4.1.1 The Expected Value of a Continuous Random Variable

If X is a continuous random variable and $f(x)$ is its probability density [redacted] function, then the expected value of X is defined as

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx.$$

discrete case
$E(X) = \sum_x x \cdot f(x)$

The integral must exist in order for the expected value to have meaning.

Example 4.1.4 *A contractor's profit on a construction job can be considered as a continuous random variable having probability density*

$$f(x) = \begin{cases} \frac{1}{18}(x+1) & \textit{for } -1 < x < 5 \\ 0 & \textit{otherwise} \end{cases}$$

(where the units are in \$1,000). What is her expected profit?

The expected value of X, where X denotes the contractor's profit in \$1,000's, is

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx \quad = \int_{-1}^{5} x \cdot \frac{1}{18}(x+1)\, dx$$

$$= \frac{1}{18} \int_{-1}^{5} x(x+1)\,dx = \frac{1}{18}\int_{-1}^{5} x^2 + x\, dx = \frac{1}{18}\left(\frac{x^3}{3} + \frac{x^2}{2}\right)\Bigg|_{x=-1}^{x=5}$$

$$= \frac{1}{18}\left[\left(\frac{125}{3} + \frac{25}{2}\right) - \left(\frac{-1}{3} + \frac{1}{2}\right)\right] = \frac{1}{18}\left[\left(\frac{250}{6} + \frac{75}{6}\right) - \left(\frac{-2}{6} + \frac{3}{6}\right)\right]$$

$$= \frac{1}{18} \cdot \left(\frac{325}{6} - \frac{1}{6}\right) = \frac{1}{18} \cdot \frac{\overset{54}{\cancel{324}}}{\underset{1}{\cancel{6}}} = 3$$

So, the exp. profit is \$3000.

4.1.2 Expectation of a Function of a Random Variable

We are not limited to considering a random variable by itself.

We can as well consider a function $g(X)$ of a random variable, and evaluate its expected value.

Theorem 4.1.5 *If X is a discrete random variable with probability distribution $f(x)$, the expected value of $g(X)$ is given by*

$$E(g(X)) = \sum_x g(x) \cdot f(x).$$

function of X

probability distribution

If X is a continuous random variable with probability density function $f(x)$, the expected value of $g(X)$ is given by

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) \cdot f(x)\, dx.$$

function of X

probability density

Example 4.1.6 *Let X be a random variable that takes the values $-1, 0, 1$, and has probability distribution given by*

$$f(-1) = 0.2, \quad f(0) = 0.5, \quad f(1) = 0.3.$$

Find $E(X^2)$.

$E(x) = -1 \cdot 0.2 + 0 \cdot 0.5 + 1 \cdot 0.3 = -0.2 + 0 + 0.3 = 0.1$

range of $X = \{-1, 0, 1\}$

range of $Y = \{0, 1\}$

Before using the theorem, let's find $E(X^2)$ directly. We view X^2 as a new random variable which we'll call Y.

$Y = X^2$

The range of Y is $\{0, 1\}$ and it has probability distribution

$$\underline{P(Y=0)} = P(X=0) = f(0) = \underline{0.5}$$

$$\underline{P(Y=1)} = P(X=1) + P(X=-1) = \underbrace{f(1)}_{0.3} + \underbrace{f(-1)}_{0.2} = \underline{0.5}$$

Then,

$$E(X^2) = E(Y) = 0 \cdot P(Y=0) + 1 \cdot P(Y=1) = 0 \cdot (0.5) + 1 \cdot (0.5) = \underline{0.5}.$$

X	$P(X=x)$	$X^2 = Y$
-1	0.2	$(-1)^2 = 1$
0	0.5	$0^2 = 0$
1	0.3	$1^2 = 1$

$P(Y=1) = 0.2 + 0.3 = 0.5$

$P(Y=0) = 0.5$

We can find the same result using the theorem as well.

Let $g(X) = X^2$, and let $x_1 = -1, x_2 = 0, x_3 = 1$. Then according to the theorem

$f(-1) = 0.2$
$f(0) = 0.5$
$f(1) = 0.3$

$$E(g(X)) = \sum_{i=1}^{3} g(x_i)f(x_i)$$

$= E(X^2)$

$$= g(x_1)f(x_1) + g(x_2)f(x_2) + g(x_3)f(x_3)$$

$$= g(-1)f(-1) + g(0)f(0) + g(1)f(1)$$

$$= (-1)^2 \cdot (0.2) + (0)^2 \cdot (0.5) + (1)^2 \cdot (0.3)$$

$= 1 \cdot 0.2 + \cancel{0 \cdot 0.5} + 1 \cdot 0.3 = 0.2 + 0.3 = 0.5$

$$= 0.5.$$

Note that: $E(X^2) = 0.5 \neq (E(X))^2 = 0.01$

Example 4.1.7 *Suppose X has probability density*

$$f(x) = \begin{cases} e^{-x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases} \quad (0, \infty)$$

Find the expected value of $g(X) = e^{3X/4}$.

By our theorem

$$E(g(X)) = \int_{-\infty}^{\infty} \underbrace{g(x)}_{e^{3x/4}} \cdot \underbrace{f(x)}_{\text{prob. density}} \, dx$$

$$= \int_0^{\infty} e^{3x/4} \cdot e^{-x} \, dx = \int_0^{\infty} e^{(3x/4)-x} \, dx$$

$$= \int_0^{\infty} e^{-\frac{x}{4}} \, dx = \lim_{c \to \infty} \int_0^{c} e^{-\frac{x}{4}} \, dx$$

$$= \lim_{c \to \infty} \left(-4 \cdot e^{-\frac{1}{4}x} \Big|_{x=0}^{x=c} \right) = \lim_{c \to \infty} \left(-4 e^{-\frac{1}{4} \cdot c} - (-4 e^{0}) \right)$$

$$= \lim_{c \to \infty} \left(\cancel{-4 e^{-\frac{c}{4}}}^{\,0} + 4 \right) = 4$$

4.1.3 Properties of Expected Value

A useful special case of the theorem is:

Theorem 4.1.8 *If a and b are constants, then*

$$E(\underbrace{aX+b}_{g(X)}) = aE(X) + b.$$

"linearity of the expected value"

In particular if $a = 0$, then $E(b) = b$ and if $b = 0$ then $E(aX) = aE(X)$.

$E(0\cdot X+b) = 0\cdot E(X)+b = b$

$E(aX+0) = aE(X)+0 = aE(X)$

To prove this (for the discrete case) let $g(X) = aX + b$. Then

$$
\begin{aligned}
E(\underbrace{aX+b}_{g(x)}) &= E(g(x)) \\
&= \sum_x g(x)\cdot f(x) \\
&= \sum_x (ax+b)\cdot f(x) \\
&= \sum_x (ax\cdot f(x) + b\cdot f(x)) \\
&= \sum_x ax\cdot f(x) + \sum_x b\cdot f(x) \\
&= a\underbrace{\sum_x x\cdot f(x)}_{E(X)} + b\underbrace{\sum_x f(x)}_{=1} \\
&= aE(X) + b \qquad (\text{since } \sum_x f(x) = 1).
\end{aligned}
$$

p. 114

prob. distribution

Theorem : $E(aX+b) = aE(X)+b$

Example 4.1.9 *Returning to out slot machine example, we chose our random variable X to be the expected payout, and not the expected profit. Then we calculated the expected payout.* We found that $E(X) = 0.23$

Suppose this time that we want the expected profit.

calculate $E(Y)$ without using the theorem

If Y is our expected profit then Y has range $\{-0.25, 19.75, 99.75, 499.75\}$

X: 0 20 100 500

So then $P(Y = y) = P(X = y + 0.25)$, and

cost is 0.25

$$E(Y) = (-0.25) \cdot P(X = 0) + (19.75) \cdot P(X = 20) \\ + (99.75) \cdot P(X = 100) + (499.75) \cdot P(X = 500)$$

$$= -0.25 \cdot \left(1 - \frac{27}{8000} - \frac{8}{8000} - \frac{1}{8000}\right) + 19.75 \cdot \frac{27}{8000} + 99.75 \cdot \frac{8}{8000} + 499.75 \cdot \frac{1}{8000} = -0.02$$

On the other hand since $Y = g(X) = X - 0.25$, we can compute

$$E(Y) = E(X - 0.25) = E(X) - 0.25$$

using the theorem (which, in this case is nicer calculation).