MATH 1350,
Exercise Set 9 - Solutions

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1. (a) What is the kernel (also called null space) of a linear transformation?

   (b) How can we check if a given vector $\mathbf{v}$ belongs to the kernel of a linear transformation T?

   (c) Explain how to find a basis for the kernel of a linear transformation.

   (d) What is the range (also called the image) of a linear transformation. What is the difference between the range and the codomain?

   (e) How can we find the range of a linear transformation?

   (f) Are either the kernel or the range vector spaces? Are they subspaces? If they are subspaces, to which vector spaces are they subspaces?

   (g) What terms do we use for the dimensions of the kernel and range respectively?

   (h) What does it mean if a linear transformation is injective (also called one-to-one)?

   (i) How can we show that a linear transformation is injective, or conversely that it is not injective?

   (j) What does it mean if a linear transformation is surjective (also called onto)?

   (k) How can we show that a linear transformation is surjective, or conversely that it is not surjective?

   (l) What does it mean if a linear transformation is bijective?

   (m) How do we relate the kernel and range of a linear transformation T to its matrix representation A?

   (n) Let $T : V \to W$ be a linear transformation where V is a finite dimensional vector space. State the Rank-Nullity Theorem for T.

   solution:

   (a) The kernel, or null space, of a linear transformation is the set of all vectors in the domain which map to the additive identity, or zero vector, of the codomain.

   (b) If $T(\mathbf{v})$ is the zero vector then $\mathbf{v}$ belongs to the kernel of T, otherwise it does not.

   (c) Set up the equation $T(\mathbf{v}) = \mathbf{0}$ and solve the resulting system for all $\mathbf{v}$. We can do this by finding the null space of the standard matrix for T.

   (d) The range of a linear map $T : V \to W$ is the set of all output vectors in W that result from applying T to all vectors in V. More explicitly

   $$\text{Im } T = \{T(v) | v \in V\} = \{w \in W | w = T(v) \text{ for some } v \in V\}.$$

   The range of T is a subset of the codomain, and not necessarily equal to the entire codomain. The codomain is the vector space where the output vectors generally lie.

   (e) Given a basis $\{v_1, \dots, v_n\}$ for the domain V of T, the range of T is the span of T applied to all basis vectors, i.e.

   $$\text{Im } T = \text{span}\{T(v_1), \dots, T(v_n)\}.$$

   If $T : \mathbb{R}^n \to \mathbb{R}^m$, then the range of T is the column space of the standard matrix for T.

(f) The kernel of $T : V \to W$ is a subspace (and hence a vector space) of the domain V, and the range of T is subspace (and hence a vector space) of the codomain W.

(g) The dimension of the kernel is called the nullity, and the dimension of the range is called the rank.

(h) A map T is injective if $T(v_1) \neq T(v_2)$ whenever $v_1 \neq v_2$, or equivalently if $T(v_1) = T(v_2)$ implies $v_1 = v_2$.

(i) Sometimes we may demonstrate injectivity from the definition (given above), however we typically determine injectivity by looking at the kernel. A linear map T is injective if and only if $\ker T = \{\mathbf{0}\}$; i.e. if the only vector which maps to the zero vector of the codomain, is the zero vector of the domain.

To show that a map is not injective, we can simply provide an example of two different vectors in the domain which map to the same vector in the codomain, or show that the kernel contains nonzero vectors (in which case we say that the kernel in *nontrivial*).

(j) A map is surjective when its range equals its codomain; i.e. each vector in the codomain occurs at the image of some vector in the domain under T.

(k) We may solve for the range of T directly by applying T to any basis for the domain and considering its span. A matrix representation can also be used to find the range. In particular if T maps from $\mathbb{R}^n$ to $\mathbb{R}^m$, then the column space of the standard matrix for T is the range of T. The rank tells us whether or not T is surjective; T is surjective if and only if the rank of T equals the dimension of the codomain. The rank of T can be found from the standard matrix of T, or we can find it using the Rank-Nullity Theorem, if we already know the nullity of T.

Conversely, to show that T is not surjective we can give an example of a vector in the codomain which has no pre-image in the domain under T; i.e. find a vector in the codomain which never occurs as the output of T. If the rank of T does not equal the dimension of the codomain, then T is not surjective.

(l) A linear transformation is bijective if it is both injective and surjective.

(m) The kernel of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is the null space of its standard matrix representation A, and the range of T is the column space of A.

(n) The Rank-Nullity Theorem in this context is $\dim V = \dim \operatorname{Im} T + \dim \ker T$.

□

2. For each linear transformation T below:

(i) Find $\ker T$ and give a basis for $\ker T$.

(ii) Find $\operatorname{Im} T$ (or range$T$), and give a basis for $\operatorname{Im} T$.

(iii) Verify that the Rank-Nullity Theorem for T is true.

(iv) Is T injective, surjective, bijective or neither?

(v) If T is not injective, give an example which demonstrates this.

(vi) If T is not surjective, find a vector which does not lie in the range of T.

(a) $T : \mathbb{R}^2 \to \mathbb{R}^3$, where $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \\ x+2y \end{pmatrix}$.

(b) $T : \mathbb{R}^3 \to \mathbb{R}^2$, where $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y \\ y+z \end{pmatrix}$.

(c) $T : \mathbb{R}^3 \to \mathbb{R}^3$, where $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-2z \\ 3x+y-2z \\ -5x-y+9z \end{pmatrix}$.

(d) $T : \mathbb{R}^3 \to \mathbb{R}^2$, where $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

(e) $T : \mathbb{R}^3 \to \mathbb{R}^3$, where $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ y \end{pmatrix}$.

(f) $T : \mathcal{M}_{2\times 2}(\mathbb{R}) \to \mathbb{R}^3$, where $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \\ a+b+2c \\ d-a \end{pmatrix}$.

(g) $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_1(\mathbb{R})$, where $T(ax^2+bx+c) = 2ax+b$.

solution:

(a) (i)

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} x \\ y \\ x+2y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad x=0, y=0$$

Thus

$$\ker T = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

(ii) The standard matrix for T is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix}$. Its column space, and hence the range of T is

$$\text{Im } T = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}.$$

Note that

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}$$

is a basis for the range of T.

(iii) We have seen that the nullity of T is 0 (dimension of $\ker T$),the rank of T is 2, and the dimension of the domain is $2 = \dim \mathbb{R}^2$. Thus $\dim \mathbb{R}^2 = \dim \text{Im } T + \dim \ker T$.

(iv) Since the nullity of T is 0, it follows that T is injective. Since the rank of T is 2 and not equal to the dimension of the codomain $\mathbb{R}^3$, we see that T is not surjective.

(v) In this case T is injective.

(vi) The vector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ does not lie in the range of T, since the system

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{array}\right)$$

is inconsistent (row reduce to make this obvious).

(b) (i) A basis for $\ker T$ is

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

(ii) A basis for ImT is

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

It follows that $\text{Im } T = \mathbb{R}^2$.

(iii) We have seen that the nullity of T is 1 ,the rank of T is 2, and the dimension of the domain is $3 = \dim \mathbb{R}^3$. Thus $\dim \mathbb{R}^3 = \dim \text{Im } T + \dim \ker T$.

(iv) Since T has nontrivial kernel (i.e. the kernel is not only the zero vector) it is not injective. We have seen that T is surjective since its range equals the codomain.

(v) Note that $T \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $T \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

(vi) In this case T is surjective.

(c) (i)

$$\ker T = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

(ii) Since T has nullity 0, it follows from the Rank-Nullity Theorem that $\dim \text{Im } T = \dim \mathbb{R}^3 - \dim \ker T = 3 - 0 = 3$. Since the codomain $\mathbb{R}^3$ is of dimension 3 this implies $\text{Im } T = \mathbb{R}^3$. We can take the standard basis for example as a basis for Im T.

(iii) We have used the Rank-Nullity Theorem to find the rank in part (ii). To find the rank more directly we can row reduce the standard matrix for T,

$$\begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{pmatrix}$$

and count the number of leading 1's (this matrix reduces to the identity matrix). Thus $\dim \mathbb{R}^3 = \dim \text{Im } T + \dim \ker T$.

(iv) Since T has trivial kernel, and T is surjective, wit follows that that T is bijective.

(v) In this case T is injective.

(vi) In this case T is surjective.

(d) (i) It is easy to see that $\ker T = \mathbb{R}^3$, as everything in the domain maps to the zero vector.

(ii)

$$\operatorname{Im} T = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

(iii) As we have seen, $\dim \mathbb{R}^3 = 3 = 0 + 3 = \dim \operatorname{Im} T + \dim \ker T$.

(iv) In this case T is neither injective nor surjective.

(v) Clearly T maps more than one vector to $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

(vi) No nonzero vectors lie in the range of T, e.g. $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

(e) (i) A basis for $\ker T$ is

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

(ii) We see that

$$\operatorname{Im} T = \left\{ \begin{pmatrix} x \\ 0 \\ y \end{pmatrix} \middle| x, y \in \mathbb{R} \right\} = \left\{ x \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \middle| x, y \in \mathbb{R} \right\},$$

and hence the range of T has basis

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

(iii) We have seen that the nullity of T is 1 ,the rank of T is 2, and the dimension of the domain is $3 = \dim \mathbb{R}^3$. Thus $\dim \mathbb{R}^3 = \dim \operatorname{Im} T + \dim \ker T$.

(iv) This map T is neither injective nor surjective.

(v) Note that $T \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = T \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

(vi) Clearly $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ does not lie in the range of T.

(f) (i) A basis for $\ker T$ is

$$\left\{ \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix} \right\}.$$

(ii) We see that

$$\operatorname{Im} T = \left\{ \begin{pmatrix} a \\ a+b+2c \\ d-a \end{pmatrix} \middle| a, b, c, d \in \mathbb{R} \right\}$$

$$= \left\{ a \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + b \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c \cdot \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + d \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \middle| a, b, c, d \in \mathbb{R} \right\}.$$

Thus

$$\left\{\begin{pmatrix}1\\1\\-1\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\2\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\},$$

is a spanning set for the range of T, however it is easy to notice that these vectors are not linearly independent; *we can tell this immediately since there are 4 vectors which belong to* $\mathbb{R}^3$*, or for example that the third vector is just twice the second vector, but generally given any spanning set we will need to use our checks to determine whether the set will be linearly independent.* To cast out redundant vectors we use the row reduction method:

$$\begin{pmatrix}1&0&0&0\\1&1&2&0\\-1&0&0&1\end{pmatrix}\xrightarrow[R_3+R_1]{R_2-R_1}\begin{pmatrix}1&0&0&0\\0&1&2&0\\0&0&0&1\end{pmatrix}$$

Since leading 1's appear in columns 1,2 and 4 it follows that a basis for the range is,

$$\left\{\begin{pmatrix}1\\1\\-1\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}.$$

In this case we see that the range of T equals the codomain $\mathbb{R}^3$, and so any known basis for$\mathbb{R}^3$ would work here, including the standard basis. This could also have been obtained indirectly by the Rank-Nullity Theorem, and knowing that the nullity of T is 1.

(iii) We have seen that the nullity of T is 1 ,the rank of T is 3, and the dimension of the domain is $4 = \dim \mathcal{M}_{2\times 2}(\mathbb{R})$. Thus $\dim \mathcal{M}_{2\times 2}(\mathbb{R}) = \dim \mathrm{Im}\ T + \dim \ker T$.

(iv) Since T has nontrivial kernel, T is not injective. We have seen that T is surjective.

(v) Note that $T\begin{pmatrix}0&0\\0&0\end{pmatrix} = T\begin{pmatrix}0&-2\\1&0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$.

(vi) In this case T is surjective.

(g) (i) Note that T is the familiar derivative map defined on $\mathcal{P}_2(\mathbb{R})$. We can see that the polynomials which map to 0 are precisely the constant polynomials. Thus a basis for $\ker T$ is

$$\{1\}.$$

(ii) We see that

$$\mathrm{Im}T = \{2ax + b | a, b \in \mathbb{R}\} = \mathrm{span}\{2x, 1\} = \mathrm{span}\{x, 1\}.$$

Since x and 1 are not scalar multiples of each other, a basis for the range of T is

$$\{x, 1\}.$$

(iii) We have seen that the nullity of T is 1 ,the rank of T is 2, and the dimension of the domain is $3 = \dim \mathcal{P}_2(\mathbb{R})$. Thus $\dim \mathcal{P}_2(\mathbb{R}) = \dim \mathrm{Im}\ T + \dim \ker T$.

(iv) Since T has nontrivial kernel T is not injective, however, since the codomain of T is $\mathcal{P}_1(\mathbb{R})$ (the polynomials of degree at most 1) we see that T is surjective.

(v) Note that $T(0) = T(1) = 0$.

(vi) In this case T is surjective.

$\square$

3. For each linear transformation T below:

   (i) Find the standard matrix A for T.

   (ii) Use A to find a basis for the range of T, and the rank of T.

   (iii) Use A to find a basis for the kernel of T, and the nullity of T.

   (iv) Deduce whether T injective, surjective, bijective or neither?

   (a) $T : \mathbb{R}^4 \to \mathbb{R}^3$, where $T\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a+b+2c+d \\ 2a+c+2d \\ 2a+b+3c+4d \end{pmatrix}$.

   (b) $T : \mathbb{R}^3 \to \mathbb{R}^3$, where $T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a+2b-c \\ 2a+4b+6c \\ -8c \end{pmatrix}$.

   (c) $T : \mathbb{R}^3 \to \mathbb{R}^3$, where $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-2z \\ 3x+y-2z \\ -5x-y+9z \end{pmatrix}$.

   (d) $T : \mathbb{R}^4 \to \mathbb{R}^4$, where $T\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a+2b-c+4d \\ 3a+b+2c-d \\ -4a-3b-c-3d \\ -a-2b+c+d \end{pmatrix}$.

   (e) $T : \mathbb{R}^5 \to \mathbb{R}^4$, where $T\begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} a+3b+d-e \\ 2a+b+3c+d \\ -a-2c+e \\ 2d+8e \end{pmatrix}$.

solution:

(a) (i)

$$A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & 0 & 1 & 2 \\ 2 & 1 & 3 & 4 \end{pmatrix}$$

(ii) We row reduce A to find a basis for its column space, which is a basis for the range of T and determines the rank of T. For this we only need to reduce to REF to locate the leading variables, however, we will reduce to RREF in order to solve for the kernel of T which we will need next.

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & 0 & 1 & 2 \\ 2 & 1 & 3 & 4 \end{pmatrix} \xrightarrow[R_3-2R_1]{R_2-2R_1} \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -2 & -3 & 0 \\ 0 & -1 & -1 & 2 \end{pmatrix} \xrightarrow{-\frac{1}{2}R_2} \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 3/2 & 0 \\ 0 & -1 & -1 & 2 \end{pmatrix}$$

$$\xrightarrow[R_3+R_2]{R_1-R_2} \begin{pmatrix} 1 & 0 & 1/2 & 1 \\ 0 & 1 & 3/2 & 0 \\ 0 & 0 & 1/2 & 2 \end{pmatrix} \xrightarrow{2R_3} \begin{pmatrix} 1 & 0 & 1/2 & 1 \\ 0 & 1 & 3/2 & 0 \\ 0 & 0 & 1 & 4 \end{pmatrix} \xrightarrow[R_2-\frac{3}{2}R_3]{R_1-\frac{1}{2}R_3} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

Since leading 1's appear in columns 1,2 and 3 of the RREF, columns 1,2, and 3 are guaranteed to form a basis for the column space of A. However, since these 3 linearly independent vectors are in $\mathbb{R}^3$, they must form a basis for $\mathbb{R}^3$, and it follows that the range of T is $\mathbb{R}^3$. Thus

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \right\}$$

is a basis for the range of T (of course any basis for $\mathbb{R}^3$, including the standard basis would work). The rank of T is 3.

(iii) To find the kernel of T we solve for the null space of A,

$$\left(\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 0 \\ 2 & 0 & 1 & 2 & 0 \\ 2 & 1 & 3 & 4 & 0 \end{array} \right),$$

however, the row reduction is already done in part (ii), and the right hand side are all 0's so they are unaffected by the row operations. Thus the kernel of A is the solution set to

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -6 & 0 \\ 0 & 0 & 1 & 4 & 0 \end{array} \right),$$

which is

$$\left\{ t \begin{pmatrix} 1 \\ 6 \\ -4 \\ 1 \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\}.$$

Hence the nullity of T is 1 and the kernel of T has basis

$$\left\{ \begin{pmatrix} 1 \\ 6 \\ -4 \\ 1 \end{pmatrix} \right\}.$$

(iv) In this case T is not injective since it has nontrivial kernel, and T is surjective since it's rank equals the dimension of the codomain.

(b) (i)

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & 6 \\ 0 & 0 & -8 \end{pmatrix}$$

(ii) In this case T has rank 2, and a basis for the range of T is

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 6 \\ -8 \end{pmatrix} \right\}.$$

(iii) A basis for the kernel of T is

$$\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

(iv) In this case T is neither injective nor surjective.

(c) (i)

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{pmatrix}$$

(ii) In this case T has rank 3, and is therefore onto thus

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is a basis for the range of T.

(iii) The kernel of T is

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

(iv) In this case T is bijective.

(d) (i)

$$A = \begin{pmatrix} 1 & 2 & -1 & 4 \\ 3 & 1 & 2 & -1 \\ -4 & -3 & -1 & -3 \\ -1 & -2 & 1 & 1 \end{pmatrix}$$

(ii) In this case T has rank 3, and a basis for the range of T is

$$\left\{ \begin{pmatrix} 1 \\ 3 \\ -4 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -3 \\ -2 \end{pmatrix}, \begin{pmatrix} 4 \\ -1 \\ -3 \\ 1 \end{pmatrix} \right\}.$$

(iii) A basis for the kernel of T is

$$\left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

(iv) In this case T is neither injective nor surjective.

(e) (i)

$$A = \begin{pmatrix} 1 & 3 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{pmatrix}$$

(ii) In this case T has rank 4, and a basis for the range of T is

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix} \right\}$$

(of course since T in surjective, any basis for $\mathbb{R}^4$ would work).

(iii) A basis for the kernel of T is

$$\left\{ \begin{pmatrix} 5 \\ 0 \\ -2 \\ -4 \\ 1 \end{pmatrix} \right\}.$$

(iv) In this case T is surjective but not injective.

$\square$