Vector Spaces and Subspace Practice Problems
Solutions

1. Let V be the set of all 2×2 matrices with real entries, i.e.

$$V = \mathcal{M}_{2\times 2}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{R} \right\}.$$

Re-define addition and scalar multiplication on V as follows:

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 & b_2 \\ c_2 & d_1 \end{pmatrix},$$

$$r \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \begin{pmatrix} r & r \\ r & r \end{pmatrix}.$$

(a) Using these definitions for addition and scalar multiplication, which of the following vector space axioms does this structure pass or fail? Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $r, s \in \mathbb{R}$:

VS1 The set V is <u>closed under vector addition</u>, that is, $\mathbf{u} + \mathbf{v} \in V$ for
VS2 Vector addition is <u>commutative</u>, $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
VS3 Vector addition is <u>associative</u>, $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
VS4 There is a <u>zero vector</u> (or <u>additive identity element</u>) $\mathbf{0} \in V$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in V$.
VS5 Each $\mathbf{v} \in V$ has an <u>additive inverse</u> $\mathbf{w} \in V$, so that $\mathbf{w} + \mathbf{v} = \mathbf{0}$
VS6 The set V is <u>closed under scalar multiplication</u>, that is, $r \cdot \mathbf{v} \in V$
VS7 Addition of scalars <u>distributes</u> over scalar multiplication, $(r + s) \cdot \mathbf{v} = r \cdot \mathbf{v} + s \cdot \mathbf{v}$
VS8 Scalar multiplication distributes over vector addition, $r \cdot (\mathbf{v} + \mathbf{w}) = r \cdot \mathbf{v} + r \cdot \mathbf{w}$
VS9 Ordinary multiplication of scalars associates with scalar multiplication, $(rs) \cdot \mathbf{v} = r \cdot (s \cdot \mathbf{v})$
VS10 Multiplication by the scalar 1 is the <u>identity</u> operation, $1 \cdot \mathbf{v} = \mathbf{v}$

(b) Is V a vector space under these vector addition and scalar multiplication operations?

solution:

(a) VS1 The result of the addition operation lies in V, no matter which two matrices are added (since V includes all 2×2 matrices). So V is closed under this addition operation.

VS2 Let $\mathbf{u} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$. Then

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 7 & 4 \end{pmatrix}$$

whereas

$$\mathbf{v} + \mathbf{u} = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 3 & 8 \end{pmatrix}.$$

In this case $\mathbf{u} + \mathbf{v} \neq \mathbf{v} + \mathbf{u}$, which shows that this addition operation is not commutative.

VS3 Let $\mathbf{u} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} \in V$ *(i.e. take arbitrary elements of V).*

Then

$$\begin{aligned}
(\mathbf{u}+\mathbf{v})+\mathbf{w} &= \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right) + \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} \\
&= \begin{pmatrix} a_1 & b_2 \\ c_2 & d_1 \end{pmatrix} + \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} \\
&= \begin{pmatrix} a_1 & b_3 \\ c_3 & d_1 \end{pmatrix} \\
&= \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_3 \\ c_3 & d_2 \end{pmatrix} \\
&= \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \left(\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} + \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}\right) \\
&= \mathbf{u} + (\mathbf{v}+\mathbf{w})
\end{aligned}$$

Since $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are arbitrary, we see that the addition operation is associative.

VS4 Suppose by way of contradiction that $\begin{pmatrix} w & x \\ y & z \end{pmatrix}$ is an additive identity for V. Then we must have (for example) that

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

By definition

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} 1 & x \\ y & 4 \end{pmatrix}$$

which would force $x = 2$ and $y = 3$ by comparing the two equations. But then (again as an example)

$$\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} + \begin{pmatrix} w & 2 \\ 3 & z \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 3 & 8 \end{pmatrix} \neq \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$$

which shows that there can be no single matrix $\begin{pmatrix} w & x \\ y & z \end{pmatrix}$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

holds for choice of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V$. Therefore there is no additive identity (or zero vector) for this addition operation. *(Note that the matrix of all zeros would not work here.)*

VS5 Since there is no additive identity in this case, we cannot define additive inverses.

VS6 The result of the scalar multiplication operation lies in V, no matter which scalar and matrix are multiplied together. So V is closed under this scalar multiplication operation.

VS7 Let $\mathbf{v} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $r = s = 1$ for example. Then

$$(r+s)\cdot\mathbf{v} = (1+1)\cdot\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 2\cdot\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}.$$

On the other hand

$$r\cdot\mathbf{v} + s\cdot\mathbf{v} = 1\cdot\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + 1\cdot\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

which shows that scalar multiplication does not distribute over addition of scalars.

VS8 Let $\mathbf{v} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in V$, and $r \in \mathbb{R}$. Then

$$\begin{aligned} r \cdot (\mathbf{v} + \mathbf{w}) &= r \cdot \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \right) \\ &= r \cdot \begin{pmatrix} a_1 & b_2 \\ c_2 & d_1 \end{pmatrix} \\ &= \begin{pmatrix} r & r \\ r & r \end{pmatrix} \\ &= \begin{pmatrix} r & r \\ r & r \end{pmatrix} + \begin{pmatrix} r & r \\ r & r \end{pmatrix} \\ &= r \cdot \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + r \cdot \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \\ &= r \cdot \mathbf{v} + r \cdot \mathbf{w} \end{aligned}$$

Therefore scalar multiplication distributes over vector addition.

VS9 This axiom fails, for example

$$(2 \cdot 2) \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 4 \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix},$$

however

$$2 \cdot \left(2 \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \right) = 2 \cdot \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}.$$

Therefore scalar multiplication is not associative.

VS10 Multiplication by the scalar 1 is not the identity operation, for example taking $\mathbf{v} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ we have

$$1 \cdot \mathbf{v} = 1 \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \mathbf{v}.$$

(b) Since V does not satisfy all axioms, it fails to be a vectors space. *(Providing just one example where an axioms fails is sufficient to show that V is not a vector space).*

□

2. Let V be the set of all polynomials of degree at most 2 with real coefficients, i.e.

$$V = \mathcal{P}_2(\mathbb{R}) = \{ax^2 + bx + c | a, b, c \in \mathbb{R}\}.$$

Re-define addition and scalar multiplication on V as follows:

$$(a_1x^2 + b_1x + c_1) + (a_2x^2 + b_2x + c_2) = a_1a_2x^2 + b_1b_2x + c_1c_2,$$

$$r(a_1x^2 + b_1x + c_1) = 0.$$

(a) Using these definitions for addition and scalar multiplication, which of the following vector space axioms does this structure pass or fail? (Use the list from 1.(a))

(b) Is V a vector space under these vector addition and scalar multiplication operations?

solution:

(a) VS1 It is clear that the result of the addition operation again lies in V; i.e. V is closed under the addition operation.

VS2 Using commutativity of multiplication in $\mathbb{R}$ we have,

$$\begin{aligned}(a_1x^2+b_1x+c_1)+(a_2x^2+b_2x+c_2) &= a_1a_2x^2+b_1b_2x+c_1c_2\\ &= a_2a_1x^2+b_2b_1x+c_1c_1\\ &= (a_2x^2+b_2x+c_2)+(a_1x^2+b_1x+c_1)\end{aligned}$$

which shows that the addition operation is commutative.

VS3 Using associativity of multiplication in $\mathbb{R}$ we have,

$$\begin{aligned}&((a_1x^2+b_1x+c_1)+(a_2x^2+b_2x+c_2))+(a_3x^2+b_3x+c_3)\\ &= (a_1a_2x^2+b_1b_2x+c_1c_2)+(a_3x^2+b_3x+c_3)\\ &= (a_1a_2)a_3x^2+(b_1b_2)b_3x+(c_1c_2)c_3\\ &= a_1(a_2a_3)x^2+b_1(b_2b_3)x+c_1(c_2c_3)\\ &= (a_1x^2+b_1x+c_1)+(a_2a_3x^2+b_2b_3x+c_2c_3)\\ &= (a_1x^2+b_1x+c_1)+((a_2x^2+b_2x+c_2)+(a_3x^2+b_3x+c_3))\end{aligned}$$

which shows that the addition operation is associative.

VS4 The additive identity element of V, relative to the addition operation is x^2+x+1 (all coefficients equal to 1) since

$$(a_1x^2+b_1x+c_1)+(x^2+x+1)=a_1x^2+b_1x+c_1.$$

VS5 No polynomial ax^2+bx+c where either a, b or c equals 0 has an additive inverse. For example x has not additive inverse since

$$x+(px^2+qx+r)=x^2+x+1$$

implies

$$qx=x^2+x+1$$

which is false *(i.e. the polynomial on the left does not equal the polynomial on the right).*

VS6 The set V is closed under this operation of scalar multiplication since $0\in V$.

VS7 Since

$$(r+s)\cdot\mathbf{v}=0=0+0=r\cdot\mathbf{v}+s\cdot\mathbf{v}$$

for any $\mathbf{v}\in V$ and $r,s\in\mathbb{R}$, we see that addition of scalars distributes over scalar multiplication.

VS8 Since

$$r\cdot(\mathbf{v}+\mathbf{w})=0=0+0=r\cdot\mathbf{v}+r\cdot\mathbf{w}$$

for any $\mathbf{v},\mathbf{w}\in V$ and $r\in\mathbb{R}$, we see that scalar multiplication distributes over vector addition.

VS9 Since

$$(rs)\cdot\mathbf{v}=0=r\cdot 0=r\cdot(s\cdot\mathbf{v})$$

for any $\mathbf{v}\in V$ and $r,s\in\mathbb{R}$ we see that ordinary multiplication of scalars associates with scalar multiplication.

VS10 Since $1\cdot\mathbf{v}=0$ for any $\mathbf{v}\in V$ we see that multiplication by 1 is not the identity operation.

(b) The set V with these operations does not form a vector space as it does not satisfy all axioms.

□

When we refer to "The vector spaces" $\mathbb{R}^n$ (of column vectors), or $\mathcal{M}_{n\times m}(\mathbb{R})$ ($n\times m$ matrices), or $\mathcal{P}_n(\mathbb{R})$ (polynomials of degree at most n), we mean those sets with their *usual* operations of vector addition and scalar multiplication (not, for example, the "re-defined" operation used above or any other operations).

Use the subspace criterion to determine which of the following set are subspaces of these familiar vector spaces.

3. (a)
$$S_1 = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \middle| \, a+b+c=0 \right\}$$
(i.e. is S_1 a subspace of $\mathbb{R}^3$ under its usual operations of vector addition and scalar multiplication?)

(b)
$$S_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{R}) \middle| \, a+b=0 \right\}.$$
(i.e. is S_2 a subspace of $\mathcal{M}_{2\times 2}(\mathbb{R})$ under its usual operations of vector addition and scalar multiplication?)

(c)
$$S_3 = \left\{ ax^2 + b \in \mathcal{P}_2(\mathbb{R}) \middle| \, a+b=0 \right\}$$
(i.e. is S_3 a subspace of $\mathcal{P}_2(\mathbb{R})$ under its usual operations of vector addition and scalar multiplication?)

(d)
$$S_4 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{R}) \middle| \, a+d=5 \right\}.$$

(e)
$$S_5 = \left\{ x^2 + bx + c \in \mathcal{P}_2(\mathbb{R}) \middle| \, b, c \in \mathbb{R} \right\}.$$

(f)
$$S_6 = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \middle| \, ab=0 \right\}$$

(g)
$$S_7 = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \mathbb{R}^4 \middle| \, a-c=0 \right\}$$

(h)
$$S_8 = \left\{ ax^3 + bx + c \in \mathcal{P}_3(\mathbb{R}) \middle| \, a+b=0 \in \mathbb{R} \right\}.$$

(i)
$$S_9 = \left\{ \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \in \mathcal{M}_{2\times 3}(\mathbb{R}) \middle| \, e = a + 2a - b \right\}.$$

solution:

In each case we will apply the subspace criterion: Either show that the given set is closed under vector addition and scalar multiplication, or give an example where either of these fails. We will give a full example for each of theses cases, and only the result for the rest.

(a) Let $\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}, \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} \in S_1$. This implies $a_1 + b_1 + c_1 = 0$ and $a_2 + b_2 + c_2 = 0$. Then
$$\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 \\ b_1 + b_2 \\ c_1 + c_2 \end{pmatrix} \in S_1$$

since

$$(a_1 + a_2) + (b_1 + b_2) + (c_1 + c_2) = (a_1 + b_1 + c_1) + (a_2 + b_2 + c_2) = 0 + 0 = 0.$$

This shows that S_1 is closed under vector addition. Let $r \in \mathbb{R}$. Then

$$r \cdot \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} ra_1 \\ rb_1 \\ rc_1 \end{pmatrix} \in S_1$$

since

$$ra_1 + rb_1 + rc_1 = r(a_1 + b_1 + c_1) = r(0) = 0.$$

Thus S_1 is also closed under scalar multiplication, and is therefore a subspace of $\mathbb{R}^3$ by the subspace criterion.

(b) Is a subspace.

(c) Is a subspace.

(d) Note that matrices $\begin{pmatrix} 5 & 0 \\ 3 & 0 \end{pmatrix}$ and $\begin{pmatrix} -2 & 3 \\ 1 & 7 \end{pmatrix}$ both belong to S_4, however their sum

$$\begin{pmatrix} 5 & 0 \\ 3 & 0 \end{pmatrix} + \begin{pmatrix} -2 & 3 \\ 1 & 7 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 4 & 7 \end{pmatrix}$$

does not belong to S_4, since the sum of the diagonal entries is not equal to 5. Thus S_4 is not closed under vector addition, and therefore is not a subspace of $\mathcal{M}_{2\times 2}(\mathbb{R})$.

(e) Is not a subspace.

(f) Is not a subspace.

(g) Is a subspace.

(h) Is a subspace.

(i) Is a subspace.

□

4. Give examples of 2 different elements belonging to each of the sets in question 3.

solution:

(a) $(1, 0, -1), (2, -1, -1)$

(b) $\begin{pmatrix} 3 & -3 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -2 & 2 \\ 5 & 1 \end{pmatrix}$

(c) $x^2 - 1, 0$

(d) $\begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix}, \begin{pmatrix} 8 & 1 \\ 3 & -3 \end{pmatrix}$

(e) $x^2, x^2 + 1$

(f) $(1, 0, 1), (0, 2, 5)$

(g) $(1, 1, 1, 1), (2, 0, 2, 4)$

(h) $5x^3 - 5x + 1, 3$

(i) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

□