MATH 1350,
Exercise Set 4 - Solutions

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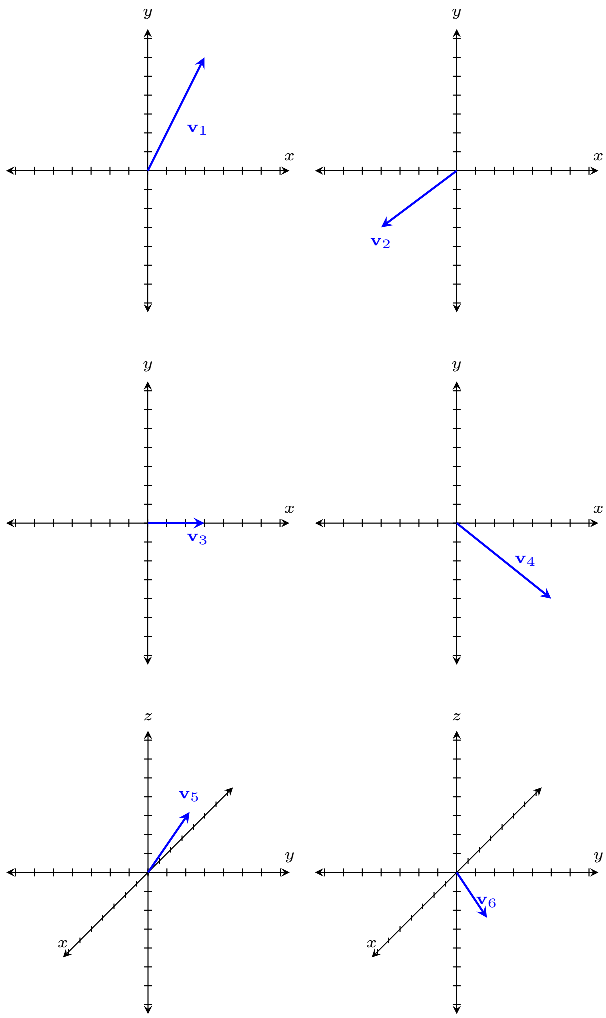
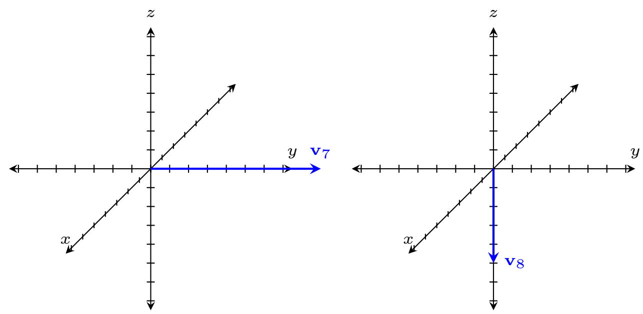
1. Sketch the following vectors on a set of axes with the initial points located at the origin (i.e. in standard position).

$$\mathbf{v}_1 = \begin{pmatrix} 3 \\ 6 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -4 \\ -3 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 5 \\ -4 \end{pmatrix},$$

$$\mathbf{v}_5 = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}, \quad \mathbf{v}_6 = \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix}, \quad \mathbf{v}_7 = \begin{pmatrix} 0 \\ 9 \\ 0 \end{pmatrix}, \quad \mathbf{v}_8 = \begin{pmatrix} 0 \\ 0 \\ -5 \end{pmatrix}.$$

solution:





$\square$

2. Write the vector $\overrightarrow{PQ}$, in column matrix form, for the following points P and Q.
   (a) $P(4, 8)$, $Q(3, 7)$
   (b) $P(-5, 0)$, $Q(-3, 1)$
   (c) $P(3, -7, 2)$, $Q(-2, 5, -4)$
   (d) $P(a, b, c)$, $Q(0, 0, 0)$
   (e) $P(0, 0, 0)$, $Q(a, b, c)$

   solution:

   (a) $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$

   (b) $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

   (c) $\begin{pmatrix} -5 \\ 12 \\ -6 \end{pmatrix}$

   (d) $\begin{pmatrix} -a \\ -b \\ -c \end{pmatrix}$

   (e) $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$\square$

3. Find a point Q that creates a nonzero vector $\overrightarrow{PQ}$, with initial point $P(-1, 3, -5)$, which points in the same direction as the vector $\begin{pmatrix} 6 \\ 7 \\ -3 \end{pmatrix}$.

Let $Q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$. Then

$$\begin{pmatrix} 6 \\ 7 \\ -3 \end{pmatrix} = \overrightarrow{PQ} = \begin{pmatrix} q_1 - (-1) \\ q_2 - 3 \\ q_3 - (-5) \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} - \begin{pmatrix} -1 \\ 3 \\ -5 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \\ -3 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \\ -5 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \\ -8 \end{pmatrix}$$

$\square$

4. Let

$$\mathbf{u} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 4 \\ 0 \\ -8 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} 6 \\ -1 \\ -4 \end{pmatrix}.$$

Write the vector (i.e. the column matrix) representing the following:

(a) $\mathbf{u} + \mathbf{v}$

(b) $\mathbf{v} - \mathbf{w}$

(c) $3\mathbf{u} + 4\mathbf{w}$

(d) $5(\mathbf{v} - 4\mathbf{u})$

(e) $(2\mathbf{u} - 7\mathbf{w}) - (8\mathbf{v} + \mathbf{u})$

(f) The vector $\mathbf{x}$ such that $2\mathbf{u} - \mathbf{v} + \mathbf{x} = 7\mathbf{x} + \mathbf{w}$.

solution:

(a) $\begin{pmatrix} 1 \\ 1 \\ -6 \end{pmatrix}$

(b) $\begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix}$

(c) $\begin{pmatrix} 15 \\ -1 \\ -10 \end{pmatrix}$

(d) $\begin{pmatrix} 80 \\ -20 \\ -80 \end{pmatrix}$

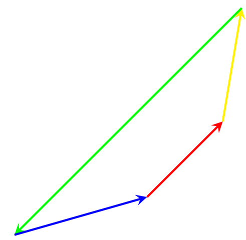
(e) $\begin{pmatrix} -77 \\ 9 \\ 94 \end{pmatrix}$

(f) $\mathbf{x} = \frac{1}{6}(2\mathbf{u} - \mathbf{v} - \mathbf{w}) = \begin{pmatrix} -8/3 \\ 1/2 \\ 8/3 \end{pmatrix}$

$\square$

5. Draw a picture that shows four nonzero vectors whose sum is the vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

solution:



$\square$

6. Find the lengths of the following vectors

$$\mathbf{v}_1 = \begin{pmatrix} 3 \\ 6 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -4 \\ -3 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 5 \\ -4 \end{pmatrix},$$

$$\mathbf{v}_5 = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}, \quad \mathbf{v}_6 = \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix}, \quad \mathbf{v}_7 = \begin{pmatrix} 0 \\ 9 \\ 0 \end{pmatrix}, \quad \mathbf{v}_8 = \begin{pmatrix} 0 \\ 0 \\ -5 \end{pmatrix}.$$

solution:

$$||\mathbf{v}_1|| = 3\sqrt{5}, \quad ||\mathbf{v}_2|| = 5, \quad ||\mathbf{v}_3|| = 3, \quad ||\mathbf{v}_4|| = \sqrt{41},$$
$$||\mathbf{v}_5|| = 5\sqrt{2}, \quad ||\mathbf{v}_6|| = 4\sqrt{2}, \quad ||\mathbf{v}_7|| = 9, \quad ||\mathbf{v}_8|| = 5.$$

$\square$

7. Show that if $\mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is a nonzero vector, then $\frac{1}{||\mathbf{v}||}\mathbf{v}$ is a unit vector.

solution:

Let's first prove the property that $||r\mathbf{v}|| = |r| \; ||\mathbf{v}||$ for $r \in \mathbb{R}$.

$$||r\mathbf{v}|| = \left|\left| \begin{pmatrix} ra \\ rb \\ rc \end{pmatrix} \right|\right| = \sqrt{(ra)^2 + (rb)^2 + (rc)^2} = \sqrt{r^2(a^2 + b^2 + c^2)} = \sqrt{r^2}\sqrt{a^2 + b^2 + c^2} = |r| \; ||\mathbf{v}||$$

Using this property we have

$$\left|\left| \frac{1}{||\mathbf{v}||}\mathbf{v} \right|\right| = \left| \frac{1}{||\mathbf{v}||} \right| \; ||\mathbf{v}|| = \frac{1}{||\mathbf{v}||}||\mathbf{v}|| = 1$$

$\square$

8. Show that the vectors
$$\mathbf{u} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 5 \\ -2 \end{pmatrix},$$
satisfy the Cauchy-Schwarz inequality, $|\mathbf{u} \cdot \mathbf{v}| \leq ||\mathbf{u}|| \, ||\mathbf{v}||$.

solution:

For these vectors we have
$$|\mathbf{u} \cdot \mathbf{v}| = |-9| = 9$$
whereas
$$||\mathbf{u}|| \, ||\mathbf{v}|| = \sqrt{5}\sqrt{29} = \sqrt{145} > \sqrt{81} = 9.$$

$\square$

9. Show that the vectors
$$\mathbf{u} = \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix},$$
satisfy the triangle inequality, $||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$.

solution:

For these vectors we have
$$||\mathbf{u} + \mathbf{v}|| = \left|\left| \begin{pmatrix} 0 \\ 5 \\ 12 \end{pmatrix} \right|\right| = 13$$
whereas
$$||\mathbf{u}|| + ||\mathbf{v}|| = \sqrt{30} + \sqrt{59} \approx 13.16.$$

$\square$

10. Find the angle between vectors $\mathbf{u}$ and $\mathbf{v}$ for the following.

(a) $\mathbf{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 5 \\ -7 \end{pmatrix}$

(b) $\mathbf{u} = \begin{pmatrix} -6 \\ -2 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$

(c) $\mathbf{u} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

solution:

Apply the formula
$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}|| \, ||\mathbf{v}||}\right)$$

(a)

$$\theta = \cos^{-1}\left(\frac{-11}{\sqrt{13}\sqrt{74}}\right) \approx 110.77^\circ$$

(b)

$$\theta = \cos^{-1}\left(\frac{-24}{4\sqrt{40}}\right) \approx 161.57^\circ$$

(c)

$$\theta = \cos^{-1}\left(\frac{3}{\sqrt{6}\sqrt{6}}\right) = 60^\circ$$

$\square$