MATH1550
Practice Set 3

These exercises are suited to Chapter 2, Conditional Probability to the end.
Topics Covered:

- Conditional probability
- The multiplication rule probability
- Independent events
- The rule of total probability
- Bayes' theorem

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1. (a) If A and B are events in a common sample space with $P(B) \neq 0$, how do we define the *conditional probability* of A given B?

   (b) What is the multiplication rule for finding the probability $P(A \cap B)$, if A and B are events in a common sample space with $P(A) \neq 0$?

   (c) What is the multiplication rule for finding the probability $P(A \cap B \cap C)$, if A, B and C are events in a common sample space with $P(A \cap B) \neq 0$?

   (d) What does it mean if two events are *independent*? Give an example of independent events in some sample space.

   (e) What does it mean if events are *dependent*? Give an example of dependent events in some sample space.

   (f) State the rule for total probability.

   (g) State Bayes' Theorem.

Solution. (a) The *conditional probability* of A given B is denoted $P(A|B)$ and

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

(b)

$$P(A \cap B) = P(A) \cdot P(B|A).$$

(c)

$$P(A \cap B \cap C) = P(A) \cdot P(B|A) \cdot P(C|A \cap B).$$

(d) Two events A and B are said to be *independent* if $P(A \cap B) = P(A) \cdot P(B)$. For example, suppose a coin is tossed twice and each of the 4 outcomes $\{HH, HT, TH, TT\}$ has a probability of $\frac{1}{4}$. If $H_1 = \{HH, HT\}$ is the event of getting heads on the first toss, and $H_2 = \{HH, TH\}$ is the event of getting heads on the second toss, then

$$P(H_1 \cap H_2) = P(HH) = \frac{1}{4} = \frac{2}{4} \cdot \frac{2}{4} = P(H_1) \cdot P(H_2),$$

which shows that H_1 and H_2 are independent events.

(e) Events A and B are *dependent* if they are not independent, and so $P(A \cap B) \neq P(A) \cdot P(B)$. Another way to express this is that $P(B|A) \neq P(B)$. For example, suppose two cards are drawn in succession from a deck of 52, and assume that any outcome (that is any pair of cards counting the order in which they are drawn) has an equally likely probability of $\frac{1}{52 \cdot 51}$. Let A_1 be the event of drawing an ace as the first card, and A_2 be the event of drawing an ace as the second card. Then

$$P(A_1) = \frac{\binom{4}{1}\binom{51}{1}}{52 \cdot 51} = \frac{1}{13}, \quad P(A_2) = \frac{\binom{51}{1}\binom{4}{1}}{52 \cdot 51} = \frac{1}{13}$$

(number of "successful" hands, over total number of 2 card hands counting the order). Since there are $4 \cdot 3 = 12$ was two draw 2 aces, including order, we have

$$P(A_1 \cap A_2) = \frac{12}{52 \cdot 51} = \frac{1}{221} \neq \frac{1}{169} = \frac{1}{13} \cdot \frac{1}{13} = P(A_1) \cdot P(A_2)$$

and so these events are not independent.

(f) If events $B_1, B_2, \ldots B_n$ form a partition of the sample space and $P(B_i) \neq 0$ for each i, and if A is any event, then

$$P(A) = P(B_1) \cdot P(A|B_1) + P(B_2) \cdot P(A|B_2) \cdot \ldots \cdot P(B_n) \cdot P(A|B_n).$$

(g) If events $B_1, B_2, \ldots B_n$ form a partition of the sample space and $P(B_i) \neq 0$ for each i, and if A is any event with $P(A) \neq 0$, then

$$P(B_i|A) = \frac{P(B_i) \cdot P(A|B_i)}{P(B_1) \cdot P(A|B_1) + P(B_2) \cdot P(A|B_2) \cdot \ldots \cdot P(B_n) \cdot P(A|B_n)}.$$

$\square$

2. If A and B are events in a common sample space with $P(A) = 0.3$, $P(B) = 0.35$, $P(A \cap B) = 0.1$.

(a) Find $P(A|B)$ and $P(B|A)$.

(b) Are A and B independent?

Solution. (a)

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.1}{0.35} = \frac{2}{7}$$

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A \cap B)}{P(A)} = \frac{0.1}{0.3} = \frac{1}{3}$$

(b) Since

$$P(A \cap B) = 0.1 \neq 0.105 = (0.3) \cdot (0.35) = P(A) \cdot P(B)$$

we see that events A and B are not independent.

$\square$

3. If A and B are events in a common sample space with $P(A) = 0.4$, $P(B) = 0.5$, $P(A \cup B) = 0.7$.

(a) Find $P(A|B)$.

(b) Are A and B independent?

Solution. (a)

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.4 + 0.5 - 0.7 = 0.2$$

$$\Rightarrow \quad P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.2}{0.5} = \frac{2}{5}$$

(b) Since

$$P(A \cap B) = 0.2 = (0.4) \cdot (0.5) = P(A) \cdot P(B)$$

we see that events A and B are independent.

$\square$

4. Today you go to the pet store and adopt either a cat (C) or a dog (D). Since you like pets so much, you go again tomorrow and adopt either a cat or a dog. The sample space for this experiment can be expressed as $S = \{CC, CD, DC, DD\}$. Assuming that each of these outcomes is equally likely, find the probability that both pets are cats given that

   (a) At least one pet is a cat.

   (b) The first pet adopted is a cat.

   Solution. Let $A =$ (at least one cat) $= \{CC, CD, DC\}$ and $B =$ (first pet adopted is a cat) $= \{CC, CD\}$. Then

   (a)
$$P(\{CC\}|A) = \frac{P(\{CC\} \cap A)}{P(A)} = \frac{P(\{CC\})}{P(A)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}.$$

   (b)
$$P(\{CC\}|B) = \frac{P(\{CC\} \cap B)}{P(B)} = \frac{P(\{CC\})}{P(B)} = \frac{\frac{1}{4}}{\frac{2}{4}} = \frac{1}{2}.$$

□

5. There are 12 ice pops in a box, of which four are cherry flavour. Three ice pops are randomly drawn from the box one after the other. Find the probability that all three will be cherry flavour.

   Solution. Let C_1,C_2 and C_3 be the events that a cherry ice pop is chosen on the first, second and third draws respectively. Note that after one cherry ice pop is drawn, there is 1 fewer to choose and 1 less in total to draw from the next time. Thus the probability that all three will be cherry is given by

$$P(C_1 \cap C_2 \cap C_3) = P(C_1) \cdot P(C_2|C_1) \cdot P(C_3|C_1 \cap C_2) = \frac{4}{12} \cdot \frac{3}{11} \cdot \frac{2}{10} = \frac{1}{55}.$$

□

6. Four cards are dealt from a shuffled deck of 52 cards. Find the probability that all four are hearts.

   Solution. By similar reasoning to question 5, we see that the probability is

$$\frac{4}{52} \cdot \frac{3}{51} \cdot \frac{2}{50} \cdot \frac{1}{49} = \frac{1}{270725} \approx 0.000003694$$

□

7. Find $P(A|B)$ if

   (a) $A \subset B$.

   (b) $B \subset A$.

   (c) A and B are independent.

   (d) A and B are mutually exclusive.

   Solution. (a) If $A \subset B$ then $A \cap B = A$ so

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)}.$$

(b) If $B \subset A$ then $A \cap B = B$ so

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1.$$

(c) If A and B are independent then

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A) \cdot P(B)}{P(B)} = P(A).$$

(d) If A and B are mutually exclusive then

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(\emptyset)}{P(B)} = \frac{0}{P(B)} = 0.$$

□

8. There are 7 red tickets and 3 blue tickets in a box. Three tickets are drawn from the box, one after another, at random. Find the probability that the first two tickets are red and third ticket is blue.

Solution. By similar reasoning to question 5, we see that the probability is

$$\frac{7}{10} \cdot \frac{6}{9} \cdot \frac{3}{8} = \frac{7}{40}$$

□

9. Aras and John are playing darts. Aras can hit the bullseye with probability $\frac{1}{3}$ and John can hit the bullseye with probability $\frac{1}{5}$. Each takes a single shot at the dart board. Assume that the events of Aras hitting the bullseye is independent of John hitting the bullseye. Find the probability that

(a) Aras does not hit the bullseye.
(b) Both hit the bullseye.
(c) One of them hits the bullseye.
(d) Neither of them hits the bullseye.

Solution. Let A and J be the events that Aras and John hit the bullseye respectively.

(a)

$$P(A') = 1 - P(A) = 1 - \frac{1}{3} = \frac{2}{3}$$

(b) Since events are independent,

$$P(A \cap J) = P(A) \cdot P(J) = \frac{1}{3} \cdot \frac{1}{5} = \frac{1}{15}$$

(c)

$$P(A \cup J) = P(A) + P(J) - P(A \cap J) = P(A) + P(J) - P(A) \cdot P(J) = \frac{1}{3} + \frac{1}{5} - \frac{1}{15} = \frac{7}{15}$$

(d) By De Morgan's law,

$$P(A' \cap J') = P((A \cup J)') = 1 - P(A \cup J) = 1 - \frac{7}{15} = \frac{8}{15}$$

□

10. A box contains two coins. One of the coins is a regular fair coin and the other coin has heads on both sides. One of the coins is chosen at random and then tossed twice.

    (a) If heads appears twice, find the probability that the "two-headed" coin was the one chosen.

    (b) If tails appears twice, find the probability that the "two-headed" coin was the one chosen.

    Solution. (a) Let C_F be the event that the fair coin is chosen and C_{2H} the event that the "two-headed" coin was the one chosen. Each of these has a probability of $\frac{1}{2}$. Then

    $$P(C_{2H}|\{HH\}) = \frac{P(C_{2H} \cap \{HH\})}{P(\{HH\})}$$

    By the rule of total probability

    $$P(\{HH\}) = P(C_F) \cdot P(\{HH\}|C_F) + P(C_{2H}) \cdot P(\{HH\}|C_{2H}) = \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot 1 = \frac{5}{8}.$$

    Since both tosses must be heads if C_{2H} occurs, $C_{2H} \cap \{HH\} = C_{2H}$ and so $P(C_{2H} \cap \{HH\}) = \frac{1}{2}$. Thus

    $$P(C_{2H}|\{HH\}) = \frac{\frac{1}{2}}{\frac{5}{8}} = \frac{4}{5}.$$

    (b) Clearly if tails appears, then the "two-headed" coin could not have been chosen. Indeed,

    $$P(C_{2H}|\{TT\}) = \frac{P(C_{2H} \cap \{TT\})}{P(\{TT\})} = \frac{P(\emptyset)}{P(\{TT\})} = 0.$$

    □

11. There are three candidates (A, B and C) running in a mayoral election in a certain city. It is known that 40 percent of the city population support candidate A, 35 percent support B, and 25 percent support C. On election day 45 percent of A supporters turn out to vote, 40 percent of B supporters voted, and 60 percent of C supporters voted. A citizen is selected at random.

    (a) What is the probability that the selected person has voted?

    (b) If the person selected did vote, what is the probability that the person supports

        i. Candidate A?

        ii. Candidate B?

        iii. Candidate C?

    (c) Who won the election?

    Solution. Let A, B and C denote the events that the person chosen supports candidate A, B and C respectively. Let V be the event that the person selected has voted.

    (a) By the rule of total probability,

    $$P(V) = P(A) \cdot P(V|A) + P(B) \cdot P(V|B) + P(C) \cdot P(V|C)$$
    $$= (0.4)(0.45) + (0.35)(0.4) + (0.25)(0.6) = 0.47$$

    (b) i.

    $$P(A|V) = \frac{P(A \cap V)}{P(V)} = \frac{P(A) \cdot P(V|A)}{P(V)} = \frac{(0.4)(0.45)}{0.47} = \frac{18}{47}$$

    ii.

    $$P(B|V) = \frac{P(B) \cdot P(V|B)}{P(V)} = \frac{(0.35)(0.4)}{0.47} = \frac{14}{47}$$

iii.

$$P(C|V) = \frac{P(C) \cdot P(V|C)}{P(V)} = \frac{(0.25)(0.6)}{0.47} = \frac{15}{47}$$

(c) Candidate A had $(0.4)(0.45) = 0.18$ of the population vote for them, Candidate B had $(0.35)(0.4) = 0.14$ and Candidate C had $(0.25)(0.6) = 0.15$, thus Candidate A had the greatest number of votes.

□

12. A box contains 10 coins where 5 coins are "two-headed", 3 coins are "two-tailed" and 2 are regular fair coins. A coin is chosen at random and tossed once.

    (a) Find the probability that heads appears on the toss.

    (b) If heads appears, find the probability that it came from the fair coin.

    Solution. Let C_{2H}, C_{2T} and C_F be the event that the two-headed, two-tailed and fair coins are chosen respectively.

    (a) By the rule of total probability

$$\begin{aligned} P(H) &= P(C_{2H}) \cdot P(H|C_{2H}) + P(C_{2T}) \cdot P(H|C_{2T}) + P(C_F) \cdot P(H|C_F) \\ &= \frac{5}{10} \cdot 1 + \frac{3}{10} \cdot 0 + \frac{2}{10} \cdot \frac{1}{2} = \frac{3}{5}. \end{aligned}$$

    (b)

$$P(C_F|H) = \frac{P(C_F \cap H)}{P(H)} = \frac{P(C_F) \cdot P(H|C_F)}{P(H)} = \frac{\frac{2}{10} \cdot \frac{1}{2}}{\frac{3}{5}} = \frac{1}{6}.$$

□