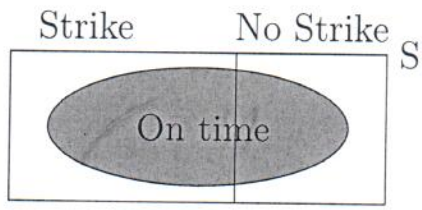
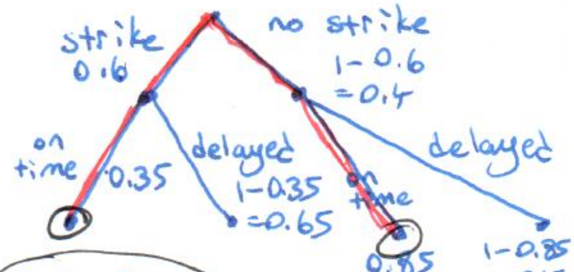
2.4 Rule of Total Probability and Bayes' Theorem

Example 2.4.1 *The completion of a construction job may be delayed because of a strike. The probabilities are* 0.60 *that there will be a strike,* 0.85 *that the construction job will be completed on time if there is no strike, and* 0.35 *that the construction job will be completed on time if there is a strike. What is the probability that the construction job will be completed on time?*





$0.6 \cdot 0.35$
$+ \; 0.4 \cdot 0.85 = 0.55$

Let A be the event that the job will be completed on time,

B the event of a strike, therefore

B' is the event of no strike.

We want $P(A)$ and are given

$$P(B) = 0.60, \quad P(A|B) = 0.35, \quad P(A|B') = 0.85.$$

Using the fact that $A = (A \cap B) \cup (A \cap B')$ (union of mutually exclusive events) and the multiplicative rule, we have

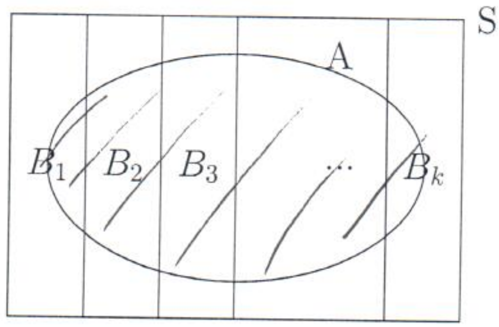
$$P(A) = P(A \cap B) + P(A \cap B') = P(B) \cdot P(A|B) + P(B') \cdot P(A|B').$$

Thus $P(A) =$ $0.6 \cdot 0.35 + (1 - 0.6) \cdot 0.85$

$= 0.21 + 0.34$

$= 0.55$

We can generalize the idea from the last example to obtain a formula for the probability of any event, given that we have a partition of our sample space into events of known probability.



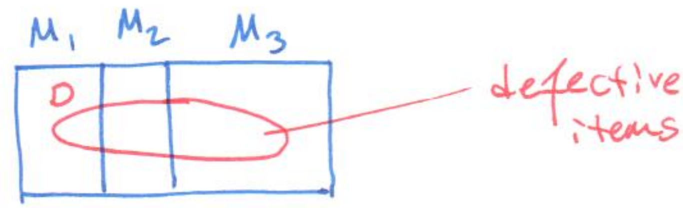
(a partition of a set S is a collection of pairwise disjoint subsets whose union is S)

Theorem 2.4.2 ***(Rule of Total Probability)***

Suppose events $B_1, B_2, \ldots B_k$ form a partition of the sample space S, and $P(B_i) \neq 0$ for $i = 1, \ldots, k$. Then for any event A in S,

$$P(A) = \sum_{i=1}^{k} P(B_i) \cdot P(A|B_i).$$

$$P(A) = P(B_1) \cdot P(A|B_1) + P(B_2) \cdot P(A|B_2) + \ldots + P(B_k) \cdot P(A|B_k)$$



Example 2.4.3 *Three machines M_1, M_2 and M_3 produce respectively 40, 10 and 50 percent of the items in a factory. The percentage of defective items produced by each respective machine is 2, 3 and 4 percent.*

Find the probability that a randomly selected item from the factory is defective. M_i : the event that item is produced by machine i.

Let D denote the event that a randomly selected item is defective.

Then,

$$P(D) = \sum_{i=1}^{3} P(M_i)\cdot P(D|M_i)$$

$$= P(M_1)\cdot P(D|M_1) + P(M_2)\cdot P(D|M_2) + P(M_3)\cdot P(D|M_3)$$

$$= 0.4\cdot 0.02 + 0.1\cdot 0.03 + 0.5\cdot 0.04$$

$$= \underbrace{0.008}_{M_1} + \underbrace{0.003}_{M_2} + \underbrace{0.02}_{M_3} = 0.031$$

(3.1%)

2.4.1 Bayes' Theorem

Example 2.4.4 *With reference to the last example: if the randomly selected item is defective, what is the probability that the item was produced by*

(a) machine M_1,

(b) machine M_2, or

(c) machine M_3?

We calculated earlier that the probability of the item being defective is 0.031.

To answer the current question (part (a)), we first ask ourselves

"What is the contribution of the defectives from M_1 to the probability 0.031?"

for M_1: $0.4 \cdot 0.02 = 0.008$

for M_2: $0.1 \cdot 0.03 = 0.003$

for M_3: $0.5 \cdot 0.04 = 0.02$

[redacted]

Theorem 2.4.5 ***(Bayes' Theorem)*** *Suppose events $B_1, B_2, \ldots B_k$ form a partition of the sample space S, and $P(B_i) \neq 0$ for $i = 1, \ldots, k$. Then for any event A in S with $P(A) \neq 0$*

$$P(B_r|A) = \frac{P(B_r) \cdot P(A|B_r)}{\underbrace{\sum_{i=1}^{k} P(B_i) \cdot P(A|B_i)}_{\text{total probability for } A}}$$

for r this is the contribution for B_r to the total probability

for $r = 1, \ldots, k$.

Proof 2.4.6

$$P(B_r|A) = \frac{P(B_r \cap A)}{P(A)} \quad \textit{(by definition)}$$

$$= \frac{P(B_r) \cdot P(A|B_r)}{P(A)} \quad \textit{(multiplication rule)}$$

$$= \frac{P(B_r) \cdot P(A|B_r)}{\sum_{i=1}^{k} P(B_i) \cdot P(A|B_i)} \quad \textit{(rule of total probability)}$$

Bayes' Theorem: $P(B_r|A) = \dfrac{P(B_r)\cdot P(A|B_r)}{\sum_{i=1}^{k} P(B_i)\cdot P(A|B_i)}$

where $B_1, \dots, B_k$ form a partition of S (sample space).

Back to the example:

Example 2.4.7 *Three machines M_1, M_2 and M_3 produce respectively 40, 10 and 50 percent of the items in a factory. The percentage of defective items produced by each respective machine is 2, 3 and 4 percent. If the randomly selected item is defective, what is the probability that the item was produced by*

(a) machine M_1,

(b) machine M_2, or

(c) machine M_3?

Using Bayes' Theorem,

(a)

$$P(M_1|D) = \frac{P(M_1)\cdot P(D|M_1)}{P(D)} \cancel{= \frac{(0.40)(0.02)}{0.031} \approx 0.2581}$$

where $P(D) = \sum_{i=1}^{3} P(M_i)\cdot P(D|M_i)$

~~(b)~~

$$P(M_1|D) = \frac{P(M_1)\cdot P(D|M_1)}{\sum_{i=1}^{3} P(M_i)\cdot P(D|M_i)} = \frac{0.4\cdot 0.02}{0.4\cdot 0.02 + 0.1\cdot 0.03 + 0.5\cdot 0.04} = \frac{0.008}{0.031} \approx 0.2581$$

b) $$P(M_2|D) = \frac{P(M_2)\cdot P(D|M_2)}{\sum_{i=1}^{3} P(M_i)\cdot P(D|M_i)} = \frac{0.1\cdot 0.03}{0.031} \approx 0.097$$

c) $$P(M_3|D) = \frac{P(M_3)\cdot P(D|M_3)}{\sum_{i=1}^{3} P(M_i)\cdot P(D|M_i)} = \frac{0.5\cdot 0.04}{0.031} \approx 0.645$$