MATH 1350
Exercise Set 3 Solutions

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1. (a) Let A be a square matrix. What does the determinant of A tell us about the solution set for the homogeneous system $A\mathbf{x} = \mathbf{0}$?
   (b) How does the determinant tell us whether or not A is invertible?
   (c) Is A singular or nonsingular if $\det A = 0$?
   (d) How do the 3 elementary row operations affect the determinant?
   (e) How can we find the determinant of an upper triangular matrix? What about a lower triangular matrix?
   (f) Let $A, B \in \mathcal{M}_{n\times n}(\mathbb{R})$. Compute the following in terms of $\det A$ and $\det B$.
   - $\det(A^{-1})$ (assuming A is invertible).
   - $\det(A^T)$
   - $\det(kA)$ for $k \in \mathbb{R}$.
   - $\det(AB)$
   - $\det(A^m)$ for $m \in \mathbb{N}$.

   solution:

   (a) If $\det A \neq 0$, its only solution is $\mathbf{x} = \mathbf{0}$, and if $\det A = 0$ there exist nonzero solutions for $\mathbf{x}$ (there are free variables and hence infinitely many solutions).
   (b) Matrix A is invertible if and only if $\det A \neq 0$.
   (c) If $\det A = 0$ then A is singular.
   (d) Row combinations ($R_i + CR_j$) do not cause a change in the determinant, row swaps ($R_i \leftrightarrow R_j$) cause the determinant to change sign (from positive to negative and vice versa) and scaling a row by a factor of k (kR_i) multiplies the determinant by k.
   (e) The determinant of a triangular matrix, upper or lower, is simply the product of its diagonal entries.
   (f)
   - $\det(A^{-1}) = \frac{1}{\det A}$
   - $\det(A^T) = \det A$
   - $\det(kA) = k^n \det A$
   - $\det(AB) = \det A \det B$
   - $\det(A^m) = (\det A)^m$

   □

2. Write the expression for the determinant of a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

   solution:

$$\det A = ad - bc.$$

□

3. Write the expression for the determinant of a 3×3 matrix $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$.

solution:

$$\det A = aei + bfg + cdh - ceg - afh - bdi.$$

We can recall this 3×3 formula by the *Rule of Sarrus*: Form the 3×5 matrix whose first 3 columns are the columns of A, and columns 4 and 5 are columns 1 and 2 of A respectively (see the diagram below). Now add the 3 products of entries along the solid lines, and subtract the 3 products along the dashed lines. *(Caution: This rule only applies to* 3×3 *matrices.)*

$$\begin{array}{ccc:cc} + & + & + & & \\ a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \\ - & - & - & & \end{array}$$

We can also recover this formula by Laplace Expansion:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg)$$

□

4. Find the determinants of the following matrices. Conclude whether or not the matrix is invertible.

(a) $\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

solution:

(a) We have $\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -5$, so this matrix is invertible.

(b) We have $\begin{vmatrix} 2 & 4 \\ 3 & 6 \end{vmatrix} = 0$, so this matrix is not invertible.

(c) We have $\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2$, so this matrix is invertible.

□

5. Find The determinant of matrix B, based on this row reduction and the definition of a determinant function.

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 6 & 7 & 1 \\ -3 & -3 & 0 \end{pmatrix} \xrightarrow{R_2 - 6R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -5 \\ -3 & -3 & 0 \end{pmatrix} \xrightarrow{R_3 + 3R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -5 \\ 0 & 0 & 3 \end{pmatrix} \xrightarrow{\frac{1}{3}R_3} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{pmatrix}$$

solution:

Start by calling the final matrix C (the one in REF)

$$C = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{pmatrix}.$$

The first two row operations do not change the value of the determinant, however the last row operation changes the determinant by a factor of $\frac{1}{3}$. Therefore

$$\det C = \frac{1}{3} \det B$$

Since C is upper triangular, its determinant is the product of the diagonal entries, which is 1. This gives us

$$\det B = 3 \det C = 3.$$

□

6. Find the determinants of the following matrices using Gauss' method.

(a) $\begin{pmatrix} 3 & 3 \\ 4 & 9 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 & 4 \\ 3 & 6 & -1 \\ 0 & 1 & 9 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 0 & 0 & 5 \\ 2 & 1 & -1 & 8 \\ 0 & 1 & -1 & 0 \\ 1 & 1 & 3 & 4 \end{pmatrix}$

solution:

(a) We have

$$A = \begin{pmatrix} 3 & 3 \\ 4 & 9 \end{pmatrix} \xrightarrow{\frac{1}{3}R_1} \begin{pmatrix} 1 & 1 \\ 4 & 9 \end{pmatrix} \xrightarrow{R_2 - 4R_1} \begin{pmatrix} 1 & 1 \\ 0 & 5 \end{pmatrix} = B.$$

Based on theses row operations, and the fact that B is in REF, we have

$$\det A = 3 \det B = 15.$$

(b) We have

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 6 & -1 \\ 0 & 1 & 9 \end{pmatrix} \xrightarrow{R_2 - 3R_1} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 0 & -13 \\ 0 & 1 & 9 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 9 \\ 0 & 0 & -13 \end{pmatrix} = B$$

so

$$\det A = -\det B = 13.$$

(c) We have

$$A = \begin{pmatrix} 2 & 0 & 0 & 5 \\ 2 & 1 & -1 & 8 \\ 0 & 1 & -1 & 0 \\ 1 & 1 & 3 & 4 \end{pmatrix} \xrightarrow[R_4 - \frac{1}{2}R_1]{R_2 - R_1} \begin{pmatrix} 2 & 0 & 0 & 5 \\ 0 & 1 & -1 & 3 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 3 & 3/2 \end{pmatrix} \xrightarrow[R_4 - R_2]{R_3 - R_2} \begin{pmatrix} 2 & 0 & 0 & 5 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 4 & -3/2 \end{pmatrix}$$

$$\xrightarrow{R_3 \leftrightarrow R_4} \begin{pmatrix} 2 & 0 & 0 & 5 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 4 & -3/2 \\ 0 & 0 & 0 & -3 \end{pmatrix} = B$$

so

$$\det A = -\det B = 24.$$

$\square$

7. Find the determinants of the following matrices using Laplace Expansion.

(a) $\begin{pmatrix} 3 & 3 \\ 4 & 9 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 & 4 \\ 3 & 6 & -1 \\ 0 & 1 & 9 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 0 & 0 & 5 \\ 2 & 1 & -1 & 8 \\ 0 & 1 & -1 & 0 \\ 1 & 1 & 3 & 4 \end{pmatrix}$

solution:

Note that we may expand along any row or column we choose.

(a) Along row 1,

$$\begin{vmatrix} 3 & 3 \\ 4 & 9 \end{vmatrix} = 3(9) - 3(4) = 15.$$

(b) Along column 1,

$$\begin{vmatrix} 1 & 2 & 4 \\ 3 & 6 & -1 \\ 0 & 1 & 9 \end{vmatrix} = 1\begin{vmatrix} 6 & -1 \\ 1 & 9 \end{vmatrix} - 3\begin{vmatrix} 2 & 4 \\ 1 & 9 \end{vmatrix} + 0\begin{vmatrix} 2 & 4 \\ 6 & -1 \end{vmatrix} = 55 - 3(14) = 13$$

(c) Along row 1,

$$\begin{aligned}
\begin{vmatrix} 2 & 0 & 0 & 5 \\ 2 & 1 & -1 & 8 \\ 0 & 1 & -1 & 0 \\ 1 & 1 & 3 & 4 \end{vmatrix} &= 2\begin{vmatrix} 1 & -1 & 8 \\ 1 & -1 & 0 \\ 1 & 3 & 4 \end{vmatrix} - 5\begin{vmatrix} 2 & 1 & -1 \\ 0 & 1 & -1 \\ 1 & 1 & 3 \end{vmatrix} \\
&= 2\left(8\begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix} + 4\begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix}\right) + 5\left(2\begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix}\right) \\
&= 2\left(8(4) + 4(0)\right) - 5\left(2(4) + (0)\right) \\
&= 24
\end{aligned}$$

(We have expanded along columns 3 and 1 respectively for the 3×3 determinants.)

□

8. Suppose matrix $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ has a determinant of 2. Find the following determinants.

(a) $\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix}$

(b) $\begin{vmatrix} a+2d & b+2e & c+2f \\ d & e & f \\ g & h & i \end{vmatrix}$

(c) $\begin{vmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{vmatrix}$

(d) $\begin{vmatrix} 3a & 3b & 3c \\ 3d & 3e & 3f \\ 3g & 3h & 3i \end{vmatrix}$

(e) $\begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix}$

(f) $\begin{vmatrix} b & a & c \\ e & d & f \\ h & g & i \end{vmatrix}$

(g) $\begin{vmatrix} a & b-a & c \\ d & e-d & f \\ g & h-g & i \end{vmatrix}$

(h) $\begin{vmatrix} a & 0 & 0 \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} 0 & b & 0 \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} 0 & 0 & c \\ d & e & f \\ g & h & i \end{vmatrix}$

solution:

(a) Rows 1 and 2 have swapped, so this determinant is -2.

(b) Since row combinations do not change the determinant, this matrix has determinant 2.

(c) Row 5 has been multiplied by 5, so this determinant is $5(2) = 10$.

(d) This determinant is $3^3(2) = 54$.

(e) Swapping rows 1 and 3, then rows 2 and 3 brings us back to A, so the determinant is $(-1)(-1)(2) = 2$.

(f) Columns 1 and 2 have swapped, so this determinant is -2.

(g) Column 1 has been subtracted from column 2, and column operations do not change the determinant, so this matrix has determinant 2.

(h) Using Laplace Expansion along row 1 we have,

$$\begin{vmatrix} a & 0 & 0 \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} 0 & b & 0 \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} 0 & 0 & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 2$$

(This follows from the more general "multilinear" property of determinants.)

□

9. Give an example of a 3×3 matrix whose determinant is zero, but whose entries are all nonzero.

solution:

Any matrix with 2 proportional rows, or columns, will have determinant zero, for example

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & -5 & 7 \end{pmatrix},$$

where row 2 is twice row 1, or

$$\begin{pmatrix} 2 & 2 & 5 \\ 2 & 2 & -3 \\ 2 & 2 & 1 \end{pmatrix},$$

where columns 1 and 2 are equal. If one of the rows (or columns), is linear combination of the other rows (columns), the determinant is zero, for example

$$\begin{pmatrix} 1 & 1 & 1 \\ 5 & 2 & 7 \\ 8 & 5 & 10 \end{pmatrix},$$

where row 3 is 3 times row 1 plus row 2, or

$$\begin{pmatrix} 2 & 1 & 6 \\ 3 & -1 & 4 \\ 5 & 1 & 12 \end{pmatrix},$$

where the second column is half of the third column minus the first column.

□

10. Let E_1, E_2 and E_3 be the elementary matrices corresponding to the elementary row operations

$$R_3 - 6R_2, \quad R_4 \leftrightarrow R_5, \quad 4R_3$$

respectively.

(a) Find $\det E_1$, $\det E_2$ and $\det E_3$.

(b) Use the examples above to find a general rule that gives the determinant for any of the 3 types of elementary matrix.

(a) Note that the size of the elementary matrix does not matter here. We know that the determinant of the identity matrix is 1 by our definition of a determinant function. Since $R_3 - 6R_2$ does not change the value of the determinant, we have that

$$\det E_1 = \det I = 1.$$

The operation $R_4 \leftrightarrow R_5$ changes the value of the determinant by a factor of -1, so

$$\det E_2 = -\det I = -1.$$

The operation $4R_3$ changes the determinant by a factor of 3, so

$$\det E_3 = 3 \det I = 3.$$

(b) More generally:

- An elementary matrix corresponding to a row operation of the form $R_i + CR_j$, for $C \in \mathbb{R}$, has determinant 1.
- An elementary matrix corresponding to a row operation of the form $R_i \leftrightarrow R_j$ has determinant -1.
- An elementary matrix corresponding to a row operation of the form kR_i, for any $k \in \mathbb{R}$, has determinant k.

solution:

□

11. In this question we prove the multiplicative property of determinants.

(a) Let E be an elementary matrix, and A a matrix such that the product EA is defined. Show that

$$\det(EA) = (\det E)(\det A)$$

by applying the properties in the definition of a determinant function. Hint: Consider 3 cases for the 3 types of matrix that E could be and use the result of the previous question.

(b) Let A and B be square matrices of the same size. Show that $\det(AB) = (\det A)(\det B)$. Hint: Consider the cases when A is singular and non singular, and make use of part (a).

solution:

(a) Recall that multiplication by an elementary matrix on the left is the same as applying the associated row operation. Using the rules found above, and our definition of a determinant function we have:

- If E corresponds to a row operation of the form $R_i + CR_j$, then

$$\det(EA) = \det A = 1 \cdot \det A = \det E \det A.$$

- If E corresponds to a row operation of the form $R_i \leftrightarrow R_j$, then

$$\det(EA) = (-1) \cdot \det A = \det E \det A.$$

- If E corresponds to a row operation of the form kR_i for nonzero $k \in \mathbb{R}$, then

$$\det(EA) = k \cdot \det A = \det E \det A.$$

Since these exhaust all possibilities, we can now see that

$$\det(EA) = \det E \det A$$

when E is an elementary matrix.

(b) The result above can be extended to show that

$$\det(AB) = \det A \det B$$

for any square matrices A and B. We break this into two cases. Case 1: If A is invertible, then A is a product of elementary matrices

$$A = E_1 E_2 \dots E_n$$

and so

$$\det(AB) = \det(E_1 E_2 \dots E_n B).$$

The property proven above can now by applied repeatedly (more formally using a proof by induction on n) to obtain

$$\det(E_1 E_2 \dots E_n B) = \det E_1 \det E_2 \dots \det E_n \det B$$

and then repeated again to get that

$$\det E_1 \det E_2 \dots \det E_n \det B = \det(E_1 E_2 \dots E_n) \det B = \det A \det B.$$

Case 2: If A is not invertible, we can show that AB is also not invertible and hence

$$\det(AB) = 0 = 0 \cdot \det B = \det A \det B.$$

Now to show that AB is also not invertible. If $B\mathbf{x} = \mathbf{0}$ for some nonzero $\mathbf{x}$ (i.e. if B is singular) then

$$(AB)\mathbf{x} = A(B\mathbf{x}) = A(\mathbf{0}) = \mathbf{0}$$

which shows that $(AB)\mathbf{x} = \mathbf{0}$ for some nonzero $\mathbf{x}$ and hence AB is singular (equivalently non-invertible). On the other hand suppose $B\mathbf{x} \neq \mathbf{0}$ whenever $\mathbf{x} \neq \mathbf{0}$; i.e. B is nonsingular and hence invertible. Since A is singular there exists a nonzero $\mathbf{y}$ such that $A\mathbf{y} = \mathbf{0}$. Let $\mathbf{v} = B^{-1}\mathbf{y}$. Then $\mathbf{v} \neq \mathbf{0}$ since $B\mathbf{v} = \mathbf{y} \neq \mathbf{0}$, and

$$(AB)\mathbf{v} = A(B\mathbf{v}) = A\mathbf{y} = \mathbf{0}.$$

This shows $(AB)\mathbf{v} = \mathbf{0}$ where $\mathbf{v} \neq \mathbf{0}$, and again that AB is non-invertible.

(A question like this would not be given on a test.)

□

12. (a) What kinds of systems of equations does Cramer's Rule apply to?
    (b) State Cramer's Rule.

solution:

(a) Cramer's Rule applies to systems whose coefficient matrix is invertible (and hence a square matrix).

(b) Suppose $A\mathbf{x} = \mathbf{b}$ is a system of n equations in n unknowns $x_1, \ldots, x_n$, where $\det A \neq 0$. Then
$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}$$
for each $i \in \{1, \ldots, n\}$ where $A_i(\mathbf{b})$ is the matrix obtained by replacing column i of A with $\mathbf{b}$.

□

13. Use Cramer's Rule to solve the following systems of equations.

(a)
$$\begin{aligned} 4x_1 + 2x_2 &= 5 \\ 3x_1 - 9x_2 &= 11 \end{aligned}$$

(b)
$$\begin{aligned} 4x_1 + 2x_2 &= 3 \\ 3x_1 - 9x_2 &= 0 \end{aligned}$$

(c)
$$\begin{aligned} x_1 + 2x_2 + 4x_3 &= 2 \\ 3x_1 + 6x_2 - x_3 &= 1 \\ x_2 + 9x_3 &= 0 \end{aligned}$$

solution:

(a) For this system we have
$$A = \begin{pmatrix} 4 & 2 \\ 3 & -9 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 5 \\ 11 \end{pmatrix}$$
By Cramer's Rule
$$x_1 = \frac{\det A_1(\mathbf{b})}{\det A} = \frac{\begin{vmatrix} 5 & 2 \\ 11 & -9 \end{vmatrix}}{\begin{vmatrix} 4 & 2 \\ 3 & -9 \end{vmatrix}} = \frac{(-67)}{(-42)} = \frac{67}{42}$$
$$x_2 = \frac{\det A_2(\mathbf{b})}{\det A} = \frac{\begin{vmatrix} 4 & 5 \\ 3 & 11 \end{vmatrix}}{\begin{vmatrix} 4 & 2 \\ 3 & -9 \end{vmatrix}} = \frac{(29)}{(-42)} = \frac{29}{42}$$

(b) For this system we have

$$A = \begin{pmatrix} 4 & 2 \\ 3 & -9 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

By Cramer's Rule

$$x_1 = \frac{\det A_1(\mathbf{b})}{\det A} = \frac{\begin{vmatrix} 3 & 2 \\ 0 & -9 \end{vmatrix}}{\begin{vmatrix} 4 & 2 \\ 3 & -9 \end{vmatrix}} = \frac{(-27)}{(-42)} = \frac{9}{14}$$

$$x_2 = \frac{\det A_2(\mathbf{b})}{\det A} = \frac{\begin{vmatrix} 4 & 3 \\ 3 & 0 \end{vmatrix}}{\begin{vmatrix} 4 & 2 \\ 3 & -9 \end{vmatrix}} = \frac{(-9)}{(-42)} = \frac{3}{14}$$

(c) For this system we have

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 6 & -1 \\ 0 & 1 & 9 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

By Cramer's Rule

$$x_1 = \frac{\det A_1(\mathbf{b})}{\det A} = \frac{\begin{vmatrix} 2 & 2 & 4 \\ 1 & 6 & -1 \\ 0 & 1 & 9 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 4 \\ 3 & 6 & -1 \\ 0 & 1 & 9 \end{vmatrix}} = \frac{96}{13}$$

$$x_2 = \frac{\det A_2(\mathbf{b})}{\det A} = \frac{\begin{vmatrix} 1 & 2 & 4 \\ 3 & 1 & -1 \\ 0 & 0 & 9 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 4 \\ 3 & 6 & -1 \\ 0 & 1 & 9 \end{vmatrix}} = -\frac{45}{13}$$

$$x_3 = \frac{\det A_3(\mathbf{b})}{\det A} = \frac{\begin{vmatrix} 1 & 2 & 2 \\ 3 & 6 & 1 \\ 0 & 1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 4 \\ 3 & 6 & -1 \\ 0 & 1 & 9 \end{vmatrix}} = \frac{5}{13}$$

$\square$

14. Find inverses for the following matrices by using the adjoint (adjugate) method.

(a) $\begin{pmatrix} 2 & -5 \\ 1 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} -3 & 3 & 2 \\ 2 & -1 & -2 \\ 2 & -2 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 2 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}$

(d) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

solution:

Let C be the matrix of cofactors of the matrix A. Then $A^{-1} = \frac{1}{|A|}C^T$.

(a) The determinant is

$$\begin{vmatrix} 2 & -5 \\ 1 & 1 \end{vmatrix} = 7,$$

and the matrix of cofactors is

$$\begin{pmatrix} 1 & -1 \\ 5 & 2 \end{pmatrix}$$

thus

$$\begin{pmatrix} 2 & -5 \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{7}\begin{pmatrix} 1 & -1 \\ 5 & 2 \end{pmatrix}^T = \begin{pmatrix} 1/7 & 5/7 \\ -1/7 & 2/7 \end{pmatrix}$$

(b) The determinant is

$$\begin{vmatrix} -3 & 3 & 2 \\ 2 & -1 & -2 \\ 2 & -2 & 0 \end{vmatrix} = -4,$$

and the matrix of cofactors is

$$\begin{pmatrix} \begin{vmatrix} -1 & -2 \\ -2 & 0 \end{vmatrix} & -\begin{vmatrix} 2 & -2 \\ 2 & 0 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 2 & -2 \end{vmatrix} \\ -\begin{vmatrix} 3 & 2 \\ -2 & 0 \end{vmatrix} & \begin{vmatrix} -3 & 2 \\ 2 & 0 \end{vmatrix} & -\begin{vmatrix} -3 & 3 \\ 2 & -2 \end{vmatrix} \\ \begin{vmatrix} 3 & 2 \\ -1 & -2 \end{vmatrix} & -\begin{vmatrix} -3 & 2 \\ 2 & -2 \end{vmatrix} & \begin{vmatrix} -3 & 3 \\ 2 & -1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} -4 & -4 & -2 \\ -4 & -4 & 0 \\ -4 & -2 & -3 \end{pmatrix}$$

thus

$$\begin{pmatrix} -3 & 3 & 2 \\ 2 & -1 & -2 \\ 2 & -2 & 0 \end{pmatrix}^{-1} = \frac{1}{(-4)}\begin{pmatrix} -4 & -4 & -2 \\ -4 & -4 & 0 \\ -4 & -2 & -3 \end{pmatrix}^T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1/2 \\ 1/2 & 0 & 3/4 \end{pmatrix}$$

(c) The determinant is

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 2 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{vmatrix} = 1,$$

and the matrix of cofactors is

$$\begin{pmatrix} \begin{vmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 0 \end{vmatrix} & -\begin{vmatrix} 2 & -1 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix} & -\begin{vmatrix} 2 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix} \\ -\begin{vmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix} \\ \begin{vmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 & 1 \\ 2 & 0 & 0 \\ 1 & -1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 0 & 0 \\ 2 & 0 & -1 \\ 1 & -1 & 0 \end{vmatrix} \\ -\begin{vmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ 0 & -1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 & 0 \\ 2 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & -1 & -2 & 1 \\ -1 & -2 & -2 & 1 \end{pmatrix}$$

thus

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 2 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}^{-1} = \frac{1}{1}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & -1 & -2 & 1 \\ -1 & -2 & -2 & 1 \end{pmatrix}^T = \begin{pmatrix} 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & -2 \\ 0 & 1 & -2 & -2 \\ 1 & -1 & 1 & 1 \end{pmatrix}$$

(A matrix this large would not be given on a test.)

(d) The determinant is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$

and the matrix of cofactors is

$$\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

thus

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}^T = \frac{1}{ad-bc}\begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

□