So, $\sigma^2 = \frac{1}{k}\sum_{i=1}^{k} x_i^2 - \left(\frac{\sum_{i=1}^{k} x_i}{k}\right)^2$

$$= \frac{\sum_{i=1}^{k} x_i^2}{k} - \left(\frac{\sum_{i=1}^{k} x_i}{k}\right)^2$$

variance for uniform distribution

5.2 The Bernoulli Distribution

Consider an experiment with two possible outcomes, either success or failure. (For example, a single coin toss.)

Assign random variable X the value 1 for success and 0 for failure.

If the probability of success is θ, then the probability of a failure is $1-\theta$.

In this case X is called a **Bernoulli random variable** and has **Bernoulli distribution** given by

$$f(x;\theta) = \theta^x(1-\theta)^{1-x} \quad \text{for } x = 0, 1.$$

$x=0$: $f(0;\theta) = \theta^0(1-\theta)^{1-0} = 1\cdot(1-\theta)^1 = 1-\theta$

$x=1$: $f(1;\theta) = \theta^1(1-\theta)^{1-1} = \theta\cdot(1-\theta)^0 = \theta$

Exercise: Show that the Bernoulli distribution has

$$\mu = \theta, \quad \sigma^2 = \theta(1-\theta).$$

$\mu = E(X) = 0\cdot f(0;\theta) + 1\cdot f(1;\theta) = 1\cdot f(1;\theta) = 1\cdot\theta = \theta$

$\sigma^2 = \mu_2' - \mu^2 = E(X^2) - \mu^2 = 0^2\cdot f(0;\theta) + 1^2\cdot f(1;\theta) - \mu^2$

$= 0 + f(1;\theta) - \mu^2 = \theta - \mu^2$

$= \theta - \theta^2 = \theta(1-\theta)$

shortcut formula

The mean of the Bernoulli distr. is $\mu = \theta$;
and the variance is $\theta(1-\theta)$.

5.3 Binomial Distribution

Now consider an experiment with repeated trials, in which the outcome of each trial is either a success or failure.

Random variable X will denote the number of successes, the probability of success is known to be θ, and n is the given number of trials in the experiment.

Then X has **binomial distribution** which is given by

\# successes   \# trials   ~~\# success~~ prob of success in a single trial

$\binom{n}{x}$ = number of ways of obtaining x successes in n trials

$$\underbrace{b(x;n,\theta)}_{\text{binomial distribution}} = \binom{n}{x}\theta^x(1-\theta)^{n-x} \quad \text{for } x = 0, 1, \ldots, n.$$

"n choose x"

prob. of success for each trial

prob. for failure for each trial

Random variable X is called a **binomial random variable** if and only if it has this distribution.

The Bernoulli distribution is the special case of the binomial distribution when $n = 1$; a single trial experiment.

for $n=1$: $b(x;n,\theta) = b(x;1,\theta) = \binom{1}{x}\theta^x \cdot (1-\theta)^{1-x}$

$x = 0$ or 1

So, for $x=0$: $b(0;1,\theta) = \binom{1}{0}\cdot\theta^0\cdot(1-\theta)^{1-0}$

$= 1\cdot 1\cdot(1-\theta) = 1-\theta$

for $x=1$: $b(1;1,\theta) = \binom{1}{1}\cdot\theta^1\cdot(1-\theta)^{1-1}$

$= 1\cdot\theta\cdot 1 = \theta$

$$b(x;n,\theta) = \binom{n}{x}\theta^x(1-\theta)^{n-x} \quad \text{for } x = 0, 1, \ldots, n$$

Example 5.3.1 *Some examples of binomial random variables:*

- *Number of heads in 35 flips of a coin with 0.63 probability of heads and 0.37 probability of tails.*

  $1-0.63=0.37$

  $P(17 \text{ heads}) = b(\underset{x}{17}; \underset{n}{35}, \underset{\theta}{0.63}) = \binom{35}{17}\cdot(0.63)^{17}\cdot(0.37)^{35-17}$

  $= \binom{35}{17}\cdot(0.63)^{17}\cdot(0.37)^{18}$

  Use a calculator to find a decimal number.

- *There is a %6.6 chance that a person has O- blood type. In a selection of 20 people what is the probability that 5 of them will have O- blood.* $=0.066$

  $P(5 \text{ people have O-}) = b(\underset{x}{5}; \underset{n}{20}, \underset{\theta}{0.066}) = \binom{20}{5}\cdot(0.066)^5\cdot(1-0.066)^{15}$

  $= \binom{20}{5}\cdot(0.066)^5\cdot(0.934)^{15}$

  We're interested in ~~the~~ some probability that relates to O- blood type; so, consider O- blood type as "success".

  Use a calculator to find the corresponding decimal number.

Values for $b(x; n, \theta)$ can be found in tables (see the textbook for example). These tables are usually computed for $n = 1, 2, \ldots, 20$ and $\theta = 0.5, 0.10, 0.15, \ldots, 0.50$.

To evaluate $b(x; n, \theta)$ from these tables for when $\theta > 0.50$ we can use the following property:

Theorem 5.3.2

probability of x successes in n trials where success prob. is θ

probability of n-x successes in n trials where prob. of success in 1-θ

$$\underbrace{b(x; n, \theta)}_{} = \underbrace{b(n - x; n, 1 - \theta)}_{}$$

For example,

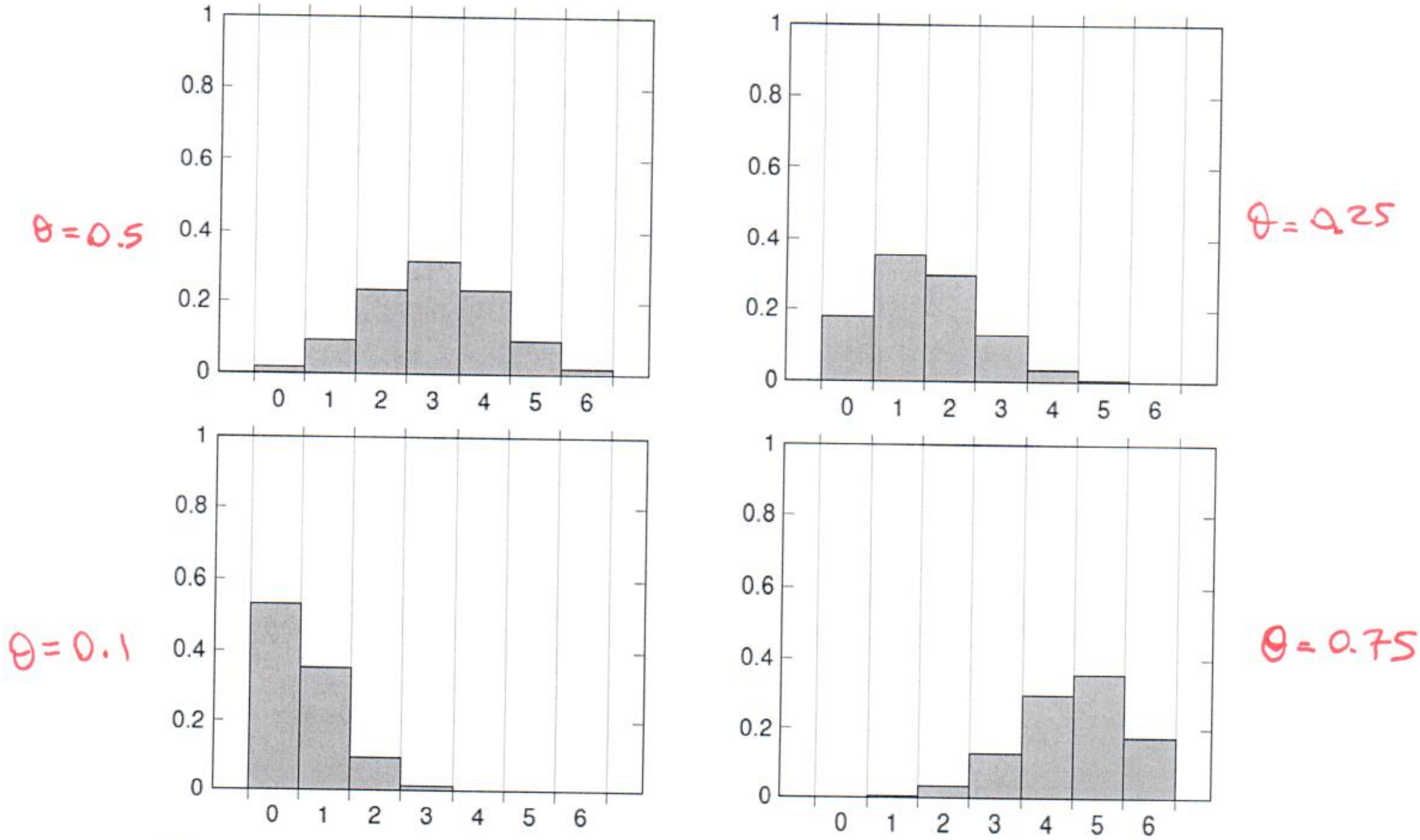
$$b(\overset{x}{7}; \overset{n}{11}, \overset{\theta}{0.75}) = b(\overset{n-x}{4}; \overset{n}{11}, \overset{1-\theta}{0.25}) \approx 0.1721$$

Exercise: Show that the theorem holds.

$$b(x; n, \theta) = \binom{n}{x} \cdot \theta^x \cdot (1-\theta)^{n-x}$$

$$b(n-x; n, 1-\theta) = \binom{n}{n-x} \cdot (1-\theta)^{n-x} \cdot (1-(1-\theta))^{n-(n-x)}$$

$$= \binom{n}{n-x} \cdot (1-\theta)^{n-x} \cdot \theta^x$$



For each graph of $b(x; n, \theta)$ we have $n = 6$. Determine which of these has $\theta = 0.1, 0.25, 0.5$, and 0.75