MATH1550
Practice Set 7 - Solutions

These exercises are suited to Chapter 3, Marginal Distributions to Independent Random Variables
Topics Covered:

- Marginal distributions
- Marginal densities
- Conditionals distributions
- Conditionals densities
- Independent Random Variables

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1. (a) Suppose X and Y are jointly distribution discrete random variables with joint distribution $f(x, y)$. How do we find the marginal distributions for X and Y? What do the marginal distributions represent?

   (b) Suppose X and Y are jointly distribution continuous random variables with joint density $f(x, y)$. How do we find the marginal densities for X and Y? What do the marginal densities represent?

   (c) How are conditional distributions/densities defined? (Write the defining formula.)

   (d) Suppose X and Y are jointly distributed random variables. What is the definition for X and Y to be *independent*?

   Solution. (a) The marginal distribution for X is probability distribution for X (alone) and is given by

   $$g(x) = \sum_y f(x, y)$$

   where the sum is taken over all values for Y in the range of Y. Similarly the marginal distribution for Y is probability distribution for Y and is given by

   $$h(y) = \sum_x f(x, y)$$

   where the sum is taken over all values for X in the range of X.

   (b) The marginal density for X is probability density for X (alone) and is given by

   $$g(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$$

   (i.e. integrating the joint density over all $\mathbb{R}$ with respect to y). Similarly the marginal density for Y is given by

   $$h(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$$

   (i.e. integrating the joint density over all $\mathbb{R}$ with respect to x).

   (c) If X and Y are jointly distributed random variables with joint distribution/density $f(x, y)$ and respective marginal distributions/densities $g(x)$ and $h(y)$, then the conditional distribution/density for X given $Y = y$ is

   $$f(x|y) = \frac{f(x, y)}{h(y)}$$

   and the conditional distribution/density for Y given $X = x$ is

   $$f(y|x) = \frac{f(x, y)}{g(x)}.$$

(d) Let X and Y be jointly distributed random variables with joint distribution/density $f(x,y)$ and respective marginal distributions/densities $g(x)$ and $h(y)$. Then X and Y are called *independent* if

$$f(x,y) = g(x)h(y)$$

(i.e. if the joint distribution/density is the product of the marginal distributions/densities).

□

2. Let X and Y be jointly distribution discrete random variables with joint distribution given below.

		x		
		-5	0	10
y	1	0.12	0.3	0.18
	2	0.08	0.2	0.12

(a) Find the marginal distributions for X and Y.

(b) Find the conditional distributions $f(x|2)$ and $f(y|0)$.

(c) Are X and Y independent?

Solution. (a) The marginal distribution for X is obtain by summing the columns and is given by

x	-5	0	10
$g(x)$	0.2	0.5	0.3

The marginal distribution for Y is obtain by summing the rows and is given by

y	1	2
$h(y)$	0.6	0.4

(b) The conditional distribution for X given $Y = 0.4$ is:

$$f(-5|2) = \frac{f(-5,2)}{h(2)} = \frac{0.08}{0.4} = 0.2$$

$$f(0|2) = \frac{f(0,2)}{h(2)} = \frac{0.2}{0.4} = 0.5$$

$$f(10|2) = \frac{f(10,2)}{h(2)} = \frac{0.12}{0.4} = 0.3$$

The conditional distribution for Y given $X = 0$ is:

$$f(1|0) = \frac{f(1,0)}{g(0)} = \frac{0.3}{0.5} = 0.6$$

$$f(2|0) = \frac{f(2,0)}{g(0)} = \frac{0.2}{0.5} = 0.4$$

(c) The random variables X and Y are independent since $f(x,y) = g(x)h(y)$ for all (x,y) pairs. This can be seen in the table below.

		x		
		-5	0	10
y	1	$0.12 = (0.2)(0.6)$	$0.3 = (0.5)(0.6)$	$0.18 = (0.3)(0.6)$
	2	$0.08 = (0.2)(0.4)$	$0.2 = (0.5)(0.4)$	$0.12 = (0.3)(0.4)$

$\square$

3. Let X and Y be jointly distributed continuous random variables with joint density

$$f(x,y) = \begin{cases} \frac{1}{6}(3x^2+4y) & -1 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the marginal densities for X and Y.

(b) Find the conditional probability density of X given $Y = y$ for any $y \in \mathbb{R}$.

(c) Compute the following probabilities

  i. $P(0.5 < X < 1 | Y = 0.25)$.

  ii. $P(X < 0.5)$

  iii. $P(0.5 < X < 1, Y > 0.25)$

  iv. $P(0.5 < Y < 1)$

  v. $P(0.5 < Y < 1 | X = 0.25)$.

(d) Are X and Y independent?

Solution. (a) For $-1 \le x \le 1$,

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\, dy = \int_0^1 \frac{1}{6}(3x^2+4y)\, dy = \frac{1}{6}\left[3x^2y + 2y^2\right]_0^1 = \frac{x^2}{2} + \frac{1}{3}$$

and $g(x) = 0$ elsewhere. Thus the marginal density for X is

$$g(x) = \begin{cases} \frac{x^2}{2} + \frac{1}{3} & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

For $0 \le y \le 1$,

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\, dx = \int_{-1}^1 \frac{1}{6}(3x^2+4y)\, dx = \frac{1}{6}\left[x^3 + 4xy\right]_{-1}^1 = \frac{1}{3} + \frac{4y}{3}$$

and $h(y) = 0$ elsewhere. Thus the marginal density for Y is

$$h(y) = \begin{cases} \frac{1}{3} + \frac{4y}{3} & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(b) For $-1 \le x \le 1$, and $0 \le y \le 1$ The conditional probability density of X given $Y = y$ is

$$f(x|y) = \frac{f(x,y)}{h(y)} = \frac{\frac{1}{6}(3x^2+4y)}{\frac{1}{3} + \frac{4y}{3}} = \frac{3x^2+4y}{2+8y}$$

and $f(x|y) = 0$ otherwise.

(c) i.

$$P(0.5 < X < 1|Y = 0.25) = \int_{0.5}^{1} f(x|0.25)\ dx = \int_{0.5}^{1} \frac{3x^2+1}{4}\ dx = \frac{1}{4}\left[x^3 + x\right]_{0.5}^{1} = \frac{11}{32}$$

ii.

$$P(X < 0.5) = \int_{0}^{0.5} g(x)\ dx = \int_{0}^{0.5} \frac{x^2}{2} + \frac{1}{3}\ dx = \frac{x^3}{6} + \frac{x}{3}\Big|_{0}^{0.5} = \frac{3}{16}$$

iii.

$$\begin{aligned} P(0.5 < X < 1, Y > 0.25) &= \int_{0.25}^{1}\int_{0.5}^{1} f(x,y)\ dx\ dy \\ &= \int_{0.25}^{1}\int_{0.5}^{1} \frac{1}{6}(3x^2+4y)\ dx\ dy \\ &= \frac{1}{6}\int_{0.25}^{1} \left[x^3 + 4xy\right]_{0.5}^{1}\ dy \\ &= \frac{1}{6}\int_{0.25}^{1} \frac{7}{8} + 2y\ dy \\ &= \frac{1}{6}\left[\frac{7y}{8} + y^2\right]_{0.25}^{1} \\ &= \frac{17}{64} \end{aligned}$$

iv.

$$P(0.5 < Y < 1) = \int_{0.5}^{1} h(y)\ dy = \int_{0.5}^{1} \frac{1}{3} + \frac{4y}{3}\ dy = \frac{y}{3} + \frac{2y^2}{3}\Big|_{0.5}^{1} = \frac{2}{3}$$

v.

$$\begin{aligned} P(0.5 < Y < 1|X = 0.25) &= \int_{0.5}^{1} f(y|0.25)\ dy \\ &= \int_{0.5}^{1} \frac{f(y,0.25)}{g(0.25)}\ dy \\ &= \int_{0.5}^{1} \frac{\frac{1}{6}(\frac{3}{16} + 4y)}{\frac{35}{96}}\ dy \\ &= \frac{16}{35}\int_{0.5}^{1} \frac{3}{16} + 4y\ dy \\ &= \frac{16}{35}\left[\frac{3y}{16} + 2y^2\right]_{0.5}^{1} \\ &= \frac{51}{64} \end{aligned}$$

(d) For $-1 \leq x \leq 1$, $0 \leq y \leq 1$,

$$g(x)h(y) = \left(\frac{x^2}{2} + \frac{1}{3}\right)\left(\frac{1}{3} + \frac{4y}{3}\right) = \frac{x^2 + 4x^2y}{6} + \frac{1+4y}{9}$$

wheres

$$f(x,y) = \frac{1}{6}(3x^2 + 4y).$$

In particular, $g(1)h(0) = \frac{5}{18}$ but $f(1,0) = \frac{1}{2}$. Since $f(x,y) \neq g(x)h(y)$ for all (x,y) pairs, X and Y are not independent.

$\square$

4. let X and Y be joint continuous random variables with marginal densities $g(x)$ and $h(y)$ respectively. Show that the conditional densities $f(x|y)$ and $f(y|x)$ are indeed valid probability densities. (Assume $g(x), h(y) \neq 0$.)

   Solution. Since $f(x,y) \geq 0$ and $g(x), h(y) \geq 0$ (as they are valid densities) it follows that $f(x|y) = \frac{f(x,y)}{h(y)} \geq 0$ and $f(y|x) = \frac{f(x,y)}{g(x)} \geq 0$. Furthermore, for any y with $h(y) \neq 0$,

$$\int_{-\infty}^{\infty} f(x|y)\, dx = \int_{-\infty}^{\infty} \frac{f(x,y)}{h(y)}\, dx = \frac{1}{h(y)} \int_{-\infty}^{\infty} f(x,y)\, dx = \frac{1}{h(y)} h(y) = 1.$$

   Similarly

$$\int_{-\infty}^{\infty} f(y|x)\, dx = 1.$$

   Therefore the conditional densities $f(x|y)$ and $f(y|x)$ are valid probability density functions. $\square$

5. Determine whether or not X and Y are independent random variables for each joint probability distribution.

(a)

y \ x	0	1	2
0	$\frac{1}{4}$	0	0
1	0	$\frac{1}{4}$	0
2	0	$\frac{1}{4}$	0
3	0	0	$\frac{1}{4}$

(b)

y \ x	0	1
0	$\frac{1}{4}$	$\frac{1}{4}$
1	$\frac{1}{4}$	$\frac{1}{4}$

(c)

y \ x	0	1
0	$\frac{1}{2}$	$\frac{1}{4}$
1	0	$\frac{1}{4}$

(d)

y \ x	0	1	2	3
0	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$
1	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$	0

Solution. In each case let $g(x)$ be the marginal distribution for X and $h(y)$ be the marginal distribution for Y.

(a) These random variables are not independent since (for example)

$$f(0,0) = \frac{1}{4}$$

but

$$g(0) \cdot h(0) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}.$$

(b) These random variables are independent since, for any (x,y) pair

$$g(x) \cdot h(y) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = f(x,y).$$

(c) These random variables are not independent since (for example)

$$f(0,1) = 0$$

but

$$g(0) \cdot h(1) = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}.$$

(d) These random variables are independent since, for any (x,y) pair

$$g(x) \cdot h(y) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = f(x,y).$$

(e) These random variables are not independent since (for example)

$$f(0,0) = 0$$

but

$$g(0) \cdot h(0) = \frac{1}{8} \cdot \frac{1}{2} = \frac{1}{16}.$$

□

6. Let X and Y and be discrete random variables with the following probability distributions

x	1	3
$g(x)$	0.1	0.9

y	-3	2	4
$h(y)$	0.4	0.3	0.3

If X and Y are independent random variables defined on a common sample space, find their joint distribution.

Solution. Since X and Y are independent, $f(x,y) = g(x) \cdot h(y)$. The joint distribution is:

		x = 1	x = 3
y	-3	0.04	0.36
	2	0.03	0.27
	4	0.03	0.27

□