MATH 1350,
Exercise Set 6 - Solutions

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1. (a) What is a vector?
   (b) What does a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ mean?
   (c) What is the span of a set of vectors?
   (d) Use set-builder notation to write the set $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$.
   (e) How do we determine whether or not a vector $\mathbf{u}$ belongs to $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$?
   (f) By definition, what does it mean for the set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ to be linearly independent? What does it mean if they are linearly dependent?
   (g) How do we determine whether a given set of vectors is linearly independent or linearly dependent?
   (h) What is the determinant test for linear independence (when does it apply)?
   (i) How can we cast out "redundant" vectors to reduce a linearly dependent set to a linearly independent one with the same span?
   (j) What is a basis for a vector space?
   (k) What is the dimension of a vector space?

   solution:

   (a) A vector is an element of a vector space.
   (b) A linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is a sum

   $$C_1\mathbf{v}_1 + C_2\mathbf{v}_2 + \cdots + C_n\mathbf{v}_n$$

   where $C_1, C_2, \ldots, C_n$ belong to the set of scalars (e.g. $\mathbb{R}$).
   (c) The span of a set of vectors is the set of all linear combinations of those vectors.
   (d) In the case where $\mathbb{R}$ is our set of scalars, we have

   $$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = \{C_1\mathbf{v}_1 + C_2\mathbf{v}_2 + \cdots + C_n\mathbf{v}_n | C_1, C_2, \ldots, C_n \in \mathbb{R}\}.$$

   (e) To determine whether or not a vector $\mathbf{u} \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ we attempt to solve

   $$\mathbf{u} = C_1\mathbf{v}_1 + C_2\mathbf{v}_2 + \cdots + C_n\mathbf{v}_n$$

   for $C_1, C_2, \ldots, C_n$.

   In the case where $\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ belong to $\mathbb{R}^m$, we can put the vectors into an $m \times (n+1)$ augmented matrix:

   $$(\mathbf{v}_1 | \mathbf{v}_2 | \ldots | \mathbf{v}_n | \mathbf{u})$$

   and row reduce to solve.
   (f) By definition, the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent if

   $$C_1\mathbf{v}_1 + C_2\mathbf{v}_2 + \cdots + C_n\mathbf{v}_n = \mathbf{0}$$

   implies $C_1 = C_2 = \cdots = C_n = 0$.

The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly dependent, if they are not linearly independent; i.e. $C_1\mathbf{v}_1 + C_2\mathbf{v}_2 + \cdots + C_n\mathbf{v}_n = \mathbf{0}$ for some $C_1, C_2, \ldots, C_n$, which are not all zero.

(g) To determine whether a given set of vectors is linearly independent we solve the equation

$$C_1\mathbf{v}_1 + C_2\mathbf{v}_2 + \cdots + C_n\mathbf{v}_n = \mathbf{0}$$

for $C_1, C_2, \ldots, C_n$. If the only solution is $C_1 = C_2 = \cdots = C_n = 0$ (i.e. the *trivial solution*) then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent. If nontrivial solutions exist then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly dependent.

In the case where $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ belong to $\mathbb{R}^m$, we can put the vectors into an $m \times (n+1)$ augmented matrix:

$$(\mathbf{v}_1|\mathbf{v}_2|\ldots|\mathbf{v}_n|\mathbf{0})$$

and row reduce to solve. If there are no free variables, then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent.

(h) The determinant test applies when $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ belong to $\mathbb{R}^n$, and hence form an $n \times n$ (square) matrix

$$M = (\mathbf{v}_1|\mathbf{v}_2|\ldots|\mathbf{v}_n),$$

in which case $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent if and only if $\det M \neq 0$.

(i) To reduce a linearly dependent set $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^m$ to a linearly independent one with the same span we

1. Form the $m \times n$ matrix
$$M = (\mathbf{v}_1|\mathbf{v}_2|\ldots|\mathbf{v}_n),$$
2. Row reduce M to REF, and identity the leading entries; i.e. identify the columns where leading 1's occur in the RREF.
3. Eliminate the vectors $\mathbf{v}_i$ from the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ for which column i does not have a leading 1 in the RREF of M. The remaining vectors are linearly independent and have the same span.

(j) A basis for a vector space V is a linearly independent spanning set for V.

(k) The dimension of a vector space V is the number of vectors in any basis for V.

□

2. Consider the following subset of $\mathbb{R}^3$.

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

(a) Write 3 different linear combinations from this set.

(b) Is this a spanning set for $\mathbb{R}^3$?

(c) Is this a linearly independent set?

(d) Is this a basis for $\mathbb{R}^3$?

solution:

(a)

$$2 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + (-1) \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + 5 \cdot \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -6 \\ 7 \end{pmatrix}$$

$$0 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 0 \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + 10 \cdot \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -10 \\ -10 \\ 10 \end{pmatrix}$$

$$2 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$$

(b) Let $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$. Then

$$x \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + y \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z \cdot \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

gives the system of equations

$$\begin{aligned} x - y - z &= a \\ y - z &= b \\ x + z &= c \end{aligned} \quad \rightarrow \quad \left(\begin{array}{ccc|c} 1 & -1 & -1 & a \\ 0 & 1 & -1 & b \\ 1 & 0 & 1 & c \end{array}\right)$$

We solve by row reduction to see if this system is consistent.

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & a \\ 0 & 1 & -1 & b \\ 1 & 0 & 1 & c \end{array}\right) \xrightarrow{R_3 - R_1} \left(\begin{array}{ccc|c} 1 & -1 & -1 & a \\ 0 & 1 & -1 & b \\ 0 & 1 & 2 & c - a \end{array}\right) \xrightarrow{R_3 - R_2} \left(\begin{array}{ccc|c} 1 & -1 & -1 & a \\ 0 & 1 & -1 & b \\ 0 & 0 & 3 & c - a - b \end{array}\right)$$

We can stop here because we see that there is a leading variable in each column, and hence there will be a unique solution to this system for any $a, b, c \in \mathbb{R}^3$. It follows that this set of vectors is s spanning set for $\mathbb{R}^3$.

(c) Suppose

$$C_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + C_2 \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + C_3 \cdot \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{1}$$

This gives the system of equations

$$\begin{aligned} C_1 - C_2 - C_3 &= 0 \\ C_2 - C_3 &= 0 \\ C_1 + C_3 &= 0 \end{aligned} \quad \rightarrow \quad \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right)$$

(same system as in part (b) but with $a = b = c = 0$). Using the row reduction started above we have

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow{R_3 - R_1} \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right) \xrightarrow{R_3 - R_2} \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{1}{3}R_3} \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow[R_2+R_3]{R_1+R_3} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{R_1+R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

This shows that the only solution to 1 is $C_1 = C_2 = C_3 = 0$, and hence this set of vectors is linearly independent.

Another way we can show this is by the determinant test, since these 3 vectors, when put together, form a square matrix. We have

$$\begin{vmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} - (-1) \begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix} + (-1) \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 1 + 1 + 1 = 3$$

(can also use the REF in the row reduction above) and since this determinant is not zero, this also shows that the 3 vectors are linearly independent.

(d) Yes, since this set of vectors is linearly independent and spans $\mathbb{R}^3$, it is a basis for $\mathbb{R}^3$.

$\square$

3. Consider the following subset of $\mathcal{M}_{2\times 2}(\mathbb{R})$.

$$\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \right\}.$$

(a) Write 3 different linear combinations from this set.

(b) Is this a spanning set for $\mathcal{M}_{2\times 2}(\mathbb{R})$?

(c) Is this a linearly independent set?

(d) Is this a basis for $\mathcal{M}_{2\times 2}(\mathbb{R})$?

solution:

(a)

$$1 \cdot \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} + (-1) \cdot \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ -2 & -3 \end{pmatrix}.$$

$$(-2) \cdot \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$0 \cdot \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} + 0 \cdot \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

(b) Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{R})$. Then

$$x \cdot \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + y \cdot \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} + z \cdot \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

gives the following system of equations (by equating corresponding matrix entries on both sides)

$$\begin{aligned} x+2y+z &= a \\ x+y+2z &= b \\ -y &= c \\ -y+z &= d \end{aligned} \quad \to \quad \left(\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 1 & 1 & 2 & b \\ 0 & -1 & 0 & c \\ 0 & -1 & 1 & d \end{array}\right)$$

Solve by row reduction:

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 1 & 1 & 2 & b \\ 0 & -1 & 0 & c \\ 0 & -1 & 1 & d \end{array}\right) \xrightarrow{R_2-R_1} \left(\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 0 & -1 & 1 & b-a \\ 0 & -1 & 0 & c \\ 0 & -1 & 1 & d \end{array}\right) \xrightarrow[R_4-R_2]{R_3-R_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & a \\ 0 & -1 & 1 & b-a \\ 0 & 0 & -1 & c+a-b \\ 0 & 0 & 0 & d+a-b \end{array}\right)$$

We can stop here because the bottom row reveals an inconsistent system whenever $d+a-b \neq 0$. For example if $d=1$ and $a=b=c=0$ (c can be any number) this shows that there are no values for x, y and z such that

$$x \cdot \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + y \cdot \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} + z \cdot \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore this set is not a spanning set for $\mathcal{M}_{2\times 2}(\mathbb{R})$.

(Later we will be able to conclude that this is true simply by the fact that any spanning set for $\mathcal{M}_{2\times 2}(\mathbb{R})$ must have at least 4 vectors.)

(c) Suppose

$$C_1 \cdot \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + C_2 \cdot \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} + C_3 \cdot \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{2}$$

Again this gives the system

$$\begin{aligned} C_1+2C_2+C_3 &= 0 \\ C_1+C_2+2C_3 &= 0 \\ -C_2 &= 0 \\ -C_2+C_3 &= 0 \end{aligned} \quad \to \quad \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right)$$

Solving (making use of what we started above)

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right) \xrightarrow{R_2-R_1} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right) \xrightarrow[R_4-R_2]{R_3-R_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We can stop here, since there is a leading variable in each column for this homogeneous system, and therefore we see that the only solution to 2 is $C_1 = C_2 = C_3 = 0$. Thus this set is linearly independent.

(Note that in this set up we can't make use of the determinant test.)

(d) Since this set of matrices does not span $\mathcal{M}_{2\times 2}(\mathbb{R})$, it is not a basis for $\mathcal{M}_{2\times 2}(\mathbb{R})$.

$\square$

4. Consider the following subset of $\mathcal{P}_2(\mathbb{R})$.

$$\left\{x^2+3x+1, 5x+2, 2x^2-x-3\right\}.$$

(a) Write 3 different linear combinations from this set.
(b) Is this a spanning set for $\mathcal{P}_2(\mathbb{R})$?
(c) Is this a linearly independent set?
(d) Is this a basis for $\mathcal{P}_2(\mathbb{R})$?

solution:

(a)

$$1\cdot(x^2+3x+1)+2\cdot(5x+2)+(-4)\cdot(2x^2-x-3) = -7x^2+17x+17$$
$$(-2)\cdot(x^2+3x+1)+0\cdot(5x+2)+1\cdot(2x^2-x-3) = -7x-5$$
$$(-2)\cdot(x^2+3x+1)+\frac{7}{5}\cdot(5x+2)+1\cdot(2x^2-x-3) = -\frac{11}{5}$$

(b) Let $a^2+bx+c \in \mathcal{P}_2(\mathbb{R})$. We solve

$$r(x^2+3x+1)+s(5x+2)+t(2x^2-x-3) = a^2+bx+c$$

By equating coefficients on like powers of x on both sides, this give the system of equations

$$\begin{aligned} r+2t &= a \\ 3r+5s-t &= b \\ r+2s-3t &= c \end{aligned} \quad \to \quad \left(\begin{array}{ccc|c} 1 & 0 & 2 & a \\ 3 & 5 & -1 & b \\ 1 & 2 & -3 & c \end{array}\right)$$

Row reducing we get

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & a \\ 3 & 5 & -1 & b \\ 1 & 2 & -3 & c \end{array}\right) \xrightarrow[R_3-R_1]{R_2-3R_1} \left(\begin{array}{ccc|c} 1 & 0 & 2 & a \\ 0 & 5 & -7 & b-3a \\ 0 & 2 & -5 & c-a \end{array}\right) \xrightarrow{R_3-\frac{2}{5}R_2} \left(\begin{array}{ccc|c} 1 & 0 & 2 & a \\ 0 & 5 & -7 & b-3a \\ 0 & 0 & -11/5 & c-(2/5)b-(1/5)a \end{array}\right)$$

We see that the system is consistent for any values for a, b and c, and therefore we conclude that this is a spanning set for $\mathcal{P}_2(\mathbb{R})$.

(c) Suppose

$$C_1(x^2+3x+1)+C_2(5x+2)+C_3(2x^2-x-3) = 0$$

This yields the system

$$\begin{aligned} C_1+2C_3 &= 0 \\ 3C_1+5C_2-C_3 &= 0 \\ C_1+2C_2-3C_3 &= 0 \end{aligned} \quad \to \quad \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 3 & 5 & -1 & 0 \\ 1 & 2 & -3 & 0 \end{array}\right)$$

Using the row reduction above we have

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 3 & 5 & -1 & 0 \\ 1 & 2 & -3 & 0 \end{array}\right) \xrightarrow[R_3-R_1]{R_2-3R_1} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 5 & -7 & 0 \\ 0 & 2 & -5 & 0 \end{array}\right) \xrightarrow{R_3-\frac{2}{5}R_2} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 5 & -7 & 0 \\ 0 & 0 & -11/5 & 0 \end{array}\right)$$

from which we can conclude that $C_1 = C_2 = C_3 = 0$, and therefore that the vectors in this set are linearly independent.

(d) Yes, this set is a basis for $\mathcal{P}_2(\mathbb{R})$ as it is both linearly independent and a spanning set.

$\square$

5. In each of the following, determine if the given vector $\mathbf{v}$ belongs to the span of the given set S. If it does, express v as a linear combination of the vectors in S.

(a)
$$\mathbf{v} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \quad S = \left\{ \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \right\}$$

(b)
$$\mathbf{v} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S = \left\{ \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 4 & -2 \\ -1 & 4 \end{pmatrix} \right\}$$

(c)
$$\mathbf{v} = x^2 + x + 1, \quad S = \left\{ x^2 + 2x - 3, x + 2, 1 \right\}$$

solution:

In each case set up a system $x\mathbf{v}_1 + y\mathbf{v}_2 + z\mathbf{v}_3 = \mathbf{v}$ and solve.

(a)
$$\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + 2 \cdot \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}.$$

(b)
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix} + (-2) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} + 1 \cdot \begin{pmatrix} 4 & -2 \\ -1 & 4 \end{pmatrix}$$

(c)
$$x^2 + x + 1 = 1 \cdot (x^2 + 2x - 3) + (-1) \cdot (x + 2) + 6 \cdot (1).$$

$\square$

6. In each of the following, determine whether $\text{span}S_1 \subseteq \text{span}S_2$, $\text{span}S_2 \subseteq \text{span}S_1$, $\text{span}S_1 = \text{span}S_2$ or neither of these.

(a)
$$S_1 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix} \right\}, S_2 = \left\{ \begin{pmatrix} 2 \\ 1 \\ 4 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \\ 2 \end{pmatrix} \right\}$$

(b)
$$S_1 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 4 \\ 2 \end{pmatrix} \right\}, S_2 = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix} \right\}$$

(c)

$$S_1 = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\}, S_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix} \right\}$$

solution:

In each part the strategy is the same. If the vectors which span S_1 belong to S_2 then their $S_1 \subseteq S_2$. Vice versa if the vectors which span in S_2 belong to S_1 then $S_2 \subseteq S_1$. If both of these are true then $S_1 = S_2$.

(a) We solve

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix} = A \begin{pmatrix} 2 \\ 1 \\ 4 \\ -1 \end{pmatrix} + B \begin{pmatrix} -1 \\ 1 \\ 1 \\ 2 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix} = C \begin{pmatrix} 2 \\ 1 \\ 4 \\ -1 \end{pmatrix} + D \begin{pmatrix} -1 \\ 1 \\ 1 \\ 2 \end{pmatrix}$$

This gives us two systems of equations

$$\begin{aligned} 2A - B &= 1 \\ A + B &= 0 \\ 4A + Bt &= 1 \\ -A + 2B &= -1 \end{aligned} \qquad \rightarrow \qquad \left(\begin{array}{cc|c} 2 & -1 & 1 \\ 1 & 1 & 0 \\ 4 & 1 & 1 \\ -1 & 2 & 1 \end{array}\right)$$

and

$$\begin{aligned} 2C - D &= 0 \\ C + D &= 1 \\ 4C + Dt &= 2 \\ -C + 2D &= 1 \end{aligned} \qquad \rightarrow \qquad \left(\begin{array}{cc|c} 2 & -1 & 0 \\ 1 & 1 & 1 \\ 4 & 1 & 2 \\ -1 & 2 & 1 \end{array}\right)$$

Since the coefficients are the same in each system, we can solve these simultaneously with the augmented matrix

$$\left(\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 4 & 1 & 1 & 2 \\ -1 & 2 & 1 & 1 \end{array}\right)$$

$$\xrightarrow[R_4+R_2]{\substack{R_1-2R_2 \\ R_3-4R_2}} \left(\begin{array}{cc|cc} 0 & -3 & 1 & -2 \\ 1 & 1 & 0 & 1 \\ 0 & -3 & 1 & -2 \\ 0 & 3 & 1 & 2 \end{array}\right) \xrightarrow[R_1+R_4]{R_3+R_4} \left(\begin{array}{cc|cc} 0 & 0 & 2 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 1 & 2 \end{array}\right)$$

Rows 1 and 3 show that this system is inconsistent, and hence there is no solution for A and B to make the first equation work. Thus S_1 is not a subset of S_2.

Similarly, by swapping sides we can use the matrix

$$\left(\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 4 & 1 \\ 1 & 1 & -1 & 2 \end{array}\right)$$

to determine whether $\text{span}S_2 \subseteq \text{span}S_1$ (this solves for C and D above).

$$\left(\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 4 & 1 \\ 1 & 1 & -1 & 2 \end{array}\right) \xrightarrow[R_4-R_1]{R_3-R_1} \left(\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 1 & -3 & 3 \end{array}\right) \xrightarrow[R_4-R_2]{R_3-2R_2} \left(\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 2 \end{array}\right)$$

Again we get an inconsistent system, thus S_2 is not a subset of S_1.

(b) As we did in part (a), to determine if the vectors which span S_1 belong to S_2 we can solve

$$\left(\begin{array}{cc|ccc} 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & -1 & -2 \\ 0 & 2 & 2 & 1 & 4 \\ 1 & 0 & 1 & 1 & 2 \end{array}\right)$$

however, we immediately see that this system is inconsistent, so S_1 is not a subset of S_2. On the other hand to determine whether S_2 is a subset of S_1 we solve

$$\left(\begin{array}{ccc|cc} 1 & 1 & 3 & 0 & 1 \\ 0 & -1 & -2 & 0 & 0 \\ 2 & 1 & 4 & 0 & 2 \\ 1 & 1 & 2 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[R_4-R_1]{R_3-2R_1} \left(\begin{array}{ccc|cc} 1 & 1 & 3 & 0 & 1 \\ 0 & -1 & -2 & 0 & 0 \\ 0 & -1 & -2 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 \end{array}\right) \xrightarrow{R_3-R_2} \left(\begin{array}{ccc|cc} 1 & 1 & 3 & 0 & 1 \\ 0 & -1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 \end{array}\right)$$

Since this system is consistent, it follows that $S_2 \subset S_1$.

(c) To determine whether S_1 is a subset of S_2 we solve

$$\left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 2 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{array}\right)$$

$$\xrightarrow[R_4-R_1]{R_3-2R_1} \left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \end{array}\right) \xrightarrow{R_4+R_2} \left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We see that the system is consistent, which shows that $S_1 \subseteq S_2$. On the other hand since the system

$$\left(\begin{array}{cc|cc} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & -1 & 1 & 0 \end{array}\right)$$

is inconsistent we see that S_2 is not a subset of S_1.

□

7. For each of the following, find a linearly independent subset of S that has the same span as S.

(a)

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix} \right\}$$

(b)

$$S = \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

(c)

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \\ 3 \end{pmatrix} \right\}$$

(d)

$$S = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\}$$

(e)

$$S = \left\{ \begin{pmatrix} -1 & 1 \\ 4 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \right\}$$

(f)

$$S = \left\{ \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \right\}$$

(g)

$$S = \{x^2 + 3x, x + 1, -3\}$$

(h)

$$S = \{x^2 + 2x + 1, x^2 + 1, x\}$$

(i)

$$S = \{-x^2 + 3x + 2, x^2 - 1, 2x^2 + 3x + 1, x + 1\}$$

solution:

(a) Form the matrix whose columns are the vectors in S, and row reduce:

$$\begin{pmatrix}1 & 1 & 2 & 1\\0 & 1 & 1 & 2\\1 & -1 & 0 & 3\\0 & -1 & -1 & 1\end{pmatrix} \xrightarrow{R_2-R_1} \begin{pmatrix}1 & 1 & 2 & 1\\0 & 1 & 1 & 2\\0 & -2 & -2 & 2\\0 & -1 & -1 & 1\end{pmatrix} \xrightarrow[R_4+R_1]{R_3+2R_2} \begin{pmatrix}1 & 1 & 2 & 1\\0 & 1 & 1 & 2\\0 & 0 & 0 & 4\\0 & 0 & 0 & 3\end{pmatrix}$$

$$\xrightarrow{\frac{1}{4}R_3} \begin{pmatrix}1 & 1 & 2 & 1\\0 & 1 & 1 & 2\\0 & 0 & 0 & 1\\0 & 0 & 0 & 3\end{pmatrix} \xrightarrow{R_4-3R_3} \begin{pmatrix}1 & 1 & 2 & 1\\0 & 1 & 1 & 2\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\end{pmatrix}$$

We stop here since we can see that leading 1's appear in columns 1,2, and 4. It follows that columns 1, 2 and 4 of the initial matrix, i.e.

$$\left\{\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}, \begin{pmatrix}1\\2\\3\\1\end{pmatrix}\right\}$$

form a linearly independent subset of

(b) Answer:

$$\left\{\begin{pmatrix}1\\-1\\1\\2\end{pmatrix}, \begin{pmatrix}1\\1\\0\\1\end{pmatrix}\right\}$$

(c) Answer:

$$S = \left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\end{pmatrix}\right\}$$

(d) The strategy used above can be applied to this case. Solving

$$x\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix} + y\begin{pmatrix}1 & 2\\1 & -1\end{pmatrix} + z\begin{pmatrix}0 & 0\\1 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$$

yields the system

$$\left(\begin{array}{ccc|c}1 & 1 & 0 & 0\\1 & 2 & 0 & 0\\0 & 1 & 1 & 0\\1 & -1 & 1 & 0\end{array}\right).$$

As above, we row reduce, then eliminate those vectors which correspond to columns with free variables:

$$\left(\begin{array}{ccc|c}1 & 1 & 0 & 0\\1 & 2 & 0 & 0\\0 & 1 & 1 & 0\\1 & -1 & 1 & 0\end{array}\right) \xrightarrow[R_4-R_1]{R_2-R_1} \left(\begin{array}{ccc|c}1 & 1 & 0 & 0\\0 & 1 & 0 & 0\\0 & 1 & 1 & 0\\0 & -2 & 1 & 0\end{array}\right) \xrightarrow[R_4+2R_2]{R_3-R_2} \left(\begin{array}{ccc|c}1 & 1 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 1 & 0\end{array}\right)$$

$$\xrightarrow{R_4-R_3}\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

There are no free variables, hence S is linearly independent, and we cannot eliminate any vectors from S to get a set with the same span.

(e) Answer:

$$\left\{\begin{pmatrix} -1 & 1 \\ 4 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\right\}$$

(f) Answer: S is linearly independent.

(g) Use the same strategy as for matrices. Solve

$$a(x^2+3x)+b(x+1)+c(-3)=0$$

for a, b, c by equating coefficients on the powers of x.

$$ax^2+(3a+b)x+(b-3c)=0.$$

This yields the system

$$\begin{array}{rcl} a & = & 0 \\ 3a+b & = & 0 \\ b-3c & = & 0 \end{array} \quad \to \quad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right).$$

Row reducing:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right) \xrightarrow{R_2-3R_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right) \xrightarrow{R_3-R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right).$$

There are no free variables, and hence S is linearly independent.

(h) Answer:

$$\{x^2+2x+1, x^2+1\}$$

(i) Answer:

$$\{-x^2+3x+2, x^2-1, 2x^2+3x+1\}$$

$\square$

8. Let V be a vector space with $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V$. Show that $S = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a subspace of V.

proof:

Let $\mathbf{x}, \mathbf{y} \in S$ so that $\mathbf{x} = C_1\mathbf{v}_1 + C_2\mathbf{v}_2 + \cdots + C_n\mathbf{v}_n$ and $\mathbf{y} = D_1\mathbf{v}_1 + D_2\mathbf{v}_2 + \cdots + D_n\mathbf{v}_n$ for some scalars $C_1, D_1, C_2, D_2, \ldots, C_n, D_n$. Then

$$\mathbf{x}+\mathbf{y} = (C_1+D_1)\mathbf{v}_1 + (C_2+D_2)\mathbf{v}_2 + \cdots + (C_n+D_n)\mathbf{v}_n \in S,$$

(since this is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$). So S is closed under vector addition. Let r be a scalar. Then

$$r \cdot \mathbf{x} = rC_1\mathbf{v}_1 + rC_2\mathbf{v}_2 + \cdots + rC_n\mathbf{v}_n \in S.$$

So S is also closed under scalar multiplication. By the subspace criterion S is a subspace of V.

$\square$

9. Find the dimension of the vector space spanned by the following set of vectors.

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

solution:

Put the vectors as columns in a single matrix and row reduce to REF.

$$\begin{pmatrix} 1 & 3 & -1 & 0 & 1 \\ 2 & 1 & -1 & 1 & 1 \\ 3 & 4 & 2 & 2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & -1 & 0 & 1 \\ 0 & -5 & 1 & 1 & -1 \\ 0 & 0 & 4 & 1 & -1 \end{pmatrix}$$

Since the rank of this matrix is 3, the dimension of its column space is 3.

$\square$

10. Let $A = \begin{pmatrix} 1 & -3 & 4 & -1 & 9 \\ -2 & 6 & -6 & -1 & -10 \\ -3 & 9 & -6 & -6 & -3 \\ 3 & -9 & 4 & 9 & 0 \end{pmatrix}$.

(a) Find a basis for the row space of A.
(b) Find a basis for the column space of A.
(c) What is the rank of A?

solution:

The RREF of A is $\begin{pmatrix} 1 & -3 & 0 & 5 & 0 \\ 0 & 0 & 1 & -3/2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

(a) The nonzero rows of the RREF,

$$\left\{ \begin{pmatrix} 1 & -3 & 0 & 5 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & -3/2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\}$$

form a basis for the row space of A.

(b) Since there are three leading 1's, in columns 1, 3 and 5, the rank of A is 3. Basis for col(A):

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ -3 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ -6 \\ -6 \\ 4 \end{pmatrix}, \begin{pmatrix} 9 \\ -10 \\ -3 \\ 0 \end{pmatrix} \right\}$$

(take columns 1, 3, and 5 from A)

(c) The rank of A is the dimension of the row, or column space of A which is 3.

$\square$

11. Let $A = \begin{pmatrix} 1 & -3 & 4 & -1 & 9 \\ -2 & 6 & -6 & -1 & -10 \\ -3 & 9 & -6 & -6 & -3 \\ 3 & -9 & 4 & 9 & 0 \end{pmatrix}$.

   (a) Find the null space of A and give a basis.

   (b) What is the nullity of A?

solution:

(a) By definition, the null space of A is the set of solutions to $A\mathbf{x} = \mathbf{0}$, i.e. the solution set to:

$$\left(\begin{array}{ccccc|c} 1 & -3 & 4 & -1 & 9 & 0 \\ -2 & 6 & -6 & -1 & -10 & 0 \\ -3 & 9 & -6 & -6 & -3 & 0 \\ 3 & -9 & 4 & 9 & 0 & 0 \end{array}\right)$$

We can use the RREF above to get $\left(\begin{array}{ccccc|c} 1 & -3 & 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & -3/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$. The solution set, and hence the null space of A is:

$$\left\{ s \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -5 \\ 0 \\ 3/2 \\ 1 \\ 0 \end{pmatrix} \middle| s, t \in \mathbb{R} \right\}, \quad \text{which has basis} \quad \left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 0 \\ 3/2 \\ 1 \\ 0 \end{pmatrix} \right\}$$

(b) Knowing the rank of A from the previous question, we can find the nullity immediately by the relationship $\text{rank}(A) + \text{nullity}(A) = \text{number of columns of } A$.

$\square$

12. Let $C = \begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & -1 & -4 & -2 \\ 3 & 4 & 11 & 8 \end{pmatrix}$.

   (a) Find the rank and nullity of C

   (b) Give a basis for the columns space of C.

   (c) Give a basis for the null space of C

solution:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & -1 & -4 & -2 \\ 3 & 4 & 11 & 8 \end{pmatrix} \xrightarrow[R_3 - 3R_2]{r_2 + r_1} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & -2 & 2 & -4 \end{pmatrix} \xrightarrow[R_1 - 2R_2]{R_3 + R_2} \begin{pmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) Since there are 2 leading 1's in the RREF of C we see that the rank of C is 2. Since the rank plus the nullity equals the number of columns, we have that the nullity of C is 2.

(b) Using the position of the leading 1's in the RREF we have that

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} \right\}$$

is a basis for the column space of C.

(c) The null space is the solution set to the system

$$\left(\begin{array}{cccc|c} 1 & 0 & 5 & 0 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

which is

$$\left\{ s \begin{pmatrix} -5 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \end{pmatrix} \middle| \, s, t \in \mathbb{R} \right\}, \quad \text{and has basis} \quad \left\{ \begin{pmatrix} -5 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

□

13. The following set W is a subspace of $\mathbb{R}^3$. Find a basis, and determine the dimension of W.

$$W = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \middle| \, a + b + c = 0 \right\}$$

solution:

Note that the condition $a + b + c = 0$ is equivalent to $c = -a - b$, so we have

$$\begin{aligned} W &= \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \middle| \, a + b + c = 0 \right\} \\ &= \left\{ \begin{pmatrix} a \\ b \\ -a - b \end{pmatrix} \middle| \, a, b \in \mathbb{R} \right\} \\ &= \left\{ a \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \middle| \, a, b \in \mathbb{R} \right\} \\ &= \operatorname{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\} \end{aligned}$$

We now have a spanning set for W, and since these two vectors make a linearly independent set (since one is not a scalar multiple of the other), a basis for W is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\},$$

and hence $\dim W = 2$.

□

14. The following set W is a subspace of $\mathbb{R}^4$. Find a basis, and determine the dimension of W.

$$W = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \mathbb{R}^4 \middle| a - c = 0 \right\}$$

solution:

Note that the condition $a - c = 0$ is equivalent to $c = a$, so we have

$$\begin{aligned} W &= \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \mathbb{R}^4 \middle| a - c = 0 \right\} \\ &= \left\{ \begin{pmatrix} a \\ b \\ a \\ d \end{pmatrix} \middle| a, b, d \in \mathbb{R} \right\} \\ &= \left\{ a \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + d \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \middle| a, b, d \in \mathbb{R} \right\} \\ &= \operatorname{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \end{aligned}$$

We now have a spanning set for W. We now check that these three vectors make a linearly independent set *(although it is easy to see that neither vector is a linear combination of the others in this case, a simple row reduction will demonstrate this)*:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_3 - R_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

A leading 1 in each column indicates that these vectors are linearly independent. Thus a basis for W is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\},$$

and hence $\dim W = 3$.

□

15. The following set W is a subspace of $\mathcal{M}_{2\times 2}(\mathbb{R})$. Find a basis, and determine the dimension of W.

$$W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{R}) \middle| \, a + 2b = 0 \right\}.$$

solution:

Note that the condition $a + 2b = 0$ is equivalent to $b = -\frac{a}{2}$, so we have

$$\begin{aligned} W &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{R}) \middle| \, a + 2b = 0 \right\} \\ &= \left\{ \begin{pmatrix} a & -a/2 \\ c & d \end{pmatrix} \middle| \, a, c, d \in \mathbb{R} \right\} \\ &= \left\{ a \begin{pmatrix} 1 & -1/2 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \middle| \, a, c, d \in \mathbb{R} \right\} \\ &= \operatorname{span} \left\{ \begin{pmatrix} 1 & -1/2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \end{aligned}$$

We now have a spanning set for W, and it is easy to check that these three vectors make a linearly independent set. Thus a basis for W is

$$\left\{ \begin{pmatrix} 1 & -1/2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

and hence $\dim W = 3$.

$\square$

16. The following set W is a subspace of $\mathcal{P}_3(\mathbb{R})$. Find a basis, and determine the dimension of W.

$$W = \left\{ ax^3 + bx + c \in \mathcal{P}_3(\mathbb{R}) \middle| \, a + b = 0 \in \mathbb{R} \right\}.$$

solution:

Note that the condition $a + b = 0$ is equivalent to $b = -a$, so we have

$$\begin{aligned} W &= \left\{ ax^3 + bx + c \in \mathcal{P}_3(\mathbb{R}) \middle| \, a + b = 0 \in \mathbb{R} \right\} \\ &= \left\{ ax^3 - ax + c \middle| \, a, c \in \mathbb{R} \right\} \\ &= \left\{ a(x^3 - x) + c(1) \middle| \, a, c \in \mathbb{R} \right\} \\ &= \operatorname{span} \left\{ x^3 - x, 1 \right\} \end{aligned}$$

We now have a spanning set for W, and since these two vectors make a linearly independent set (since one is not a scalar multiple of the other), a basis for W is

$$\left\{ x^3 - x, 1 \right\},$$

and hence $\dim W = 2$.

$\square$