Warm-up questions

- What is the purpose of LU-factorization? What do L and U represent?
- How do you find the determinant of a (square) matrix that is in RREF?
- What is the determinant of a non-invertible matrix?

Solution.

- LU-factorization is another method to solve a system of equations. It is particularly useful if asked to find the unique solutions for a system with several different **b** vectors. The L represents a lower-triangular matrix and the U is an upper-triangular matrix.
- The determinant of a matrix in RREF is the product of the diagonal entries.
- The determinant of a non-invertible, or *singular*, matrix is zero.

□

Seminar questions

<u>Process of LU-factorization:</u>

- Write the system as the matrix equation $A\mathbf{x} = \mathbf{b}$.
- Substitute $A = LU$ so that $LU\mathbf{x} = \mathbf{b}$, and then let $\mathbf{y} = U\mathbf{x}$.
- Use forward substitution to solve $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$.
- Use backward substitution to solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$, now that we have $\mathbf{y}$.

1. Reduce matrix A to REF (using *only* row cominations - no swaps or scalars) and find the inverse elementary matrix for each step. Then find the product of these matrices to create L.

$$A = \begin{pmatrix} 1 & -3 & 4 \\ -1 & 1 & 0 \\ 2 & -6 & 2 \end{pmatrix}$$

Solution.

$$\begin{pmatrix} 1 & -3 & 4 \\ -1 & 1 & 0 \\ 2 & -6 & 2 \end{pmatrix} \xrightarrow{R_3-2R_1} \begin{pmatrix} 1 & -3 & 4 \\ -1 & 1 & 0 \\ 0 & 0 & -6 \end{pmatrix} \xrightarrow{R_2+R_1} \begin{pmatrix} 1 & -3 & 4 \\ 0 & -2 & 4 \\ 0 & 0 & -6 \end{pmatrix}$$

The inverse row operations with their corresponding elementary matrices form matrix L:

$$R_3 + 2R_1 : \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \qquad R_2 - R_1 : \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$L = E_1^{-1}E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

□

2. Find $\mathbf{x}_1$ and $\mathbf{x}_2$ if $\mathbf{b}_1 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$ and $\mathbf{b}_2 = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$. Start by writing the system in form $A\mathbf{x} = LU\mathbf{x} = \mathbf{b}$.

Solution.

$$A\mathbf{x} = LU\mathbf{x} = \begin{pmatrix} 1 & -3 & 4 \\ -1 & 1 & 0 \\ 2 & -6 & 2 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \\ x_{13} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -3 & 4 \\ 0 & -2 & 4 \\ 0 & 0 & -6 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \\ x_{13} \end{pmatrix} = \begin{pmatrix} b_{11} \\ b_{12} \\ b_{13} \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$$

Now we set $\mathbf{y} = U\mathbf{x}$ so that $L\mathbf{y} = \mathbf{b}$ and we can solve for $\mathbf{y}$ by forward substitution.

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{11} \\ y_{12} \\ y_{13} \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \quad \rightarrow \quad y_{11} = 0, \quad y_{12} = -2 + 1(0) = -2, \quad y_{13} = 1 - 2(0) = 1$$

Finally, we solve for $\mathbf{x}$ using back substitution with the sytem $U\mathbf{x} = \mathbf{y}$.

$$\begin{pmatrix} 1 & -3 & 4 \\ 0 & -2 & 4 \\ 0 & 0 & -6 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \\ x_{13} \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \quad \rightarrow \quad \begin{aligned} x_{13} &= -1/6 \\ x_{12} &= -1/2(-2 - 4(-1/6)) = 2/3 \\ x_{11} &= 3(2/3) - 4(-1/6) = 8/3 \end{aligned} \quad \rightarrow \quad \begin{pmatrix} x_{11} \\ x_{12} \\ x_{13} \end{pmatrix} = \begin{pmatrix} 8/3 \\ 2/3 \\ -1/6 \end{pmatrix}.$$

Using a similar process, we get the solution for $\mathbf{x}_2$ as $\begin{pmatrix} 1/8 \\ 3/2 \\ -5/2 \end{pmatrix}$. □

3. Use the LU-factorization method to solve the system $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b}$ is given as:

$$\begin{aligned} 5x_1 - 5x_2 &= 20 \\ -2x_1 + x_2 + 5x_3 &= 8 \\ 3x_1 - 2x_2 - x_3 &= 0 \end{aligned}$$

Solution. The linear system can be expressed as $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 5 & -5 & 0 \\ -2 & 1 & 5 \\ 3 & -2 & -1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 20 \\ 8 \\ 0 \end{pmatrix}.$$

Next we can establish our L and U matrices through row reduction.

$$\begin{pmatrix} 5 & -5 & 0 \\ -2 & 1 & 5 \\ 3 & -2 & -1 \end{pmatrix} \xrightarrow[R_3 - \frac{3}{5}R_1]{R_2 + \frac{2}{5}R_1} \begin{pmatrix} 5 & -5 & 0 \\ 0 & -1 & 5 \\ 0 & 1 & -1 \end{pmatrix} \xrightarrow{R_3 + R_2} \begin{pmatrix} 5 & -5 & 0 \\ 0 & -1 & 5 \\ 0 & 0 & 4 \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -2/5 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3/5 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2/5 & 1 & 0 \\ 3/5 & -1 & 1 \end{pmatrix}$$

Since $A = LU$ we have $LU\mathbf{x} = \mathbf{b}$. Let $\mathbf{y} = U\mathbf{x}$, where $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$. Now we solve $L\mathbf{y} = \mathbf{b}$:

$$\begin{pmatrix} 1 & 0 & 0 \\ -2/5 & 1 & 0 \\ 3/5 & -1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 20 \\ 8 \\ 0 \end{pmatrix} \quad \rightarrow \quad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 20 \\ -2/5 & 1 & 0 & 8 \\ 3/5 & -1 & 1 & 0 \end{array} \right).$$

By forward substitution (starting with row 1) we see that $y_1 = 20$. Subbing into row 2 we get $y_2 = 16$, and subbing both in to row 3 we get $y_3 = 0 - 3(4) + 1(16) = 4$. Now we solve $U\mathbf{x} = \mathbf{y}$:

$$\begin{pmatrix} 5 & -5 & 0 \\ 0 & -1 & 5 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 20 \\ 16 \\ 4 \end{pmatrix} \quad \to \quad \left(\begin{array}{ccc|c} 5 & -5 & 0 & 20 \\ 0 & -1 & 5 & 16 \\ 0 & 0 & 4 & 4 \end{array}\right).$$

By backward substitution (starting with row 3) we see that $x_3 = 1$. Subbing into row 2 we get $x_2 = -(16 - 5(1)) = -11$, and subbing both into into row 1, $x_1 = -7$.

$\square$

4. Find the determinants of M_1, M_2, and M_3 using Guass' method.

$$M_1 = \begin{pmatrix} 1 & 2 \\ 2 & -3 \end{pmatrix} \qquad M_2 = \begin{pmatrix} 1 & -3 & 4 \\ -1 & 1 & 0 \\ 2 & -6 & 2 \end{pmatrix} \qquad M_3 = \begin{pmatrix} -2 & 3 & -1 & 0 & 1/2 \\ 0 & -3 & 1/5 & 9 & 9 \\ 0 & 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 1/6 & 3 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix}$$

Solution.

$$|M_1| = \begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix} \xrightarrow{R_2 - 2R_1} \begin{vmatrix} 1 & 2 \\ 0 & -7 \end{vmatrix} = 1(1)(-7) = -7$$

The REF version of M_2 was already found in question 1 using only row combinations, so the determinant is the product down the diagonal: 1(-2)(-6)=12.

Similarly, M_3 is already in REF so the determinant is: -2(-3)(1)(1/6)(7) = 7. $\square$

Extra practice problems

5. What are the determinants of the three elementary matrices?

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Solution.

The determinant of the first matrix is -1 because it is representing a row swap operation. The second matrix was scaled by 5, so its determinant is also 5. The third elementary matrix represents a row combination, which does not change the determinant from the identity matrix, so it is 1. $\square$

6. Use Laplace expansion to find the determinant of the 3×3 matrix below. Is it invertible?

$$\begin{pmatrix} -2 & 2 & 0 \\ 1 & -3 & 1 \\ 3 & 2 & 1 \end{pmatrix}$$

Solution.

$$\begin{vmatrix} -2 & 2 & 0 \\ 1 & -3 & 1 \\ 3 & 2 & 1 \end{vmatrix} = -2\begin{vmatrix} -3 & 1 \\ 2 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} + 0 = -2(-3-2) - 2(1-(-3)) = 10 - 8 = 2$$

The matrix is invertible because it has a non-zero determinant. $\square$