EE 583 Probabilistic Graphical Models

Take-Home Final

Due on Sunday, May 29, 2016

1. **[15 pts]** Consider a set of variables $\boldsymbol{X} = \{X_1, \dots, X_n\}$, where each X_i has $|Val(X_i)| = l$.
   a) How many independent parameters are in the full joint distribution over $\boldsymbol{X}$?
   b) Consider a Bayesian network over $\boldsymbol{X}$, such that each node has at most k parents. How many independent parameters can there be at most?
   c) Now, assume that each variable X_i has the parents $X_1, \dots, X_{i-1}$. How many independent parameters are there in the resulting Bayesian network?

2. **[15 pts]** Consider a Markov network over $\boldsymbol{X} = \{X_1, \dots, X_n\}$. Let $\mathcal{C}$ denote the set of cliques. Show that any Gibbs distribution associated to this Markov network can be written as a log-linear model $P(\boldsymbol{X} = \boldsymbol{x}) = \frac{1}{Z}\exp\{\sum_{c \in \mathcal{C}} w_c f_c(\boldsymbol{x}_c)\}$, where $\{f_c(\boldsymbol{x}_c): c \in \mathcal{C}\}$ is a set of features, and $\{w_c: c \in \mathcal{C}\}$ is the set of corresponding linear weights.

3. **[20 pts]** Implement an efficient algorithm of your choice that takes a Bayesian network over a set of variables $\boldsymbol{X}$ (you can specify the BN by its nodes, edges and CPTs, OR as in Homework 3, by edges and factors), and a full instantiation $\boldsymbol{x}$, and returns the probability $P(\boldsymbol{X} = \boldsymbol{x})$. Run your algorithm on the two networks of Homework 3 (non-loopy and loopy versions) to evaluate the joint probability of $(D = 0, I = 1, G = 3, S = 0, L = 1)$.

4. **[50 pts]** Consider a directed acyclic graph $G = (V, E)$, where nodes are given by the doubly indexed set
$$V = \{s_{ij}: i = 1, \dots, n, j = 1, \dots, n - i + 1\},$$
and edges are given as
$$E = \{(s_{i-1j}, s_{ij}), (s_{i-1j+1}, s_{ij}): i = 2, \dots, n, j = 1, \dots, n - i + 1\}.$$
That is, G looks like the upper-triangular part of an $n \times n$ square matrix, where for each entry in the rows $2, \dots, n$, the parents are the north and north-east neighbors. Now, consider a Bayesian network over $\boldsymbol{X} = \{X_s: s \in V\}$ with structure G, where each $X_s \in \{0,1\}$ is binary valued. Suppose conditional probabilities for $i > 1$ are given by
$$p_{s_{ij}}\left(X_{s_{ij}} = 1 \middle| x_{s_{i-1j}}, x_{s_{i-1j+1}}\right) = \begin{cases} a & \text{if } x_{s_{i-1j}} = x_{s_{i-1j+1}} \\ b & \text{otherwise} \end{cases}$$
and by
$$p_{s_{ij}}\left(X_{s_{ij}} = 1\right) = c,$$
for $i = 1$ and all j, where $a, b, c \in (0,1)$. Set $a = 0.7$, $b = 0.5$, $c = 0.3$.
   a) Use direct sampling (top-down sampling) to approximate the marginal probability $P(X_{s_{n1}} = 1)$ of the final variable $X_{s_{n1}}$, as a function of $n = 2,3, \dots, 10$.
   b) Compute the same marginal $P(X_{s_{n1}} = 1)$ with belief propagation (sum-prod algorithm). You can use your previous implementation for Homework 3.

c) Show that, for $s = s_{ij}$ and $x, y, z \in \{0,1\}$ we can write for some numbers $\alpha, \beta, \gamma, \delta$, and a constant C_s (independent from x, y, z),

$$p_s(x|y,z) = C_s \exp(\alpha x(y-z)^2 + \beta(y-z)^2 + \gamma x)$$

when $i > 1$; and

$$p_s(x) = C_s \exp(\delta x)$$

when $i = 1$. Give the expressions for $C_s, \alpha, \beta, \gamma, \delta$ as a function of a, b, c.

d) Use the result in part (c) to write down functions $H_\alpha, H_\beta, H_\gamma, H_\delta$ such that the joint distribution over all variables $\boldsymbol{X}$ can be written as

$$P(\boldsymbol{X} = \boldsymbol{x}) \doteq \pi(\boldsymbol{x}) = \frac{1}{Z} \exp\big(\alpha H_\alpha(\boldsymbol{x}) + \beta H_\beta(\boldsymbol{x}) + \gamma H_\gamma(\boldsymbol{x}) + \delta H_\delta(\boldsymbol{x})\big)$$

e) Implement Gibbs sampling algorithm to sample from π. For $n = 5$ and $n = 10$ plot, as a function of time, the empirical probability $P\big(X_{s_{n1}} = 1\big)$, as an average of corresponding Gibbs samples generated so far, where a full sweep of the network is taken as the time step. Compare the same curve obtained using the direct simulation in part (a).

Remarks

A. Submit your finals online (by e-mail or Moodle) before Sunday, May 29, 2016, 23:59, preferably everything written in a single MS-word or LaTeX format, with figures labeled and captioned, proofs, empirical and analytical results decently provided. Put your code as an Appendix.

B. If you really need to submit hand-written parts and cannot scan them, please use Office Lens (a free app on Google Play and Apple Store) to take more readable photos of your material.

C. Feel free to discuss with each other, research through relevant material, or ask me any questions (either over Moodle or email), which I will answer by replying to all (if necessary, we can even set up a day for an in-person meeting with me during the take-home period). But make sure you submit your own work.

D. Good Luck.