Lines and Planes in $\mathbb{R}^2$ and $\mathbb{R}^3$

Equations of Lines:

For each of the lines described below, give the

(a) normal form,
(b) general form,
(c) vector form,
(d) parametric form,

equations.

1. The line through the point $(2, -1)$ with direction vector $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$.

   (a) We obtain a normal vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ for this line (by inspection) since $\begin{pmatrix} 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 2 \end{pmatrix} = 0$. The normal form is then
   $$\begin{pmatrix} 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

   (b) Compute the dot products in the normal form to get the general form
   $$2x + 3y = 1.$$

   (c) The vector form is
   $$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + t \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

   (d) Equating components in the vector form gives the parametric form
   $$x = 2 - 3t, \quad y = -1 + 2t.$$

2. The line which passes through the points $(2, 4)$ and $(1, 2)$.

   (a) We can start by finding a direction vector, obtained by subtracting components of these points: $\begin{pmatrix} 1-2 \\ 2-4 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$. A normal vector is then given by $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$, and so the normal form is
   $$\begin{pmatrix} 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

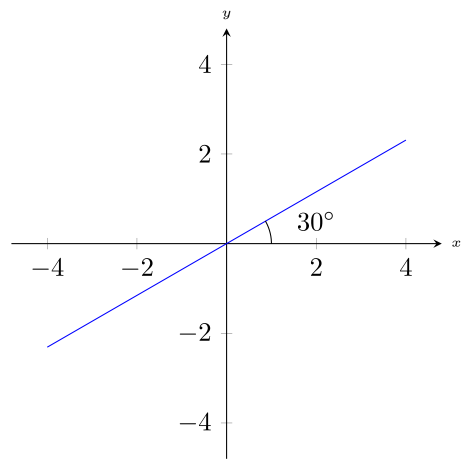
   (b) Compute the dot products in the normal form to get the general form
   $$2x - y = 0.$$

   (c) The vector form is
   $$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

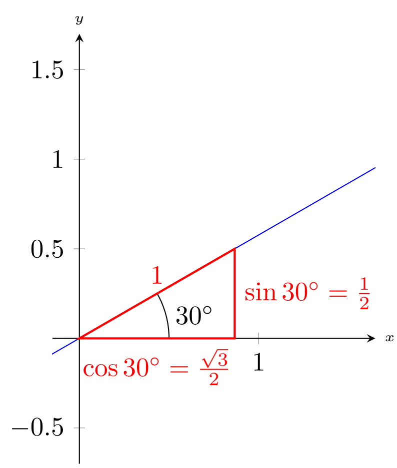
(d) The parametric form is

$$x = 1 - t, \quad y = 2 - 2t.$$

3. The line in $\mathbb{R}^2$ shown in this figure:



(a) We can find a direction vector, by taking the horizontal and vertical components of the right triangle shown, with hypotenuse 1:



Therefore $\begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix}$, or more simply $\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$, are direction vectors for this line. A normal vector is then given by $\begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix}$, and taking the origin as our point, the normal form is

$$\begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

(b) The general form

$$-x + \sqrt{3}y = 0.$$

(c) The vector form is

$$\begin{pmatrix} x \\ y \end{pmatrix} = t \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$$

(d) The parametric form is

$$x = \sqrt{3}t, \quad y = t.$$

4. The line $y = x$ in $\mathbb{R}^2$.

   (a) This line passes through the origin and has direction vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (for every unit we go in the x direction, we go one in the y direction). A normal vector is then given by $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, and so the normal form is
$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

   (b) The general form is given
$$y = x \quad \text{or} \quad x - y = 0.$$

   (c) The vector form is
$$\begin{pmatrix} x \\ y \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

   (d) The parametric form is
$$x = t, \quad y = t.$$

5. The line through the point $(4, 4)$ that is perpendicular to $\begin{pmatrix} 7 \\ 9 \end{pmatrix}$.

   (a) The normal form is
$$\begin{pmatrix} 7 \\ 9 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 9 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 4 \end{pmatrix}.$$

   (b) The general form is given
$$7x + 9y = 64.$$

   (c) The vector $\begin{pmatrix} 9 \\ -7 \end{pmatrix}$ is orthogonal to the given normal vector, and therefor is a direction vector for this line. The vector form is then
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} + t \begin{pmatrix} 9 \\ -7 \end{pmatrix}$$

   (d) The parametric form is
$$x = 4 + 9t, \quad y = 4 - 7t.$$

6. The line through the point $(12, 0, 1)$ with direction vector $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

   (a) To obtain the normal form for this line in $\mathbb{R}^3$, we will need two non-parallel vectors which are orthogonal to the given direction vector. These can be easily found by inspection, by taking one of the components to zero. For example
$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$
   are two such vectors. The normal form is the system of equations
$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 12 \\ 0 \\ 1 \end{pmatrix},$$
$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 12 \\ 0 \\ 1 \end{pmatrix}.$$

(b) The general form is given by the system

$$x + z = 13,$$

$$y + z = 1.$$

(c) The vector form is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

(d) The parametric form is

$$x = 12 + t, \quad y = t, \quad z = 1 - t$$

7. The line which passes through the points $(0, 4, 5)$ and $(1, 6 - 3)$.

(a) We can find a direction vector for this line by subtracting components of the given points:

$$\begin{pmatrix} 1-0 \\ 6-4 \\ -3-5 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -8 \end{pmatrix}$$

Then two non-parallel normal vectors are given by

$$\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 8 \\ 0 \\ 1 \end{pmatrix}.$$

The normal form is the system of equations

$$\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 4 \\ 5 \end{pmatrix},$$

$$\begin{pmatrix} 8 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 6 \\ -3 \end{pmatrix}.$$

(b) The general form is given by the system

$$2x - y = -4,$$

$$8x + z = 5.$$

(c) The vector form is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ -3 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ -8 \end{pmatrix}$$

(d) The parametric form is

$$x = 1 + t, \quad y = 6 + 2t, \quad z = -3 - 8t$$

8. The line $y = x$ and $z = 2$ in $\mathbb{R}^3$.

(a) We are given the general form system of equations for the line.

$$x - y = 0,$$

$$z = 2.$$

Noting that $(0,0,2)$ lies on the line, we can work backwards to see that the normal form is the system of equation

$$\begin{pmatrix}1\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\0\\2\end{pmatrix}$$

$$\begin{pmatrix}0\\0\\1\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}\cdot\begin{pmatrix}0\\0\\2\end{pmatrix}$$

(b) The general form is given by the system

$$x-y=0,$$

$$z=2.$$

(c) Solving the system of equations in the general form gives the vector form equation of the line:

$$x-y=0,$$

$$z=2.$$

The free variable is y, so the solution is

$$\left\{\begin{pmatrix}0\\0\\2\end{pmatrix}+t\begin{pmatrix}1\\1\\0\end{pmatrix}\middle|\, t\in\mathbb{R}\right\}.$$

The vector form is

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\2\end{pmatrix}+t\begin{pmatrix}1\\1\\0\end{pmatrix}$$

(d) The parametric form is

$$x=t,\quad y=t,\quad z=2$$

9. The line of intersection of the planes $7x-4y=5$ and $-x+4y+3z=2$.

(a) We are given the general form system of equations for the line.

$$7x-4y=5,$$

$$-x+4y+3z=2.$$

Solving this system of equations gives the vector form equation of the line:

$$\left(\begin{array}{ccc|c}7&-4&0&5\\-1&4&3&2\end{array}\right)\xrightarrow{R_1+7R_2}\left(\begin{array}{ccc|c}0&24&21&19\\-1&4&3&2\end{array}\right)\xrightarrow{\frac{1}{24}R_1}\left(\begin{array}{ccc|c}0&1&7/8&19/24\\-1&4&3&2\end{array}\right)$$

$$\xrightarrow{R_2-4R_1}\left(\begin{array}{ccc|c}0&1&7/8&19/24\\-1&0&-1/2&-7/6\end{array}\right)\xrightarrow{R_2-4R_1}\left(\begin{array}{ccc|c}0&1&7/8&19/24\\1&0&1/2&7/6\end{array}\right)\xrightarrow{R_1\leftrightarrow R_2}\left(\begin{array}{ccc|c}1&0&1/2&7/6\\0&1&7/8&19/24\end{array}\right)$$

Solution:

$$\left\{\begin{pmatrix}7/6\\19/24\\0\end{pmatrix}+t\begin{pmatrix}-1/2\\-7/8\\1\end{pmatrix}\middle|\, t\in\mathbb{R}\right\}$$

So the line passes through $(\frac{7}{6}, \frac{19}{24}, 0)$ and has direction vector $\begin{pmatrix} -1/2 \\ -7/8 \\ 1 \end{pmatrix}$. We can scale the direction vector by 8 to clear the fractions, $\begin{pmatrix} -4 \\ -7 \\ 8 \end{pmatrix}$. We can use this direction vector to find two non-parallel normal vectors as we did above, or we can simply take the coefficients on x, y, and z in the general form. It follows that the normal form is the system of equations

$$\begin{pmatrix} 7 \\ -4 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ -4 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 7/6 \\ 19/24 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 7/6 \\ 19/24 \\ 0 \end{pmatrix}$$

(b) The general form is

$$7x - 4y = 5,$$

$$-x + 4y + 3z = 2.$$

(c) The vector form is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7/6 \\ 19/24 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1/2 \\ -7/8 \\ 1 \end{pmatrix}$$

(d) The parametric form is

$$x = \frac{7}{6} - \frac{1}{2}t, \quad y = \frac{19}{24} - \frac{7}{8}t, \quad z = t$$

10. The line through the origin which is perpendicular to both $\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$.

(a) The normal form is the system of equations

$$\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

(b) The general form is

$$x - 3z = 0,$$

$$2x + 2y - z = 0.$$

(c) To find a direction vector for the this line we can take the cross product of the two given normal vectors

$$\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} \times \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 6 \\ -5 \\ 2 \end{pmatrix}.$$

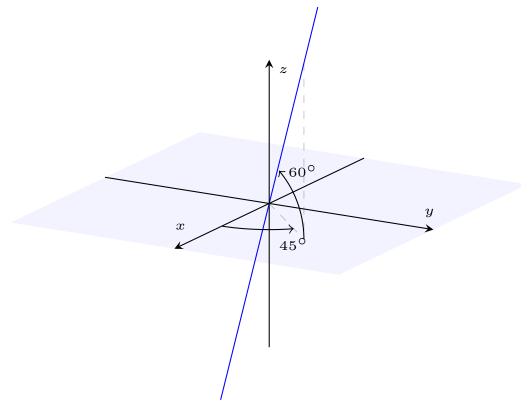
The vector form is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} 6 \\ -5 \\ 2 \end{pmatrix}$$

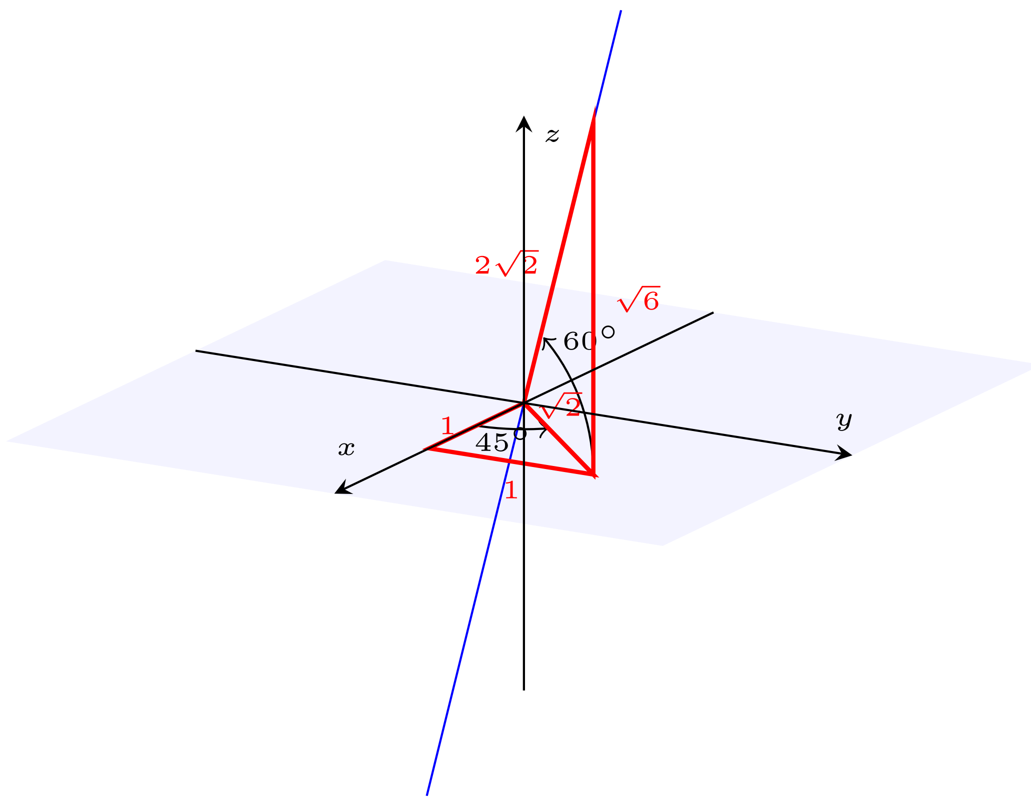
(d) The parametric form is

$$x = 6t, \quad y = -5t, \quad z = 2t$$

11. The (blue) line in $\mathbb{R}^3$ show in this figure:



(a) Using special triangles we can obtain another point lying on the line.



This shows that $(1, 1, \sqrt{6})$ is another point on the line, along with the origin. It follows that this line has direction vector $\begin{pmatrix} 1 \\ 1 \\ \sqrt{6} \end{pmatrix}$. Having this now we can find normal vectors

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \sqrt{6} \\ 0 \\ -1 \end{pmatrix}.$$

The normal form is the system of equations

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \sqrt{6} \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \sqrt{6} \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

(b) The general form is

$$x - y = 0,$$
$$\sqrt{6}x - z = 0.$$

(c) The vector form is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \\ \sqrt{6} \end{pmatrix}$$

(d) The parametric form is

$$x = t, \quad y = t, \quad z = \sqrt{6}t$$

Equations of Planes:

For each of the planes described below, give the

(a) normal form,
(b) general form,
(c) vector form,
(d) parametric form,

equations.

1. The plane through the point $(2, 2, 5)$ which is perpendicular to $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$.

   (a) The normal form is

$$\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix}$$

   (b) The general form is

$$2x + 2y - z = 3$$

   (c) The solution set of the general form gives the vector form. Taking y and z as free variables we have

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3/2 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix}$$

(d) The parametric form is

$$x = \frac{3}{2} - s - \frac{1}{2}t, \quad y = s, \quad z = t$$

2. The plane through $(-1, 0, 0)$ parallel to the vectors $\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}$.

   (a) The normal vector can be found by taking the cross product of these direction vectors:

   $$\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ -9 \\ -6 \end{pmatrix}.$$

   The normal form is

   $$\begin{pmatrix} 5 \\ -9 \\ -6 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -9 \\ -6 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

   (b) The general form is

   $$5x - 9y - 6z = -5$$

   (c) The vector form is

   $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}$$

   (d) The parametric form is

   $$x = -1 + 3s, \quad y = s - 2t, \quad z = s + 3t$$

3. The plane which passes through the points $(1, 2, 3)$, $(4, 5, 6)$, $(-3, 4, 5)$.

   (a) We start by finding two non-parallel direction vectors between any two pairs of points:

   $$\begin{pmatrix} 4-1 \\ 5-2 \\ 6-3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} -3-1 \\ 4-2 \\ 5-3 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \\ 2 \end{pmatrix}$$

   The normal vector can be found by taking the cross product of these direction vectors:

   $$\begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} \times \begin{pmatrix} -4 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -18 \\ 18 \end{pmatrix}.$$

   The normal form is

   $$\begin{pmatrix} 0 \\ -18 \\ 18 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -18 \\ 18 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

   (b) The general form is

   $$-18y + 18z = 18$$

   (c) The vector form is

   $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + s \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} + t \begin{pmatrix} -4 \\ 2 \\ 2 \end{pmatrix}$$

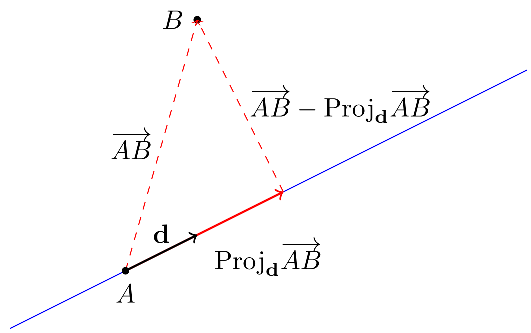
   (d) The parametric form is

   $$x = 1 + 3s - 4t, \quad y = 2 + 3s + 2t, \quad z = 3 + 3s + 2t$$

Distance from a point to a line or plane.

1. Find the distance from the point $(-2, 5)$ to the line with vector equation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$



In this case we are looking for $||\overrightarrow{AB} - \text{Proj}_{\mathbf{d}}\overrightarrow{AB}||$, where A is the point $(3, 0)$ on the line, B is the given point $(-2, 5)$, and $\mathbf{d}$ is the line's direction vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. We have

$$\overrightarrow{AB} = \begin{pmatrix} -2-3 \\ 5-0 \end{pmatrix} = \begin{pmatrix} -5 \\ 5 \end{pmatrix}$$

$$\text{Proj}_{\mathbf{d}}\overrightarrow{AB} = \frac{\overrightarrow{AB} \cdot \mathbf{d}}{||\mathbf{d}||^2}\mathbf{d} = \frac{0}{(\sqrt{2})^2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

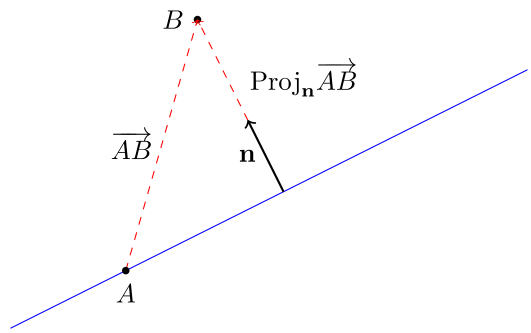
(note that $\overrightarrow{AB}$ is orthogonal to the line, so the projection is the zero vector) and hence

$$||\overrightarrow{AB} - \text{Proj}_{\mathbf{d}}\overrightarrow{AB}|| = ||\overrightarrow{AB}|| = \sqrt{50}$$

is the distance from the point B to the line.

2. Find the distance from the origin to the line

$$2x - 3y - 7 = 0.$$



In this setup the distance is calculated as $|\text{Proj}_{\mathbf{n}}\overrightarrow{AB}|$, since we given the general form $ax + by = c$, and hence a normal vector $\mathbf{n} = \begin{pmatrix} a \\ b \end{pmatrix}$. This can be calculated directly, but instead we will derive a convenient formula. If A is the point (x_1, x_2) and B is the point (x_0, y_0) then

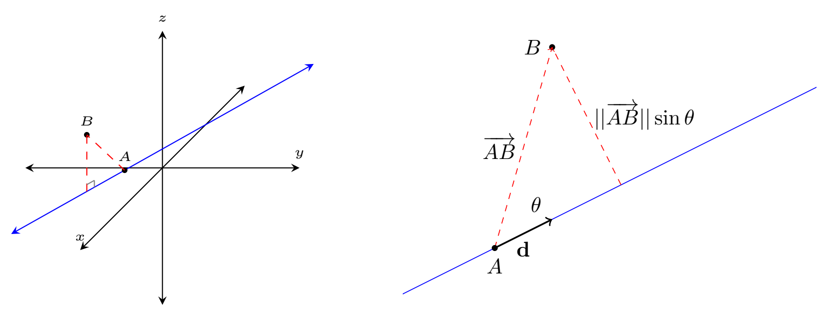
$$||\text{Proj}_{\mathbf{n}}\overrightarrow{AB}|| = \frac{|\overrightarrow{AB} \cdot \mathbf{n}|}{||\mathbf{n}||} = \frac{|a(x_0 - x_1) + b(y_0 - y_1)|}{\sqrt{a^2 + b^2}} = \frac{|ax_0 + by_0 - (ax_1 + by_1)|}{\sqrt{a^2 + b^2}} = \frac{|ax_0 + by_0 - c|}{\sqrt{a^2 + b^2}},$$

where $ax_1 + by_1 = c$ since (x_1, y_1) lies on the line (note that this eliminates the need for finding a point A). Applying this formula to our case, with line $2x - 3y = 7$ and point $B = (x_0, y_0) = (0, 0)$, we have

$$||\text{Proj}_{\mathbf{n}}\overrightarrow{AB}|| = \frac{|2(0) + (-3)(0) - 7|}{\sqrt{2^2 + (-3)^2}} = \frac{7}{\sqrt{13}}.$$

3. Find the distance from the point $(-1, 1, 1)$ to the line with direction vector equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} + t \begin{pmatrix} 6 \\ 7 \\ -1 \end{pmatrix}.$$



In this set up we could calculate $||\overrightarrow{AB} - \text{Proj}_{\mathbf{d}}\overrightarrow{AB}||$ (as was done above), but instead let's try a different approach. The distance is given by $||\overrightarrow{AB}|| \sin\theta$, and we have that

$$||\overrightarrow{AB} \times \mathbf{d}|| = ||\overrightarrow{AB}||\ ||\mathbf{d}|| \sin\theta$$

therefore the distance can be found by

$$||\overrightarrow{AB}|| \sin\theta = \frac{||\overrightarrow{AB} \times \mathbf{d}||}{||\mathbf{d}||}.$$

Here we have given $B = (-1, 1, 1)$, $A = (-1, 2, 4)$ and $\mathbf{d} = \begin{pmatrix} 6 \\ 7 \\ -1 \end{pmatrix}$. So

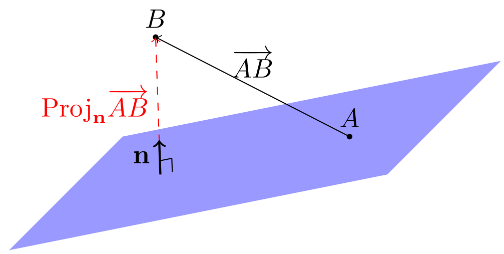
$$\overrightarrow{AB} = \begin{pmatrix} 0 \\ -1 \\ -3 \end{pmatrix} \quad \text{and} \quad \overrightarrow{AB} \times \mathbf{d} = \begin{pmatrix} 0 \\ -1 \\ -3 \end{pmatrix} \times \begin{pmatrix} 6 \\ 7 \\ -1 \end{pmatrix} = \begin{pmatrix} 22 \\ -18 \\ 6 \end{pmatrix}$$

Therefore the distance is

$$\frac{||\overrightarrow{AB} \times \mathbf{d}||}{||\mathbf{d}||} = \frac{\sqrt{22^2 + (-18)^2 + 6^2}}{\sqrt{6^2 + 7^2 + (-1)^2}} = \frac{2\sqrt{211}}{\sqrt{86}} \approx 3.1327.$$

4. Find the distance from the point $(-1, 5, 6)$ to the plane

$$3x - 4y + 10z = 5.$$



A plane with general form $ax + by + cz = d$ has normal vector $\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. If $A = (x_1, y_1, z_1)$ lies on the plane and $B = (x_0, y_0, z_0)$ is a given point, then the distance from B to the plane is $||\mathrm{Proj}_{\mathbf{n}}\overrightarrow{AB}||$. Similar to the derivation above for the distance between a point and a line, we have that

$$||\mathrm{Proj}_{\mathbf{n}}\overrightarrow{AB}|| = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}.$$

Here we have plane $3x - 4y + 10 = 5$ and point $B = (-1, 5, 6)$, so the distance is

$$\frac{|3(-1) + (-4)(5) + 10(6) - 5|}{\sqrt{3^2 + (-4)^2 + 10^2}} = \frac{32}{5\sqrt{5}} \approx 2.8622.$$

5. Find the distance from the point $(2, 4, 4)$ to the plane

$$6x - 5y + z = 4.$$

The distance is given by

$$\frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|6(2) + (-5)(4) + (4) - 4|}{\sqrt{6^2 + (-5)^2 + 1^2}} = \frac{8}{\sqrt{62}} \approx 1.0160.$$

6. Find the distance from the point $(0, 3, 6)$ to the plane through the origin with normal vector $\begin{pmatrix} 1 \\ 2 \\ -9 \end{pmatrix}$.

This plane has general form $x + 2y - 9z = 0$, and so the distance is given by

$$\frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|1(0) + 2(3) + (-9)(6) - 0|}{\sqrt{1^2 + 2^2 + (-9)^2}} = \frac{48}{\sqrt{86}} \approx 5.1760.$$

7. Find the distance from the point $(2, -10, 1)$ to the plane through the origin with vector equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} -3 \\ 2 \\ 8 \end{pmatrix}.$$

A normal vector is given by

$$\begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} -3 \\ 2 \\ 8 \end{pmatrix} = \begin{pmatrix} 6 \\ -43 \\ 13 \end{pmatrix},$$

and $(-1, 0, 0)$ is a point on the plane, so it has general form

$$6x - 43y + 13z = -6.$$

The distance is from $(2, -10, 1)$ to the plane is

$$\frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|6(2) + (-43)(-10) + 13(1) - 6|}{\sqrt{1^2 + 2^2 + (-9)^2}} = \frac{449}{\sqrt{14}} \approx 120.0003.$$