Example of Laplace Expansion for a 4×4 matrix.

Find the determinant of

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 1 & 0 & 3 & 1 \\ 2 & 1 & 4 & 3 \end{pmatrix}.$$

Using Laplace expansion (or cofactor expansion) along row 1:

$$\begin{vmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 1 & 0 & 3 & 1 \\ 2 & 1 & 4 & 3 \end{vmatrix} = (1)\begin{vmatrix} 0 & 1 & 0 \\ 0 & 3 & 1 \\ 1 & 4 & 3 \end{vmatrix} - (2)\begin{vmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 2 & 4 & 3 \end{vmatrix} + (0)\begin{vmatrix} 2 & 0 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{vmatrix} - (1)\begin{vmatrix} 2 & 0 & 1 \\ 1 & 0 & 3 \\ 2 & 1 & 4 \end{vmatrix}$$

Now compute each 3×3 determinant (we will expand along row 1 for these, but you may choose):

$$\begin{vmatrix} 0 & 1 & 0 \\ 0 & 3 & 1 \\ 1 & 4 & 3 \end{vmatrix} = (0)\begin{vmatrix} 3 & 1 \\ 4 & 3 \end{vmatrix} - (1)\begin{vmatrix} 0 & 1 \\ 1 & 3 \end{vmatrix} + (0)\begin{vmatrix} 0 & 3 \\ 1 & 4 \end{vmatrix} = (0)(5) - (1)(-1) + (0)(-3) = 1.$$

$$\begin{vmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 2 & 4 & 3 \end{vmatrix} = (2)\begin{vmatrix} 3 & 1 \\ 4 & 3 \end{vmatrix} - (1)\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} + (0)\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = (2)(5) - (1)(1) + (0)(-2) = 9.$$

$$\begin{vmatrix} 2 & 0 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{vmatrix} = (2)\begin{vmatrix} 0 & 1 \\ 1 & 3 \end{vmatrix} - (0)\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} + (0)\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = (2)(-1) - (0)(1) + (0)(1) = -2.$$

$$\begin{vmatrix} 2 & 0 & 1 \\ 1 & 0 & 3 \\ 2 & 1 & 4 \end{vmatrix} = (2)\begin{vmatrix} 0 & 3 \\ 1 & 4 \end{vmatrix} - (0)\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} + (1)\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = (2)(-3) - (0)(-2) + (1)(1) = -5.$$

We have used the 2×2 determinant formula, "$ad - bc$," here. Of course, there is no need to compute the determinant that is multiplied by a zero, but it is done here any way to show the pattern.

Plug this back into our expression above to get

$$\begin{vmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 1 & 0 & 3 & 1 \\ 2 & 1 & 4 & 3 \end{vmatrix} = (1)(1) - (2)(9) + (0)(-2) - (1)(-5) = -12.$$