**MATH 1350** **Name (Print):** ______________________
**Winter 2025**
**Test 4**
**2025-03-18**
**Time Limit: 50 Minutes** **ID number:** ______________________

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You are required to **show your work** on each problem on this test.

1. (2 points) Find the general equation of a line in $\mathbb{R}^2$ which passes through the point $(-1, 5)$ and has normal vector $\mathbf{n} = \begin{pmatrix} 5 \\ -4 \end{pmatrix}$.

   The normal equation for the line, $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$ is

$$\begin{pmatrix} 5 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 5 \end{pmatrix}$$

   Computing the dot product on both sides give the general equation of the line:

$$5x - 4y = -25.$$

2. (2 points) Give both the vector form, and parametric equations for the plane in $\mathbb{R}^3$ which passes through the point $(-2, 0, 1)$ and is parallel to the vectors $\begin{pmatrix} 7 \\ -6 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}$.

   The vector equation for the plane is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} 7 \\ -6 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}, \quad s, t \in \mathbb{R}$$

   Equating components give the parametric equations:

$$x = -2 + 7s, \quad y = -6s + 2t, \quad z = 1 + s - 3t, \quad s, t \in \mathbb{R}$$

3. (a) (2 points) Consider the subset $S_1$ of $\mathbb{R}^3$ given by

$$S_1 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, xyz = 0 \right\}.$$

Show that $S_1$ is not closed under vector addition.

The vectors $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ both belong to $S_1$, however their sum $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ does not. Therefore $S_1$ is not closed under addition.

(b) (3 points) Determine whether the following subset is a subspace of $\mathbb{R}^3$:

$$S_2 = \left\{ \begin{pmatrix} s - 6t \\ t \\ s \end{pmatrix} \,\middle|\, s, t \in \mathbb{R} \right\}.$$

(fully justify your answer)

Let $\mathbf{u} = \begin{pmatrix} a - 6b \\ b \\ a \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} c - 6d \\ d \\ c \end{pmatrix}$ be vectors in $S_2$. Then

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} (a - 6b) + (c - 6d) \\ b + d \\ a + c \end{pmatrix} = \begin{pmatrix} a + c - 6(b + d) \\ b + d \\ a + c \end{pmatrix},$$

which also belongs to $S_2$. Hence $S_2$ is closed under addition.

Now let $r \in \mathbb{R}$. Then

$$r \cdot \mathbf{u} = \begin{pmatrix} r(a - 6b) \\ rb \\ ra \end{pmatrix} = \begin{pmatrix} ra - 6rb \\ rb \\ ra \end{pmatrix},$$

which also belongs to $S_2$. Hence $S_2$ is closed under scalar multiplication. Therefore (since $S_2 \neq \emptyset$) by the subspace criterion $S_2$ is a subspace of $\mathbb{R}^3$.

4. (4 points) Determine whether $\mathbf{v} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$ belongs to span$(W)$ where $W = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix} \right\}$.
If it does, express $\mathbf{v}$ as a linear combination of those three vectors, if not then show why.

Our aim to to determine whether there exist scalars $C_1, C_2, C_3 \in \mathbb{R}$ such that

$$C_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + C_2 \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix} + C_3 \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}.$$

Solving this system:

$$\left( \begin{array}{ccc|c} 1 & 4 & 0 & 3 \\ 0 & 2 & 1 & 1 \\ -1 & 6 & 5 & 2 \end{array} \right) \xrightarrow{R_3 + R_1} \left( \begin{array}{ccc|c} 1 & 4 & 0 & 3 \\ 0 & 2 & 1 & 1 \\ 0 & 10 & 5 & 5 \end{array} \right) \xrightarrow{R_3 - 5R_2} \left( \begin{array}{ccc|c} 1 & 4 & 0 & 3 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

(at this point we see that a solution does exist since the system is consistent, we continue to solve for $C_1, C_2, C_3$)

$$\xrightarrow{R_1 - 2R_2} \left( \begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right) \xrightarrow{\frac{1}{2}R_2} \left( \begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$\text{Solution:} \quad \left\{ \begin{pmatrix} 1 \\ 1/2 \\ 0 \end{pmatrix} + C_3 \begin{pmatrix} 1 \\ -1/2 \\ 1 \end{pmatrix} \middle| \, C_3 \in \mathbb{R} \right\}$$

Letting $C_3 = 0$, for example, gives $C_1 = 1, C_2 = \frac{1}{2}$, and so

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix} + (0) \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}.$$

5. (4 points) Determine whether the following set of vectors is linearly independent or dependent. (justify your answer)

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix}, \begin{pmatrix} -3 \\ -5 \\ 1 \end{pmatrix} \right\}.$$

Our aim to to determine whether there exist scalars $C_1, C_2, C_3 \in \mathbb{R}$, not all zero, such that

$$C_1 \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + C_2 \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix} + C_3 \begin{pmatrix} -3 \\ -5 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Solving this system:

$$\left(\begin{array}{ccc|c} 1 & 3 & -3 & 0 \\ 0 & 2 & -5 & 0 \\ -2 & 4 & 1 & 0 \end{array}\right) \xrightarrow{R_3+2R_1} \left(\begin{array}{ccc|c} 1 & 3 & -3 & 0 \\ 0 & 2 & -5 & 0 \\ 0 & 10 & -5 & 0 \end{array}\right) \xrightarrow{R_3-5R_2} \left(\begin{array}{ccc|c} 1 & 3 & -3 & 0 \\ 0 & 2 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Since there are free variables in the system, there will be nonzero solutions. Therefore this set is linear dependent.

*Extra:*

If we continue to solve we can find an explicit dependence relation among these vectors

$$\xrightarrow{\frac{1}{2}R_2} \left(\begin{array}{ccc|c} 1 & 3 & -3 & 0 \\ 0 & 1 & -5/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1-3R_2} \left(\begin{array}{ccc|c} 1 & 0 & 9/2 & 0 \\ 0 & 1 & -5/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\text{Solution:} \qquad \left\{ +C_3 \begin{pmatrix} -9/2 \\ 5/2 \\ 1 \end{pmatrix} \middle| \, C_3 \in \mathbb{R} \right\}$$

Letting $C_3 = 2$, for example, gives $C_1 = -9, C_2 = 5$, and so

$$(-9) \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + 5 \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix} + 2 \begin{pmatrix} -3 \\ -5 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

*Alternative:*

By the determinant test (using Laplace expansion along column 1):

$$\begin{vmatrix} 1 & 3 & -3 \\ 0 & 2 & -5 \\ -2 & 4 & 1 \end{vmatrix} = \begin{vmatrix} 2 & -5 \\ -4 & 1 \end{vmatrix} - 2 \begin{vmatrix} 3 & -3 \\ 2 & -5 \end{vmatrix} = -18 - 2(-9) = 0.$$

Since the determinant of this matrix is zero, its columns form a linearly dependent set.