Chapter 1

COMBINATORIAL METHODS

1.1 Counting

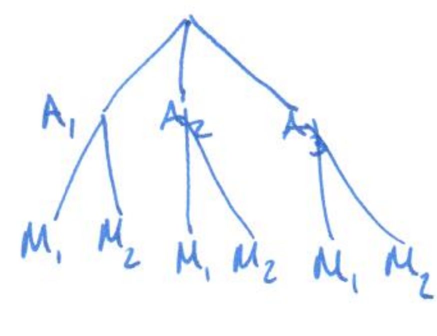
In this chapter we learn counting techniques that will be useful ~~to~~ when calculating discrete probabilities.

Theorem 1.1.1 (Counting Rule for Compound Events) *If a process/operation/choice consists of two steps where the first can be done in n_1 ways and the second can be done in n_2 ways, then the entire process can be done in $n_1 \cdot n_2$ ways.*

Example 1.1.2 *Q: How many different meal options can be made from a choice of ~~1~~3 appetizers and 2~~5~~ main dishes?*

$n_1 = 3$ appetizers
$n_2 = 2$ main dishes

$n_1 \cdot n_2 = 3 \cdot 2 = 6$ different meal options



Definition: If A and B are sets, we may form a new set

elements in $A \times B$ are ordered pairs (x,y)

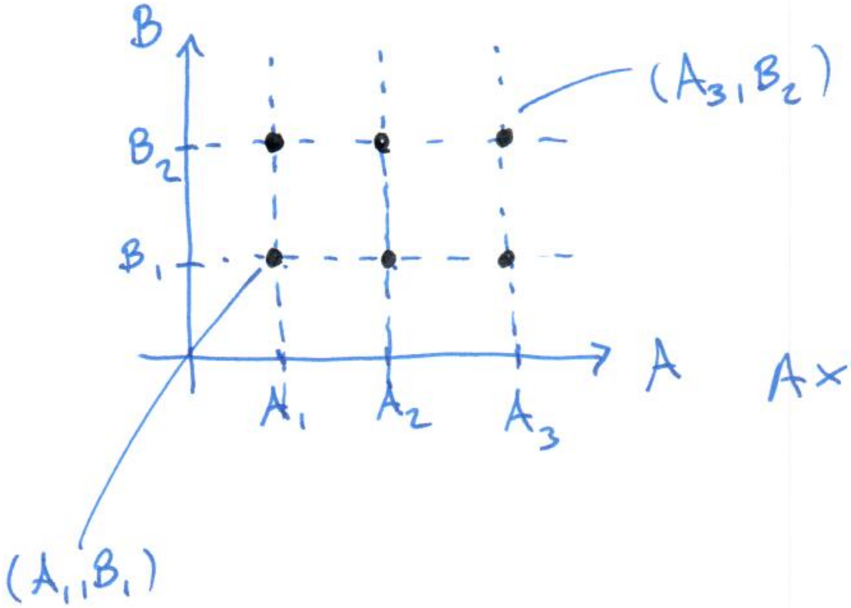
$$A \times B = \{(x,y) | x \in A, y \in B\}$$

of all (ordered) pairs of elements from A and B. $A \times B$ is called the **Cartesian product** of A and B.

$x \in A$
$y \in B$

If A and B are finite and have respectively m and n elements, then $A \times B$ has mn elements.

Example: ~~A~~ If A has 3 elements and B has 2 elements, then $A \times B$ has $3 \cdot 2 = 6$ elements.



$A = \{A_1, A_2, A_3\}$

$B = \{B_1, B_2\}$

$A \times B = \{(A_1, B_1), (A_1, B_2), (A_2, B_1), (A_2, B_2), (A_3, B_1), (A_3, B_2)\}$

Theorem 1.1.3 *If a process consists of k steps where each can be done in n_i ways (for $i = 1, 2, \ldots, k$) then the entire process can be done in $n_1 \cdot n_2 \cdot \cdots \cdot n_k$ ways.*

Definition: Generalize the Cartesian product to k sets, $A_1, \ldots, A_k$, by

an ordered k-tuple

$$A_1 \times \cdots \times A_k = \{(x_1, \ldots, x_k) | x_i \in A_i \text{ for } i \in \{1, \ldots k\}\}.$$

Thus if $A_1, \ldots, A_k$ are finite and have $n_1, \ldots, n_k$ elements respectively, then $A_1 \times \cdots \times A_k$ has $\underline{n_1 \cdots n_k}$ elements.

Example 1.1.4 *A university room number is an ordered triple $(f, h, n) \in F \times H \times N$ where*

$$F = \{1, 2\}, \quad H = \{A, B, C\}, \quad N = \{8, 9\}.$$

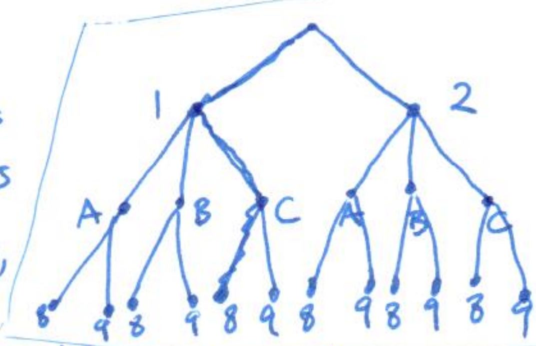
Q: How many room numbers are there?

The following *tree diagram* demonstrates the general counting rule for this example.

The set F has 2 elements
The set H has 3 elements
The set N has 2 elements

To find the number of room numbers, we find the number of elements in $F \times H \times N$. This number is $2 \cdot 3 \cdot 2 = 12$. $\boxed{108}$

There are 12 room numbers.



1.2 Permutations and Combinations

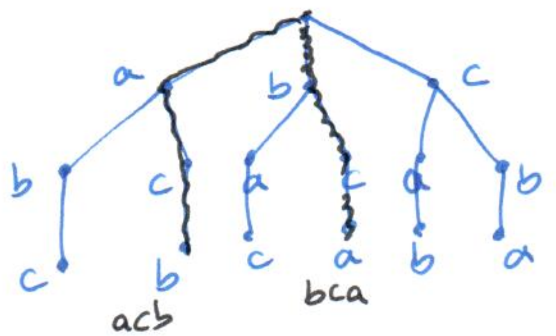
An ordered arrangement of all elements of a set S, in which no element occurs more than once, is called a **permutation** of S.

Example: Let $S = \{a, b, c\}$.

Q: How many permutations of S are there?

abc
acb
bac
bca
cab
cba

there are 6 permutations of S



$$\underset{3 \text{ options}}{\underline{3}} \cdot \underset{2 \text{ options}}{\underline{2}} \cdot \underset{1 \text{ options}}{\underline{1}} = 6 \text{ permut. of } S$$

The example hints at the following theorem:

Theorem 1.2.1 *The number of permutations of n distinct objects is*

$$n! = n \cdot (n-1) \cdot \ldots \cdot 3 \cdot 2 \cdot 1.$$

"n factorial"

ex: $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

Example: Let $\mathcal{S} = \{R, O, Y, G, B, I, V\}$ be the seven rainbow colours.

7 elements

1. How many permutations of $\mathcal{S}$ are there?

$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$

2. How many different flags, composed of three vertical bars with distinct colours can be made from $\mathcal{S}$?

Theorem 1.2.2 *The number of permutations of n distinct objects taken r at a time is*

$$_nP_r = \frac{n!}{(n-r)!}$$

$7 \cdot 6 \cdot 5$

1st bar, 2nd bar, 3rd bar

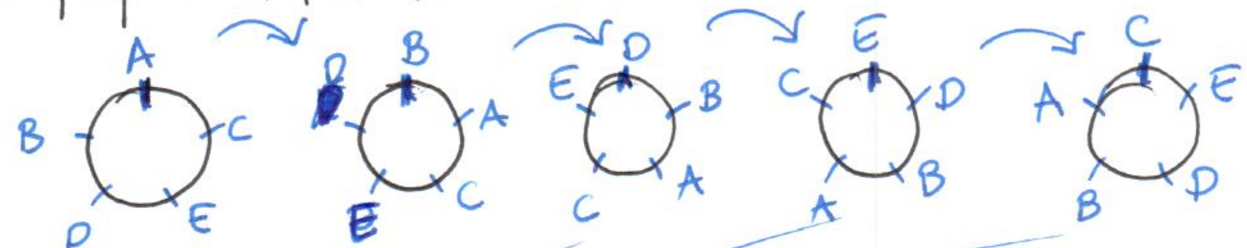
$= 7 \cdot 6 \cdot 5 = 210$ flags with the given property

$$_7P_3 = \frac{7!}{(7-3)!} = \frac{7 \cdot 6 \cdot 5 \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}$$

$$= \frac{7 \cdot 6 \cdot 5 \cdot \cancel{4!}}{\cancel{4!}}$$

Example: (Circular Permutations) How many different ~~neck-laces can be made with 2 coloured beads (one of each colour)?~~ ways are there to arrange 5 people in a circle?

5 people : A, B, C, D, E



We consider these 5 circular arrangements to be the same.

ACEDB BACED DBACE EDBAC CEDBA

In a non-circular setting (linear setting), these are 5 distinct arrangements. Each distinct circular arrangement appears in 5 copies within the set of $5! = 120$ linear arrangements. So, there are

$$\frac{5!}{5} = \frac{\cancel{5}\cdot 4\cdot 3\cdot 2\cdot 1}{\cancel{5}} = 4\cdot 3\cdot 2\cdot 1 = 4! = 24$$ distinct circular arrangements.

(24 distinct circular permut.)

Theorem 1.2.3 ***(Circular permutations)*** *The number of permutations of n distinct objects arranged in a circle is $(n-1)!$.*

$$\frac{n!}{n} = \frac{\cancel{n}\cdot(n-1)\cdot\ldots\cdot 2\cdot 1}{\cancel{n}} = (n-1)\cdot(n-2)\cdot\ldots\cdot 2\cdot 1$$
$$= (n-1)!$$

Example: Permutations with Repeated Elements

Ex: How many different permutations of the word "coolio" are there?

c, o, o, l, i, o

Objects are c, o, o, l, i, o
Note that not all objects are distinct.

(the letter o appears in 3 copies)

- Index each "o" to get the set {c,o_1,o_2,o_3, l, i}. Now we can distinguish the 3 o's from one another.
- There are 6! permutations of these distinct objects.

Considering these 6 objects as distinct objects, we obtain $6! = 720$ permutations

- In each permutation, the 3! arrangements of o_1,o_2,o_3 give the same word. e.g. the following are the same:

Consider the word $\underline{\text{ocolio}}$

$o_3 c o_2 l i o_1$ $o_3 c o_1 l i o_2$
$o_1 c o_2 l i o_3$ $o_1 c o_3 l i o_2$
$o_2 c o_1 l i o_3$ $o_2 c o_3 l i o_1$

all 6 form the same word ocolio

$6 = 3!$ The 3 copies of the letter o can be arranged among themselves in 3! ways.

- Therefore there are $\frac{6!}{3!} = 120$ different permutations of the word "coolio".

$$\frac{6!}{3!} = \frac{6\cdot5\cdot4\cdot\cancel{3\cdot2\cdot1}}{\cancel{3\cdot2}\cdot1} = 6\cdot5\cdot4 = 120$$ different permutations

So there are 120 different words one can form using the letters coolio.

Ex: How many different words can you form using the letters in BANANA.

6 letters in total

3 A's

2 N's

1 B.

$$\frac{6!}{\underset{A's}{3!}\cdot\underset{N's}{2!}\cdot\underset{B}{1!}} = \frac{\cancel{6}\cdot 5\cdot 4\cdot 3\cdot \cancel{2}\cdot \cancel{1}}{\cancel{(3\cdot 2\cdot 1)\cdot(2\cdot 1)\cdot 1}} = 60$$

Theorem 1.2.4 ***(Permutations with Repeated Elements)*** *The number of permutations of n objects of which n_1 are of one kind, n_2 are of a second kind, $\ldots$, n_k are of a kth kind and $n = n_1 + n_2 + \cdots + n_k$ is*

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1! \cdot n_2! \cdot \cdots \cdot n_k!}.$$

Example: How many different numbers can be formed with the single digits $1, 2, 2, 3, 3, 3, 3$?