The Probability of an Event:

A **probability**, or **probability measure**, is a function P which maps events in the sample space S to real numbers.

In order to assign probabilities in a meaningful way, P must satisfy the following called the **postulates** (or **axioms**) **of probability**.

P1: The probability of any event A in S is a non-negative real number, i.e. $P(A) \geq 0$.

P2: $P(S) = 1$.
sample space

P3: If $A_1, A_2, A_3, \ldots$, is a finite or infinite sequence of (pairwise) mutually exclusive events in S then

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots) = P(A_1) + P(A_2) + P(A_3) + \cdots$$

(P is **countably additive**)

2.1.5 Postulates of Probability

- Interpreting a probability as a frequency, or a proportion of time, it makes sense that $P(A) \geq 0$; in fact we will show that $0 \leq P(A) \leq 1$ for any event A.

- P2 says that the probability that outcome of the experiment lies in S must be assigned value 1. Since this is certain to happen, we interpret $P(A) = 1$ as "A happens 100 percent of the time."

- P3 is for consistency. For example, if events A_1 and A_2 share no common outcomes, then the probability that either event occurs, $P(A_1 \cup A_2)$, is the sum of their individual probabilities.

A technical detail has been overlooked in the postulates of probability presented above. In a discrete sample space S, an "event" can be any subset of S, however in the continuous case one has to be more careful about which subsets of S are allowed as events. A precise definition for these allowable events comes in a course on **measure theory**. In this course we won't require that level of detail; i.e. the subsets we assign probabilities to will be allowable events.

Single Die Roll:
Let S be the sample space for rolling a die once.

Example 2.1.8 *Each outcome in S is its own event, call these $A_1, \ldots, A_6$.*

$S = \{1, 2, 3, 4, 5, 6\}$

Events $A_1, \ldots, A_6$ are mutually exclusive, and any event E in S is a union of these, for example let $\underline{E = A_2 \cup A_4 \cup A_5}$.

$A_1 = \{1\}$ $A_2 = \{2\}$
$A_3 = \{3\}$ $A_4 = \{4\}$
$A_5 = \{5\}$
$A_6 = \{6\}$

By the classical probability concept, $P(E) = \frac{3}{6}$ (successes/number of outcomes), and $P(A_i) = \frac{1}{6}$ for each i.

$P(E) = \frac{3}{6} = \frac{1}{2}$ $E = \{2, 4, 5\}$

It follows that this satisfies the postulates of probability:

- $P(B) \geq 0$ *for any* $B \subset S$.
- $P(S) = \frac{6}{6} = 1$.

$P(S) = \frac{6}{6}$ # of successes / # outcomes

- *P3 is satisfied: for example* $P(E) = \frac{3}{6} = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = P(A_2) + P(A_4) + P(A_5)$.

$\frac{3}{6} = P(E)$ $E = A_2 \cup A_4 \cup A_5$

E is a union of mutually exclusive events

$P(E) = P(A_2) + P(A_4) + P(A_5)$

$\frac{3}{6}$ = $\frac{1}{6}$ + $\frac{1}{6}$ + $\frac{1}{6}$

Example 2.1.9 *Suppose we assigned probabilities in this experiment in a different way. Using the same notation as before say for any event* B *we specify that*

$$P(B) = \sum_{A_i \in B} P(A_i), \text{ and}$$

$$P(A_1) = \tfrac{1}{2}, P(A_2) = \tfrac{1}{4}, P(A_3) = \tfrac{1}{8},$$
$$P(A_4) = 0, P(A_5) = \tfrac{1}{16}, P(A_6) = \tfrac{1}{16}$$

Are the postulates (1 and 2 (on p.26); axioms) *of probability still satisfied?*

P1) Let $B \subseteq S$. $P(B) = \sum_{A_i \in B} P(A_i)$ and since each $P(A_i) \geq 0$, we have $P(B) \geq 0$ ✓

P2) $P(S) = P(A_1) + P(A_2) + P(A_3) + P(A_4) + P(A_5) + P(A_6)$
$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + 0 + \frac{1}{16} + \frac{1}{16} = \frac{16}{16} = 1$ ✓

Example 2.1.10 *An experiment has four possible outcomes* A, B, C, D *that are mutually exclusive. Explain why the following assignments of probabilities are not permissible.*

(a) $P(A) = 0.12, P(B) = 0.63, P(C) = 0.45, P(D) = -0.20$

$P(D) < 0$; so, this probability assignment is not permissible.

(b) $P(A) = \frac{9}{120}, P(B) = \frac{45}{120}, P(C) = \frac{27}{120}, P(D) = \frac{46}{120}$

First, observe that all numbers are non-negative.

$P(S) = P(A \cup B \cup C \cup D) = P(A) + P(B) + P(C) + P(D)$
$= \frac{9}{120} + \frac{45}{120} + \frac{27}{120} + \frac{46}{120} = \frac{127}{120} \neq 1$

So, P2 (second postulate) fails; therefore this probability assignment is not permissible.

2.1.6 The Probability of an Event

Theorem 2.1.11 *If A is an event in a discrete sample space S, then $P(A)$ is the sum of the probabilities of the individual outcomes (elements) of A.*

(Note that the theorem assumes that P is a probability measure, and hence satisfies the postulates.)

Example 2.1.12 *Experiment: Tossing a coin three times.*
Sample space: {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

Event A: Getting at least two heads $\{HHH, HHT, HTH, THH\}$

Event B: Getting exactly two tails {HTT, THT, TTH}

Event C: Getting two consecutive heads $\{HHH, HHT, THH\}$

Assuming this is a ***balanced (fair)*** *coin, i.e. equal likely heads or tails, what are the probabilities of the events above?*

$P(A) = P(HHH) + P(HHT) + P(HTH) + P(THH) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$

$P(B) = P(HTT) + P(THT) + P(TTH) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$

$P(C) = P(HHH) + P(HHT) + P(THH) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$

Example 2.1.13 *Suppose our six sided die is weighted so that each odd number is twice as likely to occur than each even number.*

What is the probability of rolling a number greater than 3?

$S = \{1,2,3,4,5,6\}$ $\qquad$ $A = \{4,5,6\} \to$ rolling a number greater than 3

$P(1)=2x \quad P(2)=x$

$P(3)=2x \quad P(4)=x$

$P(5)=2x \quad P(6)=x$

$P(S) = 1$

$P(S) = P(1)+P(2)+P(3)+P(4)+P(5)+P(6)$

$1 = 2x + x + 2x + x + 2x + x$

$1 = 9x \Rightarrow x = \frac{1}{9}$

$P(1)=\frac{2}{9} \quad P(2)=\frac{1}{9}$

$P(3)=\frac{2}{9} \quad P(4)=\frac{1}{9}$

$P(5)=\frac{2}{9} \quad P(6)=\frac{1}{9}$

Then, $P(A) = P(4)+P(5)+P(6)$

$= \frac{1}{9} + \frac{2}{9} + \frac{1}{9} = \frac{4}{9}$

What if instead each even number is four times as likely to occur than each odd number?

$S = \{1,2,3,4,5,6\}$ $\qquad$ $A = \{4,5,6\} \to$ rolling a number greater than 3

$P(1)=x \quad P(2)=4x$

$P(3)=x \quad P(4)=4x$

$P(5)=x \quad P(6)=4x$

$P(S) = 1$

$P(S) = P(1)+P(2)+P(3)+P(4)+P(5)+P(6)$

$1 = x + 4x + x + 4x + x + 4x$

$1 = 15x \Rightarrow x = \frac{1}{15}$

$P(1)=\frac{1}{15} \quad P(2)=\frac{4}{15}$

$P(3)=\frac{1}{15} \quad P(4)=\frac{4}{15}$

$P(5)=\frac{1}{15} \quad P(6)=\frac{4}{15}$

Then, ~~Ab~~ $P(A) = P(4)+P(5)+P(6)$

$= \frac{4}{15} + \frac{1}{15} + \frac{4}{15}$

$= \frac{9}{15} = \frac{3}{5}$