MATH1550
Practice Set 9

These exercises are suited to Chapter 4, from Moments to Moment Generating Functions.
Topics Covered:

- Moments about the origin and moments about the mean.
- Mean, variance and standard deviation.
- Chebyshev's Theorem.
- Moment generating functions.

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1. (a) Give the definition for the *rth moment about the origin* of a random variable.
   (b) What is the *mean* of a probability distribution? What symbol is used for the mean?
   (c) Give the definition for the *rth moment about the mean* of a random variable.
   (d) What are the *variance* and *standard deviation* of a random variable? What symbols are used for the variance and standard deviation?
   (e) Give the "shortcut" formula for finding the variance.
   (f) State Chebyshev's Theorem.
   (g) How do we find the *moment generating function* for a random variable?
   (h) How is the moment generating function used?

   Solution. (a) The *rth moment about the origin* for a discrete random variable X with probability distribution $f(x)$ is

   $$E(X^r) = \sum_x x^r f(x)$$

   and the *rth moment about the origin* for a continuous random variable X with probability density $f(x)$ is

   $$E(X^r) = \int_{-\infty}^{\infty} x^r f(x)\, dx$$

   (b) The *mean* of a probability distribution for random variable X is the expected value $E(X)$. We use μ to denote the mean.

   (c) The *rth moment about the mean* for a discrete random variable X with probability distribution $f(x)$ is

   $$E((X-\mu)^r) = \sum_x (x-\mu)^r f(x)$$

   and the *rth moment about the mean* for a continuous random variable X with probability density $f(x)$ is

   $$E((X-\mu)^r) = \int_{-\infty}^{\infty} (x-\mu)^r f(x)\, dx$$

   (d) The *variance* of a random variable X is the second moment about the mean or $E((X-\mu)^2)$, and it is denoted by $\mathrm{var}(X)$ or σ^2. The *standard deviation* is the positive square root of the variance, i.e. $\sqrt{E((X-\mu)^2)}$, and hence denoted by σ.

   (e) Using properties of expected value we may compute the variance by

   $$E((X-\mu)^2) = E(X^2) - \mu^2$$

(f) Let X be a random variable with mean μ and standard deviation σ. Chebyshev's Theorem says that for and $k > 0$,

$$P(|X - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}.$$

In words: The probability that X lies within k standard deviations of the mean is at least $1 - \frac{1}{k^2}$.

(g) The *moment generating function* for a random variable X is defined as the function of t given by

$$M_X(t) = E(e^{tX}).$$

(h) The rth moment about the origin can be extracted from the moment generating function by

$$E(X^r) = \frac{d^r}{dt^r} M_X(t)\Big|_{t=0}$$

i.e. by taking its rth derivative with respect to t and setting t equal to zero. Note that by using appropriate "short cut" formulas like the one for variance, we can express rth moments about the mean in terms of moments about the origin, which are obtained from the moment generating function.

□

2. Let X be a discrete random variable with the given probability distribution.

(a)

x	0	1	2	3	4	5	6
$P(X = x)$	$\frac{1}{64}$	$\frac{6}{64}$	$\frac{15}{64}$	$\frac{20}{64}$	$\frac{15}{64}$	$\frac{6}{64}$	$\frac{1}{64}$

(b)

x	0	1	2	3	4	5	6
$P(X = x)$	$\frac{1}{64}$	$\frac{1}{64}$	$\frac{6}{64}$	$\frac{6}{64}$	$\frac{15}{64}$	$\frac{15}{64}$	$\frac{20}{64}$

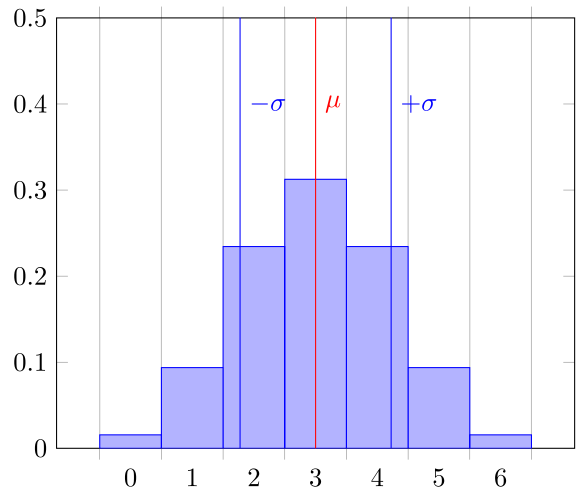
(c)

x	0	1	2	3	4	5	6
$P(X = x)$	$\frac{20}{64}$	$\frac{15}{64}$	$\frac{15}{64}$	$\frac{6}{64}$	$\frac{6}{64}$	$\frac{1}{64}$	$\frac{1}{64}$

In each case

i. Sketch the probability histogram for X.
ii. Find the mean of X.
iii. Find the variance and standard deviation of X.
iv. Find the third moment about the mean of X.

Solution. (a) i. Histogram for X. The mean is shown as well as one standard deviation to the left and to the right of the mean.



ii. The mean of X is

$$\mu = E(X) = 0 \cdot \frac{1}{64} + 1 \cdot \frac{6}{64} + 2 \cdot \frac{15}{64} + 3 \cdot \frac{20}{64} + 4 \cdot \frac{15}{64} + 5 \cdot \frac{6}{64} + 6 \cdot \frac{1}{64} = 3$$

iii. The variance of X is

$$\begin{aligned}\sigma^2 &= E((X-\mu)^2) \\ &= (0-3)^2 \cdot \frac{1}{64} + (1-3)^2 \cdot \frac{6}{64} + (2-3)^2 \cdot \frac{15}{64} + (3-3)^2 \cdot \frac{20}{64} \\ &\quad + (4-3)^2 \cdot \frac{15}{64} + (5-3)^2 \cdot \frac{6}{64} + (6-3)^2 \cdot \frac{1}{64} \\ &= \frac{3}{2}\end{aligned}$$

This can also be computed with the formula $\sigma^2 = E(X^2) - \mu^2$. In this case

$$E(X^2) = 0^2 \cdot \frac{1}{64} + 1^2 \cdot \frac{6}{64} + 2^2 \cdot \frac{15}{64} + 3^2 \cdot \frac{20}{64} + 4^2 \cdot \frac{15}{64} + 5^2 \cdot \frac{6}{64} + 6^2 \cdot \frac{1}{64} = \frac{21}{2}$$

so

$$\sigma^2 = E(X^2) - \mu^2 = \frac{21}{2} - 3^2 = \frac{3}{2}.$$

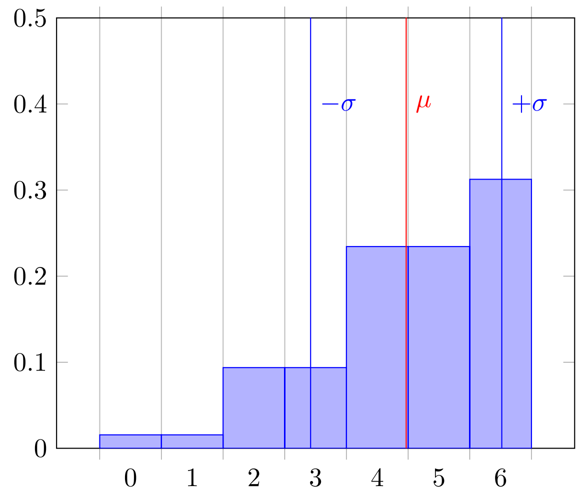
Then the standard deviation is

$$\sigma = \sqrt{\frac{3}{2}} \approx 1.2247$$

iv. The third moment about the mean of X is

$$\begin{aligned}E((X-\mu)^3) &= (0-3)^3 \cdot \frac{1}{64} + (1-3)^3 \cdot \frac{6}{64} + (2-3)^3 \cdot \frac{15}{64} + (3-3)^3 \cdot \frac{20}{64} \\ &\quad + (4-3)^3 \cdot \frac{15}{64} + (5-3)^3 \cdot \frac{6}{64} + (6-3)^3 \cdot \frac{1}{64} \\ &= 0\end{aligned}$$

(b) i. Histogram for X. The mean is shown as well as one standard deviation to the left and to the right of the mean.



ii. The mean of X is

$$\mu = E(X) = 0 \cdot \frac{1}{64} + 1 \cdot \frac{1}{64} + 2 \cdot \frac{6}{64} + 3 \cdot \frac{6}{64} + 4 \cdot \frac{15}{64} + 5 \cdot \frac{15}{64} + 6 \cdot \frac{20}{64} = \frac{143}{32} = 4.46875$$

iii. The second moment about the origin is

$$E(X^2) = 0^2 \cdot \frac{1}{64} + 1^2 \cdot \frac{1}{64} + 2^2 \cdot \frac{6}{64} + 3^2 \cdot \frac{6}{64} + 4^2 \cdot \frac{15}{64} + 5^2 \cdot \frac{15}{64} + 6^2 \cdot \frac{20}{64} = \frac{707}{32}$$

so

$$\sigma^2 = E(X^2) - \mu^2 = \frac{707}{32} - \left(\frac{143}{32}\right)^2 = \frac{2175}{1024}.$$

The standard deviation is

$$\sigma = \sqrt{\frac{2175}{1024}} \approx 1.4574$$

iv. The third moment about the mean of X is

$$\begin{aligned} E((X-\mu)^3) &= \left(0 - \frac{143}{32}\right)^3 \cdot \frac{1}{64} + \left(1 - \frac{143}{32}\right)^3 \cdot \frac{1}{64} + \left(2 - \frac{143}{32}\right)^3 \cdot \frac{6}{64} + \left(3 - \frac{143}{32}\right)^3 \cdot \frac{6}{64} \\ &\quad + \left(4 - \frac{143}{32}\right)^3 \cdot \frac{15}{64} + \left(5 - \frac{143}{32}\right)^3 \cdot \frac{15}{64} + \left(6 - \frac{143}{32}\right)^3 \cdot \frac{20}{64} \\ &\approx -2.6212 \end{aligned}$$

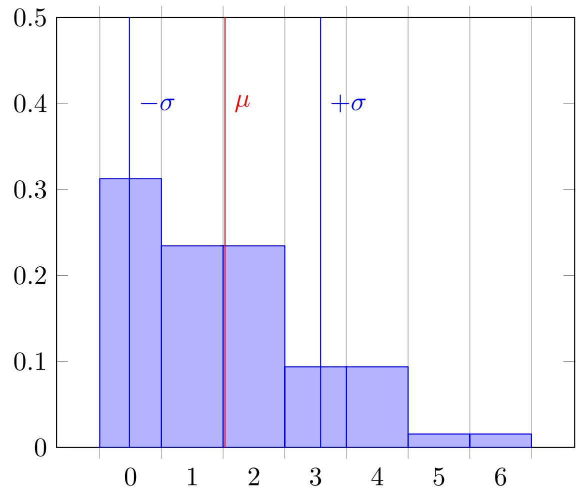
This can also be computed using the formula $E(X-\mu)^3) = E(X^3) - 3\mu E(X^2) + 2\mu^3$ (prove this as an exercise). Since we have already found μ and $E(X^2)$ we just need $E(X^3)$. In this case

$$E(X^3) = 0^3 \cdot \frac{1}{64} + 1^3 \cdot \frac{1}{64} + 2^3 \cdot \frac{6}{64} + 3^3 \cdot \frac{6}{64} + 4^3 \cdot \frac{15}{64} + 5^3 \cdot \frac{15}{64} + 6^3 \cdot \frac{20}{64} = \frac{3683}{32},$$

thus

$$\begin{aligned} E(X-\mu)^3 &= E(X^3) - 3\mu E(X^2) + 2\mu^3 \\ &= \frac{3683}{32} - 3\left(\frac{143}{32}\right)\left(\frac{707}{32}\right) + 2\left(\frac{143}{32}\right)^3 \\ &= -\frac{85890}{32768} \\ &\approx -2.6212 \end{aligned}$$

(c) i. Histogram for X. The mean is shown as well as one standard deviation to the left and to the right of the mean.



ii. The mean of X is

$$\mu = E(X) = 0 \cdot \frac{20}{64} + 1 \cdot \frac{15}{64} + 2 \cdot \frac{15}{64} + 3 \cdot \frac{6}{64} + 4 \cdot \frac{6}{64} + 5 \cdot \frac{1}{64} + 6 \cdot \frac{1}{64} = \frac{49}{32} = 1.53125$$

iii. The second moment about the origin is

$$E(X^2) = 0^2 \cdot \frac{20}{64} + 1^2 \cdot \frac{15}{64} + 2^2 \cdot \frac{15}{64} + 3^2 \cdot \frac{6}{64} + 4^2 \cdot \frac{6}{64} + 5^2 \cdot \frac{1}{64} + 6^2 \cdot \frac{1}{64} = \frac{143}{32}$$

so

$$\sigma^2 = E(X^2) - \mu^2 = \frac{143}{32} - \left(\frac{49}{32}\right)^2 = \frac{2175}{1024}.$$

The standard deviation is

$$\sigma = \sqrt{\frac{2175}{1024}} \approx 1.4574$$

iv. The third moment about the origin is

$$E(X^3) = 0^3 \cdot \frac{20}{64} + 1^3 \cdot \frac{15}{64} + 2^3 \cdot \frac{15}{64} + 3^3 \cdot \frac{6}{64} + 4^3 \cdot \frac{6}{64} + 5^3 \cdot \frac{1}{64} + 6^3 \cdot \frac{1}{64} = \frac{511}{32}$$

thus

$$\begin{aligned} E(X-\mu)^3 &= E(X^3) - 3\mu E(X^2) + 2\mu^3 \\ &= \frac{511}{32} - 3\left(\frac{49}{32}\right)\left(\frac{143}{32}\right) + 2\left(\frac{49}{32}\right)^3 \\ &= -\frac{85890}{32768} \\ &\approx 2.6212 \end{aligned}$$

□

3. Let X be a continuous random variable with probability density

$$f(x) = \begin{cases} \frac{2(x+1)}{9} & -1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the mean, variance and standard deviation of X.
(b) Find the probability that X lies within 2.5 standard deviations of the mean, and compare with the lower bound given by Chebyshev's Theorem.
(c) Find the mean and variance $Y = X^2$.
(d) Find the probability density for $Y = X^2$.

Solution. (a) The mean is

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx = \int_{-1}^{2} \frac{2x^2 + 2x}{9}\, dx = \frac{2x^3}{27} + \frac{x^2}{9}\Big|_{-1}^{2} = 1.$$

The second moment about the origin is

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_{-1}^{2} \frac{2x^3 + 2x^2}{9}\, dx = \frac{x^4}{18} + \frac{2x^3}{27}\Big|_{-1}^{2} = \frac{3}{2}.$$

The variance is

$$\sigma^2 = E(X^2) = \mu^2 = \frac{3}{2} - 1^2 = \frac{1}{2}.$$

The standard deviation is

$$\sigma = \sqrt{\frac{1}{2}} \approx 0.7071.$$

(b) Note that $\mu - 2.5\sigma = 1 - \frac{2.5}{\sqrt{2}} = \frac{4-5\sqrt{2}}{4} \approx -0.7678$ and $\mu + 2.5\sigma = 1 + \frac{2.5}{\sqrt{2}} = \frac{4+5\sqrt{2}}{4} \approx 2.7678$. Then

$$\begin{aligned} P(|X - \mu| < 2.5\sigma) &= \int_{\mu-2.5\sigma}^{\mu+2.5\sigma} f(x)\, dx \\ &= \int_{\mu-2.5\sigma}^{2} \frac{2(x+1)}{9}\, dx \\ &= \frac{x^2+2x}{9}\Big|_{\mu-2.5\sigma}^{2} \\ &= \frac{8}{9} - \frac{49 - 40\sqrt{2}}{72} \\ &= \frac{15 + 40\sqrt{2}}{72} \\ &\approx 0.9940 \end{aligned}$$

By Chebyshev's Theorem,

$$P(|X - \mu| < 2.5\sigma) \geq 1 - \frac{1}{(2.5)^2} = \frac{21}{25} = 0.84.$$

(c) Using properties of expected value we have

$$E(Y) = E(X^2) = \frac{3}{2}.$$

Using the "short cut" formula

$$\operatorname{var}(Y) = E(Y^2) - E(Y)^2 = E(X^4) - E(Y)^2.$$

Here we have

$$E(X^4) = \int_{-\infty}^{\infty} x^4 f(x)\, dx = \int_{-1}^{2} \frac{2x^5 + 2x^4}{9}\, dx = \frac{x^6}{27} + \frac{2x^5}{45}\Big|_{-1}^{2} = \frac{19}{5}.$$

so

$$\text{var}(Y) = \frac{19}{5} - \left(\frac{3}{2}\right)^2 = \frac{31}{20}$$

(d) Let $g(y)$ be the probability density for Y. Since $Y = X^2$ we see that $X = \sqrt{Y}$ for $Y \in (1, 4)$ and $X = \pm\sqrt{Y}$ for $Y \in [0, 1]$. Suppose $a, b \in (1, 4)$ with $a < b$, then

$$\begin{aligned} P(a < Y < b) &= P(a < X^2 < b) \\ &= P(\sqrt{a} < X < \sqrt{b}) \\ &= \int_{\sqrt{a}}^{\sqrt{b}} f(x)\, dx \\ &= \int_a^b f(\sqrt{y})(\sqrt{y})'\, dy \end{aligned}$$

where the last line was obtained by applying the substitution rule with $x = \sqrt{y}$. This shows that

$$g(y) = f(\sqrt{y})(\sqrt{y})' = \frac{2(\sqrt{y}+1)}{9}\left(\frac{1}{2\sqrt{y}}\right) = \frac{1}{9} + \frac{1}{9\sqrt{y}}$$

Actually, we can obtain a more general rule from this: Suppose $Y = p(X)$ where p is an invertible function and X has probability density $f(x)$, then $X = p^{-1}(Y)$ and Y has probability density $f(p^{-1}(y))(p^{-1}(y))'$ where $(p^{-1}(y))'$ is the derivative of $(p^{-1}(y))$.
Now if $a, b \in [0, 1]$ with $a < b$, then

$$\begin{aligned} P(a < Y < b) &= P(a < X^2 < b) \\ &= P(-\sqrt{b} < X < -\sqrt{a}) + P(\sqrt{a} < X < \sqrt{b}) \\ &= \int_{-\sqrt{b}}^{-\sqrt{a}} f(x)\, dx + \int_{\sqrt{a}}^{\sqrt{b}} f(x)\, dx \\ &= \int_b^a f(-\sqrt{y})(-\sqrt{y})'\, dy + \int_a^b f(\sqrt{y})(\sqrt{y})'\, dy \\ &= -\int_a^b f(-\sqrt{y})(-\sqrt{y})'\, dy + \int_a^b f(\sqrt{y})(\sqrt{y})'\, dy \\ &= \int_a^b f(\sqrt{y})(\sqrt{y})' - f(-\sqrt{y})(-\sqrt{y})'\, dy \end{aligned}$$

This shows that

$$g(y) = f(\sqrt{y})(\sqrt{y})' - f(-\sqrt{y})(-\sqrt{y})' = \frac{2(\sqrt{y}+1)}{9}\left(\frac{1}{2\sqrt{y}}\right) + \frac{2(-\sqrt{y}+1)}{9}\left(\frac{1}{2\sqrt{y}}\right) = \frac{2}{9\sqrt{y}}$$

Therefore the probability density for Y is

$$g(y) = \begin{cases} \frac{2}{9\sqrt{y}} & 0 \le y \le 1 \\ \frac{1}{9} + \frac{1}{9\sqrt{y}} & 1 < y < 4 \\ 0 & \text{otherwise} \end{cases}$$

Note: This problem was intended to be challenging to show the convenience of being able to obtain the mean and variance of Y in part (c) without having its probability density function.

□

4. In a certain manufacturing process, the (Fahrenheit) temperature never varies by more than 2° from

$62°$. The temperature is a random variable F with distribution

x	60	61	62	63	64
$P(F = x)$	1/10	2/10	4/10	2/10	1/10

(a) Find the mean of and variance of F.

(b) To convert to the measurements to degrees Celsius we let $C = \frac{5}{9}(F - 32)$. Find the mean and variance of C.

Solution. (a) Mean:

$$\mu = E(F) = \sum_x xf(x) = (60) \cdot \frac{1}{10} + (61) \cdot \frac{2}{10} + (62) \cdot \frac{4}{10} + (63) \cdot \frac{2}{10} + (64) \cdot \frac{1}{10} = 62.$$

Variance:

$$\begin{aligned}
\sigma^2 &= E((F - \mu)^2) \\
&= \sum_x (x - \mu)^2 f(x) \\
&= (60 - 62)^2 \cdot \frac{1}{10} + (61 - 62)^2 \cdot \frac{2}{10} + (62 - 62)^2 \cdot \frac{4}{10} + (63 - 62)^2 \cdot \frac{2}{10} + (64 - 62)^2 \cdot \frac{1}{10} \\
&= \frac{6}{5}
\end{aligned}$$

(b) Mean:

$$E(C) = E\left(\frac{5}{9}(F - 32)\right) = E\left(\frac{5}{9}F - \frac{160}{9}\right) = \frac{5}{9}E(F) - \frac{160}{9} = \frac{5}{9}\left(\frac{6}{5}\right) - \frac{160}{9} = -\frac{151}{9}.$$

Variance:

$$\mathrm{var}(C) = \mathrm{var}\left(\frac{5}{9}(F - 32)\right) = \mathrm{var}\left(\frac{5}{9}F - \frac{160}{9}\right) = \left(\frac{5}{9}\right)^2 \mathrm{var}(F) = \left(\frac{25}{81}\right)\left(\frac{6}{5}\right) = \frac{10}{27}.$$

$\square$

5. For the given probability distribution, find the moment generating function for the discrete random variable X and use it to compute the mean and variance of X.

(a)

x	-3	-1	2	5
$P(X = x)$	$\frac{4}{10}$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{2}{10}$

(b)

$$f(x) = \begin{cases} 3\left(\frac{1}{4}\right)^x & x = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Solution. (a) The moment generating function is

$$M_X(t) = E(e^{tX}) = e^{-3t}\frac{4}{10} + e^{-t}\frac{1}{10} + e^{2t}\frac{3}{10} + e^{5t}\frac{2}{10} = \frac{4e^{-3t} + e^{-t} + 3e^{2t} + 2e^{5t}}{10}$$

The mean of X is

$$\mu = \frac{d}{dt} M_X(t)\Big|_{t=0} = \frac{-12e^{-3t} - e^{-t} + 6e^{2t} + 10e^{5t}}{10}\Big|_{t=0} = \frac{3}{10}.$$

The second moment about the origin is

$$\begin{aligned} E(X^2) &= \frac{d^2}{dt^2} M_X(t)\Big|_{t=0} \\ &= \frac{d}{dt} \frac{-12e^{-3t} - e^{-t} + 6e^{2t} + 10e^{5t}}{10}\Big|_{t=0} \\ &= \frac{36e^{-3t} + e^{-t} + 12e^{2t} + 50e^{5t}}{10}\Big|_{t=0} \\ &= \frac{99}{10}. \end{aligned}$$

Thus the variance is

$$\sigma^2 = E(X^2) - \mu^2 = \frac{99}{10} - \left(\frac{3}{10}\right)^2 = \frac{981}{100}.$$

(b) The moment generating function is

$$\begin{aligned} M_X(t) &= E(e^{tX}) \\ &= \sum_{x=1}^{\infty} e^{tx} \cdot 3 \left(\frac{1}{4}\right)^x \\ &= 3 \sum_{x=1}^{\infty} \left(\frac{e^t}{4}\right)^x \\ &= \frac{3e^t}{4} \sum_{x=1}^{\infty} \left(\frac{e^t}{4}\right)^{x-1} \\ &= \frac{3e^t}{4} \sum_{x=0}^{\infty} \left(\frac{e^t}{4}\right)^x \\ &= \frac{3e^t}{4} \left(\frac{1}{1 - \frac{e^t}{4}}\right) \qquad \left(\text{geometric series with } r = \frac{e^t}{4}\right) \\ &= \frac{3e^t}{4} \left(\frac{4}{4 - e^t}\right) \\ &= \frac{3e^t}{4 - e^t}. \end{aligned}$$

The mean of X is

$$\mu = \frac{d}{dt} M_X(t)\Big|_{t=0} = \frac{(4 - e^t)(3e^t) - 3e^t(-e^t)}{(4 - e^t)^2}\Big|_{t=0} = \frac{12e^t}{(4 - e^t)^2}\Big|_{t=0} = \frac{12}{9} = \frac{4}{3}.$$

The second moment about the origin is

$$\begin{aligned} E(X^2) &= \frac{d^2}{dt^2} M_X(t)\Big|_{t=0} \\ &= \frac{d}{dt} \frac{12e^t}{(4-e^t)^2}\Big|_{t=0} \\ &= \frac{d}{dt} \frac{(4-e^t)^2(12e^t) - 12e^t(2(4-e^t)(-e^t)}{(4-e^t)^4}\Big|_{t=0} \\ &= \frac{180}{81} \\ &= \frac{20}{9}. \end{aligned}$$

Thus the variance is

$$\sigma^2 = E(X^2) - \mu^2 = \frac{20}{9} - \left(\frac{4}{3}\right)^2 = \frac{4}{9}.$$

□

6. Suppose X is a continuous random variable with probability density

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

for all $x \in \mathbb{R}$. It can be shown that X has moment generating function

$$M_X(t) = e^{\frac{t^2}{2}}.$$

Find the mean and variance of X.

Solution. Mean:

$$\mu = E(X) = \frac{d}{dt} M_X(t)\Big|_{t=0} = \frac{d}{dt} e^{\frac{t^2}{2}}\Big|_{t=0} = te^{\frac{t^2}{2}}\Big|_{t=0} = 0.$$

Variance:

$$E(X^2) = \frac{d^2}{dt^2} M_X(t)\Big|_{t=0} = \frac{d}{dt} te^{\frac{t^2}{2}}\Big|_{t=0} = e^{\frac{t^2}{2}} + t^2 e^{\frac{t^2}{2}}\Big|_{t=0} = 1.$$

□

7. There are 1000 people applying for 70 new jobs opening up at a manufacturing plant. The company administers a test to select the best 70 applicants. The mean score turns out to be 60, and the scores have a standard deviation of 6. If a person scores 84 on the test are they guaranteed a job? To determine this, use Chebyshev's Theorem and assume that the probability distribution is symmetric about to mean.

Solution. To obtain one of the new jobs, the person must have one of the top 70 test scores, which means their score lies in the top 0.07. Equivalently, they do not want a test score in bottom 0.07 or the middle 0.86. Since their score of 84 is above the mean, they do not lie in the bottom 0.07, so we determine if they lie in the middle 0.86. By Chebyshev's Theorem,

$$0.86 = P(|X - 60| < 6k) \geq 1 - \frac{1}{k^2} \quad \Rightarrow \quad \frac{1}{k^2} \geq 0.14 \quad \Rightarrow \quad k^2 \leq \frac{50}{7} \quad \Rightarrow \quad k \leq \sqrt{\frac{50}{7}} \approx 2.6726$$

This says (using $k = 2.6726$) that $P(43.9643 < X < 76.0357) \geq 0.86$ which implies that the score of 84 lies in the top 0.07. So the person is guaranteed to get a job.

□