4.1.1 The Expected Value of a Continuous Random Variable

If X is a continuous random variable and $f(x)$ is its probability distribution function, then the expected value of X is defined as

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx.$$

The integral must exist in order for the expected value to have meaning.

Example 4.1.4 *A contractor's profit on a construction job can be considered as a continuous random variable having probability density*

$$f(x) = \begin{cases} \frac{1}{18}(x+1) & \text{for } -1 < x < 5 \\ 0 & \text{otherwise} \end{cases}$$

(where the units are in \$1,000). What is her expected profit?

The expected value of X, where X denotes the contractor's profit in \$1,000's, is

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx \qquad = \int_{-1}^{5} x \cdot \frac{1}{18}(x+1)\, dx$$

$$=$$

4.1.2 Expectation of a Function of a Random Variable

We are not limited to considering a random variable by itself.

We can as well consider a function $g(X)$ of a random variable, and evaluate its expected value.

Theorem 4.1.5 *If X is a discrete random variable with probability distribution $f(x)$, the expected value of $g(X)$ is given by*

$$E(g(X)) = \sum_x g(x) \cdot f(x).$$

If X is a continuous random variable with probability density function $f(x)$, the expected value of $g(X)$ is given by

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) \cdot f(x)\, dx.$$

Example 4.1.6 *Let X be a random variable that takes the values $-1, 0, 1$, and has probability distribution given by*

$$f(-1) = 0.2, \quad f(0) = 0.5, \quad f(1) = 0.3.$$

Find $E(X^2)$.

Before using the theorem, let's find $E(X^2)$ directly. We view X^2 as a new random variable which we'll call Y.

The range of Y is $\{0, 1\}$ and it has probability distribution

$$P(Y = 0) = P(X = 0) = f(0) = 0.5$$

$$P(Y = 1) = P(X = 1) + P(X = -1) = f(1) + f(-1) = 0.5$$

Then,

$$E(X^2) = E(Y) = 0 \cdot P(Y = 0) + 1 \cdot P(Y = 1) = 0 \cdot (0.5) + 1 \cdot (0.5) = 0.5.$$

We can find the same result using the theorem as well.

Let $g(X) = X^2$, and let $x_1 = -1, x_2 = 0, x_3 = 1$. Then according to the theorem

$$
\begin{aligned}
E(g(X)) &= \sum_{i=1}^{3} g(x_i) f(x_i) \\
&= g(x_1)f(x_1) + g(x_2)f(x_2) + g(x_3)f(x_3) \\
&= g(-1)f(-1) + g(0)f(0) + g(1)f(1) \\
&= (-1)^2 \cdot (0.2) + (0)^2 \cdot (0.5) + (1)^2 \cdot (0.3) \\
&= 0.5.
\end{aligned}
$$

Note that: $E(X^2) = 0.5 \neq (E(X))^2 = 0.01$

Example 4.1.7 *Suppose X has probability density*

$$f(x) = \begin{cases} e^{-x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the expected value of $g(X) = e^{3X/4}$.

By our theorem

$$E(g(X)) = \int_{-\infty}^{\infty} g(x)f(x)\, dx$$

$$= \int_{0}^{\infty} e^{3x/4} \cdot e^{-x}\, dx$$

$$=$$

4.1.3 Properties of Expected Value

A useful special case of the theorem is:

Theorem 4.1.8 *If a and b are constants, then*

$$E(aX + b) = aE(X) + b.$$

In particular if $a = 0$, then $E(b) = b$ and if $b = 0$ then $E(aX) = aE(X)$.

To prove this (for the discrete case) let $g(X) = aX + b$. Then

$$\begin{aligned}
E(aX + b) &= E(g(x)) \\
&= \sum_x g(x) \cdot f(x) \\
&= \sum_x (ax + b) \cdot f(x) \\
&= \sum_x \left(ax \cdot f(x) + b \cdot f(x)\right) \\
&= \sum_x ax \cdot f(x) + \sum_x b \cdot f(x) \\
&= a \sum_x x \cdot f(x) + b \sum_x f(x) \\
&= aE(X) + b \qquad (\text{since } \sum_x f(x) = 1).
\end{aligned}$$

Example 4.1.9 *Returning to out slot machine example, we chose our random variable X to be the expected payout, and not the expected profit. Then we calculated the expected payout.*

Suppose this time that we want the expected profit.

If Y is our expected profit then Y has range $\{-0.25, 19.75, 99.75, 499.75\}$

So then $P(Y = y) = P(X = y + 0.25)$, and

$$\begin{aligned} E(Y) &= (-0.25) \cdot P(X = 0) + (19.75) \cdot P(X = 20) \\ &\quad + (99.75) \cdot P(X = 100) + (499.75) \cdot P(X = 500) \end{aligned}$$

On the other hand since $Y = g(X) = X - 0.25$, we can compute

$$E(Y) = E(X - 0.25) = E(X) - 0.25$$

using the theorem (which, in this case is nicer calculation).

We can extend the theorem above to more expressions:

Theorem 4.1.10 *If $c_1, c_2, \ldots, c_n$ are constants, then*

$$E\left(\sum_{i=1}^{n} c_i g_i(X)\right) = \sum_{i=1}^{n} c_i E(g_i(X)),$$

where the g_i are functions.

Proof (continuous case): Suppose X has p.d.f. $f(x)$.
Let $h(x) = \sum_{i=1}^{n} c_i g_i(X)$. Then

$$\begin{aligned}
E(h(X)) &= \int_{-\infty}^{\infty} h(x) \cdot f(x)\, dx \\
&= \int_{-\infty}^{\infty} \left(\sum_{i=1}^{n} c_i g_i(X)\right) \cdot f(x)\, dx \\
&= \sum_{i=1}^{n} c_i \int_{-\infty}^{\infty} g_i(X) \cdot f(x)\, dx \\
&= \sum_{i=1}^{n} c_i E(g_i(X)).
\end{aligned}$$