1. Are subsets S_1 and S_2 subspaces of $\mathbb{R}^3$ and $\mathbb{R}^2$ (respectively) under the usual addition and scalar multiplication? If so, prove these two qualities hold for any vector in the subspace. If they aren't, provide a counterexample.

$$S_1 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \, x = 2y - z \right\} \qquad\qquad S_2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| \, |x| = |y| \right\}$$

Solution. Subset S_1 is also a subspace. Using arbitrary values $r, a, b, c \in \mathbb{R}$ (where $a = 2b - c$) we observe that S_1 is closed under scalar multiplication.

$$r \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} ra \\ rb \\ rc \end{pmatrix} \text{ which is still a vector in } \mathbb{R}^3 \text{ because } ra = 2(rb) - rc = r(2b) - rc\,.$$

For addition, we use arbitrary vectors $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\begin{pmatrix} e \\ f \\ g \end{pmatrix}$ in $\mathbb{R}^3$ such that $a = 2b - c$ and $e = 2f - g$.

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} e \\ f \\ g \end{pmatrix} = \begin{pmatrix} a+e \\ b+f \\ c+g \end{pmatrix} \quad \rightarrow \quad a + e = 2(b+f) - (c+g) = 2b + 2f - c - g = 2b - c + 2f - g\,.$$

S_2 is not a subspace of $\mathbb{R}^2$ because although it is closed under scalar multiplication, vector addition does not work in every case:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \notin S_2$$

□

2. Execute $\begin{pmatrix} 0 & 4 & -3 \\ 2 & 3 & 3 \end{pmatrix} + 2\begin{pmatrix} 1 & 1 & 1 \\ -5 & 3 & 2 \end{pmatrix}$ within some vector subset $V_1 = \mathcal{M}_{2\times 3}(\mathbb{R})$ when addition and scalar multiplication are defined as follows:

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} + \begin{pmatrix} u & v & w \\ x & y & z \end{pmatrix} = \begin{pmatrix} 0 & b+v & 0 \\ d & 0 & z \end{pmatrix} \qquad\qquad r \cdot \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = \begin{pmatrix} ra & rb & 1 \\ rd & re & 1 \end{pmatrix}$$

Solution.

$$\begin{pmatrix} 0 & 4 & -3 \\ 2 & 3 & 3 \end{pmatrix} + 2\begin{pmatrix} 1 & 1 & 1 \\ -5 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 4 & -3 \\ 2 & 3 & 3 \end{pmatrix} + \begin{pmatrix} 2 & 2 & 1 \\ -10 & 6 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 6 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

□

3. Using the vector space axioms written below, provide at least two reasons of why subset V_1 from the previous question is not a valid vector space.

Solution. Various axioms are violated by V_1 as it's defined above. VS10 can be our first example. A vector $\mathbf{v}$ from a subspace that verifies VS10 should demonstrate $1 \cdot \mathbf{v} = \mathbf{v}$.

$$1 \cdot \begin{pmatrix} 1 & 1 & 1 \\ -5 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -5 & 3 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 1 & 1 \\ -5 & 3 & 2 \end{pmatrix} \quad \text{so this axiom fails.}$$

Under VS4, $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for some zero vector in the subspace. However, if we add an arbitrary 2×3 matrix in V_1 to the vector $\begin{pmatrix} 0 & 4 & -3 \\ 2 & 3 & 3 \end{pmatrix} \in V_1$,

$$\begin{pmatrix} 0 & 4 & -3 \\ 2 & 3 & 3 \end{pmatrix} + \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = \begin{pmatrix} 0 & 4+b & 0 \\ 2 & 0 & f \end{pmatrix} \neq \begin{pmatrix} 0 & 4 & -3 \\ 2 & 3 & 3 \end{pmatrix} \quad \text{and the fourth axiom fails as well.}$$

In total; axioms 2, 3, 4, 5, 7, 8, and 10 are violated and axioms 1, 6 and 9 are upheld.

□

4. Can vector $\mathbf{u}$ be written as a linear combination of the other two vectors given below? Would a set of all three of the vectors be linearly independent or dependent?

$$\mathbf{u} = \begin{pmatrix} -3 \\ 3 \\ 5 \end{pmatrix} ; \qquad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

Solution.

$$3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} + \begin{pmatrix} -6 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ 5 \end{pmatrix} = \mathbf{u}$$

The vector can be written as a linear combination of the other two, therefore a set of all three would be linearly *dependent*.

□

5. With the vectors below, determine if $\mathbf{y} \in \operatorname{span}(S_3)$ and whether S_3 spans $\mathbb{R}^3$.

$$\mathbf{y} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad S_3 = \left\{ \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix} \right\}$$

Solution. We solve for a, b and c in vector equation

$$a \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} + b \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} + c \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

which yields the system

$$\left(\begin{array}{ccc|c} 4 & 2 & -2 & 1 \\ 2 & 1 & -1 & 1 \\ -3 & -2 & 0 & 1 \end{array} \right) \longrightarrow \left(\begin{array}{ccc|c} 4 & 2 & -2 & 1 \\ 2 & 1 & -1 & 1 \\ -3 & -2 & 0 & 1 \end{array} \right) \xrightarrow{R_2 - \frac{1}{2}R_1} \left(\begin{array}{ccc|c} 4 & 2 & -2 & 1 \\ 0 & 0 & 0 & 1/2 \\ -3 & -2 & 0 & 1 \end{array} \right)$$

After this first step we see that the system is inconsistent. No solutions exist so $\mathbf{y} \notin \operatorname{span}(S)$.

From the inconsistent row in the matrix, we also know that the set of three vectors is not linearly independent. At least one of the vectors in S_3 is redundant. A set which spans $\mathbb{R}^3$ must have at least three linearly independent vectors, therefore S_3 does not span $\mathbb{R}^3$.

□

Vector space axioms

VS1: The set V is closed under vector addition, that is, $\mathbf{u} + \mathbf{v} \in V$ for any $\mathbf{u}, \mathbf{v} \in V$

VS2: Vector addition is commutative, $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

VS3: Vector addition is associative, $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

VS4: There is a zero vector $\mathbf{0} \in V$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in V$.

VS5: Each $\mathbf{v} \in V$ has an additive inverse $\mathbf{w} \in V$, so that $\mathbf{w} + \mathbf{v} = \mathbf{0}$

VS6: The set V is closed under scalar multiplication, that is, $r \cdot \mathbf{v} \in V$

VS7: Addition of scalars distributes over scalar multiplication, $(r + s) \cdot \mathbf{v} = r \cdot \mathbf{v} + s \cdot \mathbf{v}$

VS8: Scalar multiplication distributes over vector addition, $r \cdot (\mathbf{v} + \mathbf{w}) = r \cdot \mathbf{v} + r \cdot \mathbf{w}$

VS9: Ordinary multiplication of scalars associates with scalar multiplication, $(rs) \cdot \mathbf{v} = r \cdot (s \cdot \mathbf{v})$

VS10: Multiplication by the scalar 1 is the identity operation, $1 \cdot \mathbf{v} = \mathbf{v}$