MATH1550
Exercise Set 5

- Continuous random variables
- Probability density functions
- Cumulative distributions for continuous random variables

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1. Let X be a continuous random variable with probability density function

$$f(x) = \begin{cases} \frac{1}{10}(3x^2+1) & \text{for } 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Verify that $f(x)$ is a valid probability density function

   (b) Find $P(X \ge 1)$

Solution. (a) A density function must satisfy:

- $f(x) \ge 0$ for all $x \in \mathbb{R}$.
- $\displaystyle\int_{-\infty}^{\infty} f(x)\, dx = 1$.

We see that the first condition is satisfied, we need only to verify the second.

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\, dx &= \int_0^2 \frac{1}{10}(3x^2+1)\, dx \\ &= \frac{1}{10}\left[x^3+x\right]_0^2 \\ &= \frac{1}{10}[10] \\ &= 1. \end{aligned}$$

(b)

$$P(X \ge 1) = \int_1^2 \frac{1}{10}(3x^2+1)\, dx = \frac{1}{10}\left(x^3+x\right)\Big|_1^2 = \frac{8}{10} = 0.8.$$

□

2. Let Y be a continuous random variable. Let $f(x) = k(1+x)$ for $x \in [0,2]$ and $f(x) = 0$ elsewhere. For which values of k is f a valid probability density function for Y?

Solution. A density function must satisfy $f(x) \ge 0$ for all $x \in \mathbb{R}$, and $\displaystyle\int_{-\infty}^{\infty} f(x)\, dx = 1$. The first condition forces k to be non-negative. For the second condition we compute

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\,dx &= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{2} f(x)\,dx + \int_{2}^{\infty} f(x)\,dx \\
&= \int_{-\infty}^{0} 0\,dx + \int_{0}^{2} k(1+x)\,dx + \int_{2}^{\infty} 0\,dx \\
&= k\int_{0}^{2} 1+x\,dx \\
&= k\cdot\left[x+\frac{x^2}{2}\right]_0^2 \\
&= k\cdot\left[\left(2+\frac{4}{2}\right)-\left(0+\frac{0}{2}\right)\right] \\
&= 4k.
\end{aligned}$$

In order for f to be a probability density function we must have $4k = 1$. Thus, $k = \frac{1}{4}$. □

3. Which of the following are allowable as probability density functions for some continuous random variable? (show why or why not)

   (a)
$$f(x) = \begin{cases} 4x^3 & \text{for } 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

   (b)
$$g(x) = \begin{cases} 6x^2 - 2x & \text{for } 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

   (c)
$$h(x) = \begin{cases} \frac{1}{6}(1+x)^5 & \text{for } 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

   (d)
$$p(x) = \begin{cases} \frac{3}{4}(1-x^2) & \text{for } -1 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Solution. (a) Yes; $f(x) \ge 0$ for all x and

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{1} 4x^3\,dx = x^4\Big|_0^1 = 1.$$

(b) No; for example $g(\frac{1}{6}) = -\frac{1}{6} < 0$.

(c) No;

$$\int_{-\infty}^{\infty} h(x)\,dx = \int_{0}^{1} \frac{1}{6}(1+x)^5\,dx = \frac{(1+x)^6}{36}\Big|_0^1 = \frac{63}{36} \ne 1.$$

(d) Yes; $p(x) \ge 0$ for all x (since $x^2 < 1$ for $-1 < x < 1$, we have $1 - x^2 > 0$), and

$$\int_{-\infty}^{\infty} p(x)\,dx = \int_{-1}^{1} \frac{3}{4}(1-x^2)\,dx = \frac{3}{4}x - \frac{1}{4}x^3\Big|_{-1}^{1} = 1.$$

□

4. Find the cumulative distribution function $F(x)$ for the random variable X in question 1, and use $F(x)$ to compute $P(-1 \le X \le 1)$ and $P(0.5 \le X \le 1.5)$.

Solution. The cumulative distribution function is defined

$$F(x) = \int_{-\infty}^{x} f(t)\, dt.$$

Based on the piecewise definition of $f(x)$ we consider three regions: $x < 0$, $0 \le x \le 2$ and $x > 2$. For $x < 0$

$$F(x) = \int_{-\infty}^{x} f(t)\, dt = \int_{-\infty}^{x} 0\, dt = 0.$$

For $0 \le x \le 2$

$$F(x) = \int_{-\infty}^{x} f(t)\, dt = \int_{-\infty}^{0} 0\, dt + \int_{0}^{x} \frac{1}{10}(3t^2+1)\, dt = \frac{1}{10}\left[t^3+t\right]_0^x = \frac{1}{10}(x^3+x).$$

For $x > 2$

$$F(x) = \int_{-\infty}^{x} f(t)\, dt = \int_{-\infty}^{0} 0\, dt + \int_{0}^{2} \frac{1}{10}(3t^2+1)\, dt + \int_{2}^{x} 0\, dt = 0 + \frac{1}{10}\left[t^3+t\right]_0^2 + 0 = 1.$$

In summary

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{1}{10}(x^3+x) & \text{for } 0 \le x \le 2 \\ 1 & \text{for } x > 2 \end{cases}$$

Now

$$\begin{aligned} P(-1 \le X \le 1) &= F(1) - F(-1) \\ &= \frac{1}{10}(1^3+1) - 0 \\ &= \frac{1}{5}, \end{aligned}$$

and

$$\begin{aligned} P(0.5 \le X \le 1.5) &= F(1.5) - F(0.5) \\ &= \frac{1}{10}((1.5)^3+1.5) - \frac{1}{10}((0.5)^3+0.5) \\ &= \frac{39}{80} - \frac{5}{80} \\ &= \frac{17}{40}. \end{aligned}$$

□

5. Find a probability density function for the random variable whose cumulative distribution function is given by

$$F(x) = \begin{cases} 0 & \text{for } x \le 0 \\ x & \text{for } 0 < x < 1 \\ 1 & \text{for } x \ge 1 \end{cases}.$$

Solution. Using the fact that $f(x) = \frac{d}{dx}F(x)$ we have

$$f(x) = \begin{cases} 0 & \text{for } x \le 0 \\ 1 & \text{for } 0 < x < 1 \\ 0 & \text{for } x \ge 1 \end{cases}.$$

$\square$

6. The probability density function of a random variable X is given by

$$f(x) = \begin{cases} \frac{c}{\sqrt{x}} & \text{for } 0 < x < 4 \\ 0 & \text{elsewhere} \end{cases}$$

Find the value of c, and compute $P(X < \frac{1}{4})$ and $P(X > 1)$.

Solution. A density function must satisfy:

- $f(x) \ge 0$ for all $x \in \mathbb{R}$.
- $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1$.

The first condition forces the value of c to be non-negative. Applying the second condition:

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} f(x)\,dx \\ &= \int_0^4 \frac{c}{\sqrt{x}}\,dx \\ &= c\left[2\sqrt{x}\right]_0^4 \\ &= 4c. \end{aligned}$$

This forces $c = \frac{1}{4}$. Thus

$$\begin{aligned} P\left(X < \frac{1}{4}\right) &= \int_{-\infty}^{\frac{1}{4}} f(x)\,dx \\ &= \int_0^{\frac{1}{4}} \frac{1}{4\sqrt{x}}\,dx \\ &= \frac{1}{4}\left[2\sqrt{x}\right]_0^{\frac{1}{4}} \\ &= \frac{1}{4}, \end{aligned}$$

and

$$\begin{aligned} P(X > 1) &= \int_1^{\infty} f(x)\,dx \\ &= \int_1^4 \frac{1}{4\sqrt{x}}\,dx \\ &= \frac{1}{4}\left[2\sqrt{x}\right]_1^4 \\ &= \frac{1}{2}. \end{aligned}$$

$\square$

7. Suppose the probability density of continuous random variable X is given by

$$f(x) = \begin{cases} 4x^3 & \text{for } 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the cumulative distribution function $F(x)$ for X, and use it to compute $P(0.5 < X < 1)$.

Solution.

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ x^4 & \text{for } 0 \le x \le 1 \\ 1 & \text{for } x > 1 \end{cases}$$

Recall that for a continuous random variable $P(0.5 < X < 1) = P(0.5 < X \le 1)$. Thus

$$\begin{aligned} P(0.5 < X < 1) &= P(0.5 < X \le 1) \\ &= F(1) - F(0.5) \\ &= 1 - (0.5)^4 \\ &= 0.9375 \end{aligned}$$

□

8. Suppose the probability density of continuous random variable X is given by

$$f(x) = \begin{cases} \frac{x}{2} & \text{for } 0 < x \le 1 \\ \frac{1}{2} & \text{for } 1 < x \le 2 \\ \frac{3-x}{2} & \text{for } 2 < x < 3 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find the cumulative distribution function $F(x)$ for X.

(b) Use the cumulative distribution to compute the following probabilities
- $P(0.25 < x < 0.5)$
- $P(0.5 < x < 1.5)$
- $P(0.5 < x < 2.25)$

Solution. (a) Recall that the cumulative distribution function $F(x)$, for a continuous random variable X, is defined by

$$F(x) = \int_{-\infty}^{x} f(t)\, dt$$

for any $x \in \mathbb{R}$, where $f(x)$ is the density function for X. Since $f(x)$ is defined piecewise we solve for $F(x)$ in pieces as well. For $x \le 0$:

$$F(x) = \int_{-\infty}^{x} f(t)\, dt = \int_{-\infty}^{x} 0\, dt = 0.$$

For $x \in (0, 1]$:

$$\begin{aligned}
F(x) &= \int_{-\infty}^{x} f(t)\ dt \\
&= \int_{-\infty}^{0} f(t)\ dt + \int_{0}^{x} f(t)\ dt \\
&= F(0) + \int_{0}^{x} \frac{t}{2}\ dt \\
&= 0 + \frac{t^2}{4}\Big|_{0}^{x} \\
&= \frac{x^2}{4}
\end{aligned}$$

For $x \in (1, 2]$:

$$\begin{aligned}
F(x) &= \int_{-\infty}^{x} f(t)\ dt \\
&= \int_{-\infty}^{1} f(t)\ dt + \int_{1}^{x} f(t)\ dt \\
&= F(1) + \int_{1}^{x} \frac{1}{2}\ dt \\
&= \frac{(1)^2}{4} + \left[\frac{t}{2}\right]_{1}^{x} \\
&= \frac{1}{4} + \left[\frac{x}{2} - \frac{1}{2}\right] \\
&= \frac{x}{2} - \frac{1}{4}
\end{aligned}$$

For $x \in (2, 3)$:

$$\begin{aligned}
F(x) &= \int_{-\infty}^{x} f(t)\ dt \\
&= \int_{-\infty}^{2} f(t)\ dt + \int_{2}^{x} f(t)\ dt \\
&= F(2) + \int_{1}^{x} \frac{3-t}{2}\ dt \\
&= \frac{(2)}{2} - \frac{1}{4} + \left[\frac{3t}{2} - \frac{t^2}{4}\right]_{2}^{x} \\
&= \frac{3}{4} + \left[\left(\frac{3x}{2} - \frac{x^2}{4}\right) - (3-1)\right] \\
&= \frac{3x}{2} - \frac{x^2}{4} - \frac{5}{4}
\end{aligned}$$

For $x \geq 3$ we have $F(x) = 1$, since we will have integrated over all nonzero pieces of the density function. Verify this directly or see that,

$$
\begin{aligned}
F(x) &= \int_{-\infty}^{x} f(t)\ dt \\
&= \int_{-\infty}^{3} f(t)\ dt + \int_{3}^{x} f(t)\ dt \\
&= \int_{-\infty}^{3} f(t)\ dt + 0 \quad (\text{as } f(t) = 0 \text{ for } x \geq 3) \\
&= \int_{-\infty}^{3} f(t)\ dt + \int_{3}^{\infty} f(t)\ dt \\
&= \int_{-\infty}^{\infty} f(t)\ dt \\
&= 1.
\end{aligned}
$$

Putting these pieces together we have

$$
F(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{x^2}{4} & \text{for } 0 \leq x < 1 \\ \frac{x}{2} - \frac{1}{4} & \text{for } 1 \leq x < 2 \\ \frac{3x}{2} - \frac{x^2}{4} - \frac{5}{4} & \text{for } 2 \leq x < 3 \\ 1 & \text{for } x \geq 3 \end{cases}
$$

(b)

- $$
\begin{aligned}
P(0.25 < x < 0.5) &= F(0.5) - F(0.25) \\
&= \frac{(0.5)^2}{4} - \frac{(0.25)^2}{4} \\
&= \frac{3}{64} \\
&= 0.046875
\end{aligned}
$$

- $$
\begin{aligned}
P(0.5 < x < 1.5) &= F(1.5) - F(0.5) \\
&= \frac{1.5}{2} - \frac{1}{4} - \frac{(0.5)^2}{4} \\
&= \frac{7}{16} \\
&= 0.4375
\end{aligned}
$$

- $$
\begin{aligned}
P(0.5 < x < 2.25) &= F(2.25) - F(0.5) \\
&= \frac{6.75}{2} - \frac{(2.25)^2}{4} - \frac{5}{4} - \frac{(0.5)^2}{4} \\
&= \frac{51}{64} \\
&= 0.796875
\end{aligned}
$$

□

9. The continuous random variable X has cumulative distribution function given by

$$F(x) = \begin{cases} 0 & \text{for } x \leq -1 \\ \frac{x+1}{2} & \text{for } -1 \leq x < 1 \\ 1 & \text{for } x \geq 1 \end{cases}$$

(a) Compute the following probabilities
- $P\left(-\frac{1}{2} < X < \frac{1}{2}\right)$
- $P\left(2 < X < 3\right)$

(b) Determine the probability density function for X.

Solution. (a)
- $$P\left(-\frac{1}{2} < X < \frac{1}{2}\right) = F(0.5) - F(-0.5) = \frac{1.5}{2} - \frac{0.5}{2} = \frac{1}{2}$$
- $$P\left(2 < X < 3\right) = F(3) - F(2) = 1 - 1 = 0$$

(b)

$$f(x) = \begin{cases} \frac{1}{2} & \text{for } -1 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

□

10. Find the probability density function for continuous random variable Y with cumulative distribution function given by

$$F(y) = \begin{cases} 0 & \text{for } y \leq 0 \\ \frac{1}{4}y^2 & \text{for } 0 \leq y \leq 2 \\ 1 & \text{for } y > 2 \end{cases}$$

Solution. To find the probability density function we take the derivative of the cumulative distribution function. We do this separately on each interval that is it defined. So

$$f(y) = \begin{cases} 0 & \text{for } y \leq 0 \\ \frac{y}{2} & \text{for } 0 \leq y \leq 2 \\ 0 & \text{for } y > 2 \end{cases}$$

□

11. Can the following function serve as a valid probability density for a continuous random variable?

$$f(x) = \begin{cases} \frac{2}{3}(x+1) & \text{for } x \in [0,1] \\ \\ 0 & \text{otherwise} \end{cases}$$

Solution. Yes, since $f(x) \geq 0$ for all x and

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_0^1 \frac{2}{3}(x+1)\, dx = \frac{x^2}{3} + \frac{2x}{3}\bigg|_0^1 = \frac{1}{3} + \frac{2}{3} = 1.$$

□

12. Can the following function serve as a valid probability density for a continuous random variable?

$$f(x) = \begin{cases} \frac{1}{4}(x+1) & \text{for } x \in [2,4] \\ \\ 0 & \text{otherwise} \end{cases}$$

Solution. No, note that

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_2^4 \frac{1}{4}(x+1)\, dx = \frac{x^2}{8} + \frac{x}{4}\bigg|_2^4 = (2+1) - \left(\frac{1}{2} + \frac{1}{2}\right) = 2.$$

□

13. Let X be a continuous random variable with probability density function given by

$$f(x) = \begin{cases} \frac{x+1}{8} & \text{for } x \in (2,4) \\ \\ 0 & \text{otherwise} \end{cases}$$

Find $P(1.5 < X < 3)$.

Solution.

$$P(1.5 < X < 3) = \int_{1.5}^{3} \frac{x+1}{8}\, dx = \frac{x^2}{16} + \frac{x}{8}\bigg|_{1.5}^{3} = \left(\frac{9}{16} + \frac{3}{8}\right) - \left(\frac{2.25}{16} + \frac{1.5}{8}\right) = 0.609375.$$

□

14. Determine the appropriate value for k so that the following function is a valid probability density

$$f(x) = \begin{cases} \frac{k}{\sqrt{x}} & \text{for } x \in (0,4] \\ \\ 0 & \text{otherwise} \end{cases}$$

Solution. We need

$$1 = \int_{-\infty}^{\infty} f(x)\, dx = \int_0^4 \frac{k}{\sqrt{x}}\, dx = 2k\sqrt{x}\bigg|_0^4 = 4k,$$

which implies $k = \frac{1}{4}$. $\square$

15. The probability density for a continuous random variable X is given below. Find $P(X > \frac{1}{2})$.

$$f(x) = \begin{cases} 6x(1-x) & \text{for } x \in (0,1) \\ \\ 0 & \text{otherwise} \end{cases}$$

Solution.

$$P\left(X > \frac{1}{2}\right) = \int_{\frac{1}{2}}^{\infty} f(x)\,dx = \int_{\frac{1}{2}}^{1} 6x(1-x)\,dx = 3x^2 - 2x^3\Big|_{\frac{1}{2}}^{1} = (3-2) - \left(\frac{3}{4} - \frac{2}{8}\right) = 0.5.$$

$\square$

16. The probability density for a continuous random variable X is given below. Find $P(-0.5 < X \leq 0.25)$.

$$f(x) = \begin{cases} x+1 & \text{for } x \in [-1,0) \\ \\ 1-x & \text{for } x \in [0,1] \\ \\ 0 & \text{otherwise} \end{cases}$$

Solution.

$$\begin{aligned} P(-0.5 < X \leq 0.25) &= \int_{-0.5}^{0.25} f(x)\,dx \\ &= \int_{-0.5}^{0} x+1\,dx + \int_{0}^{0.25} 1-x\,dx \\ &= \left[\frac{x^2}{2} + x\right]_{-0.5}^{0} + \left[x - \frac{x^2}{2}\right]_{0}^{0.25} \\ &= \left(-\frac{1}{8} + \frac{1}{2}\right) + \left(\frac{1}{4} - \frac{1}{32}\right) \\ &= \frac{19}{32}. \end{aligned}$$

$\square$

17. Let X be a continuous random variable with probability density given by

$$f(x) = \begin{cases} \frac{1}{2}x & \text{for } 0 \leq x \leq 2 \\ \\ 0 & \text{otherwise} \end{cases}$$

Find the cumulative distribution function for X.

Fill in blank:

$$F(x) = \underline{\quad 0 \quad} \text{ for } x < 0$$

$$F(x) = \underline{\qquad\quad} \text{ for } 0 \le x \le 2$$

$$F(x) = \underline{\quad 1 \quad} \text{ for } x > 2$$

Solution. By definition,

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\, dt.$$

For $x < 0$:

$$F(x) = \int_{-\infty}^{x} f(t)\, dt = \int_{-\infty}^{x} 0\, dt = 0.$$

For $0 \le x \le 2$:

$$F(x) = \int_{-\infty}^{x} f(t)\, dt = \int_{-\infty}^{0} f(t)\, dt + \int_{0}^{x} f(t)\, dt = \int_{-\infty}^{0} 0\, d + \int_{0}^{x} \frac{1}{2} t\, dt = 0 + \frac{1}{4} t^2 \Big|_0^x = \frac{1}{4} x^2.$$

For $x > 2$:

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(t)\, dt \\ &= \int_{-\infty}^{0} f(t)\, dt + \int_{0}^{2} f(t)\, dt + \int_{2}^{\infty} f(t)\, dt \\ &= \int_{-\infty}^{0} 0\, d + \int_{0}^{2} \frac{1}{2} t\, dt + \int_{2}^{\infty} 0\, dt \\ &= 0 + \frac{1}{4} t^2 \Big|_0^2 + 0 \\ &= 1. \end{aligned}$$

□

18. The cumulative distribution function for a continuous random variable X is given below. Find $P(\frac{1}{4} \le X \le 1)$.

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ \sin(\pi x) & \text{for } 0 \le x \le \frac{1}{2} \\ 1 & \text{for } x > \frac{1}{2} \end{cases}$$

Solution.

$$P\left(\frac{1}{4} \le X \le 1\right) = F(1) - F\left(\frac{1}{4}\right) = 1 - \sin\left(\frac{\pi}{4}\right) = 1 - \frac{\sqrt{2}}{2} = \frac{2-\sqrt{2}}{2} \approx 0.2929$$

□

19. The cumulative distribution function for a continuous random variable X is given below. Find its

probability density function $f(x)$ for $0 \le x \le 1$.

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ \\ x^5 & \text{for } 0 \le x \le 1 \\ \\ 1 & \text{for } x > 1 \end{cases}$$

Solution. For $0 \le x \le 1$,

$$f(x) = \frac{d}{dx}F(x) = \frac{d}{dx}x^5 = 5x^4.$$

$\square$

20. The number of years that a certain model of car will remain on the road (i.e. before it is scrapped), given that it has been on the road for 5 years, is a continuous random variable X with cumulative distribution given by

$$F(x) = \begin{cases} 0 & \text{for } x \le 5 \\ \\ 1 - \dfrac{25}{x^2} & \text{for } x > 5 \end{cases}$$

What is the probability that such a car will last longer than 10 years?

Solution.

$$P(X > 10) = 1 - P(X \le 10) = 1 - F(10) = 1 - \left(1 - \frac{25}{100}\right) = \frac{1}{4}.$$

$\square$