MATH1550
Practice Set 11

These exercises are suited to Chapter 5.
Topics Covered:

- The Discrete Uniform Distribution
- The Bernoulli Distribution
- The Binomial Distribution
- The Negative Binomial Distribution

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1. (a) Give an example of a probability experiment with a random variable which has the discrete uniform distribution.
   (b) Give an example of a probability experiment with a random variable which has the Bernoulli distribution.
   (c) Write the probability distribution function for a random variable with binomial distribution.
   (d) Give an example of a probability experiment with a random variable which has the binomial distribution.
   (e) Write the probability distribution function for a random variable with negative binomial distribution.
   (f) Give an example of a probability experiment with a random variable which has the negative binomial distribution.
   (g) What is the relationship between the binomial distribution and the negative binomial distribution?

   Solution. (a) Some examples are
   - Rolling a regular 6-sided die where X is the number that is rolled.
   - Drawing a raffle ticket where X is the ticket number that is drawn.
   - Making a 4 digit number by drawing a numbered ball from a bag with balls numbered 0 to 9 four times with replacement. Let X be the 4 digit number that appears from 0000 to 9999.

   (b) Some examples are
   - Tossing a balanced coin once. Let $X = 1$ if heads appears and $X = 0$ if tails appears.
   - Drawing a raffle ticket. Let $X = 1$ if your ticket is drawn and $X = 0$ if it is not.
   - Asking a friend to go to a movie. Let $X = 1$ if they answer "yes" and $X = 0$ if they answer "no."

   (c) For $x = 0, 1, \ldots, n$,
   $$P(X = x) = b(x; n, \theta) = \binom{n}{x} \theta^x (1 - \theta)^{n-x}$$

   (d) Any experiment with repeated, independent Bernoulli trials has binomial distribution. For example, tossing a coin 10 times, let X be the number of heads that appears.

   (e) For $x = k, k + 1, k + 2, \ldots,$
   $$P(X = x) = b^*(x; k, \theta) = \binom{x-1}{k-1} \theta^k (1 - \theta)^{x-k}$$

   (f) Tossing a coin until a certain total number of heads appears. Let X be the number of tosses required until exactly k heads appear.

   (g)
   $$b^*(x; k, \theta) = \frac{k}{x} b(k; x, \theta)$$

   □

2. Let $b(x; n, \theta)$, be the binomial distribution where x is the number of "successes," n is the number of trials and θ is the probability of success at each trial. Compute the following:

   (a) $b(2; 5, \frac{1}{3})$.

   (b) $b(7; 10, \frac{1}{2})$.

   (c) $b(3; 4, \frac{1}{4})$.

Solution. (a)

$$b\left(2; 5, \frac{1}{3}\right) = \binom{5}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^3 = 10 \cdot \frac{1}{9} \cdot \frac{8}{27} = \frac{80}{243} \approx 0.3292.$$

(b)

$$b\left(7; 10, \frac{1}{2}\right) = \binom{10}{7}\left(\frac{1}{2}\right)^7\left(\frac{1}{2}\right)^3 = 120 \cdot \frac{1}{128} \cdot \frac{1}{8} = \frac{15}{128} \approx 0.1172.$$

(c)

$$b\left(3; 4, \frac{1}{4}\right) = \binom{4}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^1 = 4 \cdot \frac{1}{64} \cdot \frac{3}{4} = \frac{3}{64} \approx 0.0469.$$

□

3. A certain sports team has probability $\frac{2}{3}$ of winning a game whenever it plays.

   (a) Suppose 4 games are played, what is the probability that this team wins more than half of its games?

   (b) What is the probability they they lose all their games if they play 4 games?

   (c) Suppose they play 12 games. What is the probability they win more than 2 games?

   (d) If they play 15 games how many do they expect to win?

   (e) What is the probability that they get their 3rd win on exactly the 5th game?

   (f) What is the probability that they get their 10th win on exactly the 15th game?

   (g) How many games should they expect to play in order to have 10 wins?

Solution. (a) We can model this situation with the binomial distribution. The probability that they win more than half of their games if they play 4 games is

$$b\left(3; 4, \frac{2}{3}\right) + b\left(4; 4, \frac{2}{3}\right) = \binom{4}{3}\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^1 + \binom{4}{4}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^0 = \frac{32}{81} + \frac{16}{81} = \frac{16}{27} \approx 0.5926$$

(b) The probability that they lose all 4 games is

$$b\left(0; 4, \frac{2}{3}\right) = \binom{4}{0}\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^4 = \frac{1}{81} \approx 0.0123$$

(c)

$$\begin{aligned}
P(X > 2) &= 1 - P(X \leq 2) \\
&= 1 - b\left(0; 12, \frac{2}{3}\right) - b\left(1; 12, \frac{2}{3}\right) - b\left(2; 12, \frac{2}{3}\right) \\
&= 1 - \binom{12}{0}\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^{12} - \binom{12}{1}\left(\frac{2}{3}\right)^1\left(\frac{1}{3}\right)^{11} - \binom{12}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^{10} \\
&= 1 - \frac{1}{3^{12}} - 12 \cdot \frac{2}{3^{12}} - 66 \cdot \frac{4}{3^{12}} \\
&= \frac{531152}{531441} \\
&\approx 0.9995
\end{aligned}$$

(d) If X has binomial distribution with $n = 15$ and $\theta = \frac{2}{3}$ we have

$$E(X) = n\theta = 15 \cdot \frac{2}{3} = 10.$$

(e) We can model this situation with the negative binomial distribution. The probability that they win their 3rd game on exactly the 5th game is

$$b^*\left(5; 3, \frac{2}{3}\right) = \binom{4}{2}\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^2 = \frac{16}{81} \approx 0.1975.$$

(f) The probability that they win their 10th game on exactly the 15th game is

$$b^*\left(15; 10, \frac{2}{3}\right) = \binom{14}{9}\left(\frac{2}{3}\right)^{10}\left(\frac{1}{3}\right)^5 = \frac{2050048}{14348907} \approx 0.1429.$$

(g) The mean μ for a negative binomial distribution with $k = 10$ and $\theta = \frac{2}{3}$ is given by

$$\mu = \frac{k}{\theta} = \frac{10}{\frac{2}{3}} = 15.$$

□

4. Find the least number of dice that must be thrown so that there is a better than 0.5 chance of rolling at least one 6. Assume these are fair 6-sided dice. (Hint: Let X be the number of 6's that appear in n rolls.)

Solution. Let X be the number of 6's that appear in n rolls. The binomial distribution $b(k; n, \frac{1}{6})$, gives the the probability of rolling k 6's in n rolls. We are interesting in finding the smallest $n \in \mathbb{N}$ so that

$$0.5 < P(X \geq 1) = 1 - P(X = 0) = 1 - b\left(0; n, \frac{1}{6}\right) = 1 - \binom{n}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^n = 1 - \left(\frac{5}{6}\right)^n.$$

We can solve by trial and error to see that

$$\left(\frac{5}{6}\right)^3 = \frac{125}{216} > 0.5$$

whereas

$$\left(\frac{5}{6}\right)^4 = \frac{625}{1296} < 0.5$$

and thus $P(X \geq 1) > 0.5$ for $n = 4$, and this is the least such n.

Alternative: To avoid the trial and error part, note that

$$1 - \left(\frac{5}{6}\right)^n > 0.5 \quad \Rightarrow \quad \left(\frac{5}{6}\right)^n < 0.5 \quad \Rightarrow \quad n > \log_{\frac{5}{6}}(0.5) = 3.8,$$

(as $x < y$ implies $\log_a x > \log_a y$ for $0 < a < 1$) and hence $n = 4$.

□

5. In an office building for a certain company, 8 employees will have to share an office. Each person has an equally likely chance of working from home versus working in the office. What is the minimum number of desks needed to be put in the office so that each person has a desk at least 90 percent of the time? (Hint: Let X be the number of people who come to the office on a given day.)

   Solution. Letting X be the number of people working in the office, this situation can be modeled with the binomial distribution with $n = 8$ and $\theta = \frac{1}{2}$. The probability that k people will work in the office is given by $b(k; 8, \frac{1}{2})$ for $k = 0, \ldots, 8$.

   Let d be the number of desks in the office, where $d = 0, \ldots, 8$. The probability that each person gets a desk is given by

$$P(X \leq d) = \sum_{k=0}^{d} b\left(k; 8, \frac{1}{2}\right) = \sum_{k=0}^{d} \binom{8}{k} \left(\frac{1}{2}\right)^8,$$

   or by complement,

$$P(X \leq d) = 1 - P(X > d) = 1 - \sum_{k=d+1}^{8} b\left(k; 8, \frac{1}{2}\right) = 1 - \sum_{k=d+1}^{8} \binom{8}{k} \left(\frac{1}{2}\right)^8.$$

   We want the least value for $d \in \{0, \ldots, 8\}$ so that $P(X \leq d) \geq 0.9$, which requires that $\sum_{k=d+1}^{8} \binom{8}{k} \left(\frac{1}{2}\right)^d < 0.1$. Note that

$$b\left(8; 8, \frac{1}{2}\right) = \binom{8}{8} \left(\frac{1}{2}\right)^8 = \frac{1}{256}$$

$$b\left(8; 7, \frac{1}{2}\right) = \binom{8}{7} \left(\frac{1}{2}\right)^8 = \frac{8}{256}$$

$$b\left(8; 6, \frac{1}{2}\right) = \binom{8}{6} \left(\frac{1}{2}\right)^8 = \frac{28}{256}$$

   and so

$$\sum_{k=7}^{8} \binom{8}{k} \left(\frac{1}{2}\right)^8 = \frac{9}{256} \approx 0.0352 < 0.1$$

   whereas

$$\sum_{k=6}^{8} \binom{8}{k} \left(\frac{1}{2}\right)^8 = \frac{37}{256} \approx 0.1445 > 0.1.$$

   Therefore taking $d = 6$, yields $P(X \leq d) \geq 0.9$, and this is the least such d which ensure this. □

6. A discrete random variable X is said to have *Poisson distribution* if its probability distribution is given by

$$p(x; \lambda) = \begin{cases} \frac{\lambda^x e^{-\lambda}}{x!} & x \in \{0, 1, 2, \ldots, \} \\ 0 & \text{othewise} \end{cases}$$

for some $\lambda > 0$. The Poisson distribution $p(x, \lambda)$ may be used as an approximation for the binomial distribution $b(x; n, \theta)$, particularly when n is large and θ is small, by taking $\lambda = n\theta$.

(a) It is known that one million cars cross over a certain bridge every day. The probability that a car will get a flat tire is 0.00001. Write the expression for finding the probability that exactly 20 cars will have a flat tire using the binomial distribution.

(b) Use the Poisson distribution to approximate the probability in part (a).

Solution. (a) Using the binomial distribution, with $n = 1000000$ and $\theta = 0.00001$, we have

$$b(20; 1000000, 0.00001) = \binom{1000000}{20}(0.00001)^{20}(0.99999)^{9999970}$$

(b) To approximate part (a) with the Poisson distribution we take $\lambda = n\theta = (1000000)(0.00001) = 10$. Thus

$$p(20; 10) = \frac{10^{20}e^{-10}}{20!} \approx 0.0019$$

□