Vector Spaces and Subspace Practice Problems

1. Let V be the set of all 2×2 matrices with real entries, i.e.

$$V = \mathcal{M}_{2\times 2}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{R} \right\}.$$

Re-define addition and scalar multiplication on V as follows:

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 & b_2 \\ c_2 & d_1 \end{pmatrix},$$

$$r \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \begin{pmatrix} r & r \\ r & r \end{pmatrix}.$$

(a) Using these definitions for addition and scalar multiplication, which of the following vector space axioms does this structure pass or fail? Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $r, s \in \mathbb{R}$:

VS1 The set V is closed under vector addition, that is, $\mathbf{u} + \mathbf{v} \in V$ for
VS2 Vector addition is commutative, $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
VS3 Vector addition is associative, $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
VS4 There is a zero vector (or additive identity element) $\mathbf{0} \in V$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in V$.
VS5 Each $\mathbf{v} \in V$ has an additive inverse $\mathbf{w} \in V$, so that $\mathbf{w} + \mathbf{v} = \mathbf{0}$
VS6 The set V is closed under scalar multiplication, that is, $r \cdot \mathbf{v} \in V$
VS7 Addition of scalars distributes over scalar multiplication, $(r + s) \cdot \mathbf{v} = r \cdot \mathbf{v} + s \cdot \mathbf{v}$
VS8 Scalar multiplication distributes over vector addition, $r \cdot (\mathbf{v} + \mathbf{w}) = r \cdot \mathbf{v} + r \cdot \mathbf{w}$
VS9 Ordinary multiplication of scalars associates with scalar multiplication, $(rs) \cdot \mathbf{v} = r \cdot (s \cdot \mathbf{v})$
VS10 Multiplication by the scalar 1 is the identity operation, $1 \cdot \mathbf{v} = \mathbf{v}$

(b) Is V a vector space under these vector addition and scalar multiplication operations?

2. Let V be the set of all polynomials of degree at most 2 with real coefficients, i.e.

$$V = \mathcal{P}_2(\mathbb{R}) = \{ax^2 + bx + c | a, b, c \in \mathbb{R}\}.$$

Re-define addition and scalar multiplication on V as follows:

$$(a_1x^2 + b_1x + c_1) + (a_2x^2 + b_2x + c_2) = a_1a_2x^2 + b_1b_2x + c_1c_2,$$

$$r(a_1x^2 + b_1x + c_1) = 0.$$

(a) Using these definitions for addition and scalar multiplication, which of the following vector space axioms does this structure pass or fail? (Use the list from 1.(a))

(b) Is V a vector space under these vector addition and scalar multiplication operations?

3. When we refer to "The vector spaces" $\mathbb{R}^n$ (of column vectors), or $\mathcal{M}_{n\times m}(\mathbb{R})$ ($n \times m$ matrices), or $\mathcal{P}_n(\mathbb{R})$ (polynomials of degree at most n), we mean those sets with their *usual* operations of vector addition and scalar multiplication (not, for example, the "re-defined" operation used above or any other operations).

   Use the subspace criterion to determine which of the following set are subspaces of these familiar vector spaces.

   (a)
$$S_1 = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, a+b+c=0 \right\}$$

   (i.e. is S_1 a subspace of $\mathbb{R}^3$ under its usual operations of vector addition and scalar multiplication?)

   (b)
$$S_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{R}) \,\middle|\, a+b=0 \right\}.$$

   (i.e. is S_2 a subspace of $\mathcal{M}_{2\times 2}(\mathbb{R})$ under its usual operations of vector addition and scalar multiplication?)

   (c)
$$S_3 = \left\{ ax^2 + b \in \mathcal{P}_2(\mathbb{R}) \,\middle|\, a+b=0 \right\}$$

   (i.e. is S_3 a subspace of $\mathcal{P}_2(\mathbb{R})$ under its usual operations of vector addition and scalar multiplication?)

   (d)
$$S_4 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{R}) \,\middle|\, a+d=5 \right\}.$$

   (e)
$$S_5 = \left\{ x^2 + bx + c \in \mathcal{P}_2(\mathbb{R}) \,\middle|\, b, c \in \mathbb{R} \right\}.$$

   (f)
$$S_6 = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, ab=0 \right\}$$

   (g)
$$S_7 = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \mathbb{R}^4 \,\middle|\, a-c=0 \right\}$$

   (h)
$$S_8 = \left\{ ax^3 + bx + c \in \mathcal{P}_3(\mathbb{R}) \,\middle|\, a+b=0 \in \mathbb{R} \right\}.$$

   (i)
$$S_9 = \left\{ \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \in \mathcal{M}_{2\times 3}(\mathbb{R}) \,\middle|\, e = a + 2a - b \right\}.$$

4. Give examples of 2 different elements belonging to each of the sets in question 3.