Mathematics 1110H (Section A) – Calculus I: Limits, Derivatives, and Integrals
TRENT UNIVERSITY, Fall 2024

Solutions to Quiz #7
Qualitative Analysis/Curve Sketching

1. Find the domain and any and all intercepts, horizontal and vertical asymptotes, intervals of increase and decrease, local maximum and minimum points, intervals of concavity, and inflection points of $y = x^2e^{-x}$, and sketch its graph based on this information. *[5]*

SOLUTION. We run through the indicated checklist.

i. Domain. x^2 and e^{-x} are both defined (and continuous and differentiable) for all x and hence so is their product, $y = x^2e^{-x}$. Thus the domain is $\mathbb{R} = (-\infty, \infty)$.

ii. Intercepts. Plugging in $x = 0$, we get $y = 0^2e^{-0} = 0 \cdot 1 = 0$, so the y-intercept is 0.

Solving for $y = x^2e^{-x} = 0$, we must have $x^2 = 0$ (since $e^t > 0$ for all t), and hence $x = 0$. Thus the x-itercept is 0.

Note that the only x-intercept is also the only y-intercept.

iii. Horizontal asymptotes. Since

$$\begin{aligned}
\lim_{x\to-\infty} x^2e^{-x} &= \lim_{x\to-\infty} \frac{x^2 \to \infty}{e^x \to 0} = \infty \quad \text{and} \\
\lim_{x\to\infty} x^2e^{-x} &= \lim_{x\to\infty} \frac{x^2 \to \infty}{e^x \to \infty} \quad \text{(So we can apply l'Hôpital's Rule.)} \\
&= \lim_{x\to\infty} \frac{\frac{d}{dx}x^2}{\frac{d}{dx}e^x} = \lim_{x\to\infty} \frac{2x \to \infty}{e^x \to \infty} \quad \text{(Apply l'Hôpital's Rule again.)} \\
&= \lim_{x\to\infty} \frac{\frac{d}{dx}2x}{\frac{d}{dx}e^x} = \lim_{x\to\infty} \frac{2 \to 2}{e^x \to \infty} = 0^+,
\end{aligned}$$

$y = x^2e^{-x}$ has a horizontal asymptote of $y = 0$ as $x \to \infty$ only, which it approaches from above.

iv. Vertical asymptotes. Since $y = x^2e^{-x}$ is defined and continuous for all x, it has no vertical asymptotes.

v. Increase/decrease and max/min. We first compute $\frac{dy}{dx}$, using the Product, Power, and Chain Rules:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left(x^2e^{-x}\right) = \left[\frac{d}{dx}x^2\right]e^{-x} + x^2\left[\frac{d}{dx}e^{-x}\right] = 2xe^{-x} + x^2\left[e^{-x}\frac{d}{dx}(-x)\right] \\
&= 2xe^{-x} + x^2\left[-e^{-x}\right] = 2xe^{-x} - x^2e^{-x} = \left(2x - x^2\right)e^{-x} = x(2-x)e^{-x}
\end{aligned}$$

Since $e^{-x} > 0$ for all x, it follows that $\frac{dy}{dx} = 0$ exactly when $x = 0$ or $x = 2$. Similarly, $\frac{dy}{dx} > 0$ exactly when either both $x > 0$ and $2 - x > 0$, *i.e.* when $0 < x < 2$, or both

$x < 0$ and $2 - x < 0$, *i.e.* when $2 < x < 0$, which last never happens. This means that $y = x^2 e^{-x}$ is increasing for $0 < x < 2$. Also similarly, $\dfrac{dy}{dx} < 0$ exactly when either $x > 0$ and $2 - x < 0$, *i.e.* when $2 < x$, or $x < 0$ and $2 - x > 0$, *i.e.* when $x < 0$. This means that $y = x^2 e^{-x}$ is decreasing for $x < 0$ and $x > 2$. In turn this means that $x = 0$ is a local minimum and $x = 2$ is a local maximum.

As usual we summarize all this in a table:

x	$(-\infty, 0)$	0	$(0, 2)$	2	$(0, \infty)$
$\frac{dy}{dx}$	$-$	0	$+$	0	$-$
y	$\downarrow$	min	$\uparrow$	max	$\downarrow$

vi. Intervals of concavity and inflection points. We first compute $\dfrac{d^2y}{dx^2}$, using the Product, Power, and Chain Rules:

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(2x - x^2\right)e^{-x} = \left[\frac{d}{dx}\left(2x - x^2\right)\right]e^{-x} + \left(2x - x^2\right)\left[\frac{d}{dx}e^{-x}\right] \\ &= [2 - 2x]\,e^{-x} + \left(2x - x^2\right)\left[-e^{-x}\right] = 2e^{-x} - 2xe^{-x} - 2xe^{-x} + x^2e^{-x} \\ &= \left(x^2 - 4x + 2\right)e^{-x}\end{aligned}$$

Since $e^{-x} > 0$ for all x, $\dfrac{d^2y}{dx^2} = 0$ exactly when $x^2 - 4x + 2 = 0$. Applying the quadratic formula, this occurs exactly when

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 2}}{2 \cdot 1} = \frac{4 \pm \sqrt{4 \cdot 2}}{2} = \frac{4 \pm 2\sqrt{2}}{2} = 2 \pm \sqrt{2}.$$

We check what happens on either side of and between $x = 2 \pm \sqrt{2}$ by plugging suitable x values into $\dfrac{d^2y}{dx^2} = \left(x^2 - 4x + 2\right)e^{-x}$. Observe that $0 < 2 - \sqrt{2} < 2 < 2 + \sqrt{2} < 4$.

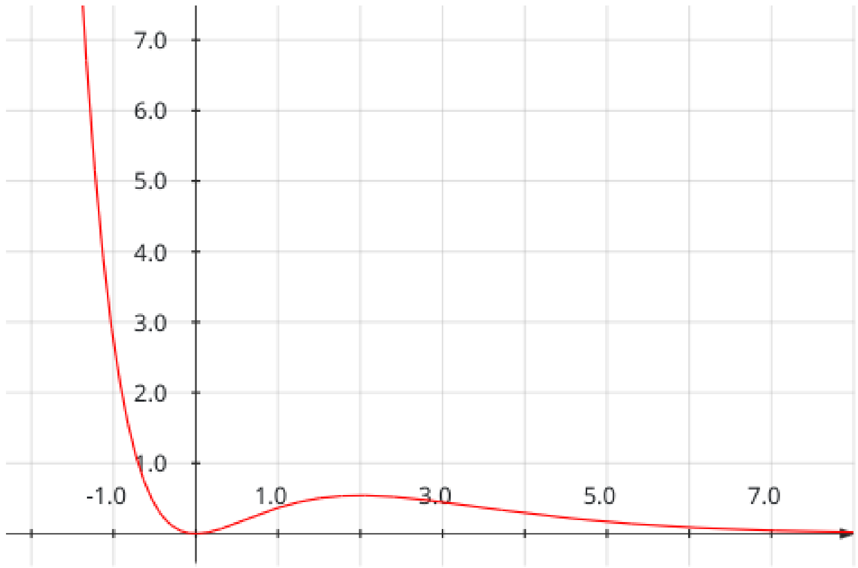
$$\begin{aligned} x = 0&: \quad \left(0^2 - 4 \cdot 0 + 2\right)e^{-2} = (0 - 0 + 2)\,e^{-2} = 2e^{-2} > 0 \\ x = 2&: \quad \left(2^2 - 4 \cdot 2 + 2\right)e^{-2} = (4 - 8 + 2)\,e^{-2} = -2e^{-2} < 0 \\ x = 4&: \quad \left(4^2 - 4 \cdot 4 + 2\right)e^{-2} = (16 - 16 + 2)\,e^{-2} = 2e^{-2} > 0 \end{aligned}$$

It follows that $\dfrac{d^2y}{dx^2} > 0$ for $x < 2 - \sqrt{2}$ and for $x > 2 + \sqrt{2}$, and then the function is concave up, and $\dfrac{d^2y}{dx^2} < 0$ for $2 - \sqrt{2} < x < 2 + \sqrt{2}$, and then the function is concave down. The function therefore has inflection points at $x = 2 \pm \sqrt{2}$.

As usual, we summarize all this in a table:

x	$\left(-\infty, 2 - \sqrt{2}\right)$	$2 - \sqrt{2}$	$\left(2 - \sqrt{2}, 2 + \sqrt{2}\right)$	$2 + \sqrt{2}$	$\left(2 + \sqrt{2}, \infty\right)$
$\frac{d^2y}{dx^2}$	$+$	0	$-$	0	$+$
y	$\smile$	infl. pt.	$\frown$	infl. pt.	$\smile$

vii. Sketch. We cheat a bit and let software do the work:



This was drawn by a program called `kmplot`. ■