ex: (yesterday) BANANA   3 A's, 2 N's, 1 B

$$\frac{\cancel{6}^{3}\cdot 5\cdot 4\cdot \cancel{3\cdot 2\cdot 1}}{\cancel{(3\cdot 2\cdot 1)}\cdot(\cancel{2}\cdot 1)\cdot 1} = 60 \qquad \frac{6!}{3!\cdot 2!\cdot 1!} = 60$$

$n = 6$
$n_1 = 3$
$n_2 = 2$
$n_3 = 1$

Theorem 1.2.4 ***(Permutations with Repeated Elements)*** *The number of permutations of n objects of which n_1 are of one kind, n_2 are of a second kind, ..., n_k are of a kth kind and $n = n_1 + n_2 + \cdots + n_k$ is*

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1! \cdot n_2! \cdot \cdots \cdot n_k!}.$$

Example: How many different numbers can be formed with the single digits 1, 2, 2, 3, 3, 3, 3?

7 digits in total

one 1
two 2's
four 3's

$$\frac{7!}{1!\cdot 2!\cdot 4!} = \frac{7\cdot \cancel{6}^{3}\cdot 5\cdot \cancel{4\cdot 3\cdot 2\cdot 1}}{1\cdot(\cancel{2}\cdot 1)\cdot \cancel{(4\cdot 3\cdot 2\cdot 1)}} = 105$$

say A, B are the offices with 4 desks each

Example: In how many ways can 20 grad students be put into 4 different offices; two with 6 desks and two with 4 desks?

Offices: A B C D (2 offices with 4 desks each, and 2 offices with 6 desks each)

Students: 1 2 3 ... 20
B C B ... A → corresponds to a word with 20 letters where A and B appear 4 times each, and C and D appear 6 times each.

So, we are basically forming 20-letter words using 4 A's, 4 B's, 6 C's and 6 D's.

We apply the formula from p.7

$$\frac{20!}{\underset{4A}{4!}\cdot\underset{4B}{4!}\cdot\underset{6C}{6!}\cdot\underset{6D}{6!}} = \ldots \text{ (compute this)}$$

1.3 Combinations and the Binomial Theorem

A **combination** of n objects taken r at a time is any subset of size r taken from a set of size n. The order of the selection does not matter (in contrast to a "combination" lock).

Example: A photographer is allowed to choose three photos to display at an upcoming art exhibit. How many possible arrangements can be made from a selection of six photos?

6 photos

a b c d e f

acd — any (unordered) choice of 3 photos results in $3! = 6$ ordered choices

adc

cad

cda

dac

dca

- Choosing in order, there are ${}_6P_3 = 6 \cdot 5 \cdot 4 = 120$ permutations.
- Each set of three appears $3! = 6$ times in a different order.
- Ignoring order there are $\frac{120}{6} = 20$ ways to choose three photos from six.

Theorem 1.3.1 ***(Combinations)*** *The number of ways to choose r objects from n distinct objects is*

# of ways to choose r objects from a set of n objects

$n = 6$

$r = 3$

"n choose r"

$$\binom{n}{r} = \frac{{}_nP_r}{r!} = \frac{n!}{r!(n-r)!}$$

for $r = 0, 1, \ldots, n.$

$0! = 1$

$${}_nP_r = \frac{n!}{(n-r)!}$$

$$\frac{{}_nP_r}{r!} = \frac{n!}{r!(n-r)!}$$

Example: Q: In the card game *cribbage*, one pair scores two points. How many points are scored with four nines?

9♠ 9♡ 9♢ 9♣

So, we need to find the number of pairs of 9's formed by a set of four 9's.

Equivalently, we want to find the number of 2-element subsets from a set of 4 elements.

$$\binom{4}{2} = \frac{4!}{2! \cdot (4-2)!} = \frac{4!}{2! \cdot 2!} = \frac{\overset{2}{\cancel{4}} \cdot 3 \cdot \cancel{2 \cdot 1}}{\cancel{(2 \cdot 1)} \cdot \cancel{(2 \cdot 1)}} = 6$$

pairs of 9's

Since we obtain 2 points for each pair, we score $2 \cdot 6 = 12$ points.

1.3.1 Binomial Coefficients

Powers of a binomial $x+y$, are computed using properties of real numbers (distributivity, associativitiy, commutativity). e.g.

$$\begin{aligned}(x+y)^2 = (x+y)(x+y) = (x+y)x + (x+y)y &= xx + yx + xy + yy \\ &= x^2 + 2xy + y^2.\end{aligned}$$

Expanding $(x+y)^n$ for large n is impractical. Instead we compute the coefficients of each $x^k y^{n-k}$ term in the result with counting techniques.

Example:

$$\begin{aligned}(x+y)^3 &= (x+y)(x+y)(x+y) \\ &= xxx + xxy + xyx + yxx + xyy + yxy + yyx + yyy \\ &= x^3 + 3x^2y^1 + 3xy^2 + y^3.\end{aligned}$$

To obtain the second line:

$\binom{3}{0}=1$ $1x^3$

$$xxx \leftrightsquigarrow (\boxed{x}+y)(\boxed{x}+y)(\boxed{x}+y)$$

$\binom{3}{1}=3$ $3x^2y^1$

$$\begin{aligned}xxy &\leftrightsquigarrow (\boxed{x}+y)(\boxed{x}+y)(x+\boxed{y}) \\ xyx &\leftrightsquigarrow (\boxed{x}+y)(x+\boxed{y})(\boxed{x}+y) \\ yxx &\leftrightsquigarrow (x+\boxed{y})(\boxed{x}+y)(\boxed{x}+y)\end{aligned}$$

$\binom{3}{2}=3$ $3x^1y^2$

$$\begin{aligned}xyy &\leftrightsquigarrow (\boxed{x}+y)(x+\boxed{y})(x+\boxed{y}) \\ yxy &\leftrightsquigarrow (x+\boxed{y})(\boxed{x}+y)(x+\boxed{y}) \\ yyx &\leftrightsquigarrow (x+\boxed{y})(x+\boxed{y})(\boxed{x}+y)\end{aligned}$$

$\binom{3}{3}=1$ $1y^3$

$$yyy \leftrightsquigarrow (x+\boxed{y})(x+\boxed{y})(x+\boxed{y})$$

Choose k factors (of the three) to provide y to get a $x^{3-k}y^k$ term for $k=0,1,2,3$.

For example, there are $\binom{3}{2}=3$ ways to obtain an xy^2 term by choosing y from two of the factors and x from the remaining one.

$$\begin{aligned}(x+y)^5 &= \binom{5}{0}x^5 + \binom{5}{1}x^4y^1 + \overset{10}{\binom{5}{2}}x^3y^2 + \binom{5}{3}x^2y^3 + \binom{5}{4}xy^4 + \binom{5}{5}y^5 \\ &= 1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5\end{aligned}$$

$0! = 1$

$\binom{n}{0} = 1$

$\binom{n}{n} = 1$

$x^0 = 1$
$(x \neq 0)$

Ex: $n = 4$

Theorem 1.3.2 *(The Binomial Theorem)*
For $n \in \mathbb{N}$

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r.$$

$$(x+y)^4 = \sum_{r=0}^{4} \binom{4}{r} x^{4-r} y^r$$
$$= \binom{4}{0} x^4 y^0 + \binom{4}{1} x^3 y^1 + \binom{4}{2} x^2 y^2 + \binom{4}{3} x y^3 + \binom{4}{4} x^0 y^4$$
$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

- Expressions $\binom{n}{r}$ are called **binomial coefficients**.
- Choosing r things from n things indirectly chooses $n-r$ things to leave behind. We have the following result:

Theorem 1.3.3 *For $n \in \mathbb{N}$ and $r = 0, 1, \ldots, n$*

$$\binom{n}{r} = \binom{n}{n-r}.$$

$n = 7$
$r = 2$

$$\binom{7}{2} = \binom{7}{7-2} = \binom{7}{5}$$

$$\binom{7}{2} = \frac{7!}{2!\,(7-2)!}$$

$$\binom{7}{5} = \frac{7!}{5!\,(7-5)!}$$