Warm-up questions

- What is required about the dimensions of matrices in order to add them? to multiply them?

  - For example, if we had an $a \times b$ matrix M_1 where a and b are positive integers, what would the dimensions of a second matrix M_2 have to be in order to take the product $M_1 M_2$?

- What is an identity matrix?
- How do we find the inverse of a matrix using Gauss-Jordan reduction? *Hint: See slide 8 in the notes.*
- How do we obtain an elementary matrix from an identity matrix? Give an example of an elementary matrix in $\mathbb{R}^4$.

Solution.

- To add two matrices, their dimensions must be identical. To multiply two matrices, the number of columns of the first matrix must match the number of rows in the second matrix. So M_2 would have dimensions $b \times c$ (where c is a positive integer). The resulting matrix would have dimensions $a \times c$.
- An identity matrix is a square matrix with 1's along the diagonal (from top left to bottom right) and zero's in every other entry.
- The inverse of a (square) matrix M is found by setting it up with the appropriately sized identity matrix on the right-hand side $(M|I)$ and row reducing until the identity matrix is on the left-hand side: $(I|M^{-1})$. The inverse M^{-1} will appear on the right.
- An elementary matrix is an identity matrix that has had one row operation (row swap, combination, or scaling) applied to it. Here is an example in $\mathbb{R}^4$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

□

Seminar questions

1. Noting the dimensions of each matrix below, determine which of the following pairs of matrices (including a matrix with itself) can be multiplied. If a pair can multiplied, write the dimensions of the resulting matrix.

$$M = \begin{pmatrix} -9 & 3 \\ -1 & 8 \end{pmatrix} \qquad N = \begin{pmatrix} 0 & 3 \\ -1 & 1 \\ 4 & -2 \end{pmatrix} \qquad P = \begin{pmatrix} -5 \\ 3 \\ 8 \end{pmatrix} \qquad Q = \begin{pmatrix} 2 & -2 & 0 \end{pmatrix}$$

Solution.

The matrices have the following dimensions: $M_{2\times 2}$, $N_{3\times 2}$, $P_{3\times 1}$, $Q_{1\times 3}$.
All possible pairs of multiplied matrices are below with the dimensions of the resulting matrix:

- MM would create another 2×2 matrix
- NM would create a 3×2 matrix
- QN would create a 1×2 matrix
- PQ would create a 3×3 matrix
- QP would create a 1×1 matrix

□

2. Calculate the product of the matrices $\begin{pmatrix} 4 & -3 & 0 \\ 1 & -1 & -5 \\ 3 & 2 & 3 \end{pmatrix}\begin{pmatrix} -1 & -3 \\ -4 & 2 \\ 0 & 3 \end{pmatrix}$.

Solution.

$$\begin{pmatrix} 4 & -3 & 0 \\ 1 & -1 & -5 \\ 3 & 2 & 3 \end{pmatrix}\begin{pmatrix} -1 & -3 \\ -4 & 2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 4(-1)-3(-4)+0(0) & 4(-3)-3(2)+0(3) \\ 1(-1)-2(-4)-5(0) & 1(-3)-1(2)-5(3) \\ 3(-1)+2(-4)+3(0) & 3(-3)+2(2)+3(3) \end{pmatrix} = \begin{pmatrix} 8 & -18 \\ 7 & -20 \\ -11 & 4 \end{pmatrix}$$

□

3. Find the inverse M^{-1} of matrix $M = \begin{pmatrix} 2 & 1 & 5 \\ 2 & 2 & 6 \\ 0 & 2 & 4 \end{pmatrix}$.

Solution.

$$MI = \left(\begin{array}{ccc|ccc} 2 & 1 & 5 & 1 & 0 & 0 \\ 2 & 2 & 6 & 0 & 1 & 0 \\ 0 & 2 & 4 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2-R_1} \left(\begin{array}{ccc|ccc} 2 & 1 & 5 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 2 & 4 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3-2R_2} \left(\begin{array}{ccc|ccc} 2 & 1 & 5 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 2 & -2 & 1 \end{array}\right)$$

$$\xrightarrow[\frac{1}{2}R_3]{\frac{1}{2}R_1} \left(\begin{array}{ccc|ccc} 1 & 1/2 & 5/2 & 1/2 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1/2 \end{array}\right) \xrightarrow[R_2-R_3]{R_1-\frac{5}{2}R_3} \left(\begin{array}{ccc|ccc} 1 & 1/2 & 0 & -2 & 5/2 & -5/4 \\ 0 & 1 & 0 & -2 & 2 & -1/2 \\ 0 & 0 & 1 & 1 & -1 & 1/2 \end{array}\right) \xrightarrow{R_1-\frac{1}{2}R_2}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 3/2 & -1 \\ 0 & 1 & 0 & -2 & 2 & -1/2 \\ 0 & 0 & 1 & 1 & -1 & 1/2 \end{array}\right) \qquad M^{-1} = \begin{pmatrix} -1 & 3/2 & -1 \\ -2 & 2 & -1/2 \\ 1 & -1 & 1/2 \end{pmatrix}.$$

□

4. For what value of k is the matrix singular?

$$\begin{pmatrix} 3 & -1 & 5 \\ 3 & k & -5 \\ -1 & 2 & -3 \end{pmatrix}$$

Solution. The matrix will be singular if there are free variables in the solution set. This can be revealed by putting the matrix in RREF.

$$\begin{pmatrix} 3 & -1 & 5 \\ 3 & k & -5 \\ -1 & 2 & -3 \end{pmatrix} \xrightarrow[-R_3]{R_2-R_1} \begin{pmatrix} 3 & -1 & 5 \\ 0 & k+1 & -10 \\ 1 & -2 & 3 \end{pmatrix} \xrightarrow{R_1-3R_3} \begin{pmatrix} 0 & 5 & -4 \\ 0 & k+1 & -10 \\ 1 & -2 & 3 \end{pmatrix} \xrightarrow{R_1\leftrightarrow R_3}$$

$$\begin{pmatrix} 1 & -2 & 3 \\ 0 & k+1 & -10 \\ 0 & 5 & -4 \end{pmatrix} \xrightarrow[5R_3]{2R_2} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 2(k+1) & -20 \\ 0 & 25 & -20 \end{pmatrix} \xrightarrow{R_2-R_3} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 2(k+1)-25 & 0 \\ 0 & 25 & -20 \end{pmatrix}$$

Therefore, $2(k+1)-25 = 0 \to k = \frac{23}{2}$ is the value for which the matrix is singular. There would only be zeros in R_2, making it impossible to reduce the matrix to I_3.

□

5. What are the elementary row operations associated with each of the following elementary matrices?

$$(a) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \qquad (b) \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \qquad (c) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad (d) \begin{pmatrix} 1 & 0 & -6 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Solution. (a) $R_3 + 2R_1$, (b) $4R_2$, (c) $R_2 \leftrightarrow R_3$, (d) $R_1 - 6R_3$ □

Additional practice problems

6. Create the inverse elementary matrices to those given in question 5. Label them with their (inverse) operations.

Solution. The inverse elementary matrices are:

(a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$ $R_3 - 2R_2$

(b) $\begin{pmatrix} 1 & 0 \\ 0 & 1/4 \end{pmatrix}$ $\frac{1}{4}R_2$

(c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $R_2 \leftrightarrow R_3$ (the inverse of this matrix is the same)

(d) $\begin{pmatrix} 1 & 0 & 6 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $R_1 + 6R_3$

□

7. Find the inverse of the matrix $\begin{pmatrix} 1 & 4 \\ -3 & 3 \end{pmatrix}$ using Gauss-Jordan reduction.

Solution.

$$\left(\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ -3 & 3 & 0 & 1 \end{array}\right) \xrightarrow{R_2+3R_1} \left(\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 15 & 3 & 1 \end{array}\right) \xrightarrow{\frac{1}{15}R_2} \left(\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 1 & 1/5 & 1/15 \end{array}\right) \xrightarrow{R_1-4R_2}$$

$$\left(\begin{array}{cc|cc} 1 & 0 & 1/5 & -4/15 \\ 0 & 1 & 1/5 & 1/15 \end{array}\right)$$

The inverse of the matrix is $\begin{pmatrix} 1/5 & -4/15 \\ 1/5 & 1/15 \end{pmatrix}$.

□