4.2 Moments

• μ = "mu"

The rth **moment about the origin** of a random variable X, denoted μ'_r, is defined as the expected value of X^r.

In the discrete case this is,

$$\mu'_r = E(X^r) = \sum_x x^r \cdot f(x).$$

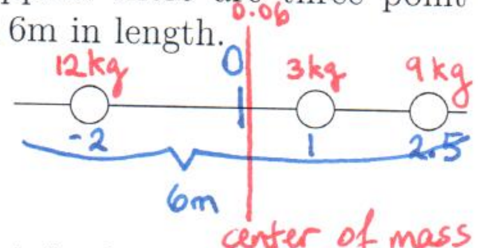
In the continuous case this is

$$\mu'_r = E(X^r) = \int_{-\infty}^{\infty} x^r \cdot f(x)\, dx$$

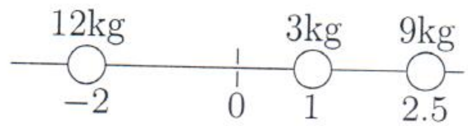
($r = 0, 1, 2, 3, \ldots$, here $f(x)$ is the probability distribution/density of X)

The Latin origin of the word is the verb "to move." In physics a moment is the product of the distance to some point from the origin (raised to the power r), times a physical quantity such as mass, force, or charge.

For example, suppose there are three point masses sitting on a massless rigid beam 6m in length.



Object 1 lies 2m left of centre and has a mass of 12kg, objects 2 and 3 lie respectively 1m and 2.5m right of centre and have respective masses 3kg and 9 kg.



Let $x_1 = -2, x_2 = 1, x_3 = 2.5$ be the positions of the objects, and let $f(x)$ be the mass distribution (i.e. $f(x_i)$ is the mass of object i).

The rth **moments of mass** of this physical system are

$$\sum_{i=1}^{3} x_i^r \cdot f(x_i)$$

$r=0$: 0^{th} moment of mass

$x_1^0 \cdot f(x_1) + x_2^0 \cdot f(x_2) + x_3^0 \cdot f(x_3)$

$= (-2)^0 \cdot 12 + (1)^0 \cdot 3 + (2.5)^0 \cdot 9$

$= 12 + 3 + 9 = 24$

total mass of the system

$r=1$: 1^{st} moment of mass

$x_1^1 \cdot f(x_1) + x_2^1 \cdot f(x_2) + x_3^1 \cdot f(x_3)$

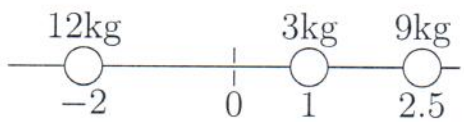
$= (-2)^1 \cdot 12 + 1^1 \cdot 3 + (2.5)^1 \cdot 9$

$= -24 + 3 + 22.5 = 1.5$

r=0 0th moment of mass: $\sum_{i=1}^{3} x_i^0 \cdot f(x_i) = (1)(12) + (1)(3) + (1)(9) = \cancel{18}.$ 24

r=1 1st moment of mass: $\sum_{i=1}^{3} x_i \cdot f(x_i) = (-2)(12) + (1)(3) + (2.5)(9) = 1.5.$

r=2 2nd moment of mass: $\sum_{i=1}^{3} x_i^2 \cdot f(x_i) = (4)(12) + (1)(3) + (6.25)(9) = 107.25.$



The 0th moment of mass is the total mass, ~~18kg~~. 24kg

The 1st moment of mass divided by the total mass is the **centre of mass**, or the balance point of the beam which is $\frac{1.5\text{kg}\cdot\text{m}}{\cancel{18}\,24\,\text{kg}} = 0.0\cancel{8}6\text{m}$ right of centre.

The 2nd moment of mass 107.25 kg·m^2, is the **moment of inertia**, which is ~~give~~ the amount of torque (in kg m^2/s^2) required to cause an angular acceleration of 1 rad/s^2, around the line $y = 0$.

In a similar way, moments of random variables describe their probability distribution.

about the origin:

$(x-0)^r$

"μ" for moment of something

"prime" to denote the moment about the origin

"0" for 0^{th} moment

The 0th **moment** about the origin **of a random variable** X **is equal to 1**, since

$$\mu_0' = \sum_x x^0 \cdot f(x) = \sum_x f(x) = 1$$

in the discrete case, and

$$\mu_0' = \int_{-\infty}^{\infty} x^0 \cdot f(x)\, dx = \int_{-\infty}^{\infty} f(x)\, dx = 1.$$

and in the continuous case.

The 1st moment about the origin is the expected value:

$$\mu_1' = \underbrace{\sum_x x \cdot f(x)}_{E(X)=},\qquad \mu_1' = \underbrace{\int_{-\infty}^{\infty} x \cdot f(x)\, dx}_{=E(X)}$$

$(x-0)^1$ $(x-0)^1$

Because of its importance, it is called the **mean of the distribution of** X (or the **mean of** X) and is denoted simply by μ.

$E(X) = \mu_1' = \mu = \text{mean}$

In every day language, "mean" and "average" (of a set of values) are used interchangeably to mean the sum of all values divided by the number of values; example, average test score.

We can view the test score as a discrete random variable X, for the experiment of selecting a test at random from the class. In this case some test scores may appear repeatedly (e.g. 5 students may have scored %80, and 3 scored %75), so the probability of drawing particular scores may not be equally likely. The expected value of X, also called the mean of X, is the sum of each score times their respective probabilities.

Assuming an equally likely chance of any <u>test</u> being drawn, the probability of drawing a particular <u>score</u> is the number of tests with that score divided by the total number of tests, and it follows that the mean and average, for equally likely outcomes, is the same.

ex: test scores: 70, 80, 80, 80, 70

X is the random variable associated to the event of choosing one test score at random

$$E(X) = 70 \cdot \frac{2}{5} + 80 \cdot \frac{3}{5} = \frac{140}{5} + \frac{240}{5} = \frac{380}{5} = 76$$

mean of the test score distribution

r^{th} moment about the origin: $\mu_r' = E(X^r) = E((X-0)^r)$

r^{th} moment about the mean: $\mu_r = E((X-\mu)^r)$

Let X be a random variable with probability distribution/density $f(x)$ having mean μ.

The **rth moment about the mean** of X, denoted μ_r, is defined as the expected value of $(X-\mu)^r$.

$$\text{discrete:} \qquad \mu_r = E((X-\mu)^r) = \sum_x (x-\mu)^r \cdot f(x),$$

$$\text{continuous:} \qquad \mu_r = E((X-\mu)^r) = \int_{-\infty}^{\infty} (x-\mu)^r \cdot f(x)\, dx$$

Note that $\mu_0 = 1$ and $\mu_1 = 0$ (provided μ exists).

discrete case: $\mu_0 = \sum_x (x-\mu)^0 \cdot f(x) = \sum_x 1 \cdot f(x) = 1$

continuous case: similar, try it

$\mu_1 = E((X-\mu)^1) = E(X-\mu) = \underbrace{E(X)}_{=\mu \text{ (p.128)}} - \mu = 0$

by the theorem on linearity of $E(X)$

Variance, σ^2, and Standard Deviation, σ:

The 2nd moment about the mean, μ_2, is called the **variance of the distribution for** X, or simply the **variance of** X, and is denoted by σ^2, or $\mathrm{var}(X)$.

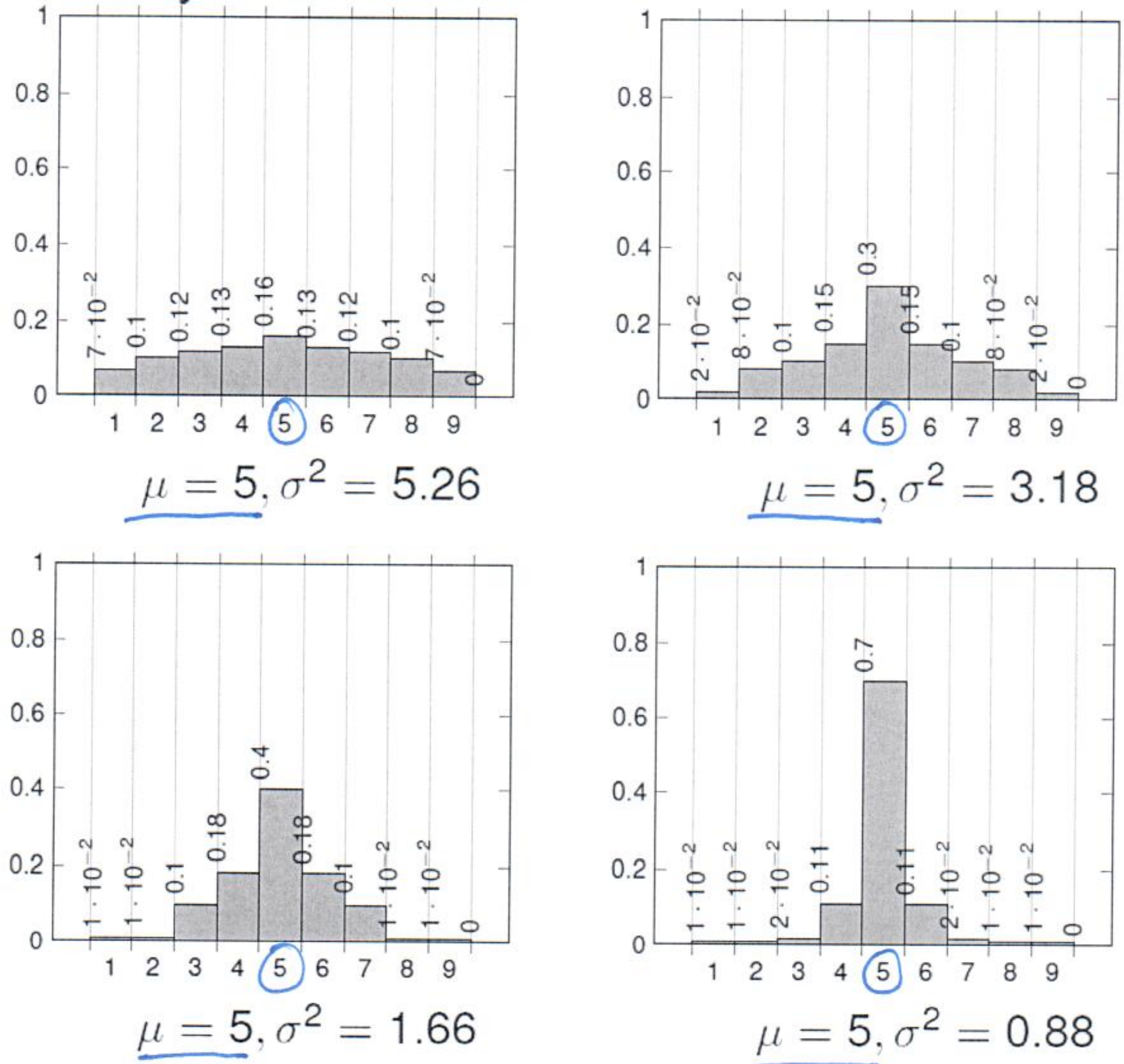
variance $= \mu_2 = E((X-\mu)^2) = \sigma^2 = \mathrm{var}(X) = 2^{nd}$ moment about the mean

The positive square root of the variance, σ, is called the **standard deviation**.

σ = standard deviation (positive square root of variance)

These values describe the dispersion of the probability distribution of X.

Probability distributions with different variance



$\mu = 5, \sigma^2 = 5.26$

$\mu = 5, \sigma^2 = 3.18$

$\mu = 5, \sigma^2 = 1.66$

$\mu = 5, \sigma^2 = 0.88$

Example 4.2.1 *Let X and Y be discrete random variables with the following distributions*

x	$P(X = x)$
1	0
2	1/4
3	1/2
4	0
5	0
6	1/4

y	$P(Y = y)$
1	1/4
2	0
3	0
4	1/2
5	1/4
6	0

Show that these distributions have the same mean and variance.