MATH1550
Exercise Set 7

- Conditional distributions
- Conditional densities
- Independent random variables

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1. Let X and Y be discrete random variables with joint probability distribution given by the following table:

$y \backslash x$	1	2	3	4
0	$\frac{4}{84}$	$\frac{18}{84}$	$\frac{12}{84}$	$\frac{1}{84}$
1	$\frac{12}{84}$	$\frac{24}{84}$	$\frac{6}{84}$	
2	$\frac{4}{84}$	$\frac{3}{84}$		

(a) Find the marginal distributions for X and Y.

(b) Find the conditional distribution for X given $Y = 1$.

Solution. (a) The marginal distribution $g(x)$ for X is given by

$$\begin{aligned} g(1) &= \frac{4}{84} + \frac{12}{84} + \frac{4}{84} = \frac{20}{84}, \\ g(2) &= \frac{18}{84} + \frac{24}{84} + \frac{3}{84} = \frac{45}{84}, \\ g(3) &= \frac{12}{84} + \frac{6}{84} = \frac{18}{84}, \\ g(4) &= \frac{1}{84}. \end{aligned}$$

The marginal distribution $h(y)$ for Y is given by

$$\begin{aligned} h(0) &= \frac{4}{84} + \frac{18}{84} + \frac{12}{84} + \frac{1}{84} = \frac{35}{84}, \\ h(1) &= \frac{12}{84} + \frac{24}{84} + \frac{6}{84} = \frac{42}{84}, \\ h(2) &= \frac{4}{84} + \frac{3}{84} = \frac{7}{84}. \end{aligned}$$

(b) If we let $f(x, y)$ denote the joint distribution of X and Y, then the conditional distribution for X given $Y = 1$ is defined as,

$$f(x|1) = \frac{f(x,1)}{h(1)} = \frac{f(x,1)}{\left(\frac{42}{84}\right)}.$$

So we have

$$f(1|1) = \frac{\left(\frac{12}{84}\right)}{\left(\frac{42}{84}\right)} = \frac{12}{42},$$
$$f(2|1) = \frac{\left(\frac{24}{84}\right)}{\left(\frac{42}{84}\right)} = \frac{24}{42},$$
$$f(3|1) = \frac{\left(\frac{6}{84}\right)}{\left(\frac{42}{84}\right)} = \frac{6}{42},$$
$$f(4|1) = \frac{0}{\left(\frac{42}{84}\right)} = 0.$$

□

2. A fair coin is tossed twice. Let X and Y be random variables such that
   - $X = 1$ if the first toss is heads, and $X = 0$ otherwise.
   - $Y = 1$ if both tosses are heads, and $Y = 0$ otherwise

   (a) Give the joint probability distribution for X and Y

   (b) Find the marginal distributions for X and Y.

   (c) Determine whether or not X and Y are independent.

Solution. (a)

		x = 0	x = 1
y	0	0.5	0.25
	1	0	0.25

(b) The marginal distribution $g(x)$ for X is given by

$$g(0) = 0.5 + 0 = 0.5,$$
$$g(1) = 0.25 + 0.25 = 0.5,$$

The marginal distribution $h(y)$ for Y is given by

$$h(0) = 0.5 + 0.25 = 0.75,$$
$$h(1) = 0 + 0.25 = 0.25.$$

(c) They are not independent, for if $f(x, y)$ is the joint distribution, then for example

$$f(0, 0) = 0.5 \neq (0.5)(0.75) = g(0) \cdot h(0).$$

□

3. Let X and Y be discrete random variables with joint probability distribution given by the following table:

y \ x	2	3	4
1	0.06	0.15	0.09
2	0.14	0.35	0.21

(a) Find the marginal distributions for X and Y.

(b) Find the conditional distribution for X given $Y = 2$.

(c) Determine whether or not X and Y are independent.

Solution. (a) The marginal distribution $g(x)$ for X is given by

$$
\begin{aligned}
g(2) &= 0.06 + 0.14 = 0.2,\\
g(3) &= 0.15 + 0.35 = 0.5,\\
g(4) &= 0.09 + 0.21 = 0.3.
\end{aligned}
$$

The marginal distribution $h(y)$ for Y is given by

$$
\begin{aligned}
h(1) &= 0.06 + 0.15 + 0.09 = 0.3,\\
h(2) &= 0.14 + 0.35 + 0.21 = 0.7.
\end{aligned}
$$

(b) If we let $f(x, y)$ denote the joint distribution of X and Y, then the conditional distribution for X given $Y = 2$ is defined as,

$$f(x|2) = \frac{f(x,2)}{h(2)} = \frac{f(x,2)}{0.7}.$$

So we have

$$
\begin{aligned}
f(2|2) &= \frac{0.14}{0.7} = 0.2,\\
f(3|2) &= \frac{0.35}{0.7} = 0.5,\\
f(4|2) &= \frac{0.21}{0.7} = 0.3.
\end{aligned}
$$

(c) Recall that X and Y are independent if $f(x, y) = g(x) \cdot h(y)$ for all x, y. We need to check six cases:

$$
\begin{aligned}
f(2,1) &= 0.06 = (0.2)(0.3) = g(2)h(1), & f(2,2) &= 0.14 = (0.2)(0.7) = g(2)h(2),\\
f(3,1) &= 0.15 = (0.5)(0.3) = g(3)h(1), & f(3,2) &= 0.35 = (0.5)(0.7) = g(3)h(2),\\
f(4,1) &= 0.09 = (0.3)(0.3) = g(4)h(1), & f(4,2) &= 0.21 = (0.3)(0.7) = g(4)h(2).
\end{aligned}
$$

Thus X and Y are independent.

$\square$

4. Let X be a random variable with the following distribution

x	-2	-1	1	2
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

Let $Y = X^2$.

(a) Find the distribution $g(y)$ of Y.

(b) Find the joint distribution $f(x, y)$ of X and Y.

(c) Find the marginal distributions of X and Y.

(d) Determine whether or not X and Y are independent.

Solution. (a) Since $Y = X^2$, the range of Y is $\{1, 4\}$, and

$$P(Y = 1) = P(X = -1) + P(X = 1), \quad P(Y = 4) = P(X = -2) + P(X = 2).$$

In summary, the distribution for Y is

y	1	4
$P(Y = y)$	$\frac{1}{2}$	$\frac{1}{2}$

(b) The joint distribution is

		x			
		-2	-1	1	2
y	1	0	$\frac{1}{4}$	$\frac{1}{4}$	0
	4	$\frac{1}{4}$	0	0	$\frac{1}{4}$

(c) Marginal distribution for X, $g(x) = \sum_y f(x, y)$:

$$g(-2) = \frac{1}{4}, \quad g(-1) = \frac{1}{4}, \quad g(1) = \frac{1}{4}, \quad g(2) = \frac{1}{4}$$

Marginal distribution for Y, $h(y) = \sum_x f(x, y)$:

$$h(1) = \frac{1}{2}, \quad h(4) = \frac{1}{2}$$

(d) Note that $f(-2, 1) = 0 \neq g(-2) \cdot h(1) = \frac{1}{8}$. Therefore X and Y are not independent.

$\square$

5. The joint density function of X and Y is given by

$$f(x, y) = \begin{cases} x + y & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the marginal densities for X and Y, and determine whether X and Y are independent.

Solution. The marginal density for X is

$$\begin{aligned} g(x) &= \int_{-\infty}^{\infty} f(x, y)\, dy \\ &= \int_0^1 x + y\, dy \\ &= xy + \frac{y^2}{2}\Big|_0^1 \\ &= x + \frac{1}{2} \end{aligned}$$

The marginal density for Y is

$$\begin{aligned} h(y) &= \int_{-\infty}^{\infty} f(x,y)\,dx \\ &= \int_0^1 x + y\,dx \\ &= \frac{x^2}{2} + xy\Big|_0^1 \\ &= \frac{1}{2} + y \end{aligned}$$

Then

$$g(x)h(y) = \left(x + \frac{1}{2}\right)\left(y + \frac{1}{2}\right) = xy + \frac{x}{2} + \frac{y}{2} + \frac{1}{4} \neq x + y = f(x,y)$$

(for example $g\left(\frac{3}{4}\right)h\left(\frac{3}{4}\right) = \frac{25}{16} \neq \frac{3}{2} = f\left(\frac{3}{4}, \frac{3}{4}\right)$). Therefore the random variables are not independent. $\square$

6. Find the marginal densities of X and Y given their joint probability density

$$f(x,y) = \begin{cases} \frac{2}{5}(x + 4y) & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Solution. The marginal density of X is

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{-\infty}^{0} f(x,y)\,dy + \int_0^1 f(x,y)\,dy + \int_1^{\infty} f(x,y)\,dy$$

$$= \int_{-\infty}^{0} 0\,dy + \int_0^1 \frac{2}{5}(x + 4y)\;dy + \int_1^{\infty} 0\,dy = \left(\frac{2xy}{5} + \frac{4y^2}{5}\right)\Big|_0^1 = \frac{2x}{5} + \frac{4}{5}.$$

The marginal density of Y is

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_{-\infty}^{0} f(x,y)\,dx + \int_0^1 f(x,y)\,dx + \int_1^{\infty} f(x,y)\,dx$$

$$= \int_{-\infty}^{0} 0\,dx + \int_0^1 \frac{2}{5}(x + 4y)\;dx + \int_1^{\infty} 0\,dx = \left(\frac{x^2}{5} + \frac{8xy}{5}\right)\Big|_0^1 = \frac{1}{5} + \frac{8y}{5}.$$

$\square$

7. Let X and Y be jointly continuous random variables with joint probability density given by

$$f(x,y) = \begin{cases} \frac{12}{5}(2x - x^2 - xy) & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the marginal densities for X and Y.

(b) Find the conditional density for X given $Y = y$ and the conditional density for Y given $X = x$.

(c) Compute the probability $P(\frac{1}{2} < X < 1 | Y = \frac{1}{4})$.

(d) Determine whether or not X and Y are independent.

Solution. (a) The marginal density $g(x)$ for X is

$$\begin{aligned}
g(x) &= \int_{-\infty}^{\infty} f(x,y)\,dy \\
&= \int_0^1 \frac{12}{5}(2x - x^2 - xy)\,dy \\
&= \frac{12}{5}\left(2xy - x^2y - \frac{xy^2}{2}\right)\Bigg|_0^1 \\
&= \frac{12}{5}\left(2x - x^2 - \frac{x}{2}\right) \\
&= \frac{18x}{5} - \frac{12x^2}{5},
\end{aligned}$$

for $0 < x < 1$ and 0 elsewhere.

The marginal density $h(y)$ for Y is

$$\begin{aligned}
h(y) &= \int_{-\infty}^{\infty} f(x,y)\,dx \\
&= \int_0^1 \frac{12}{5}(2x - x^2 - xy)\,dx \\
&= \frac{12}{5}\left(x^2 - \frac{x^3}{3} - \frac{x^2y}{2}\right)\Bigg|_0^1 \\
&= \frac{12}{5}\left(1 - \frac{1}{3} - \frac{y}{2}\right) \\
&= \frac{8}{5} - \frac{6y}{5},
\end{aligned}$$

for $0 < y < 1$ and 0 elsewhere.

(b) The conditional density for X given $Y = y$ when $0 < x < 1, 0 < y < 1$ is given by,

$$f(x|y) = \frac{f(x,y)}{h(y)} = \frac{\frac{12}{5}(2x - x^2 - xy)}{\frac{8}{5} - \frac{6y}{5}} = \frac{12x - 6x^2 - 6xy}{4 - 3y},$$

and $f(x|y) = 0$ elsewhere.

The conditional density for Y given $X = x$ when $0 < x < 1, 0 < y < 1$ is given by,

$$f(y|x) = \frac{f(x,y)}{g(x)} = \frac{\frac{12}{5}(2x - x^2 - xy)}{\frac{18x}{5} - \frac{12x^2}{5}} = \frac{4 - 2x - 2y}{3 - 2x},$$

and $f(x|y) = 0$ elsewhere.

(c)

$$\begin{aligned}
P\left(\frac{1}{2} < X < 1 \middle| Y = \frac{1}{4}\right) &= \int_{\frac{1}{2}}^{1} f\left(x \middle| \frac{1}{4}\right) \, dx \\
&= \int_{\frac{1}{2}}^{1} \frac{12x - 6x^2 - 6x(\frac{1}{4})}{4 - 3(\frac{1}{4})} \, dx \\
&= \int_{\frac{1}{2}}^{1} \frac{42x}{13} - \frac{24x^2}{13} \, dx \\
&= \frac{21x^2}{13} - \frac{8x^3}{13} \Big|_{\frac{1}{2}}^{1} \\
&= (\frac{21(1)^2}{13} - \frac{8(1)^3}{13}) - (\frac{21(\frac{1}{2})^2}{13} - \frac{8(\frac{1}{2})^3}{13}) \\
&= \frac{35}{52} \\
&\approx 0.6731
\end{aligned}$$

(d) They are not independent. For example

$$f(0.25, 0.25) = 0.9 \neq 0.975 = g(0.25) \cdot h(0.25).$$

□

8. Let X and Y be discrete random variables with joint probability distribution given by the following table:

		x		
		-3	2	4
y	1	0.1	0.2	0.2
	3	0.3	0.1	0.1

(a) Find the conditional distribution for X given $Y = 1$.

(b) Are X and Y independent? Justify your answer.

Solution. (a) If we let $f(x, y)$ denote the joint distribution of X and Y, then the conditional distribution for X given $Y = 1$ is defined as,

$$f(x|1) = \frac{f(x,1)}{h(1)} = \frac{f(x,1)}{0.5}.$$

So we have

$$f(-3|1) = \frac{0.1}{0.5} = 0.2, \quad f(2|1) = \frac{0.2}{0.5} = 0.4, \quad f(4|1) = \frac{0.2}{0.5} = 0.4.$$

(b) No, X and Y are not independent. For example

$$f(-3, 1) = 0.1 \neq (0.4)(0.5) = g(-3) \cdot h(1)$$

□

9. Given the joint probability density

$$f(x,y) = \begin{cases} \frac{2}{3}(x+2y) & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the conditional distribution of X given $Y = y$ and use it to evaluate $P(X \leq \frac{1}{2} | Y = \frac{1}{2})$.

Solution. The definition for conditional distribution of X given $Y = y$ is

$$f(x|y) = \frac{f(x,y)}{h(y)}$$

where $h(y)$ is the marginal distribution for Y. Then

$$\begin{aligned} h(y) &= \int_{-\infty}^{\infty} f(x,y)\, dx \\ &= \int_0^1 \frac{2}{3}(x+2y)\, dx \\ &= \frac{x^2}{3} + \frac{4}{3}xy \Big|_0^1 \\ &= \frac{1}{3}(1+4y) \end{aligned}$$

So for $0 < x < 1$ we have

$$f(x|y) = \frac{\frac{2}{3}(x+2y)}{\frac{1}{3}(1+4y)} = \frac{2x+4y}{1+4y}$$

and $f(x|y) = 0$ elsewhere. In particular

$$f\left(x \,\Big|\, \frac{1}{2}\right) = \frac{2x + 4(\frac{1}{2})}{1 + 4(\frac{1}{2})} = \frac{2x+2}{3}.$$

Thus

$$\begin{aligned} P\left(X \leq \frac{1}{2} \,\Big|\, Y = \frac{1}{2}\right) &= \int_0^{\frac{1}{2}} \frac{2x+2}{3}\, dx \\ &= \frac{x^2}{3} + \frac{2x}{3} \Big|_0^{\frac{1}{2}} \\ &= \frac{5}{12}. \end{aligned}$$

□

10. The joint probability density function for continuous random variables is given below. Let $f(x|y)$ be the conditional density for X given $Y = y$. Find $P(0 \leq X \leq \frac{1}{2} | Y = 1)$.

$$f(x,y) = \begin{cases} \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) & \text{for } 0 < x < 1, 0 < y < 2 \\ 0 & \text{elsewhere} \end{cases}$$

Solution. The marginal density for Y is

$$\begin{aligned}
h(y) &= \int_{-\infty}^{\infty} f(x,y)\,dx \\
&= \int_0^1 \frac{6}{7}\left(x^2 + \frac{xy}{2}\right)\,dx \\
&= \frac{6}{7}\left(\frac{x^3}{3} + \frac{x^2 y}{4}\right)\Bigg|_0^1 \\
&= \frac{6}{7}\left(\frac{1}{3} + \frac{y}{4}\right)
\end{aligned}$$

so the conditional density is

$$f(x|y) = \frac{f(x,y)}{g(y)} = \frac{\frac{6}{7}\left(x^2 + \frac{xy}{2}\right)}{\frac{6}{7}\left(\frac{1}{3} + \frac{y}{4}\right)} = \frac{12x^2 + 6xy}{4 + 3y}.$$

Thus

$$\begin{aligned}
P\left(0 \le X \le \frac{1}{2}\,\middle|\, Y = 1\right) &= \int_0^{\frac{1}{2}} f(x|1)\,dx \\
&= \int_0^{\frac{1}{2}} \frac{12x^2 + 6x}{7}\,dx \\
&= \frac{4x^3}{7} + \frac{3x^2}{7}\Bigg|_0^{\frac{1}{2}} \\
&= \frac{5}{28}.
\end{aligned}$$

□