HW 1

2) Let X and Y be multivariate random variables. Let $Z = X + Y$ the

a) sum of two multivariate random variables. We can then write

$$\Sigma_{ZZ} = E\left((Z-\mu_Z)(Z-\mu_Z)^T\right) \quad \text{where} \quad \mu_Z = E[Z] = E[X+Y]$$

using the linearity of the expectation $\mu_Z = E[X] + E[Y] = \mu_X + \mu_Y$

then we can rewrite the covariance as

$$\Sigma_{ZZ} = E\left(\left((X+Y)-(\mu_X+\mu_Y)\right)\left((X+Y)-(\mu_X-\mu_Y)\right)^T\right)$$ rearranging we can get

$$= E\left(\left((X-\mu_X)+(Y-\mu_Y)\right)\left((X-\mu_X)^T+(Y-\mu_Y)^T\right)\right)$$ now we can compute

$$= E\left((X-\mu_X)(X-\mu_X)^T + (X-\mu_X)(Y-\mu_Y)^T + (Y-\mu_Y)(X-\mu_X)^T + (Y-\mu_Y)(Y-\mu_Y)^T\right)$$

$$\Sigma_{ZZ} = E\left[(X-\mu_X)(X-\mu_X)^T\right] + E\left[(X-\mu_X)(Y-\mu_Y)^T\right] + E\left[(Y-\mu_Y)(X-\mu_X)^T\right] + E\left[(Y-\mu_Y)(Y-\mu_Y)^T\right]$$

$$\boxed{\therefore \Sigma_{ZZ} = \Sigma_{XX} + K_{XY} + K_{XY}^T + \Sigma_{YY}}$$

b) Let X and Y be two independent multivariate random variables

using the linear independent propery $E(A \cdot B) = E(A)E(B)$ we can then

rewrite $K_{XY} = E\left[(X-\mu_X)(Y-\mu_Y)^T\right] = E[X-\mu_X]\,E\left[(Y-\mu_Y)^T\right]$

Examining $E[X-\mu_X] = E[X] - \mu_X$ and by definition $\mu_X = E[X]$

$E[X-\mu_X] = E[X] - E[X] = 0$ which means

$\therefore K_{XY} = E[X-\mu_X]\,E\left[(Y-\mu_Y)^T\right] = \vec{0}\cdot\vec{0}^T = 0$ (matrix)

c) From the above results, we can deduce that the covariance

of the sum of two independent multivariate random variables will be

$Z = X + Y$, X, Y independent

$$\Sigma_{ZZ} = \Sigma_{XX} + \cancel{K_{XY}}^{0} + \cancel{K_{XY}^T}^{0} + \Sigma_{YY} = \Sigma_{XX} + \Sigma_{YY}$$

$$\boxed{\therefore \Sigma_{ZZ} = \Sigma_{XX} + \Sigma_{YY}}$$

3) 10 independent sensors for object detection. $p(\text{detected}) = 0.1$

Our system fits the requirements of a binomial distribution

where we have $n=10$ independent experiments on how many sensors detected the object with propability $p=0.1$

a) Let X be a random variable such that

$$X \sim B(n,p) = B(10, 0.1)$$

$$P(X=x) = \binom{10}{x} \cdot (0.1)^x (1-0.1)^{10-x}$$

b) $P(X \geq 1) = 1 - P(X=0)$

$$P(X=0) = \binom{10}{0} \cdot (0.1)^0 \cdot (0.9)^{10} = \frac{10!}{0!(10-0)!} \times 1 \times 0.3487$$

$$\boxed{\therefore P(X \geq 1) = 1 - 0.3487 = 0.6513}$$

65% chance at least one sensor detects the object

4) Two dimensional normally distributed random variable, $\mu = \begin{bmatrix} 3.0 \\ 4.0 \end{bmatrix}$

Principal axis (eigenvector) forms a 30 degree angle with the x-axis

The variance of that axis is 1.0, other is 0.25

PDF is in the form

$$p(x) = \frac{1}{2\pi\sqrt{|\Sigma|}} \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)\right)$$

We can compute the covariance matrix using the information we have on the eigenvectors (principal axis) and eigenvalues (variance) of the uncertainty ellipse

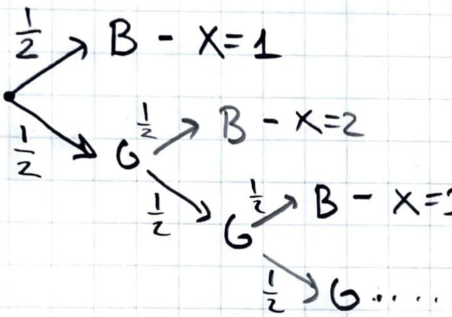
$$\Sigma = U \Lambda U^T \quad \text{where} \quad U = Rot_x(30°) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 0.25 \end{bmatrix}$$

running the operations on Python we get $\Sigma = \begin{bmatrix} 0.8125 & 0.3248 \\ 0.3248 & 0.4375 \end{bmatrix}$

$$|\Sigma| = 0.25 \qquad \Sigma^{-1} = \begin{bmatrix} 1.75 & -1.299 \\ -1.299 & 3.25 \end{bmatrix}$$

$$\boxed{\therefore p(x) = \underbrace{\frac{1}{2\pi\sqrt{0.25}}}_{=0.3183} \exp\left(-\frac{1}{2}[x_1-3.0, x_2-4.0] \begin{bmatrix} 1.75 & -1.299 \\ -1.299 & 3.25 \end{bmatrix} \begin{bmatrix} x_1-3.0 \\ x_2-4.0 \end{bmatrix}\right)}$$

5) Let X be the number of children a family has until they have a boy, inclusive. Assume $p(\text{boy}) = 0.5 = p(\text{girl})$



so $P(X=1) = \frac{1}{2}$

$P(X=2) = \frac{1}{4}$

$P(X=3) = \frac{1}{8}$

$\vdots$

We see that the series is converging to a geometric distribution

$X \sim$ $$P(X=k) = (1-p)^{k-1} p \quad \text{where } p = \frac{1}{2}$$

$$P(X=k) = \left(\frac{1}{2}\right)^{k-1} \cdot \frac{1}{2} = \frac{1}{2}^{k}$$

The expected value of a geometrically distributed random variable defined over $\mathbb{N}$ is

$$E[X] = \frac{1}{p} \rightarrow \frac{1}{\frac{1}{2}} = 2$$

So the average expected children per family in this village is 2. The python script attached simulates N many samples and converges to also 2