MATH 1350,
Exercise Set 5 - Solutions

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1. Consider the line given by the equation $y = 5x - 3$ in $\mathbb{R}^2$. Give a vector equation for this line.

   solution:

   Note that the points $P(0, -3)$ and $Q(1, 2)$ lie on this line. Therefore the line is parallel to (i.e. has direction vector)
$$\overrightarrow{PQ} = \begin{pmatrix} 1-0 \\ 2-(-3) \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}.$$
   Alternatively, we see that this line has slope 5 from the given equation, and therefore direction vector $\begin{pmatrix} 1 \\ 5 \end{pmatrix}$. The vector form equation for this line is therefore
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \end{pmatrix} + t \begin{pmatrix} 1 \\ 5 \end{pmatrix}.$$
   (Note that it does not matter which point we choose to use for the equation of the line.)

   $\square$

2. Consider the line in $\mathbb{R}^2$ which passes through the point $(1, 2)$ and has direction vector $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$. Give an equation for this line in:
   (a) Vector form,
   (b) Normal form,
   (c) General form,
   (d) Parametric form.

   solution:

   (a) The vector form equation of this line is
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t \begin{pmatrix} 2 \\ -5 \end{pmatrix}.$$

   (b) By inspection we see that this line has normal vector $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$ since
$$\begin{pmatrix} 5 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -5 \end{pmatrix} = 5(2) + 2(-5) = 0.$$
   Thus the normal form equation of this is line is
$$\begin{pmatrix} 5 \\ 2 \end{pmatrix} \cdot \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right) = 0,$$
   or
$$\begin{pmatrix} 5 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

(c) Computing the dot product in either normal form yields the general form,

$$5x + 2y - 9 = 0,$$

or

$$5x + 2y = 9.$$

(d) Separating x and y components in the vector form yields the parametric form,

$$x = 1 + 2t$$
$$y = 2 - 5t$$

□

3. Consider the line in $\mathbb{R}^2$ which passes through the point $(2, 9)$ and has normal vector $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$. Give an equation for this line in:
   (a) Vector form,
   (b) Normal form,
   (c) General form,
   (d) Parametric form.

solution:

(a) By inspection we see that this line is parallel to $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$ since

$$\begin{pmatrix} 1 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ 1 \end{pmatrix} = 1(-4) + 4(1) = 0.$$

Thus the vector form equation of this line is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 9 \end{pmatrix} + t \begin{pmatrix} -4 \\ 1 \end{pmatrix}.$$

(b) The normal form equation of the line is

$$\begin{pmatrix} 1 \\ 4 \end{pmatrix} \cdot \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 2 \\ 9 \end{pmatrix} \right) = 0,$$

or

$$\begin{pmatrix} 1 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 9 \end{pmatrix}.$$

(c) Computing the dot product in either normal form yields the general form,

$$x + 4y - 38 = 0,$$

or

$$x + 4y = 38.$$

(d) Separating x and y components in the vector form yields the parametric form,

$$x = 2 - 4t$$
$$y = 9 + t$$

□

4. Give the vector form equation of the line in $\mathbb{R}^3$ which passes through the point $(5, 12, -7)$ and has direction vector $\begin{pmatrix} 3 \\ 11 \\ 0 \end{pmatrix}$.

solution:

The vector form equation of this line is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 12 \\ -7 \end{pmatrix} + t \begin{pmatrix} 3 \\ 11 \\ 0 \end{pmatrix}.$$

$\square$

5. Give the vector form equation of the line in $\mathbb{R}^3$ which passes through the points $(-4, 23, 6)$ and $(2, 0, 54)$.

solution:

The direction vector for this line is the vector with initial point $(-4, 23, 6)$ and terminal point $(2, 0, 54)$, which is

$$\begin{pmatrix} 2-(-4) \\ 0-23 \\ 54-6 \end{pmatrix} = \begin{pmatrix} 6 \\ -23 \\ 48 \end{pmatrix}$$

Thus the vector form equation of this line is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ 23 \\ 6 \end{pmatrix} + t \begin{pmatrix} 6 \\ -23 \\ 48 \end{pmatrix}.$$

$\square$

6. Consider the line in $\mathbb{R}^3$ which passes through the point $(2, 0, -1)$ and has direction vector $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

   (a) Give vector form and parametric form equations for this line.

   (b) Find two non-parallel vectors which are orthogonal to this line.

   (c) Give a system for the normal form equations of this line (i.e. express this line as the intersection of two planes in $\mathbb{R}^3$).

solution:

(a) The vector form equation of this line is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix},$$

and the parametric equations are

$$\begin{aligned} x &= 2 + t \\ y &= t \\ z &= -1 - t \end{aligned}$$

(b) By inspection, we see that the vectors $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ are orthogonal to this line since

$$\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 1(1) + 1(0) + (-1)(1) = 0$$

and

$$\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 1(0) + 1(1) + (-1)(1) = 0.$$

(c) The system of equations

$$\begin{aligned} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \end{aligned}$$

yields the normal form for this line. Computing the dots products, we get the general form

$$\begin{aligned} x + z &= 1 \\ y + z &= -1 \end{aligned}$$

(i.e. the solution to this system makes up the set of all points on the line).

$\square$

7. Find the general equation of the plane through the point $(1, 1, 0)$ with normal vector $\mathbf{n} = \begin{pmatrix} 3 \\ 0 \\ -5 \end{pmatrix}$

solution:

Normal form:

$$\begin{pmatrix} 3 \\ 0 \\ -5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

General form:

$$3x - 5z = 3 \quad \text{or} \quad 3x - 5z - 3 = 0$$

□

8. Find the general equation of the plane with the vector equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + s \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

solution:

Normal vector

$$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 1 & 1 & 0 \\ -2 & 1 & 2 \end{vmatrix} = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$$

Normal form:

$$\begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

General form:

$$2x - 2y + 3z = -5$$

□

9. Calculate the cross product of $\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$.

solution:

Using the "determinant formula" for the cross product:

$$\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 2 & 3 & 5 \\ -1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 5 \\ 1 & 2 \end{vmatrix} \mathbf{e}_1 - \begin{vmatrix} 2 & 5 \\ -1 & 2 \end{vmatrix} \mathbf{e}_2 + \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} \mathbf{e}_3$$

$$= 1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 9 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 5 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -9 \\ 5 \end{pmatrix}$$

□

10. Find the

(a) Vector form,

(b) Normal form,

(c) General form,

(d) Parametric form,

of the plane which passes through the point $(3, -1, 3)$ with normal vector $\mathbf{n} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$.

solution:

Normal form:

$$\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix}$$

General form:

$$x - 2y + 4z = 21$$

This has solution set

$$\left\{ \begin{pmatrix} 21 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix} \middle| s, t \in \mathbb{R} \right\}$$

Vector form:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 21 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}$$

Parametric form:

$$\begin{aligned} x &= 21 + 2s - 4t \\ y &= s \\ z &= t \end{aligned}$$

$\square$

11. Find the vector equation for the line of intersection of the two planes $x + y - 2z = 3$ and $2x - y + 3z = 6$.

solution:

This line is the set of points in the solution to the system

$$\begin{aligned} x + y - 2z &= 3 \\ 2x - y + 3z &= 6 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 3 \\ 2 & -1 & 3 & 6 \end{array} \right) \xrightarrow{R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 3 \\ 0 & -3 & 7 & 0 \end{array} \right)$$

$$\xrightarrow{-\frac{1}{3}R_2} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 3 \\ 0 & 1 & -7/3 & 0 \end{array} \right) \xrightarrow{R_1 - R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1/3 & 3 \\ 0 & 1 & -7/3 & 0 \end{array} \right)$$

$$\left\{ \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1/3 \\ 7/3 \\ 1 \end{pmatrix} \middle| t \in \mathbb{R} \right\}$$

The vector form of the line is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1/3 \\ 7/3 \\ 1 \end{pmatrix}$$

Alternative: Since the line of intersection lies in both planes, it has direction vector which is parallel to both planes, and hence orthogonal to the normal vectors for both planes. Thus

$$\mathbf{d} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 1 & 1 & -2 \\ 2 & -1 & 3 \end{vmatrix} = \begin{pmatrix} 1 \\ -7 \\ -3 \end{pmatrix}$$

(This is just -3 times the direction vector found above). By inspection we see that the point $(3, 0, 0)$ lies on both planes since it satisfies both equations. Thus the vector form of this line is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -7 \\ -3 \end{pmatrix}$$

$\square$

12. Find the distance from the point $(0, 0)$ to the line

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} + t \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

solution:

Let A be the point $(1, 4)$, which lies on the line, and B be the point $(0, 0)$. Then

$$\overrightarrow{AB} = \begin{pmatrix} -1 \\ -4 \end{pmatrix}, \mathbf{d} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

$$\text{proj}_{\mathbf{d}}\overrightarrow{AB} = \frac{\mathbf{d} \cdot \overrightarrow{AB}}{\mathbf{d} \cdot \mathbf{d}}\mathbf{d} = \frac{(-6)}{8} \begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 3/2 \\ -3/2 \end{pmatrix}$$

$$\overrightarrow{AB} - \text{proj}_{\mathbf{d}}\overrightarrow{AB} = \begin{pmatrix} -1 \\ -4 \end{pmatrix} - \begin{pmatrix} 3/2 \\ -3/2 \end{pmatrix} = \begin{pmatrix} -5/2 \\ -5/2 \end{pmatrix} = -\frac{5}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The distance from B to the line is

$$||\overrightarrow{AB} - \text{proj}_{\mathbf{d}}\overrightarrow{AB}|| = \frac{5}{\sqrt{2}}$$

Alternative: We can see by inspection that

$$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is orthogonal to the line (check that its dot product with $\mathbf{d}$ is zero). Then

$$\text{proj}_{\mathbf{n}} \overrightarrow{AB} = \frac{\mathbf{n} \cdot \overrightarrow{AB}}{\mathbf{n} \cdot \mathbf{n}} \mathbf{n} = \frac{-5}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The distance from B to the line is

$$||\text{proj}_{\mathbf{n}} \overrightarrow{AB}|| = \frac{5}{\sqrt{2}}$$

□

13. Find the distance from the point $(2, 3, 1)$ to the plane $3x - y + 4z = 5$.

solution:

We can make use of the formula

$$\text{distance} = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|3(2) - (3) + 4(1) - 5|}{\sqrt{3^2 + (-1)^2 + 4^2}} = \frac{2}{\sqrt{26}}$$

□

14. Find the distance from the point $(2, 3, 2)$ to the line

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}$$

solution:

Let A be the point $(1, 1, -1)$, which lies on the line, and B be the point $(2, 3, 2)$. Then

$$\overrightarrow{AB} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{d} = \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}$$

$$\text{proj}_{\mathbf{d}} \overrightarrow{AB} = \frac{\mathbf{d} \cdot \overrightarrow{AB}}{\mathbf{d} \cdot \mathbf{d}} \mathbf{d} = \frac{12}{18} \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} = \frac{2}{3} \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}$$

$$\overrightarrow{AB} - \text{proj}_{\mathbf{d}} \overrightarrow{AB} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \frac{2}{3} \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 \\ -2 \\ 7 \end{pmatrix}$$

The distance from B to the line is

$$||\overrightarrow{AB} - \text{proj}_{\mathbf{d}} \overrightarrow{AB}|| = \frac{\sqrt{55}}{3}$$

□

15. Find a nonzero vector $\mathbf{u}$ in $\mathbb{R}^4$ which is orthogonal to $\mathbf{v} = \begin{pmatrix} 2 \\ -2 \\ 4 \\ 1 \end{pmatrix}$.

solution:

Here are some examples which can be found by inspection:

$$\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1/2 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -4 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 0 \\ 0 \end{pmatrix}.$$

If we wanted to find all possibilities for $\mathbf{u} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$ we solve

$$\mathbf{u} \cdot \mathbf{v} = 0 \quad \Rightarrow \quad 2u_1 - 2u_2 + 4u_3 + u_4 = 0,$$

(a system of 1 equation with 4 unknowns, and hence 3 free variables) which has solution set

$$\left\{ r \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1/2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \middle| \, r, s, t \in \mathbb{R} \right\}.$$

□

16. Compute the cross products $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ for the following pairs:

(a)

$$\mathbf{u} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}.$$

(b)

$$\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

(c)

$$\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(d)

$$\mathbf{u} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(e)

$$\mathbf{u} = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix}.$$

(f)

$$\mathbf{u} = \begin{pmatrix} 0 \\ 6 \\ -7 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 0 \\ -12 \\ 14 \end{pmatrix}.$$

solution:

Note that $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$.

(a)

$$\mathbf{u} \times \mathbf{v} = \begin{pmatrix} 3 \\ -1 \\ -4 \end{pmatrix}, \quad \mathbf{v} \times \mathbf{u} = \begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix}.$$

(b)

$$\mathbf{u} \times \mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v} \times \mathbf{u} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

(c)

$$\mathbf{u} \times \mathbf{v} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{v} \times \mathbf{u} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

(d)

$$\mathbf{u} \times \mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v} \times \mathbf{u} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}.$$

(e)

$$\mathbf{u} \times \mathbf{v} = \begin{pmatrix} 12 \\ 0 \\ -20 \end{pmatrix}, \quad \mathbf{v} \times \mathbf{u} = \begin{pmatrix} -12 \\ 0 \\ 20 \end{pmatrix}.$$

(f)

$$\mathbf{u} \times \mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v} \times \mathbf{u} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$\square$

17. Let

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

Verify that the cross product $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

solution:

$$\mathbf{u} \times \mathbf{v} = \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix}$$

$$\begin{aligned} (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} &= (u_2v_3 - u_3v_2)u_1 + (u_3v_1 - u_1v_3)u_2 + (u_1v_2 - u_2v_1)u_3 \\ &= u_1u_2v_3 + u_2u_3v_1 + u_1u_3v_2 - u_1u_3v_2 - u_1u_2v_3 - u_2u_3v_1 \\ &= 0 \end{aligned}$$

Similarly show $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = 0$

$\square$

18. Find a nonzero vector $\mathbf{w}$ in $\mathbb{R}^3$ which is orthogonal to both $\mathbf{u} = \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

solution:

The cross product $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, so take

$$\mathbf{w} = \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix}.$$

$\square$

19. Show that $\mathbf{u} \times \mathbf{u} = \mathbf{0}$ for any $\mathbf{u} \in \mathbb{R}^3$.

solution:

Let $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \in \mathbb{R}^3$, then

$$\mathbf{u} \times \mathbf{u} = \begin{pmatrix} u_2u_3 - u_3u_2 \\ u_3u_1 - u_1u_3 \\ u_1u_2 - u_2u_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$\square$

20. Let $r \in \mathbb{R}$ be a real number. Show that $(r\mathbf{u}) \times \mathbf{v} = r(\mathbf{u} \times \mathbf{v})$ for any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$.

solution:

Let $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{R}^3$. Then

$$r\mathbf{u} = \begin{pmatrix} ru_1 \\ ru_2 \\ ru_3 \end{pmatrix}$$

and so

$$(r\mathbf{u}) \times \mathbf{v} = \begin{pmatrix} (ru_2)v_3 - (ru_3)v_2 \\ (ru_3)v_1 - (ru_1)v_3 \\ (ru_1)v_2 - (ru_2)v_1 \end{pmatrix} = r \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix} = r(\mathbf{u} \times \mathbf{v}).$$

$\square$

21. Show that $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$ for any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$.

solution:

Let $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \in \mathbb{R}^3$. Then

$$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \begin{pmatrix} u_2(v_3 + w_3) - u_3(v_2 + w_2) \\ u_3(v_1 + w_1) - u_1(v_3 + w_3) \\ u_1(v_2 + w_2) - u_2(v_1 + w_1) \end{pmatrix} = \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix} + \begin{pmatrix} u_2w_3 - u_3w_2 \\ u_3w_1 - u_1w_3 \\ u_1w_2 - u_2w_1 \end{pmatrix} = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}.$$

$\square$