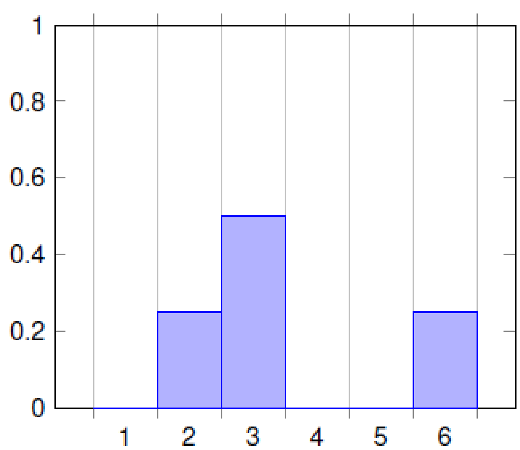
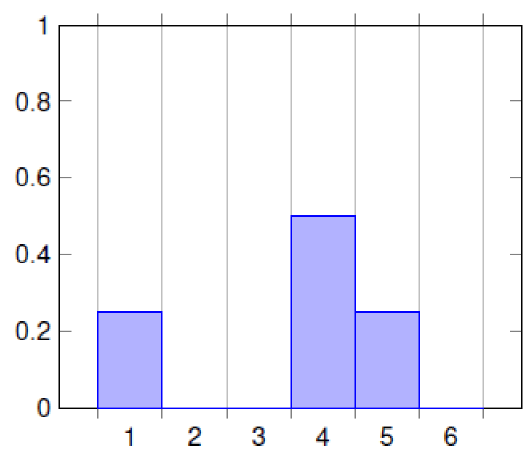
Example 4.2.1 *Let X and Y be discrete random variables with the following distributions*

x	$P(X = x)$
1	0
2	1/4
3	1/2
4	0
5	0
6	1/4

y	$P(Y = y)$
1	1/4
2	0
3	0
4	1/2
5	1/4
6	0

Show that these distributions have the same mean and variance.





For X:

$$\mu = E(X) = 1 \cdot 0 + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{2} + 4 \cdot 0 + 5 \cdot 0 + 6 \cdot \frac{1}{4} = \frac{7}{2}$$

$$\sigma^2 = E((X-\mu)^2) = \left(1 - \frac{7}{2}\right)^2 \cdot 0 + \dots$$

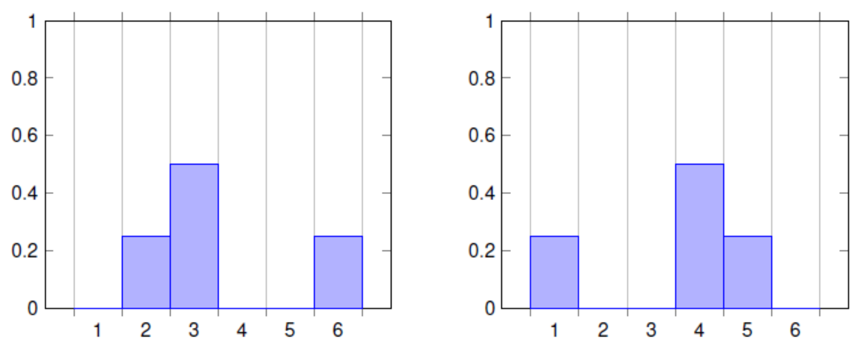
For Y:

Let us now compute the 3rd moment about the mean.

For X:

$$\mu_3 = E((X-\mu)^3) = \left(1 - \frac{7}{2}\right)^3 \cdot 0 + \dots$$

For Y:



The 3rd moment about the mean describes the symmetry of the graph about the mean.

Notice that the distribution on the left has a higher proportion of its probabilities to the left of the mean $\mu = \frac{7}{2}$ and its 3rd moment, μ_3, about the mean, is positive 3.

The opposite is true for the distribution on the right, and its 3rd moment about the mean is negative 3.

Example 4.2.2 *Let random variable X be the number of points on a regular 6-sided die. Compute mean and variance of X.*

The mean is

$$\mu = E(X) = 1 \cdot \frac{1}{6} + \dots$$

The variance is

$$\sigma^2 = E((X - \mu)^2) = (1 - 3.5)^2 \cdot \frac{1}{6} + \dots$$

Consider the variance. Using properties of expected values we have

$$\begin{aligned}
E((X-\mu)^2) &= E(X^2 - 2\mu X + \mu^2) \\
&= E(X^2) - 2E(\mu X) + E(\mu^2) \\
&= E(X^2) - 2\mu E(X) + \mu^2 \\
&= E(X^2) - 2\mu \cdot \mu + \mu^2 \\
&= E(X^2) - \mu^2
\end{aligned}$$

We summarize this in a theorem.

Theorem 4.2.3

$$\sigma^2 = \mu_2' - \mu^2$$

Example 4.2.4 *Redo the previous die rolling problem with theorem.*

We first need to find the mean μ:

$$\mu = E(X) = 1 \cdot \frac{1}{6} + \dots$$

Next we find $\mu_2' = E(X^2)$:

$$\mu_2' = E(X^2) = 1^2 \cdot \frac{1}{6} + 2^2 \cdot \frac{1}{6} + \dots$$

Then using the theorem, the variance is

$$\sigma^2 = \mu_2' - \mu^2 = \frac{91}{6} - (3.5)^2 \approx 2.9167.$$

Theorem 4.2.5 *If X has variance σ^2, then for constants a and b*

$$var(aX + b) = a^2\sigma^2.$$

Proof: Let $Y = aX + b$, and let $\mu = E(X)$.

Then

$$E(Y) = E(aX + b) = aE(X) + b = a\mu + b.$$

For the variance of Y, we have

$$\begin{aligned}
\text{var}(Y) &= E((Y - (a\mu + b))^2) \\
&= E((aX + b - a\mu - b)^2) \\
&= E((aX - a\mu)^2) \\
&= E(a^2X^2 - 2a^2X\mu + a^2\mu^2) \\
&= a^2E(X^2) - 2a^2\mu E(X) + a^2\mu^2 \\
&\qquad\qquad \text{(by Theorem 4.1.4)} \\
&= a^2(E(X^2) - 2\mu^2 + \mu^2) \\
&= a^2(E(X^2) - \mu^2) \\
&= a^2\sigma^2 \qquad \text{(by Theorem 4.2.3)}
\end{aligned}$$

4.2.1 Chebyshev's Theorem

The next important theorem shows how σ describes the spread of the probability distribution.

Theorem 4.2.6 (Chebyshev's Theorem) *Let X be a random variable with mean μ and standard deviation σ. Then for any $k > 0$,*

$$P(|X - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}.$$

In words, the probability that values for X lie within k standard deviations of the mean is at least $1 - \frac{1}{k^2}$.

(The proof of this theorem can be found in the lecture slides.)

Example 4.2.7 *The mean score of an exam is 70, with a standard deviation of 5. At least what percentage of the data set lies between 40 and 100?*

Example 4.2.8 *The mean age of a flight attendant is 40, with a standard deviation of 8. At least what percent of the data set lies between 20 and 60?*

For what comes next, we need to know the **Maclaurin Series for** e^x,

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots = \sum_{i=0}^{\infty} \frac{1}{i!}x^i,$$

and term-by-term differentiation

$$\frac{d}{dx}\left(\sum_{i=0}^{\infty} f_i(x)\right) = \sum_{i=0}^{\infty} \frac{d}{dx}\left(f_i(x)\right).$$

Now we can talk about **moment generating functions**.

The **moment generating function of a random variable** X, where it exists, is given by

$$\text{discrete case:} \quad M_X(t) = E(e^{tX}) = \sum_x e^{tx} \cdot f(x)$$

$$\text{continuous case:} \quad M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} \cdot f(x)\, dx$$

where $f(x)$ is the probability distribution/density of X.

We will see why this name is appropriate.

Expanding the expression for $M_X(t)$,

$M_X(t)$

$$= \sum_x e^{tx} \cdot f(x)$$

$$= \sum_x \left(1 + (tx) + \frac{1}{2!}(tx)^2 + \frac{1}{3!}(tx)^3 + \dots\right) \cdot f(x)$$

$$= \sum_x f(x) + (tx)f(x) + \frac{(tx)^2}{2!}f(x) + \frac{(tx)^3}{3!}f(x) + \dots$$

$$= \sum_x f(x) + t\sum_x xf(x) + \frac{t^2}{2!}\sum_x x^2 f(x) + \frac{t^3}{3!}\sum_x x^3 f(x) + \dots$$

We see the rth moments about the origin appearing in the terms of the series.

$$M_X(t) = \sum_x f(x) + t \sum_x x f(x) + \frac{t^2}{2!} \sum_x x^2 f(x) + \frac{t^3}{3!} \sum_x x^3 f(x) + \ldots$$

To extract the ith moment, we take the ith derivative with respect to t, and evaluate at $t = 0$.

For example, to get the 2nd moment: $\frac{d^2}{dt^2} M_X(t) \Big|_{t=0} =$

$$\frac{d^2}{dt^2} \left(\sum_x f(x) + t \sum_x x f(x) + \frac{t^2}{2!} \sum_x x^2 f(x) + \frac{t^3}{3!} \sum_x x^3 f(x) + \ldots \Big|_{t=0} \right)$$

Take 2nd derivative of each term with respect to t,

$$= \left(0 + 0 + \sum_x x^2 f(x) + t \sum_x x^3 f(x) + \frac{t^2}{2} \sum_x x^4 f(x) + \ldots \Big|_{t=0} \right)$$

Letting $t = 0$ gives $\frac{d^2}{dt^2} M_X(t) \Big|_{t=0} = \sum_x x^2 f(x) = E(X^2)$.