MATH1550
Practice Set 2

Do these exercises after completing Chapter 2, up to Rules of Probability.
Topics Covered:

- Probability experiments
- Classical probability concept
- Sample spaces
- Events
- Operations with sets: Union, complement, intersection, Venn diagrams, algebra of sets
- Probability measure, and postulates of probability
- Basic rules of probability

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1. (a) What is the *sample space* of a probabilty experiment? What are *events* in a probabilty experiment?
   (b) What does the classical probability concept say?
   (c) How have we defined *discrete* and *continuous* sample spaces?
   (d) With regard to sets, describe the *union*, *intersection* and *complement*. Give examples of each.
   (e) What does it mean if events are *mutually exclusive*? How can we use this when calculating probabilities?
   (f) State the distributive laws for unions and intersections of sets.
   (g) State De Morgan's laws.
   (h) What are the postulates of probability; i.e. what conditions must a permissible probability measure satisfy?
   (i) How could one find the probability of an event if one knows the probability of its complement? Give an example.
   (j) State the inclusion-exclusion principle for finding probabilities in the case of two sets.

   Solution. (a) The sample space is the set of all outcomes of the probability experiment. Events are subsets of the sample space.
   (b) The classical probability concept is a way to assign a probability measure to a probability experiment with a finite sample space. It says that if a sample space has size N (i.e. consists of N outcomes) then the probability of any event of size n (i.e. if n outcomes are considered "successful") then the probability of that event occurring is $\frac{n}{N}$ (i.e. the probability of a success is $\frac{n}{N}$).
   (c) Discrete sample spaces are those that are finite, or countably infinite. Continuous sample spaces are intervals in $\mathbb{R}$ or products of intervals in $\mathbb{R}^n$.
   (d) The union of two sets A and B is the set of all elements belonging to A or B (or both), while the intersection of A and B is the set of elements belonging to both A and B (i.e. elements common to both). The complement of set A are all elements in the sample space which do not belong to A (i.e. all outcomes other than those in A).
   (e) Events A and B are mutually exclusive if they are disjoint sets (i.e. $A \cap B = \emptyset$). If A and B are mutually exclusive, the postulate of countable additivity says that $P(A \cup B) = P(A) + P(B)$.
   (f) The distributive laws for sets are $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
   (g) De Morgan's laws are $(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$.

(h) The postulates of probability are: (P1) $P(A) \geq 0$ for any event A, (P2) $P(S) = 1$ where S is the sample space, and (P3) P is countably additive for mutually exclusive sets, i.e. if $A_1, A_2, A_3 \ldots$ is a countable collection of pairwise mutually exclusive sets, then $P(A_1 \cup A_2 \cup A_3 \cup \ldots) = P(A_1) + P(A_2) + P(A_3) + \ldots$.

(i) If $P(A')$ is known, then $P(A) = 1 - P(A')$. For example, the probability of rolling a 6 with a standard 6-sided die is $\frac{1}{6}$, and so the probability of rolling either 1,2,3,4 or 5 is

$$P(\{1,2,3,4,5\}) = 1 - P(\{1,2,3,4,5\}') = 1 - P(\{6\}) = 1 - \frac{1}{6} = \frac{5}{6}.$$

(j) If A and B are events in a common sample space, the inclusion-exclusion principle states

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

□

2. (a) Write the sample space for the experiment of tossing a coin 4 times.
   (b) Describe the sample space for the experiment of rolling 3, regular 6-sided dice.
   (c) Describe the sample space for the experiment of rolling 3, regular 6-sided dice and taking their sum.
   (d) Consider the experiment of drawing a single card from each of 3 standard decks of playing cards. How many elements are there in this sample space?
   (e) A can of paint falls from a crane and lands in parking lot below. From this scenario, create a probability experiment. What are the sample space, outcomes and events? How might one assign a probability to this experiment?

Solution. (a)

$$S = \{HHHH, HHHT, HHTH, HTHH, THHH, HHTT, HTHT, THHT, HTTH, THTH,$$
$$TTHH, HTTT, THTT, TTHT, TTTH, TTTT\}$$

(b)

$$S = \{(d_1, d_2, d_3) | d_1, d_2, d_3 \in \{1,2,3,4,5,6\}\}$$

(c)

$$S = \{3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18\}$$

(d) Since the three decks are distinct (e.g. suppose they have different coloured backs) there are $52^3 = 140608$ outcomes this sample space.

(e) Example 1: The sample space is the parking lot, and events are regions of the parking lot. An event occurs if paint lands anywhere in that region. The probability that an event occurs is the ratio of the area of that region to the area of the entire parking lot.

Example 2: The sample space is the entire volume of paint in the can. We consider 5 possible outcomes based on the amount of paint that spills out of the can:

- $A_1 = \{$up to $\frac{1}{5}$ of the can spills out$\}$,
- $A_2 = \{\frac{1}{5}$ to $\frac{2}{5}$ of the can spills out$\}$,
- $A_3 = \{\frac{2}{5}$ to $\frac{3}{5}$ of the can spills out$\}$,
- $A_4 = \{\frac{3}{5}$ to $\frac{4}{5}$ of the can spills out$\}$,
- $A_5 = \{\frac{4}{5}$ up to the entire can spills out$\}$.

We specify that $P(A_1) = 0.1$, $P(A_1) = 0.2$, $P(A_3) = 0.3$, $P(A_4) = 0.3$, $P(A_5) = 0.1$.

Example 3: We consider 2 possible outcomes, either the lid comes off or it does not, and we assume that either can occur with equal probability.

Example 4: We consider 3 possible outcomes; the can lands either on its bottom, on its top or on its side. We might assume that it has equal probability of landing on its top or bottom, and the probability that it is 10 times as likely to land on its side.

$\square$

3. In each case determine whether the function P is a valid probability measure for the given sample space S.

   (a) Let $S = \{A_1, A_2, A_3, A_4, A_5\}$ where $P(A_1) = 0.6$, $P(A_2) = 0.2$, $P(A_3) = 0.1$, $P(A_4) = 0.05$, $P(A_5) = 0.05$. Assume the probability of any event is the sum of the probabilities of the outcomes is contains.

   (b) Let $S = \{A_1, A_2, A_3, A_4\}$ where $P(A_1) = 0.3$, $P(A_2) = -0.1$, $P(A_3) = 0.4$, $P(A_4) = 0.4$. Assume the probability of any event is the sum of the probabilities of the outcomes is contains.

   (c) Let $S = \{A_1, A_2, A_3, A_4, A_5, A_6\}$ where $P(A_1) = 0.15$, $P(A_2) = 0.55$, $P(A_3) = 0.15$, $P(A_4) = 0.1$, $P(A_5) = 0.05$, $P(A_6) = 0.05$. Assume the probability of any event is the sum of the probabilities of the outcomes is contains.

Solution. (a) The postulate of countable additivity is automatically satisfied by the assumption. We see that $P(A_i) \geq 0$ for each outcome and hence this will be true for each event by the additivity assumption. We also have that

$$P(S) = P(A_1) + P(A_2) + P(A_3) + P(A_4) + P(A_5) = 0.6 + 0.2 + 0.1 + 0.05 + 0.05 = 1.$$

Therefore this is a valid probability measure.

(b) This fails to be a valid probability immediately by the fact that $P(A_2) = -0.1$.

(c) Since

$$\begin{aligned} P(S) &= P(A_1) + P(A_2) + P(A_3) + P(A_4) + P(A_5) + P(A_6) \\ &= 0.15 + 0.55 + 0.15 + 0.1 + 0.05 + 0.05 = 1.05 \neq 1 \end{aligned}$$

P fails to be a valid probability measure.

$\square$

4. Let A_1, A_2, A_3 be events in the sample space S such that $S = A_1 \cup A_2 \cup A_3$. Let P be a function on S such that

   - $P(A_1) = 0.65$, $P(A_2) = 0.4$, $P(A_3) = 0.3$,
   - $P(A_1 \cup A_2) = 0.8$, $P(A_1 \cup A_3) = 0.8$, $P(A_2 \cup A_3) = 0.8$,
   - $P(A_1 \cap A_2) = 0.15$, $P(A_1 \cap A_3) = 0.15$, $P(A_2 \cap A_3) = 0.1$,
   - $P(A_1 \cap A_2 \cap A_3) = 0.05$, and $P(A_1 \cup A_2 \cup A_3) = 1$.

   Give a reason why P fails to be a valid probability measure.

Solution. Since $P(A \setminus B) = P(A) - P(A \cap B)$ we have $P(A_1 \setminus A_2) = 0.65 - 0.15 = 0.5$ and $P(A_2 \setminus A_1) = 0.4 - 0.15 = 0.25$. Now

$$A_1 \cup A_2 = (A_1 \setminus A_2) \cup (A_2 \setminus A_1) \cup (A_1 \cap A_2)$$

is a disjoint union of sets. However

$$P(A_1 \cup A_2) = 0.8 \neq 0.9 = 0.5 + 0.25 + 0.15 = P(A_1 \setminus A_2) + P(A_2 \setminus A_1) + P(A_1 \cap A_2)$$

which shows that P is not countably additive, and hence not a probability measure.

Alternatively, note that $P(A_1 \cup A_2) = 0.8$ but

$$P(A_1) + P(A_2) - P(A_1 \cap A_2) = 0.65 + 0.4 - 0.15 = 0.9.$$

Since the inclusion-exclusion principle does not hold, this fails to be a valid probability measure. $\square$

5. A bag contains 15 billiard balls numbered 1 through 15. A ball is drawn at random. Assuming that each ball is equally likely to be drawn, what is the probability that its number is

   (a) Even?

   (b) Less than 5?

   (c) Even and less than 5?

   (d) Even or less than 5?

   Solution. It is assumed that each ball (outcome) has an equally likely probability of $\frac{1}{15}$. Thus

   (a)
   $$P(\text{Even}) = P(2,4,6,8,10,12,14) = P(2) + P(4) + P(6) + P(8) + P(10) + P(12) + P(14) = \frac{7}{15}.$$

   (b)
   $$P(\text{Less than } 5) = P(1,2,3,4) = \frac{4}{15}.$$

   (c)
   $$P(\text{Even and less than } 5) = P(2,4) = \frac{2}{15}.$$

   (d)
   $$\begin{aligned} P(\text{Even or less than } 5) &= P(\text{Even}) + P(\text{Less than } 5) - P(\text{Even and less than } 5) \\ &= \frac{7}{15} + \frac{4}{15} - \frac{2}{15} = \frac{9}{15}. \end{aligned}$$

   $\square$

6. There are 5 horses in a race, and you choose 2. What is the probability that one of the horses you chose is the winner? Assume that each horse has an equally likely chance of winning.

   Solution. There are $\binom{5}{2} = 10$ ways in which one can choose 2 horses from the 5, and 4 of these pairs will contain the winner. Therefore there is a $\frac{4}{10}$ probability of selecting the winning horse. $\square$

7. A 6-sided die (with sides numbers 1 through 6) is rolled. The die is weighted so that it has the following probabilities:

   $$P(1) = 0.1, P(2) = 0.3, P(3) = 0.2, P(4) = 0.1, P(5) = 0.1, P(6) = 0.2.$$

   Consider the following events:

   $$A = \{\text{number rolled is even}\}, B = \{2,3,4,5\}, C = \{x|x < 3\}, D = \{x|x > 7\}.$$

   Find the following probabilities.

   (a) $P(A)$, $P(B)$, $P(C)$, $P(D)$.

   (b) $P(A \cap B)$

(c) $P(A \cup C)$

(d) $P(B \cap C)$

Solution. (a)

$$P(A) = P(2) + P(4) + P(6) = 0.3 + 0.1 + 0.2 = 0.6,$$
$$P(B) = P(2) + P(3) + P(4) + P(5) = 0.3 + 0.2 + 0.1 + 0.1 = 0.7,$$
$$P(C) = P(1) + P(2) = 0.1 + 0.3 = 0.4,$$
$$P(D) = P(\emptyset) = 0.$$

(b)
$$P(A \cap B) = P(2, 4) = P(2) + P(4) = 0.3 + 0.1 = 0.4.$$

(c)
$$P(A \cup C) = P(1, 2, 4, 6) = P(1) + P(2) + P(4) + P(6) = 0.1 + 0.3 + 0.1 + 0.2 = 0.7.$$

(d)
$$P(B \cap C) = P(2) = 0.3.$$

$\square$

8. Let A and B be events in a common sample space such that

$$P(A \cup B) = 0.8, P(A) = 0.4, P(A \cap B) = 0.3$$

Find the following probabilities.

(a) $P(A')$ (where A' denotes the complement of A).

(b) $P(B)$

(c) $P(A \cap B')$

(d) $P(A' \cap B')$

Solution. (a)
$$P(A') = 1 - P(A) = 1 - 0.4 = 0.6.$$

(b)
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$\Rightarrow \quad P(B) = P(A \cup B) - P(A) + P(A \cap B) = 0.8 - 0.4 + 0.3 = 0.7.$$

(c) The set $A \cap B'$ is all elements in A which are also not in B, so $A \cap B' = A \setminus B$ (it may help to look at a Venn diagram). Since
$$A = (A \setminus B) \cup (A \cap B)$$
is a disjoint union
$$P(A \cap B') = P(A \setminus B) = P(A) - P(A \cap B) = 0.4 - 0.3 = 0.1.$$

(d) By De Morgan's Law
$$P(A' \cap B') = P((A \cup B)') = 1 - P(A \cup B) = 1 - 0.8 = 0.2.$$

$\square$