### 1.3.1 Binomial Coefficients

Powers of a binomial $x + y$, are computed using properties of real numbers (distributivity, associativitiy, commutativity). e.g.

$$(x+y)^2 = (x+y)(x+y) = (x+y)x + (x+y)y = xx + yx + xy + yy$$
$$= x^2 + 2xy + y^2.$$

Expanding $(x+y)^n$ for large $n$ is impractical. Instead we compute the coefficients of each $x^k y^{n-k}$ term in the result with counting techniques.

**Example**:

$$\begin{aligned}(x+y)^3 &= (x+y)(x+y)(x+y)\\ &= xxx + xxy + xyx + yxx + xyy + yxy + yyx + yyy\\ &= x^3 + 3x^2y^1 + 3xy^2 + y^3.\end{aligned}$$

To obtain the second line:

$\binom{3}{0} = 1$ $1x^3$

$$xxx \leftrightsquigarrow (\boxed{x} + y)(\boxed{x} + y)(\boxed{x} + y)$$

$\binom{3}{1} = 3$ $3x^2y^1$

$$xxy \leftrightsquigarrow (\boxed{x} + y)(\boxed{x} + y)(x + \boxed{y})$$
$$xyx \leftrightsquigarrow (\boxed{x} + y)(x + \boxed{y})(\boxed{x} + y)$$
$$yxx \leftrightsquigarrow (x + \boxed{y})(\boxed{x} + y)(\boxed{x} + y)$$

$\binom{3}{2} = 3$ $3x^1y^2$

$$xyy \leftrightsquigarrow (\boxed{x} + y)(x + \boxed{y})(x + \boxed{y})$$
$$yxy \leftrightsquigarrow (x + \boxed{y})(\boxed{x} + y)(x + \boxed{y})$$
$$yyx \leftrightsquigarrow (x + \boxed{y})(x + \boxed{y})(\boxed{x} + y)$$

$\binom{3}{3} = 1$ $1y^3$

$$yyy \leftrightsquigarrow (x + \boxed{y})(x + \boxed{y})(x + \boxed{y})$$

Choose $k$ factors (of the three) to provide $y$ to get a $x^{3-k}y^k$ term for $k = 0, 1, 2, 3$.

For example, there are $\binom{3}{2} = 3$ ways to obtain an $xy^2$ term by choosing $y$ from two of the factors and $x$ from the remaining one.

$$(x+y)^5 = \binom{5}{0}x^5 + \binom{5}{1}x^4y^1 + \overset{10}{\binom{5}{2}}x^3y^2 + \binom{5}{3}x^2y^3 + \binom{5}{4}xy^4 + \binom{5}{5}y^5$$
$$= 1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5$$

$0! = 1$

$\binom{n}{0} = 1$

$\binom{n}{n} = 1$

$x^0 = 1$
$(x \neq 0)$

By the Binomial Theorem, the coefficient of $x^{n-r}y^r$ in $(x+y)^n$ is $\binom{n}{r}$.
Similarly, the coef. of $x^r y^{n-r}$ in $(x+y)^n$ is $\binom{n}{n-r}$.

**Theorem 1.3.2** *(The Binomial Theorem)*
*For $n \in \mathbb{N}$*

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r.$$

Ex: $n = 4$

$$(x+y)^4 = \sum_{r=0}^{4} \binom{4}{r} x^{4-r} y^r$$

$$= \binom{4}{0}x^4y^0 + \binom{4}{1}x^3y^1 + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}x^0y^4$$

$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

- Expressions $\binom{n}{r}$ are called **binomial coefficients**.
- Choosing $r$ things from $n$ things indirectly chooses $n-r$ things to leave behind. We have the following result:

**Theorem 1.3.3** *For $n \in \mathbb{N}$ and $r = 0, 1, \ldots, n$*

$$\binom{n}{r} = \binom{n}{n-r}.$$

$n = 7$
$r = 2$

$$\binom{7}{2} = \binom{7}{7-2} = \binom{7}{5}$$

$$\binom{7}{2} = \frac{7!}{2! \cdot (7-2)!}$$

$$\binom{7}{5} = \frac{7!}{5! \cdot (7-5)!}$$

The next result is best understood through Pascal's Triangle.

**Theorem 1.3.4** *For $n \in \mathbb{N}$ and $r = 0, 1, \ldots, n-1$*

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}.$$

ex: $n=6$, $r=4$

$$\binom{6}{4} = \binom{5}{4} + \binom{5}{3}$$

$\dfrac{6!}{4!\cdot(6-4)!} = 15$ $\quad = 5 \quad = 10$

**Pascal's triangle**:

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \; + \; \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \; + \; \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \; + \; \binom{5}{4} \quad \binom{5}{5}$$

$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$

**Proof 1.3.5 (proof: $(\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1})$)** *Consider,*

$$(1+x)^n = (1+x)(1+x)^{n-1} = (1+x)^{n-1} + x(1+x)^{n-1}.$$

*Both sides are polynomials in $x$, and two polynomials are equal if and only they have the same coefficients, so we may equate coefficients of $x^r$ for any $r = 0, 1, \ldots, n$.*

$=\binom{n}{n-r}$

*On the left, the coefficient of $x^r$ is $\binom{n}{r}$ (by binomial theorem).*

*On the right, the coefficient of $x^r$ in $(1+x)^{n-1}$ is $\binom{n-1}{r}$, and in $x(1+x)^{n-1}$, the coefficient on $x^r$ is $\binom{n-1}{r-1}$.*

*Thus* $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$.

Ex: Find the coeff. of $x^4y^2$ in $(x+y)^6$.

Use the Binomial Theorem.

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$$

$n=6$

so, $r=2$

So, the coefficient of $x^4y^2$ is $\binom{6}{2}$.

$$\binom{6}{2} = \frac{6!}{2!\cdot(6-2)!} = \frac{6\cdot5\cdot\cancel{4\cdot3\cdot2\cdot1}}{(2\cdot1)\cdot\cancel{(4\cdot3\cdot2\cdot1)}} = 15$$

**Theorem 1.3.6** *(**Multinomial Coefficients**) Let $r_1 + r_2 + \cdots + r_k = n$. The coefficient of $x_1^{r_1} x_2^{r_2} \cdots x_k^{r_k}$ in $(x_1 + x_2 + \cdots + x_k)^n$ (such coefficients are called **multinomial coefficients**) is*

ex: BANANA $n=6$
3 A's
2 N's
1 B

$\binom{6}{3,2,1}$

$$\binom{n}{r_1, r_2, \ldots, r_k} = \frac{n!}{r_1! \cdot r_2! \cdot \ldots \cdot r_k!}.$$

$\dfrac{6!}{3! \cdot 2! \cdot 1!}$

**Example 1.3.7** *What is the coefficient of $x_1^3 x_3^4 x_4^2$ in the expansion of $(x_1 + x_2 + x_3 + x_4 + x_5)^9$?*

$n = 9$ $r_1 = 3$ $r_2 = 0$ $r_3 = 4$
$r_4 = 2$ $r_5 = 0$

$$\binom{9}{3,0,4,2,0} = \frac{9!}{3! \cdot 0! \cdot 4! \cdot 2! \cdot 0!}$$

$(0! = 1)$

$$= \frac{9!}{3! \cdot 4! \cdot 2!} = 1260$$

**Proof 1.3.8** *Hint: Similar to permutations with repeated elements.*

Why does this formula work (for finding multinomial coefficients)?

$x_1^3 x_3^4 x_4^2$

$(x_1 + x_2 + x_3 + x_4 + x_5) \cdot (x_1 + x_2 + x_3 + x_4 + x_5) \cdots (x_1 + x_2 + x_3 + x_4 + x_5)$

$x_1$ $x_1$ $x_1$ $x_3$ $x_3$ $x_3$ $x_3$ $x_4$ $x_4$

$x_1^3 x_3^4 x_4^2$

Q: How many words of length 9 can be formed using $3 x_1$'s, $4 x_3$'s and $2 x_4$'s?

$x_1 x_1 x_1 x_3 x_3 x_3 x_3 x_4 x_4$



$$\frac{9!}{3! \cdot 4! \cdot 2!} = 1260$$

Therefore, this number is the coeff. of $x_1^3 x_3^4 x_4^2$ in the expansion.

# Chapter 2

# Probability

## 2.1 Probability Concepts and Rules

Mathematics is used to model real world phenomena.

*Deterministic model* (ideal situation): Predicts the outcome of an experiment with certainty based on given initial conditions. e.g. velocity of a falling object

$$v = gt.$$

*Probabilistic, or stochastic, model* (randomness): When the same initial conditions can lead to a variety of outcomes, these models provide a value (probability) to the possible outcomes. e.g. rolling a die results in one of six numbers facing up.

Assign each outcome the value $\frac{1}{6}$.

### 2.1.1 Classical Probability Concept:

When there are $N$ possible (equally likely) outcomes of which $k$ are considered successful, then the *probability* of a success is the ratio $\frac{k}{N}$.

**Example 2.1.1** *Probability of...*

- *tossing tails with a balanced coin:*

tails is the only success outcome ← $\frac{1}{2}$ $k = 1$ $N = 2$

2 possible outcomes heads/tails (equally likely)

- *drawing an ace from deck of cards:*

$\frac{4}{52} = \frac{1}{13}$ $N = 52$ → 52 cards, $k = 4$ → 4 aces

each card is equally likely to be drawn

- *rolling either 3 or 5 with a (fair, six-sided) die:*

all outcomes : 1, 2, 3, 4, 5, 6 $N = 6$

successes : 3, 5 $k = 2$ $\frac{2}{6} = \frac{1}{3}$

- *rolling a total of 5 with a (fair) pair of (six-sided) dice:*

die 1: 1, 2, 3, 4, 5, 6 $N = 36$

die 2: 1, 2, 3, 4, 5, 6 $k = 4$

$\frac{4}{36} = \frac{1}{9}$

(1,1), (1,2), (1,3), (1,4), (1,5), (1,6)
(2,1), (2,2), (2,3), (2,4), (2,5), (2,6)
(3,1), (3,2) ...
(4,1)

These values describe the frequency of the successful outcome; the proportion of time the event occurs in the **long run**.

# LONG RUN
# LONG RUUUUUUUNN

### 2.1.2 Sample Spaces

The set of all possible outcomes of an experiment is called the **sample space**.

**Example**:

| Experiment | Sample space |
|---|---|
| single coin toss | heads — tails<br>$\{H, T\}$ |
| roll of two dice | $\{(d_1, d_2) \mid d_1, d_2 \in \{1, 2, 3, 4, 5, 6\}\}$ — the sample space has 36 equally likely outcomes that are given as ordered pairs. |
| sum of two dice | (not equally likely)<br>$\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ |
| drawing a card from a standard deck of cards | all $(n, s)$ with $n \in \{2, \ldots, 10, J, Q, K, A\}$ and $s \in \{\spadesuit, \heartsuit, \diamondsuit, \clubsuit\}$<br>$13 \times 4 = 52$ outcomes |
| three coin tosses | $\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$<br>8 equally likely outcomes (assuming the coin is balanced) |

All previous examples were *finite* sample spaces. The following experiments have infinite sample spaces.

**Example 2.1.2** • *Tossing a coin until heads is reached:*

$$\{H, TH, TTH, TTTH, TTTH, \ldots\}$$

• *Playtime for two AA alkaline batteries in a Wii remote:*

$$\{t \ hours \mid t \in [0, 50]\}$$

the outcome is any real number between 0 and 50

**Continuous and Discrete Sample Spaces**:

There is an important distinction between the sample spaces in the previous example; the outcomes of the first example (coin toss) may be listed, whereas the outcomes in the other ~~two~~ belong to a continuum of values.

A **discrete sample space** has only finitely many, or a countably infinite number of elements.

A **continuous sample space** is an interval in $\mathbb{R}$, or a product of intervals lying in $\mathbb{R}^n$.

The important distinction is how probabilities are assigned.

**Events**: While individual elements of a sample space are called **outcomes**, subsets of a sample space are called **events**. If the outcome of an experiment lies in an event, we say that event has occurred.

**Example 2.1.3** *Experiment: Tossing a coin three times.*

*Sample space:* $\{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

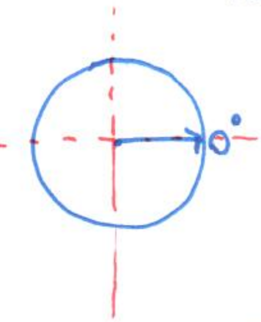
*Event A: Getting at least two heads:* $\{HHH, HHT, HTH, THH\}$

*Event B: Getting exactly two tails* : $\{HTT, THT, TTH\}$

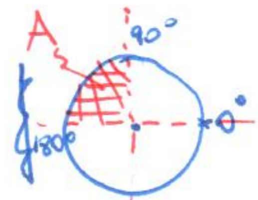
*Event C: Getting two consecutive heads*

{HHH, HHT, THH}

**Example 2.1.4** *Experiment: Spinning a probability spinner.* 

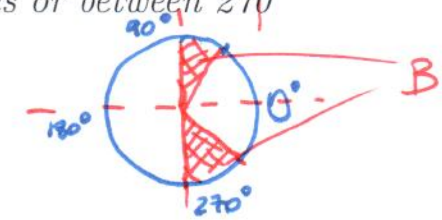


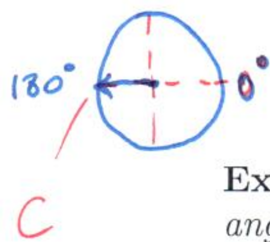

*Sample space:* $\{\theta \text{ degrees} | \theta \in [0, 360)\}$




*Event A: Landing between 90 and 180 degrees,* $[90, 180]$

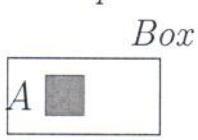
*Event B: Landing either between 45 and 90 degrees or between 270 and 315 degrees,* $[45, 90] \cup [270, 315]$



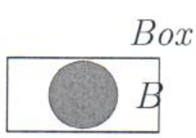



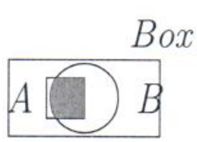

*Event C: Landing precisely on 180 degrees.*

**Example 2.1.5** *Experiment: Dropping a pencil head first into a rectangular box.*
*Sample space: All points on the bottom of the box.*

*Event A:* 

*(pencil lands in shaded region)*

*Event B:* 


*Event* $A \cap B$*:* 


intersection

### 2.1.3 Union, Intersection, Complement

Let $A$ and $B$ be events in sample space $S$; i.e. $A$ and $B$ are subsets of a set $S$.

The **union** of $A$ and $B$ is the set of outcomes that are in either $A$ or $B$ or both.

union    element of    such that

$$A \cup B = \{x \in S | x \in A \text{ or } x \in B\}.$$

The **intersection** of $A$ and $B$ is the set of outcomes that are in both $A$ and $B$.

intersection

$$A \cap B = \{x \in S | x \in A \text{ and } x \in B\}.$$

The **complement** of $A$ in $S$ is the set of outcomes in $S$ that are not in $A$.

complement

$$A' = \{x \in S | x \notin A\} = S \setminus A.$$

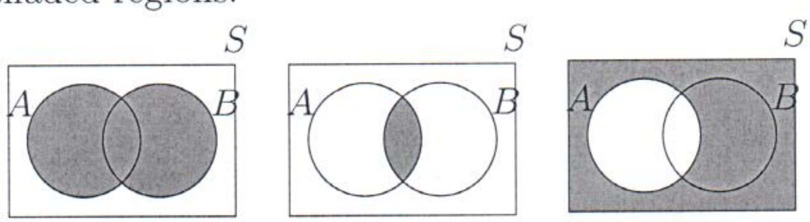
read: "A complement" or "A prime"

"set difference"

Other books/texts use $\overline{A}$ for "A complement"

**Venn Diagrams:**

A **Venn diagram** is a visual depiction of subsets of some "universal" set. Subsets are represented (usually by disks) lying within a rectangle representing the universal set. Sets of interest are represented by shaded regions.




$A \cup B$    $A \cap B$    $A'$

union    intersection    A complement

In the pencil dropping example, Venn diagrams gave a literal representation of the events in that experiment, but these representations can be used in more general situations to help visualize relationships between different subsets of a sample space.

**Mutually Exclusive Events**

A set which has no elements is called the **empty set**, denoted $\emptyset$.

$\phi$ empty set

$\{\ \}$ empty set

**Example 2.1.6** *Experiment: Rolling two dice.*

*Event A: Rolling at least one six.*

$$A = \{(d_1, 6), (6, d_2) | d_1, d_2 \in \{1, 2, 3, 4, 5, 6\}\}$$

$A = \{(1,6), (2,6), (3,6), (4,6), (5,6), (6,6), (6,5), (6,4), (6,3), (6,2), (6,1)\}$

11 outcomes in A

*Event B: Sum of dice equals 4.*

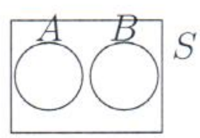
$B = \{(d_1, 4-d_1) | d_1 \in \{1,2,3\}\}$

$B = \{(1,3), (2,2), (3,1)\}$
3 outcomes in B

*Event C: Rolling at least one six and having a sum of 4.*

$$C = A \cap B = \emptyset.$$

$C = A \cap B = \phi$




$A \cap B = \emptyset$

Sets with empty intersection are called disjoint, and the events in this case are called **mutually exclusive**.

In this example, A and B are mutually exclusive events.

### 2.1.4 Algebra of Sets

Let $A, B$ and $C$ be subsets of a universal set $S$.

- Idempotent laws:

  $$A \cup A = A, \qquad A \cap A = A$$

- Associative laws:

  $$(A \cup B) \cup C = A \cup (B \cup C), \qquad (A \cap B) \cap C = A \cap (B \cap C)$$

- Commutative laws:

  $$A \cup B = B \cup A, \qquad A \cap B = B \cap A$$

- Distributive laws:

Venn diagram representation is on the next page

  $$A \cup (B \cap C) = (A \cup B) \cap (A \cup C), \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

- Identity laws:

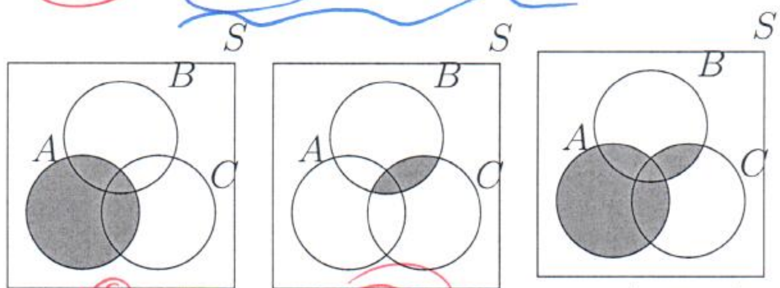
  $$A \cup \emptyset = A, \quad A \cup S = S, \quad A \cap S = A, \quad A \cap \emptyset = \emptyset$$

- Complement laws:
  $(A')' = A, \quad A \cup A' = S, \quad A \cap A' = \emptyset, \quad S' = \emptyset, \quad \emptyset' = S$

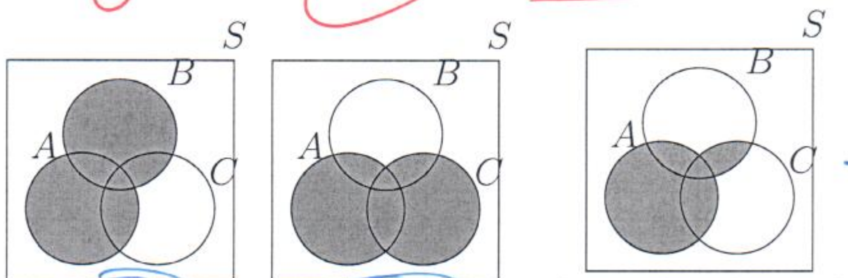
- DeMorgan's Laws:
  $(A \cup B)' = A' \cap B', \quad (A \cap B)' = A' \cup B'$

**Set Operations:**

**Example 2.1.7** *Use Venn diagrams to verify the distributive law* $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.




$A$ $\qquad$ $B \cap C$ $\qquad$ $A \cup (B \cap C)$




$A \cup B$ $\qquad$ $A \cup C$ $\qquad$ $(A \cup B) \cap (A \cup C)$

the same

So,
$A \cup (B \cap C)$
$= (A \cup B) \cap (A \cup C)$

**Proof:** (of $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$)
[How does one even "prove" such a statement mathematically?]

We need to show:

1) $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

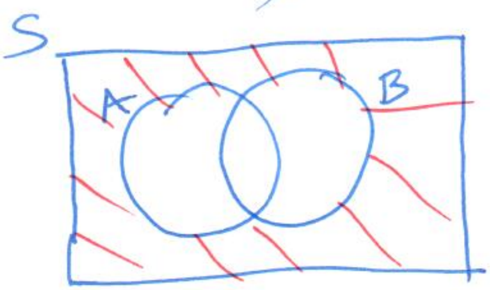
2) $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$

Try this.

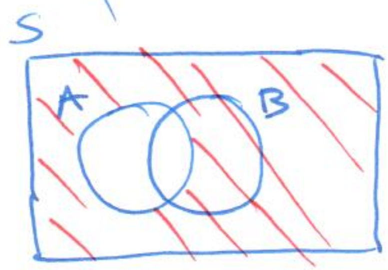
**Exercise**: Use Venn diagrams to verify DeMorgan's Laws.

$$(A \cup B)' = A' \cap B', \quad (A \cap B)' = A' \cup B'$$

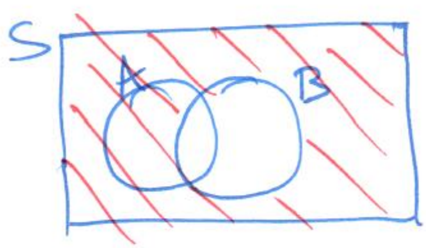
exercise



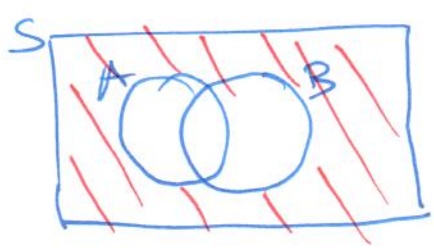

$(A \cup B)'$




$A'$




$B'$




$A' \cap B'$

Observe that $(A \cup B)' = A' \cap B'$