Mathematics 1110H (Section A) – Calculus I: Limits, Derivatives, and Integrals
TRENT UNIVERSITY, Fall 2024

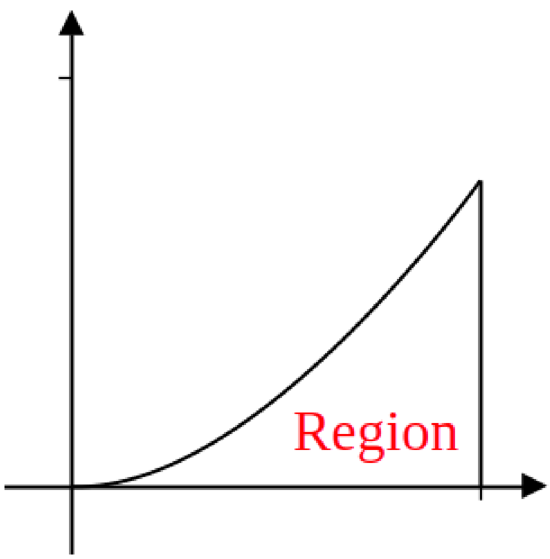
Solutions to Optional Quiz #11†
Crank up the volume a little bit more!

Please do the following problems by hand, showing all your work. Feel free to *check* your work by turning SageMath or other software loose.

Consider the region between $y = \arctan\left(x^2\right)$ and the x-axis for $0 \leq x \leq 1$, and the solid obtained by revolving this region about the y-axis.

1. Sketch the region. *[0.5]*

SOLUTION. Note that $\arctan\left(0^2\right) = \arctan(0) = 0$, since $\tan(0) = 0$, and $\arctan\left(1^2\right) = \arctan(1) = \frac{\pi}{4}$, since $\tan\left(\frac{\pi}{4}\right) = 1$. Less essentially, at least for our eventual application, $\frac{d}{dx}\arctan\left(x^2\right) = \frac{1}{1+\left(x^2\right)^2} \cdot \frac{d}{dx}x^2 = \frac{2x}{1+x^4}$, so at $x = 0$, the graph of $y = \arctan\left(x^2\right)$ has slope 0, and at $x = 1$, it has slope 1. Moreover, between $x = 0$ and $x = 1$, the graph has a positive slope. This gives us a region that looks more or less as follows:



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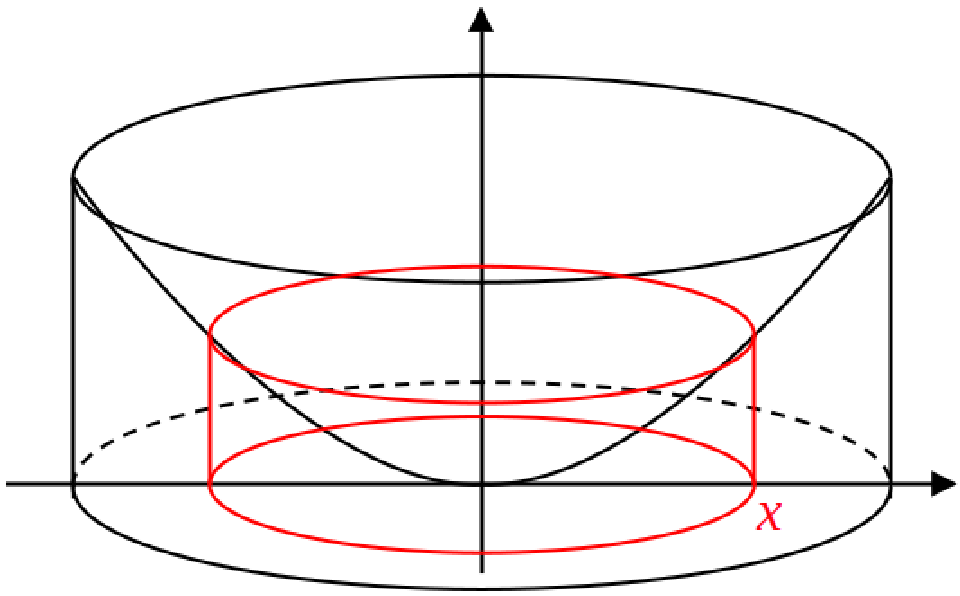
† Best nine quizzes still count for the final mark; doing this quiz may let you improve your mark by dropping the lowest two quizzes instead if just one.

2. Sketch the solid. *[0.5]*

Solution. Here is a sketch of the solid obtained by revolving the region about the y-axis, with a cylindrical shell drawn in for the sake of question **3**.



$\square$

3. Find the volume of the solid. *[4]*

Solution. We will use the method of cylindrical shells to find the volume of this solid of revolution. (Using the disk/washer method would force us to deal with the likes of $x = \sqrt{\tan(y)}$, if only temporarily.) Since we revolved the region about the y-axis, the cylindrical shells are perpendicular to the x-axis, so we use x as the basic variable.

The shell at x, for an x between 0 and 1, has radius $r = x - 0 = x$ and height $h = y - 0 = \arctan\left(x^2\right)$, and thus has area $2\pi rh = 2\pi x \arctan\left(x^2\right)$. It follows that the volume of the solid is given by:

$$V = \int_0^1 2\pi rh\,dx = \int_0^1 2\pi x \arctan\left(x^2\right)\,dx$$

Substitute $w = x^2$, so $dw = 2x\,dx$, and change the limits as we go along: $\begin{smallmatrix} x & 0 & 1 \\ w & 0 & 1 \end{smallmatrix}$

$$= \int_0^1 \pi \arctan(w)\,dw$$

Integration by parts, with $u = \arctan(w)$ and $v' = 1$, so $u' = \frac{1}{1+w^2}$ and $v = w$, just like in class.

$$= \pi \left[w \arctan(w)\Big|_0^1 - \int_0^1 \frac{w}{1+w^2}\,dw \right]$$

Substitute $z = 1 + w^2$, so $dz = 2w\,dw$ and $w\,dw = \frac{1}{2}\,dz$, and change the limits: $\begin{smallmatrix} w & 0 & 1 \\ z & 1 & 2 \end{smallmatrix}$

$$= \pi \left[1\arctan(1) - 0\arctan(0) - \int_1^2 \frac{1}{z} \cdot \frac{1}{2}\,dz \right] = \pi \left[\frac{\pi}{4} - 0 - \frac{1}{2}\ln(z)\Big|_1^2 \right]$$

$$= \pi \left[\frac{\pi}{4} - \left(\frac{1}{2}\ln(2) - \frac{1}{2}\ln(1) \right) \right] = \pi \left[\frac{\pi}{4} - \frac{1}{2}\ln(2) + \frac{1}{2} \cdot 0 \right] = \frac{\pi^2}{4} - \frac{\pi \ln(2)}{2} \quad \blacksquare$$