Warm-up questions

- What are the different kinds of elementary row operations? Give an example of each.
- What is a leading variable? What is a free variable? How are they identified?
- Give an example of a system of equations in Row Echelon Form.

Seminar questions

1. Solve the system using elementary row operations. Which of the variables (if any) are free variables?

$$\begin{aligned} x+3y &= 5 \\ -2x+y &= -3 \end{aligned}$$

Solution.

$$\begin{array}{rcl} x+3y &=& 5 \\ -2x+y &=& -3 \end{array} \quad R_2{+}2R_1 \rightarrow \quad \begin{array}{rcl} x+3y &=& 5 \\ 7y &=& 7 \end{array} \quad \frac{1}{7}R_2 \rightarrow \quad \begin{array}{rcl} x+3y &=& 5 \\ y &=& 1 \end{array} \quad R_1{-}3R_2 \rightarrow \quad \begin{array}{rcl} x &=& 2 \\ y &=& 1 \end{array}$$

There are no free variables in the system. □

2. Is the given tuple is a solution to the linear system? If not, find a valid solution.

$$(0,3,-2) \text{ for the system} \qquad \begin{aligned} x+y+z &= 1 \\ -2x+2y+5z &= 0 \end{aligned}$$

Solution.

In the first equation, $(0)+(3)+(-2)=1$ is consistent. In the second row, $-2(0)+2(3)+5(-2) = 6-10 = -4 \neq 0$ so $(0,3,-2)$ is not a solution for the linear system.
All solutions can be found by row reducing the system to REF. $(-1,4,-2)$ is one such solution. □

3. Which of the following systems of equations are in row echelon form (REF)? For each system in REF, identify which variables are leading, and which (if any) are free variables.

$$A: \begin{array}{rcl} x_2 &=& 3 \\ -x_1+2x_2 &=& 2 \end{array} \qquad B: \begin{array}{rcl} x &=& 1 \\ y &=& 1 \\ z &=& 0 \end{array} \qquad C: \begin{array}{rcl} x-y+3z &=& -3 \\ w+2y &=& -2 \\ x &=& 2 \end{array} \qquad D: \begin{array}{rcl} x_1+x_2+x_4 &=& -1 \\ x_3-4x_4 &=& 0 \\ x_5 &=& \pi \end{array}$$

Solution.

Only B and D are in REF. System B has three leading variables and no free variables. System D has three leading variables (x_1,x_3,x_5) and two free (x_2,x_4). □

4. Solve the system of equations. You may solve it as an augmented matrix if you wish. How many solutions are there?

$$\begin{aligned} x_1+6x_2-4x_3 &= -7 \\ -2x_1+3x_2-5x_3 &= 3 \\ x_1+x_2 &= -4 \end{aligned}$$

Solution.

$$\begin{aligned} x_1+6x_2-4x_3 &= -7 \\ -2x_1+3x_2-5x_3 &= 3 \\ x_1+x_2 &= -4 \end{aligned} \quad \xrightarrow[R_3-R_1]{R_2+2R_1} \quad \begin{aligned} x_1+6x_2-4x_3 &= -7 \\ 15x_2-13x_3 &= -11 \\ -5x_2+4x_3 &= 3 \end{aligned} \quad \xrightarrow{R_2+3R_3} \quad \begin{aligned} x_1+6x_2-4x_3 &= -7 \\ -x_3 &= -2 \\ -5x_2+4x_3 &= 3 \end{aligned}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \quad \begin{aligned} x_1+6x_2-4x_3 &= -7 \\ -5x_2+4x_3 &= 3 \\ -x_3 &= -2 \end{aligned} \quad \xrightarrow{-R_3} \quad \begin{aligned} x_1+6x_2-4x_3 &= -7 \\ -5x_2+4x_3 &= 3 \\ x_3 &= 2 \end{aligned}$$

At this point we can begin to back substitute to find values for x_2 and x_1. Continuing on R_2,

$$-5x_2+4(2)=3 \implies -5x_2+8=3 \implies -5x_2=-5 \implies x_2=1.$$

Then we find the value of x_1 by returning to R_1:

$$x_1+6(1)-4(2)=-7 \implies x_1+6-8=-7 \implies x_1=-5$$

□

Additional practice problem

5. Consider a situation where an advertising agency is trying to determine how many times ads A, B, and C should appear on an app. To save money, the agency has decided to spend exactly \$600 on advertising overnight (from 9pm to 9am) while they've budgeted for \$1,000 of ads from 9am to 9pm.

- During the day, it costs \$1.20 for each ad A, \$1.80 for each ad B, and \$2.10 for each ad C.
- Overnight, the rates to run ads A, B, and C are reduced to \$0.90, \$1.30, and \$1.70 respectively.

Set up a system of equations to represent this scenario.

Without solving, what is the maximum number of leading variables the system can have?

Solution.

$$\begin{aligned} 1.2A+1.8B+2.1C &= 1000 \\ .9A+1.3B+1.7C &= 600 \end{aligned}$$

This system represents the word problem above. Because there are three variables and only two equations, we know we can have at most two leading variables. At least one will have to be free. □