① $\frac{d}{dt}\left(\frac{dR(t)}{dt}\right) = \frac{d}{dt}\left(-aJ(t)\right) \Rightarrow \frac{d^2}{dt^2}R(t) = -a\underbrace{\frac{dJ(t)}{dt}}$ $\quad \frac{dJ(t)}{dt} = bR(t)$

$$\frac{d^2R(t)}{dt} = -abR(t) \Rightarrow \frac{d^2R(t)}{dt} + abR(t) = 0$$

$$\lambda^2 + abi^2 = 0 \Rightarrow \lambda^2 - (i\sqrt{ab})^2 = 0 \Rightarrow \quad (\lambda - i\sqrt{ab})(\lambda + i\sqrt{ab}) = 0$$

$$\lambda_1 = i\sqrt{ab} \qquad \lambda_2 = -i\sqrt{ab}$$

$$R(t) = c_1 e^{i\sqrt{ab}t} + c_2 e^{-i\sqrt{ab}t}$$

$$R'(t) = i\sqrt{ab}\,c_1 e^{i\sqrt{ab}t} - i\sqrt{ab}\,c_2 e^{-i\sqrt{ab}t}$$

$$R(0) = c_1 + c_2 = 1$$

$$R'(0) = i\sqrt{ab}\,c_1 - i\sqrt{ab}\,c_2 = 0$$

$$i\sqrt{ab}\,(c_1 - c_2) = 0$$

$\sqrt{-1} \neq 0$ both positive must be equal to 0

$$\begin{array}{r} c_1 + \cancel{c_2} = 1 \\ + \; c_1 - \cancel{c_2} = 0 \\ \hline 2c_1 = 1 \Rightarrow c_1 = \frac{1}{2} \end{array}$$

$$c_1 - c_2 = 0 \Rightarrow c_1 = c_2 \Rightarrow c_2 = \frac{1}{2}$$

$$R(t) = \frac{1}{2}e^{i\sqrt{ab}t} + \frac{1}{2}e^{-i\sqrt{ab}t} = \frac{1}{2}\left(e^{i\sqrt{ab}t} + \frac{1}{2}e^{-i\sqrt{ab}t}\right)$$

$$R(t) = \frac{1}{2}\left(\cos(\sqrt{ab}t) + \cancel{i\cdot\sin(\sqrt{ab}t)} + \cos(\sqrt{ab}t) \cancel{- i\sin(\sqrt{ab}t)}\right)$$

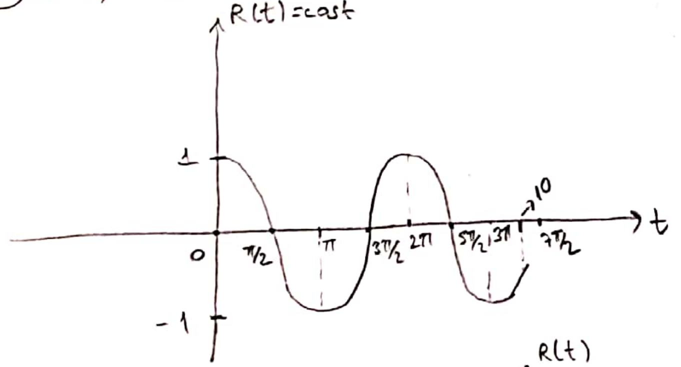
$$= \frac{1}{\cancel{2}}\,\cancel{2}\cos(\sqrt{ab}t)$$

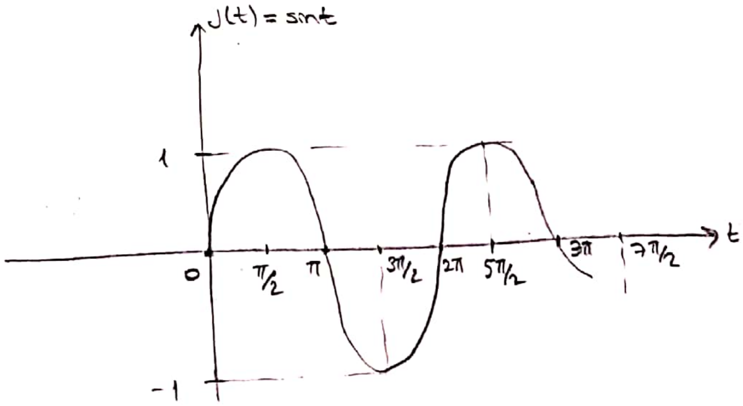
$$\underline{R(t) = \cos(\sqrt{ab}t)}$$

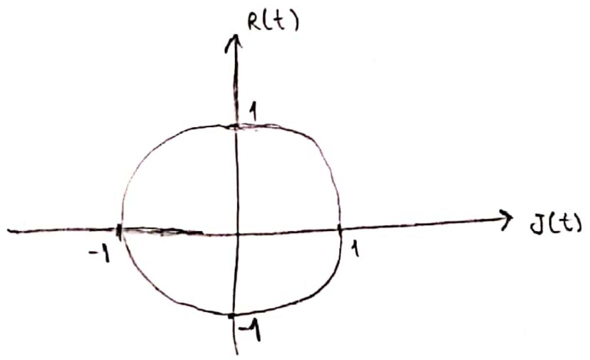
$$\frac{dR(t)}{dt} = -aJ(t) \Rightarrow -\frac{(\cos(\sqrt{ab}t))'}{a} = J(t)$$

$$J(t) = \underline{\frac{\sqrt{ab}\sin(\sqrt{ab}t)}{a}} = \cancel{\sqrt{\frac{b}{a}}\sin(\sqrt{ab}t)}$$

② $a=1, b=1 \Rightarrow R(t) = \cos t, \; J(t) = \sin t$







Juliet's love is directly proportional with Romeo's love. As stated in the question, when Romeo loves her, she begins to love her; when Romeo hates her, she begins to hate her. On the other hand, Romeo's love is inversely proportional to Juliet's love. when she loves him, he begin to lose interest; when she loses interest, he begins to love her. However, the most important thing in their relationship. is they are always related to each other. Their relationship is always connected, and always affects the other's emotion.

(3) $\frac{R(t_{k+1})-R(t_k)}{\Delta t} = \frac{\Delta R}{\Delta t} \Rightarrow$ average rate of change over Δt

$$\lim_{\Delta t \to 0} \frac{R(t_{k+1})-R(t_k)}{\Delta t} = \frac{dR(t)}{dt}$$

$$\underbrace{\frac{R[k+1]-R[k]}{\Delta t}}_{\frac{-AJ(k)}{\Delta t}} = \underbrace{\frac{dR(t)}{dt}}_{-aJ(t) \text{ (from (1))}}$$

Since the differential equation is approximated such that $R(t)$ and $J(t)$ remain constant from t_k until t_{k+1} and then undergo a step change, $J(t) = J(k)$.

$$\frac{-AJ(k)}{\Delta t} = -aJ(k) \Rightarrow \underline{A = a.\Delta t}$$

Similarly,

$$\underbrace{\frac{J[k+1]-J[k]}{\Delta t}}_{\frac{BR[k]}{\Delta t}} = \underbrace{\frac{dJ(t)}{dt}}_{bR(t) = bR[k]}$$

$$\frac{BR[k]}{\Delta t} = bR[k] \Rightarrow \underline{B = b\Delta t}$$

(4) $R[k+2]-R[k+1] = -AJ[k+1]$

$J[k+1]-J[k] = BR[k] \Rightarrow J[k+1] = BR[k]+J[k]$

$R[k+2]-R[k+1] = -ABR[k]-AJ[k]$

$R[k+1]-R[k] = -AJ[k] \Rightarrow \quad J[k] = \dfrac{R[k+1]-R[k]}{-A}$

$R[k+2]-R[k+1] = -ABR[k]+R[k+1]-R[k]$

$R[k+2]-2R[k+1]+(AB+1)R[k]=0$

$\lambda^2 - 2\lambda + (AB+1) = 0$

$$\lambda_{1,2} = \frac{2 \mp \sqrt{4-4AB-4}}{2} = \frac{2\mp 2\sqrt{-AB}}{2} = 1 \mp \sqrt{-AB}$$

$R[k] = \alpha_1(1+\sqrt{-AB})^k + \alpha_2(1-\sqrt{-AB})^k$

$R(0)=1 \Rightarrow \quad 1 = \alpha_1+\alpha_2$

$J(0) = \dfrac{\sqrt{ab}\,\sin(\sqrt{ab}.0)}{a} = 0 \Rightarrow \quad R(1) - \overbrace{R(0)}^{1} = 0$

$R(1) = 1$

$\alpha_1(1+\sqrt{-AB}) + \alpha_2(1-\sqrt{-AB}) = 1$

$\underbrace{\alpha_1+\alpha_2}_{1} + \sqrt{-AB}(\alpha_1-\alpha_2) = 1$

$\sqrt{-AB}\,\underbrace{(\alpha_1-\alpha_2)}_{0} = 0$

from Q3)
both are positive
since they are Δt multiplies
of a and b respectively,
and a and b are positive constants
so $\sqrt{-AB}$ is nonzero

$$\begin{array}{r} \alpha_1 - \alpha_2 = 0 \\ + \quad \alpha_1 + \alpha_2 = 1 \\ \hline 2\alpha_1 = 1 \Rightarrow \alpha_1 = 1/2,\ \alpha_2 = 1/2 \end{array}$$

$$R[k] = \frac{1}{2}(1+\sqrt{-AB})^k + \frac{1}{2}(1-\sqrt{-AB})^k = \frac{1}{2}(1+i\sqrt{AB})^k + \frac{1}{2}(1-i\sqrt{AB})^k$$

(5) $R[k+1]-R[k] = -AJ[k]$

$$\frac{1}{2}\left[(1+i\sqrt{AB})^{k+1} + (1-i\sqrt{AB})^{k+1} - (1+i\sqrt{AB})^k - (1-i\sqrt{AB})^k\right] = -AJ[k]$$

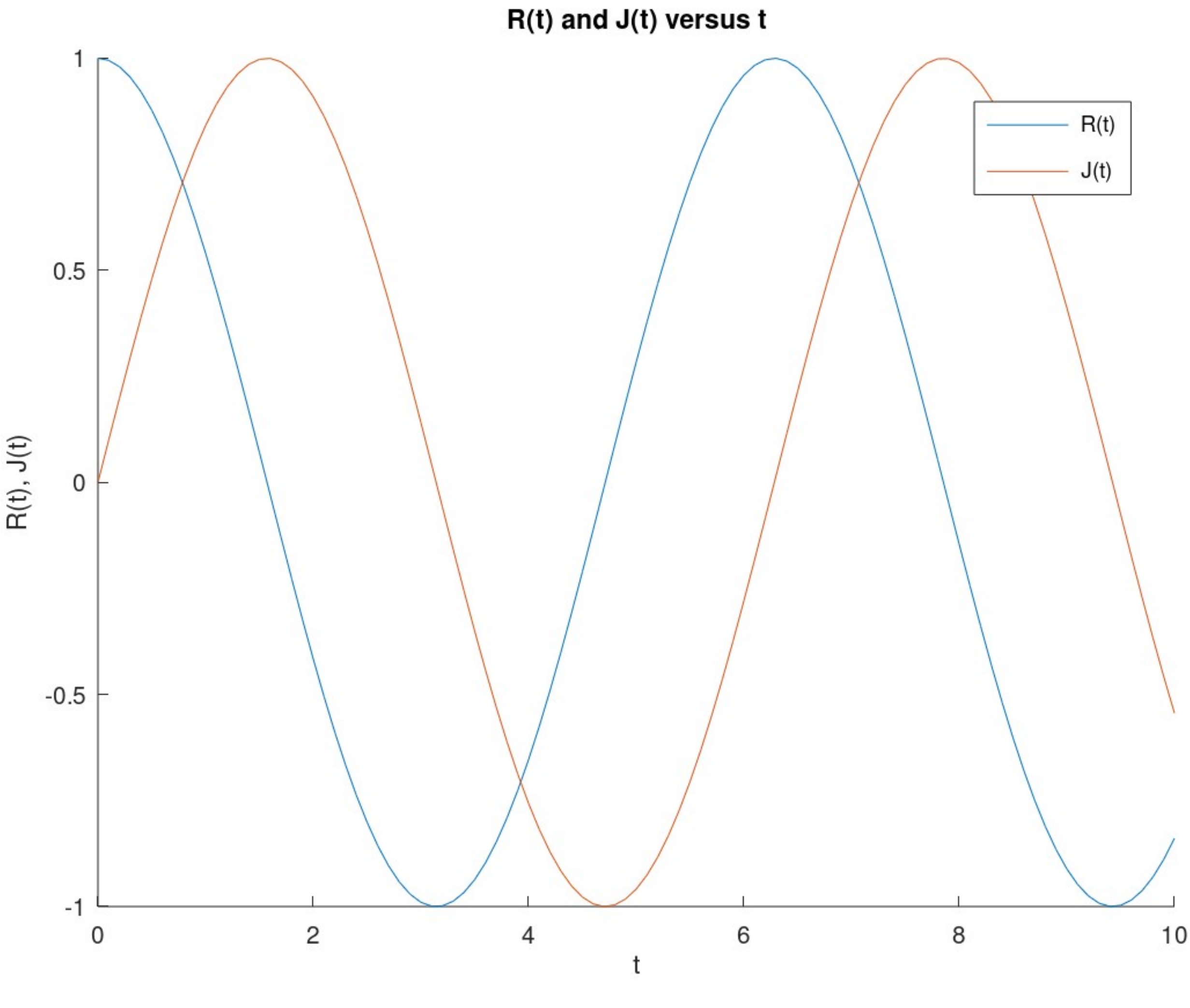
$$J[k] = \frac{1}{2A}\left[(1+i\sqrt{AB})^k + (1-i\sqrt{AB})^k - (1+i\sqrt{AB})^{k+1} - (1-i\sqrt{AB})^{k+1}\right]$$

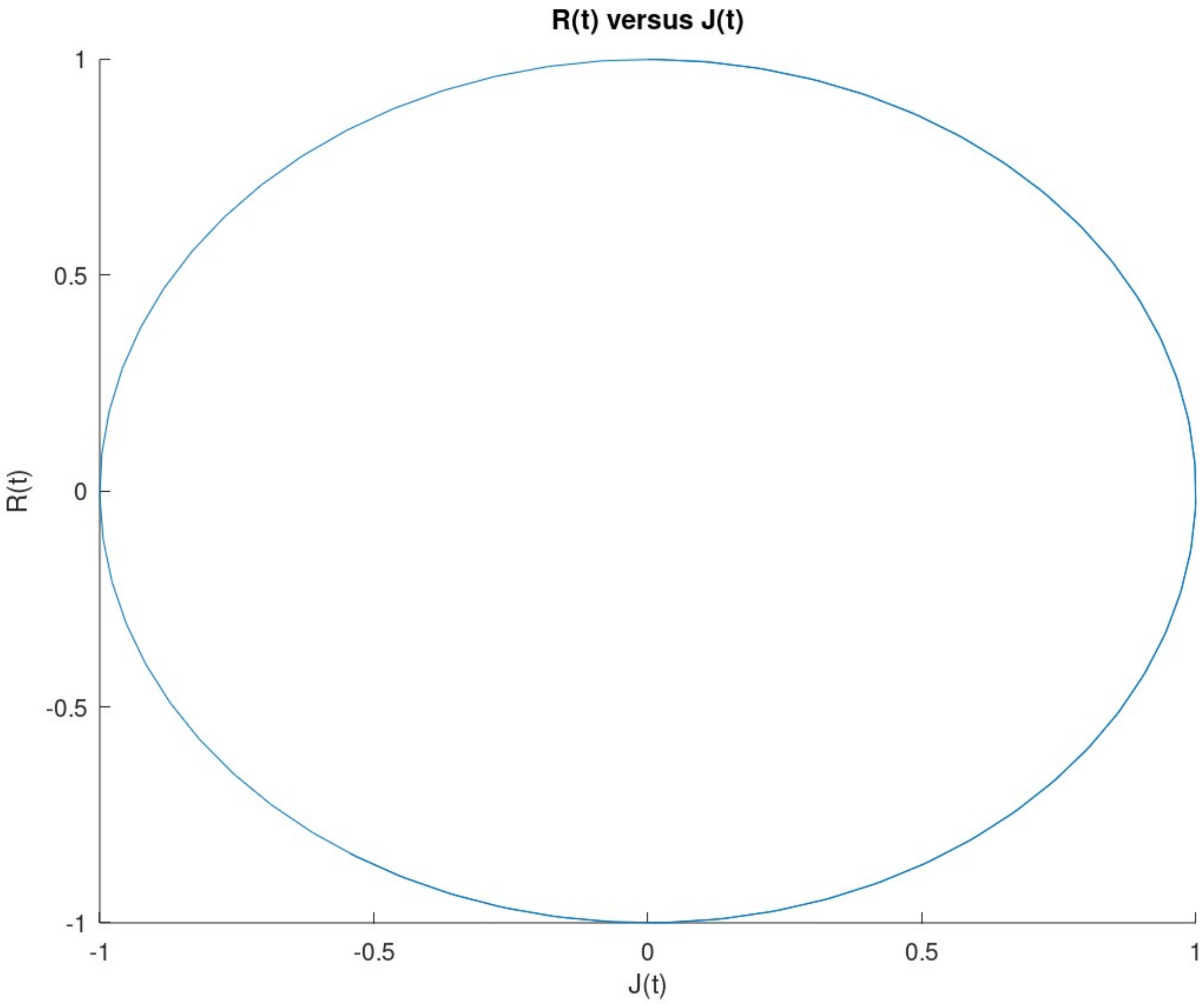
$A = \hat{a}\Delta t = \Delta t$
$B = \hat{b}\Delta t = \Delta t$

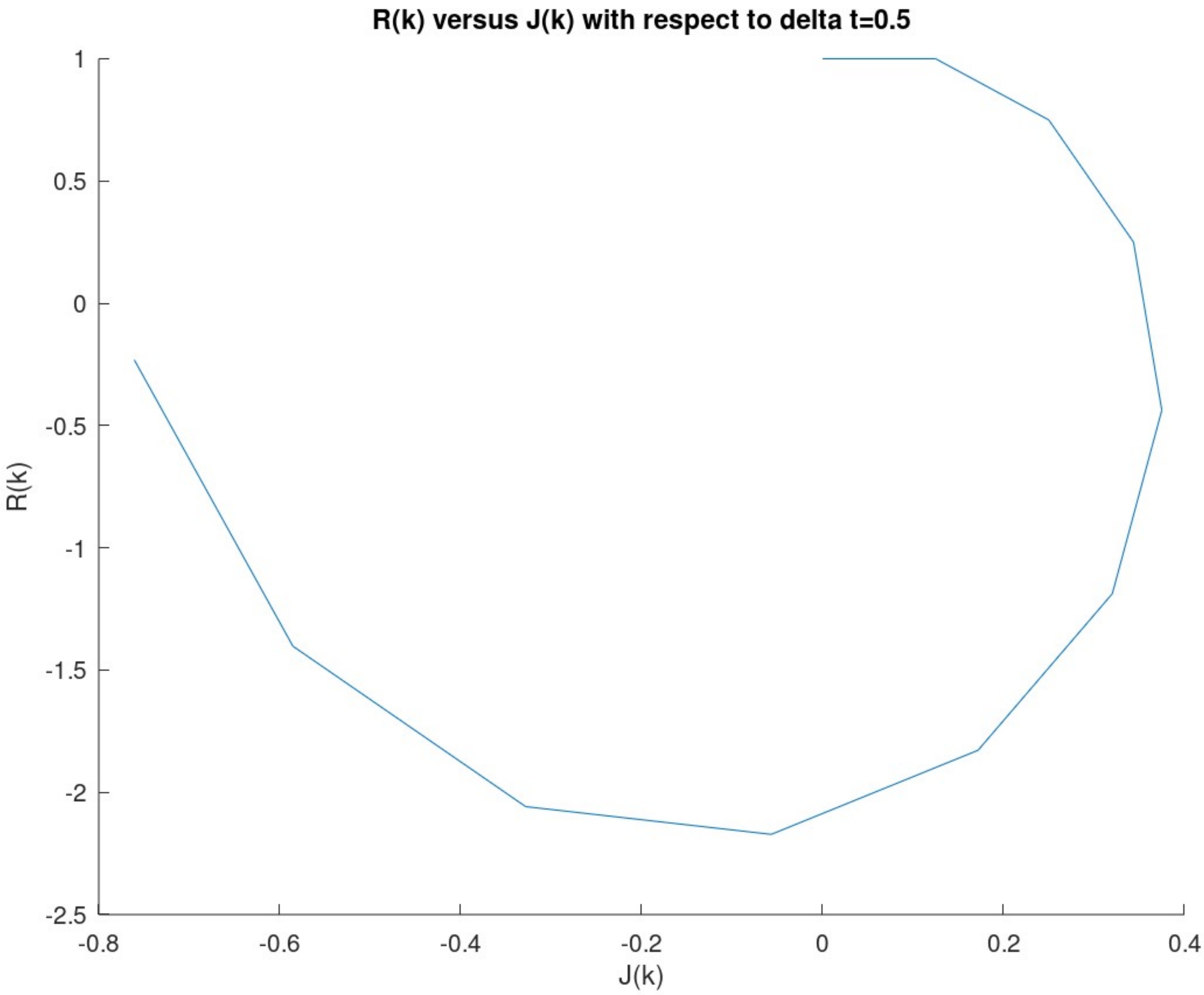
$$J[k] = \frac{1}{2\Delta t}\left[(1+i\Delta t)^k + (1-i\Delta t)^k - (1+i\Delta t)^{k+1} - (1-i\Delta t)^{k+1}\right]$$

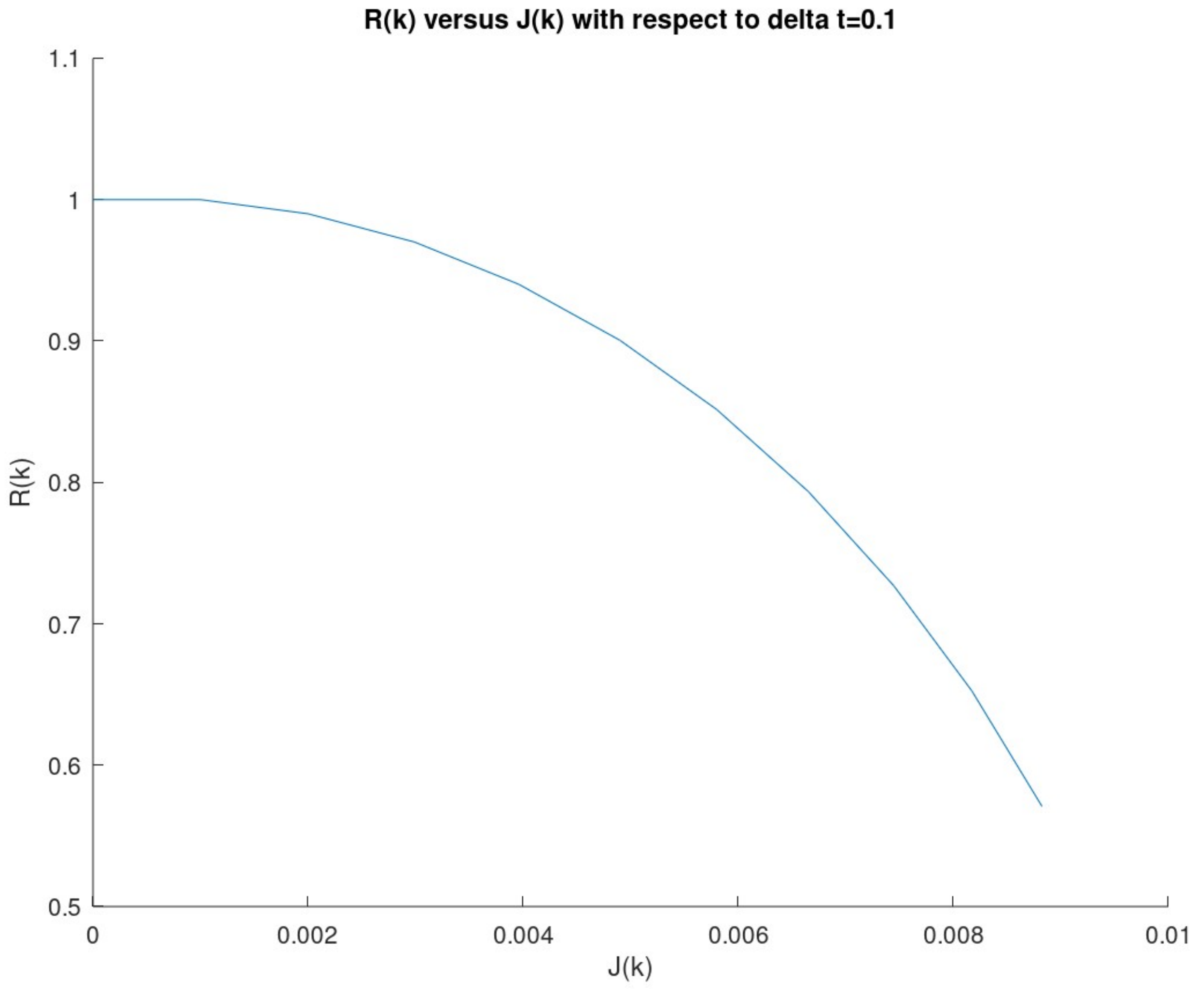
$$R[k] = \frac{1}{2}\left[(1+i\Delta t)^k + (1-i\Delta t)^k\right]$$

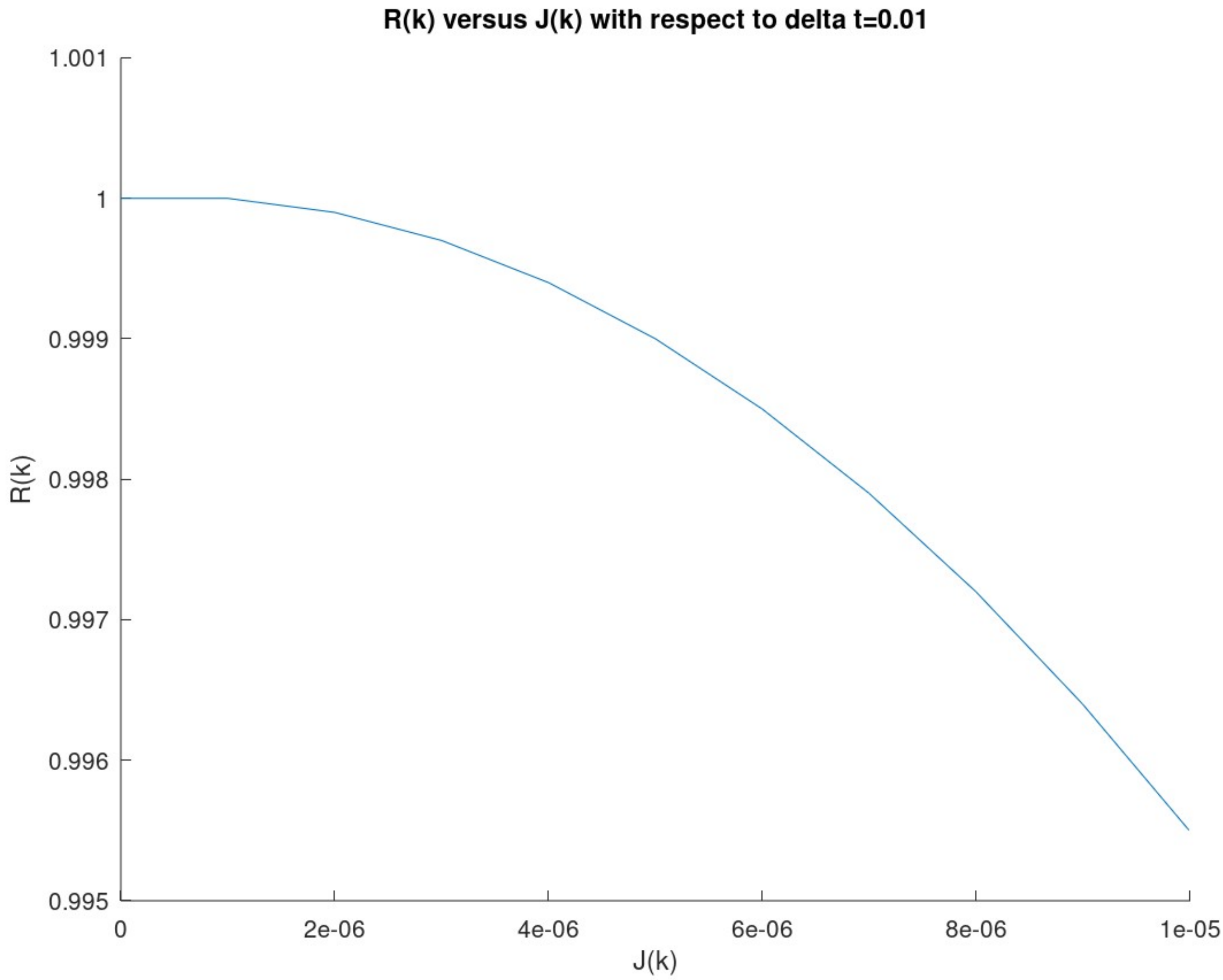
These graphs are similar to the graph that I obtained in Question 2. However, the graph with $\Delta t = 0.01$ is the most similar, and the graph with $\Delta t = 0.5$ is the least similar one. As Δt goes smaller and smaller, we approximate the continuous case closer and closer. Hence, it is a good idea to use the discrete approximation to the continuous dynamic system sometimes to understand it better; but if we use the discrete approximation we have to choose Δt small to get a closer approximation.


R(t) and J(t) versus t
R(t)
J(t)
1
0.5
0
-0.5
-1
R(t), J(t)
0
2
4
6
8
10
t


R(t) versus J(t)
1
0.5
0
-0.5
-1
R(t)
-1
-0.5
0
0.5
1
J(t)


R(k) versus J(k) with respect to delta t=0.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
R(k)
J(k)


R(k) versus J(k) with respect to delta t=0.1
1.1
1
0.9
0.8
0.7
0.6
0.5
R(k)
0
0.002
0.004
0.006
0.008
0.01
J(k)


R(k) versus J(k) with respect to delta t=0.01
1.001
1
0.999
0.998
0.997
0.996
0.995
R(k)
0
2e-06
4e-06
6e-06
8e-06
1e-05
J(k)