MATH 1350, Winter 2025
Mini-Assignment 5

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1. Find all values of $k \in \mathbb{R}$ such that $\det(A - kI_3) = 0$ where

$$A = \begin{pmatrix} 2 & -2 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

How many different k values are there?

Answer: 3

$$\det(A - KI_3) = \det\left(A = \begin{pmatrix} 2 & -2 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} - k \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) = \det \begin{pmatrix} 2-k & -2 & 0 \\ -3 & 1-k & 0 \\ 0 & 0 & 3-k \end{pmatrix}$$

Using Laplace expansion along row 3 we have

$$\begin{vmatrix} 2-k & -2 & 0 \\ -3 & 1-k & 0 \\ 0 & 0 & 3-k \end{vmatrix} = (3-k) \begin{vmatrix} 2-k & -2 \\ -3 & 1-k \end{vmatrix} = (3-k)((2-k)(1-k)-6) = (3-k)(k^2-3k-4)$$

$$= (3-k)(k-4)(k+1)$$

Therefore $\det(A - kI_3) = 0$ implies $k = -1$, 3 or 4, so there are 3 suitable k values.

2. Consider the system $A\mathbf{x} = \mathbf{b}$ where $A = \begin{pmatrix} 3 & -1 \\ 5 & 5 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -2 \\ 10 \end{pmatrix}$.

Find the ordered triple, $(|A|, |A_1(\mathbf{b})|, |A_2(\mathbf{b})|)$, which are the determinants used in Cramer's rule.

A: $(10, 0, 40)$ B: $(20, 10, 40)$ C: $(20, 0, 20)$ D: $(20, 0, 40)$ E: Neither

Answer: D

Determinants: $|A| = \begin{vmatrix} 3 & -1 \\ 5 & 5 \end{vmatrix} = 20$, $|A_1(\mathbf{b})| = \begin{vmatrix} -2 & -1 \\ 10 & 5 \end{vmatrix} = 0$, $|A_2(\mathbf{b})| = \begin{vmatrix} 3 & -2 \\ 5 & 10 \end{vmatrix} = 40$.

By Cramer's rule: $x_1 = \frac{|A_1(\mathbf{b})|}{|A|} = 0$, $x_2 = \frac{|A_2(\mathbf{b})|}{|A|} = 2$.

3. Consider the system $A\mathbf{x} = \mathbf{b}$ where $A = \begin{pmatrix} 1 & 5 \\ 2 & -3 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

If $x_1 = 2$ find $|A_1(\mathbf{b})|$ via Cramer's rule.

Answer: -26

Since $|A| = \begin{vmatrix} 1 & 5 \\ 2 & -3 \end{vmatrix} = -13$ and $2 = x_1 = \frac{|A_1(\mathbf{b})|}{|A|}$, we can rearrange to get that

$$|A_1(\mathbf{b})| = 2|A| = -26.$$

4. Consider the system $A\mathbf{x} = \mathbf{b}$ where $A = \begin{pmatrix} 2 & 5 \\ 2 & 3 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

   If $x_1 = 3$ and $|A_2(b)| = 4$ find $\mathbf{b}$.

   A: $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$, B: $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$, C: $\begin{pmatrix} -4 \\ 3 \end{pmatrix}$, D: $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$, E: $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ F: Neither of these

   Answer: $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

   Since $|A| = \begin{vmatrix} 2 & 5 \\ 2 & 3 \end{vmatrix} = -4$, Cramer's rule yields $x_2 = \frac{|A_2(b)|}{|A|} = \frac{4}{-4} = -1$. Thus

$$\mathbf{b} = A\mathbf{x} = \begin{pmatrix} 2 & 5 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

5. Find the Adjoint of $\begin{pmatrix} 1 & -1 & 2 \\ 4 & 0 & 6 \\ 0 & 1 & -1 \end{pmatrix}$.

   A: $\begin{pmatrix} -6 & 1 & -6 \\ 4 & -1 & 2 \\ 4 & -1 & 4 \end{pmatrix}$ B: $\begin{pmatrix} -6 & 1 & 0 \\ 4 & -1 & 2 \\ 4 & -1 & 4 \end{pmatrix}$ B: $\begin{pmatrix} -6 & 1 & -6 \\ 4 & -1 & 0 \\ 4 & -1 & 4 \end{pmatrix}$ D: $\begin{pmatrix} -6 & 1 & -6 \\ 4 & -1 & 2 \\ 4 & 0 & 4 \end{pmatrix}$

   Answer: A

$$\text{Matrix of cofactors: } \begin{pmatrix} \begin{vmatrix} 0 & 6 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 4 & 6 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} 4 & 0 \\ 0 & 1 \end{vmatrix} \\ -\begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} -1 & 2 \\ 0 & 6 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 4 & 6 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 4 & 0 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} -6 & 4 & 4 \\ 1 & -1 & -1 \\ -6 & 2 & 4 \end{pmatrix}$$

$$\text{Adjoint: } \begin{pmatrix} -6 & 4 & 4 \\ 1 & -1 & -1 \\ -6 & 2 & 4 \end{pmatrix}^T = \begin{pmatrix} -6 & 1 & -6 \\ 4 & -1 & 2 \\ 4 & -1 & 4 \end{pmatrix}.$$

   (we could also have eliminated incorrect answers by checking the differing cofactor in each row)

6. If $M = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{pmatrix}$ then $\text{adj}(M) = \begin{pmatrix} 24 & -12 & -2 \\ 5 & 3 & -5 \\ -4 & 2 & 4 \end{pmatrix}$. Use this to find $\det M$.

Answer: 22

Since $M^{-1} = \frac{1}{\det M}\text{adj}(M)$, we have $I = MM^{-1} = \frac{1}{\det M}M\text{adj}(M)$. Rearrange this to get

$$\det MI = M\text{adj}(M) = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{pmatrix} \begin{pmatrix} 24 & -12 & -2 \\ 5 & 3 & -5 \\ -4 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 22 & 0 & 0 \\ 0 & 22 & 0 \\ 0 & 0 & 22 \end{pmatrix}.$$

Therefore $\det M = 22$.