Matrix

- Identity Matrix (IM)
- Scalar Matrix
- IM but off by one
- IM but with negative entries
- Diagonal Matrix
- Square Zero Matrix
- Shear Matrix
- Orthogonal Matrix
- Projection Matrix
- Invertible Matrix

• Identity Matrix

$$\begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots & 1 \end{bmatrix}$$ → no transformation.

• Scalar Matrix

$$\begin{bmatrix} k & & 0 \\ & k & \\ 0 & & \ddots & k \end{bmatrix}$$ → scaling uniformly (bigger or smaller)

• IM but off by one

$$\begin{bmatrix} k & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix} \text{ OR } \begin{bmatrix} 1 & & 0 \\ & k & \\ 0 & & 1 \end{bmatrix} \cdots$$

→ scale just one axis
→ the other axes are unchanged.
→ not preserving the original ratio.

• IM but with negative entries

$$\begin{bmatrix} -1 & & 0 \\ & -1 & \\ 0 & & 1 \end{bmatrix} \text{ OR } \begin{bmatrix} 1 & & 0 \\ & -1 & \\ 0 & & -1 \end{bmatrix} \cdots$$

→ reflection by one/more axes
→ preserving original proportion.

• Diagonal Matrix

$$\begin{bmatrix} a & & 0 \\ & b & \\ 0 & & c \ddots \end{bmatrix}$$

→ scale each axis according to the diagonal entries
→ not preserving original ratio

Matrix-Matrix Multiplication

- Is not a multiplication actually.

$$B \cdot A = C$$

B: some other transformation
A: some transformation
C: combined transformation
"compose" the overall transformation

$$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

Scale x axis by 3 | Scale y axis by 2 | Combination

- Composition of sequential transformations.

$$B \cdot A \underset{\text{not always}}{=} A \cdot B$$

• Square Zero Matrix

$$\begin{bmatrix} 0 & 0 \cdots \\ 0 & 0 \\ \vdots & \end{bmatrix}$$

→ makes every vector the zero vector.
→ moves all dots to origin.

• Shear Matrix

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \cdots$$

→ slanting the object.
→ preserving area/volume/...
→ shear transformation.

• Orthogonal Matrix

$$\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

- square matrix
- all column vectors are unit vectors
- all column vectors are orthogonal.

$v_1 \cdot v_2 = 0$ — v_1 and v_2 are perpendicular to each other.

→ A transformation o
rotation
→ Rotation Matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \quad \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \quad \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

around x-axis | around y-axis | around z-axis.

Matrix

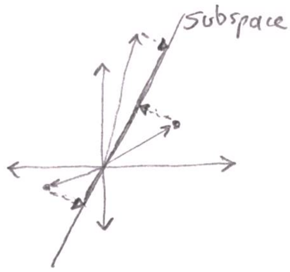
• Projection Matrix

Space → subspace

3D → 2D

2D → 1D

⋮



→ projection to subspace

→ compression to lower dimension

→ vector moves to closest point

• Invertible Matrix

Inverse of a matrix : untransformation

scale → unscale

reflect → reflect back

rotate → rotate back

shear → unshear

- - - - - - - - - - - -

Some transformation → revert

$A^{-1}A = I$

• zero matrix and projection matrix cannot be untransform.

(loss of information)

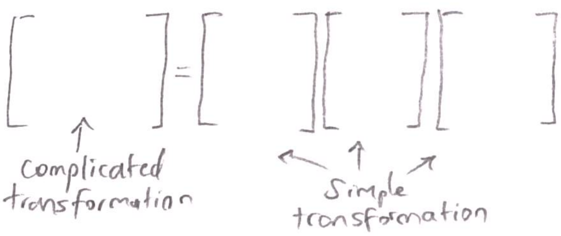
→ to untransform

→ to undo, revert, invert

→ not every matrix has an inverse.

Spectral Decomposition

$\underbrace{\text{Diagonal}}_{\text{Scale}}$ & $\underbrace{\text{Orthogonal}}_{\text{Rotation.}}$



Symmetric:

$$\begin{bmatrix} a & k_1 & k_2 \\ k_1 & b & k_3 \\ k_2 & k_3 & c \end{bmatrix}$$

Transpose:

$$\begin{bmatrix} 2 & 3 \\ 5 & 7 \\ 11 & 13 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 5 & 11 \\ 3 & 7 & 13 \end{bmatrix}$$

Composition : multiplication of some matrices → one matrix

Decomposition : one matrix → multiplication of some matrices.

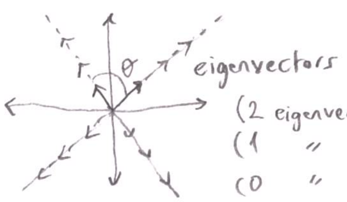
Easy. Difficult.

$$\boxed{\begin{matrix} S = S^T \\ Q^T = Q^{-1} \end{matrix}}$$

★ S : Symmetric Matrix

Q : Orthogonal Matrix

↳ transpose of orthogonal matrix : rotation in the reverse direction.



(2 eigenvectors)

(1 ″)

(0 ″)

$\vec{e}_1 \ \lambda_1$

$\vec{e}_2 \ \lambda_2$

eigenvalues.

$$A \underset{\text{eigenvector}}{\vec{v}} = \lambda \vec{v}$$

eigenvector

linear transformation

scale

original direction

★ eigenvectors stay on their original line.

If S is symmetric → eigenvectors are orthogonal (perpendicular)

Q → standard basis to eigenvectors

Q^{-1} → eigen vectors to standard basis (x, y)

★ orthogonal eigenvectors

Spectral Decomposition

(4)

$S = Q \lambda Q^T$

symmetric — Diagonal — Orthogonal — Orthogonal

$$\underset{2\times 2}{\begin{bmatrix} S \end{bmatrix}} = \underbrace{\begin{bmatrix} | & | \\ \vec{e}_1 & \vec{e}_2 \\ \downarrow & | \end{bmatrix}}_{\text{eigenvectors of } S} \underbrace{\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}}_{\text{eigenvalues of } S} \underbrace{\begin{bmatrix} | & | \\ \vec{e}_1 & \vec{e}_2 \\ | & | \end{bmatrix}^T}_{\text{transpose of eigenvectors of } S}$$

Rotation. $Q^T = Q^{-1}$ → rotate eigenvectors to the standard basis.

Scaling. λ scale

Rotation. Q → rotate standard basis to the eigen vector directions.

$[S]$ → Symmetric matrix (loss of generality

1. Set up characteristic polynomial
2. Solve for eigen values.
3. Use eigenvalues to solve eigen vectors
4. Normalize eigenvectors
5. Arrange eigen vectors into Matrix Q
6. Arrange eigen values into Matrix λ
7. Transpose Matrix Q to get Q^T

Singular Value Decomposition (SVD)

(5)

No restriction on: Symmetry, Dimension, Rank

- Dimension Eraser
- Creating Symmetric Matrices
- Singular Values.

$A = U \Sigma V^T$

Diagonal Matrix : Scale (Strech)
Orthogonal Matrix : Rotation

R^2 $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ R^3 $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

$$\underset{2\times 3}{\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix} \cdot \\ \cdot \end{bmatrix}}$$

rectangular matrix } 3D to 2D transformation from 3 dimension to 2 dimension.

* $\boxed{\begin{array}{l} 2\times 3 \rightarrow \text{dimension eraser} \\ 3\times 2 \rightarrow \text{dimension adder.} \end{array}}$

Not Symmetrical to symmetric

A : not symmetrical matrix $(m\times n)$

$\underbrace{A^TA\ (n\times n),\ AA^T\ (m\times m)}_{\text{symmetric matrices}}$

$AA^T = S_{Left} \rightarrow \begin{bmatrix} | & | \\ u_1 & u_2 \\ | & | \end{bmatrix}$ Left singular vectors

$A^TA = S_{Right} \rightarrow \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \end{bmatrix}$ Right singular vectors

} PSD matrices. $\lambda_i \geq 0$

$\lambda_1 \geq \lambda_2$
$\Updownarrow\ \lambda_1 = \lambda_1,\ \lambda_2 = \lambda_2$
$\lambda_1 \geq \lambda_2 \geq \lambda_3$

* $\sqrt{\lambda_1} = \sigma_1$, $\sqrt{\lambda_2} = \sigma_2$ } singular values of matrix

Singular Value Decomposition

$A = U \Sigma V^T \Rightarrow$ A is unconditionally decomposed.

U: Orthogonal, Σ: Rectangular Diagonal, V^T: Orthogonal

$$\underbrace{\begin{bmatrix} \cdot & \cdot & \cdot \\ & A & \\ \cdot & \cdot & \cdot \end{bmatrix}}_{2\times3} = \underbrace{\overset{U}{\begin{bmatrix} | & | \\ \vec{u}_1 & \vec{u}_2 \\ | & | \end{bmatrix}}}_{2\times2} \underbrace{\overset{\Sigma}{\begin{bmatrix} \sigma_1 & \cdot & \cdot \\ \cdot & \sigma_2 & \cdot \end{bmatrix}}}_{2\times3} \underbrace{\overset{V^T}{\begin{bmatrix} - & v_1^T & - \\ - & v_2^T & - \\ - & v_3^T & - \end{bmatrix}}}_{3\times3}$$

U: left singular vectors of A. $\sigma_1 \geq \sigma_2$. V^T: right singular vectors of A

U: contains the normalized eigenvectors of S_{Left}.

V^T: rotate right singular vectors to standard basis.

Σ: Compose of diagonal matrix with a dimension eraser.

$$\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

[scale] [erase dimension]

-OR-

[add dimension] [scale]

U: rotate standard basis to left singular vectors.

Note:

SVD: For any Matrix A → sum of rank-1 matrices

Fourier Transform: For any Funct $g(t)$ → sum of pure sinusoidal functions.

★ A: equals to the sum of rank-1 matrices

$$A = \sum_{i=1}^{k} \sigma_i u_i v_i^T = \sigma_1 \begin{bmatrix} | & - \vec{v}_1^T - \\ \vec{u}_1 & \\ | & \end{bmatrix} + \sigma_2 \begin{bmatrix} | & - \vec{v}_2 - \\ \vec{u}_2 & \\ \downarrow & \end{bmatrix} + \dots$$

Low rank approximation.

Notes:

Trace of a square matrix is sum of its diagonal entries.

$$tr(A) = \sum_{k=1}^{n} a_{kk}$$

Trace of a matrix is the sum of its eigenvalues.

$$tr(A) = \sum_{i=1} \lambda_i$$

Determinant: The n-dimensional content (length when $n=1$, area when $n=2$, volume when $n=3$... etc will scale by a factor of the determinant. ★

Ex: Determinant of a $\underbrace{\text{projection matrix}}_{\text{compression to lower dimension}}$ is zero.

Ex: Determinant of a $\underbrace{\text{orthogonal matrix}}_{\text{rotation}}$ is one or minus one.

Negative determinant: The matrix has a mirroring component.