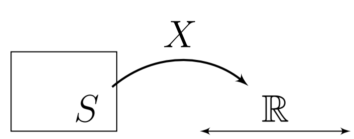
Chapter 3

Probability Distributions and Probability Densities

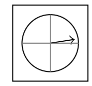
3.1 Random Variables

Let S be a sample space with a probability measure. A **random variable** is a function $X : S \to \mathbb{R}$, which maps the outcomes in the sample space to real numbers. The output of a random variable is something we can measure.



Random variables are defined when we want to focus on a particular property of the outcomes of an experiment. More than one random variable can be defined for a given sample space.

Usually capital letters like "X" are used to denote random variables; and their lower case letter like "x" are used for particular values that X can take.



Example 3.1.1 *Earlier we mentioned the experiment of spinning a probability spinner, and described the sample space as* $\{\theta \text{ degrees} | \theta \in [0, 360)\}$.

However, the actual sample space could include more information, such as multiple rotations, angular velocity at time t*, elapsed time, the color it landed on, etc.*

A random variable focuses on ***one*** *property of the outcome that can be assigned a real number.*

Some examples of random variables:

- X_1*: resting position (degrees), outputs values in* $[0, 360)$.
- X_2*: resting position (radians), outputs values in* $[0, 2\pi)$.
- X_3*: number of full rotations, can take values* $0, 1, 2, 3, \ldots$.

Example 3.1.2 *Two socks are selected at random and removed in succession from a drawer containing five brown socks and three green socks.*

List the elements of the sample space, the corresponding probabilities, and the corresponding values of the random variable X, where X is the number of brown socks selected.

Element of sample space	*Probability*	*value of X*
BB	$\frac{20}{56}$	*2*
BG		
GB		
GG		

We write: $P(X=2)=\frac{20}{56}, \quad P(X=1)=\frac{30}{56}, \quad P(X=0)=\frac{6}{56}.$

Example 3.1.3 *Three balls are randomly chosen (without replacement) from a bag of 20 balls numbered 1-20. We bet that at least one of the numbers drawn is equal to or greater than 17. What is the probability of winning the bet?*

Outcomes in the sample space are subsets of three numbered balls, and they are all equally likely to occur.

Let random variable X denote the largest number of the three selected. Thus X takes values $3, 4, \ldots, 20$, and we want $P(X \geq 17)$.

Since total probability equals 1, $P(X \geq 17) = 1 - P(X \leq 16)$

Let us calculate $P(X \leq 16)$.

How many 3-element subsets of balls 1-16 are there?

How many 3-element subsets of balls 1-20 are there?

Recall that the set of all possible output values of a function is called its **range**.

If the range of a random variable X is a finite or countably infinite set, then we say that X is a **discrete random variable**.

In contrast, a **continuous random variables** is one whose range is a continuum of values, like an interval or a union of intervals in $\mathbb{R}$.

We will deal with this type later. The important difference to notice is in how the probabilities are assigned.

3.2 Probability Distributions

Example 3.2.1 *Experiment: Rolling two dice*

Let random variable X denote the sum of a roll. The range of X is $\{2, 3, \ldots, 12\}$.

Knowing that each outcome in the sample space has probability $\frac{1}{36}$, we can easily find the probability that X takes on any value in its range. e.g. $P(X = 7) = \frac{6}{36}$, $P(X = 11) = \frac{2}{36}$.

This information can be summarized in a table.

x	$P(X = x)$
2	1/36
3	2/36
4	3/36
5	4/36
6	5/36
7	6/36
8	5/36
9	4/36
10	3/36
11	2/36
12	1/36

We would like a rule $f(x)$ which gives $P(X = x)$ for each value x in the range of random variable X.

In this case the probabilities are given by the function

$$f(x) = \ldots.$$

Such a function is called a **probability distribution of X**.

If X is a discrete random variable, the function f given by

$$f(x) = P(X = x)$$

for each x in the range of X, is called the **probability distribution of** X. (Also called the **probability mass function** of X.)

Theorem 3.2.2 *A function f is allowable as a probability distribution for X if and only if its values, $f(x)$, satisfy*

1. $f(x) \geq 0$ *for any x,*
2. $\sum_{x} f(x) = 1$, *(sum taken over all x in the range of X)*

Example 3.2.3 *Let X be the random variable that counts the number of heads obtained in tossing a balanced coin 4 times.*

(a) What is the range of X?

(b) What is $P(X = x)$ for each x in the range of X?

(c) Find formula for the probability distribution of X.

Example 3.2.4 *Return to the dice rolling experiment.*

Let Y be the maximum that either die shows in a single roll: $Y(a,b) = \max(a,b)$.

For example, $Y(3,5) = 5$.

(a) What is the range of Y?

(b) What is $P(Y = y)$ for each y in the range of Y?

(c) Find a formula for the probability distribution of Y.

Example 3.2.5 *Check whether the function given by*

$$f(x) = \frac{x+2}{25},$$

for $x = 1, 2, 3, 4, 5$ *can serve as the probability distribution of a discrete random variable.*

Probability distributions for a random variable, say X, may be represented graphically by means of a **probability histogram**.

Each rectangle corresponds to a value for X, its height is $P(X = x)$, and its width is 1, so that the area of each rectangle equals $P(X = x)$. *The total area of the histogram is 1.*

The probability histogram below is for the number of heads in 4 coin flips.