Example 4.1.9 *Returning to out slot machine example, we chose our random variable X to be the expected payout, and not the expected profit. Then we calculated the expected payout.*

Suppose this time that we want the expected profit.

If Y is our expected profit then Y has range $\{-0.25, 19.75, 99.75, 499.75\}$

So then $P(Y = y) = P(X = y + 0.25)$, and

$$\begin{aligned} E(Y) = (-0.25) \cdot P(X = 0) + (19.75) \cdot P(X = 20) \\ + (99.75) \cdot P(X = 100) + (499.75) \cdot P(X = 500) \end{aligned}$$

On the other hand since $Y = g(X) = X - 0.25$, we can compute

$$E(Y) = E(X - 0.25) = E(X) - 0.25$$

using the theorem (which, in this case is nicer calculation).

We can extend the theorem above to more expressions:

Theorem 4.1.10 *If* $c_1, c_2, \ldots, c_n$ *are constants, then*

$$E\left(\sum_{i=1}^{n} c_i g_i(X)\right) = \sum_{i=1}^{n} c_i E(g_i(X)),$$

where the g_i *are functions.*

Proof (continuous case): Suppose X has p.d.f. $f(x)$. Let $h(x) = \sum_{i=1}^{n} c_i g_i(X)$. Then

$$\begin{aligned} E(h(X)) &= \int_{-\infty}^{\infty} h(x) \cdot f(x)\, dx \\ &= \int_{-\infty}^{\infty} \left(\sum_{i=1}^{n} c_i g_i(X)\right) \cdot f(x)\, dx \\ &= \sum_{i=1}^{n} c_i \int_{-\infty}^{\infty} g_i(X) \cdot f(x)\, dx \\ &= \sum_{i=1}^{n} c_i E(g_i(X)). \end{aligned}$$

4.1.4 Multivariate Expected Value

Suppose X and Y are random variables with a joint probability distribution/density $f(x, y)$.

Then $Z = g(X, Y)$ is a random variable defined by the function g depending on X and Y.

The expected value of Z may be computed in the following way.

Theorem 4.1.11 *With notation as above if X and Y are discrete random variables, then*

$$E(g(X,Y)) = \sum_x \sum_y g(x,y) \cdot f(x,y)$$

where sums are taken over x and y in the ranges of X and Y respectively.

In the continuous case, we have the following result.

$$E(g(X,Y)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) \cdot f(x,y)\, dx\, dy$$

Back to the caplet example (yet again!):

Two caplets are randomly selected from a bottle containing 3 aspirin, 2 sedative, and 4 laxative.

Let X be the number of aspirin selected, and Y be the number of sedative selected.

$$
\begin{array}{cc|c|c|c|}
 & & & x & \\
 & & 0 & 1 & 2 \\
\hline
 & 0 & \frac{6}{36} & \frac{12}{36} & \frac{3}{36} \\
\mathrm{y} & 1 & \frac{8}{36} & \frac{6}{36} & \\
 & 2 & \frac{1}{36} & & \\
\hline
\end{array}
$$

Let $Z = X + Y$. Then Z is the random variable which gives the total number of aspirin or sedative when two caplets are drawn.

What is the expected value of Z?

$$E(X+Y) = \sum_{x=0}^{2}\sum_{y=0}^{2}(x+y)\cdot f(x,y)$$

$$=$$

Example 4.1.12 *The joint probability density of X and Y is given by*

$$f(x,y) = \begin{cases} x+y & \textit{for } 0 < x < 1, 0 < y < 1 \\ 0 & \textit{otherwise} \end{cases}$$

Find the expected value of XY.

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \cdot f(x,y)\, dx\, dy$$

$$= \int_0^1 \int_0^1 xy \cdot (x+y)\, dx\, dy$$

$$= \int_0^1 \int_0^1 x^2y + xy^2\, dx\, dy$$

$$=$$