Mathematics 1110H (Section A) – Calculus I: Limits, Derivatives, and Integrals
Trent University, Fall 2024

Solutions to Quiz #9
Integration

Evaluate each of the following definite integrals by hand using antiderivatives. Please show all your work. *Hint*: Simplify first ...

1. $\displaystyle\int_0^{\pi/2} 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\,dx$ *[1]*

Solution. We will use the double angle formula $\sin(2t) = 2\sin(t)\cos(t)$ and the fact that the antiderivative of $\sin(x)$ is $-\cos(x)$, which follows from $\frac{d}{dx}\cos(x) = -\sin(x)$. Then:

$$\int_0^{\pi/2} 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\,dx = \int_0^{\pi/2} \sin\left(2\cdot\frac{x}{2}\right)\,dx = \int_0^{\pi/2} \sin(x)\,dx = -\cos(x)\big|_0^{\pi/2}$$
$$= \left[-\cos\left(\frac{\pi}{2}\right)\right] - [-\cos(0)] = [-0] - [-1] = 1 \quad \square$$

2. $\displaystyle\int_{-1}^{1} \frac{2}{1+x^2}\,dx$ *[1]*

Solution. Here we will exploit the fact that $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$, so the antiderivative of $\frac{1}{1+x^2}$ is $\arctan(x)$:

$$\int_{-1}^{1} \frac{2}{1+x^2}\,dx = 2\int_{-1}^{1} \frac{1}{1+x^2}\,dx = 2\arctan(x)\big|_{-1}^{1} = 2\arctan(1) - 2\arctan(-1)$$
$$= 2\cdot\frac{\pi}{4} - 2\cdot\left(-\frac{\pi}{4}\right) = \frac{\pi}{2} + \frac{\pi}{2} = \pi \quad \square$$

3. $\displaystyle\int_{2}^{3} \frac{x^2-5x+4}{x-4}\,dx$ *[1.5]*

Solution. The key to this one are the observations that $x^2-5x+4 = (x-1)(x-4)$ and that $4 \notin [2,3]$, so for x between 2 and 3 we have $\frac{x^2-5x+4}{x-4} = x-1$. Then, with the help of the Power Rule for integration:

$$\int_2^3 \frac{x^2-5x+4}{x-4}\,dx = \int_2^3 \frac{(x-4)(x-1)}{x-4}\,dx = \int_2^3 (x-1)\,dx = \left[\frac{x^2}{2} - x\right]_2^3$$
$$= \left[\frac{3^2}{2} - 3\right] - \left[\frac{2^2}{2} - 2\right] = \left[\frac{9}{2} - \frac{6}{2}\right] - [2-2] = \frac{3}{2} - 0 = \frac{3}{2} \quad \square$$

4. $\displaystyle\int_0^{1/\sqrt{\ln(41)}} \ln\left(41^x\right)\, dx$ *[1.5]*

Solution. Here we use the property of logarithms that $\ln\left(a^b\right) = b \cdot \ln(a)$ to simplify the integrand, and then apply the Power Rule:

$$\int_0^{1/\sqrt{\ln(41)}} \ln\left(41^x\right)\, dx = \int_0^{1/\sqrt{\ln(41)}} x \cdot \ln(41)\, dx = \left[\ln(41) \cdot \frac{x^2}{2}\right]_0^{1/\sqrt{\ln(41)}}$$

$$= \left[\frac{\ln(41)}{2} \cdot \left(\frac{1}{\sqrt{\ln(41)}}\right)^2\right] - \left[\frac{\ln(41)}{2} \cdot 0^2\right]$$

$$= \left[\frac{\ln(41)}{2} \cdot \frac{1}{\ln(41)}\right] - 0 = \frac{1}{2} - 0 = \frac{1}{2} \quad \blacksquare$$