two dice are rolled

Example 3.2.4 *Return to the dice rolling experiment.*

Let Y be the maximum that either die shows in a single roll: $Y(a,b) = \max(a,b)$.

For example, $Y(3,5) = 5$.

$Y(2,6) = 6$
$Y(4,4) = 4$

(a) What is the range of Y? $\{1,2,3,4,5,6\}$

(b) What is $P(Y = y)$ for each y in the range of Y?

$P(Y=1) = \frac{1}{36}$
$P(Y=2) = \frac{3}{36}$
$P(Y=3) = 5/36$
$P(Y=4) = 7/36$
$P(Y=5) = 9/36$
$P(Y=6) = 11/36$

11	12	13	14	15	16
21	22	23	24	25	26
31	32	33	34	35	36
41	42	43	44	45	46
51	52	53	54	55	56
61	62	63	64	65	66

(c) Find a formula for the probability distribution of Y.

$$P(Y=y) = \frac{2y-1}{36}$$

Example 3.2.5 *Check whether the function given by*

$$f(x) = \frac{x+2}{25},$$

range

for $x = 1, 2, 3, 4, 5$ *can serve as the probability distribution of a discrete random variable.*

Check two things: 1) Are all $f(x)$ values non-negative? 2) Do the values of $f(x)$ add up to 1?

1) Yes, they are all non-negative.

2) $f(1)+f(2)+f(3)+f(4)+f(5) = \frac{3}{25}+\frac{4}{25}+\frac{5}{25}+\frac{6}{25}+\frac{7}{25} = \frac{25}{25} = 1$

So, yes the given function can serve as the probability distribution for a discrete random variable.

Probability distributions for a random variable, say X, may be represented graphically by means of a **probability histogram**.

Each rectangle corresponds to a value for X, its height is $P(X = x)$, and its width is 1, so that the area of each rectangle equals $P(X = x)$. *The total area of the histogram is 1.*

The probability histogram below is for the number of heads in 4 coin flips.

f:

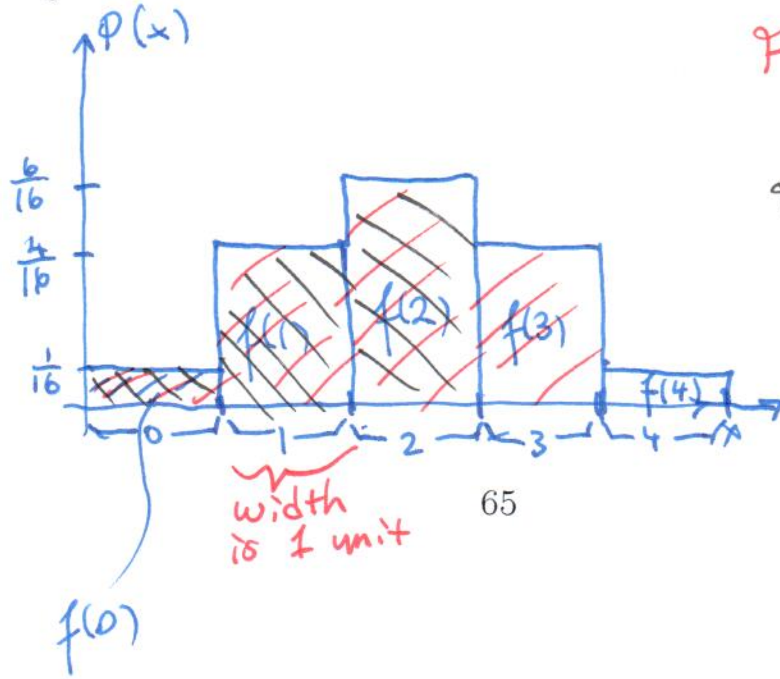
$P(X=0) = \frac{1}{16}$

$P(X=1) = \frac{4}{16}$

$P(X=2) = \frac{6}{16}$

$P(X=3) = \frac{4}{16}$

$P(X=4) = \frac{1}{16}$



$F(3) = f(0)+f(1)+f(2)+f(3)$

$F(2) = f(0)+f(1)+f(2)$

$F(3) - F(2) = f(3)$

3.3 Cumulative Distribution

In many problems we are interested in the probability that the value of a random variable is less than or equal to (or "at most") some real number x. i.e. $P(X \leq x)$.

If X is a discrete random variable with probability distribution f, the function given by

$$F(x) = P(X \leq x) = \sum_{t \leq x} f(t)$$

for $x \in (-\infty, \infty)$, is called the **cumulative distribution of** X(also called the **distribution function**).

Example 3.3.1 *Let X be the random variable that counts the number of heads in 4 coin flips.*

$f(0) = \frac{1}{16}$ $f(1) = \frac{4}{16}$ $f(2) = \frac{6}{16}$

$f(3) = \frac{4}{16}$ $f(4) = \frac{1}{16}$

$f(2) = \dfrac{6}{16}$ *while*

$F(2) = f(0) + f(1) + f(2) = \dfrac{1}{16} + \dfrac{4}{16} + \dfrac{6}{16} = \dfrac{11}{16}$

[redacted]

Back to a previous example:

Example 3.3.2 *Two socks are selected at random and removed in succession from a drawer containing five brown socks and three green socks.*

Let X be the random variable that counts the number of brown socks selected.

We found these values earlier on:

Element of sample space	*Probability*	x
BB	$\frac{20}{56}$	*2*
BG	$\frac{15}{56}$	*1*
GB	$\frac{15}{56}$	*1*
GG	$\frac{6}{56}$	*0*

The probability distribution f is given by

$$f(x) = \begin{cases} \frac{20}{56} & \text{for } x = 2 \\ \frac{30}{56} & \text{for } x = 1 \\ \frac{6}{56} & \text{for } x = 0 \end{cases}$$

$F(0) = f(0) = \frac{6}{56}$

$F(1) = f(0) + f(1) = \frac{6}{56} + \frac{30}{56} = \frac{36}{56}$

$F(2) = f(0) + f(1) + f(2) = \frac{5}{56} + \frac{30}{56} + \frac{20}{56} = 1$

Find $F(0), F(1), F(2)$, and express the cumulative distribution (distribution function) $F(x)$ as a piece-wise defined function.

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{6}{56} & 0 \le x < 1 \\ \frac{36}{56} & 1 \le x < 2 \\ \frac{56}{56} = 1 & 2 \le x \end{cases}$$

Example 3.3.3 *Suppose a random variable X has range $\{1, 2, 3, 4\}$. Define f by*

$$f(1) = \frac{1}{4}, \quad f(2) = \frac{1}{2}, \quad f(3) = \frac{1}{8}, \quad f(4) = \frac{1}{8}$$

(a) Show that f is a valid probability distribution for X.

All values of f are non-negative.

$$f(1)+f(2)+f(3)+f(4) = \frac{1}{4} + \frac{1}{2} + \frac{1}{8} + \frac{1}{8} = \frac{8}{8} = 1$$

So, f is a valid prob. distribution.

(b) Find the cumulative distribution (distribution function) for X.

$$F(1) = f(1) = \frac{1}{4}$$

$$F(2) = f(1) + f(2) = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$$

$$F(3) = f(1) + f(2) + f(3) = \frac{1}{4} + \frac{1}{2} + \frac{1}{8} = \frac{7}{8}$$

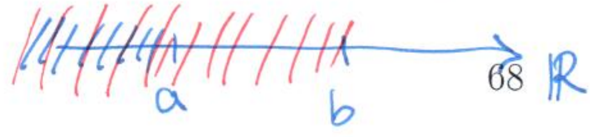
$$F(4) = f(1) + f(2) + f(3) + f(4) = \frac{1}{4} + \frac{1}{2} + \frac{1}{8} + \frac{1}{8} = 1$$

$$F(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{4} & 1 \le x < 2 \\ 3/4 & 2 \le x < 3 \\ 7/8 & 3 \le x < 4 \\ 1 & 4 \le x \end{cases}$$

Theorem 3.3.4 *The cumulative distribution $F(x)$ satisfies*

1. $\lim_{x \to -\infty} F(x) = 0$ *and* $\lim_{x \to \infty} F(x) = 1$.

2. *If $a < b$ then $F(a) \le F(b)$ for any $a, b \in \mathbb{R}$.*



Theorem 3.3.5 *If the range of a random variable X consists of the values $x_1 < x_2 < \cdots < x_n$, then $f(x_1) = F(x_1)$ and*

$$f(x_i) = F(x_i) - F(x_{i-1})$$

for $i = 2, 3, \ldots, n$.

[redacted]

$F(x_2) = f(x_1) + f(x_2)$

$F(x_3) = f(x_1) + f(x_2) + f(x_3)$

$F(x_3) - F(x_2) = f(x_3)$

Example 3.3.6 *The cumulative distribution for a discrete random variable X is given by*

$(F(-3) = 0)$

$$F(x) = \begin{cases} 0 & \text{for } x < -2 \\ \frac{4}{18} & \text{for } -2 \leq x < -1 \\ \frac{7}{18} & \text{for } -1 \leq x < 0 \\ \frac{12}{18} & \text{for } 0 \leq x < 1 \\ 1 & \text{for } x \geq 1 \end{cases}$$

$F(-2) = \frac{4}{18}$

$F(-1) = \frac{7}{18}$

$F(0) = \frac{12}{18}$

$F(1) = \frac{18}{18}$

Find the probability distribution for X.

$f(1) = F(1) - F(0) = \frac{18}{18} - \frac{12}{18} = \frac{6}{18}$

$f(0) = F(0) - F(-1) = \frac{12}{18} - \frac{7}{18} = \frac{5}{18}$

$f(-1) = F(-1) - F(-2) = \frac{7}{18} - \frac{4}{18} = \frac{3}{18}$

$f(-2) = F(-2) - F(-3) = \frac{4}{18} - 0 = \frac{4}{18}$

$$f(x) = \begin{cases} \frac{4}{18} & x = -2 \\ \frac{3}{18} & x = -1 \\ \frac{5}{18} & x = 0 \\ \frac{6}{18} & x = 1 \end{cases}$$

3.4 Continuous Random Variables

3.4.1 Probability Density Function (p.d.f)

On a 100 km stretch of rural road we are concerned with the possibility that a deer might cross.

We are interested in the probability that it will occur at a given location or stretch of the road. The sample space for this experiment consists of all points in the interval from 0-100.

Suppose the probability that a deer crosses in any stretch of road is the length of that section divided by 100.

So, from point a to point b with $0 \leq a, b \leq 100$, is the interval $[a, b]$ and its length is given by $b - a$. So, its probability is

$$P([a, b]) = \frac{b - a}{100}.$$

→ length of the stretch from a to b

total length

$\frac{b-a}{100}$ $\frac{d-c}{100}$

$b-a$ $d-c$

$\frac{b-a}{100} + \frac{d-c}{100}$

The probability of any two or more non-overlapping intervals can be found by summing the probabilities of the connected components.

Thus the probability measure proposed here has non-negative values, assigns the entire sample space a probability of 1, and is countably additive; hence it satisfies our postulates of probability.

$$\frac{100-0}{100} = 1$$

We have taken the sample space to be any point on this stretch of road, and the random variable X here is the function that assigns that point to a real number in the interval $[0, 100]$. This is an example of a **continuous random variable**.

We can give the probability that X lies within an interval by

$$P(a \leq x \leq b) = \frac{b-a}{100}$$ $[a,b]$

for $a < b$, however the probability that X is any single point is zero.

In the case of a continuous random variable, probabilities cannot simply be assigned to every outcome as is done with a discrete random variable.

Therefore a **continuous random variable** must be accompanied by a **probability density function** in order to compute probabilities.

A positive-valued function f defined on $\mathbb{R}$ is called a **probability density function** for continuous random variable X, if and only if

$$P(a \leq X \leq b) = \int_a^b f(x)\, dx$$

for any $a, b \in \mathbb{R}$ with $a \leq b$. These are also called "**p.d.f**'s" for short.

In the deer crossing example, the p.d.f. for X is $f(x) = \frac{1}{100}$.

For example

$$P(35 \leq X \leq 50) = \int_{35}^{50} \frac{1}{100}\, dx$$

$$= \frac{x}{100}\Big|_{35}^{50} = \frac{50-35}{100} = \frac{15}{100}.$$

$$\int_{\underset{=a}{35}}^{\overset{=b}{50}} f(x)\,dx = \int_{35}^{50} \frac{1}{100}\,dx = \frac{x}{100}\Big|_{35}^{50} = \frac{50}{100} - \frac{35}{100} = \frac{15}{100}$$

Notice that $f(r)$ does not give the probability that $X = r$.

0 35 50 100

50 - 35 = 15

Prob. that a deer might cross between 35 and 50 is 15/100

Let X be a continuous random variable. By properties of integrals it follows that

Theorem 3.4.1 *If $a, b \in \mathbb{R}$ with $a \leq b$ then*

$$P(a \leq X \leq b) = P(a \leq X < b) = P(a < X \leq b) = P(a < X < b).$$

$[a,b]$ $[a,b)$ $(a,b]$ (a,b)

(informally: because the contribution of a single point to an integral is 0)

From the postulates of probability we obtain the following result:

Theorem 3.4.2 *A function f can serve as a probability density function for X only if it satisfies*

1. $f(x) \geq 0$ *for all* $x \in \mathbb{R}$.
2. $\int_{-\infty}^{\infty} f(x)\, dx = 1.$

(very similar to the discrete case)

continuous random variable

Example 3.4.3 *If X has probability density function*

$$f(x) = \begin{cases} k \cdot e^{-3x} & \text{for } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

find k and $P(0.5 \leq X \leq 1)$.

First we need to find what k is.

Since f is given to be a probability density function, it satisfies Condition 2 of the previous theorem.

Solve for k using Condition 2. from the theorem.

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{0} 0\, dx + \int_{0}^{\infty} k \cdot e^{-3x}\, dx \\ &= \lim_{c \to \infty} k \left. \frac{e^{-3x}}{(-3)} \right|_0^c \\ &= \lim_{c \to \infty} k \frac{e^{-3c}}{(-3)} - k \frac{e^{-3(0)}}{(-3)} \\ &= \frac{k}{3}. \qquad \text{(since } \lim_{r \to \infty} e^{-r} = 0) \end{aligned}$$

Thus $k = 3$. Now we can compute

$$P(0.5 \leq X \leq 1) = \int_{0.5}^{1} f(x)\, dx = \int_{0.5}^{1} 3e^{-3x}\, dx = -e^{-3x}\Big|_{0.5}^{1}$$

$$= -e^{-3} - (-e^{-1.5}) \approx 0.1733$$

From Theorem 3.4.2 on page 74 (part 2.) we know that

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

p.d.f

$$f(x) = \begin{cases} k \cdot e^{-3x} & x > 0 \\ 0 & x \le 0 \end{cases}$$

$$\Rightarrow \cancel{\int_{-\infty}^{0} 0\,dx} + \int_{0}^{\infty} k \cdot e^{-3x}\,dx = 1$$

improper integrals

$$\Rightarrow \int_{0}^{\infty} k \cdot e^{-3x}\,dx = 1 \quad \Rightarrow \lim_{c\to\infty} \int_{0}^{c} k \cdot e^{-3x}\,dx = 1$$

an antiderivative for $k \cdot e^{-3x}$ is $\frac{k}{-3} e^{-3x}$

$$\Rightarrow \lim_{c\to\infty} \left(\frac{k}{-3} e^{-3x} \Big|_{0}^{c} \right) = 1$$

$$\Rightarrow \lim_{c\to\infty} \left(\cancel{\frac{k}{-3} e^{-3c}} \to 0 - \frac{k}{-3} e^{-3\cdot 0} \right) = 1$$

$$\Rightarrow -\frac{k}{-3} e^{0} = 1 \quad \Rightarrow \frac{k}{3} \cdot 1 = 1 \quad \Rightarrow \underline{k = 3}$$

Next, we find $P(0.5 \le X \le 1)$.

That is, we calculate $\int_{0.5}^{1} f(x)\,dx = \int_{0.5}^{1} 3 \cdot e^{-3x}\,dx$

an antiderivative for $3 \cdot e^{-3x}$ is

$$3 \cdot \frac{e^{-3x}}{-3} = \underline{-e^{-3x}}$$

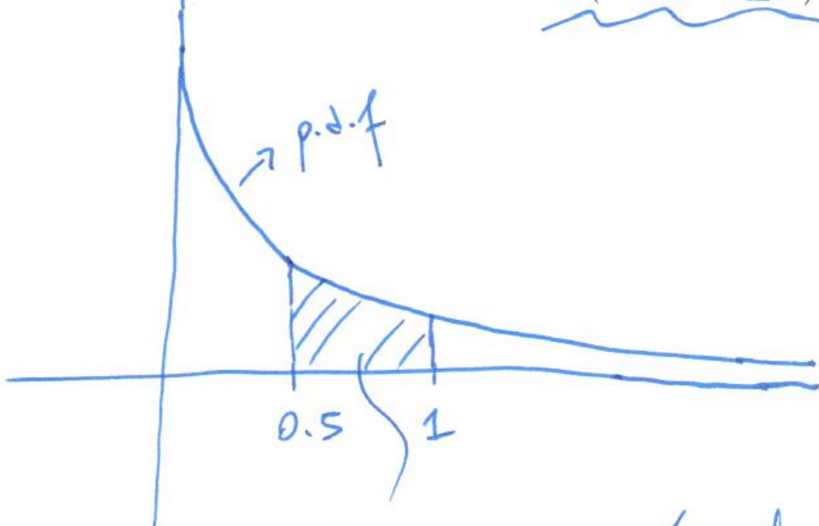
$$= -e^{-3x} \Big|_{0.5}^{1} = -e^{-3\cdot 1} - \left(-e^{-1.5}\right)$$

$$= -e^{-3} + e^{-1.5} \approx 0.1733$$

probability
density function

Graph of $3e^{-3x}$ is given below.

The shaded area denotes $P(0.5 \leq X \leq 1)$.



the area of the
shaded region is $= P(0.5 \leq x \leq 1)$

3.4.2 Cumulative Distribution Function of a Continuous Random Variable

Let X ~~is~~ be a continuous random variable with probability density function f. Then the function

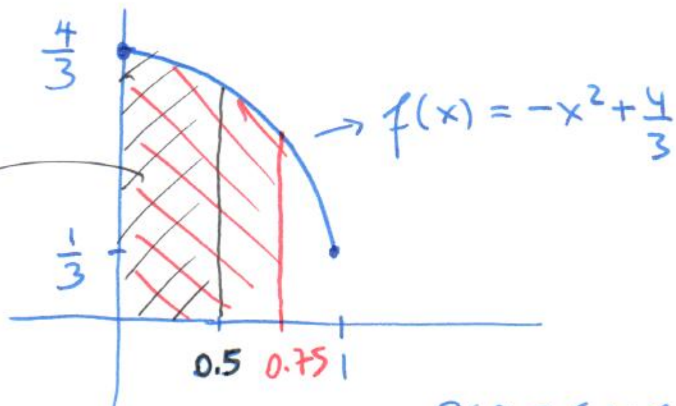
$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\, dt$$

for all $x \in \mathbb{R}$, is called the **cumulative distribution function of** X.

Example 3.4.4 *Random variable X with p.d.f. $f(x) = -x^2 + \frac{4}{3}$ for $0 \leq x \leq 1$ and 0 elsewhere. (p.d.f. plotted in ~~red~~ blue)*

cumulative distribution function

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$



area is $F(0.5)$

$$P(0.5 \leq X \leq 0.75) = F(0.75) - F(0.5)$$

Cumulative distribution function is $F(x) = \int_{-\infty}^{x} f(t)\, dt$.

Shade the areas representing the values $F(0.5)$ and $F(0.75)$ respectively.

From the properties of integrals we have the following.

Theorem 3.4.5 *If continuous random variable X has probability density function $f(x)$ and cumulative distribution function $F(x)$ then*

$$P(a \leq X \leq b) = F(b) - F(a)$$

for any $a, b \in \mathbb{R}$ with $a \leq b$, and

$$f(x) = \frac{d}{dx}F(x)$$

where the derivative exists.

Using the previous example with p.d.f. $f(x) = -x^2 + \frac{4}{3}$ for $0 \leq x \leq 1$, and 0 elsewhere, we have:

$$P(0.25 \leq X \leq 0.75) = F(0.75) - F(0.25)$$

Let's see this considering the relevant shaded areas on the corresponding graphs.

done on the previous page

p.d.f

$$f(x) = \begin{cases} -x^2 + \frac{4}{3} & 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

The cumulative distribution function is

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \underbrace{\int_{0}^{x} -t^2 + \frac{4}{3}\,dt}_{\text{because } f \text{ is } 0 \text{ on } (-\infty,0)} = \ldots \left(\frac{-t^3}{3} + \frac{4}{3}t\right)\Bigg|_{t=0}^{t=x}$$

$$= \frac{-x^3}{3} + \frac{4}{3}x - 0 = -\frac{x^3}{3} + \frac{4}{3}x$$

and its derivative is the probability density function

$$\frac{d}{dx}F(x) = \frac{d}{dx}\left(-\frac{x^3}{3} + \frac{4x}{3}\right) = \ldots - \frac{\cancel{3}x^2}{\cancel{3}} + \frac{4}{3} = -x^2 + \frac{4}{3}$$