1. Below a transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ is defined. Show that T is a linear map. Then create its standard matrix.

$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a+c \\ 2b \end{pmatrix}$$

Solution. To be a linear map, T must be closed under addition and scalar multiplication.

$$T\left(\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} d \\ e \\ f \end{pmatrix}\right) =$$

$\therefore$ the map is closed under addition.

$$r \cdot T\begin{pmatrix} a \\ b \\ c \end{pmatrix} =$$

$\therefore$ the map is closed under scalar multiplication.
We find the standard matrix for T by applying it to the standard basis vectors

$\square$

2. Find the standard matrix for the linear map $T : \mathbb{R}^4 \to \mathbb{R}^3$ where

$$T\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a+2b-c+d \\ a+c+d \\ 2a-4b+6c+2d \end{pmatrix}.$$

Then find a basis for the kernel and image of the linear map T.

Solution. The standard matrix is given by

$$\left(\begin{array}{c|c|c|c} T\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} & T\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} & T\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} & T\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \end{array}\right) = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 1 & 0 & 1 & 1 \\ 2 & -4 & 6 & 2 \end{pmatrix}.$$

The kernel of T is the null space of this matrix, and the image of T is its column space. Both are found by row reduction.

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 2 & -4 & 6 & 2 & 0 \end{array}\right) \xrightarrow[R_3-2R_1]{R_2-R_1} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 0 \\ 0 & -2 & 2 & 0 & 0 \\ 0 & -8 & 8 & 0 & 0 \end{array}\right) \xrightarrow{-\frac{1}{2}R_2} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 0 \\ 0 & -2 & 2 & 0 & 0 \\ 0 & -8 & 8 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[R_1-2R_2]{R_3+8R_2} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Leading 1's appear in columns 1 and 2, therefore a basis for the column space (and hence the image of T) is

$$\left\{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}\right\}.$$

The null space (and hence the kernel of T) is

$$\left\{ s\begin{pmatrix}-1\\1\\1\\0\end{pmatrix} + t\begin{pmatrix}-1\\0\\0\\1\end{pmatrix} \middle| s,t\in\mathbb{R} \right\}$$

which has basis

$$\left\{ \begin{pmatrix}-1\\1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\0\\1\end{pmatrix} \right\}.$$

□

3. Are T_A and T_B one-to-one and onto?

$$T_A\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}-a & 0\\a & 0\\a & 0\end{pmatrix} \qquad T_B\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}b\\-a\\a+c\end{pmatrix}$$

Solution.

The standard matrix $A = \begin{pmatrix}-1 & 0\\1 & 0\\1 & 0\end{pmatrix}$ can be reduced to $\begin{pmatrix}1 & 0\\0 & 0\\0 & 0\end{pmatrix}$, implying that the second variable is a free variable. This makes the kernel $s\left\{\begin{pmatrix}0\\1\end{pmatrix}\middle| s\in\mathbb{R}\right\}$ with basis $\left\{\begin{pmatrix}0\\1\end{pmatrix}\right\}$ which is nontrivial (it contains more than just the zero vector). T_A is not one-to-one by default.
A transformation is onto only when everything in the codomain can be created from its image. T_A has the image $\left\{ s\begin{pmatrix}1 & 0\\0 & 0\\0 & 0\end{pmatrix} \middle| s\in\mathbb{R}\right\}$ which does not span all 3×2 matrices, so the transformation is not onto.

The standard matrix for $B = \begin{pmatrix}0 & 1 & 0\\-1 & 0 & 0\\1 & 0 & 1\end{pmatrix}$ representing T_B can be reduced to $I^3 = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$ whose kernel only contains the zero vector. Therefor T_B is one-to-one. The transformation is also onto as the basis for the image spans the whole codomain, $\mathbb{R}^3$.

□

4. Using the characteristic equation $\det(A-\lambda I)=0$, find the eigenvalues for matrices C and D.

$$C = \begin{pmatrix}5 & 3\\0 & -3\end{pmatrix} \qquad D = \begin{pmatrix}2 & 1 & 0\\4 & 3 & 4\\-1 & 0 & 2\end{pmatrix}$$

Solution. To find eigenvalues for a matrix M we solve the characteristic equation $\det(M-\lambda I)=0$.

$$0 = \begin{vmatrix}5-\lambda & 3\\0 & -3-\lambda\end{vmatrix} = (5-\lambda)(-3-\lambda)-0 \quad\Rightarrow\quad \lambda=5 \text{ or } \lambda=-3.$$

$$0 = \begin{vmatrix}2-\lambda & 1 & 0\\4 & 3-\lambda & 4\\-1 & 0 & 2-\lambda\end{vmatrix} = (2-\lambda)\begin{vmatrix}3-\lambda & 4\\0 & 2-\lambda\end{vmatrix} - 1\begin{vmatrix}4 & 4\\-1 & 2-\lambda\end{vmatrix} = (2-\lambda)((3-\lambda)(2-\lambda)-0)-(4(2-\lambda)-1(4))$$

$$= (3-\lambda)(2-\lambda)^2 - 4(2-\lambda+1) = (3-\lambda)((2-\lambda)^2-4) \quad \Rightarrow \quad \lambda = 0,3,4$$

□

5. Find the eigenspaces for matrix D (defined above) such that $(D-\lambda I)\mathbf{x} = \mathbf{0}$.

Solution.

Eigenvalues $\lambda = 0,3,4$ were obtained in Question 4. For each of these, we must find an eigenspace.
For $\lambda = 0$:

$$\left(\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 4 & 3 & 4 & 0 \\ -1 & 0 & 2- & 0 \end{array}\right) - \left(\begin{array}{ccc|c} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 2-\lambda & 1 & 0 & 0 \\ 4 & 3-\lambda & 4 & 0 \\ -1 & 0 & 2-\lambda & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 2-0 & 1 & 0 & 0 \\ 4 & 3-0 & 4 & 0 \\ -1 & 0 & 2-0 & 0 \end{array}\right) =$$

$$\left(\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 4 & 3 & 4 & 0 \\ -1 & 0 & 2 & 0 \end{array}\right) \xrightarrow[R_2+4R_3]{R_1+2R_3} \left(\begin{array}{ccc|c} 0 & 1 & 4 & 0 \\ 0 & 3 & 12 & 0 \\ -1 & 0 & 2 & 0 \end{array}\right) \xrightarrow[-R_3]{R_2-3R_1} \left(\begin{array}{ccc|c} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \quad \text{The eigenspace for } \lambda = 0 \text{ is} \quad E_0 = \left\{ t\begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} \middle|\, t \in \mathbb{R} \right\} = \text{span}\left\{ \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} \right\}.$$

For $\lambda = 3$:

$$\left(\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 4 & 0 & 4 & 0 \\ -1 & 0 & -1 & 0 \end{array}\right) \xrightarrow[\frac{1}{4}R_2]{-R_1} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 \end{array}\right) \xrightarrow[R_3+R_1]{R_2-R_1} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right) \xrightarrow[R_3+R_2]{R_1+R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \quad E_3 = \left\{ t\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \middle|\, t \in \mathbb{R} \right\} = \text{span}\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

For $\lambda = 4$:

$$\left(\begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ 4 & -1 & 4 & 0 \\ -1 & 0 & -2 & 0 \end{array}\right) \xrightarrow[R_2+4R_3]{R_1-2R_3} \left(\begin{array}{ccc|c} 0 & 1 & 4 & 0 \\ 0 & -1 & -4 & 0 \\ -1 & 0 & -2 & 0 \end{array}\right) \xrightarrow[-R_3]{R_1+R_2} \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & -1 & -4 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right) \xrightarrow[-R_2]{R_1\leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \quad E_4 = \left\{ t\begin{pmatrix} -2 \\ -4 \\ 1 \end{pmatrix} \middle|\, t \in \mathbb{R} \right\} = \text{span}\left\{ \begin{pmatrix} -2 \\ -4 \\ 1 \end{pmatrix} \right\}$$

□