Example 2.2.9 *Find the probabilities of randomly drawing two aces in succession from an ordinary deck of 52 playing cards if we sample*

(a) without replacement.

(b) with replacement.

Let A_1 be the event that card 1 is an ace, and A_2 be the event that card 2 is an ace. Then $A_1 \cap A_2$ is the event that both are aces.

(a)

(b)

Since $A \cap B \cap C = (A \cap B) \cap C)$ we have by the multiplication rule

$$P((A \cap B) \cap C)) = P(A \cap B) \cdot P(C|A \cap B).$$

Applying the multiplication rule again to $P(A \cap B)$ gives,

Theorem 2.2.10 *If A, B and C are events in S and $P(A \cap B) \neq 0$, then*

$$P(A \cap B \cap C) = P(A) \cdot P(B|A) \cdot P(C|A \cap B).$$

Example 2.2.11 *A bushel of 126 apples contains 15 rotten ones. If three apples are chosen at random, what is the probability that all three are rotten? (Solve by using the theorem.)*

If A_1, A_2, A_3 are the events that the first, second and third (resp.) choice is rotten, then

$P(A_1 \cap A_2 \cap A_3) =$

2.3 (In)dependent Events

Example 2.3.1 *Suppose a coin is tossed twice. What is the probability of getting tails on the second toss given that the first toss was tails?*

Surely the outcome of the second toss does not depend on what has previously come up.

Indeed if $S = \{HH, HT, TH, TT\}$, $A_1 = \{TH, TT\}$, and $A_2 = \{HT, TT\}$ are the events of getting tails on flips 1, and 2, then

$$P(A_2|A_1) = \frac{P(A_1 \cap A_2)}{P(A_1)} = \frac{P(\{TT\})}{P(A_1)} = \frac{\frac{1}{4}}{\frac{2}{4}} = \frac{1}{2} = P(A_2).$$

Similarly we can also see that $P(A_1|A_2) = P(A_1)$. In this case events A_1 and A_2 are called **independent**.

Remember the multiplication rule:

$$P(A \cap B) = P(A) \cdot P(B|A).$$

When the events A and B are independent, this identity simplifies to

$$P(A \cap B) = P(A) \cdot P(B).$$

The last identity can be thought of as the definition of independent events.

Definition 2.3.2 *Events A and B are called* ***independent*** *if and only if*

$$P(A \cap B) = P(A) \cdot P(B).$$

They are otherwise called ***dependent****. ($P(A) = 0$ or $P(B) = 0$ also allowed)*

Example 2.3.3 *In the initial coin toss example, the probability of getting two consecutive tails is*

$$P(T_1 \cap T_2) = P(T_1) \cdot P(T_2)$$

Example 2.3.4 *Suppose a coin a tossed 3 times. Let*
$A = \{HHH, HHT\}$ *- first two are* H
$B = \{HHT, HTT, THT, TTT\}$ *- third is* T
$C = \{HTT, THT, TTH\}$ *- exactly two* T

(a) *Show that* A *and* B *are independent.*

(b) Show that B and C are dependent.

Example 2.3.5 *Back to the example of drawing two aces from a deck of cards:*

(without replacement)

Events A_1 and A_2 are dependent because:

(with replacement)

Events A_1 and A_2 are independent because:

Theorem 2.3.6 *If A and B are independent then so are A and B'.*

Proof 2.3.7 *Since $A = (A \cap B) \cup (A \cap B')$ we have*

$$\begin{aligned} P(A) &= P(A \cap B) + P(A \cap B') \quad \textit{(mutually exclusive events)} \\ &= P(A) \cdot P(B) + P(A \cap B') \quad \textit{(A, B independent).} \end{aligned}$$

Rearrange this equation to get

$$\begin{aligned} P(A \cap B') &= P(A) - P(A) \cdot P(B) \\ &= P(A) \cdot (1 - P(B)) \\ &= P(A) \cdot P(B'). \end{aligned}$$

Definition 2.3.8 *Events* $A_1, A_2, \ldots, A_k$ *are* independent *if and only if the probability of the intersection of* ***any number*** *of these is equal to the product of their individual probabilities.*

Example 2.3.9 *Three events* A_1, A_2, A_3 *are independent if and only if*

$$\begin{aligned} P(A_1 \cap A_2) &= P(A_1) \cdot P(A_2) \\ P(A_1 \cap A_3) &= P(A_1) \cdot P(A_3) \\ P(A_2 \cap A_3) &= P(A_2) \cdot P(A_3) \\ P(A_1 \cap A_2 \cap A_3) &= P(A_1) \cdot P(A_2) \cdot P(A_3) \end{aligned}$$

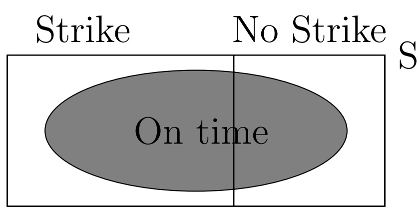
Example 2.3.10 *Let* $S = \{a, b, c, d\}$ *be the sample for space for an experiment with equally likely outcomes and define events*

$$A = \{a, d\}, \quad B = \{b, d\}, \quad C = \{c, d\}.$$

Show that A, B, C *are pairwise independent, but not independent.*

2.4 Rule of Total Probability and Bayes' Theorem

Example 2.4.1 *The completion of a construction job may be delayed because of a strike. The probabilities are* 0.60 *that there will be a strike,* 0.85 *that the construction job will be completed on time if there is no strike, and* 0.35 *that the construction job will be completed on time if there is a strike. What is the probability that the construction job will be completed on time?*



Let A be the event that the job will be completed on time,

B the event of a strike, therefore

B' is the event of no strike.

We want $P(A)$ and are given

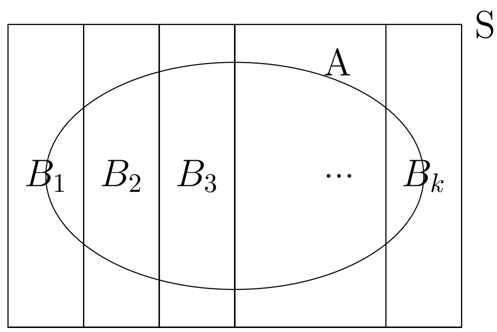
$$P(B) = 0.60, \quad P(A|B) = 0.35, \quad P(A|B') = 0.85.$$

Using the fact that $A = (A \cap B) \cup (A \cap B')$ (union of mutually exclusive events) and the multiplicative rule, we have

$$P(A) = P(A \cap B) + P(A \cap B') = P(B) \cdot P(A|B) + P(B') \cdot P(A|B').$$

Thus $P(A) =$

We can generalize the idea from the last example to obtain a formula for the probability of any event, given that we have a partition of our sample space into events of known probability.



(a partition of a set S is a collection of pairwise disjoint subsets whose union is S)

Theorem 2.4.2 ***(Rule of Total Probability)***

Suppose events $B_1, B_2, \ldots B_k$ form a partition of the sample space S, and $P(B_i) \neq 0$ for $i = 1, \ldots, k$. Then for any event A in S,

$$P(A) = \sum_{i=1}^{k} P(B_i) \cdot P(A|B_i).$$