

Example 2.2.9 Find the probabilities of randomly drawing two aces in succession from an ordinary deck of 52 playing cards if we sample

(a) without replacement.

(b) with replacement.

Let A_1 be the event that card 1 is an ace, and A_2 be the event that card 2 is an ace. Then $A_1 \cap A_2$ is the event that both are aces.

(a) left as an exercise

b) left as an exercise



p. 41 (multiplication rule)

$$P(A \cap B) = P(A) \cdot P(B|A) \quad (\text{if } P(A) \neq 0)$$

Venn diagram
↑

Since $A \cap B \cap C = (A \cap B) \cap C$ we have by the multiplication rule

$$P(A \cap B \cap C) = P((A \cap B) \cap C) = P(A \cap B) \cdot P(C|A \cap B).$$

$$= P(A) \cdot P(B|A) \cdot P(C|A \cap B) \leftarrow$$

Applying the multiplication rule again to $P(A \cap B)$ gives,

Theorem 2.2.10 If A, B and C are events in S and $P(A \cap B) \neq 0$, then

$$P(A \cap B \cap C) = P(A) \cdot P(B|A) \cdot P(C|A \cap B).$$

$$P(A \cap B \cap C) = P(A) \cdot P(B|A) \cdot P(C|A \cap B)$$

Example 2.2.11 A bushel of 126 apples contains 15 rotten ones. If three apples are chosen at random, what is the probability that all three are rotten? (Solve by using the theorem.)

If A_1, A_2, A_3 are the events that the first, second and third (resp.) choice is rotten, then

$$P(A_1 \cap A_2 \cap A_3) = P(A_1) \cdot P(A_2|A_1) \cdot P(A_3|A_1 \cap A_2)$$

$$P(A_1) = \frac{15}{126}$$

$$= \frac{15}{126} \cdot \frac{14}{125} \cdot \frac{13}{124}$$

$$P(A_2|A_1) = \frac{14}{125}$$

$$P(A_3|A_1 \cap A_2) = \frac{13}{124}$$

2.3 (In)dependent Events

Example 2.3.1 Suppose a coin is tossed twice. What is the probability of getting tails on the second toss given that the first toss was tails?

Surely the outcome of the second toss does not depend on what has previously come up.

Indeed if $S = \{HH, HT, TH, TT\}$, $A_1 = \{TH, TT\}$, and $A_2 = \{HT, TT\}$ are the events of getting tails on flips 1, and 2, then

$$P(A_2|A_1) = \frac{P(A_1 \cap A_2)}{P(A_1)} = \frac{P(\{TT\})}{P(A_1)} = \frac{\frac{1}{4}}{\frac{2}{4}} = \frac{1}{2} = P(A_2).$$

$$P(A_2) = \frac{2}{4} = \frac{1}{2}$$

Similarly we can also see that $P(A_1|A_2) = P(A_1)$. In this case events A_1 and A_2 are called independent.

Remember the multiplication rule:

$$P(A \cap B) = P(A) \cdot P(B|A).$$

When the events A and B are independent, this identity simplifies to

$$P(B|A) = P(B) \quad \text{so, } P(A \cap B) = P(A) \cdot P(B)$$

$$P(A \cap B) = P(A) \cdot P(B).$$

The last identity can be thought of as the definition of independent events.

Definition 2.3.2 Events A and B are called **independent** if and only if

$$P(A \cap B) = P(A) \cdot P(B).$$

They are otherwise called **dependent**. ($P(A) = 0$ or $P(B) = 0$ also allowed)

Example 2.3.3 In the initial coin toss example, the probability of getting two consecutive tails is

A_1 : 1st toss is tails

$$P(A_1 \cap A_2) = P(A_1) \cdot P(A_2)$$

since A_1, A_2 are independent events

$$P(A_1 \cap A_2) = P(A_1) \cdot P(A_2)$$

$$\frac{1}{4} \quad \frac{1}{2} \quad \frac{1}{2}$$

Example 2.3.4 Suppose a ^{balanced} coin is tossed 3 times. Let

$A = \{HHH, \underline{HHT}\}$ - first two are H

$B = \{\underline{HHT}, \underline{HTT}, \underline{THT}, TTT\}$ - third is T

$C = \{\underline{HTT}, \underline{THT}, TTH\}$ - exactly two T

(a) Show that A and B are independent.

To show that A and B are independent, we need to show that $P(A \cap B) = P(A) \cdot P(B)$.

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

$$P(A) = \frac{2}{8} = \frac{1}{4} \quad P(B) = \frac{4}{8} = \frac{1}{2} \quad P(A \cap B) = \frac{1}{8}$$

we see that $\frac{1}{8} = P(A \cap B) \quad \frac{1}{4} \cdot \frac{1}{2} = P(A) \cdot P(B)$

So, A and B are independent.

(b) Show that B and C are dependent.

we need to show that $P(B \cap C) \neq P(B) \cdot P(C)$.

$$P(B) = \frac{4}{8} = \frac{1}{2}$$

$$P(C) = \frac{3}{8}$$

$$P(B \cap C) = \frac{2}{8} = \frac{1}{4}$$

Observe that

$$P(B \cap C) = \frac{1}{4} \neq \frac{1}{2} \cdot \frac{3}{8} = P(B) \cdot P(C);$$

so, B and C are dependent.

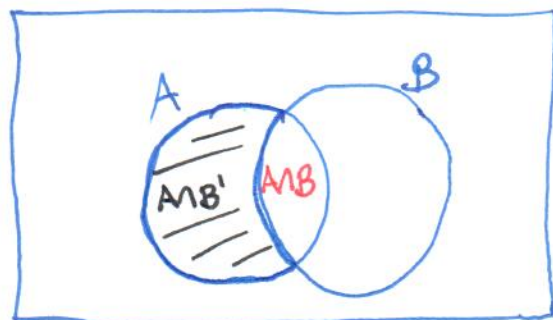
~~Example 3. Find the probability of drawing two cards from a deck of cards.~~

~~(without replacement)~~

~~Events A_1 and A_2 are dependent because~~

~~Events A and B are independent~~

~~Events A and B are independent~~



$$A = (A \cap B) \cup (A \cap B')$$

$$P(A) = P((A \cap B) \cup (A \cap B'))$$

$$= P(A \cap B) + P(A \cap B') \quad \text{By Inc-Ex Principle}$$

$$- P((A \cap B) \cap (A \cap B')) \quad \text{"B complement"}$$

Theorem 2.3.6 If A and B are independent then so are A and B' .

Proof 2.3.7 Since $A = (A \cap B) \cup (A \cap B')$ we have

$$P(A) = P(A \cap B) + P(A \cap B') \quad (\text{mutually exclusive events})$$

$$= P(A) \cdot P(B) + P(A \cap B') \quad (A, B \text{ independent}).$$

Rearrange this equation to get $P(A) = P(A) \cdot P(B) + P(A \cap B')$

$$P(A) - P(A) \cdot P(B) = P(A \cap B')$$

$$P(A \cap B') = P(A) - P(A) \cdot P(B) \Rightarrow P(A) \cdot (1 - P(B))$$

$$= P(A) \cdot (1 - P(B)) = P(A \cap B')$$

$$= P(A) \cdot P(B').$$

$$\Rightarrow P(A) \cdot P(B') = P(A \cap B')$$

Therefore, A and B' are independent.

Definition 2.3.8 Events A_1, A_2, \dots, A_k are independent if and only if the probability of the intersection of any number of these is equal to the product of their individual probabilities.

Example 2.3.9 Three events A_1, A_2, A_3 are independent if and only if

$$P(A_1 \cap A_2) = P(A_1) \cdot P(A_2)$$

$$P(A_1 \cap A_3) = P(A_1) \cdot P(A_3)$$

$$P(A_2 \cap A_3) = P(A_2) \cdot P(A_3)$$

$$P(A_1 \cap A_2 \cap A_3) = P(A_1) \cdot P(A_2) \cdot P(A_3)$$

Example 2.3.10 Let $S = \{a, b, c, d\}$ be the sample space for an experiment with equally likely outcomes and define events

$$A = \{a, d\}, \quad B = \{b, d\}, \quad C = \{c, d\}.$$

Show that A, B, C are pairwise independent, but not independent.

$$P(A) = \frac{2}{4} = \frac{1}{2} \quad P(B) = \frac{2}{4} = \frac{1}{2} \quad P(C) = \frac{2}{4} = \frac{1}{2}$$

$$P(A \cap B) = P(\{d\}) = \frac{1}{4} = P(A) \cdot P(B) \rightarrow A \& B \text{ are pairwise indep.}$$

$$P(A \cap C) = P(\{d\}) = \frac{1}{4} = P(A) \cdot P(C) \rightarrow A \& C \text{ " "}$$

$$P(B \cap C) = P(\{d\}) = \frac{1}{4} = P(B) \cdot P(C) \rightarrow B \& C \text{ " "}$$

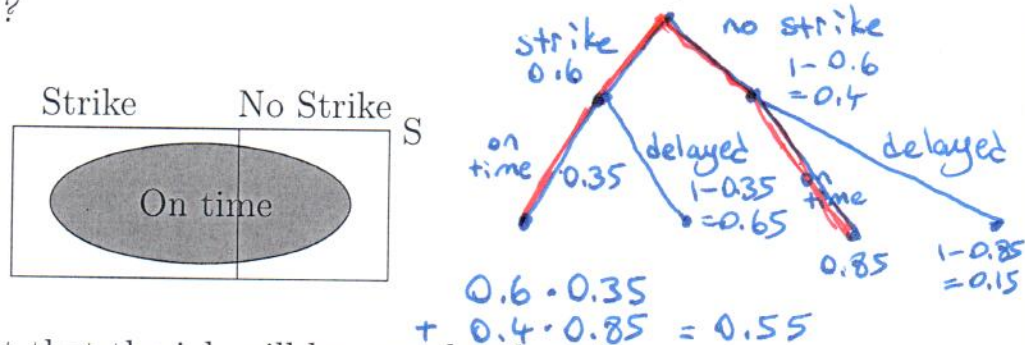
$P(A \cap B \cap C)$	$P(A) \cdot P(B) \cdot P(C)$		
"	"	"	"
$P(\{d\})$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
"			
$\frac{1}{4}$	$= \frac{1}{8}$		

50

There A, B, C are not independent

2.4 Rule of Total Probability and Bayes' Theorem

Example 2.4.1 The completion of a construction job may be delayed because of a strike. The probabilities are 0.60 that there will be a strike, 0.85 that the construction job will be completed on time if there is no strike, and 0.35 that the construction job will be completed on time if there is a strike. What is the probability that the construction job will be completed on time?



Let A be the event that the job will be completed on time,

B the event of a strike, therefore

B' is the event of no strike.

We want $P(A)$ and are given

$$P(B) = 0.60, \quad P(A|B) = 0.35, \quad P(A|B') = 0.85.$$

Using the fact that $A = (A \cap B) \cup (A \cap B')$ (union of mutually exclusive events) and the multiplicative rule, we have

$$P(A) = P(A \cap B) + P(A \cap B') = P(B) \cdot P(A|B) + P(B') \cdot P(A|B').$$

Thus $P(A) =$