MATH 1350
Exercise Set 10 - Solutions

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1. (a) What are eigenvalues and eigenvectors for a square matrix A?
   (b) What is the characteristic equation for a square matrix A?
   (c) What do we use the characteristic equation for?
   (d) What is an eigenspace for a square matrix A?
   (e) How do we solve for the eigenspaces?
   (f) How do we define algebraic multiplicity and geometric multiplicity of an eigenvalue?
   (g) What is the definition for a matrix A to be diagonalizable?
   (h) What are necessary and sufficient conditions for a matrix A to be diagonalizable?
   (i) How do we diagonalize a square matrix A, provided this is possible? Conversely, how can we show that a matrix is not diagonalizable?

   solution:

   (a) A nonzero vector $\mathbf{v}$ is an eigenvector for a square matrix A if $A\mathbf{v} = \lambda\mathbf{v}$ for some scalar λ, in which case λ is called an eigenvalue for A associated with $\mathbf{v}$.
   (b) The characteristic equation for a square matrix A is the polynomial equation (in λ) that comes from $\det(A - \lambda I) = 0$, where I is the identity matrix of appropriate size.
   (c) We solve the characteristic equation for λ to find the eigenvalues of A.
   (d) An eigenspace for A is the set of all eigenvectors associated to a given eigenvalue, along with the zero vector.
   (e) To solve for the eigenspaces of A, we take a known eigenvalue λ and solve for all vectors $\mathbf{v}$ which satisfy the system $(A - \lambda I) = 0\mathbf{v}$.
   (f) The algebraic multiplicity of an eigenvalue is the (maximum) number of times it appears as a zero in the characteristic equation; that is the number of linear factors $(x - \lambda)$ that appear in the factorization of the characteristic polynomial $\det(A - xI)$. The geometric multiplicity of an eigenvalue is the dimension of its associated eigenspace.
   (g) A matrix A to said to be diagonalizable, if it is similar to a diagonal matrix, that is, if there exists an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.
   (h) An $n \times n$ matrix A is diagonalizable if and only if the sum of the dimensions of its eigenspaces is n. Equivalently, if there exists a basis for $\mathbb{R}^n$ consisting of eigenvectors for A.
   (i) To diagonalize an $n \times n$ matrix A, we solve for the eigenvalues, use these to solve for the eigenspaces, and then find a basis for each eigenspace. If, when we put these basis vectors together they form a basis for $\mathbb{R}^n$, then A is diagonalizable, and we form the matrix P by taking its columns to be the basis of eigenvectors just obtained. Then $P^{-1}AP = D$ where D is the diagonal matrix whose diagonal entries are the eigenvalues of A, appearing in the same order that corresponds to their associated eigenvector columns of P.

   To show that A is not diagonalizable, we can solve for the eigenspaces to show that that sum of their dimensions is less than n. It can be shown that the algebraic multiplicity of an eigenvalues is always greater than or equal to its geometric multiplicity. So it suffices to find an eigenvalue whose algebraic multiplicity is strictly greater than its geometric multiplicity, in order to show that A is not diagonalizable.

$\square$

2. For each of the following matrices:

$$\begin{pmatrix} 7 & -1 \\ 6 & 2 \end{pmatrix}, \begin{pmatrix} -2 & 2 \\ 5 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 2 & 1 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) Write the characteristic equation.
(b) Solve for the eigenvalues.
(c) Solve for the eigenspaces.
(d) Determine whether the matrix is diagonalizable. If so find an invertible matrix P and a diagonal matrix D such that $D = P^{-1}MP$ (where M is the given matrix). If not, explain why.

solution:

$\begin{pmatrix} 7 & -1 \\ 6 & 2 \end{pmatrix}$

(a) Characteristic equation:

$$0 = \begin{vmatrix} 7-\lambda & -1 \\ 6 & 2-\lambda \end{vmatrix} = (7-\lambda)(2-\lambda) + 6 = \lambda^2 - 9\lambda + 20$$

(b) Eigenvalues:

$$0 = \lambda^2 - 9\lambda + 20 = (\lambda - 4)(\lambda - 5)$$

The eigenvalues are $\lambda = 4$ and $\lambda = 5$.

(c) Eigenspaces:

For $\lambda = 4$:

$$\left(\begin{array}{cc|c} 3 & -1 & 0 \\ 6 & -2 & 0 \end{array}\right) \xrightarrow{R_2 - 2R_1} \left(\begin{array}{cc|c} 3 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \xrightarrow{\frac{1}{3}R_1} \left(\begin{array}{cc|c} 1 & -1/3 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$E_4 = \text{span}\left\{ \begin{pmatrix} 1/3 \\ 1 \end{pmatrix} \right\}$$

For $\lambda = 5$:

$$\left(\begin{array}{cc|c} 2 & -1 & 0 \\ 6 & -3 & 0 \end{array}\right) \xrightarrow{R_2 - 3R_1} \left(\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \xrightarrow{\frac{1}{2}R_1} \left(\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$E_5 = \text{span}\left\{ \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} \right\}$$

(d) Since the algebraic multiplicity equals the geometric multiplicity for each of the eigenvalues, this matrix is diagonalizable, and we have

$$P = \begin{pmatrix} 1/3 & 1/2 \\ 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 4 & 0 \\ 0 & 5 \end{pmatrix}.$$

$\begin{pmatrix} -2 & 2 \\ 5 & 1 \end{pmatrix}$

(a) Characteristic equation:

$$0 = \begin{vmatrix} -2-\lambda & 2 \\ 5 & 1-\lambda \end{vmatrix} = (-2-\lambda)(1-\lambda) - 10 = \lambda^2 + \lambda - 12$$

(b) Eigenvalues:

$$0 = \lambda^2 + \lambda - 12 = (\lambda+4)(\lambda-3)$$

The eigenvalues are $\lambda = -4$ and $\lambda = 3$.

(c) Eigenspaces:

For $\lambda = -4$:

$$\left(\begin{array}{cc|c} 2 & 2 & 0 \\ 5 & 5 & 0 \end{array}\right) \xrightarrow{\frac{1}{2}R_1} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 5 & 5 & 0 \end{array}\right) \xrightarrow{R_2 - 5R_1} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$E_{-4} = \text{span}\left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}$$

For $\lambda = 3$:

$$\left(\begin{array}{cc|c} -5 & 2 & 0 \\ 5 & -2 & 0 \end{array}\right) \xrightarrow{R_2 + R_1} \left(\begin{array}{cc|c} -5 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \xrightarrow{-\frac{1}{5}R_1} \left(\begin{array}{cc|c} 1 & -2/5 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$E_3 = \text{span}\left\{\begin{pmatrix} 2/5 \\ 1 \end{pmatrix}\right\}$$

(d) Since the algebraic multiplicity equals the geometric multiplicity for each of the eigenvalues, this matrix is diagonalizable, and we have

$$P = \begin{pmatrix} -1 & 2/5 \\ 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} -4 & 0 \\ 0 & 3 \end{pmatrix}$$

$\begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix}$

(a) Characteristic equation:

$$0 = \begin{vmatrix} 3-\lambda & 2 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^2$$

(b) Eigenvalues:

$$0 = (3-\lambda)^2$$

The only eigenvalues is $\lambda = 3$.

(c) Eigenspaces:

For $\lambda = 3$:

$$\left(\begin{array}{cc|c} 0 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \xrightarrow{\frac{1}{2}R_1} \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$E_3 = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$$

(d) Since the algebraic multiplicity of $\lambda = 3$ does not equal its geometric multiplicity, this matrix is not diagonalizable.

$\begin{pmatrix} 3 & 2 & 1 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{pmatrix}$

(a) Characteristic equation:

$$0 = \begin{vmatrix} 3-\lambda & 2 & 1 \\ 0 & 4-\lambda & 0 \\ 0 & 0 & 7-\lambda \end{vmatrix} = (3-\lambda)(4-\lambda)(7-\lambda)$$

(b) Eigenvalues:

$$0 = (3-\lambda)(4-\lambda)(7-\lambda)$$

The eigenvalues are $\lambda = 3$, $\lambda = 4$ and $\lambda = 7$.

(c) Eigenspaces:

For $\lambda = 3$:

$$\left(\begin{array}{ccc|c} 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right) \xrightarrow[\frac{1}{4}R_3]{R_1 - 2R_2} \left(\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{R_3 - R_1} \left(\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$E_3 = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$$

For $\lambda = 4$:

$$\left(\begin{array}{ccc|c} -1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right) \xrightarrow[\frac{1}{3}R_3]{-R_1} \left(\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{R_1 + R_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$E_4 = \text{span}\left\{\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}\right\}$$

For $\lambda = 7$:

$$\left(\begin{array}{ccc|c} -4 & 2 & 1 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[-\frac{1}{3}R_2]{-\frac{1}{4}R_1} \left(\begin{array}{ccc|c} 1 & -1/2 & -1/4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1 + \frac{1}{2}R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1/4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$E_7 = \text{span}\left\{\begin{pmatrix} 1/4 \\ 0 \\ 1 \end{pmatrix}\right\}$$

(d) Since the algebraic multiplicity equals the geometric multiplicity for each of the eigenvalues, this matrix is diagonalizable, and we have

$$P = \begin{pmatrix} 1 & 2 & 1/4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

(a) Characteristic equation:

$$\begin{aligned} 0 &= \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & 1-\lambda \end{vmatrix} + \begin{vmatrix} 1 & 1-\lambda \\ 1 & 1 \end{vmatrix} \\ &= (1-\lambda)((1-\lambda)^2 - 1) - (1-\lambda-1) + (1-(1-\lambda)) \\ &= (1-\lambda)(\lambda^2 - 2\lambda) + \lambda + \lambda \\ &= \lambda((1-\lambda)(\lambda-2) + 2)) \\ &= \lambda(-\lambda^2 + 3\lambda) \\ &= -\lambda^2(\lambda - 3) \end{aligned}$$

(b) Eigenvalues:

$$0 = -\lambda^2(\lambda - 3)$$

The eigenvalues are $\lambda = 0$ and $\lambda = 3$.

(c) Eigenspaces:

For $\lambda = 0$:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \xrightarrow[R_3-R-1]{R_2-R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$E_0 = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

For $\lambda = 3$:

$$\left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right) \xrightarrow[R_3-R_2]{R_1+2R_2} \left(\begin{array}{ccc|c} 0 & -3 & 3 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right) \xrightarrow{-\frac{1}{3}R_1} \left(\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right)$$

$$\xrightarrow[R_2-3R_1]{R_2+2R_1} \left(\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$E_3 = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

(d) Since the algebraic multiplicity equals the geometric multiplicity for each of the eigenvalues, this matrix is diagonalizable, and we have

$$P = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$\begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

(a) Characteristic equation:

$$\begin{aligned} 0 &= \begin{vmatrix} 2-\lambda & 0 & 1 \\ 1 & 1-\lambda & -1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 1-\lambda & -1 \\ 0 & 1-\lambda \end{vmatrix} + \begin{vmatrix} 1 & 1-\lambda \\ 0 & 0 \end{vmatrix} \\ &= (2-\lambda)(1-\lambda)^2 \end{aligned}$$

(b) Eigenvalues:

$$0 = (2-\lambda)(1-\lambda)^2$$

The eigenvalues are $\lambda = 1$ and $\lambda = 2$.

(c) Eigenspaces:

For $\lambda = 1$:

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$E_1 = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

For $\lambda = 2$:

$$\left(\begin{array}{ccc|c} 0 & 0 & -1 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right) \xrightarrow[R_3 - R_1]{R_2 - R_1} \left(\begin{array}{ccc|c} 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{-R_1} \left(\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$E_2 = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

(d) Since the algebraic multiplicity equals the geometric multiplicity for each of the eigenvalues, this matrix is diagonalizable, and we have

$$P = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

□