MATH 1350,
Exercise Set 7 - Solutions

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1. (a) What is the definition of a linear transformation (also called a linear map)?
   (b) How do we show that a given mapping between vector spaces is linear?
   (c) How do we show that a given mapping between vector spaces is not linear?
   (d) Describe in your own words what the domain and codomain of a linear transformation are.

solution:

(a) A linear transformation is a mapping (or function) $T : V \to W$, from vector space V to vector space W such that for any $\mathbf{u}, \mathbf{v} \in V$,

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}),$$

i.e., T preserves vector addition, and for any scalar $r \in \mathbb{R}$ and any vector $\mathbf{v} \in V$,

$$T(r \cdot \mathbf{v}) = r \cdot T(\mathbf{v}),$$

i.e. T preserves scalar multiplication.

(b) To show that a given map T is linear (provided it is indeed linear) we must show that it preserves both vector addition and scalar multiplication. To show that T preserves vector addition, we take two arbitrary vectors $\mathbf{u}, \mathbf{v}$ belonging to the domain of T and demonstrate $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u})+T(\mathbf{v})$ by using the particular rule that defines T. To show that T preserves scalar multiplication, we take an arbitrary scalar $r \in \mathbb{R}$, and an arbitrary vector $\mathbf{v}$ belonging to the domain of T and demonstrate $T(r \cdot \mathbf{v}) = r \cdot T(\mathbf{v})$ by using the particular rule that defines T. By taking arbitrary vectors and scalars, we show that T satisfies these properties for any vectors and scalars, not only for particular cases.

(c) To show that a given mapping T is not linear (provided this is the case) we may demonstrate that it fails to preserve vector addition or scalar multiplication; demonstrating failure of only one of theses is sufficient. To show that either property fails, we give an explicit example of two specific vectors $\mathbf{u}, \mathbf{v}$ in the domain of T (in the case of vector addition), or a specific scalar r and a vector $\mathbf{v}$ (in the case of scalar multiplication), and show by example that $T(\mathbf{u} + \mathbf{v}) \neq T(\mathbf{u}) + T(\mathbf{v})$, or that $T(r \cdot \mathbf{v}) \neq r \cdot T(\mathbf{v})$.

We may also show that T is not linear by demonstrating a known property of linear maps which fails for T. For example, we know that linear maps must map the zero vector of the domain to zero vector of the codomain. So if this fails, then T is not linear.

(d) If $T : V \to W$ is a mapping from V to W, then V is the domain of T and W is the codomain of T. The vector space V, the domain of T, are the "input" vectors, and the vector space W, the codomain of T, is the vector space where the "output" vectors generally belong. Note that all vectors of V are valid inputs for T, but not all vectors in W will necessarily be outputs of T.

$\square$

2. Let T be the mapping from $\mathbb{R}^3$ to $\mathbb{R}^2$ given by

$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3b - c \\ -a \end{pmatrix}$$

(a) What are the domain and codomain of T?

(b) Apply T to each of the following vectors:

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(c) Notice that $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ -3 \end{pmatrix}$. Show that $T\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + T\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} = T\begin{pmatrix} 3 \\ 4 \\ -3 \end{pmatrix}$.

(d) Notice that $(-5) \cdot \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} -10 \\ -15 \\ 20 \end{pmatrix}$. Show that $(-5) \cdot T\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} = T\begin{pmatrix} -10 \\ -15 \\ 20 \end{pmatrix}$.

(e) Show that T preserves vector addition and scalar multiplication, and hence that T is linear.

(f) Find 5 different vectors that T maps to $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

(g) Are there any vectors in $\mathbb{R}^3$ that T maps to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$? If not show why, and if so is there more than one vector in $\mathbb{R}^3$ that maps to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$?

(h) Are there any vectors in $\mathbb{R}^3$ that T maps to $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$? If not show why, and if so is there more than one vector in $\mathbb{R}^3$ that maps to $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$?

(i) Is it possible to obtain any vector in $\mathbb{R}^2$ as the result of T applied to some vector in $\mathbb{R}^3$; i.e. for any $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}$, does there exist a vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$ such that $T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$?

solution:

(a) The domain is $\mathbb{R}^3$ and the codomain is $\mathbb{R}^2$.

(b)

$$T\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad T\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 13 \\ -2 \end{pmatrix}, \quad T\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad T\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \quad T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

(c) We have

$$T\begin{pmatrix}1\\1\\1\end{pmatrix}+T\begin{pmatrix}2\\3\\-4\end{pmatrix}=\begin{pmatrix}2\\-1\end{pmatrix}+\begin{pmatrix}13\\-2\end{pmatrix}=\begin{pmatrix}15\\-3\end{pmatrix}$$

and

$$T\begin{pmatrix}3\\4\\-3\end{pmatrix}=\begin{pmatrix}15\\-3\end{pmatrix}.$$

(d) We have

$$(-5)\cdot T\begin{pmatrix}2\\3\\-4\end{pmatrix}=(-5)\cdot\begin{pmatrix}13\\-2\end{pmatrix}=\begin{pmatrix}-65\\10\end{pmatrix},$$

and

$$T\begin{pmatrix}-10\\-15\\20\end{pmatrix}=\begin{pmatrix}-65\\10\end{pmatrix}.$$

(e) Let $\mathbf{u}=\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}, \mathbf{v}=\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}\in\mathbb{R}^3$. Then

$$\begin{aligned}
T(\mathbf{u}+\mathbf{v}) &= T\begin{pmatrix}u_1+v_1\\u_2+v_2\\u_3+v_3\end{pmatrix}\\
&= \begin{pmatrix}3(u_2+v_2)-(u_3+v_3)\\-(u_1+v_1)\end{pmatrix}\\
&= \begin{pmatrix}3u_2-u_3+3v_2-v_3\\-u_1-v_1\end{pmatrix}\\
&= \begin{pmatrix}3u_2-u_3\\-u_1\end{pmatrix}+\begin{pmatrix}3v_2-v_3\\-v_1\end{pmatrix}\\
&= T\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}+T\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}\\
&= T(\mathbf{u})+T(\mathbf{v}).
\end{aligned}$$

Thus T preserves vector addition. Let $r \in \mathbb{R}$. Then

$$\begin{aligned} T(r\mathbf{u}) &= T\begin{pmatrix} ru_1 \\ ru_2 \\ ru_3 \end{pmatrix} \\ &= \begin{pmatrix} 3ru_2 - ru_3 \\ -ru_1 \end{pmatrix} \\ &= r\begin{pmatrix} 3u_2 - u_3 \\ -u_1 \end{pmatrix} \\ &= rT\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \\ &= rT(\mathbf{u}). \end{aligned}$$

Thus T also preserves scalar multiplication and hence T is a linear transformation.

(f) For this particular map T it is not difficult to guess at 5 different vectors which map to $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$; we have already seen 2 example in part (a). Instead let's solve for all vectors in $\mathbb{R}^3$ which T sends to $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$:

$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 3b - c \\ -a \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{aligned} 3b - c &= 0 \\ -a &= 0 \end{aligned}$$

Solving this we see that $a = 0, b = \frac{1}{3}c$ and c is free, in other words we have solution set

$$\left\{ t\begin{pmatrix} 0 \\ 1/3 \\ 1 \end{pmatrix} \middle| \; t \in \mathbb{R} \right\}.$$

Taking $t = -1, 0, 1, 2, 3$ for example yields the vectors

$$\begin{pmatrix} 0 \\ -1/3 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2/3 \\ 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix},$$

giving us 5 examples of vectors which all map to $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

(g) By inspection we can see that

$$T\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Since T is linear, we may add any vector which map to $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, to $\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$, and the result will still be $\begin{pmatrix}1\\0\end{pmatrix}$. For example

$$T\begin{pmatrix}0\\4/3\\3\end{pmatrix} = T\left(\begin{pmatrix}0\\1\\2\end{pmatrix} + \begin{pmatrix}0\\1/3\\1\end{pmatrix}\right) = T\begin{pmatrix}0\\1\\2\end{pmatrix} + T\begin{pmatrix}0\\1/3\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}.$$

This example shows us that if T maps more than one vector to the zero vector, in this case $\begin{pmatrix}0\\0\end{pmatrix}$, then for any given output vector of T, there are multiple input vectors having the same output.

(h) It is easy to see that

$$T\begin{pmatrix}-1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}.$$

By what we saw in part (g), we know that there will be more than one (in fact infinitely many) vectors in $\mathbb{R}^3$ which also map to $\begin{pmatrix}0\\1\end{pmatrix}$.

(i) Yes, it is possible to obtain any vector in $\begin{pmatrix}x\\y\end{pmatrix} \in \mathbb{R}^2$ as the result of T applied to some vector in $\mathbb{R}^3$. In parts (g) and (h) we saw

$$T\begin{pmatrix}0\\1\\2\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}, \text{ and } T\begin{pmatrix}-1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}.$$

Now for any $x, y \in \mathbb{R}$, we can use the linearity of T to obtain

$$\begin{pmatrix}x\\y\end{pmatrix} = x\cdot\begin{pmatrix}1\\0\end{pmatrix} + y\cdot\begin{pmatrix}0\\1\end{pmatrix} = x\cdot T\begin{pmatrix}0\\1\\2\end{pmatrix} + y\cdot T\begin{pmatrix}-1\\0\\0\end{pmatrix} = T\begin{pmatrix}0\\x\\2x\end{pmatrix} + T\begin{pmatrix}-y\\0\\0\end{pmatrix} = T\begin{pmatrix}-y\\x\\2x\end{pmatrix}.$$

This shows that

$$T\begin{pmatrix}-y\\x\\2x\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix},$$

and since x and y are arbitrary, it follows that any vector in $\mathbb{R}^2$ may be obtained as an output vector from T.

We could have also solved for this directly:

$$T\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix} \quad\Rightarrow\quad \begin{pmatrix}3b-c\\-a\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix} \quad\Rightarrow\quad \begin{aligned}3b-c &= x\\ -a &= y\end{aligned}$$

Solving this we obtain solution set

$$\left\{ \begin{pmatrix} -y \\ x/3 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1/3 \\ 1 \end{pmatrix} \middle| \, t \in \mathbb{R} \right\},$$

which gives the set of all vectors in $\mathbb{R}^3$ which map to $\begin{pmatrix} a \\ b \end{pmatrix}$, for any given $x, y \in \mathbb{R}$; in particular we see that the system is consistent for any $a, b \in \mathbb{R}$. Letting $t = 2x$ (for example) yields the solution shown above.

$\square$

3. Let T be the mapping from $\mathbb{R}^3$ to $\mathbb{R}^2$ given by

$$T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ab \\ b \end{pmatrix}$$

(a) Apply T to each of the following vectors:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(b) Notice that $\begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$. Show that $T\begin{pmatrix} -1 \\ 1 \end{pmatrix} + T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = T\begin{pmatrix} 0 \\ 2 \end{pmatrix}$.

(c) Notice that $\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$. Show that $T\begin{pmatrix} 1 \\ 2 \end{pmatrix} + T\begin{pmatrix} 2 \\ 3 \end{pmatrix} \neq T\begin{pmatrix} 3 \\ 5 \end{pmatrix}$.

(d) Does T preserve vector addition?

(e) Does T preserve scalar multiplication? Either prove that it does, or give an example showing that it does not.

(f) Is T a linear map?

solution:

(a)

$$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad T\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \quad T\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad T\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(b) We have

$$T\begin{pmatrix} -1 \\ 1 \end{pmatrix} + T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix},$$

and

$$T\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

As we will see, the map T here does not preserve vector addition in general, and hence is not linear. This example demonstrates the importance of using arbitrary vectors, when proving linearity, since for these specific two vectors, the additive property does hold.

(c) We have

$$T\begin{pmatrix}1\\2\end{pmatrix} + T\begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix} + \begin{pmatrix}6\\3\end{pmatrix} = \begin{pmatrix}8\\5\end{pmatrix}$$

however,

$$T\begin{pmatrix}3\\5\end{pmatrix} = \begin{pmatrix}15\\5\end{pmatrix}.$$

(d) The example in part (c) shows that T fails to preserve vector addition.

(e)

$$2 \cdot T\begin{pmatrix}1\\2\end{pmatrix} = 2 \cdot \begin{pmatrix}2\\2\end{pmatrix} = \begin{pmatrix}4\\4\end{pmatrix},$$

whereas

$$T\left(2 \cdot \begin{pmatrix}1\\2\end{pmatrix}\right) = T\begin{pmatrix}2\\4\end{pmatrix} = \begin{pmatrix}8\\4\end{pmatrix}.$$

This shows that T fails to preserve scalar multiplication.

(f) Part (c) alone is sufficient to show that T is not linear, as it fails to preserve vector addition in general. (Part (e) alone would also be sufficient show that T is not linear.)

□

4. Determine whether each of the following maps are linear, and provide the appropriate justification in either case.

(a) $T : \mathbb{R}^2 \to \mathbb{R}^3$, where $T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\y\\x+2y\end{pmatrix}$.

(b) $T : \mathbb{R}^2 \to \mathbb{R}^2$, where $T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x^2\\y\end{pmatrix}$.

(c) $T : \mathbb{R}^3 \to \mathbb{R}^2$, where $T\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}xy\\yz\end{pmatrix}$.

(d) $T : \mathbb{R}^3 \to \mathbb{R}^2$, where $T\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x+y\\y+z\end{pmatrix}$.

(e) $T : \mathbb{R}^3 \to \mathbb{R}^2$, where $T\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$.

(f) $T : \mathbb{R}^3 \to \mathbb{R}^2$, where $T\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$.

(g) $T : \mathbb{R}^3 \to \mathbb{R}^2$, where $T\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2x\\y+1\end{pmatrix}$.

(h) $T : \mathbb{R}^3 \to \mathbb{R}^3$, where $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ y \end{pmatrix}$.

(i) $T : \mathcal{M}_{2\times 2}(\mathbb{R}) \to \mathbb{R}^3$, where $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \\ a+b+2c \\ d-a \end{pmatrix}$.

(j) $T : \mathcal{M}_{2\times 2}(\mathbb{R}) \to \mathcal{M}_{2\times 2}(\mathbb{R})$, where $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$.

(k) $T : \mathcal{M}_{2\times 2}(\mathbb{R}) \to \mathbb{R}$, where $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$.

(l) $T : \mathcal{M}_{2\times 2}(\mathbb{R}) \to \mathcal{M}_{2\times 2}(\mathbb{R})$, where $T(A) = A^{-1}$.

(m) $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_1(\mathbb{R})$, where $T(ax^2+bx+c) = 2ax+b$.

(n) $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$, where $T(ax^2+bx+c) = \frac{d}{dx}(ax^2+bx+c)$.

(o) $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R})$, where $T(ax^2+bx+c) = |a|x^2 - bx + c$.

solution:

(In some cases only the answer will be provided, justification either way will be similar to those examples where it is provided.)

(a) Let $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in \mathbb{R}^2$. Then

$$T\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) = T\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \end{pmatrix} = \begin{pmatrix} x_1+x_2 \\ y_1+y_2 \\ (x_1+x_2)+2(y_1+y_2) \end{pmatrix}$$

$$= \begin{pmatrix} x_1 \\ y_1 \\ x_1+2y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ x_2+2y_2 \end{pmatrix} = T\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + T\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}.$$

Thus T preserves vector addition. Let $r \in \mathbb{R}$. Then

$$T\left(r \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right) = T\begin{pmatrix} rx_1 \\ ry_1 \end{pmatrix} = \begin{pmatrix} rx_1 \\ ry_1 \\ rx_1+2(ry_1) \end{pmatrix} = r \cdot \begin{pmatrix} x_1 \\ y_1 \\ x_1+2y_1 \end{pmatrix} = r \cdot T\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}.$$

Thus T also preserves scalar multiplication, and therefore is a linear transformation.

(b) This map is not linear, for example

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} + T\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix},$$

whereas

$$T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}\right) = T\begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 9 \\ 0 \end{pmatrix}.$$

Since

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} + T\begin{pmatrix} 2 \\ 0 \end{pmatrix} \neq T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}\right),$$

this shows that T does not preserve vector addition, and hence is not a linear transformation.

We may also demonstrate that this map does not preserve scalar multiplication, for example

$$2 \cdot T\begin{pmatrix}1\\0\end{pmatrix} = 2 \cdot \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix},$$

whereas

$$T\left(2 \cdot \begin{pmatrix}1\\0\end{pmatrix}\right) = T\begin{pmatrix}2\\0\end{pmatrix} = \begin{pmatrix}4\\0\end{pmatrix},$$

thus

$$2 \cdot T\begin{pmatrix}1\\0\end{pmatrix} \neq T\left(2 \cdot \begin{pmatrix}1\\0\end{pmatrix}\right).$$

(c) Not linear.

(d) Linear.

(e) Linear.

(f) Not linear.

(g) Not linear.

(h) Linear.

(i) Let $\begin{pmatrix}a_1 & b_1\\c_1 & d_1\end{pmatrix}, \begin{pmatrix}a_2 & b_2\\c_2 & d_2\end{pmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{R})$. Then

$$T\begin{pmatrix}a_1 & b_1\\c_1 & d_1\end{pmatrix} + T\begin{pmatrix}a_2 & b_2\\c_2 & d_2\end{pmatrix} = \begin{pmatrix}a_1\\a_1+b_1+2c_1\\d_1-a_1\end{pmatrix} + \begin{pmatrix}a_2\\a_2+b_2+2c_2\\d_2-a_2\end{pmatrix}$$
$$= \begin{pmatrix}a_1+a_2\\(a_1+b_1+2c_1)+(a_2+b_2+2c_2)\\(d_1-a_1)+(d_2-a_2)\end{pmatrix}$$

and

$$T\left(\begin{pmatrix}a_1 & b_1\\c_1 & d_1\end{pmatrix} + \begin{pmatrix}a_2 & b_2\\c_2 & d_2\end{pmatrix}\right) = T\begin{pmatrix}a_1+a_2 & b_1+b_2\\c_1+c_2 & d_1+d_2\end{pmatrix} = \begin{pmatrix}a_1+a_2\\(a_1+a_2)+(b_1+b_2)+2(c_1+c_2)\\(d_1+d_2)-(a_1+a_2)\end{pmatrix}$$
$$= \begin{pmatrix}a_1+a_2\\(a_1+b_1+2c_1)+(a_2+b_2+2c_2)\\(d_1-a_1)+(d_2-a_2)\end{pmatrix}.$$

Thus T preserves vector addition. Let $r \in \mathbb{R}$. Then

$$r \cdot T\begin{pmatrix}a_1 & b_1\\c_1 & d_1\end{pmatrix} = r \cdot \begin{pmatrix}a_1\\a_1+b_1+2c_1\\d_1-a_1\end{pmatrix} = \begin{pmatrix}ra_1\\r(a_1+b_1+2c_1)\\r(d_1-a_1)\end{pmatrix} = \begin{pmatrix}ra_1\\ra_1+rb_1+2rc_1\\rd_1-ra_1\end{pmatrix}$$

and

$$T\left(r\cdot\begin{pmatrix}a_1 & b_1\\ c_1 & d_1\end{pmatrix}\right)=T\begin{pmatrix}ra_1 & rb_1\\ rc_1 & rd_1\end{pmatrix}=\begin{pmatrix}ra_1\\ ra_1+rb_1+2rc_1\\ rd_1-ra_1\end{pmatrix}.$$

Thus T also preserves scalar multiplication and therefore is a linear transformation.

(j) Linear.

(k) Not linear.

(l) Not linear.

(m) Let $a_1x^2+b_1x+c_1, a_2x^2+b_2x+c_2\in\mathcal{P}_2(\mathbb{R})\to\mathcal{P}_2(\mathbb{R})$. Then

$$T(a_1x^2+b_1x+c_1)+T(a_2x^2+b_2x+c_2)=(2a_1x+b_1)+(2a_2x+b_2)=2(a_1+a_2)x+b_1+b_2$$

and

$$\begin{aligned}T\big((a_1x^2+b_1x+c_1)+(a_2x^2+b_2x+c_2)\big)&=T((a_1+a_2)x^2+(b_1+b_2)x+c_1+c_2)\\&=2(a_1+a_2)x+b_1+b_2.\end{aligned}$$

Thus T preserves vector addition. Let $r\in\mathbb{R}$. Then

$$r\cdot T(a_1x^2+b_1x+c_1)=r\cdot(2a_1x+b_1)=2ra_1x+2b_1$$

and

$$T(r\cdot(a_1x^2+b_1x+c_1))=T(ra_1x^2+rb_1x+rc_1)=2ra_1x+2b_1.$$

Thus T also preserves scalar multiplication and therefore is a linear transformation.

(n) Linear.

(o) This map is not linear, for example

$$T(x^2)+T(-x^2)=x^2+x^2=2x^2,$$

but

$$T(x^2+(-x^2))=T(0)=0.$$

Thus T does not preserve vector addition, and hence is not linear.

□

5. Let $S:\mathbb{R}^2\to\mathbb{R}^2$ be a linear transformation with

$$S\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}2\\-5\end{pmatrix}\quad\text{and}\quad S\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}4\\4\end{pmatrix}.$$

(a) Find $S\begin{pmatrix}2\\3/2\end{pmatrix}$.

(b) Find a general rule for S; i.e. find the vector in $\mathbb{R}^2$ given by $S\begin{pmatrix}a\\b\end{pmatrix}$ for any $\begin{pmatrix}a\\b\end{pmatrix}\in\mathbb{R}^2$.

solution:

(a) Since

$$\begin{pmatrix} 2 \\ 3/2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{3}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

we have by the linearity of S that

$$S\begin{pmatrix} 2 \\ 3/2 \end{pmatrix} = 2S\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{3}{2}S\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 2 \\ -5 \end{pmatrix} + \frac{3}{2}\begin{pmatrix} 4 \\ 4 \end{pmatrix} = \begin{pmatrix} 10 \\ -4 \end{pmatrix}.$$

(b) Since

$$\begin{pmatrix} a \\ b \end{pmatrix} = a \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

we use the linearity of S to obtain

$$S\begin{pmatrix} a \\ b \end{pmatrix} = a \cdot S\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \cdot S\begin{pmatrix} 0 \\ 1 \end{pmatrix} = a \cdot \begin{pmatrix} 2 \\ -5 \end{pmatrix} + b \cdot \begin{pmatrix} 4 \\ 4 \end{pmatrix} = \begin{pmatrix} 2a+4b \\ -5a+4b \end{pmatrix}.$$

$\square$

6. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with

$$T\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad T\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 8 \end{pmatrix}.$$

(a) Find $T\begin{pmatrix} 1 \\ -4 \end{pmatrix}$.

(b) Find a general rule for T.

solution:

(a) Since

$$\begin{pmatrix} 1 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

we have by the linearity of T that

$$T\begin{pmatrix} 1 \\ -4 \end{pmatrix} = T\begin{pmatrix} 1 \\ -4 \end{pmatrix} - T\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 8 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ 9 \end{pmatrix}$$

(b) Begin by solving

$$\begin{pmatrix} a \\ b \end{pmatrix} = x \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix} + y \cdot \begin{pmatrix} 3 \\ -1 \end{pmatrix} \quad \Rightarrow \quad \begin{array}{rcl} 2x+3y & = & a \\ 3x-y & = & b \end{array} \quad \longrightarrow \quad \left(\begin{array}{cc|c} 2 & 3 & a \\ 3 & -1 & b \end{array}\right)$$

$$\xrightarrow{\frac{1}{2}R_1} \left(\begin{array}{cc|c} 1 & 3/2 & a/2 \\ 3 & -1 & b \end{array}\right) \xrightarrow{R_2-3R_1} \left(\begin{array}{cc|c} 1 & 3/2 & a/2 \\ 0 & -11/2 & b-3a/2 \end{array}\right)$$

$$\xrightarrow{-\frac{2}{11}R_2} \left(\begin{array}{cc|c} 1 & 3/2 & a/2 \\ 0 & 1 & 3a/11-2b/11 \end{array}\right) \xrightarrow{R_1-\frac{3}{2}R_2} \left(\begin{array}{cc|c} 1 & 0 & a/11+3b/11 \\ 0 & 1 & 3a/11-2b/11 \end{array}\right)$$

Thus $x = a/11 + 3b/11$ and $y = 3a/11 - 2b/11$. Using the linearity of T we obtain

$$T\begin{pmatrix} a \\ b \end{pmatrix} = x \cdot T\begin{pmatrix} 2 \\ 3 \end{pmatrix} + y \cdot T\begin{pmatrix} 3 \\ -1 \end{pmatrix} = (a/11 + 3b/11) \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} + (3a/11 - 2b/11) \cdot \begin{pmatrix} 8 \\ 8 \end{pmatrix}$$

$$= \begin{pmatrix} (25a - 13b)/11 \\ (23a - 19b)/11 \end{pmatrix}$$

□

7. Consider the maps,

$$T_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix} \quad \text{and} \quad T_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}.$$

(a) Show that T_1 and T_2 are both linear.

(b) Compute $(T_1 + T_2)\begin{pmatrix} 3 \\ -2 \end{pmatrix}$.

(c) Compute $(5T_1)\begin{pmatrix} 9 \\ 4 \end{pmatrix}$.

(d) Compute $(T_2 \circ T_1)\begin{pmatrix} 2 \\ 5 \end{pmatrix}$.

(e) Find a rule for the map $T_3 = T_1 + T_2$, and show that this map is linear.

(f) Find a rule for the map $T_4 = (5T_1)$, and show that this map is linear.

(g) Find a rule for the composite linear transformation $T_5 = T_2 \circ T_1$, and show that this map is linear.

(h) Does the reverse composition $T_1 \circ T_2$ make sense in this case? If so, is $T_1 \circ T_2 = T_2 \circ T_1$?

solution:

(a) Let $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in \mathbb{R}^2$. Then

$$T_1\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + T_1\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ -y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ -y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ -y_1 - y_2 \end{pmatrix}$$

and

$$T_1\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) = T_1\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ -(y_1 + y_2) \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ -y_1 - y_2 \end{pmatrix}.$$

We also have

$$T_2\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + T_2\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ x_1 \end{pmatrix} + \begin{pmatrix} y_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 + y_2 \\ x_1 + x_2 \end{pmatrix}$$

and

$$T_2\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) = T_2\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} = \begin{pmatrix} y_1 + y_2 \\ x_1 + x_2 \end{pmatrix}.$$

Thus T_1 and T_2 both preserve vector addition. Let $r \in \mathbb{R}$. Then

$$r \cdot T_1\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = r \cdot \begin{pmatrix} x_1 \\ -y_1 \end{pmatrix} = \begin{pmatrix} rx_1 \\ -ry_1 \end{pmatrix},$$

and

$$T_1\left(r\cdot\begin{pmatrix}x_1\\y_1\end{pmatrix}\right)=T_1\begin{pmatrix}rx_1\\ry_1\end{pmatrix}=\begin{pmatrix}rx_1\\-ry_1\end{pmatrix}.$$

We also have,

$$r\cdot T_2\begin{pmatrix}x_1\\y_1\end{pmatrix}=r\cdot\begin{pmatrix}y_1\\x_1\end{pmatrix}=\begin{pmatrix}ry_1\\rx_1\end{pmatrix},$$

and

$$T_2\left(r\cdot\begin{pmatrix}x_1\\y_1\end{pmatrix}\right)=T_2\begin{pmatrix}rx_1\\ry_1\end{pmatrix}=\begin{pmatrix}ry_1\\rx_1\end{pmatrix}.$$

Thus T_1 and T_2 both preserve scalar multiplication, hence they are linear transformations.

(b)

$$(T_1+T_2)\begin{pmatrix}3\\-2\end{pmatrix}=T_1\begin{pmatrix}3\\-2\end{pmatrix}+T_2\begin{pmatrix}3\\-2\end{pmatrix}=\begin{pmatrix}3\\2\end{pmatrix}+\begin{pmatrix}-2\\3\end{pmatrix}=\begin{pmatrix}1\\5\end{pmatrix}.$$

(c)

$$(5T_1)\begin{pmatrix}9\\4\end{pmatrix}=5\cdot T_1\begin{pmatrix}9\\4\end{pmatrix}=5\cdot\begin{pmatrix}9\\-4\end{pmatrix}=\begin{pmatrix}45\\-20\end{pmatrix}.$$

(d)

$$(T_2\circ T_1)\begin{pmatrix}2\\5\end{pmatrix}=T_2\left(T_1\begin{pmatrix}2\\5\end{pmatrix}\right)=T_2\begin{pmatrix}2\\-5\end{pmatrix}=\begin{pmatrix}-5\\2\end{pmatrix}.$$

(e)

$$T_3\begin{pmatrix}x\\y\end{pmatrix}=(T_1+T_2)\begin{pmatrix}x\\y\end{pmatrix}=T_1\begin{pmatrix}x\\y\end{pmatrix}+T_2\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x\\-y\end{pmatrix}+\begin{pmatrix}y\\x\end{pmatrix}=\begin{pmatrix}x+y\\x-y\end{pmatrix}.$$

(Showing that T_3 is linear is left as an exercise, but this follows immediately from the more general fact that any sum of linear maps is again a linear map.)

(f)

$$T_4\begin{pmatrix}x\\y\end{pmatrix}=(5T_1)\begin{pmatrix}x\\y\end{pmatrix}=5\cdot T_1\begin{pmatrix}x\\y\end{pmatrix}=5\cdot\begin{pmatrix}x\\-y\end{pmatrix}=\begin{pmatrix}5x\\-5y\end{pmatrix}.$$

(Showing that T_4 is linear is left as an exercise, but this follows immediately from the more general fact that the scalar multiple of a linear map is again a linear map.)

(g)

$$T_5\begin{pmatrix}x\\y\end{pmatrix}=(T_2\circ T_1)\begin{pmatrix}x\\y\end{pmatrix}=T_2\left(T_1\begin{pmatrix}x\\y\end{pmatrix}\right)=T_2\begin{pmatrix}x\\-y\end{pmatrix}=\begin{pmatrix}-y\\x\end{pmatrix}.$$

(Showing that T_5 is linear is left as an exercise, but this follows immediately from the more general fact that the composition of two linear maps is again a linear map.)

(h) The reverse composition $T_1\circ T_2$ makes sense in this case *(although generally this won't always be the case)*. This is because the codomain of T_2 is the same as the domain of T_1, and hence there maps ca be applies in succession. The rule for $T_1\circ T_2:\mathbb{R}^2\to\mathbb{R}^2$ is

$$(T_1\circ T_2)\begin{pmatrix}x\\y\end{pmatrix}=T_1\left(T_2\begin{pmatrix}x\\y\end{pmatrix}\right)=T_1\begin{pmatrix}y\\x\end{pmatrix}=\begin{pmatrix}y\\-x\end{pmatrix},$$

which we can see is not the same as the rule for $T_2 \circ T_1$. For example

$$(T_1 \circ T_2)\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}2\\-1\end{pmatrix}$$

whereas

$$(T_2 \circ T_1)\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}-2\\1\end{pmatrix}.$$

$\square$