Linear Independence Practice Problems
Solutions

Determine whether the given sets of vectors in $\mathbb{R}^2$ are linearly independent or linearly dependent (over $\mathbb{R}$) by the row reduction method, and verify this using the determinant method.

1. $S_1 = \left\{ \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}$
2. $S_2 = \left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\}$
3. $S_3 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$
4. $S_4 = \left\{ \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 8 \end{pmatrix} \right\}$
5. $S_5 = \left\{ \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \end{pmatrix} \right\}$
6. $S_6 = \left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 4 \end{pmatrix} \right\}$

Question 1 by row-reduction method: We want to solve

$$C_1 \begin{pmatrix} 3 \\ 4 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1}$$

for C_1 and C_2. If the only solution is $C_1 = C_2 = 0$, the vectors are linearly independent, and if there are nonzero solutions they are linearly dependent. Equation (1) yields a system of equations which we solve by row-reduction.

$$\left(\begin{array}{cc|c} 3 & 1 & 0 \\ 4 & 3 & 0 \end{array} \right) \xrightarrow{R_2 - \frac{4}{3}R_1} \left(\begin{array}{cc|c} 3 & 1 & 0 \\ 0 & 5/3 & 0 \end{array} \right) \xrightarrow[\frac{3}{5}R_2]{\frac{1}{3}R_1} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \end{array} \right) \xrightarrow{R_1 - R_2} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right)$$

The solution set is

$$\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

which says $C_1 = C_2 = 0$. Therefore equation (1) only has the trival solution, and hence the set S_1 is linearly independent.

Question 1 by determinant method: We saw that equation (1) yields the system

$$\left(\begin{array}{cc|c} 3 & 1 & 0 \\ 4 & 3 & 0 \end{array} \right)$$

which, in matrix form, is $A\mathbf{x} = \mathbf{0}$ where

$$A = \begin{pmatrix} 3 & 1 \\ 4 & 3 \end{pmatrix}.$$

The system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution (i.e. $\mathbf{x} = \mathbf{0}$) when $\det A \neq 0$, and has nonzero solutions (i.e. there are free variables) when $\det A = 0$. Note that this only applies when A is a square matrix (and hence determinants apply). Then

$$\det A = \begin{vmatrix} 3 & 1 \\ 4 & 3 \end{vmatrix} = 9 - 4 = 5 \neq 0$$

and hence the only solution to $A\mathbf{x} = \mathbf{0}$, and hence equation (1), is $C_1 = C_2 = 0$. Therefore S_1 is linearly independent.

The same 2 strategies apply to the sets of vectors below, however since there are only 2 vectors in each set, it suffices to check whether one of the vectors is a scalar multiple of the other. More generally, if any one of the vectors in a set (of any size) is a linearly combination of the others, then the set is linearly dependent, and if this is not the case, they are independent.

Answers:

1. $S_1 = \left\{ \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}$ - linearly independent
2. $S_2 = \left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\}$ - linearly independent
3. $S_3 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ - linearly independent
4. $S_4 = \left\{ \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 8 \end{pmatrix} \right\}$ - linearly dependent
5. $S_5 = \left\{ \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \end{pmatrix} \right\}$ - linearly dependent
6. $S_6 = \left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 4 \end{pmatrix} \right\}$ - linearly independent

---

Determine whether the given sets in $\mathbb{R}^3$ are linearly independent by the row reduction method.

1. $S_7 = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \right\}$
2. $S_8 = \left\{ \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix} \right\}$
3. $S_9 = \left\{ \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

Question 1: As with the example above for S_1 we look at the vector equation

$$C_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} + C_3 \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{2}$$

which yields the following system of equations,

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \end{array} \right).$$

At this point we already see that the system has a free variable, which means that there will be nonzero solutions to equation (2), and hence the set S_7 is linearly dependent. This answers the question and is all that is required.

Suppose we ask for an explicit example of values for C_1, C_2 and C_3, which are not all zero, such that equation (2) holds. Then we solve the system above.

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \end{array} \right) \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) x \to \frac{1}{3} R_2 \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \xrightarrow{R_1 - R_2} \left(\begin{array}{ccc|c} 1 & 0 & 8/3 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

The variable C_3 is free and the solution set is

$$\left\{ t \begin{pmatrix} -8/3 \\ -1/3 \\ 1 \end{pmatrix} \middle| \, t \in \mathbb{R} \right\}.$$

Letting $t = 3$, for example, yields the solution $C_1 = -8$, $C_2 = -1$ and $C_3 = 3$, and hence

$$(-8) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + (-1) \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The same strategy applies to the other sets. Solve the systems of equations to see whether $C_1 = C_2 = C_3 = 0$ (which means there are no free variables) or if there are nontrivial solutions like the example above (this means you will have at least one free variable).

Answers:

1. $S_7 = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \right\}$ - linearly dependent
2. $S_8 = \left\{ \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix} \right\}$ - linearly independent
3. $S_9 = \left\{ \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ - linearly independent

---

Determine whether the given sets in $\mathbb{R}^3$ are linearly independent by determinant method.

1. $S_{10} = \left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 7 \\ 3 \end{pmatrix} \right\}$ 2. $S_{11} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \right\}$ 3. $S_{12} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \right\}$

Question 1: As was described for set S_1 above, when the vectors form a square matrix, the determinant of this matrix tells us whether or not the columns are linearly independent; if determinant is zero they are dependent, if determinant is nonzero they are independent.

$$\begin{vmatrix} 2 & 3 & 5 \\ 1 & 2 & 7 \\ 1 & 0 & 3 \end{vmatrix} = 2\begin{vmatrix} 2 & 7 \\ 0 & 3 \end{vmatrix} - 3\begin{vmatrix} 1 & 7 \\ 1 & 3 \end{vmatrix} + 5\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = 2(6) - 3(-4) + 5(-2) = 14.$$

Since the determinant is nonzero, the set S_{10} is linearly independent.

The same strategy applies to the other sets here.

Answers:

1. $S_{10} = \left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 7 \\ 3 \end{pmatrix} \right\}$ - linearly independent. 2. $S_{11} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \right\}$ - linearly independent.
3. $S_{12} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \right\}$ - linearly dependent.

Determine whether the given sets of vectors are linearly independent or linearly dependent.

1. $S_{13} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} \right\}$
2. $S_{14} = \left\{ \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right\}$
3. $S_{15} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ 2 \end{pmatrix} \right\}$
4. $S_{16} = \left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 2 \\ 3 \end{pmatrix} \right\}$
5. $S_{17} = \{x^2 - 3, 6x^2 + 2\}$
6. $S_{18} = \{2x^2 - x + 3, 4x^2 + x + 2, 8x^2 - x + 8\}$
7. $S_{19} = \{x^2 - x + 1, x + 1, -3\}$
8. $S_{20} = \{x^2 + 4x - 1, 3x^2 + x + 4, 2x^2 - 3x + 5, 5x^2 - 2x + 9\}$
9. $S_{21} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \right\}$
10. $S_{22} = \left\{ \begin{pmatrix} -1 & 4 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 4 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$

We can use a combination of strategies/theorems to answer these questions.

1. $S_{13} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} \right\}$ - linearly dependent: These 4 vectors belong to $\mathbb{R}^3$. No set of vectors from $\mathbb{R}^3$, which is larger than $3 = \dim \mathbb{R}^3$, can be linearly independent.

2. $S_{14} = \left\{ \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right\}$ - linearly independent: There are only 2 vectors here, and they are not scalar multiples of each other.

3. $S_{15} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ 2 \end{pmatrix} \right\}$ - linearly independent: Use the row reduction method here.

4. $S_{16} = \left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 2 \\ 3 \end{pmatrix} \right\}$ - linearly dependent: Use the row reduction method here.

5. $S_{17} = \{x^2 - 3, 6x^2 + 2\}$ - linearly independent: Note that neither vector *(in this case the vectors are polynomials)* is a scalar multiple of the other.

6. $S_{18} = \{2x^2 - x + 3, 4x^2 + x + 2, 8x^2 - x + 8\}$ - linearly independent: Set up the equation

$$C_1(2x^2 - x + 3) + C_2(4x^2 + x + 2) + C_3(8x^2 - x + 8) = 0. \tag{3}$$

Combining coefficients on like powers of x gives

$$(2C_1 + 4C_2 + 8C_3)x^2 + (-C_1 + C_2 - C_3)x + (3C_1 + 2C_2 + 8C_3)(1) = 0.$$

The only way this can be true is if $(2C_1 + 4C_2 + 8C_3) = 0$, $(-C_1 + C_2 - C_3) = 0$ and $(3C_1 + 2C_2 + 8C_3) = 0$, which yields a linear system with 3 equations and 3 unknowns. We solve.

$$\left(\begin{array}{ccc|c} 2 & 4 & 8 & 0 \\ -1 & 1 & -1 & 0 \\ 3 & 2 & 8 & 0 \end{array}\right) \xrightarrow{\frac{1}{2}R_1} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ -1 & 1 & -1 & 0 \\ 3 & 2 & 8 & 0 \end{array}\right) \xrightarrow[R_3 - 3R_1]{R_2 + R_1} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & -4 & -4 & 0 \end{array}\right) \xrightarrow{R_3 + \frac{4}{3}R_2} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

At this point we see that there are free variables in the system, and hence there exist trivial solutions (for C_1, C_2, and C_3) in equation (3). Therefore the set S_{18} is linearly dependent.

To see this more explicitly we can continue to solve the system.

$$\xrightarrow{\frac{1}{3}R_2} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1 - 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The solution set is

$$\left\{ t \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \middle| \, t \in \mathbb{R} \right\},$$

so (for example) setting $t = 1$ we get that $C_1 = -2$, $C_2 = -1$, and $C_3 = 1$. Subbing these into equation (3) shows that

$$(-2)(2x^2 - x + 3) + (-1)(4x^2 + x + 2) + (1)(8x^2 - x + 8) = 0.$$

7. $S_{19} = \{x^2 - x + 1, x + 1, -3\}$ - linearly independent: Use the row reduction method (as seen in S_{18}).

8. $S_{20} = \{x^2 + 4x - 1, 3x^2 + x + 4, 2x^2 - 3x + 5, 5x^2 - 2x + 9\}$ - linearly dependent: There are 4 vectors in this set, but $\dim \mathcal{P}_2(\mathbb{R}) = 3$, therefore this set cannot be linearly independent. *(For practice, verify this with the row reduction method, and show that at least one of the vectors is a linear combination of the others.)*

9. $S_{21} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \right\}$ - linearly independent. linearly independent: Set up the equation

$$C_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} + C_3 \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{4}$$

By equating corresponding matrix entries on both sides of equation (4) we get that $C_1 + C_2 = 0$ (for the 1,1-entry), $-C_2 + C_3 = 0$ (for the 2,2-entry), $2C_3 = 0$ (for the 2,1-entry) and $C_1 + C_3 = 0$ (for the 2,2-entry). This yields a system of 4 equations with 3 unknowns. We solve.

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow{R_4 - R_1} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right) \xrightarrow{R_4 - R_2} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

At this point we see that there are no free variables in the system and thus the only solution to equation (4) is $C_1 = C_2 = C_3 = 0$. Therefore the set S_{21} is linearly independent.

10. $S_{22} = \left\{ \begin{pmatrix} -1 & 4 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 4 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$ - linearly independent: Use the row reduction method (as seen in S_{21}).

Find a value of k such that the given vectors are linearly dependent.

1. $S_{23} = \left\{ \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ k \\ 1 \end{pmatrix} \right\}$ 2. $S_{24} = \left\{ \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} k \\ -4 \end{pmatrix} \right\}$ 3. $S_{25} = \left\{ \begin{pmatrix} 1 \\ k \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \\ k \end{pmatrix} \right\}$

Each of these sets of vectors form a square matrix, which means the determinant test for linearly independence applies. For question 1: Put the column vectors into a matrix and compute its determinant; you can use any method you like, but Laplace expansion is convenient here.

Question 1:

$$\begin{vmatrix} -1 & 2 & 3 \\ 1 & -3 & k \\ 3 & 4 & 1 \end{vmatrix} = (-1) \begin{vmatrix} -3 & k \\ 4 & 1 \end{vmatrix} - 2 \begin{vmatrix} 1 & k \\ 3 & 1 \end{vmatrix} + 3 \begin{vmatrix} 1 & -3 \\ 3 & 4 \end{vmatrix} = -(-3 - 4k) - 2(1 - 3k) + 3(13) = 7k + 40$$

This determinant is zero only when $7k + 40 = 0$ or $k = -\frac{40}{7}$. Therefore S_{23} is linearly dependent only when $k = -\frac{40}{7}$.

Question 2:

$$\begin{vmatrix} -1 & k \\ 2 & -4 \end{vmatrix} = 4 - 2k$$

This determinant is zero when $k = 2$. Therefore S_{24} is linearly dependent when $k = 2$.

Question 3:

$$\begin{vmatrix} 1 & 0 & 4 \\ k & 2 & 4 \\ -2 & -1 & k \end{vmatrix} = \begin{vmatrix} 2 & 4 \\ -1 & k \end{vmatrix} + 4 \begin{vmatrix} k & 2 \\ -2 & -1 \end{vmatrix} = (2k + 4) + 4(-k + 4) = 20 - 2k$$

This determinant is zero when $k = 10$. Therefore S_{25} is linearly dependent when $k = 10$.