Infinite Discrete Sample Space: When a sample space has countably infinite outcomes, probabilities must be assigned via some rule/formula as opposed to listing them individually.

Example 2.1.14 *Tossing a coin until heads is reached:* $S = \{H, TH, TTH, TTTH, TTTH, \dots\}$.

If A_i is the event of i flips, then $P(A_i) = \frac{1}{2^i}$ defines a probability on S (assuming countable additivity). From the geometric series formula we get P2:

$$P(S) = \sum_{i=1}^{\infty} P(A_i) = -1 + \sum_{i=0}^{\infty} \frac{1}{2^i} = -1 + \frac{1}{1 - \frac{1}{2}} = -1 + 2 = 1.$$

Brief note on infinite series:

- Sequence: countably infinite list of real numbers; $r_1, r_2, r_3, \ldots.$

$$\text{e.g.} \qquad 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots$$

- Geometric sequence: terms occur in a common ratio r; $a, ar, ar^2, ar^3 \ldots$

$$\text{e.g.} \qquad 4, \tfrac{4}{3}, \tfrac{4}{9}, \tfrac{4}{27}, \tfrac{4}{81}, \ldots\ldots$$

- Partial sum of a sequence: $T_n = \displaystyle\sum_{i=1}^{n} r_i = r_1 + r_2 + \cdots + r_n$.

- Series: Limit of partial sums (if it exists); $\displaystyle\lim_{n\to\infty} T_n = \sum_{i=1}^{\infty} r_i$.

- Partial sum of a geometric sequence: $G_n = a + ar + \cdots + ar^n$.

$$\begin{aligned}
(1-r)G_n &= (1-r)(a + ar + ar^2 + \cdots + ar^n) \\
&= a + ar + ar^2 + \cdots + ar^n \\
&\quad - (ar + ar^2 + ar^3 + \cdots + ar^{n+1}) \\
&= a - ar^{n+1}
\end{aligned}$$

So $G_n = \frac{a-ar^{n+1}}{(1-r)}$ (for $r \neq 1$) .

- If $-1 < r < 1$ then $\displaystyle\lim_{n\to\infty} r^n = 0$, and so it follows that

$$\sum_{i=0}^{\infty} ar^i = \lim_{n\to\infty} G_n = \frac{a}{1-r}.$$

- In the coin flipping example above, $a = 1$ and $r = \frac{1}{2}$.

Theorem 2.1.15 ***(The Probability of an Event with equally likely outcomes)***

If an experiment has N equally likely outcomes and A is an event made up of k of those outcomes then

$$P(A) = \frac{k}{N}.$$

Example 2.1.16 *A five-card poker hand dealt from a deck of 52 playing cards is said to be a full house if it consists of three of a kind and a pair. If all the five-card hands are equally likely, what is the probability of being dealt a full house?*

$$8\spadesuit, J\spadesuit, 8\heartsuit, 8\clubsuit, J\diamondsuit$$

2.1.7 Rules of Probability

Theorem 2.1.17 *Let S be a sample space with probability measure P, and let A and B be events in S. Then*

1. $P(A) + P(A') = 1$, *or equivalently* $P(A') = 1 - P(A)$.
2. $P(\emptyset) = 0$.
3. *If* $A \subset B$ *then* $P(A) \leq P(B)$.
4. $0 \leq P(A) \leq 1$.
5. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. *(think of the Venn diagrams)*

Example 2.1.18 *What is the probability that among a group of r people at least two have the same birthday (ignoring leap years)?*

Let A be the event that at least two people have the same birthday.

What event does A' denote?

Theorem 2.1.19 ***(The Inclusion-Exclusion Principle - two events)*** *If A and B are any two events in sample space S, then*

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Theorem 2.1.20 ***(The Inclusion-Exclusion Principle - three events)*** *If A, B and C are any three events in sample space S, then*

$$\begin{aligned} P(A \cup B \cup C) = {} & P(A) + P(B) + P(C) \\ & - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C). \end{aligned}$$

(Note that the Inclusion-Exclusion Principle can be generalized to many more sets.)

Example 2.1.21 *Suppose the probabilities are* $0.86, 0.35,$ *and* 0.29*, respectively, that a family owns a laptop computer, a desktop computer, or both kinds. What is the probability that a family owns either or both kinds of computer and what is the probability that a family owns neither?*

Let A be the event that family owns a laptop and B the even that a family owns a desktop. Then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.86 + 0.35 - 0.29 = 0.92.$$

The probability that a family owns neither is

$$P\left((A \cup B)'\right) = 1 - P(A \cup B) = 1 - 0.92 = 0.08.$$

2.2 Conditional Probability

Example 2.2.1 *Mayoral candidate Alice receives 56 percent of the entire vote, but only 47 percent of the female vote.*

Let $P(A)$ be the probability that a randomly selected person has voted for Alice, and let $P(A|F)$ denote the probability that a randomly selected female has voted for Alice.

So,

Value $P(A|F)$ is called the ***conditional probability of*** *A* ***relative to*** *F, or the* ***conditional probability of*** *A* ***given*** *F.*

Example 2.2.2 *Let A be the event of rolling 8 with two dice; Then $P(A) = \frac{5}{36} \approx 0.1389$.*

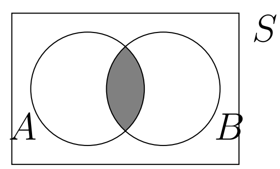
Suppose we are given that the roll of die 1 is 3. Knowing this (i.e. given that this event has occurred) what is the probability of rolling an 8?

Let $B = \{(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)\}$; the event that die 1 is 3. Since these outcomes are equally likely prior to knowing die 1 *is 3, they are still equally likely given that* B *has occurred.*

So given B *has occurred, each have probability* $\frac{1}{6}$ *(the other 30 outcomes have probability 0).*

Therefore the probability of rolling an 8 given that die 1 is a 3 is $P(A|B) = \frac{1}{6} \approx 0.1667$

In the example above, if B occurs, then in order for A to occur, the outcome must lie in both A and B. Thus $A \cap B$ becomes the event of interest, and B is considered the new sample space.



The conditional probability of A given B is the probability of $A \cap B$ relative to the probability of B.

Definition 2.2.3 *If* A *and* B *are events in* S *and* $P(B) \neq 0$*, then the* ***conditional probability*** *of* A *given* B *is*

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Example 2.2.4 *Results of a survey of 50 car dealerships:*

	Good service under warranty	*Poor service under warranty*
In business 10 years or more	*16*	*4*
In business less than 10 years	*10*	*20*

If a person randomly chooses one dealership, what is the probability that

(a) they get one who provides good service under warranty?

(b) they get one who provides good service under warranty if they select from dealers in business 10 years or more?

Let G be the event of getting good service and T the event that a dealer has been in business 10 year or more. Let $n(A)$ denote the number of element in event A.

Example 2.2.5 *A coin is tossed twice. Assuming all points in the sample space*

$$S = \{HH, HT, TH, TT\}$$

are equally likely, what is the probability that both flips land on heads given that

(a) the first flip is heads?

(b) at least one flip is heads?

$S = \{HH, HT, TH, TT\}$

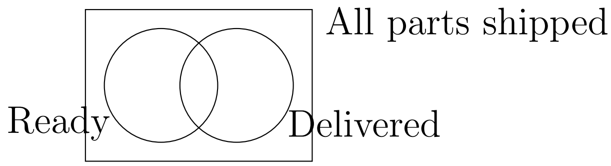
$A = \{HH\}$ - event that both flips are heads,

$B = \{HH, HT\}$ event that first flip is heads,

$C = \{HH, HT, TH\}$ - event that at least one flip is heads.

Example 2.2.6 *A manufacturer of airplane parts knows from past experience that the probability is* 0.80 *that an order will be ready for shipment on time, and it is* 0.72 *that an order will be ready for shipment on time and will also be delivered on time.*

What is the probability that such an order will be delivered on time given that it was ready for shipment on time?



Conditional Probability Multiplication Rule:

Theorem 2.2.7 *If A and B are events in S and $P(A) \neq 0$, then*

$$P(A \cap B) = P(A) \cdot P(B|A).$$

This means that the probability that both A and B will occur is the product of the probability of A and the probability of B given A.

Now back to the problem:

Example 2.2.8 *A pot contains 8 red balls and 4 green balls. We draw 2 balls without replacement. If each ball has an equally likely chance of being chosen, what is the probability that both balls are red?*

(*without replacement means that the first ball is not returned to the pot before the second ball is drawn*)

Let R_1 be the event that ball 1 is red, and R_2 be the event that ball 2 is red. Then $R_1 \cap R_2$ is the event that both are red.

Since the the outcomes are equally likely, we could also have computed the probability as the number of successful outcomes over total number of outcomes:

$$P(R_1 \cap R_2) = \frac{\binom{8}{2}}{\binom{12}{2}} = \frac{28}{66} = \frac{14}{33} \approx 0.4242.$$

Example 2.2.9 *Find the probabilities of randomly drawing two aces in succession from an ordinary deck of 52 playing cards if we sample*

(a) without replacement.

(b) with replacement.

Let A_1 be the event that card 1 is an ace, and A_2 be the event that card 2 is an ace. Then $A_1 \cap A_2$ is the event that both are aces.

(a)

(b)

Since $A \cap B \cap C = (A \cap B) \cap C)$ we have by the multiplication rule

$$P((A \cap B) \cap C)) = P(A \cap B) \cdot P(C|A \cap B).$$

Applying the multiplication rule again to $P(A \cap B)$ gives,

Theorem 2.2.10 *If A, B and C are events in S and $P(A \cap B) \neq 0$, then*

$$P(A \cap B \cap C) = P(A) \cdot P(B|A) \cdot P(C|A \cap B).$$

Example 2.2.11 *A bushel of 126 apples contains 15 rotten ones. If three apples are chosen at random, what is the probability that all three are rotten? (Solve by using the theorem.)*

If A_1, A_2, A_3 are the events that the first, second and third (resp.) choice is rotten, then

$$P(A_1 \cap A_2 \cap A_3) =$$

2.3 (In)dependent Events

Example 2.3.1 *Suppose a coin is tossed twice. What is the probability of getting tails on the second toss given that the first toss was tails?*

Surely the outcome of the second toss does not depend on what has previously come up.

Indeed if $S = \{HH, HT, TH, TT\}$, $A_1 = \{TH, TT\}$, and $A_2 = \{HT, TT\}$ are the events of getting tails on flips 1, and 2, then

$$P(A_2|A_1) = \frac{P(A_1 \cap A_2)}{P(A_1)} = \frac{P(\{TT\})}{P(A_1)} = \frac{\frac{1}{4}}{\frac{2}{4}} = \frac{1}{2} = P(A_2).$$

Similarly we can also see that $P(A_1|A_2) = P(A_1)$. In this case events A_1 and A_2 are called **independent**.

Remember the multiplication rule:

$$P(A \cap B) = P(A) \cdot P(B|A).$$

When the events A and B are independent, this identity simplifies to

$$P(A \cap B) = P(A) \cdot P(B).$$

The last identity can be thought of as the definition of independent events.

Definition 2.3.2 *Events A and B are called **independent** if and only if*

$$P(A \cap B) = P(A) \cdot P(B).$$

*They are otherwise called **dependent**. ($P(A) = 0$ or $P(B) = 0$ also allowed)*

Example 2.3.3 *In the initial coin toss example, the probability of getting two consecutive tails is*

$$P(T_1 \cap T_2) = P(T_1) \cdot P(T_2)$$

Example 2.3.4 *Suppose a coin a tossed 3 times. Let*
$A = \{HHH, HHT\}$ *- first two are* H
$B = \{HHT, HTT, THT, TTT\}$ *- third is* T
$C = \{HTT, THT, TTH\}$ *- exactly two* T

(a) Show that A *and* B *are independent.*

(b) Show that B and C are dependent.

Example 2.3.5 *Back to the example of drawing two aces from a deck of cards:*

(without replacement)

Events A_1 and A_2 are dependent because: