MATH 1350
Exercise Set 2 Solutions

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1. In this question $a, b, c, d, e, f, g, h, i$ represent arbitrary real numbers. Compute the following matrix products

$$\begin{pmatrix}1 & 0 & 0\end{pmatrix}\begin{pmatrix}a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}, \qquad \begin{pmatrix}0 & 1 & 0\end{pmatrix}\begin{pmatrix}a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}, \qquad \begin{pmatrix}0 & 0 & 1\end{pmatrix}\begin{pmatrix}a & b & c\\ d & e & f\\ g & h & i\end{pmatrix},$$

$$\begin{pmatrix}a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}, \qquad \begin{pmatrix}a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}, \qquad \begin{pmatrix}a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix},$$

$$\begin{pmatrix}1 & 0 & 0\end{pmatrix}\begin{pmatrix}a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}, \qquad \begin{pmatrix}0 & 1 & 0\end{pmatrix}\begin{pmatrix}a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}, \qquad \begin{pmatrix}0 & 0 & 1\end{pmatrix}\begin{pmatrix}a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}.$$

solution:

$$\begin{pmatrix}1 & 0 & 0\end{pmatrix}\begin{pmatrix}a & b & c\\ d & e & f\\ g & h & i\end{pmatrix} = \begin{pmatrix}a & b & c\end{pmatrix}, \qquad \begin{pmatrix}0 & 1 & 0\end{pmatrix}\begin{pmatrix}a & b & c\\ d & e & f\\ g & h & i\end{pmatrix} = \begin{pmatrix}d & e & f\end{pmatrix},$$

$$\begin{pmatrix}0 & 0 & 1\end{pmatrix}\begin{pmatrix}a & b & c\\ d & e & f\\ g & h & i\end{pmatrix} = \begin{pmatrix}g & h & i\end{pmatrix}, \qquad \begin{pmatrix}a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}a\\d\\g\end{pmatrix},$$

$$\begin{pmatrix}a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}b\\e\\h\end{pmatrix}, \qquad \begin{pmatrix}a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}c\\f\\i\end{pmatrix},$$

$$\begin{pmatrix}1 & 0 & 0\end{pmatrix}\begin{pmatrix}a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}b\end{pmatrix} \qquad \begin{pmatrix}0 & 1 & 0\end{pmatrix}\begin{pmatrix}a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}e\end{pmatrix},$$

$$\begin{pmatrix}0 & 0 & 1\end{pmatrix}\begin{pmatrix}a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}g\end{pmatrix}.$$

$\square$

2. Find the $3, 4$ entry and the $4, 3$ entry of the matrix product AA^T where

$$A = \begin{pmatrix} 9 & 7 & 10 & -3 & 1 & 1 & 0 & 8 & 5 & 4 & 10 & -3 \\ 0 & 2 & 6 & 4 & -1 & 1 & 0 & 5 & 12 & 4 & 6 & 2 \\ 3 & 3 & 1 & -7 & 0 & 2 & 0 & 1 & -1 & 1 & 5 & 6 \\ 1 & 2 & 2 & 4 & 0 & 0 & 1 & 6 & 2 & 7 & 3 & 5 \\ 4 & 0 & 0 & 1 & -1 & 2 & 6 & 5 & 12 & -3 & 2 & 2 \\ 0 & 0 & 2 & 1 & 3 & -4 & 4 & 7 & 13 & -1 & 6 & 8 \\ 6 & 2 & 1 & 1 & 0 & 0 & 5 & 0 & 4 & -3 & 6 & 6 \\ 5 & 5 & 0 & 1 & 11 & 13 & 1 & -4 & 5 & 2 & 2 & 1 \\ 3 & 1 & 0 & 2 & 0 & 0 & -1 & 6 & -7 & 8 & 9 & 2 \\ -2 & 6 & -7 & 1 & 0 & 10 & 2 & 4 & 4 & -7 & 8 & 0 \\ 0 & 0 & 0 & 5 & -1 & 3 & 3 & 1 & -5 & 7 & 6 & 9 \\ -5 & 1 & 1 & 0 & -2 & 4 & -3 & 0 & 8 & 2 & 0 & 4 \end{pmatrix}.$$

solution:

Recall that A^T, the transpose of A, is the matrix whose rows are the columns of A in the same order. The $3,4$ entry of the product AA^T is product of the third row of A with the fourth column of A^T. The fourth column of A^T is the fourth row of A. Therefore $3,4$ entry of AA^T is

$$3(1) + 3(2) + 1(2) + (-7)(4) + 0(0) + 2(0) + 0(1) + 1(6) + (-1)(2) + 1(7) + 5(3) + 6(5) = 39.$$

The matrix AA^T is symmetric. To see this, note that

$$(AA^T)^T = (A^T)^T A^T = AA^T.$$

Therefore, the i,j entry of AA^T is equal to its j,i entry. So in particular the $3,4$ entry and the $4,3$ entry of the matrix product AA^T are both 39.

□

3. A certain hardware factory produces bolts, nails and screws, and ships to 4 different stores. The following matrix can be used to summarize the number of boxes of each of the 3 products that is shipped to each of 4 stores:

$$A = \begin{pmatrix} 100 & 150 & 120 & 200 \\ 200 & 220 & 100 & 300 \\ 350 & 300 & 400 & 480 \end{pmatrix}$$

Entry a_{ij} is the number of boxes of product i which are shipped to store j.

The cost of shipping by truck is \$ 2.00 per box of bolts, \$1.00 per box of nails and \$1.50 per box of screws. The cost of shipping by train is \$ 2.25 per box of bolts, \$1.25 per box of nails and \$1.50 per box of screws. Use these shipping prices to create a cost matrix B that may be used to compare the cost of shipping by performing an appropriate matrix multiplication.

solution:

If we let

$$B = \begin{pmatrix} 2 & 1 & 1.5 \\ 2.25 & 1.25 & 1.5 \end{pmatrix},$$

then the matrix product BA is

$$C = \begin{pmatrix} 2 & 1 & 1.5 \\ 2.25 & 1.25 & 1.5 \end{pmatrix} \begin{pmatrix} 100 & 150 & 120 & 200 \\ 200 & 220 & 100 & 300 \\ 350 & 300 & 400 & 480 \end{pmatrix} = \begin{pmatrix} 925 & 970 & 940 & 1420 \\ 1000 & 1062.5 & 995 & 1545 \end{pmatrix},$$

where row 1 column j of C gives the total shipping cost by truck to store j, and row 2 column j gives the total shipping cost by train to store j.

□

4. For each of the following 2×2 matrices either write its inverse or show that one does not exist.

$$A = \begin{pmatrix} 1 & 4 \\ 5 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -4 \\ 1/2 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 3 & -4 \\ 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix}$$

solution:

Recall the inverse formula for a 2×2 matrix:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

which holds if and only if $ad - bc \neq 0$. Thus

$$A^{-1} = -\frac{1}{22} \begin{pmatrix} -2 & -4 \\ -5 & 1 \end{pmatrix} = \begin{pmatrix} 1/11 & 2/11 \\ 5/22 & -1/22 \end{pmatrix},$$

$$B \text{ is not invertible since } 2(-1) - (-4)\left(\frac{1}{2}\right) = 0,$$

$$C^{-1} = \frac{1}{3} \begin{pmatrix} 1 & 4 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1/3 & 4/3 \\ 0 & 1 \end{pmatrix},$$

$$D^{-1} = \frac{1}{36} \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} = \begin{pmatrix} 1/9 & 0 \\ 0 & 1/4 \end{pmatrix}.$$

□

5. Find the inverse of the matrix below, or show that one does not exist.

$$\begin{pmatrix} 0 & 3 & -2 \\ 1 & 5 & 2 \\ 1 & 1 & -1 \end{pmatrix}$$

solution:

$$\left(\begin{array}{ccc|ccc} 0 & 3 & -2 & 1 & 0 & 0 \\ 1 & 5 & 2 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|ccc} 1 & 5 & 2 & 0 & 1 & 0 \\ 0 & 3 & -2 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3-R_1}\left(\begin{array}{ccc|ccc}1&5&2&0&1&0\\0&3&-2&1&0&0\\0&-4&-3&0&-1&1\end{array}\right)\xrightarrow{\frac{1}{3}R_2}\left(\begin{array}{ccc|ccc}1&5&2&0&1&0\\0&1&-2/3&1/3&0&0\\0&-4&-3&0&-1&1\end{array}\right)$$

$$\xrightarrow{R_3+4R_2}\left(\begin{array}{ccc|ccc}1&5&2&0&1&0\\0&1&-2/3&1/3&0&0\\0&0&-17/3&4/3&-1&1\end{array}\right)\xrightarrow{-\frac{3}{17}R_3}\left(\begin{array}{ccc|ccc}1&5&2&0&1&0\\0&1&-2/3&1/3&0&0\\0&0&1&-4/17&3/17&-3/17\end{array}\right)$$

$$\xrightarrow[R_2+\frac{2}{3}R_3]{R_1-2R_3}\left(\begin{array}{ccc|ccc}1&5&0&8/17&11/17&6/17\\0&1&0&3/17&2/17&-2/17\\0&0&1&-4/17&3/17&-3/17\end{array}\right)\xrightarrow{R_1-5R_2}\left(\begin{array}{ccc|ccc}1&0&0&-7/17&1/17&16/17\\0&1&0&3/17&2/17&-2/17\\0&0&1&-4/17&3/17&-3/17\end{array}\right)$$

Therefore the inverse of this matrix exists and is

$$\begin{pmatrix}-7/17&1/17&16/17\\3/17&2/17&-2/17\\-4/17&3/17&-3/17\end{pmatrix}$$

□

6. Solve the following four systems of equations

$$A\mathbf{w}=\mathbf{a},\quad A\mathbf{x}=\mathbf{b},\quad A\mathbf{y}=\mathbf{c},\quad A\mathbf{z}=\mathbf{d}$$

for the solution vectors

$$\mathbf{w}=\begin{pmatrix}w_1\\w_2\\w_3\end{pmatrix},\quad \mathbf{x}=\begin{pmatrix}x_1\\x_2\\w_3\end{pmatrix},\quad \mathbf{y}=\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix},\quad \mathbf{z}=\begin{pmatrix}z_1\\z_2\\z_3\end{pmatrix}$$

given that

$$A=\begin{pmatrix}1&0&-2\\-3&1&1\\2&0&-1\end{pmatrix},\quad \mathbf{a}=\begin{pmatrix}3\\1\\6\end{pmatrix},\quad \mathbf{b}=\begin{pmatrix}0\\2\\9\end{pmatrix},\quad \mathbf{c}=\begin{pmatrix}1\\1\\1\end{pmatrix},\quad \mathbf{d}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

by first finding A^{-1}.

solution:

$$\left(\begin{array}{ccc|ccc}1&0&-2&1&0&0\\-3&1&1&0&1&0\\2&0&-1&0&0&1\end{array}\right)\xrightarrow[R_2-2R_1]{R_2+3R_1}\left(\begin{array}{ccc|ccc}1&0&-2&1&0&0\\0&1&-5&3&1&0\\0&0&3&-2&0&1\end{array}\right)$$

$$\xrightarrow{1/3R_3}\left(\begin{array}{ccc|ccc}1&0&-2&1&0&0\\0&1&-5&3&1&0\\0&0&1&-2/3&0&1/3\end{array}\right)\xrightarrow[R_2+5R_3]{R_1+2R_3}\left(\begin{array}{ccc|ccc}1&0&0&-1/3&0&2/3\\0&1&0&-1/3&1&5/3\\0&0&1&-2/3&0&1/3\end{array}\right)$$

So

$$A^{-1}=\begin{pmatrix}-1/3&0&2/3\\-1/3&1&5/3\\-2/3&0&1/3\end{pmatrix}.$$

We can solve the system $A\mathbf{w} = \mathbf{a}$ for $\mathbf{w}$ by multiplying both sides of the equation by A^{-1}:

$$A\mathbf{w} = \mathbf{a} \quad \Rightarrow \quad A^{-1}A\mathbf{w} = A^{-1}\mathbf{a} \quad \Rightarrow \quad I\mathbf{w} = A^{-1}\mathbf{a} \quad \Rightarrow \quad \mathbf{w} = A^{-1}\mathbf{a}$$

Thus

$$\mathbf{w} = \begin{pmatrix} -1/3 & 0 & 2/3 \\ -1/3 & 1 & 5/3 \\ -2/3 & 0 & 1/3 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ 10 \\ 0 \end{pmatrix}$$

Similarly

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{pmatrix} -1/3 & 0 & 2/3 \\ -1/3 & 1 & 5/3 \\ -2/3 & 0 & 1/3 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 9 \end{pmatrix} = \begin{pmatrix} 6 \\ 17 \\ 3 \end{pmatrix},$$

$$\mathbf{y} = A^{-1}\mathbf{c} = \begin{pmatrix} -1/3 & 0 & 2/3 \\ -1/3 & 1 & 5/3 \\ -2/3 & 0 & 1/3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 7/3 \\ -5/3 \end{pmatrix},$$

$$\mathbf{z} = A^{-1}\mathbf{d} = \begin{pmatrix} -1/3 & 0 & 2/3 \\ -1/3 & 1 & 5/3 \\ -2/3 & 0 & 1/3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

□

7. (a) What is the definition of an elementary matrix?
   (b) How do you obtain an elementary matrix for a given elementary row operation?
   (c) How are elementary matrices related to matrix multiplication?
   (d) Explain how to find the inverse of an elementary matrix by looking at its associated elementary row operation.

solution:

(a) An elementary matrix is the matrix which results from applying a single elementary row operation to an identity matrix.

(b) If the elementary row operation is being performed on an $m \times n$ matrix, then apply the elementary row operation to the $m \times m$ identity matrix I_m to obtain the corresponding elementary matrix.

(c) Multiplication on the left by an elementary matrix has the same result as performing its associated elementary row operation.

(d) To find the inverse of an elementary matrix, first determine the inverse of the associated elementary row operation (see table below), then form the elementary matrix for the inverse elementary row operation.

ERO	Inverse ERO
$CR_i \quad (C \in \mathbb{R} \setminus \{0\})$	$\frac{1}{C}R_i$
$R_i \leftrightarrow R_j$	$R_i \leftrightarrow R_j$
$R_i + CR_j \quad (C \in \mathbb{R} \setminus \{0\})$	$R_i - CR_j$

□

8. Determine the elementary row operation for each of the following elementary matrices.

(a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & -4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

solution:

(a) $R_3 + 3R_1$

(b) $R_1 \leftrightarrow R_2$

(c) $2R_2$

(d) $R_1 - 4R_2$

□

9. Write the elementary matrix corresponding the given elementary row operations. Assume a 4×4 matrix.

(a) $R_2 \leftrightarrow R_4$

(b) $2R_1$

(c) $R_2 - R_3$

(d) $R_3 - R_2$

(e) $R_1 \leftrightarrow R_4$

(f) $-5R_3$

(g) $R_3 + 2R_2$

(h) $R_4 - 5R_3$

solution:

(a) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(e) $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

(f) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(g) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(h) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -5 & 1 \end{pmatrix}$

$\square$

10. Find the inverse of each of the matrices in question 9.

solution:

(a) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(e) $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

(f) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1/5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(g) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(h) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & 1 \end{pmatrix}$

□

11. (a) Write the elementary matrices corresponding to each row operation in the following row reduction, and express U as the product of A and these elementary matrices.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -3 & -3 & 0 \\ 0 & 7 & 1 \end{pmatrix} \xrightarrow{R_2+3R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 7 & 1 \end{pmatrix} \xrightarrow{R_2\leftrightarrow R_3} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 7 & 1 \\ 0 & 0 & 3 \end{pmatrix} \xrightarrow{\frac{1}{3}R_3} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 7 & 1 \\ 0 & 0 & 1 \end{pmatrix} = U$$

solution:

$$\text{Let} \quad E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} \quad \text{so that} \quad U = E_3E_2E_1A.$$

□

(b) Write the inverse matrices for each of the elementary matrices in the previous question, and express A as a product of U and these elementary matrices.

solution:

$$\text{We have} \quad E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

and so

$$A = E_1^{-1}E_2^{-1}E_3^{-1}U.$$

□

12. Write A and A^{-1} as a product of elementary matrices, where

$$A = \begin{pmatrix} -1 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

solution:

$$\left(\begin{array}{ccc|ccc} -1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow{-R_1} \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3-R_1} \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & 0 & 1 \end{array}\right) \xrightarrow{-R_2} \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 2 & 1 & 1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3-2R_2} \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 3 & 1 & 2 & 1 \end{array}\right) \xrightarrow{\frac{1}{3}R_3} \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1/3 & 2/3 & 1/3 \end{array}\right)$$

$$\xrightarrow[R_2+R_3]{R_1-R_3} \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & -4/3 & -2/3 & -1/3 \\ 0 & 1 & 0 & 1/3 & -1/3 & 1/3 \\ 0 & 0 & 1 & 1/3 & 2/3 & 1/3 \end{array}\right) \xrightarrow{R_1+R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1/3 & -1/3 & 1/3 \\ 0 & 0 & 1 & 1/3 & 2/3 & 1/3 \end{array}\right)$$

Thus

$$A^{-1} = \begin{pmatrix} -1 & -1 & 0 \\ 1/3 & -1/3 & 1/3 \\ 1/3 & 2/3 & 1/3 \end{pmatrix}$$

Consider the following elementary matrices which correspond to the elementary row operations used above.

$$E_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix},$$

$$E_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}, E_6 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, E_7 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, E_8 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$E_8E_7E_6E_5E_4E_3E_2E_1A = I_3$$

which shows that

$$A^{-1} = E_8E_7E_6E_5E_4E_3E_2E_1$$

and

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}E_6^{-1}E_7^{-1}E_8^{-1}.$$

$\square$

13. For the following row reduction find L in the LU decomposition of M.

$$M = \begin{pmatrix} 2 & 4 & -4 \\ 1 & -4 & 3 \\ -6 & -9 & 10 \end{pmatrix} \xrightarrow{R_2-\frac{1}{2}R_1} \begin{pmatrix} 2 & 4 & -4 \\ 0 & -6 & 5 \\ -6 & -9 & 10 \end{pmatrix} \xrightarrow{R_3+3R_1} \begin{pmatrix} 2 & 4 & -4 \\ 0 & -6 & 5 \\ 0 & 3 & -2 \end{pmatrix} \xrightarrow{R_3+\frac{1}{2}R_2} \begin{pmatrix} 2 & 4 & -4 \\ 0 & -6 & 5 \\ 0 & 0 & 1/2 \end{pmatrix} = U$$

solution:

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -3 & -1/2 & 1 \end{pmatrix}$$

$\square$

14. (a) Show that matrix N is invertible and express N as a product of elementary matrices where

$$N = \begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{pmatrix}.$$

solution:

$$(N|I) = \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 3 & 1 & -2 & 0 & 1 & 0 \\ -5 & -1 & 9 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3+5R_1]{R_2-3R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 4 & -3 & 1 & 0 \\ 0 & -1 & -1 & 5 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3+R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 4 & -3 & 1 & 0 \\ 0 & 0 & 3 & 2 & 1 & 1 \end{array}\right) \xrightarrow{\frac{1}{3}R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 4 & -3 & 1 & 0 \\ 0 & 0 & 1 & 2/3 & 1/3 & 1/3 \end{array}\right)$$

$$\xrightarrow[R_2-4R_3]{R_1+2R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/3 & 2/3 & 2/3 \\ 0 & 1 & 0 & -17/3 & -1/3 & -4/3 \\ 0 & 0 & 1 & 2/3 & 1/3 & 1/3 \end{array}\right) = (I|N^{-1})$$

Therefore N is invertible with

$$N^{-1} = \begin{pmatrix} 7/3 & 2/3 & 2/3 \\ -17/3 & -1/3 & -4/3 \\ 2/3 & 1/3 & 1/3 \end{pmatrix}$$

We have used 6 elementary row operations to row reduce N to the identity matrix. In terms of elementary matrices we have

$$E_6E_5E_4E_3E_2E_1N = I$$

where

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{pmatrix}, E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}, E_5 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, E_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}.$$

Their inverses are

$$E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{pmatrix}, E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$$E_4^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}, E_5^{-1} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, E_6^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix},$$

and isolating N we have

$$N = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}E_6^{-1}$$

□

Find the LU-factorization of matrix N.

(b) *solution:*

Looking at the row reduction for N above, we see that the the first 3 elementary row operations put N into the REF

$$U = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{pmatrix}.$$

We use these first 3 elementary row operations to find

$$L = E_1^{-1}E_2^{-1} = E_3^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -5 & -1 & 1 \end{pmatrix}$$

Therefore the LU-factorization of N is

$$\begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -5 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{pmatrix}$$

□

(c) Use the LU-factorization of N found above to solve the system $N\mathbf{x} = \mathbf{b}$ where

$$\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

solution:

Using the LU-factorization of N converts $N\mathbf{x} = \mathbf{b}$ to $LU\mathbf{x} = \mathbf{b}$. Let $\mathbf{y} = U\mathbf{x}$ and solve $L\mathbf{y} = \mathbf{b}$ by forward substitution. From

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -5 & -1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

the first row implies $y_1 = 1$, while the second row implies $3(1) + y_2 = 0$ or $y_2 = -3$, and the third row implies $-5(1) - 1(-3) + y_3 = 1$ or $y_3 = 3$. So

$$\mathbf{y} = \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix}.$$

Now plug this into $U\mathbf{x} = \mathbf{y}$ and solve for $\mathbf{x}$ by back substitution. From

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix},$$

the bottom row gives $3x_3 = 3$ or $x_3 = 1$, the middle row implies $x_2 + 4(1) = -3$ or $x_2 = -7$, and the top row implies $x_1 + 0(-7) - 2(1) = 1$ or $x_1 = 3$. We now have the solution

$$\mathbf{x} = \begin{pmatrix} 3 \\ -7 \\ 1 \end{pmatrix},$$

$\square$