MATH1550
Exercise Set 10 - Solutions

- Bivariate Moments
- Covariance
- Conditional Expectations

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1. Use properties of expected value to prove that $\text{cov}(X, Y)$ (or σ_{XY}) is given by

$$\text{cov}(X, Y) = E(XY) - E(X)E(Y).$$

Solution.

$$\text{cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)]$$

$$= E(XY - \mu_Y X - \mu_X Y + \mu_X \mu_Y)$$

$$= E(XY) - \mu_Y E(X) - \mu_X E(Y) + E(\mu_X \mu_Y)$$

$$= E(XY) - \mu_Y \mu_X - \mu_X \mu_Y + \mu_X \mu_Y$$

$$= E(XY) - \mu_Y \mu_X = E(XY) - E(X)E(Y)$$

□

2. Show, for the case of joint discrete random variables X and Y, that if X and Y are independent then

$$E(XY) = E(X)E(X).$$

(*find an example in the notes/exercises where the converse is not true.*)

Solution. Let the joint probability distribution for the random variables X and Y be $f(x, y)$; let $g(x)$ and $h(y)$ denote the marginal distributions of X and Y, respectively. Then,

$E(XY) = \sum_x \sum_y xy \cdot f(x, y)$

$= \sum_x \sum_y xy \cdot (g(x) \cdot h(y)) \qquad (f(x, y) = g(x) \cdot h(y)$ because X and Y are independent)

$= \sum_x x \cdot g(x) \cdot \sum_y y \cdot h(y) = E(X)E(Y).$

See the Chapter 4 lecture notes (near the end) for an example where $E(XY) = E(X)E(Y)$, but X and Y are not independent.

□

3. Let X and Y be discrete random variables with joint probability distribution given by the following table:

		x		
		-1	0	1
y	0	0	1/6	1/12
	1	1/4	0	1/2

(a) Find the covariance of X and Y.
(b) Determine whether X and Y are independent (justify your answer).

Solution. (a)

$$\mu_X = E(X) = (-1)\cdot(0+\frac{1}{4}) + 0\cdot(\frac{1}{6}+0) + 1\cdot(\frac{1}{12}+\frac{1}{2}) = -\frac{1}{4}+0+\frac{7}{12} = \frac{4}{12} = \frac{1}{3}$$

$$\mu_Y = E(Y) = 0\cdot(0+\frac{1}{6}+\frac{1}{12}) + 1\cdot(\frac{1}{4}+0+\frac{1}{2}) = \frac{1}{4}+\frac{1}{2} = \frac{3}{4}$$

We also need to find $E(XY)$.

$$E(XY) = (-1)\cdot 0\cdot 0 + (-1)\cdot 1\cdot\frac{1}{4} + 0\cdot 0\cdot\frac{1}{6} + 0\cdot 1\cdot 0 + 1\cdot 0\cdot\frac{1}{12} + 1\cdot 1\cdot\frac{1}{2} = 0 - \frac{1}{4} + 0 + 0 + 0 + \frac{1}{2} = \frac{1}{4}$$

Now, we can evaluate the covariance using the formula $\sigma_{XY} = E(XY) - \mu_X\mu_Y$. We get, $\sigma_{XY} = \frac{1}{4} - \frac{1}{3}\cdot\frac{3}{4} = 0$.

(b) X and Y are not independent because, for example $P(X=0) = \frac{1}{6}$ and $P(Y=0) = \frac{3}{12}$, but $P(X=0, Y=0) = \frac{1}{6} \neq \frac{1}{6}\cdot\frac{3}{12}$.

□

4. Let X and Y be jointly continuous random variables with joint probability density given by

$$f(x,y) = \begin{cases} \frac{3}{5}x(y+x) & \text{for } 0<x<1, 0<y<2 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find μ_X and μ_Y.
(b) Find the covariance of X and Y. Are X and Y independent?

Solution. (a)

$$\mu_X = E(X) = \int_0^2\int_0^1 x\cdot\left(\frac{3}{5}x(y+x)\right)\,dx\,dy = \int_0^2\int_0^1 \frac{3x^3}{5} + \frac{3x^2y}{5}\,dx\,dy$$
$$= \int_0^2 \frac{3x^4}{20} + \frac{x^3y}{5}\Big|_0^1\,dy = \int_0^2 \frac{3}{20} + \frac{y}{5}\,dy = \frac{3y}{20} + \frac{y^2}{10}\Big|_0^2 = \frac{7}{10}.$$

$$\mu_Y = E(Y) = \int_0^2\int_0^1 y\cdot\left(\frac{3}{5}x(y+x)\right)\,dx\,dy = \int_0^2\int_0^1 \frac{3x^2y}{5} + \frac{3xy^2}{5}\,dx\,dy$$
$$= \int_0^2 \frac{x^3y}{5} + \frac{3x^2y^2}{10}\Big|_0^1\,dy = \int_0^2 \frac{y}{5} + \frac{3y^2}{10}\,dy = \frac{y^2}{10} + \frac{y^3}{10}\Big|_0^2 = \frac{6}{5}.$$

(b)

$$
\begin{aligned}
E(XY) &= \int_0^2 \int_0^1 xy \cdot \left(\frac{3}{5}x(y+x)\right)\, dx\, dy = \int_0^2 \int_0^1 \frac{3x^3y}{5} + \frac{3x^2y^2}{5}\, dx\, dy \\
&= \int_0^2 \frac{3x^4y}{20} + \frac{x^3y^2}{5}\bigg|_0^1 dy = \int_0^2 \frac{3y}{20} + \frac{y^2}{5}\, dy = \frac{3y^2}{40} + \frac{y^3}{15}\bigg|_0^2 = \frac{5}{6}
\end{aligned}
$$

$$
\operatorname{cov}(X,Y) = E(XY) - \mu_X\mu_Y = -\frac{1}{150}
$$

Since $\operatorname{cov}(X,Y) \neq 0$, it follows that X and Y are not independent.

□

5. Let X and Y be continuous random variables with joint probability density

$$
f(x,y) = \begin{cases} \frac{1}{8}(x+y) & \text{for } 0 \le x \le 2, 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}
$$

(a) Find the marginal distribution for X.

(b) Find the covariance for X and Y.

Solution. (a) For $0 \le x \le 2$ we have

$$
\begin{aligned}
g(x) &= \int_{-\infty}^{\infty} f(x,y)\, dy \\
&= \int_0^2 \frac{1}{8}(x+y)\, dy \\
&= \frac{1}{8}\left(xy + \frac{y^2}{2}\bigg|_0^2\right) \\
&= \frac{x+1}{4},
\end{aligned}
$$

and $g(x) = 0$ otherwise.

(b)

$$
\begin{aligned}
\mu_X = E(X) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x \cdot f(x,y)\, dx\, dy \\
&= \int_0^2\int_0^2 x \cdot \frac{1}{8}(x+y)\, dx\, dy \\
&= \frac{1}{8}\int_0^2 \left(\frac{x^3}{3} + \frac{x^2y}{2}\bigg|_0^2\right) dy \\
&= \frac{1}{8}\int_0^2 \left(\frac{8}{3} + 2y\right) dy \\
&= \frac{1}{8}\left(\frac{8y}{3} + y^2\bigg|_0^2\right) \\
&= \frac{7}{6}
\end{aligned}
$$

$$
\begin{aligned}
\mu_Y = E(Y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y \cdot f(x,y)\;dx\;dy \\
&= \int_0^2\int_0^2 y \cdot \frac{1}{8}(x+y)\;dx\;dy \\
&= \frac{1}{8}\int_0^2 \left(\frac{x^2 y}{2} + xy^2 \Big|_0^2 \right)\;dy \\
&= \frac{1}{8}\int_0^2 \left(2y + 2y^2\right)\;dy \\
&= \frac{1}{8}\left(y^2 + \frac{2y^3}{3} \Big|_0^2 \right) \\
&= \frac{7}{6}
\end{aligned}
$$

$$
\begin{aligned}
E(XY) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \cdot f(x,y)\;dx\;dy \\
&= \int_0^2\int_0^2 xy \cdot \frac{1}{8}(x+y)\;dx\;dy \\
&= \frac{1}{8}\int_0^2 \left(\frac{x^3 y}{3} + \frac{x^2 y^2}{2} \Big|_0^2 \right)\;dy \\
&= \frac{1}{8}\int_0^2 \left(\frac{8y}{3} + 2y^2\right)\;dy \\
&= \frac{1}{8}\left(\frac{4y^2}{3} + \frac{2y^3}{3} \Big|_0^2 \right) \\
&= \frac{4}{3}
\end{aligned}
$$

Therefore

$$
\operatorname{cov}(X,Y) = E(XY) - \mu_X\mu_Y = \frac{4}{3} - \left(\frac{7}{6}\right)\left(\frac{7}{6}\right) = -\frac{1}{36} \approx -0.02778.
$$

$\square$

6. Let X and Y be continuous random variables with joint probability density

$$
f(x,y) = \begin{cases} 2x & \text{for } 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}
$$

(a) Find the marginal distribution for Y.

(b) Find the covariance for X and Y.

Solution. (a) The marginal distribution for Y is:

$$\begin{aligned}
h(y) &= \int_{-\infty}^{\infty} f(x,y)\ dx \\
&= \int_0^1 2x\ dx \\
&= x^2\Big|_0^1 \\
&= 1
\end{aligned}$$

(b) The mean of X is:

$$\begin{aligned}
\mu_X &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xf(x,y)\ dx\ dy \\
&= \int_0^1\int_0^1 2x^2\ dx\ dy \\
&= \int_0^1 \frac{2x^3}{3}\Big|_0^1\ dy \\
&= \int_0^1 \frac{2}{3}\ dy \\
&= \frac{2}{3}y\Big|_0^1 \\
&= \frac{2}{3}
\end{aligned}$$

The mean of Y is:

$$\begin{aligned}
\mu_Y &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yf(x,y)\ dx\ dy \\
&= \int_0^1\int_0^1 2xy\ dx\ dy \\
&= \int_0^1 x^2y\Big|_0^1\ dy \\
&= \int_0^1 y\ dy \\
&= \frac{y^2}{2}\Big|_0^1 \\
&= \frac{1}{2}
\end{aligned}$$

The first product moment about the origin is:

$$\begin{aligned}
E(XY) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)\,dx\,dy \\
&= \int_0^1\int_0^1 2x^2y\,dx\,dy \\
&= \int_0^1 \left.\frac{2x^3y}{3}\right|_0^1 dy \\
&= \int_0^1 \frac{2y}{3}\,dy \\
&= \left.\frac{y^2}{3}\right|_0^1 \\
&= \frac{1}{3}
\end{aligned}$$

The covariance is:

$$\sigma_{XY} = E(XY) - \mu_X\mu_Y = \frac{1}{3} - \left(\frac{2}{3}\right)\left(\frac{1}{2}\right) = 0$$

□

7. The joint distribution, $f(x,y)$, for discrete random variables X and Y is given below. Find the covariance of X and Y.

y \ x	1	2	3	4	5	6
2	$\frac{1}{36}$					
3		$\frac{2}{36}$				
4		$\frac{1}{36}$	$\frac{2}{36}$			
5			$\frac{2}{36}$	$\frac{2}{36}$		
6			$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	
7				$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$
8				$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$
9					$\frac{2}{36}$	$\frac{2}{36}$
10					$\frac{1}{36}$	$\frac{2}{36}$
11						$\frac{2}{36}$
12						$\frac{1}{36}$

Solution. We will start computing the column sums and row sums to determine the marginal distributions.

$y \backslash x$	1	2	3	4	5	6	
2	$\frac{1}{36}$						$\frac{1}{36}$
3		$\frac{2}{36}$					$\frac{2}{36}$
4		$\frac{1}{36}$	$\frac{2}{36}$				$\frac{3}{36}$
5			$\frac{2}{36}$	$\frac{2}{36}$			$\frac{4}{36}$
6			$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$		$\frac{5}{36}$
7				$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{6}{36}$
8				$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{5}{36}$
9					$\frac{2}{36}$	$\frac{2}{36}$	$\frac{4}{36}$
10					$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$
11						$\frac{2}{36}$	$\frac{2}{36}$
12						$\frac{1}{36}$	$\frac{1}{36}$
	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$	

Then

$$\mu_X = \sum_{x=1}^{6} xg(x) = (1)\frac{1}{36} + (2)\frac{3}{36} + (3)\frac{5}{36} + (4)\frac{7}{36} + (5)\frac{9}{36} + (6)\frac{11}{36} = \frac{161}{36}$$

$$\begin{aligned}\mu_Y = \sum_{y=2}^{12} yh(y) &= (2)\frac{1}{36} + (3)\frac{2}{36} + (4)\frac{3}{36} + (5)\frac{4}{36} + (6)\frac{5}{36} + (7)\frac{6}{36} \\ &+ (8)\frac{5}{36} + (9)\frac{4}{36} + (10)\frac{3}{36} + (11)\frac{2}{36} + (12)\frac{1}{36} = 7\end{aligned}$$

$$\begin{aligned}E(XY) &= \sum_{x=1}^{6}\sum_{y=2}^{12} xyf(x,y) \\ &= (2)\frac{1}{36} + (6)\frac{2}{36} + (8)\frac{1}{36} + (12)\frac{2}{36} + (15)\frac{2}{36} + (18)\frac{1}{36} + (20)\frac{2}{36} + (24)\frac{2}{36} + (28)\frac{2}{36} \\ &\quad + (32)\frac{1}{36} + (30)\frac{2}{36} + (35)\frac{2}{36} + (40)\frac{2}{36} + (45)\frac{2}{36} + (50)\frac{1}{36} + (42)\frac{2}{36} + (48)\frac{2}{36} \\ &\quad + (54)\frac{2}{36} + (60)\frac{2}{36} + (66)\frac{2}{36} + (72)\frac{1}{36} \\ &= \frac{1232}{36}\end{aligned}$$

so

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{1232}{36} - \left(\frac{161}{36}\right)(7) = \frac{35}{12}$$

□

8. Let X and Y have joint density function given below. Find $E(X)$.

$$f(x,y) = \begin{cases} \frac{x+y}{3} & \text{for } 0 < x < 2, 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Solution.

$$\begin{aligned} E(X) &= \int_0^1 \int_0^2 x\left(\frac{x+y}{3}\right) \, dx \, dy \\ &= \int_0^1 \frac{x^3}{9} + \frac{x^2 y}{6}\Big|_0^2 \, dy \\ &= \int_0^1 \frac{8}{9} + \frac{2y}{3} \, dy \\ &= \frac{8y}{9} + \frac{y^2}{3}\Big|_0^1 \\ &= \frac{11}{9} \end{aligned}$$

$\square$

9. Let X and Y have joint density function given below. Given that $E(X) = \frac{5}{6}$ and $E(Y) = \frac{17}{6}$, find $\operatorname{Cov}(X,Y)$.

$$f(x,y) = \begin{cases} \frac{6-x-y}{8} & \text{for } 0 < x < 2, 2 < y < 4 \\ 0 & \text{elsewhere} \end{cases}$$

Solution. We have

$$\begin{aligned} E(XY) &= \int_2^4 \int_0^2 xy\left(\frac{6-x-y}{8}\right) \, dx \, dy \\ &= \int_2^4 \frac{3x^2 y}{8} - \frac{x^3 y}{24} - \frac{x^2 y^2}{16}\Big|_0^2 \, dy \\ &= \int_2^4 \frac{3y}{2} - \frac{y}{3} - \frac{y^2}{4} \, dy \\ &= \frac{3y^2}{4} - \frac{y^2}{6} - \frac{y^3}{12}\Big|_2^4 \\ &= \left(12 - \frac{16}{6} - \frac{64}{12}\right) - \left(3 - \frac{4}{6} - \frac{8}{12}\right) \\ &= \frac{7}{3} \end{aligned}$$

Thus

$$\operatorname{Cov}(X,Y) = \frac{7}{3} - \left(\frac{5}{6}\right)\left(\frac{17}{6}\right) = -\frac{1}{36}.$$

$\square$

10. Let X and Y be joint continuous random variables with joint density

$$f(x,y) = \begin{cases} \frac{2}{3}(x+2y) & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the conditional expected value of X given $Y = \frac{1}{2}$, i.e. find $E(X|\frac{1}{2})$.

Solution.
$$E\left(X|Y=\frac{1}{2}\right)=\int_x x\cdot f\left(x|\frac{1}{2}\right)\,dx=\int_0^1 x\cdot f\left(x|\frac{1}{2}\right)\,dx$$
where $f\left(x|\frac{1}{2}\right)=\dfrac{f\left(x,\frac{1}{2}\right)}{h\left(\frac{1}{2}\right)}$.

$$h(y)=\int_0^1\frac{2}{3}(x+2y)dx=\frac{1}{3}(1+4y)$$

Then,
$$h\left(\frac{1}{2}\right)=\frac{1}{3}(1+4y)\Big|_{y=\frac{1}{2}}=1,$$
and
$$f\left(x|\frac{1}{2}\right)=\frac{f\left(x,\frac{1}{2}\right)}{1}=\frac{2}{3}\left(x+2\cdot\frac{1}{2}\right)=\frac{2}{3}(x+1),$$
for $0<x<1$, and 0 otherwise.

Then,
$$E\left(X|Y=\frac{1}{2}\right)=\int_0^1 x\cdot\frac{2}{3}(x+1)\,dx=\frac{2}{3}\int_0^1 x^2+x\;dx=\frac{2}{3}\left(\frac{x^3}{3}+\frac{x^2}{2}\right)\Big|_0^1=\frac{2}{3}\cdot\left(\frac{1}{3}+\frac{1}{2}\right)=\frac{5}{9}.$$

□

11. Let X be the amount a salesperson spends on gas in a day, and Y be the amount of money for which they are reimbursed. The joint density of X and Y is
$$f(x,y)=\begin{cases}\frac{1}{25}\left(\frac{20-x}{x}\right) & \text{for } 10<x<20,\ \frac{x}{2}<y<x\\ 0 & \text{otherwise}\end{cases}$$

(gives the probability (density) that they will be reimbursed y dollars after spending x dollars)

Find, $f(y|x)$, the conditional probability of Y given $X=x$ and use it to find the probability of being reimbursed at least \$8 given that \$12 was spent. What is the expected reimbursement given that \$12 was spent?

Solution. Let $g(x)$ be the marginal density for X. Then
$$\begin{aligned}g(x)=\int_{-\infty}^{\infty}f(x,y)\,dy=\int_{\frac{x}{2}}^{x}\frac{1}{25}\left(\frac{20-x}{x}\right)\,dy&=\frac{1}{25}\left(\frac{20-x}{x}\right)y\Big|_{\frac{x}{2}}^{x}\\&=\frac{1}{25}\left(\frac{20-x}{x}\right)\left(x-\frac{x}{2}\right)=\left(\frac{20-x}{25x}\right)\left(\frac{x}{2}\right)=\frac{20-x}{50}\end{aligned}$$

Then, for $10<x<20$, $\frac{x}{2}<y<x$
$$f(y|x)=\frac{f(x,y)}{g(x)}=\frac{\frac{1}{25}\left(\frac{20-x}{x}\right)}{\left(\frac{20-x}{50}\right)}=\left(\frac{20-x}{25x}\right)\left(\frac{50}{20-x}\right)=\frac{2}{x}$$
and $f(y|x)=0$ otherwise.

Setting $x = 12$ we have $f(y|12) = \frac{1}{6}$ for $\frac{12}{2} < y < 12$ and $f(y|12) = 0$ otherwise. Then

$$P(Y \geq 8|X = 12) = \int_8^{12} \frac{1}{6}\, dy = \left.\frac{y}{6}\right|_8^{12} = \frac{2}{3}.$$

The expected reimbursement given that x dollars were spent is

$$E(Y|x) = \int_{-\infty}^{\infty} y \cdot f(y|x)\, dy.$$

In the case that $x = 12$ we have

$$E(Y|12) = \int_6^{12} \frac{y}{6}\, dy = \left.\frac{y^2}{12}\right|_6^{12} = 9$$

□

12. Let X and Y have joint density function given below. Find $E(Y|X = 1)$. Hint, the marginal density for X is $g(x) = \frac{3-x}{4}$ for $0 < x < 2$ and is 0 elsewhere.

$$f(x,y) = \begin{cases} \frac{6-x-y}{8} & \text{for } 0 < x < 2, 2 < y < 4 \\ 0 & \text{elsewhere} \end{cases}$$

Solution. The conditional density for Y given $X = x$ is

$$f(y|x) = \frac{f(x,y)}{g(x)} = \frac{\frac{6-x-y}{8}}{\frac{3-x}{4}} = \frac{6-x-y}{3-x}$$

for $0 < x < 2, 2 < y < 4$ and is 0 elsewhere. Thus

$$\begin{aligned} E(Y|1) &= \int_2^4 y f(y|1)\, dy \\ &= \int_2^4 y \left(\frac{5-y}{2}\right)\, dy \\ &= \left.\frac{5y^2}{4} - \frac{y^3}{6}\right|_2^4 \\ &= \left(20 - \frac{64}{6}\right) - \left(5 - \frac{8}{6}\right) \\ &= \frac{17}{3} \end{aligned}$$

□