Linear Combinations and Span Practice Problems

1. Write 3 different examples of linear combinations of the vectors in each of the given sets.

   (a)
   $$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

   (b)
   $$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \right\}$$

   (c)
   $$\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \right\}$$

   (d)
   $$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

   (e)
   $$\{x^2 + 2x + 1, x - 1, -x^2 - 5\}$$

   (f)
   $$\{x^2, x^2 + x, x^2 + x + 1\}$$

   (g)
   $$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \right\}$$

2. For each of the following, determine whether the given vector $\mathbf{v}$ belongs to the span of the given set of vectors. If it does, express it as a linear combination, otherwise explain why it does not belong.

   (a)
   $$\mathbf{v} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \quad , \quad \left\{ \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \right\}$$

   (b)
   $$\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad , \quad \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

(c)

$$\mathbf{v} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \quad , \quad \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \right\}$$

(d)

$$\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad , \quad \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

(e)

$$\mathbf{v} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad \left\{ \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 4 & -2 \\ -1 & 4 \end{pmatrix} \right\}$$

(f)

$$\mathbf{v} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \quad , \quad \left\{ \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 4 & 3 \\ 3 & 3 \end{pmatrix} \right\}$$

(g)

$$\mathbf{v} = \begin{pmatrix} 3 & -1 & 7 \\ 2 & 1 & 0 \\ 0 & 1 & 5 \end{pmatrix} \quad , \quad \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\}$$

(h)

$$\mathbf{v} = 2x^2 - 3x + 1 \quad , \quad \{-x^2 - x - 1, x^2 + 3x, 3x - 1\}$$

(i)

$$\mathbf{v} = x^2 + x + 1 \quad , \quad \{x^2 + 2x - 3, x + 2, 1\}$$

(j)

$$\mathbf{v} = x \quad , \quad \{x^2 + x, x^2x + 2, x^2 - 3x - 4\}$$

3. In each part, determine whether $S_1 \subseteq \mathrm{S}_2$, $S_2 \subseteq S_1$, $S_1 = S_2$, or neither.

(a)

$$S_1 = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix} \right\}, \quad S_2 = \operatorname{span}\left\{ \begin{pmatrix} 2 \\ 1 \\ 4 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \\ 2 \end{pmatrix} \right\}$$

(b)

$$S_1 = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 4 \\ 2 \end{pmatrix} \right\}, \quad S_2 = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix} \right\}$$

(c)

$$S_1 = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\}, \quad S_2 = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix} \right\}$$

(d)

$$S_1 = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}, \quad S_2 = \operatorname{span}\left\{ \begin{pmatrix} -2 \\ 0 \\ 4 \\ 8 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 6 \\ 10 \end{pmatrix} \right\}$$

(e)

$$S_1 = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}, \quad S_2 = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$