1 Disk equations

Continuity equation

$$2\pi r\Sigma\,|v_r| = \dot{M} \tag{1}$$

r component of momentum conservation equation

$$v_r\frac{\partial v_r}{\partial r} + (\Omega_k^2 - \Omega^2)r = -\frac{1}{\rho}\frac{\partial}{\partial r}\left(P + \frac{B_z^2 + B_\phi^2}{8\pi}\right) + \frac{1}{4\pi\rho}\left(B_z\frac{\partial B_r}{\partial z} - \frac{B_\phi^2}{r}\right) \tag{2}$$

φ component of momentum conservation equation

$$\frac{d}{dr}(\dot{M}r^2\Omega) = -\gamma_\phi r^2 B_z^2 \tag{3}$$

z component of momentum conservation equation

$$\begin{aligned} &v_r\frac{\partial v_z}{\partial r} + v_z\frac{\partial v_z}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} - \frac{GM_* z}{r^3}\left(1 + \frac{z^2}{r^2}\right)^{-3/2} - \frac{1}{8\pi\rho}\frac{\partial}{\partial z}\left(B_\phi^2\right) \\ &- \frac{B_r}{4\pi\rho}\left(\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r}\right) \end{aligned} \tag{4}$$

Poloidal component of induction equation

$$\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = \frac{(v_z B_r - v_r B_z)}{\eta} \tag{5}$$

where $\eta = -D_{BL} v_r \delta r_{in}$
Energy equation

$$\begin{aligned} &\rho c_v\left[v_r\frac{\partial T}{\partial r} + v_z\frac{\partial T}{\partial z}\right] + P\left[\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{\partial u_z}{\partial z}\right] + \frac{1}{r}\frac{\partial}{\partial r}(rF^r) + \frac{\partial F^z}{\partial z} \\ &= \frac{\eta}{4\pi}\left(\frac{\partial B_\phi}{\partial z}\right)^2 + \frac{\eta}{4\pi}\left(\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r}\right)^2 + \frac{\eta}{4\pi}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(rB_\phi\right)\right]^2 \end{aligned} \tag{6}$$

2 Non-dimensional conservation equations

Continuity equation

$$x\sigma u_r = -1 \tag{7}$$

r component of momentum conservation equation

$$\varepsilon^2 u_r \frac{\partial u_r}{\partial x} + \left(x^{-3} - \omega^2\right) x = -\frac{\varepsilon^2}{\rho}\frac{\partial p}{\partial x} - \frac{\varepsilon}{2\rho}\frac{\partial}{\partial x}\left(b_z^2 + b_\phi^2\right) + \frac{b_z}{\rho}\frac{\partial b_r}{\partial \zeta} - \frac{\varepsilon}{x}\frac{b_\phi^2}{\rho} \tag{8}$$

φ component of momentum conservation equation

$$\varepsilon \frac{\partial}{\partial x}\left(x^2\omega\right) = \left|\gamma_\phi\right| x^2 b_z^2 \tag{9}$$

z component of momentum conservation equation

$$\begin{aligned}&\varepsilon^2 u_r \frac{\partial u_z}{\partial x} + \varepsilon u_z \frac{\partial u_z}{\partial \zeta} = -\frac{\varepsilon}{\rho}\frac{\partial p}{\partial \zeta} - \varepsilon\frac{\zeta}{x^3}\left(1 - \frac{3}{2}\varepsilon^2\frac{\zeta^2}{x^2}\right) - \frac{1}{2\rho}\frac{\partial b_\phi^2}{\partial \zeta}\\ &- \frac{b_r}{\rho}\left(\frac{\partial b_r}{\partial \zeta} - \varepsilon\frac{\partial b_z}{\partial x}\right)\end{aligned} \tag{10}$$

Poloidal component of induction equation

$$\frac{\partial b_r}{\partial \zeta} - \varepsilon\frac{\partial b_z}{\partial x} = -\frac{\varepsilon}{\delta}\frac{\left(u_z b_r - u_r b_z\right)}{D_{BL} u_r} \tag{11}$$

Energy equation

$$\begin{aligned}&\frac{T_t c_v}{c_{s,t}^2}\rho\left[\varepsilon u_r \frac{\partial \theta}{\partial x} + u_z\frac{\partial \theta}{\partial \zeta}\right] + p\left[\frac{\varepsilon}{x}\frac{\partial}{\partial x}\left(x u_r\right) + \frac{\partial u_z}{\partial \zeta}\right] + \frac{F_t}{\rho_t c_{s,t}^3}\left[\frac{\varepsilon^2}{x}\frac{\partial}{\partial x}\left(x\Phi^r\right) + \frac{\partial \Phi^z}{\partial \zeta}\right]\\ &= \frac{\eta}{\varepsilon}\left[\left(\frac{\partial b_\phi}{\partial \zeta}\right)^2 + \left(\frac{\partial b_r}{\partial \zeta} - \varepsilon\frac{\partial b_z}{\partial x}\right)^2 + \left(\frac{\varepsilon}{x}\frac{\partial}{\partial x}\left(x b_\phi\right)\right)^2\right]\end{aligned} \tag{12}$$

3 Coordinate Stretching, Perturbative Expansion, δ=ε, $\varepsilon \to$ 0

Continuity equation

$$\sigma_0 u_{r0} = -1 \tag{13}$$

r component of momentum conservation equation

$$\rho_0 \left(1 - \omega_0^2\right) = -\frac{1}{2}\frac{\partial}{\partial X}\left(b_{z0}^2 + b_{\phi 0}^2\right) + b_{z0}\frac{\partial b_{r0}}{\partial \zeta} \tag{14}$$

φ component of momentum conservation equation

$$\frac{\partial \omega_0}{\partial X} = \left|\gamma_\phi\right| b_{z0}^2 \tag{15}$$

z component of momentum conservation equation

$$\frac{\partial p_0}{\partial \zeta} = -\frac{1}{2}\frac{\partial b_{\phi 0}^2}{\partial \zeta} - b_{r0}\left(\frac{\partial b_{r0}}{\partial \zeta} - \frac{\partial b_{z0}}{\partial X}\right) \tag{16}$$

Poloidal component of induction equation

$$\frac{\partial b_{z0}}{\partial X} + \frac{b_{z0}}{D_{BL}} = 0 \tag{17}$$

Energy equation

$$\begin{aligned} &p_0\left[\frac{\partial u_{r0}}{\partial X} + \frac{\partial u_{z0}}{\partial \zeta}\right] + \frac{F_t}{\rho_t c_{s,t}^3}\left[\frac{\partial \Phi_0^r}{\partial X} + \frac{\partial \Phi_0^z}{\partial \zeta}\right] \\ &= \eta\left[\left(\frac{\partial b_{\phi 0}}{\partial \zeta}\right)^2 + \left(\frac{\partial b_{r0}}{\partial \zeta} - \frac{\partial b_{z0}}{\partial X}\right)^2 + \left(\frac{\partial b_{\phi 0}}{\partial X}\right)^2\right] \end{aligned} \tag{18}$$

4 Vertical Integration

Continuity equation

$$\sigma_0 u_{r0} = -1 \tag{19}$$

r component of momentum conservation equation

$$\sigma_0 \left(1 - \omega_0^2\right) = -h_0 \frac{d}{dX}\left(b_{z0}^2\right) + \frac{2}{3} b_{\phi 0}^{+2} \frac{dh_0}{dX} - \frac{1}{3} h_0 \frac{db_{\phi 0}^{+2}}{dX} \tag{20}$$

φ component of momentum conservation equation

$$\frac{d\omega_0}{dX} = \left|\gamma_\phi\right| b_{z0}^2 \tag{21}$$

z component of momentum conservation equation

$$p_o = \frac{1}{2} b_{\phi 0}^{+2} \tag{22}$$

Poloidal component of induction equation

$$\frac{db_{z0}}{dX} + \frac{b_{z0}}{D_{BL}} = 0 \tag{23}$$

Energy equation

$$\begin{aligned} 2h_0 \frac{d\sigma_0}{dX} &+ \frac{F_t}{\rho_t c_{st}^3}\left[-\frac{4h_0^2}{D_{BL}^2} + \frac{2h_0}{D_{BL}}\frac{dh_0}{dX} - \frac{2h_0^2}{\sigma_0 D_{BL}}\frac{d\sigma_0}{dX} + 1\right]\sigma_0 \\ &- 4D_{BL}\left[\frac{1}{h_0} + \frac{h_0}{\gamma_\phi^2 D_{BL}^2} + \frac{h_0}{3D_{BL}}\right]\sigma_0 = 0 \end{aligned} \tag{24}$$

Here the constant μ is equal to

$$\dot{M}_t = 4\pi\varepsilon r_{in}^2 \rho_t c_{st} \tag{25}$$

5 Solutions

h_0 comes from numerical solution of Eq. 24

$$b_{z0} = b_0 e^{-X/D_{BL}} \tag{26}$$

$$b_{\phi 0} = \left|\gamma_\phi\right| b_0 e^{-X/D_{BL}} \tag{27}$$

$$\omega_0 = 1 - \frac{\left|\gamma_\phi\right| b_0^2 D_{BL}}{2} e^{-2X/D_{BL}} \tag{28}$$

$$p_0 = \frac{1}{2}\gamma_\phi^2 b_0^2 e^{-2X/D_{BL}} \tag{29}$$

$$\theta_0 = \frac{1}{2^{1/4}}\gamma_\phi^{1/2} b_0^{1/2} e^{-X/2D_{BL}} \tag{30}$$

$$\sigma_0 = \frac{4b_0\left|\gamma_\phi\right|}{3D_{BL}\left(1+\omega_0\right)}\frac{dh_0}{dX} + \frac{4}{\left(1+\omega_0\right)}\left[\frac{b_0}{\left|\gamma_\phi\right| D_{BL}^2} + \frac{b_0\left|\gamma_\phi\right|}{3D_{BL}}\right] h_0 \tag{31}$$

$$u_{ro} = -\frac{1}{\sigma_0} \tag{32}$$

$$\rho_0 = \frac{\sigma_0}{2h_0} \tag{33}$$

$$c_{s0} = \sqrt{\frac{p_0}{\rho_0}} \tag{34}$$

$$\Phi_0^r = -\frac{\theta_0^4 h_0}{\sigma_0 D_{BL}} \tag{35}$$

$$\Phi_0^z = \frac{\theta_0^4}{\sigma_0} \tag{36}$$

6 Outer Solutions

$$\omega_{out} = x^{-3/2} \tag{37}$$

$$\theta_{out} = \left(\frac{2\eta}{9}\right)^{1/4} \alpha^{-1/4} x^{-3/8} \tag{38}$$

$$\rho_{out} = \frac{8\eta^3}{81\mu_{out}^2} \alpha^{-1} \dot{m}^{-2} x^{3/2} \left(1 - Cx^{-1/2}\right)^{-2} \tag{39}$$

$$\sigma_{out} = \frac{8\eta^2}{27\mu_{out}} \alpha^{-1} \dot{m}^{-1} x^{3/2} \left(1 - Cx^{-1/2}\right)^{-1} \tag{40}$$

$$h_{out} = \frac{3\mu_{out}}{2\eta} \dot{m} \left(1 - Cx^{-1/2}\right) \tag{41}$$

$$c_{sout} = \frac{3\mu_{out}}{2\eta} \dot{m} x^{-3/2} \left(1 - Cx^{-1/2}\right) \tag{42}$$

$$p_{out} = \frac{2\eta}{9} \alpha^{-1} x^{-3/2} \tag{43}$$

$$u_{rout} = \frac{27\mu_{out}^2}{8\eta^2} \alpha \dot{m}^2 x^{-5/2} \left(1 - Cx^{-1/2}\right) \tag{44}$$

7 Brave New World

When $\dot{M}$ is radial dependent the only differences we are gonna make are the followings

Angular momentum conservation

$$\frac{d}{dX}(\dot{m}_0\omega_0) = |\gamma_\phi| b_{z0}^2 \tag{45}$$

Mass conservation

$$\sigma_0 u_{r0} = -\dot{m}_0 \tag{46}$$

Energy conservation

$$\begin{aligned} &\left[2\dot{m}_0 - \frac{F_t}{\rho_t c_{st}^3}\frac{2h_0}{D_{BL}}\right]\frac{d\sigma_0}{dX} + \frac{F_t}{\rho_t c_{st}^3}\left[-\frac{4h_0}{D_{BL}} + \frac{2}{D_{BL}}\frac{dh_0}{dX} + \frac{1}{h_0} - 2\frac{d\dot{m}_0}{dX}\right]\sigma_0 \\ &- 4\dot{m}_0 D_{BL}\left[\frac{1}{h_0^2} + \frac{1}{\gamma_\phi^2 D_{BL}^2} + \frac{1}{3D_{BL}^2}\right]\sigma_0 = 0 \end{aligned} \tag{47}$$

The rest of the equations are same with constant radial independent mass accretion rate case.

8 Boundary Region Disk Solutions

8.1 Mass accretion rate is constant

$$\sigma_0 = \frac{4b_0\,|\gamma_\phi|}{3D_{BL}(1+\omega_0(X))}\frac{dh_0}{dX} + \frac{4}{1+\omega_0}\left[\frac{b_0}{|\gamma_\phi|\,D_{BL}^2} + \frac{b_0\,|\gamma_\phi|}{3D_{BL}}\right] h_0 \tag{48}$$

$$\begin{aligned}&\left[2\dot{m} - \frac{F_t}{\rho_t c_{st}^3}\frac{2h_0}{D_{BL}}\right]\frac{d\sigma_0}{dX} + \frac{F_t}{\rho_t c_{st}^3}\left[-\frac{4h_0}{D_{BL}} + \frac{2}{D_{BL}}\frac{dh_0}{dX} + \frac{1}{h_0}\right]\sigma_0 \\ &- 4\dot{m}D_{BL}\left[\frac{1}{h_0^2} + \frac{1}{\gamma_\phi^2 D_{BL}^2} + \frac{1}{3D_{BL}^2}\right]\sigma_0 = 0\end{aligned} \tag{49}$$

At the end, we diminished the number of equations that must be solved to two. If we take a close look to eq 48 - 49, it can be seen easily that to have 3 different kind of solutions. First solution requires $\sigma_0 = 0$ which is not a physical solution. Seconds solution is none of the terms are zero and obviously can only be solved numerically. Last solution indicates that two terms in front of σ_0 and its derivative can be zero. In this case we have first zero terms states that

$$h_0 = \frac{\rho_t c_{st}^3}{F_t}\dot{m}D_{BL}. \tag{50}$$

Under the following two assumptions we can produce a solution for the second zero term.

$$\frac{F_t}{\rho_t c_{st}^3}\left[-\frac{4h_0}{D_{BL}} + \frac{2}{D_{BL}}\frac{dh_0}{dX} + \frac{1}{h_0}\right] - 4\dot{m}D_{BL}\left[\frac{1}{h_0^2} + \frac{1}{\gamma_\phi^2 D_{BL}^2} + \frac{1}{3D_{BL}^2}\right] = 0 \tag{51}$$

Mass accretion rate in the boundary region is constant with spatial coordinate X, which enable us to have almost SS73 type but constant half disk thickness solution. In this situation we replace h_0 on the second zero term of Eq 49 and write following equation

$$\frac{F_t^2}{\rho_t^2 c_{st}^6} = -\frac{4\dot{m}^2}{3}\left[D_{BL} + \frac{1}{3} + \frac{1}{\gamma_\phi^2}\right] \tag{52}$$

It is not hard to see that this solution also not a physical one. Again we are fooled by the beauty of equations. Is it destiny or punishment that comes from the skies? No one knows...

8.2 Mass accretion rate is radial dependent

Let's assume that mass accretion rate on the boundary region has a radial dependency.

$$\sigma_0 = \frac{4b_0\,|\gamma_\phi|}{3D_{BL}(1+\omega_0(X))}\frac{dh_0}{dX} + \frac{4}{1+\omega_0}\left[\frac{b_0}{|\gamma_\phi|\,D_{BL}^2} + \frac{b_0\,|\gamma_\phi|}{3D_{BL}}\right]h_0 \tag{53}$$

$$\begin{aligned}&\left[2\dot{m}_0 - \frac{F_t}{\rho_t c_{st}^3}\frac{2h_0}{D_{BL}}\right]\frac{d\sigma_0}{dX} + \frac{F_t}{\rho_t c_{st}^3}\left[-\frac{4h_0}{D_{BL}} + \frac{2}{D_{BL}}\frac{dh_0}{dX} + \frac{1}{h_0} - 2\frac{d\dot{m}_0}{dX}\right]\sigma_0 \\ &- 4\dot{m}_0 D_{BL}\left[\frac{1}{h_0^2} + \frac{1}{\gamma_\phi^2 D_{BL}^2} + \frac{1}{3D_{BL}^2}\right]\sigma_0 = 0\end{aligned} \tag{54}$$

What we suggest as a solution is

$$2\dot{m}_0 - \frac{F_t}{\rho_t c_{st}^3}\frac{2h_0}{D_{BL}} = 0 \tag{55}$$

and

$$\frac{F_t}{\rho_t c_{st}^3}\left[-\frac{4h_0}{D_{BL}} + \frac{2}{D_{BL}}\frac{dh_0}{dX} + \frac{1}{h_0} - 2\frac{d\dot{m}_0}{dX}\right] - 4\dot{m}_0 D_{BL}\left[\frac{1}{h_0^2} + \frac{1}{\gamma_\phi^2 D_{BL}^2} + \frac{1}{3D_{BL}^2}\right] = 0 \tag{56}$$

We can find a radial dependent solution for half disk thickness

$$h_0 = \frac{\rho_t c_{st}^3}{F_t} D_{BL}\dot{m}_0 \tag{57}$$

and implant it in second part of the energy conservation equation to get

$$\left(1 - \frac{F_t}{\rho_t c_{st}^3}\right)\frac{d\dot{m}_0}{dX} - 2\left(1 + \frac{1}{\gamma_\phi^2 D_{BL}} + \frac{1}{3D_{BL}}\right)\dot{m}_0 - \frac{3}{2D_{BL}}\frac{F_t^2}{\rho_t^2 c_{st}^6}\frac{1}{\dot{m}_0} = 0 \tag{58}$$

or in another words

$$A\frac{d\dot{m}_0}{dX} - B\dot{m}_0 - C\frac{1}{\dot{m}_0} = 0 \tag{59}$$

The solution of this simple differential equation is (D is integration constant)

$$\dot{m}_0 = \sqrt{\frac{1}{B}exp\left[\frac{2B}{A}(X-D)\right] - \frac{C}{B}} \tag{60}$$

$$h_0 = \frac{\rho_t c_{st}^3}{F_t} D_{BL} \dot{m}_0 \tag{61}$$

$$\frac{d}{dX}(\dot{m}_0 \omega_0)9 = |\gamma_\phi| b_{z0}^2 \tag{62}$$

$$\sigma_0 = \frac{4b_0 \left|\gamma_\phi\right|}{3D_{BL}(1+\omega_0)} \frac{dh_0}{dX} + \frac{4}{1+\omega_0} \left[\frac{b_0}{\left|\gamma_\phi\right| D_{BL}^2} + \frac{b_0 \left|\gamma_\phi\right|}{3D_{BL}} \right] h_0 \tag{63}$$

$$u_{r0} = -\frac{\dot{m}_0}{\sigma_0} \tag{64}$$

$$p_0 = \frac{1}{2} b_{z0}^2 \tag{65}$$

$$\theta_0 = p_0^{1/4} \tag{66}$$

Let's pay a little attention to the sign of A. Because one of the signs - probably plus sign- will not have a physical meaning as X goes to infinity. And integration constant D might have physical meaning in terms of determining the outer radius of boundary region. In this case limit X does not have to go to infinity but can go to D which maybe is the outer radius of our transition zone. As gorgeous as it seems, since both B and C are positive constants inner part of square root is negative if limit of X goes to D. So limit of X has to go to infinity to match the radial dependent inner solutions of disk parameters to the outer ones.

Birds&Bees: Where did fit functions come from?

It all started with an assumption to determine the inner radius of an accretion disk. When two celestial objects love each very very much they feel the urge to express it in a more physical way. But like in any part of the life all individuals need their own private spaces and sometimes even celestial objects get bored from excessive amount of attention called accretion disk. These times an invisible almost a magical wall occurs. In our case the magical wall is Alfven radius, the place where dipole magnetic field balances ram pressure.

$$r_{in} = r_A = (GM_*)^{-1/7} \dot{M}^{-2/7} B_*^{4/7} R_*^{12/7} \tag{67}$$

Once dipole magnetic field lines interact with innermost region of the disk, a special region occurs as a result. Due to the diffusivity of magnetic stresses into matter, angular velocity of the test particle at the boundary region deviates from its Keplerian value. To be able to say something about the thickness of so called boundary region, denoted as δ, we use Shakura&Sunyaev (1973) half disk thickness solution

$$H_{ss} = \frac{3}{8\pi} \frac{\kappa_T}{c} \dot{M} \left[1 - Cr^{-1/2}\right] \tag{68}$$

$$H_t = a \times H_{ss} + H_0 \tag{69}$$

to get the aspect ratio which is

$$\varepsilon = \frac{H_t}{r_{in}} = c_1 \dot{m}^{2/7} + c_2 \dot{m}^{9/7} + c_3 \dot{m}^{10/7} \tag{70}$$

where c_1 , c_2 and c_3 are constant. Now we are ready to express the dimensionless boundary region thickness δ in terms of mass accretion rate with the help of a constant (0(1)) comes from SS73 viscosity prescription.

$$\delta = \frac{D_{BL}}{2} \varepsilon = C_1 \dot{m}^{2/7} + C_2 \dot{m}^{9/7} + C_3 \dot{m}^{10/7} \tag{71}$$

describes dimensionless boundary region thickness. Because of the constraints coming from physical parameters, including mass, radius and dipole magnetic field strength of neutron star, that compose C_1 ,

C_2 and C_3 , signs of these constants are determined by aforesaid physical parameters. In that case, signs of C_1 , C_2 and C_3 must be $'\pm'$, $'+'$ and $'-'$ respectively.

Similarly mass accretion rate dependence of measure of how much the magnetic field is twisted on the equatorial region ,$|\gamma_\phi|$ comes from our disk solutions. It can be written as

$$\omega_0(r_{in}) = \frac{\Omega_*(r_{in})}{\Omega_K(r_{in})} = 1 - \frac{|\gamma_\phi| b_0^2 D_{BL}}{2} \tag{72}$$

$$|\gamma_\phi| = \frac{2}{b_0^2 D_{BL}} - \frac{4\pi \ \Delta\nu \ r_{A,18}^{3/2}}{(GM_*)^{1/2} \ b_0^2 \ D_{BL}} \dot{m}^{-3/7} \tag{73}$$

Additionally Lorentzian and Gaussian fits functions that was used for fitting observational data of frequency vs peak separation are defined as below.

First order Gaussian fit function

$$f(x) = y0 + \sqrt{\frac{2}{\pi}} \frac{A_1}{w_1} exp \left(- \frac{2\,(x - xc1)^2}{w_1^2} \right) \tag{74}$$

Second order Gaussian fit function

$$f(x) = y0 + \sqrt{\frac{2}{\pi}} \frac{A_1}{w_1} exp \left(- \frac{2\,(x - xc1)^2}{w_1^2} \right) + \sqrt{\frac{2}{\pi}} \frac{A_2}{w_2} exp \left(- \frac{2\,(x - xc2)^2}{w_2^2} \right) \tag{75}$$