Warm-up question

Consider the set of matrices of the form $M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ where $\theta \in \mathbb{R}$.

Sketch the vector $\mathbf{x}$ and the matrix product $M\mathbf{x}$ where:

(a) $\theta = \frac{\pi}{4}$ and $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(b) $\theta = \frac{\pi}{4}$ and $x = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

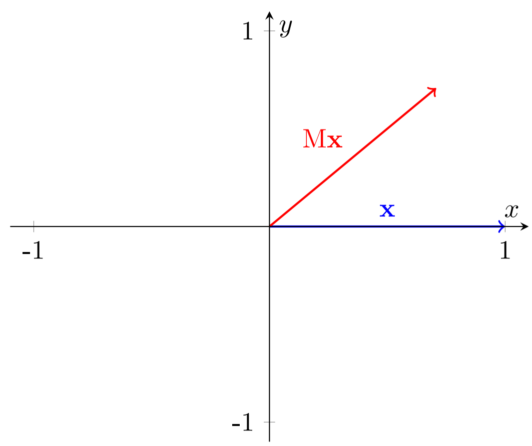
(c) $\theta = \frac{\pi}{2}$ and $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(d) $\theta = \frac{\pi}{2}$ and $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Solution.

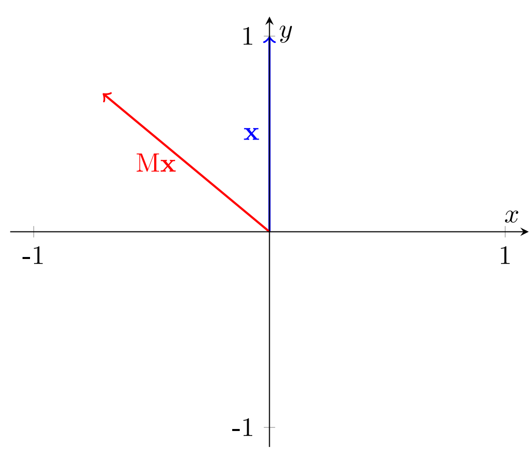
(a)

$$M\mathbf{x} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$



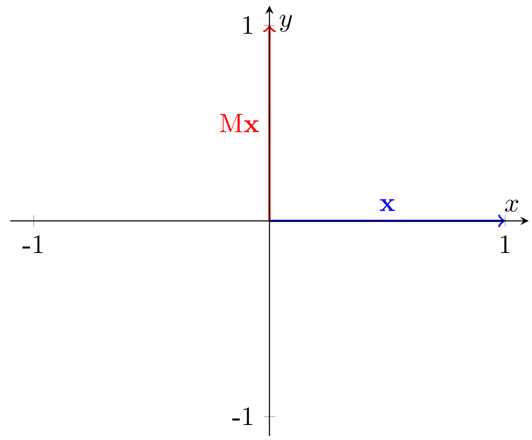
(b)

$$M\mathbf{x} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$



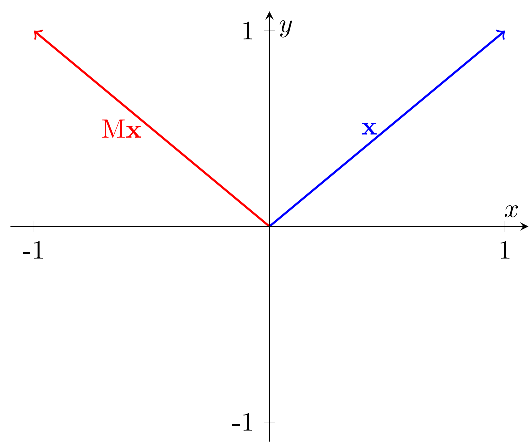
(c)

$$M\mathbf{x} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



(d)

$$M\mathbf{x} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$



□

Seminar questions

1. Using the cosine law from lecture, verify that the angle between $\mathbf{x}$ and $M\mathbf{x}$ is θ for parts (a) and (c) of the Warm-up question.

   Solution.

   (a) The angle θ between $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $M\mathbf{x} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ is given by

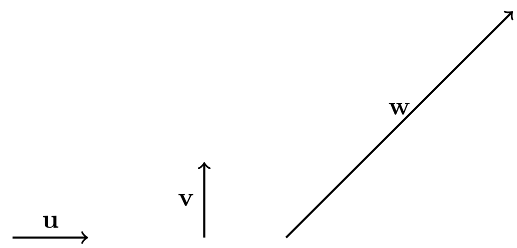
$$\theta = \cos^{-1}\left(\frac{\mathbf{x}\cdot(M\mathbf{x})}{||\mathbf{x}||\ ||M\mathbf{x}||}\right) = \cos^{-1}\left(\frac{\frac{1}{\sqrt{2}}}{(1)(1)}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}.$$

(c) The angle θ between $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $M\mathbf{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is given by

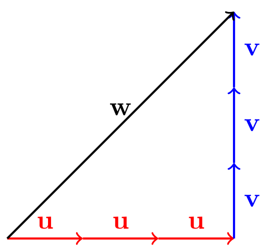
$$\theta = \cos^{-1}\left(\frac{\mathbf{x} \cdot (M\mathbf{x})}{||\mathbf{x}||\ ||M\mathbf{x}||}\right) = \cos^{-1}\left(\frac{0}{(1)(1)}\right) = \cos^{-1}(0) = \frac{\pi}{2}.$$

□

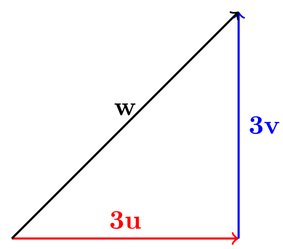
2. Three vectors, $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are given below. Show, geometrically, how the vector $\mathbf{w}$ can be written as a linear combination of $\mathbf{u}$ and $\mathbf{v}$; that is, show that $\mathbf{w} = C_1\mathbf{u} + C_2\mathbf{v}$ for some $C_1, C_2 \in \mathbb{R}$. Do this by finding the required values for C_1 and C_2 (use a ruler if you have one, or give your best estimate).



Solution.



or



So we see that

$$\mathbf{w} = 3\mathbf{u} + 3\mathbf{v}.$$

□

3. Create one vector that runs parallel to $\begin{pmatrix} 2 \\ -1 \\ 1 \\ 2 \end{pmatrix}$ and one vector that runs perpendicular to it. Find the magnitude of each.

Solution.
$\begin{pmatrix} -2 \\ 1 \\ -1 \\ -2 \end{pmatrix}$ is an example of a vector that is parallel to the given vector and has the same magnitude:

$$\sqrt{2^2 + 1^2 + 1^2 + 2^2} = \sqrt{10}.$$

$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ is perpendicular to the given vector because $\begin{pmatrix} 2 \\ -1 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = 0$. The magnitude of this vector is

$$\sqrt{0^2 + 1^2 + 1^2 + 0^2} = \sqrt{2}.$$

□

4. Create a line that passes through point $(0, -2, 2)$ and runs perpendicular to the vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Write the equation of this line in vector, parametric, normal, and general forms.

Solution.
Given a normal vector, the normal form for the line is:

$$\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p} \rightarrow \quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix}$$

General form is obtained by expanding the dot products:

$$x_1 + x_2 + x_3 = 1(0) + 1(-2) + 1(2) = 0$$

$\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ can be used as the direction vector for this line as it runs perpendicular to $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. That makes the vector form of the line:

$$\begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

And lastly parametric form:

$$\begin{aligned} x_1 &= -2t \\ x_2 &= -2 + t \\ x_3 &= 2 + t \end{aligned}$$

□

5. Verify the Cauchy-Schwarz inequality and the triangle inequality for the vectors

$$\mathbf{u} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}.$$

Solution. We have

$$\mathbf{u} \cdot \mathbf{v} = 4$$

$$||\mathbf{u}|| = \sqrt{6} \approx 2.4495$$

$$||\mathbf{v}|| = \sqrt{54} \approx 7.3485$$

$$||\mathbf{u} + \mathbf{v}|| = \sqrt{68} \approx 8.2462$$

Now

$$|\mathbf{u} \cdot \mathbf{v}| = 4 \leq 16 = \sqrt{6}\sqrt{54} = ||\mathbf{u}|| \; ||\mathbf{v}||$$

which demonstrates the Cauchy-Schwarz inequality, and

$$||\mathbf{u} + \mathbf{v}|| \approx 8.2462 < 2.449 + 7.348 < ||\mathbf{u}|| + ||\mathbf{v}||$$

which demonstrates the triangle inequality. □

Additional practice problem

6. What is the distance between the vectors $\begin{pmatrix} 1 \\ 1 \\ 4 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 5 \\ -1 \\ 1 \end{pmatrix}$?

Solution. The distance is

$$\left\| \begin{pmatrix} 1 \\ 1 \\ 4 \\ -2 \end{pmatrix} - \begin{pmatrix} -3 \\ 5 \\ -1 \\ 1 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 4 \\ -4 \\ 5 \\ -3 \end{pmatrix} \right\| = \sqrt{4^2 + (-4)^2 + 5^2 + (-3)^2} = \sqrt{66}.$$

$\square$