Sensitivity (Part 1) 31



order ⇒ $PK\varepsilon$

$$L = PK$$

$$\underset{\text{Identity Matrix}}{I}\, y = P_d d + \underbrace{PK}_{\text{matrix}}\overbrace{(r-y-n)}^{\varepsilon}$$

Goals of feedback

- Stability (design)
- Uncertainty (compensation)
- Disturbance (rejection)
- Noise (attenuation)

$$(I+PK)\,y = PKr + P_d d - PKn$$

$$y = \underbrace{(I+PK)^{-1}PKr}_{\text{reference tracking}} + \underbrace{(I+PK)^{-1}P_d d}_{\text{disturbance rejection}} - \underbrace{(I+PK)^{-1}PKn}_{\text{noise attenuation}}$$

$(I+PK)^{-1}PK$: Complimentary Sensitivity : T

$(I+PK)^{-1}$: Sensitivity : S

$$\boxed{\begin{aligned} L &= PK \\ S &= (I+L)^{-1} \\ T &= (I+L)^{-1}L \end{aligned} \quad \Big\}\ S+T=I}$$

$$\boxed{y = Tr + SP_d d - Tn}$$

Note: $y_m = Tr + SP_d d + Sn$

$\varepsilon = r - y_m = Sr - SP_d d - Sn$

→ we cannot reject high freq noise from the ε.

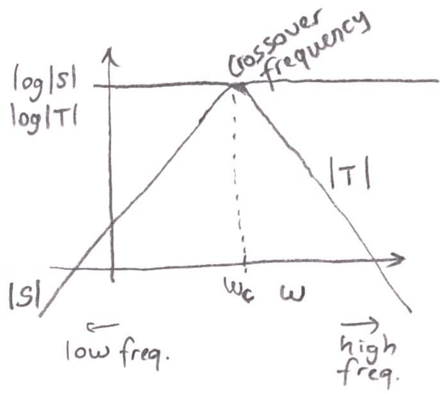
Sensitivity and Complementary Sensitivity (Part 2) 32

$$\underset{\text{minimize error}}{E} = r - y = r - (Tr + SP_d d - Tn) = r(\underbrace{1-T}_{S}) - SP_d d - Tn = Sr - SP_d d + Tn$$

$$\boxed{E = \underbrace{Sr}_{①} - \underbrace{SP_d d}_{②} + \underbrace{Tn}_{③}} \qquad \boxed{y = Tr + SP_d d - Tn}$$

For $y=r$, T must be high at low frequencies.

① For low frequencies, we want S to be small.
② For low frequency disturbance, we want S to be small.
③ For noise attenuation, we want T to be small for high frequencies.



$$S+T=I$$

If S is small, T has to be close to 1.
If T is small, S has to be close to 1.

* Usually reference tracking and disturbance rejection are kinda low frequency phenomena. So, having penalization on S for low frequencies. and noise attenuation is usually something we care about at high frequencies.
* There is a sweet spot in ω_c (crossover frequency) where I get to choose, where do I think noise is gonna start to dominate, or disturbances dominate.
* If to track faster and faster disturbance rejection, we try to move ω_c to the right, which means we have to have less noisy measuments first of
* Higher quality measurements is needed for higher performance

Loop Shaping

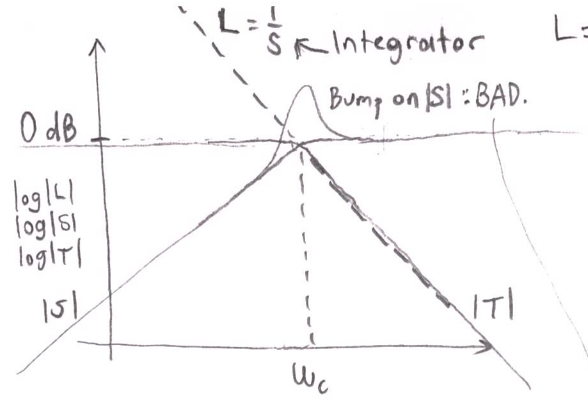
$L = PK$

$S = (I+L)^{-1}$

$T = (I+L)^{-1}L$

$\}\ S + T = I$

Design a K to get good L.



crossover frequency (0dB line)

$L = \frac{\omega_B}{s}$

$\omega_B \rightarrow$ to the right

higher bandwidth performance.

★ Bigger the peak, the worse the sensitivity and the less robust the system is.

For low frequencies

We want S to be small.
L must be big.

} Big L at low frequencies means good reference tracking and disturbance rejection

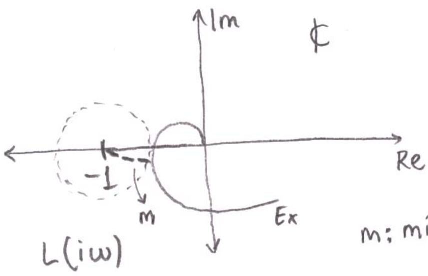
For high frequencies

We want T to be small.
L must be small.

} Small L at high frequencies means good noise attenuation

Sensitivity and Robustness

• Nyquist stability criteria



m: minimum distance

★ Closer it gets to (-1), the less robust the system when we close the loop.

$$y = \frac{PK}{1+PK} r$$

$1+PK \rightarrow = 0$ if $PK = -1$

Transfer function (simplified)

★ S : Direct measure of the robustness of the system.

$$S = \frac{1}{1+L}$$

↓ If L close to -1, then S is big.

$\max_{\omega} |S|$

max of |S| for all frequencies

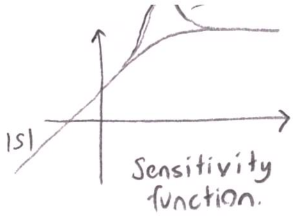
★ lower the peak |S| for robustness as low as possible.

The system becomes less robust

Time delay rotate nyquist plot.
Model uncertainty grows nyquist plot.
Increasing K grows nyquist plot.

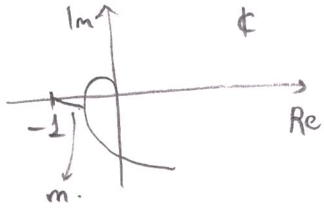
Limitations on Robustness

$$S = \frac{1}{1+L} \Bigg\} \quad \max_{\omega} |S| = \frac{1}{m}$$



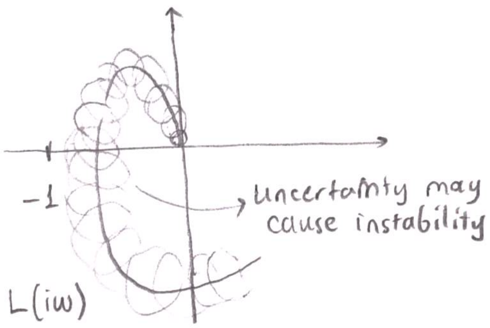
- Design the system
  - ↳ low |S| at low freq.
  - ↳ low peak.

• If the peak is 10dB,
that means it is only 0.1 away
from the instability point.



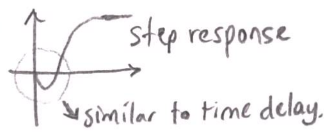
Non-robustness means.

- P is bigger or small
- K is bigger or small
- Time delay.



- Model uncertainty
- Time delays
- Right half plane zeros of P

] Fundamentally limits on how small max |S| can be.



Ex: $\frac{s-1}{s^2}$ } zero at 1

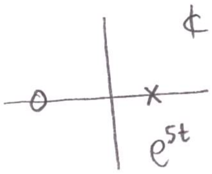
The only way to solve,
we have to lower the bandwidth
get less performance. (lower the peak)

Cautionary Tale about Inverting the Plant Dynamics

$\to \boxed{K} \to \boxed{P} \to$

$P = \frac{s+10}{s-5}$ ↖ unstable plant

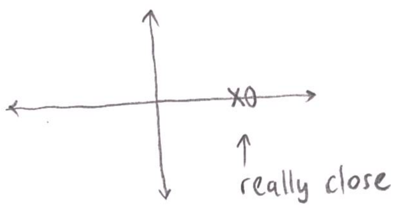
Invert $P \Rightarrow K = \frac{s-5}{s+10}$



$P_{true} = \frac{s+10}{s-5+\varepsilon}$ ↳ very small

unstable pole is a little bit off.

$$P_{true} K = \frac{s-5}{s-5+\varepsilon}$$



★ If there is an unstable (rhp) pole
Do not try to invert the plant.

Even an ε off, the system is unstable
and unobservable.

★ If there is a rhp zero,
inverting the plant (the controller)
the controller would be unstable.

★ It means this unstable mode
is nearly unobservable!

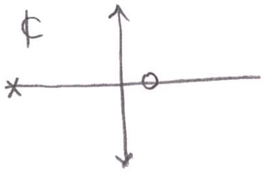
- Anything near $s=5$ is gonna be really hard to observe.
- The system is grow exponentially and barely visible in output y. (internally blowing up)

Control Systems with non-minimum phase dynamics.

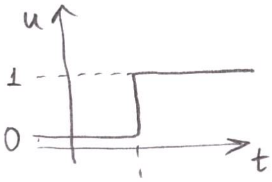
Non-minimum phase → limits on robustness. ★

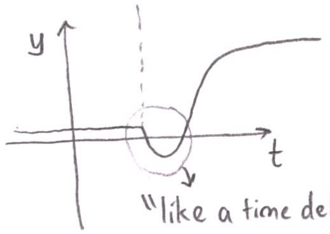
1. Right Half Plane zero

$$G(s) = \frac{s-1}{s+10}$$

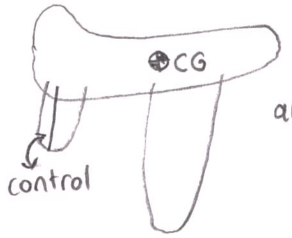


2. "Goes in the wrong direction first"

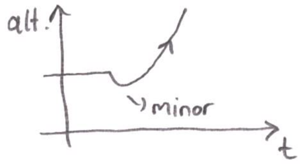




Aircraft

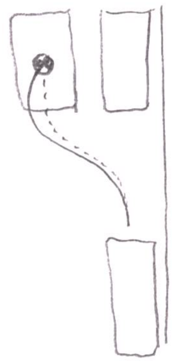


To gain altitude,



"barely non-minimum"

Car

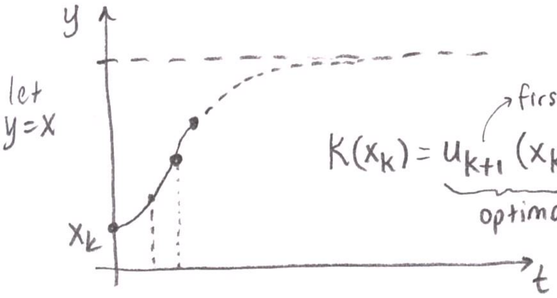


slow process: parallel parking.

★ the worse the non-minimu[m] system is (the bigger the dip) the less robust to fast changes.

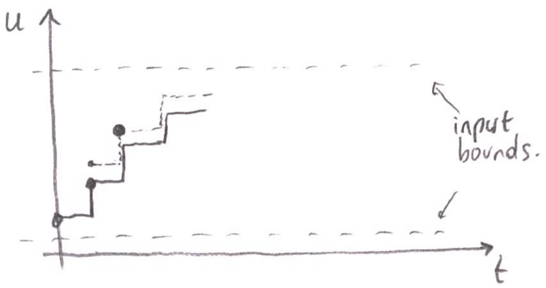
- Settle for slower control.

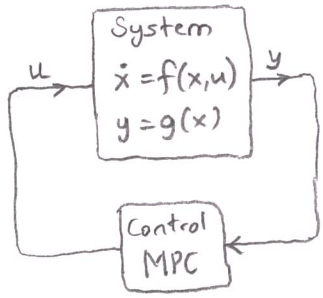
Model Predictive Control



$$K(x_k) = \underbrace{u_{k+1}}_{\text{optimal short time control starting at } x(k)}(x_k)$$

first time step





- constraints
- non-linear
- linear parameter varying

$$\dot{x} = A(\mu)x + B(\mu)u$$

Optimization

@ every time-step. → modify K.

relies on fast hardware

Linear Quadratic Regulator

$$J = \int_0^\infty (x^T Q x + u^T R u + 2x^T N u)\,dt$$

$$\dot{x} = Ax + Bu$$

$$A^T X + XA - (XB+N)R^{-1}(B^T X + N^T) + Q = 0$$

$$K = R^{-1}(B^T X + N^T)$$

$$J = \int_0^\infty (x^T Q_1 x + \bar{u}^T R \bar{u})\,dt$$

$$Q_1 = Q - N R^{-1} N^T \qquad \dot{x} = A_1 x + Bu$$

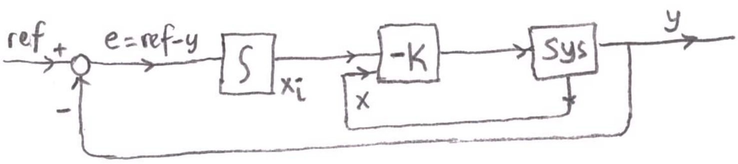
$$A_1 = A - B R^{-1} N^T$$

$$\bar{u} = u + R^{-1} N^T x$$

$A = n \times n$ $\quad Q = n \times n$

$B = n \times r$ $\quad R = r \times r$

Linear Quadratic Integrator



$$\frac{d}{dt}x = Ax + Bu \qquad u = -K\begin{bmatrix} x \\ x_i \end{bmatrix}$$

$$y = Cx + Du$$

$$J = \int_0^\infty (z^T Q z + u^T R u + 2z^T N u)\,dt$$

$$z = \begin{bmatrix} x \\ x_i \end{bmatrix}$$

$$\left.\begin{aligned} \frac{d}{dt} z &= A_a z + B_a u \\ y &= C_a z + D_a u \end{aligned}\right\}\ \text{plant with augmented integrator}$$

$A = n \times n$

$B = n \times r$

$C = m \times n$

$D = m \times r$

$u = r \times 1$

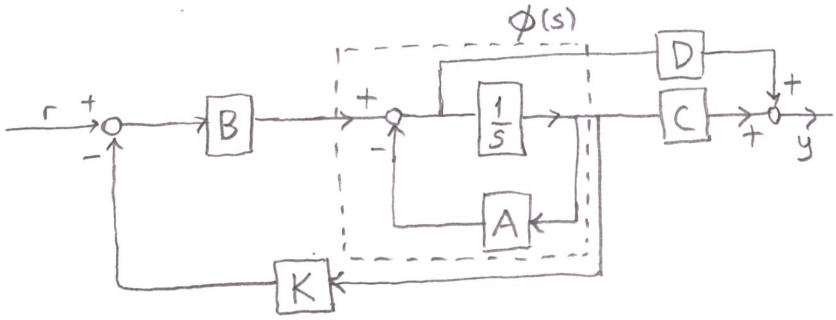
$x = n \times 1$

$y = m \times 1$

$$\begin{bmatrix} \dot{x} \\ \dot{x}_i \end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix}}_{(n+m)\times(n+m)} \underbrace{\begin{bmatrix} x \\ x_i \end{bmatrix}}_{(n+m)\times 1} + \begin{bmatrix} B \\ -D \end{bmatrix} u + \begin{bmatrix} 0 \\ I \end{bmatrix} ref$$

$$A_{aug} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} \qquad B_{aug} = \begin{bmatrix} B \\ -D \end{bmatrix}$$

Linear Quadratic Regulator (Gain & Phase Margins)



$$\dot{x} = Ax + Bu$$

$$sX(s) = AX(s) + BU(s)$$

$$(sI - A)X(s) = BU(s)$$

$$\frac{X(s)}{U(s)} = \underbrace{(sI-A)^{-1}B}_{\phi(s)}$$

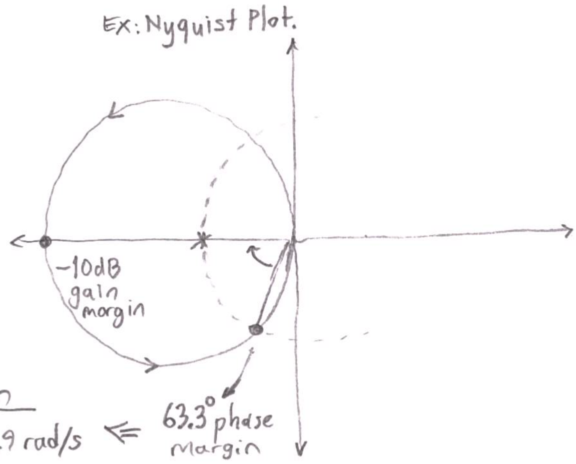
The loop transfer function

$$K\phi(s)B$$

$$\star \quad \boxed{K(sI-A)^{-1}B}$$

$\downarrow$

nyquist(.)



Delay Margin

0.0482 s @ 22.9 rad/s $\Leftarrow$ 63.3° phase margin

Bryson's Rule

Select Q and R diagonal with: (Just a starting point)

$$Q_{ii} = \frac{1}{\text{max. acceptable value of } (z_i^2)} \qquad i \in \{1, 2, 3 \ldots \ell\}$$

$\rightarrow$ states

$$R_{JJ} = \frac{1}{\text{max. acceptable value of } (u_J^2)} \qquad J \in \{1, 2, \ldots k\}$$

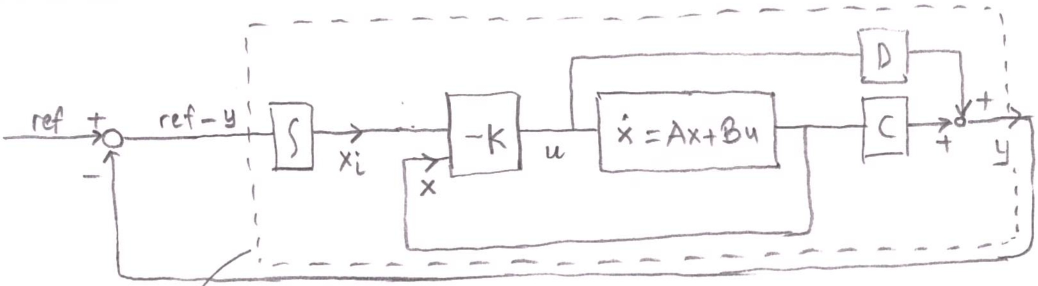
$\rightarrow$ inputs

$$J = \int_0^\infty \left(\sum_{i=1}^{\ell} Q_{ii} z_i(t)^2 + \rho \sum_{J=1}^{m} R_{JJ} u(t)^2 \right) dt$$

$\rho \downarrow$ constant

(scalar ratio between the state & the control terms)

Linear Quadratic Integrator (Gain & Phase Margins)



$$\boxed{\begin{aligned}\dot{\bar{x}} &= \hat{A}\bar{x} + \hat{B}e \\ y &= \hat{C}\bar{x} + \hat{D}e\end{aligned}}$$

$$u = -K\begin{bmatrix} x \\ x_i \end{bmatrix}$$

$$\dot{x} = Ax - \underset{1\times2}{BK}\underset{2\times1}{\begin{bmatrix} x \\ x_i \end{bmatrix}} = Ax - \begin{bmatrix} BK_s x \\ BK_o x_i \end{bmatrix}$$

Integrator

$$\dot{x}_i = ref - y = e$$
$$y_i = x_i$$

$$y = Cx - \begin{bmatrix} DK_s x \\ DK_o x_i \end{bmatrix}$$

$$\boxed{\begin{bmatrix} \dot{x} \\ \dot{x}_i \end{bmatrix} = \underbrace{\begin{bmatrix} A - BK_s & -BK_o \\ 0 & 0 \end{bmatrix}}_{\hat{A}}\begin{bmatrix} x \\ x_i \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\hat{B}} e}$$

$$\boxed{y = \underbrace{\begin{bmatrix} C - DK_s & -DK_o \end{bmatrix}}_{\hat{C}}\begin{bmatrix} x \\ x_i \end{bmatrix}}$$

★ nyquist → $\hat{A}, \hat{B}, \hat{C}, D$

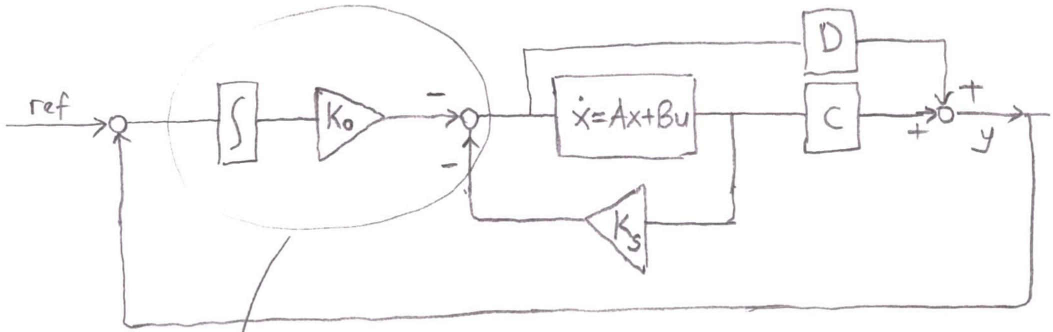
$D \rightarrow 0 \quad (e \rightarrow y)$

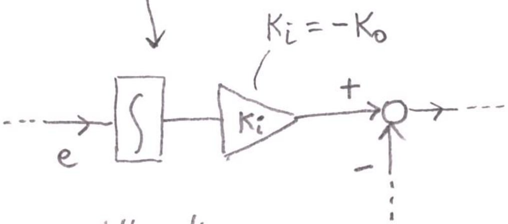
OR

nyquist$\left(\frac{y}{e}(s)\right)$

$$\boxed{\frac{y}{e}(s) = \begin{bmatrix} C - DK_s & -DK_o \end{bmatrix}(sI - \hat{A})^{-1}\hat{B}}$$

Loop transfer function ★





• Alternative

Note: Closed Loop (ref → y)

$$\begin{bmatrix} \dot{x} \\ \dot{x}_i \end{bmatrix} = \begin{bmatrix} A - BK_s & -BK_o \\ -C & 0 \end{bmatrix}\begin{bmatrix} x \\ x_i \end{bmatrix} + \begin{bmatrix} 0 \\ -D \end{bmatrix}u + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$u = -K\begin{bmatrix} x \\ x_i \end{bmatrix}$$

$$-Du = DK\begin{bmatrix} x \\ \cdots \end{bmatrix}$$

$$\boxed{\begin{bmatrix} \dot{x} \\ \dot{x}_i \end{bmatrix} = \begin{bmatrix} A - BK_s & -BK_o \\ -C + DK_s & DK_o \end{bmatrix}\begin{bmatrix} x \\ x_i \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r}$$

$$e = r - y$$
$$y = Cx + Du$$
$$e = r - Cx - Du$$

$$y = \begin{bmatrix} C - DK_s & -DK_o \end{bmatrix}\begin{bmatrix} x \\ x_i \end{bmatrix}$$

$$y = Cx - Du$$
$$\hookrightarrow -K\begin{bmatrix} x \\ \cdots \end{bmatrix}$$

Robust Control

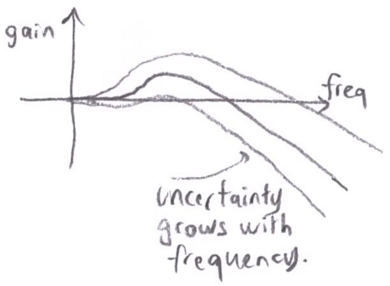
Uncertainty

↓

Mathematical model

- Complex dynamics
- Uncertain driving forces
- Intentional simplicity
- Stochastic events
- Process variations

} Add margin to solve.



- Gain and Phase margins

too little vs too much

Combination of gain and phase margin must be considered.

Robust Control ⇒ It is a design method to choose pid, full-state etc. gains.

Ex: Robust pid controller

Robust Control → Analysis: Is the system robust?

Robust Control → Synthesis: Create a robust system.

---

1. Understand uncertainty and represent it in your model
2. Analyze your system to see if its robust to the uncertanties
3. If not, design it so that is robust.

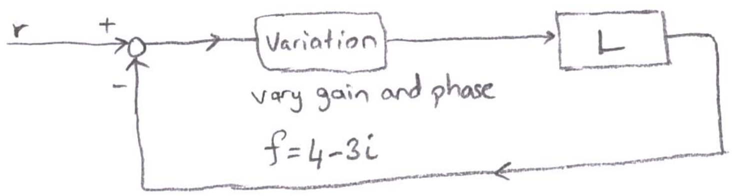
Loop shaping → difficult for MIMO
Robust control method.
difficult for system that have uncertainty that can't be bounded with simple gain and phase margin.

Other methods: difficult for systems that are highly non-linear.
H_∞, μ synthesis ...etc.

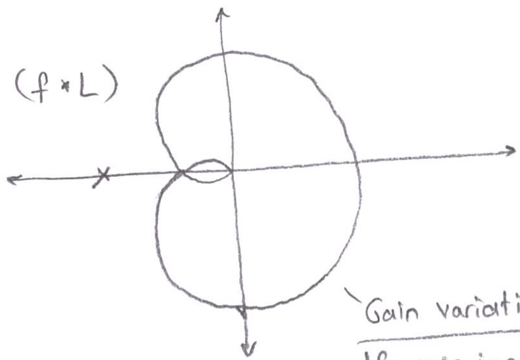
Disk Margin

Margin is a way to specify how much uncertainty the system can handle.

L → controller + plant + filter... etc.



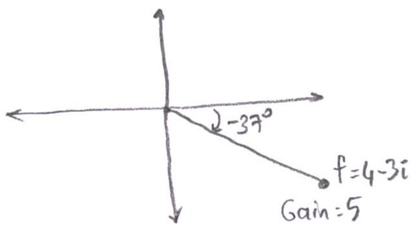
nyquist plot

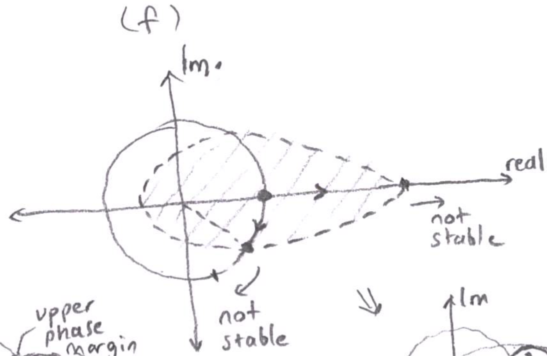


Gain variation

If gain increases, plot grows.

↓

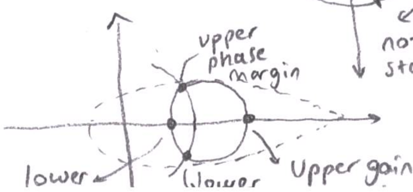


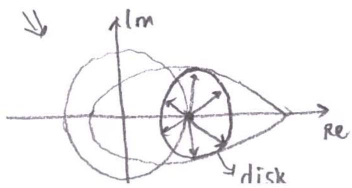


Phase variation

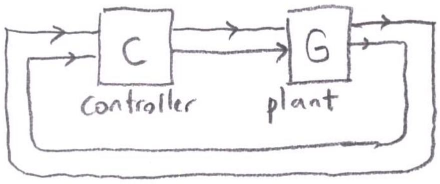
If phase increases plot rotates.

* disk = D(e, α)
↓ Disk position relative to
↳ size of the disk.





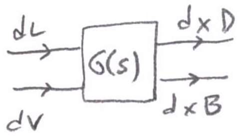
Disk margins for MIMO systems



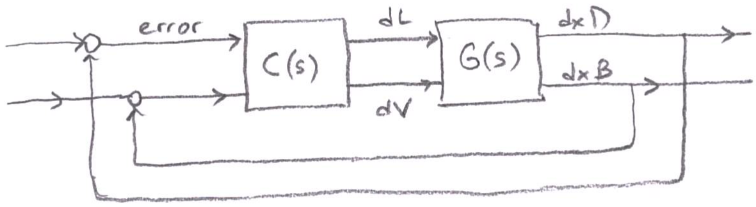
1. Create a MIMO system
2. Develop a simple controller
3. Check how robust it is.

1) 2 inputs - 2 outputs.

$$G(s) = \begin{bmatrix} \frac{0.878}{75s+1} & \frac{-0.864}{75s+1} \\ \frac{1.082}{75s+1} & \frac{-1.096}{75s+1} \end{bmatrix}$$



2)



$L(s) = C(s)\,G(s)$

⇓

let's choose this
Inversion-based PI controller → $\frac{k}{s}G^{-1}(s)$

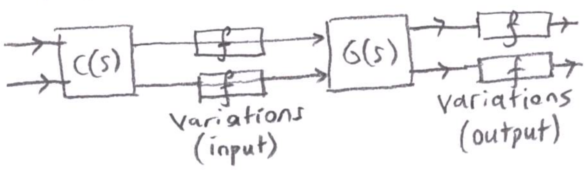
Closed loop

$$\frac{1}{\frac{1}{k}s+1} \quad \text{both channels}$$

$$L(s) = \underbrace{\frac{k}{s}G^{-1}(s)\,G(s)}$$

Is this robust?

3) Can vary plant input, output.



Multi-loop input disk margin.
Multi-loop output disk margin.

⇓

disk margin → (C*G) ← input
↘ (G*C) ← output

disk margin(G, C) Multi-loop concurrent input/output disk margins.

↓

Note: the controller designed by inverting the plant will be not robust.

Margin

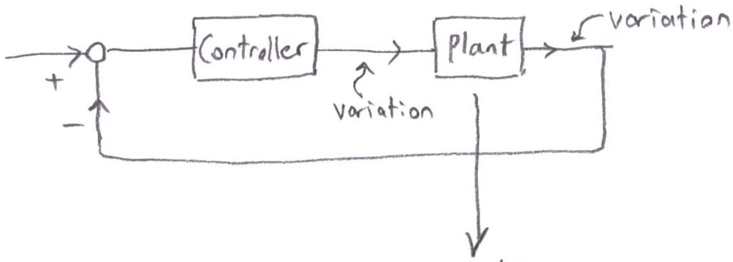
classical gain and phase margin ⇒ single channel input or output disk margin

⇓

multi-loop input/output disk margin ⇐ multi-loop input or output disk margin

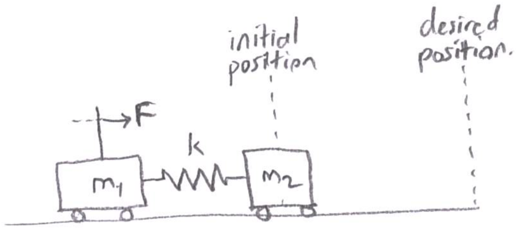
Working with Parameter Uncertainty

Quantifying model and plant uncertainty



variations in the model.

Model parameter Uncertainty.



State-space:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-k}{m_1} & 0 & \frac{k}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{m_2} & 0 & \frac{-k}{m_2} & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ \frac{1}{m_1} \\ 0 \\ 0 \end{bmatrix}$$

C = 4x4 Identity Matrix. $D = [0]_{4\times4}$

State-vector

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \begin{array}{l} \text{pos. } m_1 \\ \text{vel. } m_1 \\ \text{pos. } m_2 \\ \text{vel. } m_2 \end{array}$$

Controller

LQR

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 20 \end{bmatrix} \quad R = 7 \quad \underbrace{lqr(A, B, Q, R)}_{K}$$

MMIO = diskmargin(K, G)

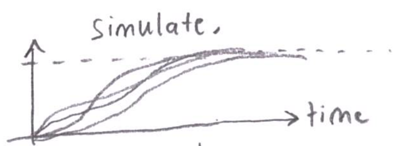
GainMargin : [0,7617 1,3129]
PhaseMargin : [-15.4 +15.4]

} But in some cases it is not straight forward to relate gain and phase margin back to physical system.

Let say, there are many different (k) springs, different (m) masses

* How much variation in k, m_1 and m_2 can the system handle before instability occurs?

different variations k, m_1, m_2 → build model →



Monte-Carlo Approach.

^ may miss by simulating the system using random parameters.

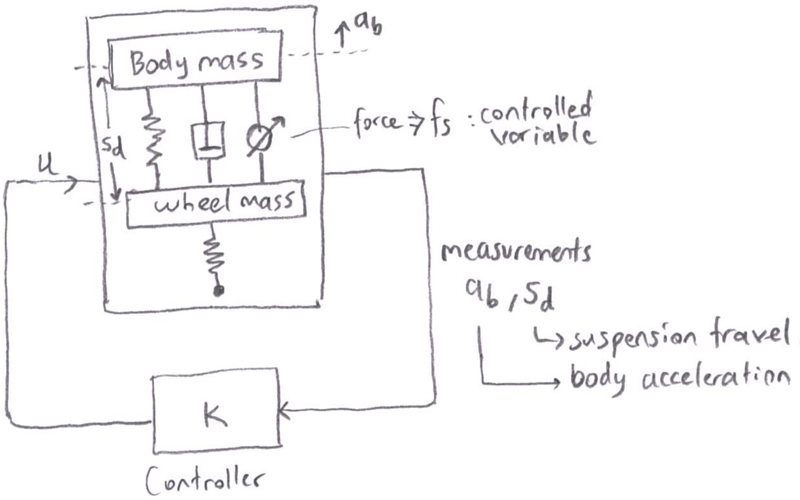
Uncertain Parameter Model

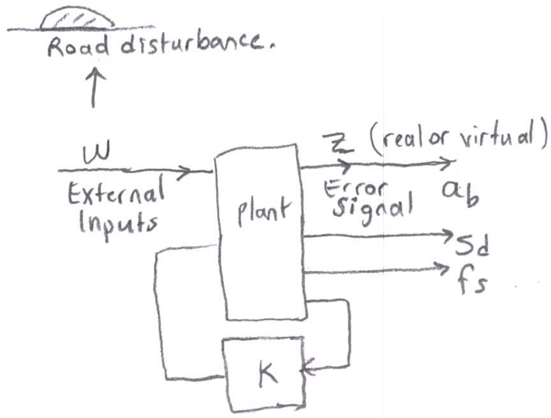
↳ deterministic approach
More reliable.

H_∞ and μ Synthesis

Control design for an Active Suspension System

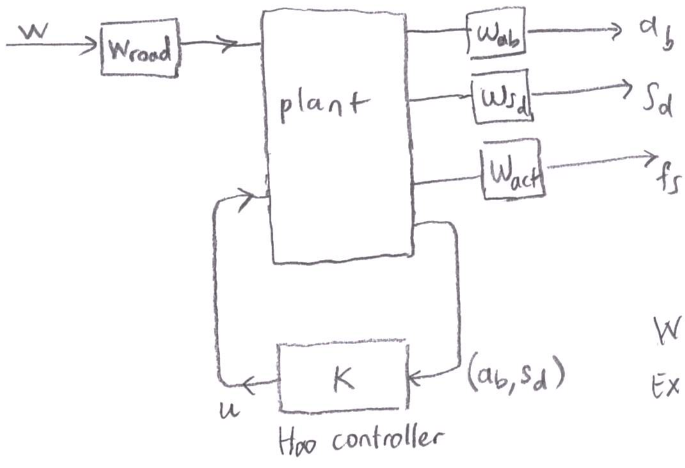
H_∞ synthesis	μ synthesis
Nominal plant model	Robust control of an uncertain model





H_∞ : Optimization process that calculates the controller that minimizes the gain between w & z

Each signal may treated not equally in the optimization problem.
Adding weighting factors.



Ex:
(W_{act}) ↓ lower the actuator weight if actuator energy is not important.

Using different weights creates different controllers like sport/comfort/balanced etc.

W can be frequency dependent.
Ex: W_{act} can have high pass filter. It means penalizing high frequency control commands.

Model-uncertainties
Sensor noise
Four wheels interacting each other
Actuator uncertainty

} Designed H_∞ controller might be unstable.

(Order reduction may be needed)

Note: Order of the generated controller can be quite high.
- more expensive / more bulky
- time-consuming to re-tuning finetuning
- low reliability. etc...

μ synthesis: Takes these uncertainties into account from the start.
↳ Extension of H_∞ : minimizes the worst case gain across the entire uncertainty space.