Linear Combinations and Span Practice Problems
Solutions

1. Write 3 different examples of linear combinations of the vectors in each of the given sets.

   (a)
   $$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

   (b)
   $$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \right\}$$

   (c)
   $$\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \right\}$$

   (d)
   $$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

   (e)
   $$\{x^2 + 2x + 1, x - 1, -x^2 - 5\}$$

   (f)
   $$\{x^2, x^2 + x, x^2 + x + 1\}$$

   (g)
   $$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \right\}$$

solution:

(a)

$$2 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + (-1) \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + 3 \cdot \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \cdot \begin{pmatrix} 0 \\ -4 \\ 5 \end{pmatrix}$$

$$0 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$1 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \cdot \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

(b)

$$1\cdot\begin{pmatrix}1\\1\\1\\1\end{pmatrix}+(-1)\cdot\begin{pmatrix}-1\\-1\\1\\1\end{pmatrix}+1\cdot\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}+(-1)\cdot\begin{pmatrix}1\\1\\1\\-1\end{pmatrix}=\begin{pmatrix}2\\2\\-2\\-2\end{pmatrix}$$

$$0\cdot\begin{pmatrix}1\\1\\1\\1\end{pmatrix}+0\cdot\begin{pmatrix}-1\\-1\\1\\1\end{pmatrix}+0\cdot\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}+0\cdot\begin{pmatrix}1\\1\\1\\-1\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$0\cdot\begin{pmatrix}1\\1\\1\\1\end{pmatrix}+1\cdot\begin{pmatrix}-1\\-1\\1\\1\end{pmatrix}+1\cdot\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}+0\cdot\begin{pmatrix}1\\1\\1\\-1\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

(c)

$$\frac{1}{2}\cdot\begin{pmatrix}1&1\\0&0\end{pmatrix}+1\cdot\begin{pmatrix}2&1\\-1&-1\end{pmatrix}+\frac{1}{2}\cdot\begin{pmatrix}1&2\\0&1\end{pmatrix}=\begin{pmatrix}3&5/2\\-1&-1/2\end{pmatrix}$$

$$0\cdot\begin{pmatrix}1&1\\0&0\end{pmatrix}+0\cdot\begin{pmatrix}2&1\\-1&-1\end{pmatrix}+4\cdot\begin{pmatrix}1&2\\0&1\end{pmatrix}=\begin{pmatrix}4&8\\0&4\end{pmatrix}$$

$$\sqrt{2}\cdot\begin{pmatrix}1&1\\0&0\end{pmatrix}+0\cdot\begin{pmatrix}2&1\\-1&-1\end{pmatrix}+0\cdot\begin{pmatrix}1&2\\0&1\end{pmatrix}=\begin{pmatrix}\sqrt{2}&\sqrt{2}\\0&0\end{pmatrix}$$

(d)

$$1\cdot\begin{pmatrix}1\\0\\0\\1\\1\end{pmatrix}+(-1)\cdot\begin{pmatrix}1\\0\\0\\0\\1\end{pmatrix}+0\cdot\begin{pmatrix}1\\0\\0\\1\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix}$$

$$(-1)\cdot\begin{pmatrix}1\\0\\0\\1\\1\end{pmatrix}+1\cdot\begin{pmatrix}1\\0\\0\\0\\1\end{pmatrix}+1\cdot\begin{pmatrix}1\\0\\0\\1\\0\end{pmatrix}=\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}$$

$$1\cdot\begin{pmatrix}1\\0\\0\\1\\1\end{pmatrix}+0\cdot\begin{pmatrix}1\\0\\0\\0\\1\end{pmatrix}+(-1)\cdot\begin{pmatrix}1\\0\\0\\1\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}$$

(e)

$$3\cdot(x^2+2x+1)+(-2)\cdot(x-1)+1\cdot(-x^2-5)=2x^2+4x$$

$$-\frac{1}{2}\cdot(x^2+2x+1)+1\cdot(x-1)+\left(-\frac{1}{2}\right)\cdot(-x^2-5)=1$$

$$-\frac{1}{2}\cdot(x^2+2x+1)+2\cdot(x-1)+\left(-\frac{1}{2}\right)\cdot(-x^2-5)=x$$

(f)

$$1\cdot(x^2)+0\cdot(x^2+x)+0\cdot(x^2+x+1)=x^2$$

$$(-1)\cdot(x^2)+1\cdot(x^2+x)+0\cdot(x^2+x+1)=x$$

$$0\cdot(x^2)+(-1)\cdot(x^2+x)+1\cdot(x^2+x+1)=1$$

(g)

$$1 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 0 \cdot \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$
$$0 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$
$$(-2) \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\square$

2. For each of the following, determine whether the given vector $\mathbf{v}$ belongs to the span of the given set of vectors. If it does, express it as a linear combination, otherwise explain why it does not belong.

(a)
$$\mathbf{v} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \quad , \quad \left\{ \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \right\}$$

(b)
$$\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad , \quad \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

(c)
$$\mathbf{v} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \quad , \quad \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \right\}$$

(d)
$$\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad , \quad \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

(e)
$$\mathbf{v} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad \left\{ \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 4 & -2 \\ -1 & 4 \end{pmatrix} \right\}$$

(f)
$$\mathbf{v} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \quad , \quad \left\{ \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 4 & 3 \\ 3 & 3 \end{pmatrix} \right\}$$

(g)
$$\mathbf{v} = \begin{pmatrix} 3 & -1 & 7 \\ 2 & 1 & 0 \\ 0 & 1 & 5 \end{pmatrix} \quad , \quad \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\}$$

(h)
$$\mathbf{v} = 2x^2 - 3x + 1 \quad , \quad \left\{ -x^2 - x - 1, x^2 + 3x, 3x - 1 \right\}$$

(i)
$$\mathbf{v} = x^2 + x + 1 \quad , \quad \left\{ x^2 + 2x - 3, x + 2, 1 \right\}$$

(j)
$$\mathbf{v} = x \quad , \quad \left\{ x^2 + x, x^2 x + 2, x^2 - 3x - 4 \right\}$$

solution:

(a) Our goal is to solve the vector equation,

$$x\begin{pmatrix}1\\-2\\0\end{pmatrix}+y\begin{pmatrix}1\\0\\-1\end{pmatrix}+z\begin{pmatrix}-1\\2\\1\end{pmatrix}=\begin{pmatrix}0\\2\\1\end{pmatrix}.$$

This yields the system of equations

$$\begin{aligned}x+y-z&=0\\-2x+2z&=2\\-y+z&=1\end{aligned}$$

which we can solve using the augmented matrix form.

$$\left(\begin{array}{ccc|c}1&1&-1&0\\-2&0&2&2\\0&-1&1&1\end{array}\right)\xrightarrow{R_2+2R_1}\left(\begin{array}{ccc|c}1&1&-1&0\\0&2&0&2\\0&-1&1&1\end{array}\right)\xrightarrow{\frac{1}{2}R_2}\left(\begin{array}{ccc|c}1&1&-1&0\\0&1&0&1\\0&-1&1&1\end{array}\right)$$

$$\xrightarrow[R_3+R_2]{R_1-R_2}\left(\begin{array}{ccc|c}1&0&-1&-1\\0&1&0&1\\0&0&1&2\end{array}\right)\xrightarrow{R_1+R_3}\left(\begin{array}{ccc|c}1&0&0&1\\0&1&0&1\\0&0&1&2\end{array}\right).$$

The unique solution is $x=1, y=1, z=2$, so

$$\begin{pmatrix}1\\-2\\0\end{pmatrix}+\begin{pmatrix}1\\0\\-1\end{pmatrix}+2\begin{pmatrix}-1\\2\\1\end{pmatrix}=\begin{pmatrix}0\\2\\1\end{pmatrix}.$$

(b) Solving with the same strategy as part (a), we see that

$$\frac{1}{2}\begin{pmatrix}1\\1\\0\end{pmatrix}+\frac{1}{2}\begin{pmatrix}1\\0\\1\end{pmatrix}+\frac{1}{2}\begin{pmatrix}0\\1\\1\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}.$$

(c) Solving with the same strategy as part (a), we obtain an inconsistent system. Thus vector $\mathbf{v}$ does not belong to the span of the given set of vectors.

(d) Solving with the same strategy as part (a), we obtain the solution set

$$\left\{\begin{pmatrix}1/6\\1/6\\1/3\\0\end{pmatrix}+t\begin{pmatrix}-1/3\\-1/3\\1/3\\1\end{pmatrix}\,\middle|\, t\in\mathbb{R}\right\}$$

Setting $t=0$ (for example) yields

$$\frac{1}{6}\begin{pmatrix}1\\2\\3\\4\end{pmatrix}+\frac{1}{6}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}+\frac{1}{3}\begin{pmatrix}2\\2\\1\\1\end{pmatrix}+0\begin{pmatrix}0\\0\\1\\1\end{pmatrix}=\begin{pmatrix}1\\1\\1\\1\end{pmatrix}.$$

(e) Our goal is to solve the vector equation,

$$x\begin{pmatrix}-1&2\\1&3\end{pmatrix}+y\begin{pmatrix}1&0\\0&3\end{pmatrix}+z\begin{pmatrix}4&-2\\-1&4\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}.$$

Equating entries on both sides yields the system

$$\begin{aligned} -x+y+4z &= 1 \\ 2x-2z &= 0 \\ x-z &= 0 \\ 3x+3y+4z &= 1 \end{aligned}$$

which we can solve using the augmented matrix form.

$$\left(\begin{array}{ccc|c} -1 & 1 & 4 & 1 \\ 2 & 0 & -2 & 0 \\ 1 & 0 & -1 & 0 \\ 3 & 3 & 4 & 1 \end{array}\right).$$

Solving yields the unique solution $x = 1, y = -2, z = 1$, so we see

$$1\begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix} + (-2)\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} + 1\begin{pmatrix} 4 & -2 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(f) Solving with the same strategy as part (e), we obtain an inconsistent system. Thus vector $\mathbf{v}$ does not belong to the span of the given set of vectors.

(g) We may solve with the same strategy as part (e), and obtain an inconsistent system (this would involve system of 9 equations with 5 unknowns). However we can see this more simply by inspection. Consider the equation

$$x_1\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + x_2\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + x_3\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + x_4\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + x_5\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & -1 & 7 \\ 2 & 1 & 0 \\ 0 & 1 & 5 \end{pmatrix}$$

The 3 in the $1,1$-entry of $\mathbf{v}$ forces the x_1 to be 3, which in turn forces x_3 to be -2 because the $2,2$-entry of $\mathbf{v}$ is 1. This makes the $3,3$-entry of the sum the left equal to 1 and not 5, since the $3,3$-entries in the other matrices are all 0. Thus vector $\mathbf{v}$ does not belong to the span of the given set of vectors.

(h)

$$\mathbf{v} = 2x^2 - 3x + 1 \quad , \quad \left\{-x^2 - x - 1, x^2 + 3x, 3x - 1\right\}$$

(i)

$$\mathbf{v} = x^2 + x + 1 \quad , \quad \left\{x^2 + 2x - 3, x + 2, 1\right\}$$

(j)

$$\mathbf{v} = x \quad , \quad \left\{x^2 + x, x^2x + 2, x^2 - 3x - 4\right\}$$

□

3. In each part, determine whether $S_1 \subseteq \mathrm{S}_2$, $S_2 \subseteq S_1$, $S_1 = S_2$, or neither.

(a)

$$S_1 = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}\right\}, \quad S_2 = \operatorname{span}\left\{\begin{pmatrix} 2 \\ 1 \\ 4 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \\ 2 \end{pmatrix}\right\}$$

(b)

$$S_1 = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 4 \\ 2 \end{pmatrix}\right\}, \quad S_2 = \operatorname{span}\left\{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}\right\}$$

(c)

$$S_1 = \text{span}\left\{\begin{pmatrix}0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\-1\end{pmatrix}\right\}, \quad S_2 = \text{span}\left\{\begin{pmatrix}1\\0\\2\\1\end{pmatrix}, \begin{pmatrix}1\\1\\2\\0\end{pmatrix}\right\}$$

(d)

$$S_1 = \text{span}\left\{\begin{pmatrix}1\\1\\2\\3\end{pmatrix}, \begin{pmatrix}-2\\-1\\0\\1\end{pmatrix}\right\}, \quad S_2 = \text{span}\left\{\begin{pmatrix}-2\\0\\4\\8\end{pmatrix}, \begin{pmatrix}3\\2\\2\\2\end{pmatrix}, \begin{pmatrix}1\\2\\6\\10\end{pmatrix}\right\}$$

(e)

$$S_1 = \text{span}\left\{\begin{pmatrix}0\\0\\0\\-1\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right\}, \quad S_2 = \text{span}\left\{\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}, \begin{pmatrix}-1\\0\\0\\1\end{pmatrix}\right\}$$

solution:

In each part the strategy is the same. If the vectors which span S_1 belong to S_2 then their $S_1 \subseteq S_2$. Vice versa if the vectors which span in S_2 belong to S_1 then $S_2 \subseteq S_1$. If both of these are true then $S_1 = S_2$.

(a) We solve

$$\begin{pmatrix}1\\0\\1\\-1\end{pmatrix} = A\begin{pmatrix}2\\1\\4\\-1\end{pmatrix} + B\begin{pmatrix}-1\\1\\1\\2\end{pmatrix}$$

and

$$\begin{pmatrix}0\\1\\2\\1\end{pmatrix} = C\begin{pmatrix}2\\1\\4\\-1\end{pmatrix} + D\begin{pmatrix}-1\\1\\1\\2\end{pmatrix}$$

This gives us two systems of equations

$$\begin{aligned} 2A - B &= 1\\ A + B &= 0\\ 4A + B &= 1\\ -A + 2B &= -1 \end{aligned} \quad \rightarrow \quad \left(\begin{array}{cc|c} 2 & -1 & 1\\ 1 & 1 & 0\\ 4 & 1 & 1\\ -1 & 2 & 1 \end{array}\right)$$

and

$$\begin{aligned} 2C - D &= 0\\ C + D &= 1\\ 4C + D &= 2\\ -C + 2D &= 1 \end{aligned} \quad \rightarrow \quad \left(\begin{array}{cc|c} 2 & -1 & 0\\ 1 & 1 & 1\\ 4 & 1 & 2\\ -1 & 2 & 1 \end{array}\right)$$

Since the coefficients are the same in each system, we can solve these simultaneously with the augmented matrix

$$\left(\begin{array}{cc|cc} 2 & -1 & 1 & 0\\ 1 & 1 & 0 & 1\\ 4 & 1 & 1 & 2\\ -1 & 2 & 1 & 1 \end{array}\right)$$

$$\xrightarrow[R_4+R_2]{\substack{R_1-2R_2\\R_3-4R_2}} \left(\begin{array}{cc|cc} 0 & -3 & 1 & -2 \\ 1 & 1 & 0 & 1 \\ 0 & -3 & 1 & -2 \\ 0 & 3 & 1 & 2 \end{array}\right) \xrightarrow[R_1+R_4]{R_3+R_4} \left(\begin{array}{cc|cc} 0 & 0 & 2 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 1 & 2 \end{array}\right)$$

Rows 1 and 3 show that this system is inconsistent, and hence there is no solution for A and B to make the first equation work. Thus S_1 is not a subset of S_2.

Similarly, by swapping sides we can use the matrix

$$\left(\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 4 & 1 \\ 1 & 1 & -1 & 2 \end{array}\right)$$

to determine whether $\text{span} S_2 \subseteq \text{span} S_1$ (this solves for C and D above).

$$\left(\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 4 & 1 \\ 1 & 1 & -1 & 2 \end{array}\right) \xrightarrow[R_4-R_1]{R_3-R_1} \left(\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 1 & -3 & 3 \end{array}\right) \xrightarrow[R_4-R_2]{R_3-2R_2} \left(\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 2 \end{array}\right)$$

Again we get an inconsistent system, thus S_2 is not a subset of S_1.

(b) As we did in part (a), to determine if the vectors which span S_1 belong to S_2 we can solve

$$\left(\begin{array}{cc|ccc} 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & -1 & -2 \\ 0 & 2 & 2 & 1 & 4 \\ 1 & 0 & 1 & 1 & 2 \end{array}\right)$$

however, we immediately see that this system is inconsistent, so S_1 is not a subset of S_2. On the other hand to determine whether S_2 is a subset of S_1 we solve

$$\left(\begin{array}{ccc|cc} 1 & 1 & 3 & 0 & 1 \\ 0 & -1 & -2 & 0 & 0 \\ 2 & 1 & 4 & 0 & 2 \\ 1 & 1 & 2 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[R_4-R_1]{R_3-2R_1} \left(\begin{array}{ccc|cc} 1 & 1 & 3 & 0 & 1 \\ 0 & -1 & -2 & 0 & 0 \\ 0 & -1 & -2 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 \end{array}\right) \xrightarrow{R_3-R_2} \left(\begin{array}{ccc|cc} 1 & 1 & 3 & 0 & 1 \\ 0 & -1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 \end{array}\right)$$

Since this system is consistent, it follows that $S_2 \subset S_1$.

(c) To determine whether S_1 is a subset of S_2 we solve

$$\left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 2 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{array}\right)$$

$$\xrightarrow[R_4-R_1]{R_3-2R_1} \left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \end{array}\right) \xrightarrow{R_4+R_2} \left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We see that the system is consistent, which shows that $S_1 \subseteq S_2$. On the other hand since the system

$$\left(\begin{array}{cc|cc} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & -1 & 1 & 0 \end{array}\right)$$

is inconsistent we see that S_2 is not a subset of S_1.

(d) To determine whether S_1 is a subset of S_2 we solve

$$\left(\begin{array}{ccc|cc} -2 & 3 & 1 & 1 & -2 \\ 0 & 2 & 2 & 1 & -1 \\ 4 & 2 & 6 & 2 & 0 \\ 8 & 2 & 10 & 3 & 1 \end{array}\right)$$

$$\xrightarrow[R_4+4R_1]{R_3+2R_1} \left(\begin{array}{ccc|cc} -2 & 3 & 1 & 1 & -2 \\ 0 & 2 & 2 & 1 & -1 \\ 0 & 8 & 8 & 4 & -4 \\ 0 & 14 & 14 & 7 & -7 \end{array}\right) \xrightarrow[R_4-7R_2]{R_3-4R_2} \left(\begin{array}{ccc|cc} -2 & 3 & 1 & 1 & -2 \\ 0 & 2 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

We see that the system is consistent, which shows that $S_1 \subseteq S_2$. On the other hand to determine whether S_2 is a subset of S_1 we solve

$$\left(\begin{array}{cc|ccc} 1 & -2 & -2 & 3 & 1 \\ 1 & -1 & 0 & 2 & 2 \\ 2 & 0 & 4 & 2 & 6 \\ 3 & 1 & 8 & 2 & 10 \end{array}\right)$$

$$\xrightarrow[R_4-3R_1]{\substack{R_2-R_1 \\ R_3-2R_1}} \left(\begin{array}{cc|ccc} 1 & -2 & -2 & 3 & 1 \\ 0 & 1 & 2 & -1 & 1 \\ 0 & 4 & 8 & -4 & 4 \\ 0 & 7 & 14 & -7 & 7 \end{array}\right) \xrightarrow[R_4-7R_2]{R_3-4R_2} \left(\begin{array}{cc|ccc} 1 & -2 & -2 & 3 & 1 \\ 0 & 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Since this system is consistent, it follows that $S_2 \subset S_1$, and hence $S_1 = S_2$.

(e) To determine whether S_1 is a subset of S_2 we solve

$$\left(\begin{array}{cc|cc} 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 \end{array}\right)$$

$$\xrightarrow{R_4+R_1} \left(\begin{array}{cc|cc} 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{array}\right)$$

Since this system is inconsistent we see that S_1 is not a subset of S_2. On the other hand to determine whether S_2 is a subset of S_1 we solve

$$\left(\begin{array}{cc|cc} 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 \end{array}\right)$$

We see that the system is consistent, which shows that $S_2 \subset S_1$.

$\square$