Example 3.2.4 *Return to the dice rolling experiment.*

Let Y *be the maximum that either die shows in a single roll:* $Y(a,b) = \max(a,b)$.

For example, $Y(3,5) = 5$.

(a) *What is the range of* Y*?*

(b) *What is* $P(Y = y)$ *for each* y *in the range of* Y*?*

(c) *Find a formula for the probability distribution of* Y.

Example 3.2.5 *Check whether the function given by*

$$f(x) = \frac{x+2}{25},$$

for $x = 1, 2, 3, 4, 5$ *can serve as the probability distribution of a discrete random variable.*

Probability distributions for a random variable, say X, may be represented graphically by means of a **probability histogram**.

Each rectangle corresponds to a value for X, its height is $P(X = x)$, and its width is 1, so that the area of each rectangle equals $P(X = x)$. *The total area of the histogram is 1.*

The probability histogram below is for the number of heads in 4 coin flips.

3.3 Cumulative Distribution

In many problems we are interested in the probability that the value of a random variable is less than or equal to (or "at most") some real number x. i.e. $P(X \leq x)$.

If X is a discrete random variable with probability distribution f, the function given by

$$F(x) = P(X \leq x) = \sum_{t \leq x} f(t)$$

for $x \in (-\infty, \infty)$, is called the **cumulative distribution of** X(also called the **distribution function**).

Example 3.3.1 *Let X be the random variable that counts the number of heads in 4 coin flips.*

$$f(2) = \frac{6}{16} \textit{ while}$$

$$F(2) = f(0) + f(1) + f(2) = \frac{1}{16} + \frac{4}{16} + \frac{6}{16} = \frac{11}{16}$$

The corresponding columns of the probability histogram are as follows:

Back to a previous example:

Example 3.3.2 *Two socks are selected at random and removed in succession from a drawer containing five brown socks and three green socks.*

Let X be the random variable that counts the number of brown socks selected.

We found these values earlier on:

Element of sample space	*Probability*	x
BB	$\frac{20}{56}$	*2*
BG	$\frac{15}{56}$	*1*
GB	$\frac{15}{56}$	*1*
GG	$\frac{6}{56}$	*0*

The probability distribution f is given by

$$f(x) = \begin{cases} \frac{20}{56} & \textit{for } x = 2 \\ \frac{30}{56} & \textit{for } x = 1 \\ \frac{6}{56} & \textit{for } x = 0 \end{cases}$$

Find $F(0), F(1), F(2)$, and express the cumulative distribution (distribution function) $F(x)$ as a piece-wise defined function.

Example 3.3.3 *Suppose a random variable X has range $\{1, 2, 3, 4\}$. Define f by*

$$f(1) = \frac{1}{4}, \quad f(2) = \frac{1}{2}, \quad f(3) = \frac{1}{8}, \quad f(4) = \frac{1}{8}$$

(a) Show that f is a valid probability distribution for X.

(b) Find the cumulative distribution (distribution function) for X.

Theorem 3.3.4 *The cumulative distribution $F(x)$ satisfies*

1. $\lim_{x\to-\infty} F(x) = 0$ *and* $\lim_{x\to\infty} F(x) = 1$.

2. *If $a < b$ then $F(a) \leq F(b)$ for any $a, b \in \mathbb{R}$.*

Theorem 3.3.5 *If the range of a random variable X consists of the values $x_1 < x_2 < \cdots < x_n$, then $f(x_1) = F(x_1)$ and*

$$f(x_i) = F(x_i) - F(x_{i-1})$$

for $i = 2, 3, \ldots, n$.

Let's see this on a probability histogram.

Example 3.3.6 *The cumulative distribution for a discrete random variable X is given by*

$$F(x) = \begin{cases} 0 & \textit{for } x < -2 \\ \\ \frac{4}{18} & \textit{for } -2 \leq x < -1 \\ \\ \frac{7}{18} & \textit{for } -1 \leq x < 0 \\ \\ \frac{12}{18} & \textit{for } 0 \leq x < 1 \\ \\ 1 & \textit{for } x \geq 1 \end{cases}$$

Find the probability distribution for X.

3.4 Continuous Random Variables

3.4.1 Probability Density Function (p.d.f)

On a 100 km stretch of rural road we are concerned with the possibility that a deer might cross.

We are interested in the probability that it will occur at a given location or stretch of the road. The sample space for this experiment consists of all points in the interval from 0-100.

Suppose the probability that a deer crosses in any stretch of road is the length of that section divided by 100.

So, from point a to point b with $0 \leq a, b \leq 100$, is the interval $[a, b]$ and its length is given by $b - a$. So, its probability is

$$P([a, b]) = \frac{b - a}{100}.$$

The probability of any two or more non-overlapping intervals can be found by summing the probabilities of the connected components.

Thus the probability measure proposed here has non-negative values, assigns the entire sample space a probability of 1, and is countably additive; hence it satisfies our postulates of probability.

We have taken the sample space to be any point on this stretch of road, and the random variable X here is the function that assigns that point to a real number in the interval $[0, 100]$. This is an example of a **continuous random variable**.

We can give the probability that X lies within an interval by

$$P(a \leq x \leq b) = \frac{b-a}{100}$$

for $a < b$, however the probability that X is any single point is zero.

In the case of a continuous random variable, probabilities cannot simply be assigned to every outcome as is done with a discrete random variable.

Therefore a **continuous random variable** must be accompanied by a **probability density function** in order to compute probabilities.

A positive-valued function f defined on $\mathbb{R}$ is call a **probability density function** for continuous random variable X, if and only if

$$P(a \le X \le b) = \int_a^b f(x)\,dx$$

for any $a, b \in \mathbb{R}$ with $a \le b$. These are also called "**p.d.f**'s" for short.

In the deer crossing example, the p.d.f. for X is $f(x) = \frac{1}{100}$.

For example

$$P(35 \le X \le 50) = \int_{35}^{50} \frac{1}{100}\,dx$$

$$= \left.\frac{x}{100}\right|_{35}^{50} = \frac{50-35}{100} = \frac{15}{100}.$$

Notice that $f(r)$ does not give the probability that $X = r$.

Let X be a continuous random variable. By properties of integrals it follows that

Theorem 3.4.1 *If $a, b \in \mathbb{R}$ with $a \leq b$ then*

$$P(a \leq X \leq b) = P(a \leq X < b) = P(a < X \leq b) = P(a < X < b).$$

From the postulates of probability we obtain the following result:

Theorem 3.4.2 *A function f can serve as a probability density function for X only if it satisfies*

1. $f(x) \geq 0$ *for all* $x \in \mathbb{R}$.
2. $\displaystyle\int_{-\infty}^{\infty} f(x)\, dx = 1.$

Example 3.4.3 *If X has probability density function*

$$f(x) = \begin{cases} k \cdot e^{-3x} & \textit{for } x > 0 \\ 0 & \textit{elsewhere} \end{cases}$$

find k and $P(0.5 \leq X \leq 1)$.

First we need to find what k is.

Since f is given to be a probability density function, it satisfies Condition 2 of the previous theorem.

Solve for k using Condition 2. from the theorem.

$$\begin{aligned} 1 = \int_{-\infty}^{\infty} f(x)\,dx &= \int_{-\infty}^{0} 0\,dx + \int_{0}^{\infty} k \cdot e^{-3x}\,dx \\ &= \lim_{c\to\infty} k\, \frac{e^{-3x}}{(-3)}\bigg|_0^c \\ &= \lim_{c\to\infty} k\frac{e^{-3c}}{(-3)} - k\frac{e^{-3(0)}}{(-3)} \\ &= \frac{k}{3}. \qquad \left(\text{since } \lim_{r\to\infty} e^{-r} = 0\right) \end{aligned}$$

Thus $k = 3$. Now we can compute

$$P(0.5 \leq X \leq 1) = \int_{0.5}^{1} f(x)\,dx = \int_{0.5}^{1} 3e^{-3x}\,dx = -e^{-3x}\Big|_{0.5}^{1}$$

$$= -e^{-3} - (-e^{-1.5}) \approx 0.1733$$

Graph of $3e^{-3x}$ is given below.

The shaded area denotes $P(0.5 \leq X \leq 1)$.

3.4.2 Cumulative Distribution Function of a Continuous Random Variable

Let X is a continuous random variable with probability density function f. Then the function

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\, dt$$

for all $x \in \mathbb{R}$, is called the **cumulative distribution function of** X.

Example 3.4.4 *Random variable X with p.d.f. $f(x) = -x^2 + \frac{4}{3}$ for $0 \leq x \leq 1$ and 0 elsewhere. (p.d.f. plotted in red)*

Cumulative distribution function is $F(x) = \int_{-\infty}^{x} f(t)\, dt$.

Shade the areas representing the values $F(0.5)$ and $F(0.75)$ respectively.

From the properties of integrals we have the following.

Theorem 3.4.5 *If continuous random variable X has probability density function $f(x)$ and cumulative distribution function $F(x)$ then*

$$P(a \leq X \leq b) = F(b) - F(a)$$

for any $a, b \in \mathbb{R}$ with $a \leq b$, and

$$f(x) = \frac{d}{dx}F(x)$$

where the derivative exists.

Using the previous example with p.d.f. $f(x) = -x^2 + \frac{4}{3}$ for $0 \leq x \leq 1$, and 0 elsewhere, we have:

$$P(0.25 \leq X \leq 0.75) = F(0.75) - F(0.25)$$

Let's see this considering the relevant shaded areas on the corresponding graphs.

The cumulative distribution function is

$$F(x) = \int_{-\infty}^{x} f(t)\, dt = \int_{0}^{x} -t^2 + \frac{4}{3}\, dt = \dots$$

and its derivative is the probability density function

$$\frac{d}{dx}F(x) = \frac{d}{dx}\left(-\frac{x^3}{3} + \frac{4x}{3}\right) = \dots$$

Example 3.4.6 *Find the cumulative distribution function $F(x)$ for*

$$f(x) = \begin{cases} 3e^{-3x} & \textit{for } x > 0 \\ 0 & \textit{elsewhere} \end{cases}$$

and use it to evaluate $P(0.5 \leq X \leq 1)$.

For $x > 0$ we have

$$F(x) = \int_{-\infty}^{\infty} f(t)\, dt = \int_0^x 3e^{-3t}\, dt = -e^{-3t}\Big|_0^x = -e^{-3x} + 1.$$

For $x \leq 0$, $f(x) = 0$.

So,

$$F(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ 1 - e^{-3x} & \text{for } x > 0 \end{cases}$$

Then using the theorem,

$$P(0.5 \leq X \leq 1) = \ldots$$