binomial distribution

$b(x; n, \theta) = \binom{n}{x} \theta^x (1-\theta)^{n-x}$

Moments of the Binomial Distribution:

Theorem 5.3.3 *Moment generating function of the binomial distribution is*

$$M_X(t) = (1 + \theta(e^t - 1))^n.$$

Proof 5.3.4

defn. of moment generating fnc.

$$M_X(t) = E(e^{tX})$$

$$= \sum_{x=0}^{n} (e^{tx}) \cdot \binom{n}{x} \theta^x (1-\theta)^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (e^t\theta)^x (1-\theta)^{n-x}$$

$$= (e^t\theta + (1-\theta))^n \quad \textit{(by the binomial theorem)}$$

$$= (1 + \theta(e^t - 1))^n.$$

$$M_X(t) = E(e^{tX}) = \sum_{x=0}^{n} (e^{tx}) \cdot \binom{n}{x} \theta^x (1-\theta)^{n-x}$$

$$= \sum_{x=0}^{n} (e^t)^x \cdot \theta^x \cdot \binom{n}{x} \cdot (1-\theta)^{n-x}$$

$$= \sum_{x=0}^{n} (e^t \cdot \theta)^x \cdot \binom{n}{x} \cdot (1-\theta)^{n-x}$$

$$= \sum_{\substack{x=0 \\ i}}^{n} \binom{n}{x} \underbrace{(e^t \cdot \theta)}_{a}{}^{x} \cdot \underbrace{(1-\theta)}_{b}{}^{n-x}$$

Binomial Theorem:

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i \cdot b^{n-i}$$

$$= (e^t \cdot \theta + (1-\theta))^n \quad \text{by Binomial Theorem}$$

$$= (e^t \cdot \theta + 1 - \theta)^n$$

$$= (1 + \theta(e^t - 1))^n$$

$M_X(t) = ((1+\theta\cdot(e^t-1))^n$ for a random variable with binomial distribution

From the moment generating function, we can find the mean.

Finding the **mean** from the moment generating function:

$$E(X) = \mu = \frac{d}{dt}M_X(t)\bigg|_{t=0}$$

$$= \frac{d}{dt}(1+\theta(e^t-1))^n\bigg|_{t=0}$$

CHAIN RULE

$$= n(1+\theta(e^t-1))^{n-1}\cdot(\theta e^t)\big|_{t=0}$$

$$= n(1+\theta(e^0-1))^{n-1}\cdot\theta\cdot e^0 = n(1+\theta\cdot 0)^{n-1}\cdot\theta\cdot 1$$

$$= n(1+\theta(e^0-1))^{n-1}\cdot(\theta e^0) \qquad = n(1)^{n-1}\cdot\theta$$

$$= n\cdot\theta$$

$$= n\theta.$$

Next we want to find the **variance**.
First, we need the second moment about the origin:

$$E(X^2) = \frac{d^2}{dt^2} M_X(t)\Big|_{t=0}$$

$$= \frac{d^2}{dt^2}(1 + \theta(e^t - 1))^n\Big|_{t=0}$$

$$= \frac{d}{dt} n\theta e^t (1 + \theta(e^t - 1))^{n-1}\Big|_{t=0}$$

$$= n\theta e^t(1 + \theta(e^t - 1))^{n-1}$$

$$+n(n-1)\theta e^t(1 + \theta(e^t - 1))^{n-2} \cdot (\theta e^t)\big|_{t=0}$$

$$= n\theta + n(n-1)\theta^2$$

Finally, we can use the formula for the variance:

$$\sigma^2 = E(X^2) - \mu^2 = n\theta + n(n-1)\theta^2 - (n\theta)^2 = n\theta - n\theta^2 = n\theta(1-\theta).$$

We obtained the following theorem:

Theorem 5.3.5 *The mean and variance of the binomial distribution:*

$$\boxed{\mu = n\theta, \quad \sigma^2 = n\theta(1-\theta)}$$

variance : $\sigma^2 = E(X^2) - \mu^2$

shortcut formula

2nd moment about the origin

mean

we need to find this

we found $\mu = n \cdot \theta$

$M_X(t)$

$= (1 + \theta \cdot (e^t - 1))^n$

moment generating fnc. for a random variable with binomial distribution

$$E(X^2) = \frac{d^2}{dt^2} M_X(t) \Big|_{t=0} = \frac{d}{dt^2} ((1+\theta \cdot (e^t - 1))^n \Big|_{t=0}$$

$$= \frac{d}{dt} \Big(\underbrace{n (1+\theta \cdot (e^t - 1))^{n-1} \cdot \theta e^t}_{\text{1st derivative of the moment generating fnc.}} \Big) \Big|_{t=0}$$

$$= \frac{d}{dt} \Big(\underbrace{n \cdot \theta \cdot e^t}_{f} \cdot \underbrace{(1+\theta \cdot (e^t - 1))^{n-1}}_{g} \Big) \Big|_{t=0}$$

$(fg)' = f \cdot g' + f' \cdot g$

$$\underset{\text{PRODUCT RULE}}{=} \left(\underbrace{n \cdot \theta \cdot e^t}_{f} \cdot \overbrace{(n-1) \cdot (1+\theta \cdot (e^t - 1))^{n-2} \cdot \theta \cdot e^t}^{g'} + \underbrace{n \cdot \theta \cdot e^t}_{f'} \cdot \underbrace{(1+\theta \cdot (e^t-1))^{n-1}}_{g} \right) \Bigg|_{t=0}$$

$$= n \cdot \theta \cdot \overset{=1}{\cancel{e^0}} \cdot (n-1) \cdot (1 + \theta \cdot (\overset{=1}{\cancel{e^0}} - 1))^{n-2} \cdot \theta \cdot \overset{=1}{\cancel{e^0}}$$
$$+ n \cdot \theta \cdot \underset{=1}{\cancel{e^0}} \cdot (1 + \theta \cdot (\underset{=1}{\cancel{e^0}} - 1))^{n-1}$$

$$= n \cdot \theta \cdot (n-1) \cdot 1^{n-2} \cdot \theta$$
$$+ n \cdot \theta \cdot 1^{n-1}$$

$$= n\theta \cdot (n-1) \cdot \theta + n\theta$$

$$= n(n-1) \cdot \theta^2 + n\theta \quad = E(X^2)$$

Then, since $\sigma^2 = E(X^2) - \mu^2$

and since $E(X^2) = n(n-1) \cdot \theta^2 + n\theta$ and $\mu = n\theta$,

we have

$$\sigma^2 = (n(n-1) \cdot \theta^2 + n\theta) - (n\theta)^2$$
$$= (n^2 - n)\theta^2 + n\theta - n^2\theta^2$$
$$= \cancel{n^2\theta^2} - n\theta^2 + n\theta - \cancel{n^2\theta^2} \quad = n\theta - n\theta^2$$
$$= n\theta(1-\theta)$$

So, a random variable with binomial distribution has variance $\sigma^2 = n\theta(1-\theta)$