1. Is the set below a basis for $\mathbb{R}^3$?

$$\left\{ \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \right\}$$

Solution.

A linearly independent set of three vectors in $\mathbb{R}^3$ is a spanning set for $\mathbb{R}^3$, therefore it is also a basis. So we check to determine whether the set of vectors is linear independence. This can easily be done by row reducing or calculating the determinant (as the vectors form a square matrix).

Row Reduction method:

$$\begin{pmatrix} -1 & 3 & 0 \\ 2 & -2 & 1 \\ -3 & 0 & -2 \end{pmatrix} \xrightarrow[R_3-3R_1]{R_2+2R_1} \begin{pmatrix} -1 & 3 & 0 \\ 0 & 4 & 1 \\ 0 & -9 & -2 \end{pmatrix} \xrightarrow{R_3+\frac{9}{4}R_2} \begin{pmatrix} -1 & 3 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$$

There are no free variables in the system, therefore the vectors are linearly independent.

Determinant method:

$$\begin{vmatrix} -1 & 3 & 0 \\ 2 & -2 & 1 \\ -3 & 0 & -2 \end{vmatrix} = -1 \begin{vmatrix} -2 & 1 \\ 0 & -2 \end{vmatrix} - 3 \begin{vmatrix} 2 & 1 \\ -3 & -2 \end{vmatrix} = -(4-0) - 3(-4+3) = -1 \neq 0$$

The set of vectors is linearly dependent, therefore it forms a basis for $\mathbb{R}^3$

□

2. Provide a basis for the subspace spanned by the linearly dependent set of vectors.

$$\left\{ \begin{pmatrix} 6 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ -2 \end{pmatrix}, \begin{pmatrix} -30 \\ -10 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

Solution.

Since the set exists within $\mathbb{R}^2$, the basis for the set should have no more than two vectors. $-5\begin{pmatrix} 6 \\ 2 \end{pmatrix} = \begin{pmatrix} -30 \\ -10 \end{pmatrix}$ so at least one of these can be removed from the set. By observation, it is clear to see that the other two vectors are linearly independent because they are not parallel. One is not a scalar multiple of the other. The following are all acceptable bases (although there are an infinite number of correct options):

$$\left\{ \begin{pmatrix} 6 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ -2 \end{pmatrix} \right\}, \quad \left\{ \begin{pmatrix} -30 \\ -10 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}, \quad \left\{ \begin{pmatrix} 5 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}, \quad \left\{ \begin{pmatrix} -30 \\ -10 \end{pmatrix}, \begin{pmatrix} 5 \\ -2 \end{pmatrix} \right\}, \quad \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

The last basis is acceptable because we know that any linearly independent set with the same number of vectors as the space it is occupying should span the set. Therefore we can also use the standard basis for $\mathbb{R}^2$. □

3. Find a basis for the subspace defined by D in $\mathbb{R}^5$. What is D's dimension?

$$D = \left\{ \begin{pmatrix} c+2d \\ b-3c \\ 6a \\ 2a-d \\ 4d \end{pmatrix} \middle| \, a,b,c,d \in \mathbb{R} \right\}$$

Solution.

$$\begin{pmatrix} c+2d \\ b-3c \\ 6a \\ 2a-d \\ 4d \end{pmatrix} = a \begin{pmatrix} 0 \\ 0 \\ 6 \\ 2 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c \begin{pmatrix} 1 \\ -3 \\ 0 \\ 0 \\ 0 \end{pmatrix} + d \begin{pmatrix} 2 \\ 0 \\ 0 \\ -1 \\ 4 \end{pmatrix}$$

These four vectors span the subspace. They are also linearly independent, which is identified through row reduction:

$$\begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & -3 & 0 \\ 6 & 0 & 0 & 0 \\ 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 4 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} 6 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 2 \\ 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 4 \end{pmatrix} \xrightarrow{R_4 - \frac{1}{3}R_1} \begin{pmatrix} 6 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 4 \end{pmatrix} \xrightarrow{R_4 - \frac{1}{3}R_1} \begin{pmatrix} 6 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

There are no free variables in the system. As it is spanning and linearly independent, basis of set D can be given by a set of these vectors. The dimension of a basis is equivalent to the number of vectors present, because those vectors are by definition independent. Thus, $\dim(D) = 4$.

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ 6 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \\ -1 \\ 4 \end{pmatrix} \right\}$$

□

4. What is the rank, dimension for the column space, and nullity of the matrix $M = \begin{pmatrix} 5 & 3 & 9 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix}$?

Solution.

$$\begin{pmatrix} 5 & 3 & 9 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix} \xrightarrow{5R_3} \begin{pmatrix} 5 & 3 & 9 \\ 0 & 2 & 1 \\ 10 & 0 & 15 \end{pmatrix} \xrightarrow{R_3 - 2R_1} \begin{pmatrix} 5 & 3 & 9 \\ 0 & 2 & 1 \\ 0 & -6 & -3 \end{pmatrix} \xrightarrow{R_3 + 3R_2} \begin{pmatrix} 5 & 3 & 9 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{aligned} \dim(M) &= 2 \\ \operatorname{rank}(M) &= 2 \\ \dim(\operatorname{null}(M)) &= 1 \end{aligned}$$

□

5. Find the basis for the null space (or kernel) of S, where $S = \left\{ \begin{pmatrix} a+2b-c+d \\ a+c+d \\ 2a-4b+6c+2d \end{pmatrix} \middle| \, a,b,c,d \in \mathbb{R} \right\}$.

Solution.

The subspace can be recreated with a matrix whose column space is givenby the vectors of coefficients in the linear combination written below. The null space of S is found by row reducing the augmented matrix.

$$\begin{pmatrix} a+2b-c+d \\ a+c+d \\ 2a-4b+6c+2d \end{pmatrix} = a\begin{pmatrix}1\\1\\2\end{pmatrix} + b\begin{pmatrix}2\\0\\-4\end{pmatrix} + c\begin{pmatrix}-1\\1\\6\end{pmatrix} + d\begin{pmatrix}2\\2\\2\end{pmatrix} \quad \rightarrow \quad \left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 2 & -4 & 6 & 2 & 0 \end{array}\right)$$

$$\xrightarrow[R_3-2R_1]{R_2-R_1} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 0 \\ 0 & -2 & 2 & 0 & 0 \\ 0 & -8 & 8 & 0 & 0 \end{array}\right) \xrightarrow{-\frac{1}{2}R_2} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 0 \\ 0 & -2 & 2 & 0 & 0 \\ 0 & -8 & 8 & 0 & 0 \end{array}\right) \xrightarrow[R_1-2R_2]{R_3+8R_2} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The null space (and hence the kernel of T) is the solution set for this system.

$$\left\{ s\begin{pmatrix}-1\\1\\1\\0\end{pmatrix} + t\begin{pmatrix}-1\\0\\0\\1\end{pmatrix} \middle| s,t \in \mathbb{R} \right\} \quad \rightarrow \quad \text{The basis is} \quad \left\{ \begin{pmatrix}-1\\1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\0\\1\end{pmatrix} \right\}.$$

□

6. Explain your answers to the conceptual questions below.
   (a) Can a set with 4 vectors in $\mathbb{R}^3$ be a spanning set for $\mathbb{R}^3$?
   (b) Can a set with 4 vectors in $\mathbb{R}^4$ be a spanning set for $\mathbb{R}^4$?
   (c) Can a set with 4 vectors in $\mathbb{R}^5$ be a spanning set for $\mathbb{R}^5$?
   (d) If the rank of a 7×5 matrix is 4, what is it's nullity?
   (e) What is the largest possible dimension of a set that is spanned by 5 vectors in $\mathbb{R}^4$?

Solution.

(a) Yes, it is possible that a set of 4 vectors in $\mathbb{R}^3$ spans $\mathbb{R}^3$ as long as no more than one of the vectors is redundant.
(b) Yes, a set with 4 vectors in $\mathbb{R}^4$ can span $\mathbb{R}^4$ as long as set is linearly *in*dependent.
(c) No, a set of 4 vectors cannot span $\mathbb{R}^5$.
(d) The nullity of the matrix is determined from the number of columns: $5 - 4 = 1$
(e) The dimension for a set in $\mathbb{R}^4$ is no more than 4 but could be as small as 1.

□