MATH 1550
Winter 2025
Test 3
2025-03-24
Time Limit: 50 Minutes

Name (Print): ______________________

ID number: ______________________

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1. (3 points) Suppose that the continuous random variable X has moment generating function given by
$$M_X(t) = (3 - 2e^t)^{-1}.$$
Find the mean, variance and standard deviation of X.

(Show your work/formula and give a rounded answer to 4 decimal places if needed.)

Solution:

The mean of X is:

$$\begin{aligned}
\mu &= \frac{d}{dt} M_X(t)\Big|_{t=0} \\
&= \frac{d}{dt}(3 - 2e^t)^{-1}\Big|_{t=0} \\
&= \left((3 - 2e^t)^{-2} \cdot 2e^t \right)\big|_{t=0} \\
&= (3 - 2)^{-2} \cdot 2 = 2.
\end{aligned}$$

The second moment about the origin is:

$$\begin{aligned}
E(X^2) &= \frac{d^2}{dt^2} M_X(t)\Big|_{t=0} \\
&= \frac{d}{dt}\left((3 - 2e^t)^{-2} \cdot 2e^t\right)\Big|_{t=0} \\
&= \frac{-2e^t(2e^t + 3)}{(2e^t - 3)^3}\Big|_{t=0} \\
&= 10.
\end{aligned}$$

The variance of X is:
$$\sigma^2 = E(X^2) - \mu^2 = 10 - 2^2 = 6.$$

Finally, the standard deviation of X is:

$$\sigma = \sqrt{6} \approx 2.449.$$

2. (1 point) Let X and Y be discrete random variables with joint probability distribution given by the following table:

$$\begin{array}{cc|ccc|} & & & x & \\ & & -5 & 1 & 4 \\ \hline & 2 & 0.2 & 0.2 & 0.3 \\ y & & & & \\ & 3 & 0.1 & 0.2 & 0 \\ \hline \end{array}$$

Here, the marginal distribution $g(x)$ for X is given by

$$\begin{aligned} g(-5) &= 0.3, \\ g(1) &= 0.4, \\ g(4) &= 0.3; \end{aligned}$$

and the marginal distribution $h(y)$ for Y is given by

$$\begin{aligned} h(2) &= 0.7, \\ h(3) &= 0.3. \end{aligned}$$

Determine if X and Y are independent. Justify your answer.

Solution: No, X and Y are not independent. For example

$$f(-5,2) = 0.2 \neq (0.3)(0.7) = g(-5) \cdot h(2)$$

3. (2 points) A fair coin is tossed until either the total number of tails is two or the total number of heads is two. What is the expected number of tosses?

Solution: Let random variable X represent the total number of tosses. The possible outcomes of this experiment are:

$$HH, HTH, HTT, TT, THT, THH$$

and therefore the range of X is $\{2, 3\}$. The probability distribution of X is

x	2	3
$P(X = x)$	$\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$	$\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$

The expected number of tosses is

$$E(X) = (2)\frac{1}{2} + (3)\frac{1}{2} = 2.5.$$

4. (2 points) If X and Y have the following joint probability density, find $E(X+2Y)$.

$$f(x,y) = \begin{cases} 4xy & \text{for } 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Solution:

$$E(X+2Y) = \int_0^1 \int_0^1 (x+2y) \cdot 4xy \, dx \, dy = \int_0^1 \int_0^1 4x^2y + 8xy^2 \, dx \, dy$$

$$= \int_0^1 \left(\frac{4x^3y}{3} + 4x^2y^2\right)\Big|_0^1 \, dy = \int_0^1 \frac{4y}{3} + 4y^2 \, dy = \left(\frac{2y^2}{3} + \frac{4y^3}{3}\right)\Big|_0^1$$

$$= \left(\frac{2}{3} + \frac{4}{3}\right) - \big(0+0\big) = 2$$

5. (2 points) Let X and Y be continuous random variables with joint probability density

$$f(x,y) = \begin{cases} \frac{9}{14}x^2 + \frac{3}{14}y^2 & \text{for } 0 \le x \le 1, 0 \le y \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

The mean of X is given as $\mu_X = E(X) = \frac{17}{28}$, and the first and first product moment about the origin is given as $E(XY) = \frac{3}{4}$. Find the covariance for X and Y.

Solution:

$$\begin{aligned} \mu_Y = E(Y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y \cdot f(x,y) \, dx \, dy \\ &= \int_0^2 \int_0^1 \frac{9}{14}x^2y + \frac{3}{14}y^3 \, dx \, dy \\ &= \int_0^2 \left(\frac{3}{14}x^3y + \frac{3}{14}xy^3\right)\Big|_0^1 \, dy \\ &= \int_0^2 \left(\frac{3}{14}y + \frac{3}{14}y^3\right) \, dy \\ &= \left(\frac{3}{28}y^2 + \frac{3}{56}y^4\right)\Big|_0^2 = \frac{3}{7} + \frac{6}{7} = \frac{9}{7}. \end{aligned}$$

Therefore

$$\text{cov}(X,Y) = E(XY) - \mu_X\mu_Y = \frac{3}{4} - \left(\frac{17}{28}\right) \cdot \left(\frac{9}{7}\right) = \frac{-6}{196} = \frac{-3}{98} \approx -0.0306.$$