**MATH 1350**
**Winter 2025**
**Test 3**
**2025-03-04**
**Time Limit: 50 Minutes**

**Name (Print):** ______________________

**ID number:** ______________________

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You are required to **show your work** on each problem on this test.

1. (4 points) Solve using Cramer's Rule: (you must demonstrate Cramer's rule to get full marks.)

$$\begin{aligned} 4x_1 + 2x_2 &= 11 \\ -3x_1 + x_2 &= -1 \end{aligned}$$

Write this system as $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{pmatrix} 4 & 2 \\ -3 & 1 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 11 \\ -1 \end{pmatrix}.$$

By Cramer's Rule

$$x_1 = \frac{|A_1(\mathbf{b})|}{|A|} = \frac{\begin{vmatrix} 11 & 2 \\ -1 & 1 \end{vmatrix}}{\begin{vmatrix} 4 & 2 \\ -3 & 1 \end{vmatrix}} = \frac{13}{10}$$

$$x_2 = \frac{|A_2(\mathbf{b})|}{|A|} = \frac{\begin{vmatrix} 4 & 11 \\ -3 & -1 \end{vmatrix}}{\begin{vmatrix} 4 & 2 \\ -3 & 1 \end{vmatrix}} = \frac{29}{10}$$

2. Let $\mathbf{u} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$, and $\mathbf{v} = \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}$.

(a) (1 point) Compute $\mathbf{u} \cdot \mathbf{v}$.

$$\mathbf{u} \cdot \mathbf{v} = (2)(-1) + (-1)(-1) + (3)(3) = 8$$

(b) (1 point) Find a unit vector that points in the same direction as $\mathbf{u}$.

The vector

$$\frac{1}{||\mathbf{u}||}\mathbf{u} = \frac{1}{\sqrt{14}} \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

is a vector that points in the same direction as $\mathbf{u}$.

(c) (1 point) Find the angle $\theta$ between $\mathbf{u}$ and $\mathbf{v}$. (Use degrees and round to 1 decimal place.)

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}||||\mathbf{v}||}\right) = \cos^{-1}\left(\frac{8}{\sqrt{14}\sqrt{11}}\right) \approx 49.9^\circ$$

(d) (1 point) Find the projection of $\mathbf{u}$ onto $\mathbf{v}$; i.e. find $\text{proj}_{\mathbf{v}}\mathbf{u}$.

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} = \frac{8}{14} \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

(e) (1 point) Find nonzero vector $\mathbf{w} \in \mathbb{R}^3$ which is orthogonal to $\mathbf{u}$.

The vector $\mathbf{w} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, for example, is orthogonal to $\mathbf{u}$ since $\mathbf{u} \cdot \mathbf{w} = 0$. the set of all such vectors is

$$\left\{ s \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3/2 \\ 0 \\ 1 \end{pmatrix} \middle| \, s, t \in \mathbb{R} \right\}$$

3. (a) (3 points) Use Laplace expansion along row 1 to find the determinant of the matrix

$$A = \begin{pmatrix} 2 & 2 & -1 \\ -3 & 1 & 1 \\ -3 & 3 & 5 \end{pmatrix}$$

$$\begin{vmatrix} 2 & 2 & -1 \\ -3 & 1 & 1 \\ -3 & 3 & 5 \end{vmatrix} = 2\begin{pmatrix} 1 & 1 \\ 3 & 5 \end{pmatrix} - 2\begin{pmatrix} -3 & 1 \\ -3 & 5 \end{pmatrix} + (-1)\begin{pmatrix} -3 & 1 \\ -3 & 3 \end{pmatrix} = 2(2) - 2(-12) - (-6) = 34.$$

(b) (1 point) Is $A$ invertible?

Yes, $A$ is invertible since $|A| \neq 0$.

4. (3 points) Use any method you like to find the determinant of the matrix $B = \begin{pmatrix} 0 & 6 & 2 & 2 & 1 \\ 0 & 2 & 4 & -6 & 2 \\ 2 & -2 & 4 & 4 & 8 \\ 0 & 3 & 5 & 0 & -1 \\ 0 & 0 & 4 & 0 & 4 \end{pmatrix}$.

You must show your work for full marks.

Using Gauss' Method:

$$B = \begin{pmatrix} 0 & 6 & 2 & 2 & 1 \\ 0 & 2 & 4 & -6 & 2 \\ 2 & -2 & 4 & 4 & 8 \\ 0 & 3 & 5 & 0 & -1 \\ 0 & 0 & 4 & 0 & 4 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} 2 & -2 & 4 & 4 & 8 \\ 0 & 2 & 4 & -6 & 2 \\ 0 & 6 & 2 & 2 & 1 \\ 0 & 3 & 5 & 0 & -1 \\ 0 & 0 & 4 & 0 & 4 \end{pmatrix} \xrightarrow[R_4 - \frac{3}{2}R_2]{R_3 - 3R_2} \begin{pmatrix} 2 & -2 & 4 & 4 & 8 \\ 0 & 2 & 4 & -6 & 2 \\ 0 & 0 & -10 & 20 & -5 \\ 0 & 0 & -1 & 9 & -4 \\ 0 & 0 & 4 & 0 & 4 \end{pmatrix}$$

$$\xrightarrow[R_5 + \frac{4}{10}R_3]{R_4 - \frac{1}{10}R_3} \begin{pmatrix} 2 & -2 & 4 & 4 & 8 \\ 0 & 2 & 4 & -6 & 2 \\ 0 & 0 & -10 & 20 & -5 \\ 0 & 0 & 0 & 7 & -7/2 \\ 0 & 0 & 0 & 8 & 2 \end{pmatrix} \xrightarrow{R_5 - \frac{8}{7}R_4} \begin{pmatrix} 2 & -2 & 4 & 4 & 8 \\ 0 & 2 & 4 & -6 & 2 \\ 0 & 0 & -10 & 20 & -5 \\ 0 & 0 & 0 & 7 & -7/2 \\ 0 & 0 & 0 & 0 & 6 \end{pmatrix} = C$$

Then

$$|C| = (2)(2)(-10)(7)(6) = -1680.$$

Since one row swap was used, and the rest row combinations,

$$|B| = -|C| = 1680.$$

*Alternative:* A combination of Gauss' Method and Laplace expansion.

$$B = \begin{pmatrix} 0 & 6 & 2 & 2 & 1 \\ 0 & 2 & 4 & -6 & 2 \\ 2 & -2 & 4 & 4 & 8 \\ 0 & 3 & 5 & 0 & -1 \\ 0 & 0 & 4 & 0 & 4 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} 2 & -2 & 4 & 4 & 8 \\ 0 & 2 & 4 & -6 & 2 \\ 0 & 6 & 2 & 2 & 1 \\ 0 & 3 & 5 & 0 & -1 \\ 0 & 0 & 4 & 0 & 4 \end{pmatrix} \xrightarrow[R_4 - \frac{3}{2}R_2]{R_3 - 3R_2} \begin{pmatrix} 2 & -2 & 4 & 4 & 8 \\ 0 & 2 & 4 & -6 & 2 \\ 0 & 0 & -10 & 20 & -5 \\ 0 & 0 & -1 & 9 & -4 \\ 0 & 0 & 4 & 0 & 4 \end{pmatrix} = C$$

Then

$$|C| = (2)(2)\begin{vmatrix} -10 & 20 & -5 \\ -1 & 9 & -4 \\ 4 & 0 & 4 \end{vmatrix} = (4)\left(4\begin{vmatrix} 20 & -5 \\ 9 & -4 \end{vmatrix} + 4\begin{vmatrix} -10 & 20 \\ -1 & 9 \end{vmatrix}\right) = 4(4(35) + 4(-70)) = -1680$$

Since one row swap was used, and the rest row combinations,

$$|B| = -|C| = 1680.$$