MATH 1350 **Name (Print):** ______________________

Winter 2025

Test 2

2025-02-11 **ID number:** ______________________

You are required to **show your work** on each problem on this test.

1. (a) (3 points) Find the inverse of the following matrix A.

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$(A|I_3) = \left(\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_2 - 2R_1} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & -3 & -2 & 1 & -2 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & -3 & -2 & 1 & -2 & 0 \end{array}\right)$$

$$\xrightarrow{R_3 + 3R_2} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 4 & 1 & -2 & 3 \end{array}\right) \xrightarrow{\frac{1}{4}R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{2} & \frac{3}{4} \end{array}\right)$$

$$\xrightarrow[R_2 - 2R_3]{R_1 - R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -\frac{1}{4} & \frac{3}{2} & -\frac{3}{4} \\ 0 & 1 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{2} & \frac{3}{4} \end{array}\right) \xrightarrow{R_1 - 2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & -\frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{2} & \frac{3}{4} \end{array}\right) = (I_3|A^{-1})$$

Thus

$$A^{-1} = \begin{pmatrix} \frac{3}{4} & -\frac{1}{2} & \frac{1}{4} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ \frac{1}{4} & -\frac{1}{2} & \frac{3}{4} \end{pmatrix}.$$

(b) (2 points) Solve the equation $AX = B$ for X, where A is as above, and $B = \begin{pmatrix} 4 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Multiply both sides by A^{-1} to get $X = A^{-1}B$. Thus

$$X = A^{-1}B = \begin{pmatrix} \frac{3}{4} & -\frac{1}{2} & \frac{1}{4} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ \frac{1}{4} & -\frac{1}{2} & \frac{3}{4} \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 3 & -1/2 \\ -2 & 1 \\ 1 & -1/2 \end{pmatrix}$$

2. (a) (1 point) Write the 3×3 elementary matrix for the elementary row operation "$R_1 + \frac{2}{3}R_2$".

$$\begin{pmatrix} 1 & \frac{2}{3} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) (1 point) Write the elementary row operation corresponding to the elementary matrix $E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$.

$$R_2 \leftrightarrow R_4$$

(c) (1 point) Find the inverse of the matrix E in part (b).

$$E^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

(d) (1 point) Find the inverse of $M_1 = \begin{pmatrix} 4 & 9 \\ 3 & 6 \end{pmatrix}$

$$M_1^{-1} = \frac{1}{(-3)} \begin{pmatrix} 6 & -9 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ 1 & -\frac{4}{3} \end{pmatrix}$$

(e) (1 point) If $M_2^{-1} = \begin{pmatrix} 12 & 0 \\ -5 & -2 \end{pmatrix}$ find $(M_1M_2)^{-1}$ (using M_1 above).

$$(M_1M_2)^{-1} = M_2^{-1}M_1^{-1} = \begin{pmatrix} 12 & 0 \\ -5 & -2 \end{pmatrix} \begin{pmatrix} -2 & 3 \\ 1 & -\frac{4}{3} \end{pmatrix} = \begin{pmatrix} -24 & 36 \\ 8 & -\frac{37}{3} \end{pmatrix}$$

3. (4 points) Find the LU-factorization of $A = \begin{pmatrix} 1 & 4 & -7 \\ -3 & 6 & 9 \\ 6 & 6 & 0 \end{pmatrix}$. Be sure to state your final answer.

$$A = \begin{pmatrix} 1 & 4 & -7 \\ -3 & 6 & 9 \\ 6 & 6 & 0 \end{pmatrix} \xrightarrow{R_2+3R_1} \begin{pmatrix} 1 & 4 & -7 \\ 0 & 18 & -12 \\ 6 & 6 & 0 \end{pmatrix} \xrightarrow{R_3-6R_1} \begin{pmatrix} 1 & 4 & -7 \\ 0 & 18 & -12 \\ 0 & -18 & 42 \end{pmatrix} \xrightarrow{R_3+R_2} \begin{pmatrix} 1 & 4 & -7 \\ 0 & 18 & -12 \\ 0 & 0 & 30 \end{pmatrix} = U$$

The EROs involved have elementary matrices

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -6 & 0 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Here $E_3E_2E_1A = U$, and so $A = E_1^{-1}E_2^{-1}E_3^{-1}U$, where

$$E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 6 & 0 & 1 \end{pmatrix}, \quad E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

Thus our matrix L is

$$L = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 6 & -1 & 1 \end{pmatrix},$$

and hence the LU-factorization of A is

$$\begin{pmatrix} 1 & 4 & -7 \\ -3 & 6 & 9 \\ 6 & 6 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 6 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 4 & -7 \\ 0 & 18 & -12 \\ 0 & 0 & 30 \end{pmatrix}.$$

4. (3 points) Suppose M is a 4×4 matrix and that

$$M \xrightarrow{R_3 - 6R_1} B \xrightarrow{R_4 + \frac{1}{2}R_2} C \xrightarrow{-2R_3} I_4$$

where I_4 is the 4×4 identity matrix. Find M^{-1}.

If E_1, E_2 and E_3 are the elementary matrices for $R_3 - 6R_1, R_4 + \frac{1}{2}R_2$ and $-2R_3$ respectively, then the row reduction above says that $E_3E_2E_1M = I_4$. Thus $M^{-1} = E_3E_2E_1$, i.e.

$$M^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -6 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 12 & 0 & -2 & 0 \\ 0 & \frac{1}{2} & 0 & 1 \end{pmatrix}$$