MATH 1350
Exercise Set 1 Solution

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1. (a) What is a linear combination? What is linear equation? What is a system of linear equations?
   (b) What is a solution to a system of equations?
   (c) What are the three elementary row operations used in Gauss' method?
   (d) How does one determine the free variables in a system of equations?
   (e) What is row echelon form (REF)? What is reduced row echelon form (RREF)?

   solution:

   (a) A linear combination of variables $x_1, x_2, \ldots, x_n$ is an expression of the form
   $$C_1x_1 + C_2x_2 + \cdots + C_nx_n$$
   where $C_1, C_2, \ldots, C_n \in \mathbb{R}$. A linear equation, in variables $x_1, x_2, \ldots, x_n$, is an equation of the form
   $$C_1x_1 + C_2x_2 + \cdots + C_nx_n = D$$
   where $C_1, C_2, \ldots, C_n, D \in \mathbb{R}$. A system of linear equations is a collection of one or more linear equations.
   (b) A solution to a system of linear equations, in n variables $x_1, x_2, \ldots, x_n$, is an n-tuple which satisfies all equations in the system.
   (c) The three elementary row operations are row swaps (interchanging two different rows), row scaling (multiplying a row by a nonzero number), and row combinations (adding a multiple of one row to another).
   (d) Put the system in REF. In each row, the left-most variable with a nonzero coefficient is a leading variable. The variables which are not leading variables are the free variables.
   (e) Row echelon form is when each leading variable is to the right of the leading variable in the row above it, and any all-zero rows are at the bottom. Reduced row echelon form is a matrix which is in REF such that each leading entry is 1, and above and below each leading entry is a zero.

   $\square$

2. For each linear system, determine whether or not the given tuple is a solution to the system.
   (a) $(1, 3)$
   $$\begin{aligned} x + y &= 4 \\ x - y &= -2 \end{aligned}$$
   (b) $(1, 2, 1)$
   $$\begin{aligned} x + y + 2z &= 5 \\ 2x - y &= 1 \end{aligned}$$

(c) $(1, 1, -1)$

$$\begin{aligned} x + y + 2z &= 5 \\ 2x - y &= 1 \end{aligned}$$

solution:

(a) $(1, 3)$

$$\begin{aligned} x + y &= 4 \\ x - y &= -2 \end{aligned}$$

This is a solution to the system; setting $x = 1$ and $y = 3$ we have that $1 + 3 = 4$ and $1 - 3 = -2$.

(b) $(1, 2, 1)$

$$\begin{aligned} x + y + 2z &= 5 \\ 2x - y &= 1 \end{aligned}$$

solution:

This is not a solution to the system; it does not satisfy the second equation.

(c) $(1, 1, -1)$

$$\begin{aligned} x + y + 2z &= 5 \\ 2x - y &= 1 \end{aligned}$$

solution:

This is not a solution to the system; it does not satisfy the first equation.

$\square$

3. Consider the following system of linear equations:

$$\begin{aligned} x + y + 2z &= 5 \\ 2x - y &= 1 \end{aligned}$$

Which of the following 3-tuples is a solution to this system?

(a) $(1, 0, 2)$
(b) $(-4, -1, 0)$
(c) $(-2, -1, 2)$
(d) $(0, -1, 4)$
(e) $(0, -1, 2)$

(f) $(2,1,0)$

solution:

We could answer this by simply subbing in each point. Instead we will find the solution set and compare. First put the system into REF

$$\begin{array}{rcl} x+y+2z &=& 5 \\ 2x-y &=& 1 \end{array} \xrightarrow{R_2-2R_1} \begin{array}{rcl} x+y+2z &=& 5 \\ -3y-4z &=& -9 \end{array}$$

Now solve for the leading variables (x and y) in terms of the free variables (z).

$$R_2 : y = 3 - \frac{4}{3}z$$
$$R_1 : x + \left(3 - \frac{4}{3}z\right) + 2z = 5 \quad \Rightarrow \quad x = 2 - \frac{2}{3}z$$

The solution set is therefore

$$\left\{ \left(2 - \frac{2}{3}z, 3 - \frac{4}{3}z, z\right) \middle| \, z \in \mathbb{R} \right\}$$

Subbing in the z value for each 3-tuple, we see that neither of these lie in this solution set.

$\square$

4. Consider the following system of linear equations:

$$x + y - 2z = 5$$
$$2x + 3y + 4z = 2$$

Which of the following is the solution set to this system?

(a) $\{(3, 8, -1)\}$
(b) $\{(5, 0, 0)\}$
(c) $\{(t, t, -2t) \mid t \in \mathbb{R}\}$
(d) $\{(13 + 10t, -8 - 8t, t) \mid t \in \mathbb{R}\}$
(e) $\{(1 - s + 2t, s, t) \mid s, t \in \mathbb{R}\}$
(f) This system has no solution.

solution:

$$\begin{array}{rcl} x+y-2z &=& 5 \\ 2x+3y+4z &=& 2 \end{array} \xrightarrow{R_2-2R_1} \begin{array}{rcl} x+y-2z &=& 5 \\ y+8z &=& -8 \end{array}$$

$$R_2 : y = -8 - 8z$$
$$R_1 : x + (-8 - 8z) - 2z = 5 \quad \Rightarrow \quad x = 13 + 10z$$

The solution set is therefore

$$\{(13 + 10z, -8 - 8z, z) \mid z \in \mathbb{R}\}.$$

This is the same set as in (d); note that the letter used for the parameter (z versus t) does not affect the 3-tuples appearing in this set.

□

5. Which of the following systems of linear equations is in row-echelon form (REF)? (Hint: It may help to rewrite the system and include spaces for the terms which have a coefficient of zero.)

   (a)
   $$\begin{aligned} w + 2x - y + 3z &= 0 \\ y + z &= 0 \end{aligned}$$

   (b)
   $$\begin{aligned} x_1 + 2x_3 + 3x_4 &= 0 \\ 4x_3 + 8x_4 &= 0 \end{aligned}$$

   (c)
   $$\begin{aligned} x + y - 2z &= 5 \\ y + z &= -1 \\ -3z &= -9 \end{aligned}$$

   (d)
   $$\begin{aligned} x_1 + 2x_2 + 3x_5 &= 2 \\ x_3 - x_5 &= 4 \\ x_4 - 2x_5 &= 3 \\ 0 &= 0 \end{aligned}$$

   (e)
   $$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0 \\ x_3 + 3x_4 &= 0 \\ x_2 + -4x_3 - 2x_4 &= 0 \end{aligned}$$

   solution:

   (a) This system is in REF. Note that the order of the variables is w, x, y, z. The leading variable of row 2, y, is to the right of the leading variable in row 1, which is w. (If we wanted to be more precise we could say the leading variable of each row comes later in the ordering than the leading variable in the row above.)

   (b) This system is in REF.

   (c) This system is in REF.

   (d) This system is in REF.

(e) This system is not in REF. (Swapping rows 2 and 3 would put this into REF.)

□

6. The following system of equations is in REF. Write the solution set.

$$\begin{aligned} x_1 - x_2 + 4x_5 &= 2 \\ x_3 - x_5 &= 2 \\ x_4 - x_5 &= 3 \end{aligned}$$

solution:

We start by expressing the leading variables, x_1, x_3 and x_4, in terms of the free variables, x_2 and x_5.

$$R_3 : x_4 = 3 + x_5$$

$$R_2 : x_3 = 2 + x_5$$

$$R_1 : x_1 = 2 + x_2 - 4x_5$$

Therefore the solution set is

$$\{(2 + x_2 - 4x_5, x_2, 2 + x_5, 3 + x_5, x_5) | x_2, x_5 \in \mathbb{R}\}.$$

□

7. Put the following system of equations into REF and solve by back substitution.

$$\begin{aligned} 2x + 8y + 4z &= 2 \\ 2x + 5y + z &= 5 \\ 4x + 10y - z &= 1 \end{aligned}$$

solution:

$$\begin{array}{rcl} 2x + 8y + 4z &=& 2 \\ 2x + 5y + z &=& 5 \\ 4x + 10y - z &=& 1 \end{array} \xrightarrow[R_3 - 2R_1]{R_2 - R_1} \begin{array}{rcl} 2x + 8y + 4z &=& 2 \\ -3y - 3z &=& 3 \\ -6y - 9z &=& -3 \end{array} \xrightarrow{R_3 - 2R_2} \begin{array}{rcl} 2x + 8y + 4z &=& 2 \\ -3y - 3z &=& 3 \\ -3z &=& -9 \end{array}$$

$$R_3 : z = 3$$

$$R_2 : -3y - 3(3) = 3 \quad \Rightarrow \quad y = -4$$

$$R_1 : 2x + 8(-4) + 4(3) = 2 \quad \Rightarrow \quad x = 11$$

Therefore the solution set is

$$\{(11, -4, 3)\}.$$

□

8. The following is an example of a system of linear equations which has no solution (i.e. an inconsistent system). Demonstrate that this is true by putting the system into REF.

$$\begin{aligned} w - 3x - 5z &= -7 \\ 3w - 12x - 2y - 27z &= -33 \\ -2w + 10x + 2y + 24z &= 29 \\ -w + 6x + y + 14z &= 17 \end{aligned}$$

solution:

$$\begin{array}{rcl} w - 3x - 5z &=& -7 \\ 3w - 12x - 2y - 27z &=& -33 \\ -2w + 10x + 2y + 24z &=& 29 \\ -w + 6x + y + 14z &=& 17 \end{array} \xrightarrow[R_4+R_1]{\substack{R_2-3R_1 \\ R_3+2R_3}} \begin{array}{rcl} w - 3x - 5z &=& -7 \\ -3x - 2y - 12z &=& -12 \\ 4x + 2y + 14z &=& 15 \\ 3x + y + 9z &=& 10 \end{array}$$

$$\xrightarrow[R_4+R_1]{R_3+\frac{4}{3}R_2} \begin{array}{rcl} w - 3x - 5z &=& -7 \\ -3x - 2y - 12z &=& -12 \\ -2/3y - 2z &=& -1 \\ -y - 3z &=& -2 \end{array} \xrightarrow{-\frac{3}{2}R_3} \begin{array}{rcl} w - 3x - 5z &=& -7 \\ -3x - 2y - 12z &=& -12 \\ y + 3z &=& 3/2 \\ -y - 3z &=& -2 \end{array}$$

$$\xrightarrow{R_4+R_3} \begin{array}{rcl} w - 3x - 5z &=& -7 \\ -3x - 2y - 12z &=& -12 \\ y + 3z &=& 3/2 \\ 0 &=& -1/2 \end{array}$$

The contradiction in the bottom row demonstrates that there is no solution to this system of equations (system is inconsistent).

□

9. Write an appropriate system of linear equations which can be used to solve for the individual masses of the hammer, saw and toolbox. Then solve the system by first putting it into REF.

solution:

Let h, s and t represent the masses of the hammer, saw and toolbox respectively. Then the three figures above yield the following linear system

$$\begin{aligned} h + s &= 10 \\ s + t &= 20 \\ h + t &= 24 \end{aligned}$$

Putting this into REF:

$$\begin{array}{rcl} h+s & = & 10 \\ s+t & = & 20 \\ h+t & = & 24 \end{array} \xrightarrow{R_3-R_1} \begin{array}{rl} h+s & = 10 \\ s+t & = 20 \\ -s+t & = 14 \end{array} \xrightarrow{R_3+R_2} \begin{array}{rl} h+s & = 10 \\ s+t & = 20 \\ 2t & = 34 \end{array}$$

Solving:

$$R_3 : t = 17$$

$$R_2 : s + 17 = 20 \quad \Rightarrow \quad s = 3$$

$$R_1 : h + 3 = 10 \quad \Rightarrow \quad h = 7$$

Therefore the hammer has a mass of 7 kg, the saw 3 kg, and the toolbox 17 kg.

□

10. Create your own examples of a system of equations having,
    (a) a single solution,
    (b) infinitely many solutions,
    (c) no solution.

solution:

Here are some examples:

(a)

$$\begin{aligned} x + y &= 1 \\ x - 2y &= 0 \end{aligned}$$

(b)

$$\begin{aligned} 2x + 6y &= 1 \\ -x - 3y &= -\frac{1}{2} \end{aligned}$$

(c)

$$\begin{aligned} x + y &= 4 \\ 2x + 2y &= 6 \end{aligned}$$

□