Covariance and Independence:

Recall that joint random variables X and Y are independent if and only if $f(x, y) = g(x) \cdot h(y)$; i.e. their joint distribution is the product of the marginal distributions.

The following theorem is a consequence of this.

Theorem 4.2.13 *If X and Y are independent, then*

$$\underbrace{E(XY)}_{\mu'_{1,1}} = \underbrace{E(X)}_{\mu_X} \cdot \overbrace{E(Y)}^{=\mu_Y}$$

and $\underbrace{\sigma_{XY}}_{\text{cov}(X,Y)} = 0$.

X, Y independent $\Rightarrow \sigma_{XY} = 0$

This theorem is only a one-way implication as seen in the next example.

From the theorem we know
if $\sigma_{XY} \neq 0$, then X and Y
are not independent.

Example 4.2.14 *Let X and Y be discrete random variables with joint distribution given by,*

marginal distr. for X

$g(-1) = \frac{1}{3}$ $g(0) = \frac{1}{3}$ $g(1) = \frac{1}{3}$

$$
\begin{array}{cc|ccc|}
 & & & x & \\
 & & -1 & 0 & 1 \\
\hline
 & -1 & \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\
y & 0 & 0 & 0 & 0 \\
 & 1 & \frac{1}{6} & 0 & \frac{1}{6} \\
\hline
\end{array}
$$

$h(-1) = \frac{4}{6} = \frac{2}{3}$

$h(0) = 0$

$h(1) = \frac{1}{3}$

marginal distr. for Y

Find cov(X, Y). Determine whether X and Y are independent.

To find $\text{cov}(X, Y)$ first we need the marginal distributions, $g(x)$ and $h(y)$.

Then we can find μ_X and μ_Y:

mean of X

$\mu_X = \sum_x x \cdot g(x) = (-1) \cdot \frac{1}{3} + (0) \cdot \frac{1}{3} + (1) \cdot \frac{1}{3} = 0$.

mean of Y

$\mu_Y = \sum_y y \cdot h(y) = (-1) \cdot \frac{2}{3} + 0 \cdot 0 + 1 \cdot \frac{1}{3} = -\frac{2}{3} + 0 + \frac{1}{3} = -\frac{1}{3}$

y \ x	-1	0	1	
-1	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{2}{3}$ $= h(-1)$
0	0	0	0	0 $= h(0)$
1	$\frac{1}{6}$	0	$\frac{1}{6}$	$\frac{1}{3}$ $= h(1)$
	$g(-1) =$ $\frac{1}{3}$	$g(0)$ $\frac{1}{3}$	$\frac{1}{3}$ $g(1)$	

Next we need $E(XY) = \sum_x \sum_y xy \cdot f(x,y)$:

$$= ((-1)\cdot(-1))\cdot\frac{1}{6} + \ldots \cancel{(-1)\cdot 0\cdot 0} + (-1)\cdot 1\cdot\frac{1}{6}$$

$$+ \cancel{0\cdot(-1)\cdot\frac{1}{3}} + \cancel{0\cdot 0\cdot 0} + \cancel{0\cdot 1\cdot 0}$$

$$+ 1\cdot(-1)\cdot\frac{1}{6} + \cancel{1\cdot 0\cdot 0} + 1\cdot 1\cdot\frac{1}{6}$$

$$= \frac{1}{6} - \frac{1}{6} - \frac{1}{6} + \frac{1}{6} = 0$$

$\rho_{XY} = cov(X,Y) = 0$

So, $\underbrace{\text{cov}(X,Y) = E(XY) - \overset{0}{\mu_X}\overset{-\frac{1}{3}}{\mu_Y}}_{=0} = 0 - (0)(-\frac{1}{3}) = 0.$

However, the random variables are not independent, as can be seen by the example that $f(-1,-1) = \frac{1}{6} \neq g(-1)\cdot h(-1) = \frac{2}{9}$.

earlier:
X, Y are independent:
$f(x,y) = g(x)\cdot h(y)$
for all x, y

ex: $x=-1$, $y=-1$
$f(-1,-1) = \frac{1}{6} \neq g(-1)\cdot h(-1)$
$= \frac{1}{3}\cdot\frac{2}{3} = \frac{2}{9}$

		$x = 0$	$x = 1$	$x = 2$	
y	0	$\frac{6}{36}$	$\frac{12}{36}$	$\frac{3}{36}$	$\frac{21}{36}$
	1	$\frac{8}{36}$	$\frac{6}{36}$		$\frac{14}{36}$
	2	$\frac{1}{36}$			$\frac{1}{36}$
		$\frac{15}{36}$	$\frac{18}{36}$	$\frac{3}{36}$	

Are the random variables X and Y of the caplet example independent?

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No. We found that $\operatorname{cov}(X, Y) = -\frac{7}{54} \neq 0$.

Therefore they are not independent.

Example 4.2.15 *Let X and Y be jointly continuous random variables with joint density*

$$f(x) = \begin{cases} \frac{3x^2y}{28} & \text{for } 1 < x < 2, 1 < y < 3 \\ 0 & \text{otherwise} \end{cases}$$

Find the covariance of X and Y and determine whether they are independent.

$$\sigma_{X,Y} = \mathrm{cov}(X,Y) = \overbrace{E(XY)}^{\mu'_{1,1}} - \mu_X \mu_Y$$

$$\mu_X = E(X) = \iint x \cdot f(x,y)\,dx\,dy = \int_1^3 \int_1^2 x \cdot \frac{3}{28} x^2 y \,dx\,dy$$

$$= \frac{3}{28} \int_1^3 \underline{\int_1^2 x^3 y \,dx}\,dy = \frac{3}{28} \int_1^3 \left(\frac{x^4}{4} y\right)\Big|_{x=1}^{x=2} dy = \frac{3}{28} \int_1^3 \left(4y - \frac{y}{4}\right) dy$$

$$= \frac{3}{28} \int_1^3 \frac{15}{4} y \,dy = \frac{3}{28} \cdot \frac{15}{4} \int_1^3 y \,dy = \frac{3}{28} \cdot \frac{15}{4} \left(\frac{y^2}{2}\right)\Big|_{y=1}^{y=3}$$

$$= \frac{3}{28} \cdot \frac{15}{4} \cdot \left(\frac{9}{2} - \frac{1}{2}\right) = \frac{3}{28} \cdot \frac{15}{\cancel{4}} \cdot \cancel{4} = \frac{45}{28}$$

$$\mu_Y = E(Y) = \iint y\cdot f(x,y)\,dx\,dy = \int_1^3\int_1^2 y\cdot\frac{3}{28}x^2y\,dx\,dy$$

$$= \frac{3}{28}\int_1^3\int_1^2 x^2y^2\,dx\,dy = \frac{3}{28}\int_1^3\left(\frac{x^3}{3}y^2\right)\Bigg|_{x=1}^{x=2} dy$$

$$= \frac{3}{28}\int_1^3\left(\frac{8}{3}y^2-\frac{1}{3}y^2\right)dy = \frac{3}{28}\int_1^3\frac{7}{3}y^2\,dy$$

$$= \frac{3}{28}\cdot\frac{7}{3}\int_1^3 y^2\,dy = \frac{3}{28}\cdot\frac{7}{3}\left(\frac{y^3}{3}\right)\Bigg|_{y=1}^{y=3}$$

$$= \frac{3}{28}\cdot\frac{7}{3}\cdot\left(\frac{27}{3}-\frac{1}{3}\right) = \frac{\cancel{3}}{\cancel{28}}_{\cancel{4}\,2}\cdot\frac{\cancel{7}}{\cancel{3}}\cdot\frac{\cancel{26}^{\,13}}{3} = \frac{13}{6}$$

$$\mu'_{1,1} = E(XY) = \int\int xy\, f(x,y)\,dx\,dy = \int_1^3\int_1^2 x \cdot y \cdot \frac{3}{28} x^2 y\, dx\, dy$$

$$= \frac{3}{28}\int_1^3 \underbrace{\int_1^2 x^3 y^2\, dx}\, dy = \frac{3}{28}\int_1^3 \left(\frac{x^4}{4} y^2\right)\Bigg|_{x=1}^{x=2} dy$$

$$= \frac{3}{28}\int_1^3 \left(\frac{16}{4}y^2 - \frac{1}{4}y^2\right) dy = \frac{3}{28}\int_1^3 \frac{15}{4} y^2\, dy$$

$$= \frac{3}{28} \cdot \frac{15}{4}\int_1^3 y^2\, dy = \frac{3}{28} \cdot \frac{15}{4}\left(\frac{y^3}{3}\right)\Bigg|_{y=1}^{y=3}$$

$$= \frac{3}{28} \cdot \frac{15}{4} \cdot \left(\frac{27}{3} - \frac{1}{3}\right) = \frac{\cancel{3}}{\cancel{28}_{14}} \cdot \frac{15}{4} \cdot \frac{\cancel{26}^{13}}{\cancel{3}} = \frac{195}{56}$$

$$E(XY) = \frac{195}{56} \qquad \mu_x = \frac{45}{28} \qquad \mu_y = \frac{13}{6}$$

Using the short-cut formula:

$$\rho_{X,Y} = \operatorname{cov}(X,Y) = E(XY) - \mu_x \cdot \mu_y = \frac{195}{56} - \frac{\cancel{45}^{15}}{28} \cdot \frac{13}{\cancel{6}_{2}}$$

$$= 0$$

Since $\operatorname{cov}(X,Y)=0$, we cannot make any conclusion about the independence of X and Y yet.

So, we need to check if $f(x,y) = g(x) \cdot h(y)$ for all x,y

where $\underbrace{g(x)}_{\text{marginal density for } X} = \int f(x,y)\,dy$ and $\underbrace{h(y)}_{\text{marginal density for } Y} = \int f(x,y)\,dx$

$$= \int_1^3 \frac{3}{28} x^2 y\,dy = \frac{3}{28} \int_1^3 x^2 y\,dy$$

$$= \frac{3}{28}\left(x^2 \frac{y^2}{2}\right)\Big|_{y=1}^{y=3}$$

$$= \frac{3}{28}\left(\frac{9}{2}x^2 - \frac{1}{2}x^2\right) = \frac{3}{28}\, 4x^2 = \underbrace{\frac{3}{7}x^2}_{g(x)}$$

$g(x) = 0$ elsewhere

$$= \int_1^2 \frac{3}{28} x^2 y\,dx = \frac{3}{28}\int_1^2 x^2 y\,dx$$

$$= \frac{3}{28}\left(\frac{x^3}{3} \cdot y\right)\Big|_{x=1}^{x=2}$$

$$= \frac{3}{28}\left(\frac{8}{3} \cdot y - \frac{1}{3}y\right) = \frac{\cancel{3}}{\cancel{28}_{4}} \cdot \frac{\cancel{7}^{1}}{\cancel{3}} y = \underbrace{\frac{y}{4}}_{h(y)}$$

$h(y) = 0$ elsewhere

Then, $g(x) \cdot h(y) = \frac{3}{7}x^2 \cdot \frac{y}{4} = \frac{3}{28}x^2 y$ for $1<x<2$ and $1<y<3$

($g(x) \cdot h(y) = 0$ elsewhere)

Note that $g(x) \cdot h(y) = f(x,y)$ for all x, y.
Therefore, X and Y are independent.

Conditional Expectations:

"x given y"

Earlier we defined conditional probability $f(x|y)$, for joint random variables X and Y, we can also talk about conditional expectation.

Let X and Y have probability distribution/density $f(x, y)$, and let $u(X)$ be some function of X. The **conditional expected value of $u(X)$ given $Y = y$** is

expectation

$$\text{discrete case: } \overset{E(u(x)|y)}{\cancel{E(X|y)}} = \sum_x u(x) \cdot f(x|y)$$

$$\text{continuous case: } \overset{E(u(x)|y)}{\cancel{E(X|y)}} = \int_x u(x) \cdot f(x|y)\, dx$$

$\overset{E(u(x)|y)}{\cancel{E(X|y)}}$ is called the **conditional mean of $\overset{u(x)}{\cancel{X}}$ given $Y = y$.**

E(x|y) is called conditional mean of X given Y = y.

$E(u(X)|y)$
$= \sum_x u(x) \cdot f(x|y)$

here $u(X) = X$:

$= \sum_x x \cdot f(x|y)$

Example 4.2.16

		$x=0$	$x=1$	$x=2$	
y	0	$\frac{6}{36}$	$\frac{12}{36}$	$\frac{3}{36}$	$\frac{21}{36} = h(0)$
	1	$\frac{8}{36}$	$\frac{6}{36}$		$\frac{14}{36} = h(1)$
	2	$\frac{1}{36}$			$\frac{1}{36} = h(2)$
		$g(0)$ $\frac{15}{36}$	$g(1)$ $\frac{18}{36}$	$g(2)$ $\frac{3}{36}$	

Find the expected value (conditional mean) of X given that $Y = 1$.

here: $u(X) = X$

By definition,

$$E(X|1) = \sum_x x f(x|1).$$

$E(X|y) = \sum_x x \cdot f(x|y)$

$E(X|1) = \sum_x x \cdot f(x|1)$

Recall that $f(x|y) = \frac{f(x,y)}{h(y)}$, and so

$$f(0|1) = \frac{8}{14}, \quad f(1|1) = \frac{6}{14}, \quad f(2|1) = 0.$$

$f(0|1) = \frac{f(0,1)}{h(1)} = \frac{\frac{8}{36}}{\frac{14}{36}} = \frac{8}{14} = \frac{4}{7}$

Therefore we have

$$E(X|1) = \underbrace{0 \cdot \frac{8}{14}}_{=0} + \underbrace{1 \cdot \frac{6}{14}}_{\frac{6}{14}} + \underbrace{2 \cdot 0}_{0} = \frac{6}{14} \approx 0.4286.$$

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$f(1|1) = \frac{f(1,1)}{h(1)} = \frac{\frac{6}{36}}{\frac{14}{36}} = \frac{6}{14} = \frac{3}{7}$

$f(2|1) = \frac{f(2,1)}{h(1)} = 0$

Chapter 5

Special Probability Distributions

This chapter presents some commonly used probability distributions for discrete random variables. Having a pre-determined probability distribution to model a chance experiment prevents from having to re-derive its properties each time (e.g. mean and variance).

The models presented depend on **parameters**; input values which tailor the probability distribution to the particular example.

In some cases the values of the distribution for a range of parameters are recorded in a table which can be used to evaluate probabilities, rather than computing the sums directly, (or integrating in the case of continuous random variables).

This is not only a convenience, in some cases it may be impractical to compute such values on the spot, or impossible if, for example, no exact expression exists for an integral.

5.1 Discrete Uniform Distribution

Suppose a random variable X has a finite range of k values, $\{x_1, x_2, \ldots, x_k\}$. Then X has **discrete uniform distribution** if

$$f(x) = \frac{1}{k}$$

for $x \in \{x_1, x_2, \ldots, x_k\}$. In other words each outcome is equally likely.

mean for the uniform distribution

Our only parameter in this case is k. For the discrete uniform distribution:

$$\mu = E(X) = x_1 \cdot \frac{1}{k} + x_2 \cdot \frac{1}{k} + \ldots + x_k \cdot \frac{1}{k}$$

$$E(X) = \text{mean} = \mu = \sum_{i=1}^{k} x_i f(x_i) = \frac{\sum_{i=1}^{k} x_i}{k}.$$

$$= \frac{1}{k} \cdot (x_1 + x_2 + \ldots + x_k)$$

$$= \frac{1}{k} \cdot \sum_{i=1}^{k} x_i = \frac{\sum_{i=1}^{k} x_i}{k}$$

variance $= E((X-\mu)^2)$

$$\sigma^2 = \sum_{i=1}^{k} (x_i - \mu)^2 f(x_i) = \boxed{\frac{\sum_{i=1}^{k}(x_i - \mu)^2}{k}} = \underbrace{\overbrace{\frac{\sum_{i=1}^{k} x_i^2}{k}}^{E(X^2)} - \overbrace{\left(\frac{\sum_{i=1}^{k} x_i}{k}\right)^2}^{(E(X))^2}}_{\text{shortcut formula}}$$

$$\sigma^2 = \mu_2' - \mu^2$$

$$= E(X^2) - (E(X))^2$$

$$\sigma^2 = \mu_2' - \mu_2 = E(X^2) - (E(X))^2$$

$$= \left(x_1^2 \cdot \frac{1}{k} + x_2^2 \cdot \frac{1}{k} + \ldots + x_k^2 \cdot \frac{1}{k}\right)$$

$$- \left(\frac{\sum_{i=1}^{k} x_i}{k}\right)^2$$

$$\text{So, } \sigma^2 = \frac{1}{k}\sum_{i=1}^{k} x_i^2 - \left(\frac{\sum_{i=1}^{k} x_i}{k}\right)^2$$

$$= \frac{\sum_{i=1}^{k} x_i^2}{k} - \left(\frac{\sum_{i=1}^{k} x_i}{k}\right)^2$$

variance for uniform distribution

5.2 The Bernoulli Distribution

Consider an experiment with two possible outcomes, either success or failure. (For example, a single coin toss.)

Assign random variable X the value 1 for success and 0 for failure.

If the probability of success is θ, then the probability of a failure is $1-\theta$.

In this case X is called a **Bernoulli random variable** and has **Bernoulli distribution** given by

$$f(x;\theta) = \theta^x(1-\theta)^{1-x} \quad \text{for } x = 0, 1.$$

Exercise: Show that the Bernoulli distribution has

$$\mu = \theta, \quad \sigma^2 = \theta(1-\theta).$$