

### 4.1.1 The Expected Value of a Continuous Random Variable

If  $X$  is a continuous random variable and  $f(x)$  is its probability ~~density~~ <sup>density</sup> function, then the expected value of  $X$  is defined as

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x) dx.$$

discrete case  
 $E(X) = \sum_x x \cdot f(x)$

The integral must exist in order for the expected value to have meaning.

**Example 4.1.4** A contractor's profit on a construction job can be considered as a continuous random variable having probability density

$$f(x) = \begin{cases} \frac{1}{18}(x+1) & \text{for } -1 < x < 5 \\ 0 & \text{otherwise} \end{cases}$$

(where the units are in \$1,000). What is her expected profit?

The expected value of  $X$ , where  $X$  denotes the contractor's profit in \$1,000's, is

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x \cdot f(x) dx = \int_{-1}^5 x \cdot \frac{1}{18}(x+1) dx \\ &= \frac{1}{18} \int_{-1}^5 x(x+1) dx = \frac{1}{18} \int_{-1}^5 (x^2 + x) dx = \frac{1}{18} \left( \frac{x^3}{3} + \frac{x^2}{2} \right) \Big|_{x=-1}^{x=5} \\ &= \frac{1}{18} \left[ \left( \frac{125}{3} + \frac{25}{2} \right) - \left( \frac{-1}{3} + \frac{1}{2} \right) \right] = \frac{1}{18} \left[ \left( \frac{250}{6} + \frac{75}{6} \right) - \left( \frac{-2}{6} + \frac{3}{6} \right) \right] \\ &= \frac{1}{18} \cdot \left( \frac{325}{6} - \frac{1}{6} \right) = \frac{1}{18} \cdot \frac{324}{6} = 3 \end{aligned}$$

So the exp. profit is \$3000.

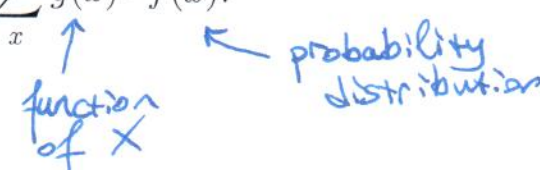
### 4.1.2 Expectation of a Function of a Random Variable

We are not limited to considering a random variable by itself.

We can as well consider a function  $g(X)$  of a random variable, and evaluate its expected value.

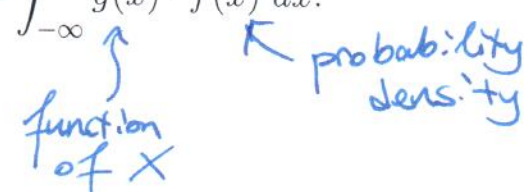
**Theorem 4.1.5** *If  $X$  is a discrete random variable with probability distribution  $f(x)$ , the expected value of  $g(X)$  is given by*

$$E(g(X)) = \sum_x g(x) \cdot f(x).$$



*If  $X$  is a continuous random variable with probability density function  $f(x)$ , the expected value of  $g(X)$  is given by*

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) \cdot f(x) dx.$$



**Example 4.1.6** Let  $X$  be a random variable that takes the values  $-1, 0, 1$ , and has probability distribution given by

$$f(-1) = 0.2, \quad f(0) = 0.5, \quad f(1) = 0.3.$$

Find  $E(X^2)$ .

$$\begin{aligned} E(X) &= -1 \cdot 0.2 \\ &\quad + 0 \cdot 0.5 \\ &\quad + 1 \cdot 0.3 \\ &= -0.2 + 0 + 0.3 \\ &= 0.1 \end{aligned}$$

range of  $X$   
 $= \{-1, 0, 1\}$

range of  $Y$   
 $= \{0, 1\}$

Before using the theorem, let's find  $E(X^2)$  directly. We view  $X^2$  as a new random variable which we'll call  $Y$ .

$$Y = X^2$$

The range of  $Y$  is  $\{0, 1\}$  and it has probability distribution

$$P(Y = 0) = P(X = 0) = f(0) = \underline{0.5}$$

$$P(Y = 1) = P(X = 1) + P(X = -1) = \underset{0.3}{f(1)} + \underset{0.2}{f(-1)} = \underline{0.5}$$

Then,

$$E(X^2) = E(Y) = 0 \cdot P(Y = 0) + 1 \cdot P(Y = 1) = 0 \cdot (0.5) + 1 \cdot (0.5) = \underline{0.5}.$$

$X$	$P(X=x)$	$X^2 = Y$
-1	0.2	$(-1)^2 = 1$
0	0.5	$0^2 = 0$
1	0.3	$1^2 = 1$

~~$P(Y=1)$~~

$$P(Y=1) = 0.2 + 0.3 = 0.5$$

$$P(Y=0) = 0.5$$

We can find the same result using the theorem as well.

Let  $g(X) = X^2$ , and let  $x_1 = -1, x_2 = 0, x_3 = 1$ . Then according to the theorem

$$E(g(X)) = \sum_{i=1}^3 g(x_i) f(x_i)$$

*(The above equation is circled in blue)*

$$\begin{aligned} f(-1) &= 0.2 \\ f(0) &= 0.5 \\ f(1) &= 0.3 \end{aligned}$$

$$= g(x_1)f(x_1) + g(x_2)f(x_2) + g(x_3)f(x_3)$$

$$= g(-1)f(-1) + g(0)f(0) + g(1)f(1)$$

$$= (-1)^2 \cdot (0.2) + (0)^2 \cdot (0.5) + (1)^2 \cdot (0.3)$$

*(Blue arrows point from the g(x\_i) terms in the previous line to the squared terms here)*

$$= 1 \cdot 0.2 + 0 \cdot 0.5 + 1 \cdot 0.3 = 0.2 + 0.3 = 0.5$$

*(The above line is written in blue ink)*

Note that:  $E(X^2) = 0.5 \neq (E(X))^2 = 0.01$

*(The above statement is circled in blue)*

**Example 4.1.7** Suppose  $X$  has probability density

$$f(x) = \begin{cases} e^{-x} & \text{if } x > 0 \quad (0, \infty) \\ 0 & \text{otherwise} \end{cases}$$

Find the expected value of  $g(X) = e^{3X/4}$ .

By our theorem

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) \cdot f(x) dx$$

$\overset{3x/4}{e}$   
 $\uparrow$   
 prob. density

$$= \int_0^{\infty} e^{3x/4} \cdot e^{-x} dx = \int_0^{\infty} e^{(3x/4)-x} dx$$

$$= \int_0^{\infty} e^{-\frac{x}{4}} dx = \lim_{c \rightarrow \infty} \int_0^c e^{-\frac{x}{4}} dx$$

$$= \lim_{c \rightarrow \infty} \left( -4 \cdot e^{-\frac{1}{4}x} \right) \Big|_{x=0}^{x=c} = \lim_{c \rightarrow \infty} \left( -4e^{-\frac{1}{4} \cdot c} - (-4e^0) \right)$$

$$= \lim_{c \rightarrow \infty} \left( -4e^{-\frac{c}{4}} + 4 \right) = 4$$

$\nearrow 0$



### 4.1.3 Properties of Expected Value

A useful special case of the theorem is:

**Theorem 4.1.8** If  $a$  and  $b$  are constants, then

$$E(aX + b) = aE(X) + b.$$

"linearity of the expected value"

In particular if  $a = 0$ , then  $E(b) = b$  and if  $b = 0$  then  $E(aX) = aE(X)$ .

$$E(0 \cdot X + b) = 0 \cdot E(X) + b = b$$

$$E(aX + 0) = aE(X) + 0 = aE(X)$$

To prove this (for the discrete case) let  $g(X) = aX + b$ . Then

$$E(aX + b) = E(g(x))$$

$$= \sum_x g(x) \cdot f(x)$$

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$$= \sum_x (ax + b) \cdot f(x)$$

$$= \sum_x (ax \cdot f(x) + b \cdot f(x))$$

$$= \sum_x ax \cdot f(x) + \sum_x b \cdot f(x)$$

$$= a \sum_x x \cdot f(x) + b \sum_x f(x)$$

$$= aE(X) + b \quad \left( \text{since } \sum_x f(x) = 1 \right).$$

prob. distribution

Theorem :  $E(aX+b) = aE(X)+b$

**Example 4.1.9** Returning to our slot machine example, we chose our random variable  $X$  to be the expected payout, and not the expected profit. Then we calculated the expected payout.

we found that  $E(X) = 0.23$

Suppose this time that we want the expected profit.

calculate  $E(Y)$  without using the theorem

If  $Y$  is our expected profit then  $Y$  has range  $\{-0.25, 19.75, 99.75, 499.75\}$

So then  $P(Y = y) = P(X = y + 0.25)$ , and  $X: 0 \quad 20 \quad 100 \quad 500$   
cost is 0.25

$$E(Y) = (-0.25) \cdot P(X = 0) + (19.75) \cdot P(X = 20) + (99.75) \cdot P(X = 100) + (499.75) \cdot P(X = 500)$$

$$= -0.25 \cdot \cancel{P(X=0)} \left( 1 - \frac{27}{8000} - \frac{8}{8000} - \frac{1}{8000} \right) + 19.75 \cdot \frac{27}{8000} + 99.75 \cdot \frac{8}{8000} + 499.75 \cdot \frac{1}{8000} = -0.02$$

On the other hand since  $Y = g(X) = X - 0.25$ , we can compute

$$E(Y) = E(X - 0.25) = E(X) - 0.25$$

using the theorem (which, in this case is nicer calculation).