1. Are subsets S_1 and S_2 subspaces of $\mathbb{R}^3$ and $\mathbb{R}^2$ (respectively) under the usual addition and scalar multiplication? If so, prove these two qualities hold for any vector in the subspace. If they aren't, provide a counterexample.

$$S_1 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \, x = 2y - z \right\} \qquad S_2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| \, |x| = |y| \right\}$$

2. Execute $\begin{pmatrix} 0 & 4 & -3 \\ 2 & 3 & 3 \end{pmatrix} + 2 \begin{pmatrix} 1 & 1 & 1 \\ -5 & 3 & 2 \end{pmatrix}$ within some vector subset $V_1 = \mathcal{M}_{2\times 3}(\mathbb{R})$ when addition and scalar multiplication are defined as follows:

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} + \begin{pmatrix} u & v & w \\ x & y & z \end{pmatrix} = \begin{pmatrix} 0 & b+v & 0 \\ d & 0 & z \end{pmatrix} \qquad r \cdot \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = \begin{pmatrix} ra & rb & 1 \\ rd & re & 1 \end{pmatrix}$$

3. Using the vector space axioms written below, provide at least two reasons of why subset V_1 from the previous question is not a valid vector space.

4. Can vector $\mathbf{u}$ be written as a linear combination of the other two vectors given below? Would a set of all three of the vectors be linearly independent or dependent?

$$\mathbf{u} = \begin{pmatrix} -3 \\ 3 \\ 5 \end{pmatrix} \; ; \qquad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

5. With the vectors below, determine if $\mathbf{y} \in \text{span}(S_3)$ and whether S_3 spans $\mathbb{R}^3$.

$$\mathbf{y} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad S_3 = \left\{ \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix} \right\}$$

Vector space axioms

VS1: The set V is closed under vector addition, that is, $\mathbf{u} + \mathbf{v} \in V$ for any $\mathbf{u}, \mathbf{v} \in V$

VS2: Vector addition is commutative, $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

VS3: Vector addition is associative, $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

VS4: There is a zero vector $\mathbf{0} \in V$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in V$.

VS5: Each $\mathbf{v} \in V$ has an additive inverse $\mathbf{w} \in V$, so that $\mathbf{w} + \mathbf{v} = \mathbf{0}$

VS6: The set V is closed under scalar multiplication, that is, $r \cdot \mathbf{v} \in V$

VS7: Addition of scalars distributes over scalar multiplication, $(r + s) \cdot \mathbf{v} = r \cdot \mathbf{v} + s \cdot \mathbf{v}$

VS8: Scalar multiplication distributes over vector addition, $r \cdot (\mathbf{v} + \mathbf{w}) = r \cdot \mathbf{v} + r \cdot \mathbf{w}$

VS9: Ordinary multiplication of scalars associates with scalar multiplication, $(rs) \cdot \mathbf{v} = r \cdot (s \cdot \mathbf{v})$

VS10: Multiplication by the scalar 1 is the identity operation, $1 \cdot \mathbf{v} = \mathbf{v}$