1 Inner radius of a magnetically truncated disk

One can estimate the magnetic truncation radius of an accretion disk around a neutron star of mass M and radius R with a surface dipolar field strength $B_* = \mu_*/R^3$ as the radius where the magnetic pressure balances the ram pressure of the disk matter, i.e.,

$$\frac{B^2}{8\pi} = -\frac{1}{2}\rho v_r v_\phi, \tag{1}$$

where ρ is the density, v_r and v_ϕ are the radial and azimuthal components of the velocity field in the accretion flow, and $B = \mu_*/r^3$ with μ_* being the magnetic dipole moment of the neutron star. For spherical accretion, the mass inflow rate is given by

$$\dot{M} = -4\pi r^2 \rho v_r. \tag{2}$$

Using equation (2) together with $v_\phi = \Omega_{\rm K} r = (GM/r)^{1/2} \approx v_{\rm ram}$, the pressure balance (equation 1) at the truncation radius $r_{\rm A}$ yields

$$r_{\rm A} = \dot{M}^{-2/7}(GM)^{-1/7} B_*^{4/7} R^{12/7} \tag{3}$$

which is also known to be the Alfvén radius for spherical accretion.

The inner radius of a magnetically truncated disk can, in principle, be expressed in terms of the Alfvén radius for spherical accretion, i.e., $r_{\rm in} = \xi r_{\rm A}$ (see, e.g., Ghosh & Lamb 1979 for $\xi \approx 0.41$; Lovelace et al. 1995 for $\xi \approx 0.24$). In our discussion, ξ is a factor which depends on the typical aspect ratio, $\varepsilon = H_t/r_{\rm in}$, with H_t being the typical half-thickness of the disk. We use the vertical hydrostatic equilibrium for the disk to define the typical sound speed as $c_{s,t} = \Omega_{\rm K}(r_{\rm in}) H_t = \varepsilon v_{\phi,t}$, where $v_{\phi,t} = \Omega_{\rm K}(r_{\rm in}) r_{\rm in}$ is the typical value for the azimuthal velocity. We estimate the mass inflow rate in the disk using the continuity equation,

$$\dot{M} = -2\pi r \Sigma v_r = -4\pi H r \rho v_r, \tag{4}$$

where $\Sigma = 2\rho H$ is the surface density. In terms of the typical values, ρ_t for the density and $c_{s,t}$ for the radial velocity, we obtain the typical mass inflow rate

$$\dot{M} = \varepsilon 4\pi r_{\rm in}^2 \rho_t c_{s,t}. \tag{5}$$

Using equation (1), we find the typical value of the plasma beta,

$$\beta_t \equiv \frac{4\pi\rho_t c_{s,t}^2}{B_t^2} = \varepsilon, \tag{6}$$

within the boundary layer at the innermost disk radius where the typical magnetic field strength can be estimated as $B_t = 4\pi\rho_t c_{s,t} v_{\phi,t}$ (see equation 1). As the magnetic dipole field of the neutron star determines the field strength at the innermost disk radius, we write $B_t \approx \mu_*/r_{\rm in}^3$ and obtain

$$r_{\rm in} = \varepsilon^{2/7} r_{\rm A} \approx 0.27 r_{\rm A} \left(\frac{\varepsilon}{0.01}\right)^{2/7} \tag{7}$$

using equations (5) and (6).

2 Magnetic boundary layer in the inner disk

As the dominant stresses are magnetic near the innermost disk radii, the conservation of angular momentum in a steady magnetic boundary layer (BL) situated at $r = r_{\rm in}$ can be written as

$$\frac{d}{dr}\left(\dot{M} r^2 \Omega\right) = -\gamma_\phi r^2 B_z^2, \tag{8}$$

where Ω is the angular velocity of the matter, B_z is the poloidal component of the magnetic field, and $\gamma_\phi \equiv B_\phi^+/B_z$ is the azimuthal pitch in the BL (see, e.g., Ghosh & Lamb 1979). Here, $B_\phi^+ \equiv B_\phi(z = H)$ is the toroidal field component at the surface. To see how the angular velocity Ω deviates from its Keplerian value $\Omega_{\rm K}$, we consider the radial-momentum conservation for the steady flow, i.e.,

$$v_r \frac{\partial v_r}{\partial r} + \left(\Omega_{\rm K}^2 - \Omega^2\right) r = -\frac{1}{\rho}\frac{\partial}{\partial r}\left(P + \frac{B_z^2 + B_\phi^2}{8\pi}\right) + \frac{1}{4\pi\rho}\left(B_z \frac{\partial B_r}{\partial z} - \frac{B_\phi^2}{r}\right). \tag{9}$$

In the absence of magnetic fields, the gradient of the thermal pressure, P, can determine the width of a non-magnetic BL. The effective screening of the poloidal field and therefore the magnetic pressure gradients can be large enough, however, in a magnetic BL due to the presence of strong toroidal currents to compensate for the second term in equation (9). To estimate the radial profile of B_z in a magnetic BL, we consider the poloidal component of the induction equation,

$$\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = \frac{(v_z B_r - v_r B_z)}{\eta}, \tag{10}$$

where $\eta = -D_{\rm BL} v_r \delta r_{\rm in}$ is the effective magnetic diffusivity with $D_{\rm BL}$ being the diffusivity coefficient of order unity. The radial extension of the BL, $\delta r_{\rm in}$, acts as the electromagnetic screening length for the poloidal magnetic field (Ghosh & Lamb 1979).

Next, we write equations (8)-(10) in a non-dimensional form. Normalizing each quantity by its typical value, we define $u_r \equiv v_r/c_{s,t}$, $u_z \equiv v_z/c_{s,t}$, $x \equiv r/r_{\rm in}$, $\zeta \equiv z/H_t$, $\omega(x) \equiv \Omega(r)/\Omega_{\rm K}(r_{\rm in})$, $b_r \equiv B_r/B_t$, $b_\phi \equiv B_\phi/B_t$, $b_z \equiv B_z/B_t$, $\varrho \equiv \rho/\rho_t$, and $p \equiv P/P_t$ with $P_t = \rho_t c_{s,t}^2$. We obtain

$$\varepsilon \frac{d}{dx}\left(x^2\omega\right) = |\gamma_\phi|\, x^2 b_z^2, \tag{11}$$

$$\varepsilon^2 u_r \frac{\partial u_r}{\partial x} + \left(x^{-3} - \omega^2\right) x = -\varepsilon^2 \frac{1}{\varrho}\frac{\partial p}{\partial x} - \varepsilon \frac{1}{2\varrho}\frac{\partial}{\partial x}\left(b_z^2 + b_\phi^2\right) + \frac{1}{\varrho}\left(b_z \frac{\partial b_r}{\partial \zeta} - \varepsilon \frac{b_\phi^2}{x}\right), \tag{12}$$

and

$$\frac{\partial b_r}{\partial \zeta} - \varepsilon \frac{\partial b_z}{\partial x} = -\frac{\varepsilon}{\delta}\frac{(u_z b_r - u_r b_z)}{D_{\rm BL} u_r} \tag{13}$$

for the angular and radial-momentum conservations and the poloidal component of the induction equation, respectively. Using the coordinate stretching $x = 1 + \delta X$ with X being the radial coordinate in the BL, it follows from equations (11)-(13) that

$$\frac{\varepsilon}{\delta}\frac{d}{dX}(\omega) = |\gamma_\phi|\, b_z^2, \tag{14}$$

$$\frac{\varepsilon^2}{\delta} u_r \frac{\partial u_r}{\partial X} + \left(1 - \omega^2\right) = -\frac{\varepsilon^2}{\delta}\frac{1}{\varrho}\frac{\partial p}{\partial X} - \frac{\varepsilon}{\delta}\frac{1}{2\varrho}\frac{\partial}{\partial X}\left(b_z^2 + b_\phi^2\right) + \frac{1}{\varrho}\left(b_z \frac{\partial b_r}{\partial \zeta} - \varepsilon b_\phi^2\right), \tag{15}$$

and

$$\frac{\partial b_r}{\partial \zeta} - \frac{\varepsilon}{\delta}\frac{\partial b_z}{\partial X} = -\frac{\varepsilon}{\delta}\frac{(u_z b_r - u_r b_z)}{D_{\rm BL} u_r}. \tag{16}$$

Note that this set of equations are consistent with the existence of a non-Keplerian BL of dimensionless radial width $\delta = \varepsilon$. Assuming $u_z \ll u_r$ and $b_r^+ \ll b_z$, equation (16) yields

$$b_z(X) = b_0 e^{-X/D_{\rm BL}} \tag{17}$$

for $\delta = \varepsilon$, where b_0 is a constant of order unity. Using equation (17), it follows from equation (14) that

$$\omega(X) = 1 - (1 - \omega_0)e^{-2X/D_{\rm BL}}, \tag{18}$$

where $\omega_0 = 1 - |\gamma_\phi| D_{\rm BL} b_0^2/2$. The angular velocity of the disk matter outside the BL is Keplerian, i.e., $\omega(x) = x^{-3/2}$. Matching the inner and outer solutions gives

$$\omega(x) = x^{-3/2} - (1 - \omega_0)\exp\left[-\frac{2(x-1)}{\varepsilon D_{\rm BL}}\right] \tag{19}$$

as the unified solution for the BL-disk angular velocity profile.

The radial epicyclic frequency in the inner disk can be estimated using

$$\kappa = \sqrt{2\Omega\left(2\Omega + r\frac{d\Omega}{dr}\right)} \tag{20}$$

which can be put into the form

$$\kappa = \Omega_{\rm K}(r_{\rm in})\sqrt{2\omega\left(2\omega + x\frac{d\omega}{dx}\right)}. \tag{21}$$

Using equation (19), we find

$$\kappa = \Omega_{\rm K}(r_{\rm in})\sqrt{2\omega\left\{2\omega + x\left[\frac{2(1-\omega_0)}{\varepsilon D_{\rm BL}}e^{-2(x-1)/\varepsilon D_{\rm BL}} - \frac{3}{2}x^{-5/2}\right]\right\}} \tag{22}$$

throughout the inner disk. At the innermost radius of the disk, the radial epicyclic frequency becomes

$$\kappa(r_{\rm in}) = \Omega_{\rm K}(r_{\rm in})\sqrt{2\omega_0\left[2\omega_0 + \frac{2(1-\omega_0)}{\varepsilon D_{\rm BL}} - \frac{3}{2}\right]}. \tag{23}$$

We can also write

$$\kappa(r_{\rm in}) = \Omega_{\rm K}(r_{\rm in})\sqrt{(2 - D_{\rm BL}b_0^2|\gamma_\phi|)\left(\frac{1}{2} - D_{\rm BL}b_0^2|\gamma_\phi| + b_0^2|\gamma_\phi|\varepsilon^{-1}\right)} \tag{24}$$

using $\omega_0 = 1 - |\gamma_\phi| D_{\rm BL} b_0^2/2$.

The lower and upper kHz QPO frequencies, ν_1 and ν_2, can be interpreted as the $\kappa - \Omega$ and κ frequency bands of oscillatory modes, respectively at the innermost disk radius. Using equations (3) and (7), it follows from equation (24) that

$$\nu_1 = \varepsilon^{-3/7} \nu_{\rm K}(r_{\rm A}) \left[\sqrt{(2 - |\gamma_\phi|) \left(\frac{1}{2} - |\gamma_\phi| + |\gamma_\phi| \, \varepsilon^{-1} \right)} - 1 + \frac{|\gamma_\phi|}{2} \right], \qquad (25)$$

where $b_0 = 1 = D_{\rm BL}$ is assumed for both b_0 and $D_{\rm BL}$ are constants of order unity. Note that $|\gamma_\phi|$ has a critical value (e.g., 2) below which the QPO frequency is observable. As the azimuthal pitch varies, in general, with the mass inflow rate in the disk, the kHz QPOs are expected to disappear when $|\gamma_\phi|$ exceeds its critical value. In equation (25),

$$\nu_{\rm K}(r_{\rm A}) = 2486.5 \, {\rm Hz} \, \dot{m}^{3/7} b^{-6/7} m^{5/7} R_6^{-18/7} \qquad (26)$$

is the Keplerian frequency at the Alfvén radius. Here, we define $\dot{m} \equiv \dot{M}/10^{18} {\rm g\,s^{-1}}$, $b \equiv B_*/10^8\,{\rm G}$, $m \equiv M/1.4\,M_\odot$, and $R_6 \equiv R/10^6\,{\rm cm}$. The typical aspect ratio of the disk is expected to vary with the mass inflow rate in the disk, i.e., $\varepsilon = \varepsilon(\dot{M})$. Since $r_{\rm A} > r_{\rm in}$ (equation 7), we estimate the $\dot{M}$ dependence of ε using

$$\varepsilon = \frac{H_t}{r_{\rm in}} = \frac{H_{\rm SS}(r_{\rm A})}{r_{\rm A}}, \qquad (27)$$

where

$$H_{\rm SS}(r_{\rm A}) = 1.591549 \times 10^6 \, {\rm cm} \; \dot{m} \left[1 - \left(\frac{3 r_{\rm S}}{r_{\rm A}} \right)^{1/2} \right] \qquad (28)$$

is the half-thickness of the Keplerian disk at the Alfvén radius (Shakura & Sunyaev 1973). The innermost stable circular orbit (ISCO) is close to $3r_{\rm S}$, where $r_{\rm S} = 2GM/c^2$ is the Schwarzschild radius. The ratio of the ISCO and the Alfvén radius can be written explicitly as

$$\frac{3 r_{\rm S}}{r_{\rm A}} = 1.3613 \, \dot{m}^{2/7} b^{-4/7} m^{8/7} R_6^{-12/7}. \qquad (29)$$

Using equation (29), we find

$$H_{\rm SS}(r_{\rm A}) = 1.591549 \times 10^6 \, {\rm cm} \; \left[\dot{m} - (1.1667) \dot{m}^{8/7} b^{-2/7} m^{4/7} R_6^{-6/7} \right]. \qquad (30)$$

The typical aspect ratio of the disk can be expressed as

$$\varepsilon = 1.7401\, \dot{m}^{9/7} b^{-4/7} m^{1/7} R_6^{-12/7} \left[1 - (1.1667)\dot{m}^{1/7} b^{-2/7} m^{4/7} R_6^{-6/7}\right] \quad (31)$$

according to equation (27).