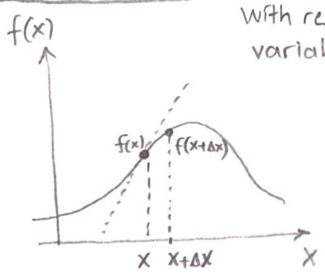
Engineering Math

Linear algebra & differential equations

1. Review Calculus
   → Taylor series
2. Simple ordinary differential equations (ODE)
   → $\dot{x} = \lambda x$
3. Systems of ODEs
   → $\dot{x} = Ax$
4. Eigenvalues and eigenvectors
   → $AT = TD$
5. Nonlinear Systems & Chaos
   → $\dot{x} = f(x)$
6. Numerics and Computations

The Derivative:

Rate of change of function $f(x)$ with respect to an independent variable x.

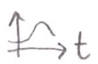


$$\frac{df}{dx} \approx \frac{f(x+\Delta x)-f(x)}{x+\Delta x - x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$\boxed{\frac{df}{dx} = \lim_{\Delta x \to 0}\left(\frac{f(x+\Delta x)-f(x)}{\Delta x}\right)}$$

$$\frac{dx}{dt} = f(x)$$

$x(t)$



Power Law

$f(x) = x^n$ we know $\boxed{\frac{df}{dx} = n x^{n-1}}$

$$\frac{df}{dx} \approx \frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{(x+\Delta x)^n - x^n}{\Delta x}$$

$$= \frac{\left(x^n + n x^{n-1}\Delta x + \frac{n(n-1)}{2}x^{n-2}\Delta x^2 + \ldots + \Delta x^n\right) - x^n}{\Delta x}$$

$$= \underbrace{n x^{n-1} + \frac{n(n-1)}{2}x^{n-2}\Delta x + \ldots O(\Delta x^2)}$$

$$\lim_{\Delta x \to 0}(\quad) = n x^{n-1}$$

Chain Rule

$f(x), g(x)$

$$f'(x) = \frac{df}{dx}($$

$$\frac{d}{dx} f(g(x)) = \frac{df}{dx}(g(x)) \cdot \frac{dg}{dx}(x) = f'(g(x))\, g'(x)$$

Ex: $f(x) = \sin x$, $g(x) = x^3$

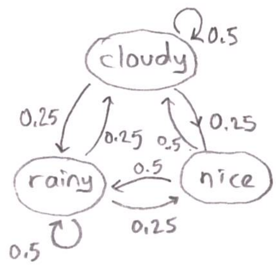
$$f(g(x)) = \sin(x^3)$$

$$\frac{d}{dx} f(g(x)) = \cos(x^3)\, 3x^2$$

Modeling with Matrices and Vectors

modeling

probabilistic   deterministic



$$\underline{x} = \begin{bmatrix} pr(R) \\ pr(N) \\ pr(C) \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \leftarrow \underline{x}_{today}$$

probability

$$\underline{x}_{tomorrow} = \underline{\underline{A}}\, \underline{x}_{today} = \begin{bmatrix} 0.5 \\ 0 \\ 0.5 \end{bmatrix}$$

$$\underline{\underline{A}} = \begin{array}{c} \begin{array}{ccc} R & N & C \end{array} \\ \begin{bmatrix} 0.5 & 0.5 & 0.25 \\ 0.25 & 0 & 0.25 \\ 0.25 & 0.5 & 0.5 \end{bmatrix} \end{array} \begin{array}{l} R_{tom} \\ N_{tom} \\ C_{tom} \end{array}$$

probability transition matrix

$$\underline{x}_{k+1} = \underline{\underline{A}}\, \underline{x}_k$$

$\Downarrow$

converges to

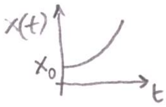
$$\underline{x} = \begin{bmatrix} 0.4 \\ 0.2 \\ 0.4 \end{bmatrix}$$

can be predicted using A matrix (eigen values, eigen vectors)

The simplest ordinary differential equation (ODE)

$\dot{x} = \lambda x$ Ex: population of an animal.

$\frac{dx}{dt} = \lambda x$



∗Method I

$\frac{dx}{dt} = \lambda x \Rightarrow \int \frac{dx}{x} = \int \lambda dt$

$\Rightarrow \ln(x(t)) = \lambda t + c$

$\Rightarrow x(t) = e^{\lambda t + c}$

$\Rightarrow x(t) = e^{\lambda t} \underbrace{e^{c}}_{k\text{: constant.}}$

$\Rightarrow x(t) = e^{\lambda t} . k$

$\downarrow$

$x(0) = k$

initial condition

$\dot{x} = \lambda x$
$x(0)$ $\Rightarrow \boxed{x(t) = e^{\lambda t} x(0)}$

First order
Linear
ordinary Differential Eqn.

First order: dx/dt Linear: λx Ordinary: only x

"e"   ④

$x(1) = (1+r)\, x(0)$

$\vdots$

$x(1) = \left(1 + \frac{r}{N}\right)^N x(0)$

$e = \lim_{N \to \infty} \left(1 + \frac{1}{N}\right)^N$

$\Rightarrow \lim_{N \to \infty} \left(1 + \frac{r}{N}\right)^N \Rightarrow e^r$

the x is continuously increasing at a rate of r proportional to the current x.

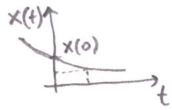
$\dot{x} = \lambda x \Rightarrow x(t) = e^{rt} x(0)$

$x(1) = e^{r} x(0)$

$x(2) = e^{2r} x(0)$

$\vdots$

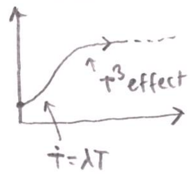
$\dot{x} = -\lambda x \Rightarrow x(t) = e^{-\lambda t} x(0)$



Ex: Radioactive Decay

Thermal runaway in electronics

$\dot{T} = \lambda T - T^3$ ← radiation



Solving DEs with Power series   ⑤

$\dot{x} = ax \Rightarrow x(t) = e^{at} x_0$

$x(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \ldots$ polynomial taylor series expansion.

$x_0 = c_0$

$\dot{x}(t) = c_1 + 2c_2 t + 3c_3 t^2 + \ldots$

$\updownarrow \quad \updownarrow \quad \updownarrow$

$a\, x(t) = ac_0 + ac_1 t + ac_2 t^2 + \ldots$

$\dot{x} = ax$ match each power of t.

$c_1 = ac_0 = ax_0$ $c_1 = ax_0$

$2c_2 = ac_1 = a^2 c_0 = a^2 x_0$ $c_2 = \frac{a^2}{2} x_0$

$3c_3 = ac_2 = \frac{a^2}{2} c_1 = \frac{a^3}{2} x_0$ $c_3 = \frac{a^3}{3.2} x_0$

$\vdots$ $\vdots$

$$\boxed{c_{N+1} = \frac{a^{N+1}}{(N+1)!} x_0}$$

$x(t) = x_0 + x_0 at + x_0 \frac{a^2 t^2}{2} + \ldots + x_0 \frac{a^N t^N}{N!}$

$x(t) = x_0 \underbrace{\left[1 + at + \frac{a^2 t^2}{2!} + \ldots + \frac{a^N t^N}{N!}\right]}_{e^{at}}$ power series expansion.

Taylor Series & Power Series

A function $f(x+\Delta x)$ may be Taylor expanded about a base point x (If f is smooth at x)

$$f(x+\Delta x) = f(x) + \frac{df}{dx}(x)\,\Delta x + \frac{d^2f}{dx^2}(x)\frac{\Delta x^2}{2!} + \dots + \frac{d^nf}{dx^n}(x)\frac{\Delta x^n}{n!}$$

Identical ⋆

Or $f(x)$ expanded about a point "a"

$$\boxed{f(x) = f(a) + \frac{df}{dx}(a)(x-a) + \frac{d^2f}{dx^2}(a)\frac{(x-a)^2}{2!} + \dots + \frac{d^nf}{dx^n}(a)\frac{(x-a)^n}{n!}}$$

$(x-a)$ a x $\quad x = a+\Delta x$

Ex: $\sin(x) = f(x)$ $\quad a=0$ (Maclaurin series)

$$f(x) = f(0) + f'(0)\,x + f''(0)\frac{x^2}{2!} + \dots = \sin(x)$$

$$= \cancel{\sin(0)} + \cos(0)\,x - \cancel{\frac{\sin(0)}{2!}}x^2 - \frac{\cos(0)}{3!}x^3 + \cancel{\frac{\sin(0)}{4!}}x^4 + \frac{\cos(0)}{5!}x^5 + \dots$$

$$\boxed{\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \dots}$$

$$\boxed{\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots}$$

Taylor Series of the Exponential Function and Euler's formula

$f(x) = e^x$

$$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots = \sum_{k=0}^{\infty}\frac{x^k}{k!}}$$

$$e^{ix} = 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \dots$$

$$= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} + \dots$$

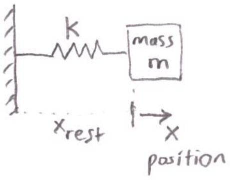
$$\boxed{e^{ix} = \cos(x) + i\sin(x)}$$ Euler's formula

$$\dot{x} = \lambda x \Rightarrow x(t) = e^{\lambda t}x(0)$$

λ ↓ imaginary complex $\qquad e^{\lambda t}$ ↓ cos & sin

Second order systems

Harmonic Oscillator



Newton's 2nd Law

$$F = ma = m\frac{d^2x}{dt^2}$$

Second derivative.

$$F_{spring} = -kx \quad , k=1, m=1 \Rightarrow \frac{d^2x}{dt^2} = -x \Rightarrow \boxed{\ddot{x} = -x}$$

2nd order OD[E]

$$\underbrace{\begin{matrix} x(0) = x_0 \\ \dot{x}(0) = v_0 = \end{matrix}}_{\text{initial conditio[ns]}}$$

Method I : Guess!

$$\left.\begin{aligned} x(t) &= \cos(t)\, x_0 \\ \dot{x}(t) &= -\sin(t)\, x_0 \\ \ddot{x}(t) &= -\cos(t)\, x_0 \end{aligned}\right] \ddot{x}(t) = -x(t)$$

for general m, k

$$x(t) = \cos\left(\sqrt{\frac{k}{m}}\, t\right) x(0)$$

$$\omega = \sqrt{\frac{k}{m}} : \text{natural frequency}$$

Method II: Taylor Series

$$x(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + \dots \Rightarrow x(0) = x_0 = c_0$$

$$\dot{x}(t) = c_1 + 2c_2 t + 3c_3 t^2 + 4c_4 t^3 + \dots \Rightarrow \dot{x}(0) = v_0 = c_1$$

$$\ddot{x}(t) = 2c_2 + 3.2\, c_3 t + 4.3\, c_4 t^2 + \dots$$

$$\ddot{x} = -x \Rightarrow \left.\begin{aligned} 2c_2 &= -c_0 \\ 3.2\, c_3 &= -c_1 \\ 4.3\, c_4 &= -c_2 \\ &\vdots \end{aligned}\right\} \quad \begin{aligned} c_2 &= \frac{-c_0}{2} = \frac{-x_0}{2!} \\ c_3 &= \frac{-c_1}{3.2} = \frac{-v_0}{3!} \\ c_4 &= \frac{-c_2}{4.3} = \frac{x_0}{4!} \end{aligned}$$

$$x(t) = x_0 + v_0 t + \frac{-x_0}{2!}t^2 + \frac{-v_0}{3!}t^3 + \frac{x_0}{4!}t^4 + \dots = v_0 \sin(t) + x_0 \cos(t) \Rightarrow \boxed{x(t) = \cos(t)\, x_0 + \sin(t)\, v_0}$$

Method III : Guess again!

what function when taking multiple derivative is similar to itself up to a constant?

$$\frac{d}{dt}(e^t) = e^t$$

$$x(t) = e^{\lambda t} \leftarrow \text{guess.}$$

$$\dot{x} = \lambda e^{\lambda t}$$

$$\ddot{x} = \lambda^2 e^{\lambda t}$$

$$\ddot{x} = -x \Rightarrow \lambda^2 e^{\lambda t} = -e^{\lambda t}$$

$$\lambda^2 = -1$$

$$\lambda = \pm\sqrt{-1} = \pm i$$

$$x(t) = c_1 e^{it} + c_2 e^{-it}$$

real, complex, complex

euler's formula:

$$e^{it} = \cos(t) + i\sin(t)$$

$$e^{-it} = \cos(t) - i\sin(t)$$

$$x(t) = (c_1 + c_2)\cos(t) + i(c_1 - c_2)\sin(t)$$

$$x(0) = c_1 + c_2 = x_0$$

$$\dot{x}(0) = i(c_1 - c_2) = v_0$$

$$\boxed{x(t) = x_0 \cos(t) + v_0 \sin(t)}$$

Method IV : "suspend variables"

Introduce a new variable $\frac{dx}{dt} = v$

$$\dot{x} = v$$

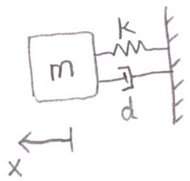
$$\dot{v} = -x$$

$$\underbrace{\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix}}_{\text{System of 1st order ODEs}} \star \qquad x(0) = \begin{bmatrix} x(0) \\ v(0) \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} \underline{\underline{A}} \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} \qquad \underline{x} = \begin{bmatrix} x \\ v \end{bmatrix} \qquad \dot{\underline{x}} = \underline{\underline{A}}\,\underline{x} \Rightarrow \underline{x}(t) = e^{\underline{\underline{A}}t}\,\underline{x}(0)$$

$\lambda = \pm i \leftarrow$ eigenvalues of A matrix.

ODE: Spring-Mass-Damper



$F = ma$

$m\ddot{x} = -kx - d\dot{x}$

$m\ddot{x} + d\dot{x} + kx = 0$

$\ddot{x} + \frac{d}{m}\dot{x} + \frac{k}{m}x = 0$

$2\zeta\omega = \frac{d}{m} \quad \frac{k}{m} = \omega^2$

$$\boxed{\ddot{x} + 2\zeta\omega\dot{x} + \omega^2 x = 0}$$

Guess: $x(t) = e^{\lambda t}$

$\dot{x}(t) = \lambda e^{\lambda t}$

$\ddot{x}(t) = \lambda^2 e^{\lambda t}$

$\lambda^2 e^{\lambda t} + 2\zeta\omega\lambda e^{\lambda t} + \omega^2 e^{\lambda t} = 0$

$$\boxed{\lambda^2 + 2\zeta\omega\lambda + \omega^2 = 0}$$

characteristic equation

λ = eigen values satisfy characteristic equation.

Note: $\det(A - \lambda I) = 0$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2\zeta\omega \pm \sqrt{4\zeta^2\omega^2 - 4\omega^2}}{2}$$

$$\lambda_{1,2} = \boxed{-\zeta\omega \pm \sqrt{\omega^2(\zeta^2 - 1)}}$$

$\zeta > 1$

$\zeta = 1$

$\zeta < 1$

$$\boxed{x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}}$$

linear superposition.

$\left.\begin{array}{l} x(0) = x_0 = c_1 + c_2 \\ \dot{x}(0) = v_0 = \lambda_1 c_1 + \lambda_2 c_2 \end{array}\right\}$ initial conditions.

Matrix system of equations. ★

$\dot{x} = v$

$\dot{v} = -\frac{d}{m}v - \frac{k}{m}x$

$\underline{\dot{x}} = \underline{\underline{A}}\,\underline{x}$

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{d}{m} \end{bmatrix}}_{\underline{\underline{A}}}\begin{bmatrix} x \\ v \end{bmatrix}$$

$\det(A - \lambda I) = 0$

$$\det\left(\begin{bmatrix} 0-\lambda & 1 \\ -\frac{k}{m} & -\frac{d}{m} - \lambda \end{bmatrix}\right) = 0$$

$-\lambda\left(-\frac{d}{m} - \lambda\right) + \frac{k}{m} = 0$

$$\boxed{\lambda^2 + \frac{d}{m}\lambda + \frac{k}{m} = 0}$$

Second Order Differential Equations

$\ddot{x} + 3\dot{x} + 2x = 0$

$x(0) = 2$

$\dot{x}(0) = -3$

Try $\left.\begin{array}{l} x(t) = e^{\lambda t} \\ \dot{x}(t) = \lambda e^{\lambda t} \\ \ddot{x}(t) = \lambda^2 e^{\lambda t} \end{array}\right\}$ $\lambda^2 e^{\lambda t} + 3\lambda e^{\lambda t} + 2e^{\lambda t} = 0$

$\lambda^2 + 3\lambda + 2 = 0$

$(\lambda+1)(\lambda+2) = 0$

$\lambda = -1, \lambda = -2$

Superposition

$x(t) = c_1 e^{-t} + c_2 e^{-2t}$

$\left.\begin{array}{l} x(0) = c_1 + c_2 = 2 \\ \dot{x}(0) = -c_1 - 2c_2 = -3 \end{array}\right\}$ Initial conditions

$c_2 = 1, c_1 = 1$

$$\boxed{x(t) = e^{-t} + e^{-2t}}$$

Matrix system

$\underline{x} = \begin{bmatrix} x \\ v \end{bmatrix} \quad \underline{\dot{x}} = \begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix}$

$\dot{x} = v$

$\dot{v} = -3\dot{x} - 2x = -3v - 2x$

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}}_{\underline{\underline{A}}}\begin{bmatrix} x \\ v \end{bmatrix}$$

$\det(A - \lambda I) = 0$

$-\lambda(-3-\lambda) + 2 = 0$

$\lambda^2 + 3\lambda + 2 = 0$

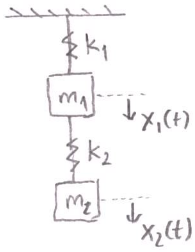
$\lambda = -1, \lambda = -2$ } eigenvalues.

Higher Order ODE Systems

$$a_n \frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \dots + a_2 \frac{d^2 x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = 0$$

nth derivative

$$a_n \overset{(n)}{x} + a_{n-1}\overset{(n-1)}{x} + \dots + a_2 \ddot{x} + a_1 \dot{x} + a_0 = 0$$



$$\sum \vec{F} = m\vec{a}$$

$$-k_1 x_1 + k_2(x_2 - x_1) = m_1 \ddot{x}_1$$

$$\left.\begin{cases} m_1\ddot{x}_1 + k_1 x_1 - k_2(x_2 - x_1) = 0 & (1) \\ m_2 \ddot{x}_2 + k_2(x_2 - x_1) = 0 & (2) \end{cases}\right\}$$ system of 2 coupled second order OD

★ single 4th order ODE

1. Solve (1) for $x_2 = f(x_1, \dot{x}_1, \ddot{x}_1)$
2. Take 2 derivatives
   $\ddot{x}_2 = f(\ddot{x}_1, \dddot{x}_1, \ddddot{x}_1)$
3. Plug $\ddot{x}_2$ to Eq (2)

$$x_2 = \frac{1}{k_2} m_1 \ddot{x}_1 + \frac{1}{k_2} k_1 x_1 + \frac{k_2}{k_2} x_1$$

$$x_2 = \frac{m_1}{k_2}\ddot{x}_1 + \frac{k_1}{k_2}\dot{x}_1 + x_1$$

$$\ddot{x}_2 = \frac{m_1}{k_2} x_1^{(4)} + \frac{k_1}{k_2}\dddot{x}_1 + \ddot{x}_1$$

$$\frac{m_2 m_1}{k_2} x_1^{(4)} + \frac{m_2 k_1}{k_2}\dddot{x}_1 + m_2 \ddot{x}_1 + m_1 \ddot{x}_1 + k_1 x_1 + k_2 x_1 - k_2 x_1 = 0$$

$$\boxed{\frac{m_1 m_2}{k_2} x_1^{(4)} + \frac{k_1 m_2}{k_2} + (m_2 + m_1)\ddot{x}_1 + k_1 x_1 = 0}$$

system of 4 coupled 1st order ODEs ★

$$\begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ \dot{v}_1 \\ \dot{x}_2 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix}$$

$$\dot{\underline{x}} = \underline{A} \cdot \underline{x}$$

$\dot{x}_1 = v_1 \quad \ddot{x}_1 = \dot{v}_1$

$\dot{x}_2 = v_2 \quad \ddot{x}_2 = \dot{v}_2$

$$m_1 \dot{v}_1 + \overbrace{k_1 x_1 - k_2 x_2 + k_2 x_1}^{(k_1+k_2)x_1} =$$

$$m_2 \dot{v}_2 + k_2 x_2 - k_2 x_1 = 0$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{(k_1+k_2)}{m_1} & 0 & \frac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & 0 & -\frac{k_2}{m_2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix}$$

Solving General High-Order Linear ODEs

$$a_n \frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \dots + a_2 \frac{d^2 x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = 0$$

Try $x(t) = e^{\lambda t}$

$\dot{x}(t) = \lambda e^{\lambda t}$

$\ddot{x}(t) = \lambda^2 e^{\lambda t}$

$\vdots$

$x^{(n)}(t) = \lambda^n e^{\lambda t}$

$$e^{\lambda t}\Big[\underbrace{a_n \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_2 \lambda^2 + a_1 \lambda + a_0}_{\text{characteristic equation.}}\Big] = 0$$

$e^{\lambda t}$: only 0 when $t \to -\infty$

characteristic equation ↳ has n roots $\lambda_1, \lambda_2 \dots \lambda_n$

Superposition (Linear)

$$\boxed{x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \dots + c_n e^{\lambda_n t}}$$ ★ Generic solution for high order linear ODE.

Need n initial conditions to uniquely determine constants $(c_1, c_2 \dots c_n)$

$$\begin{bmatrix} x(0) \\ \dot{x}(0) \\ \ddot{x}(0) \\ \vdots \\ x^{(n-1)}(0) \end{bmatrix} \text{ to solve for } \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} x(0) \\ \dot{x}(0) \\ \vdots \\ \overset{n-1}{x}(0) \end{bmatrix}}_{D} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \dots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \dots & \lambda_n^2 \\ \vdots & & & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & & \dots & \lambda_n^{n-1} \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix}}_{C}$$

$x(0) = c_1 + c_2 + c_3 + \dots + c_n$

$\dot{x}(0) = c_1\lambda_1 + c_2\lambda_2 + \dots + c_n\lambda_n$

$\vdots$

$x^{(n-1)}(0) = c_1\lambda_1^{n-1} + c_2\lambda_2^{n-1} + \dots + c_n\lambda_n^{n-1}$

$$\boxed{C = M^{-1} D}$$

M: Vandermonde Matrix. ★

Matrix Systems of Differential Equations

(13)

$$x^{(n)} + a_{n-1}x^{(n-1)} + \dots + a_2\ddot{x} + a_1\dot{x} + a_0 x = 0$$

Introduce new variables

$$\begin{aligned}
x_1 &= x & \dot{x}_1 &= x_2 = x^{(1)} \\
x_2 &= \dot{x} & \dot{x}_2 &= x_3 = x^{(2)} \\
x_3 &= \ddot{x} & \dot{x}_3 &= x_4 = x^{(3)} \\
&\vdots & &\vdots \\
x_n &= x^{(n-1)} & \dot{x}_n &= x^{(n)} = -a_0x_1 - a_1x_2 - \dots - a_{n-1}x_n
\end{aligned}$$

System of n coupled 1st order ODEs.

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 \dots & 0 & 0 \\ \vdots & \vdots & & & \vdots & \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-2} & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

↳ Eigenvalues of A are the roots of the characteristic polynomial

$$\boxed{\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x}}$$

↖ Linear Algebra & Differential Equations. ★

Eigen values of A are λ's satisfy $\det(A - \lambda I) = 0$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} \qquad \dot{\underline{x}} = \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

Ex:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}}^{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad A - \lambda I = \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -a_0 & -a_1 & -a_2 - \lambda \end{bmatrix}$$

$$\begin{aligned}
\det(A - \lambda I) &= -\lambda\left(-\lambda(-a_2 - \lambda) + a_1\right) + (+a_0) + 0 = 0 \\
&-\lambda(+\lambda a_2 + \lambda^2 + a_1) - a_0 = 0 \\
&-\lambda^3 - \lambda^2 a_2 - \lambda a_1 - a_0 = 0 \\
&\boxed{\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0}
\end{aligned}$$

$$\dddot{x} + a_2\ddot{x} + a_1\dot{x} + a_0 x = 0$$

$$\boxed{\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0}$$

Eigenvalues and Eigenvectors

(14)

Solving $\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x}$ — Necessary coordinate transformation to simplify the $\dot{x} = Ax$.

Case 1. Decoupled Dynamics.

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \underbrace{\begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}}_{D} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

D ↘ diagonal (decoupled) matrix.

$$\begin{aligned}
\dot{x}_1 &= \lambda_1 x_1 \rightarrow x_1(t) = e^{\lambda_1 t}x_1(0) \\
\dot{x}_2 &= \lambda x_2 \rightarrow x_2(t) = e^{\lambda_2 t}x_2(0) \\
&\vdots \\
\dot{x}_n &= \lambda x_n \rightarrow x_n(t) = e^{\lambda_n t}x_n(0)
\end{aligned}$$

$$\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}(t)}_{\underline{x}(t)} = \underbrace{\begin{bmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & & \\ & & & \\ 0 & 0 & \dots & e^{\lambda_n t} \end{bmatrix}}_{e^{Dt}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}(0)}_{\text{initial conditions. } \underline{x}(0)}$$

For general system $\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x}$ we want a change of coordinates $\underline{x} = \underline{\underline{T}}\,\underline{z}$ (↗ invertable (non zero determinant)) that diagonalizes the ODE. ★

$$\dot{\underline{z}} = D\underline{z} \Rightarrow \underline{z}(t) = e^{Dt}\underline{z}(0)$$

$$\underline{z} = T^{-1}x \Leftarrow x = Tz$$

$$T\dot{\underline{z}} = \dot{x} = Ax \Rightarrow \underline{\underline{T}}\dot{\underline{z}} = \underline{\underline{A}}\,\underline{\underline{T}}\,\underline{z}$$

$$\dot{\underline{z}} = \underbrace{\underline{\underline{T}}^{-1}\underline{\underline{A}}\,\underline{\underline{T}}}_{D}\,\underline{z} \qquad D = T^{-1}AT$$

we want a T matrix so that $T^{-1}AT = D$

$$AT = TD$$

$$\underline{\underline{A}}\begin{bmatrix} | & | & | & & | \\ t_1 & t_2 & t_3 & \dots & t_n \\ | & | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & | & | \\ t_1 & t_2 & t_3 & t_n \\ | & | & | & | \end{bmatrix}\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

↓ eigenvectors — ↑ eigenvalues.

$$= \begin{bmatrix} | & | & & | \\ \lambda_1 t_1 & \lambda_2 t_2 & \dots & \lambda_n t_n \\ | & | & & | \end{bmatrix}$$

$$\boxed{\underline{\underline{A}}\,\underline{\underline{T}} = \underline{\underline{T}}\,\underline{\underline{D}}}$$ ★ Eigenvalue Equation.

★ The diagonal elements of D
↳ Eigen values of A

★ The coloums of T
↳ Eigen vectors of A

$\boxed{A t_1 = \lambda_1 t_1}$

Eigenvalues & Eigenvectors

$\boxed{AT = TD}$

columns are eigenvectors.

diagonal entries are eigenvalues

$\dot{x} = \lambda x$

$\boxed{Ax = \lambda x}$ for special vector x, for special values λ

* Eigenvalue equation for pair (x, λ)

Ex: $A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$

try $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow Ax = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

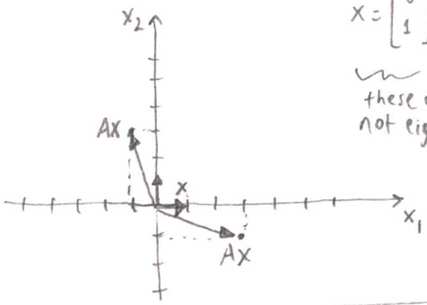
$x = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow Ax = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

rotates

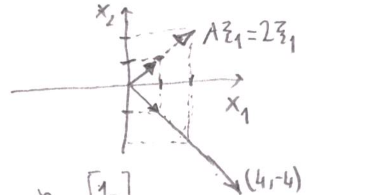
these are not eigenvectors.

special $\xi_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow Ax = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

scalar (no rotation) $\lambda_1 = 2$



Eigenvector, eigenvalue pair (ξ_1, λ_1)



special $\xi_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\lambda_2 = 4$

$A\xi_2 = 4\xi_2$

$Ax = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$

2 pair of eigenvectors & eigenvalue.

$Ax = \lambda x = \lambda I x$

$(\underline{A} - \lambda \underline{I})\underline{x} = 0$

$\underline{x}$ → vector

Case1 $x = 0$ (solution but not a vector)

Case 2 $x \neq 0$, and $\boxed{\det(A - \lambda I) = 0}$

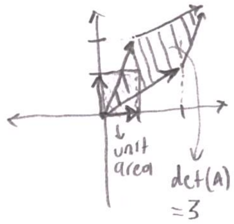
is singular

Meaning it maps some vectors to 0

polynomial equation in λ and roots are the eigenvalues. (characteristic equation)

Determinant measures the volume of a unit cube after all sides mapped through A. (nth dimension.)

Ex: $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$



$\det(A) = 0$ means no area (losing 1D)

$\det(A - \lambda I) = \det \begin{bmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{bmatrix} \Rightarrow \lambda^2 - 6\lambda + 8 = 0$ $\begin{matrix} -4 \\ -2 \end{matrix}$

$\lambda_1 = 4, \lambda_2 = 2$

$(A - \lambda I)x = 0$

↓ eigenvector

For $\lambda_2 = 2$ $\begin{bmatrix} x_1 - x_2 \\ -x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \xi_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For $\lambda_1 = 4$ $\begin{bmatrix} -x_1 - x_2 \\ -x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \xi_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

eigenvectors (not normalized)

$(A - \lambda I)$ for $\lambda_2 = 2 \Rightarrow \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - x \\ -x_1 + x \end{bmatrix}$

for $\lambda_1 = 4 \Rightarrow \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 - x \\ -x_1 - \end{bmatrix}$

Solving Systems of Differential Equations with Eigenvalues and Eigenvectors

$\dot{\underline{x}} = \underline{A}\underline{x}$ $\boxed{x(t) = e^{At} x(0)}$

$\underline{x} = \underline{T}\underline{z} \Rightarrow \dot{\underline{z}} = \underline{D}\underline{z}$

$\Downarrow$

$z(t) = e^{Dt} z(0)$

$AT = TD$

$D = T^{-1}AT$

$A = TDT^{-1}$

$A^2 = TDT^{-1}TDT^{-1} = TD^2T^{-1}$

$A^3 = TD^3T^{-1}$

$\vdots$

easy to compute D^n (diagonal entries)

$\dot{\underline{x}} = \underline{A}\underline{x}$

$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots$ ← not easy to compute

$e^{At} = TT^{-1} + TDT^{-1}t + TD^2T^{-1}\frac{t^2}{2!} + \dots$

$e^{At} = T\underbrace{\left[I + Dt + \frac{D^2t^2}{2!} + \dots\right]}T^{-1}$

$e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & & \ddots \end{bmatrix}$

$e^{At} = Te^{Dt}T^{-1}$

$\boxed{x(t) = Te^{Dt}T^{-1}x(0)}$ ← easy to compute

$\underbrace{T^{-1}x(0)}_{z(0)}$ → writing ICs in z-coordinates.

$\underbrace{e^{Dt}T^{-1}x(0)}_{z(t)}$ → Solution in z coordinates

$\underbrace{Te^{Dt}T^{-1}x(0)}_{x(t)}$ → Solution in x coordinates.

2x2 Systems of ODEs : Sources & Sinks

(17)

$\underline{\dot{x}} = \underline{\underline{A}}\,\underline{x} \qquad A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix},\ T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},\ D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$

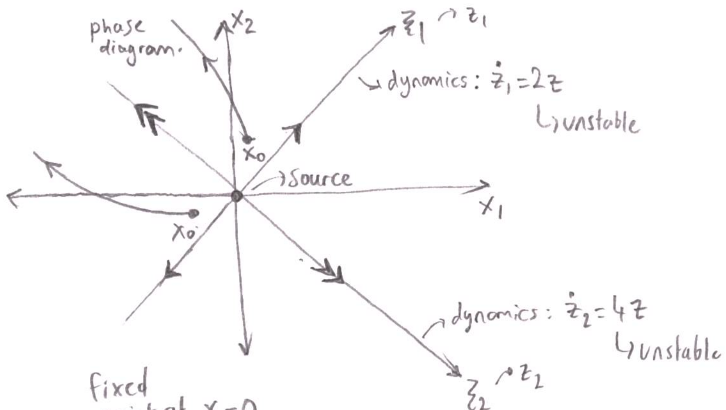
$x(t) = e^{At} x(0) = T e^{Dt} T^{-1} x(0)$

$x(t) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{4t} \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix} x_0 = \begin{bmatrix} e^{2t} & e^{4t} \\ e^{2t} & -e^{4t} \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix} x_0$

$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0.5e^{2t} + 0.5e^{4t} & 0.5e^{2t} - 0.5e^{4t} \\ 0.5e^{2t} - 0.5e^{4t} & 0.5e^{2t} + 0.5e^{4t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$

$T^{-1}: \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right] = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -2 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0.5 & -0.5 \end{bmatrix}$

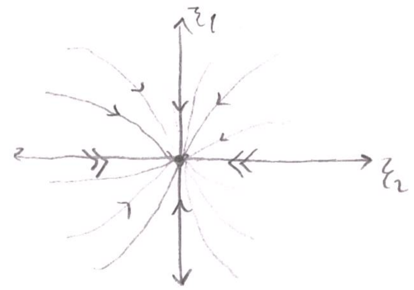
$T^{-1}: \left[\begin{array}{cc|cc} 1 & 0 & 0.5 & 0.5 \\ 0 & 1 & 0.5 & -0.5 \end{array}\right] \Rightarrow T^{-1} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix}$



fixed point at $\underline{x} = 0$

Unstable source (node)

when $\lambda_1 = -2,\ \lambda_2 = -4$



stable sink (node)

2x2 Systems of ODEs : Saddle Points and Instability

(18)

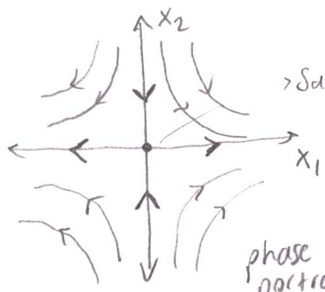
$\underline{\dot{x}} = \underline{\underline{A}}\,\underline{x} \qquad \underline{\underline{A}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$x(t) = e^{At} x(0) = T e^{Dt} T^{-1}$

already diagonal $\lambda_1 = 1, \lambda_2 = -1$

e^{t}, e^{-t}

$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$



Saddle point. (unstable)

fixed point at $\underline{x} = 0$

phase portrait.

2x2 Systems of ODEs: Imaginary Eigenvalues and Center Fixed Points.

$\dot{x} = Ax$

$x(t) = e^{At}x(0) = Te^{Dt}T^{-1}x(0)$

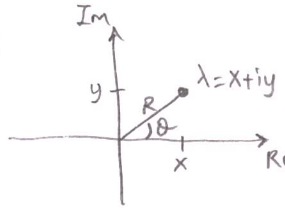
$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & +2 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\det(A - \lambda I) = 0$

$\lambda^2 + 4 = 0$

$\lambda = \pm 2i$

$\mathbb{C}$



$x = R\cos(\theta)$ $\quad e^{i\theta} = \cos\theta + i\sin\theta$

$y = R\sin(\theta)$

$\boxed{\lambda(R,\theta) = Re^{i\theta}} = \underbrace{R\cos\theta}_{x} + \underbrace{R i\sin\theta}_{iy}$

polar coordinate representation.

$\lambda = Re^{i\theta}$

$\lambda^2 = R^2e^{2i\theta}$

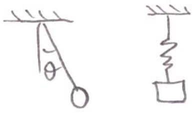
$\lambda^N = R^N e^{Ni\theta}$

ξ_1 for $\lambda_1 = 2i$, $A - 2iI = \begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix}$ $\quad \begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} -2ia + 2b = 0 \\ -2a - 2ib = 0 \end{matrix}$

$a = 1,\ b = i$

$\begin{bmatrix} 1 \\ i \end{bmatrix} \to$ eigen vector.

ξ_2 for $\lambda_2 = -2i$, ... $\begin{bmatrix} 1 \\ -i \end{bmatrix} \to$ eigen vector



$\ddot{x} = -x$

$\dot{x} = v$

$\dot{v} = -x$

$\ddot{\theta} = -\sin\theta$

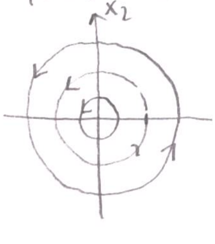
$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}\frac{1}{2}$$

$$D = \begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix}$$

$$x(t) = \underbrace{\underbrace{\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}}_{T}\underbrace{\begin{bmatrix} e^{2it} & 0 \\ 0 & e^{-2it} \end{bmatrix}}_{e^{Dt}}\underbrace{\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}\frac{1}{2}}_{T^{-1}}}_{\text{Real.}}x(0) = \underbrace{\begin{bmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{bmatrix}}_{\substack{\text{Real valued} \\ \text{Pure Rotation Matrix} \\ (R_{2t})}}x(0)$$

phase portrait.

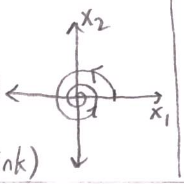


Ex: x_1 = position
x_2 = velocity
mass-spring

• center fixed point.	• stable ... marginally ... neutrally

For $A = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} \Rightarrow \lambda = -1 \pm 2i$

(decay) (oscilating)
(−)



Stable Spiral (sink)

Stability and Eigenvalues

$\dot{x} = Ax$

$\ddot{x} + 3\dot{x} + 2x = 0$

$x(t) = c_1e^{-t} + c_2e^{-2t}$

is stable because all fundamental solutions (2 solutions) e^{-t} and e^{-2t} are stable.

$AT = TD$

eigenvectors (T), eigenvalues (D)

$x(t) = Te^{Dt}T^{-1}x(0)$

$$e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & & \ddots\ e^{\lambda_n t} \end{bmatrix}$$

$\dot{x} = Ax$ is stable if all eigenvalues λ are stable.
unstable if even one eigenvalue is unstable.

$\lambda = a \pm ib$

$e^{\lambda t} = e^{at \pm ibt} = e^{at} \cdot e^{ibt}$

$$\underbrace{= \underbrace{e^{at}}_{\substack{\text{scales the} \\ \text{solution.}}}\underbrace{\left[\cos(bt) + i\sin(bt)\right]}_{\substack{\text{unit length (norm)} \\ \text{in the complex} \\ \text{plane}}}}_{\text{Real part of } \lambda \text{ determines stability.}}$$

$a < 0$ stable

$a = 0$ neutrally stable

$a > 0$ unstable.

"Linear" Differential Equation

Linear ODE $\dot{x} = Ax \Rightarrow \underline{x} = \underline{\underline{T}}\,\underline{z}$

$\dot{\underline{z}} = \underline{\underline{D}}\,\underline{z}$

$x(t) = Te^{Dt}T^{-1}x(0)$

$\downarrow$ adding all solutions.

$\ddot{x} + 3\dot{x} + 2x = 0$ is linear

Both $x(t) = e^{-t}$

$x(t) = e^{-2t}$

are solutions.

$x(t) = \alpha e^{-t} + \beta e^{-2t}$ is also a solution

* $\boxed{\text{Linear superposition}}$

$\dot{x} = -\alpha e^{-t} - 2\beta e^{-2t}$

$\ddot{x} = \alpha e^{t} + 4\beta e^{-2t}$

$\ddot{x} + 3\dot{x} + 2x = 0$

$$\underbrace{\alpha e^{-t} - 3\alpha e^{-t} + \alpha e^{-t}}_{\substack{0\\ \text{first solution}}} + \underbrace{4\beta e^{-2t} - 6\beta e^{-2t} + 2\beta e^{-2t}}_{\substack{0\\ \text{second solution}}} = 0$$

What operations are linear?

$f(x+y) = f(x) + f(y)$, $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ ← Linear*

for any numbers α, β.

Not linear

$f(x) = x^2$

$f(x_1) = x_1^2$

$f(x_2) = x_2^2$

$f(x_1 + x_2) = (x_1 + x_2)^2 \neq x_1^2 + x_2^2$

$f(x) = \dfrac{dx}{dt}$ ← Linear* Derivatives are.

$\dfrac{d}{dt}(\alpha x + \beta y) = \alpha \dfrac{dx}{dt} + \beta \dfrac{dy}{dt} = \alpha f(x) + \beta f(y)$

Matrix multiplication is linear*

$f(\underline{x}) = \underline{\underline{A}}\,x$

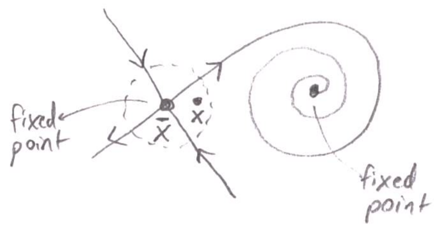
$$\underline{\underline{A}}(\alpha x_1 + \beta x_2) \Rightarrow \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}\begin{bmatrix} \alpha x_1 + \beta x_2 \\ \vdots \\ \alpha x_n + \beta x_n \end{bmatrix} = \alpha \underline{\underline{A}}\,\underline{x_1} + \beta \underline{\underline{A}}\,\underline{x_2}$$ ← Linear.*

Linearizing Nonlinear Differential Equations Near a Fixed Point

$\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x}$ from $\dot{\underline{x}} = \underline{f}(\underline{x})$ → nonlinear

$\dot{x} = f(x)$ $\quad \bar{x}$ is a fixed point if $f(\bar{x}) = 0$

For x near $\bar{x}$, $x = \bar{x} + \Delta x$, $\Delta x = x - \bar{x}$ is small.



$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2 \dots) \\ f_2(x_1, x_2 \dots) \\ \vdots \\ f_n(x_1, x_2 \dots) \end{bmatrix}$$

$\dot{x} = f(x) = f(\bar{x} + \Delta x)$ $\quad$ Taylor Series

$$= \underbrace{f(\bar{x})}_{\substack{f(\bar{x})=0\\ \text{fixed point.}}} + \underbrace{\frac{Df}{Dx}(\bar{x})}_{\substack{\text{Matrix of derivatives}\\ (\text{Jacobian matrix evaluated at } \bar{x})}}\Delta x + \underbrace{\frac{D^2f}{Dx^2}(\bar{x})\frac{\Delta x^2}{2!} + \dots + \text{h.o.t}}_{\text{very small if } \Delta x \text{ is small}}$$

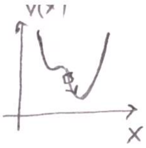
higher order terms. $\mathcal{O}(\Delta x^3)$

$$\dot{x} = \frac{d}{dt}(\underbrace{\bar{x}}_{\text{constant}} + \underbrace{\Delta x}_{x}) = \boxed{\frac{d}{dt}(\Delta x) = \frac{Df}{Dx}(\bar{x})\Delta x}$$

LINEAR ODE in Δx

$$\frac{Df}{Dx} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & & \frac{\partial f_2}{\partial x_n} \\ \vdots & & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$ ← matrix of partial derivatives (Jacobian Matrix)

Particle in a Potential Well: Nonlinear Dynamics



Consider a potential $V(x)$ and imagine dropping a particle "bead" (pos. x) can roll or slide down this potential surface.

Force is minus gradient of the potential ⇒ $\boxed{F = -\frac{\partial V}{\partial x}}$

Newton's 2nd law

$$\boxed{\ddot{x} = -\frac{\partial V}{\partial x}}$$

Could drive from Lagrange

Kinetic Energy $T = \frac{1}{2}\dot{x}^2$

Potential Energy $V(x)$

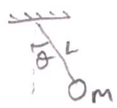
Lagrange: $L(x,\dot{x}) = T(\dot{x}) - V(x) = \frac{1}{2}\dot{x}^2 - V(x)$

Euler-Lagrange Equation:

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0} \;\star$$

$$\frac{d}{dt}\left(\dot{x}\right) - \frac{\partial}{\partial x}\left(V(x)\right) = 0 \Rightarrow \ddot{x} - \frac{\partial V}{\partial x} = 0$$

Ex:



$$\left.\begin{aligned} T &= \frac{1}{2} m L^2 \dot{\theta}^2 \\ V &= -mg\cos\theta \end{aligned}\right\} \ddot{\theta} = -\sin\theta$$

Drawing Phase Portraits for Nonlinear Systems.

$$\ddot{x} = -\frac{\partial V}{\partial x}$$

Ex: $\underbrace{\ddot{x} = -x + x^2}_{\text{nonlinear}} \Rightarrow \left(V(x) = \frac{x^2}{2} - \frac{x^3}{3}\right)$

$\dot{x} = v$

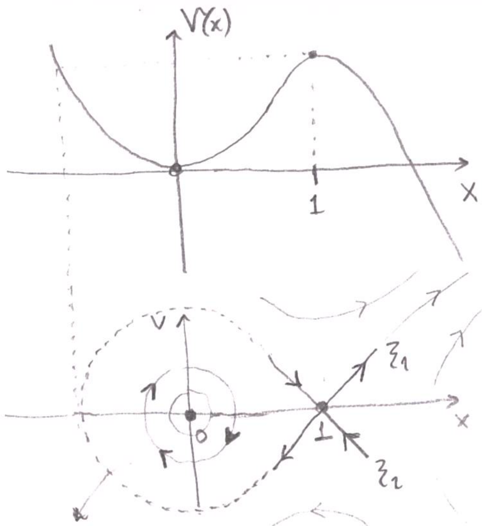
$\ddot{x} = \dot{v} = -x + x^2$

Fixed points

$\dot{\underline{x}} = 0 \Rightarrow \left.\begin{aligned} \dot{x} &= 0 \to v = 0 \\ \dot{v} &= 0 \to x = 1, x = 0 \end{aligned}\right\} \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} f_1(x,v) \\ f_2(x,v) \end{bmatrix} \Rightarrow \frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} v \\ -x+x^2 \end{bmatrix}$$

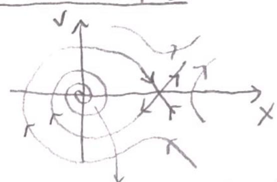
$$\frac{Df}{D\underline{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1+2x & 0 \end{bmatrix}$$



direction

Test $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ goes down.

If there is damping



Stable spiral sink

For fixed point $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\frac{Df}{D\underline{x}}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$\lambda = \pm i$ Linear

For fixed point $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\frac{Df}{D\underline{x}}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$\lambda = \pm 1$ Saddle

$\xi_1 \Rightarrow (A - \lambda I)\underline{x} = 0 \Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} = 0 \Rightarrow \xi_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \lambda = 1$

$\xi_2 \Rightarrow (A - \lambda I)\underline{x} = 0 \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} = 0 \Rightarrow \xi_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \lambda = -1$

Phase Portrait for Double Well Potential

$\ddot{x} = -\frac{\partial V}{\partial x}$ $\qquad V(x) = \frac{x^4}{4} - \frac{x^2}{2}$

$\ddot{x} = x - x^3 - \dot{x}$

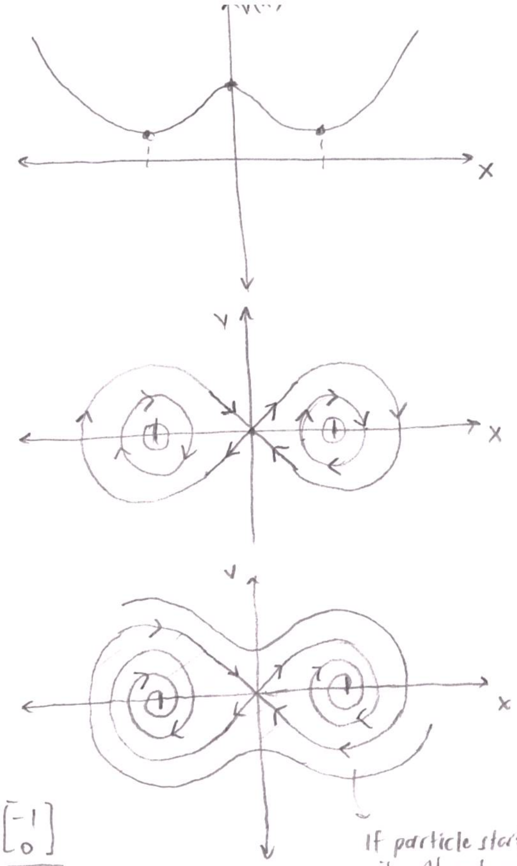
optional damping (friction)

$\dot{x} = v$
$\dot{v} = x - x^3 - v$

Fixed points

$\dot{\underline{x}} = 0$, $\begin{bmatrix}\dot{x}\\ \dot{v}\end{bmatrix} = 0$ $\rightarrow v = 0$, $x - x^3 - v = 0$

$\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\end{bmatrix}$ 3 fixed points.



Without friction

With friction.

If particle starts here it will end up at the left spiral sink.

Linearize about fixed points. (Find Jacobian)

$f(\underline{x}) = \begin{bmatrix} v \\ x - x^3 - v\end{bmatrix} \Rightarrow \frac{Df}{D\underline{x}} = \begin{bmatrix} 0 & 1 \\ 1-3x^2 & 0 \end{bmatrix}$ ($0 \to -1$ for friction)

For $\begin{bmatrix}0\\0\end{bmatrix}$

$\frac{Df}{D\underline{x}}\left(\begin{bmatrix}0\\0\end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0\end{bmatrix}$ ($0 \to -1$ for friction)

$\lambda = \pm 1$

Saddle

still saddle $\lambda = \pm 1$

For $\begin{bmatrix}1\\0\end{bmatrix}$

$\frac{Df}{D\underline{x}}\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ -2 & 0\end{bmatrix}$ ($0 \to -1$ for friction.)

center fixed point

For $\begin{bmatrix}-1\\0\end{bmatrix}$

$\frac{Df}{D\underline{x}}\left(\begin{bmatrix}-1\\0\end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ -2 & 0\end{bmatrix}$ ($0 \to -1$ for friction)

center fixed point.

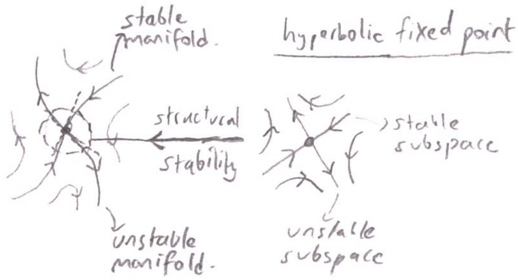
The Hartman-Grobman Theorem, Structural Stability of Linearization, and Stable/Unstable Manifolds.

$\dot{\underline{x}} = \underline{\underline{A}}\underline{x}$, $\dot{\underline{x}} = \underline{f}(\underline{x})$ near a fixed point $\bar{x}$ where $f(\underline{x}) = 0$

$A = \frac{Df}{D\underline{x}}(\bar{x})$ $\left[\begin{array}{l}\text{If all eigenvalues } \lambda \text{ of } A \\ \text{have non-zero real part} \\ \lambda = a \pm bi \\ a \neq 0\end{array}\right]$ called hyperbolic

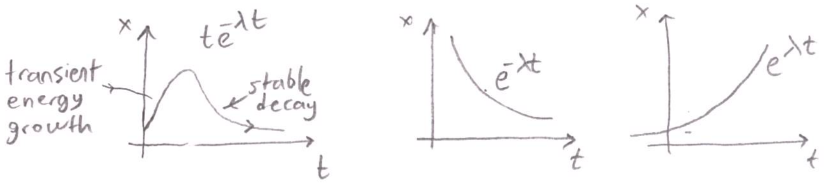
then locally for small neighborhood around $\bar{x}$ then linearized dynamics

$\dot{x} = Ax$ are faithfully representative of full non-linear system. ☆



Repeated eigenvalues and Secular Terms: Transient Growth in Non-Normal Systems.

$\dot{x} = Ax$ Non-Normal System: $A^T A \neq A A^T$
↳ A is not diagonalizable.



$$A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \Rightarrow e^{At} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}$$

repeated λ.

Secular (transient) term.

$$\begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix} = \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} = e^{st} e^{Tt}$$

Prof: If $ST = TS$ (not true in general)

then $e^{S+T} = e^S e^T$

use binomial theorem: $(S+T)^n = n! \sum_{J+k=n} \frac{S^J T^k}{J!\,k!}$

Verify:

$$A = \underbrace{\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}}_{S} + \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{T} \Rightarrow ST = \begin{bmatrix} 0 & \lambda \\ 0 & 0 \end{bmatrix} = TS \quad \Big\} \text{ so, } e^{At} = e^{(S+T)t} = e^{St} e^{Tt}$$

$$e^{St} = \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{bmatrix} \qquad e^{Tt} = I + Tt + \cancelto{0}{\frac{T^2 t^2}{2!}} + \cancelto{0}{\cdots} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

$$TT = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Non-Normal Linear Systems and Transient Energy Growth: Bypass Transition to Turbulence.

Non-normal A matrix. $\Rightarrow A^T A \neq A A^T$

(Almost degenerate)
$\lambda_1 \neq \lambda_2$ approximately repeated eigenvalues.

$$A = \begin{bmatrix} -0.009 & 1 \\ 0 & -0.01 \end{bmatrix}$$

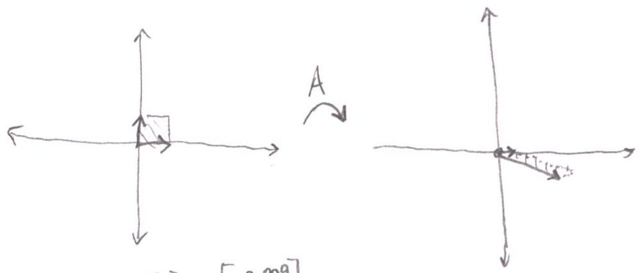
$\lambda = -0.01$
$\lambda = -0.009$

$(A - \lambda I)\xi = 0$
↓
eigenvectors

$$\begin{bmatrix} 1 & 1 \\ 0 & -0.001 \end{bmatrix}$$

↓ ↓
eigenvectors are almost in the same direction.

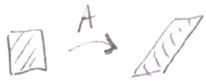
$$\xi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \xi_2 = \begin{bmatrix} 1 \\ -0.001 \end{bmatrix}$$



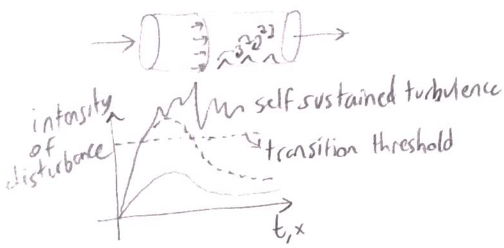
$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} -0.009 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} 1 \\ -0.01 \end{bmatrix}$

If the eigenvalues are very close and eigenvectors are almost parallel, transient energy growth occurs.



highly sheared



- actually system is stable but non-normal energy growth, this perturbation ... and excite

Systems of Differential Equations: Diagonalization and Jordan Canonical Form.

Case I.

Diagonalize A, with $\underbrace{\text{eigenvectors}}_{T}$ & $\underbrace{\text{eigenvalues}}_{D}$

$$T^{-1}AT = D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \ddots \\ & & & & \lambda_n \end{bmatrix}$$ distinct eigenvalues (not repeated)

If eigenvalues λ of A are real & distinct then eigenvectors T span $\mathbb{R}^n$

Case II.

Any complex eigenvalues λ must come in complex conjugate pairs if A real valued.

$\lambda_1 = a+ib \qquad \lambda_2 = a-ib$

$$T^{-1}AT = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$ → Not a diagonal matrix. It is structured as $\underbrace{\begin{bmatrix} a & b \\ -b & a \end{bmatrix}}_{\text{Jordan Block}}$

$$e^{At} = \begin{bmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{bmatrix} e^{at}$$

Case III

Repeated λ and not enough null space $(A-\lambda I)$

$$T^{-1}AT = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$ → Not diagonal.

$n \times n$ a matrix:

- Distinct, real, $\lambda_1, \lambda_2 \dots \lambda_m$
- Complex pair $\lambda \pm i\omega$
- Repeated pair $\mu \times 2$ (but $A-\mu I$ has 2D null space)
- Repeated pair $\gamma \times 2$ (but $A-\gamma I$ has 1D null space)

Jordan Canonical Form ★

$$\begin{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \ \lambda_n \end{bmatrix} & & & & 0 \\ & \begin{bmatrix} \lambda & \omega \\ -\omega & \lambda \end{bmatrix} & & & \\ & & \begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix} & & \\ 0 & & & \begin{bmatrix} \gamma & 1 \\ 0 & \gamma \end{bmatrix} \end{bmatrix}$$

Jordan Blocks

Universal for all A.

Normal Matrix $A^TA = AA^T$

Non-Normal Matrix $A^TA \neq AA^T$

★ Normal matrix means A is diagonalizable.

Note: $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ & a_{22} & a_{23} & \cdots \\ & & \ddots & \\ 0 & & & a_{nn} \end{bmatrix}$$ $\} \ (A-\lambda I)=0$

Thus, $a_{11} = \lambda_1$, $a_{22} = \lambda_2$, $\vdots$, $a_{nn} = \lambda_n$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ → repeated eigenvalues but still diagonalizable

$\lambda_1 = 1, \ \lambda_2 = 1$

$(A-\lambda I) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ 2D null-space

rank $=0$

any eigenvector.

$(A-\lambda I)\underline{x} = 0$

$T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\xi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \xi_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

different dynamics.

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

↳ never gonna be diagonalized.

$\lambda_1 = 1, \ \lambda_2 = 1$

$(A-\lambda I) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

rank $=1$

only has one distinct eigenvectors

$(A-\lambda I)\underline{x} = 0$

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\xi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$$e^{At} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}$$

- Transient behaviour.

$(A-\lambda I)^2 \underline{x} = 0$

↳ ξ_2 : 2nd generalized eigenvector

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ 2 dimensional null space

$\xi_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ any eigenvec

Case III.

Repeated λ eigenvalues.

$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$ rank$(A-\lambda I)=0$ (3 distinct eigenvectors)

$, te^{\lambda t}$

$\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$ rank$(A-\lambda I) = 1$, $(A-\lambda I)^2$ (2 distinct, 1 generalized)

$, t^2 e^{\lambda t}$

$\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$ rank$(A-\lambda I) = 2$, $(A-\lambda I)^2$, $(A-\lambda I)^3$ (1 distinct, 2 generalized)

Differential Equations with Forcing: Method of Undetermined Coefficients

(forcing could be a control or disturbance)

Method I: Undetermined coefficients
Method II: Variation of parameters

(*) $\ddot{x} + 3\dot{x} + 2x = 0$ "homogeneous"

(**) $\ddot{x} + 3\dot{x} + 2x = f(t)$ "inhomogeneous"

Ex: $f(t) = e^{-3t}$

$x(0) = 2$, $\dot{x}(0) = -4$

Characteristic Equation

$\lambda^2 + 3\lambda + 2 = 0$

$(\lambda + 2)(\lambda + 1) = 0$

$\lambda = -1$, $\lambda = -2$

Ex: $f(t) = \cos(wt)$

$x_p(t) = A\cos(wt) + B(\sin wt)$

- Part I: Solve the homogeneous equation (*)

$$(*) \to x(t) = c_1 e^{-t} + c_2 e^{-2t}$$

- Part II: Find a "Particular" solution to (**) by assuming $x_p(t) = ke^{-3t}$

$\dot{x}_p = -3ke^{-3t}$
$\ddot{x}_p = 9ke^{-3t}$

$$\ddot{x} + 3\dot{x} + 2x = f(t)$$

$$9ke^{-3t} + 3(-3)ke^{-3t} + 2(ke^{-3t}) = f(t)$$

$$2ke^{-3t} = e^{-3t} \Rightarrow k = 0.5$$

$$x_p(t) = 0.5\, e^{-3t}$$

- Part III: Add up all three terms

$$x(t) = \underbrace{c_1 e^{-t} + c_2 e^{-2t}} + \underbrace{0.5 e^{-3t}} \quad \text{superposition.}$$

$$\left.\begin{aligned} x(0) &= c_1 + c_2 + 0.5 = 2 \\ \dot{x}(0) &= -c_1 - 2c_2 - 1.5 = -4 \end{aligned}\right\} \begin{aligned} c_1 &= 0.5 \\ c_2 &= +1 \end{aligned}$$

$$x(t) = 0.5e^{-t} + e^{-2t} + 0.5e^{-3t}$$

Differential Equations with Forcing: Method of Variation of Parameters

$$x(t) = k_1 e^{-t} + k_2 e^{-2t}$$

$$x(t) = u_1(t) e^{-t} + u_2(t) e^{-2t}$$

$$\dot{x} = -u_1 e^{-t} + \dot{u}_1 e^{-t} + -2u_2 e^{-2t} + \dot{u}_2 e^{-2t}$$

$$\dot{x} = -u_1 e^{-t} - 2u_2 e^{-2t} + \boxed{\underbrace{\dot{u}_1 e^{-t} + \dot{u}_2 e^{-2t}}_{\text{assume that } = 0}}$$

$$\ddot{x} = (u_1 - \dot{u}_1)e^{-t} + (4u_2 - 2\dot{u}_2)e^{-2t}$$

$$\ddot{x} + 3\dot{x} + 2x = u_1 e^{-t} - \dot{u}_1 e^{-t} + 4u_2 e^{-2t} - 2\dot{u}_2 e^{-2t}$$
$$-3u_1 e^{-t} - 6u_2 e^{-2t} + 2u_1 e^{-t} + 2u_2 e^{-2t} = e^{-3t}$$

$$= e^{-t}(\cancelto{0}{u_1 - 3u_1 + 2u_1}) + e^{-2t}(\cancelto{0}{4u_2 - 6u_2 + 2u_2})$$

$$\boxed{-\dot{u}_1 e^{-t} - 2\dot{u}_2 e^{-2t} = e^{-3t}}$$

$$\dot{u}_1 e^{-t} + \dot{u}_2 e^{-2t} = 0 \Rightarrow \dot{u}_2 = -\dot{u}_1 e^{t}$$

$$-\dot{u}_1 e^{-t} - 2\dot{u}_2 e^{-2t} = e^{-3t} \Rightarrow \dot{u}_2 = -e^{-t} \Rightarrow u_2 = e^{-t} + k_2$$

$$\dot{u}_1 = e^{-2t} \Rightarrow u_1 = -0.5e^{-2t} +$$

Integrate.

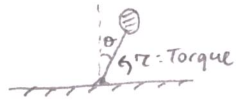
$$x(t) = u_1(t) e^{-t} + u_2(t) e^{-2t}$$

$$x(t) = (-0.5e^{-2t} + k_1)e^{-t} + (e^{-t} + k_2)e^{-2t}$$

$$\boxed{x(t) = \underbrace{0.5e^{-3t}}_{x_p(t)} + k_1 e^{-t} + k_2 e^{-2t}}$$

Systems of Differential Equations with Forcing: Example in Control Theory

$$\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\,\underline{u} \quad \Rightarrow \underline{x}(t) = e^{\underline{\underline{A}}t}\,\underline{x}(0) + \int_0^t e^{\underline{\underline{A}}(t-\tau)}\,\underline{\underline{B}}\,u(\tau)\,d\tau$$



$$\ddot{\theta} = -\sin\theta + \tau$$

$$\dot{\theta} = \omega$$
$$\dot{\omega} = -\sin\theta + \tau$$

$$\frac{d}{dt}\begin{bmatrix}\theta\\ \omega\end{bmatrix} = \underbrace{\begin{bmatrix}\omega\\ -\sin\theta\end{bmatrix}}_{f(x)} + \begin{bmatrix}0\\ \tau\end{bmatrix}$$

$$\frac{Df}{Dx} = \begin{bmatrix}0 & 1\\ -\cos\theta & 0\end{bmatrix}$$

For $\theta = \pi$ upright position

$$\left.\frac{Df}{Dx}\right|_{\theta=\pi} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \quad \lambda = \pm 1$$

$$\underbrace{\frac{d}{dt}\begin{bmatrix}\theta\\ \omega\end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}\theta\\ \omega\end{bmatrix}}_{x} + \underbrace{\begin{bmatrix}0\\ 1\end{bmatrix}}_{B}\underbrace{\tau}_{u}$$

Ex: Feedback for stability

If $\underbrace{\tau}_{u} = -2\theta - 2\omega = \underbrace{\begin{bmatrix}-2 & -2\end{bmatrix}}_{K}\underbrace{\begin{bmatrix}\theta\\ \omega\end{bmatrix}}_{u}$

$$\begin{bmatrix}0\\ 1\end{bmatrix}\tau = \begin{bmatrix}0 & 0\\ -2 & -2\end{bmatrix}\begin{bmatrix}\theta\\ \omega\end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix}\theta\\ \omega\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 1\\ -1 & -2\end{bmatrix}}_{\text{controlled system}}\begin{bmatrix}\theta\\ \omega\end{bmatrix}$$

$$\underbrace{(-2-\lambda)\lambda + 1 = 0 \Rightarrow \lambda = -1, -1}_{\text{controlled system is stable.}} \ \Big\}\ \text{stable.}$$

$u = Kx$ feedback control

$$\dot{x} = Ax + Bu$$
$$\dot{x} = Ax + BKu$$
$$\dot{x} = \underbrace{(A + BK)}_{\text{new dynamics.}}u$$

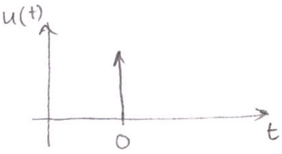
Linear Systems of Differential Equations with Forcing: Convolution and the Dirac Delta Function.

Case 1: $u(t) = 0$ and $x(0)$

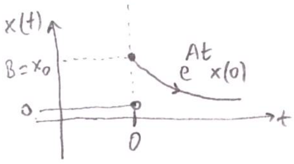
$$\dot{x} = Ax \Rightarrow x(t) = e^{At}x(0) \ \Big\}\ \text{initial condition response}$$

Case 2: $x(0) = 0$ and $u(t) = \delta(t)$ and $\underbrace{\underline{\underline{B}} = \underline{x_0}}_{\text{assume}}$

$$\dot{x} = Ax + Bu \Rightarrow x(t) = \int_{t_0}^{t}(Ax + Bu)\,d\tau$$

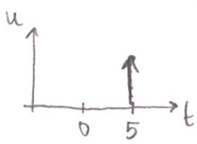


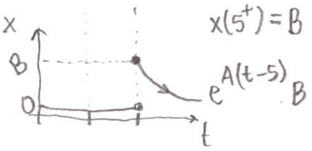
$$x(0^+) = \underbrace{\underline{\underline{B}} = \underline{x_0}}_{\text{let it equal to } x_0}$$



Ex: $\dot{x} = Ax + Bu \quad x(0) = 0$

$u = \delta(t-5)$ Impulse @ $t = 5$





$$x(5^+) = B$$

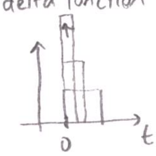
Impulse Response

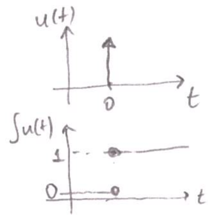
$$x(t) = \begin{cases} 0, & t < 5 \\ e^{A(t-5)}B, & t \geq 5 \end{cases}$$

$\delta(t)$ - Dirac delta function

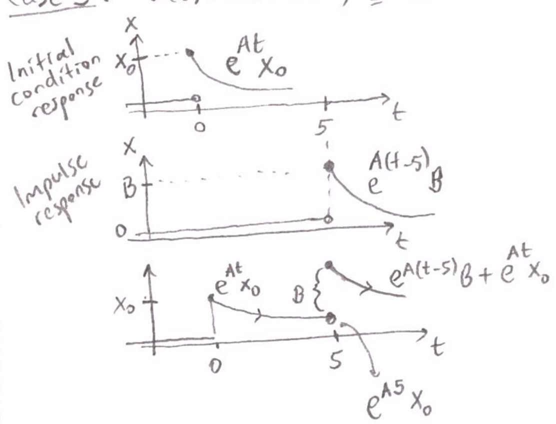
$$\int_{-\infty}^{\infty}\delta(t)\,dt = 1$$

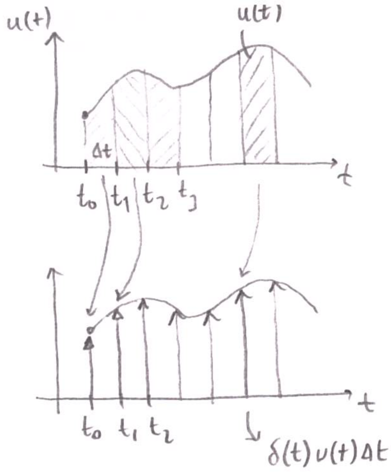
$$\int_{-\varepsilon}^{\varepsilon}\delta(t)\,dt = 1 \quad \text{for any } \varepsilon > 0$$



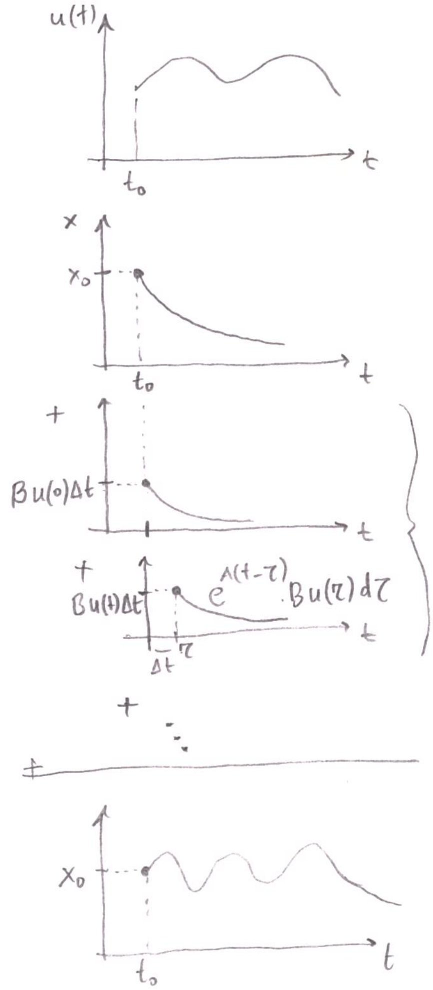


Case 3: $u(t) = \delta(t-5)$, $\underline{\underline{B}} = \underline{B}$ and $\underline{x}(0) = \underline{x_0}$





$\lim\limits_{\Delta t \to 0}$ becomes an infinite train of infinitesimal delta functions



$$\underbrace{\int_0^t e^{A(t-\tau)} Bu(\tau)\,d\tau}_{\text{Convolution Integral}}$$

$$\boxed{x(t) = e^{At}x_0 + \underbrace{\int_0^t e^{A(t-\tau)} B u(\tau)\,d\tau}_{h * u}}$$

$$\boxed{h(t) = \underbrace{e^{At}B}_{\text{impulse response}}}$$

Forced Systems of Differential Equations

$$\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\,\underline{u}$$
↳ Input

$$\underline{y} = \underline{\underline{C}}\,\underline{x} + \underline{\underline{D}}\,\underline{u}$$

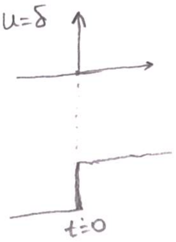
Assume $y = x$, $C = I$, $D = 0$

↓ output ↓ internal state

State-Space System

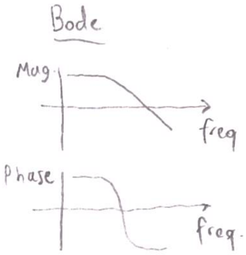
Impulse $u = \delta$

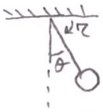
Step



$$u = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

Bode





Nonlinear

$$\ddot{\theta} = -\sin(\theta) + \tau$$

linearized at fixed point $\theta = 0$ $\quad \left.\dfrac{Df}{D\underline{x}}\right|_{\underline{x}=\begin{bmatrix}0\\0\end{bmatrix}}$ ⇒ Jacobian

$$\frac{d}{dt}\begin{bmatrix}\theta\\ \omega\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}\theta\\ \omega\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}\tau$$

$$A = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}, \quad B = \begin{bmatrix}0\\1\end{bmatrix}, \quad C = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \quad D = \begin{bmatrix}0\\0\end{bmatrix}$$

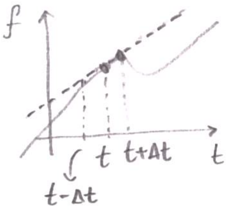
$$A = \begin{bmatrix}0 & 1\\ -1 & -0.1\end{bmatrix}$$ → adding damping

$$\frac{d}{dt}\begin{bmatrix}\theta\\ \omega\end{bmatrix} = \begin{bmatrix}f_1(\theta,\omega)\\ f_2(\theta,\omega)\end{bmatrix}$$

$$\frac{Df}{D\underline{x}} = \begin{bmatrix}\frac{\partial f_1}{\partial\theta} & \frac{\partial f_1}{\partial\omega}\\ \frac{\partial f_2}{\partial\theta} & \frac{\partial f_2}{\partial\omega}\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\cos(\theta) & 0\end{bmatrix} \underset{\theta=0}{\overset{\Rightarrow}{\text{at}}} \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$$

$$\left.\begin{aligned}\dot{\theta} &= \omega\\ \dot{\omega} &= -\sin(\theta) + \tau\end{aligned}\right\} \frac{d}{dt}\begin{bmatrix}\theta\\ \omega\end{bmatrix} = \begin{bmatrix}f_1(\theta,\omega)\\ f_2(\theta,\omega)\end{bmatrix} = \begin{bmatrix}\omega\\ -\sin\theta + \tau\end{bmatrix}$$

Numerical Differentiation



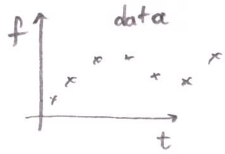
$$\frac{df}{dt} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

$$\frac{df}{dt} \approx \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

numerical integration

$$\frac{dx}{dt} = f(x) \Rightarrow \frac{x(t+\Delta t) - x(t)}{\Delta t} \approx f(t)$$

$$\Rightarrow x(t+\Delta t) \approx x(t) + \Delta t\, f(x(t))$$



$$f(t+\Delta t) = f(t) + \frac{df}{dt}(t)\Delta t + \frac{\Delta t^2}{2!}\frac{d^2f}{dt^2}(t) + \frac{\Delta t^3}{3!}\frac{d^3f}{dt^3}(t) + \dots + \theta(\Delta t^4) \quad \text{Taylor Series.}$$

$$f(t-\Delta t) = f(t) - \frac{df}{dt}(t)\Delta t + \frac{\Delta t^2}{2!}\frac{d^2f}{dt^2}(t) - \frac{\Delta t^3}{3!}\frac{d^3f}{dt^3}(t) + \dots + \theta(\Delta t^4)$$

* Forward difference

$$\frac{df}{dt} \approx \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

* Backward difference

$$\frac{df}{dt} \approx \frac{f(t) - f(t-\Delta t)}{\Delta t}$$

Forward difference

$$\frac{df}{dt} \approx \frac{f(t+\Delta t) - f(t)}{\Delta t} = \underbrace{\frac{df}{dt}}_{} + \underbrace{\overbrace{\frac{d^2f}{dt^2}\frac{\Delta t}{2!}}^{\text{leading order error term}} + \frac{d^3f}{dt^3}\frac{\Delta t^2}{3!} + \dots}_{\text{Error } \theta(\Delta t)}$$

Backward difference

$$\frac{df}{dt} \approx \frac{f(t) - f(t-\Delta t)}{\Delta t} = \underbrace{\frac{df}{dt}}_{} \underbrace{- \frac{d^2f}{dt^2}\frac{\Delta t}{2!} + \frac{d^3f}{dt^3}\frac{\Delta t^2}{3!} + \dots}_{\text{Error } \theta(\Delta t)}$$

Central difference

$$\frac{df}{dt} \approx \frac{f(t+\Delta t) - f(t-\Delta t)}{\Delta t} = \underbrace{\frac{df}{dt}}_{} + 2\left[\frac{d^3f}{dt^3}\frac{\Delta t^3}{3!} + \frac{d^5f}{dt^5}\frac{\Delta t^5}{5!} + \dots + \theta(\Delta t^7)\right.$$

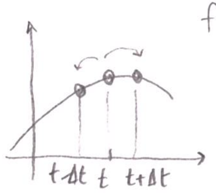
$$\frac{df}{dt} \approx \frac{f(t+\Delta t) - f(t-\Delta t)}{2\Delta t} = \underbrace{\frac{df}{dt}}_{} + \underbrace{\frac{d^3f}{dt^3}\frac{\Delta t^2}{3!} + \frac{d^5f}{dt^5}\frac{\Delta t^4}{5!} + \dots + \theta(\Delta t^6)}_{\text{Error } \theta(\Delta t^2)}$$

★

Numerical Differentiation: Second Derivatives and Differentiating Data

Second Derivative

$$\underbrace{\frac{\partial^2 f}{\partial t^2}}_{f''(t)} = \lim_{\Delta t \to 0} \frac{f'(t+\Delta t) - f'(t)}{\Delta t}$$

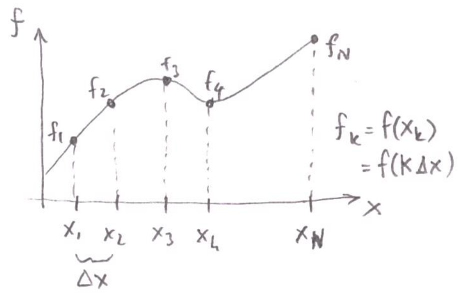


$$f(t+\Delta t) + f(t-\Delta t) = 2f(t) + \Delta t^2 f''(t) + \theta(\Delta t^4)$$

$$\boxed{f''(t) \approx \frac{f(t+\Delta t) - 2f(t) + f(t-\Delta t)}{\Delta t^2} + \underbrace{\theta(t^2)}_{\text{error}}}$$

Central difference 2nd derivative.

Differentiating Data



$$\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_N \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_N \end{bmatrix}}_{\text{Data}} \Bigg\} \frac{df}{dx} = \begin{cases} \text{FD at } x_1 \\ \text{CD middle} \\ \text{BD at } x_N \end{cases}$$

- Using $x_k - x_{k-1}$ (instead of Δx) also works for non-uniformly spaced data.
- Real time data → only BD.

$f(t+\Delta t)$ not measured yet

As $\Delta t \to 0$ does error become arbitrarily small? No.
Because numerical truncation error.
round off error $\underbrace{e_r = 10^{-16}}$ for double precision.

$$\underbrace{\sqrt{2} + 10^{-16}/2 \Leftrightarrow \sqrt{2}}_{\text{equal}}$$

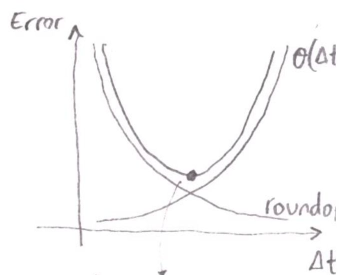
Ex: 0.5 → binary (exact, no roundoff error)

$$\frac{df}{dt} = \frac{f(t+\Delta t) - f(t-\Delta t) + 2e_r}{2\Delta t} + \theta(\Delta t^2)$$

$$\frac{df}{dt} = \frac{f(t+\Delta t) - f(t-\Delta t)}{2\Delta t} + \underbrace{\frac{e_r}{\Delta t}}_{\substack{\text{roundoff}\\ \text{error}}} + \underbrace{\theta(\Delta t^2)}_{\substack{\text{Taylor series}\\ \text{error}}}$$

$$|\text{Error}| \leq \frac{e_r}{\Delta t} + \frac{\Delta t^2 M}{6} \qquad M: \max|f'''| \quad \frac{d^3 f}{}\Delta t^2$$

Sources of Error



$$\frac{\partial E_{max}}{\partial \Delta t} = \frac{-e_r}{\Delta t^2} + \frac{\Delta t M}{3}$$

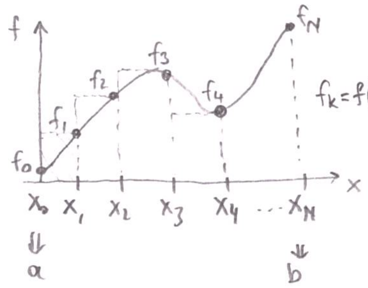
$$\boxed{\Delta t = 3\sqrt{\frac{3e_r}{M}}}$$

Assume $M = 1$
$e_r = 10^{-16}$

$$\Delta t \approx 10^{-5}$$

Numerical Integration: Discrete Riemann Integrals and Trapezoid Rule.

$\int_a^b f(x)\,dx$ Numericly computing the definite integral



N is # of rectangles.

$\Delta x = \frac{b-a}{N}$ width

$$\int_a^b f(x)\,dx = \lim_{N\to\infty} \sum_{k=1}^{N} \left(f(a + k\Delta x) \cdot \Delta x \right)$$

Riemann Integral Approximation.

Local Error Δx^2

Global/Total $\underbrace{\Delta x}_{(N \to \frac{1}{\Delta x})}$

$$= \lim_{N\to\infty} \sum_{k=1}^{N} f(x_k)\,\Delta x \quad \text{Right-sided rectangle}$$

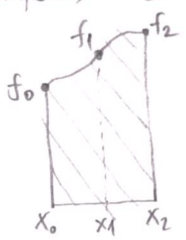
$$= \lim_{N\to\infty} \sum_{k=0}^{N-1} f(x_k)\,\Delta x \quad \text{Left-sided rectangle}$$

trapezoidal integration

$$\underbrace{= \frac{1}{2}(RS + LS)}$$

Local Error Δx^3
Total Error Δx^2

Simpson's Rule



$$\int_{x_0}^{x_2} f(x)\,dx = \frac{\Delta x}{3}\left(f_0 + 4f_1 + f_2\right) - \frac{\Delta x^5}{90} f^{(4)}(x)$$

Local Error : Δx^5
Total Error : Δx^4

Numerical Simulation of Ordinary Differential Equations : Integrating ODEs.

$\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x} \Rightarrow \dot{\underline{x}} = \underline{f}(\underline{x})$

$\underline{x}$: vector representing the state of system.

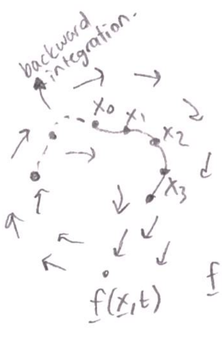
$f(\underline{x})$: may be a non-linear function. (vector field) may be solution to a PDE

$\underline{f}(\underline{x}, t)$: changing in time $\quad \frac{\partial f}{\partial t} = N(f, f_x, f_{xx})$

★ Derive Numerical Schemes to approximate $x_{k+1} = F(x_k)$
- Forward Euler
- Backward Euler
- Runge Kutta 2^{nd}, 4^{th} order
- Advanced integrators for chaos

- Error
- Stability

} Properties.



$$\underline{f}(\underline{x}) = \begin{bmatrix} x \text{ velocity} \\ y \text{ velocity} \end{bmatrix}$$

$$\frac{dx}{dt} \approx \frac{x(t+\Delta t) - x(t)}{\Delta t} \approx f(x(t))$$

$$\boxed{x(t+\Delta t) \approx x(t) + \Delta t\, f(x(t))}$$

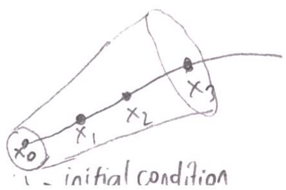
Forward Euler $\Downarrow$

$$x_{k+1} = x_k + \Delta t\, f(x_k)$$

Trajectory in a path $\underline{x}(t)$

$\underline{x}_0 \to \underline{x}_1 \to \underline{x}_2 \to \cdots \to \underline{x}_n$

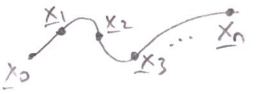
where $\underline{x}_k = \underline{x}(t_k) = \underline{x}(k\Delta t)$



Deriving Forward Euler and Backward/Implicit Euler Integration Schemes for Differential Equations.

$\dot{\underline{x}} = \underline{A}\,\underline{x} \Rightarrow \dot{\underline{x}} = \underline{f}(\underline{x})$

$x(0) = x_0$



$\underline{x}_k = \underline{x}(t_k) = \underline{x}(k\Delta t)$

$x_k \xrightarrow{F} x_{k+1}$

$\underline{x}_0 \xrightarrow{F} \underline{x}_1 \xrightarrow{F} \underline{x}_2 \rightarrow \dots \rightarrow \underline{x}_n$

$\dot{\underline{x}} = f(\underline{x}) \rightarrow \dot{\underline{x}}(t) = f(\underline{x}(t))$

$$\frac{d\underline{x}(t)}{dt} \simeq \frac{\underline{x}(t+\Delta t) - \underline{x}(t)}{\Delta t} \simeq f(\underline{x}(t))$$

$$\underline{x}(t+\Delta t) = \underline{x}(t) + \Delta t\, f(\underline{x}(t))$$

$$= F(\underline{x}(t))$$

$$x(t_{k+1}) = F(\underline{x}(t_k))$$

$$x((k+1)\Delta t) = x(k\Delta t + \Delta t) = x(t_k + \Delta t)$$

$$x(t_k + \Delta t) = x(t_k) + \Delta t\, f(x(t_k))$$

$$\star \quad \boxed{x_{k+1} = x_k + \Delta t\, f(x_k)} \leftarrow \text{Index Notation}$$

x evaluated at t_k

explicit

FORWARD EULER

$$\frac{d x_{k+1}}{dt} \simeq \frac{x_{k+1} - x_k}{\Delta t} \simeq \dot{x}(t_{k+1}) = f(x_{k+1})$$

$$\star \quad \boxed{x_{k+1} = x_k + \Delta t\, f(x_{k+1})}$$

implicit fnc. of x_{k+1}

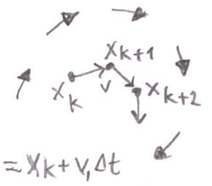
BACKWARD EULER

If $\dot{x} = Ax$

$$x_{k+1} = x_k + \Delta t\, A x_{k+1}$$

$$(\mathbb{I} - \Delta t A)\, x_{k+1} = x_k$$

$$\star \quad \boxed{x_{k+1} = (\mathbb{I} - \Delta t A)^{-1} x_k}$$



$x_{k+1} = x_k + v_1 \Delta t$

$x_{k+2} = x_{k+1} + v_2 \Delta t$

If $\dot{x} = Ax$

$$x_{k+1} = x_k + \Delta t\, A x_k$$

$$\boxed{x_{k+1} = \underbrace{(\mathbb{I} + \Delta t A)}_{\text{first 2 terms of Taylor Series of}} x_k} \quad \star$$

first 2 terms of Taylor Series of $x_{k+1} = e^{A t} x_k \leftarrow$ exact solution.

Numerical Integration of ODEs with Forward Euler and Backward Euler

Forward Euler

$$\boxed{x_{k+1} = x_k + \Delta t\, f(x_k)}$$

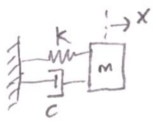
$$= (\mathbb{I} + \Delta t A)\, x_k \quad \text{if } \dot{x} = Ax$$

Backward Euler

$$\boxed{x_{k+1} = x_k + \Delta t\, f(x_{k+1})}$$

$$= (\mathbb{I} - \Delta t A)^{-1} x_k \quad \text{if } \dot{x} = Ax$$

Ex:



$\underline{x} = \begin{bmatrix} x \\ v \end{bmatrix} \quad \dot{x} = v$

$$m\ddot{x} + kx + c\dot{x} = 0 \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0 \qquad \zeta = \frac{c}{2\sqrt{km}}$$

ω_0 : natural frequency

ζ : damping ratio.

$$\boxed{\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = 0}$$

$$\lambda^2 + 2\zeta\omega_0\lambda + \omega_0^2 = 0$$

Case I: $\zeta < 1$
underdamped
the system oscillates

Case II: $\zeta = 1$
Critically damped.

Case III: $\zeta > 1$
Overdamped.

$$\left.\begin{aligned} \dot{x} &= v \\ \dot{v} &= -2\zeta\omega_0 v - \omega_0^2 x \end{aligned}\right\} \quad \frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = A\underline{x}$$

$$\underbrace{\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix}}_{\dot{\underline{x}}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{bmatrix}}_{\underline{A}} \underbrace{\begin{bmatrix} x \\ v \end{bmatrix}}_{\underline{x}}$$

$x_{k+1} = (\mathbb{I} + \Delta t A)\, x_k$ forward

$x_{k+1} = \underbrace{(\mathbb{I} - \Delta t A)^{-1}}_{\text{eigenvalues of these; stability}} x_k$ backward

eigenvalues of these; stability

Error Analysis of Euler Integration Scheme for Differential Equations using Taylor Series

Error of Forward Euler as a function of Δt

$$x_{k+1} = x_k + \Delta t\, f(x_k)$$

$$x_{k+1} \approx x(t_k + \Delta t)$$

Taylor Expand true $[x(t_k+\Delta t)]$

$$\left[x(t_k+\Delta t) = \underbrace{\underbrace{x(t_k)}_{x_k} + \underbrace{\frac{dx}{dt}(t_k)\,\Delta t}_{f(x_k)\Delta t}}_{\text{Approximation.}} + \underbrace{\frac{d^2x}{dt^2}(t_k)\frac{\Delta t^2}{2!} + \cdots}_{\text{Error}}\right]$$

$\theta(\Delta t^2)$ "locally" (every time step) — Error per time step

$\theta(\Delta t)$ "global" (over entire trajectory) — Global error

$\left(\frac{1}{\Delta t}\right)\theta(\Delta t^2)$ → total time step.

↳ For deterministic system global error approximation is valid.

★ But chaotic systems, local perturbations are gonna amplify massively.

The same for the Backward Euler.

- In chaotic systems
  Even if there is no integration error.

small epsilon difference (diverge) amplifying because of the chaotic dynamics.

Stability of Forward Euler and Backward Euler Integration Scheme for Differential Equations

Forward Euler $x_{k+1} = (I + \Delta t A)x_k$

Backward Euler $x_{k+1} = (I - \Delta t A)^{-1} x_k$

$\dot{x} = \lambda x \Rightarrow x(t) = e^{\lambda t}x(0)$

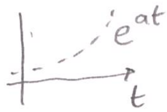
$\lambda = a \pm ib$

$e^{\lambda t} = \underset{\sim}{e^{at}}[\cos(bt) + i\sin(bt)]$

stability comes from real part of λ

if $a<0$ stable

if $a>0$ unstable



Continuous time.

$x_{k+1} = \lambda x_k$ (Discrete Time)

$x_1 = \lambda x_0$

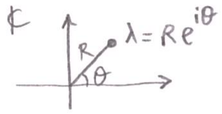
$x_2 = \lambda x_1 = \lambda^2 x_0$

$\vdots$

$x_{k+1} = \lambda x_k = \lambda^{k+1}x_0$

stable if $\lambda^{k+1} \to 0$ as $k \to \infty$

unstable if $|\lambda^{k+1}| \to \infty$ as $k \to \infty$



$\lambda^2 = R^2 e^{i2\theta}$

$\vdots$

$\lambda^k = R^k e^{ik\theta}$

If $R>1$ unstable

If $R<1$ stable

$\lambda = a \pm ib$

$R = \sqrt{a^2+b^2}$

Discrete time.

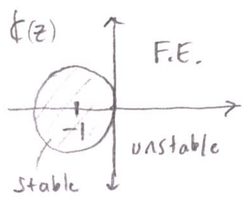
F.E: $x_{k+1} = (1+\lambda\Delta t)x_k \Rightarrow x_N = (1+\lambda\Delta t)^N x_0$

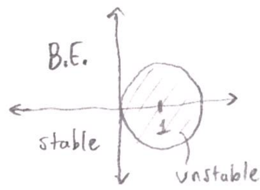
is unstable if $|1+\lambda\Delta t| > 1$

B.E: $x_{k+1} = (1-\lambda\Delta t)^{-1}x_k \Rightarrow x_N = \left(\frac{1}{1-\lambda\Delta t}\right)^N x_0$

is unstable if $\left|\frac{1}{1-\lambda\Delta t}\right| > 1$

Say $z = \lambda\Delta t$





F.E is stable when

★ $|\text{eigs}(I + \Delta t A)| < 1$

B.E is stable when

★ $|\text{eigs}((I - \Delta t A)^{-1})| < 1$

$x_{k+1} = Mx_k$

$x_N = M^N x_0$

$MT = TD$

$M = TDT^{-1}$

$x_N = TD^N T^{-1} x$

eigs(M)

stable: radius <1

Unstable: radius >1

F.E. $M = (I + \Delta t A)$

B.E. $M = (I - \Delta t A$

Runge-Kutta Integrator

$\dot{\underline{x}} = f(\underline{x}, t)$

Forward Euler

$$x_{k+1} = x_k + \Delta t\, f(x_k, t_k) = x_k + \Delta t\, f_1$$

Runge-Kutta 2nd

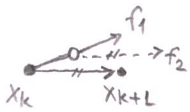
$$x_{k+1} = x_k + \Delta t\, f\Big(\underbrace{x_k + \frac{\Delta t}{2} f(x_k, t_k)}_{\text{Half Euler step}}, t_k + \frac{\Delta t}{2}\Big)$$

OR

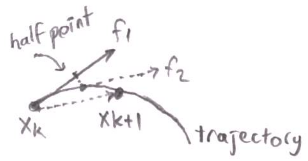
$$x_{k+1} = x_k + \Delta t\, f_2$$

$$f_1 = f(x_k, t_k)$$

$$f_2 = f\left(x_k + \frac{\Delta t}{2} f_1,\ t_k + \frac{\Delta t}{2}\right)$$



Error is locally $\theta(\Delta t^3)$ every time step.
globally $\theta(\Delta t^2)$ over entire trajectory



$f_1 = f(x_k, t_k)$ vector field at base point

$f_2 = f\Big(\underbrace{x_k + \frac{\Delta t}{2} f_1}_{\text{half F.E. step in } f_1 \text{ direction}},\ t_k + \frac{\Delta t}{2}\Big)$ take half step in f_1 direction and query vector field.

$x_{k+1} = x_k + \underbrace{\Delta t\, f_2}$ full F.E. step in f_2 direction.

Runge-Kutta 4th

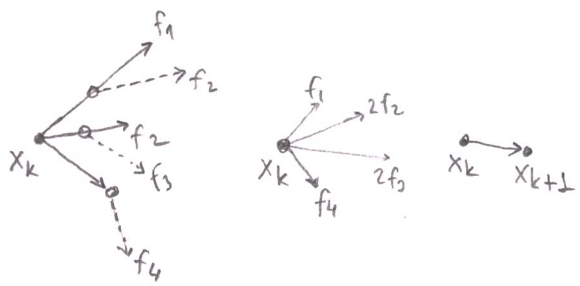
$$x_{k+1} = x_k + \frac{\Delta t}{6}\left(f_1 + 2f_2 + 2f_3 + f_4\right)$$

$$f_1 = f(x_k, t_k)$$

$$f_2 = f\left(x_k + \frac{\Delta t}{2} f_1,\ t_k + \frac{\Delta t}{2}\right)$$

$$f_3 = f\left(x_k + \frac{\Delta t}{2} f_2,\ t_k + \frac{\Delta t}{2}\right)$$

$$f_4 = f\left(x_k + \Delta t\, f_3,\ t_k + \Delta t\right)$$



$\theta(\Delta t^5)$ local

$\theta(\Delta t^4)$ global

RK4
↓
written to get rid of terms in the Taylor Series to get the error $\theta(\Delta t^5)$ locally.

$f_1 \rightarrow$ vector field at base point

$f_2 \rightarrow$ evaluate vector field after half F.E. in f_1

$f_3 \rightarrow$ evaluate vector field after half F.E. in f_2

$f_4 \rightarrow$ evaluate vector field after full F.E. in f_3

RK2 → RK4

RK4 is very accurate but computationally costly (2 times)

However RK4 Δt can be larger so it is more efficient

Coding a 4th order Runge-Kutta Integrator

$$x_{k+1} = x_k + \frac{\Delta t}{6}\left(f_1 + 2f_2 + 2f_3 + f_4\right)$$

$$f_1 = f(x_k, t_k)$$

$$f_2 = f\left(x_k + \frac{\Delta t}{2} f_1, t_k + \frac{\Delta t}{2}\right)$$

$$f_3 = f\left(x_k + \frac{\Delta t}{2} f_2, t_k + \frac{\Delta t}{2}\right)$$

$$f_4 = f\left(x_k + \Delta t f_3, t_k + \Delta t\right)$$

Lorenz 1963

$$\dot{x} = \sigma(y - x) \qquad \sigma = 10$$

$$\dot{y} = x(\rho - z) - y \qquad \beta = 8/3$$

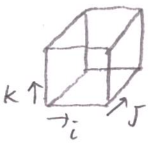
$$\dot{z} = xy - \beta z \qquad \rho = 28$$

"butterfly"

Numerical Integration of Chaotic Dynamics: Uncertainty Propagation & Vectorized Integration

x, y, z

cubic initial condition - large cubic points.



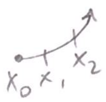
for i
  for J
    for k
      integrate

nested loops are costly in not compiled languages (Matlab, python)

Using vectorized version will be faster.

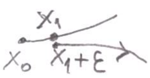
Chaotic Dynamical Systems

↳ Sensitive dependence on initial conditions



RK4 error $\theta(\Delta t^5)$ local

$\theta(\Delta t^4)$ global: Not true for chaotic systems



$\varepsilon e^{\lambda t}$

λ - Lyapunov Exp.

$$x(t) = x(0) + \underbrace{\int_0^t f(x(\tau))\, d\tau}$$

flow map ← $\Phi_0^t(x(0))$

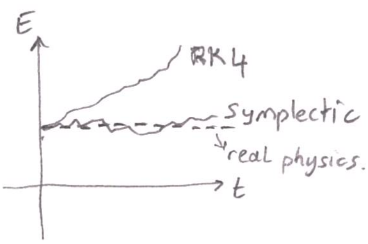
Jacobian of flow map.

$$\underbrace{\Phi_0^t(x_0 + \varepsilon) = \Phi_0^t(x_0)}_{\text{Taylor expansion}} + \underbrace{\frac{D\Phi_0^t}{DX}(x_0)\,\varepsilon}_{\text{leading error}} + \theta(\varepsilon^2)$$

$$e^{\lambda t} = \sigma_{max}\left(\frac{D\Phi}{DX}\right)$$

λ: lyapunov exponent.

RK4 (errors) → Conversation of Energy will be violated.



- Hamiltonian
  ↳ symplectic integrator

$$\dot{q} = -\frac{\partial H}{\partial p}$$

$$\dot{p} = \frac{\partial H}{\partial q}$$

- Lagrangian
  ↳ variational integrator

RKF78: nearly symplectic

better than RK4.