MATH1550
Exercise Set 3 - Solutions

- Conditional Probability
- Independent Events
- Rule of Total Probability

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1. Let A, B and C be events from a common sample space such that:

   $P(A) = 0.72, \quad P(B) = 0.56, \quad P(C) = 0.25, \quad P(A \cap B) = 0.42, \quad P(A \cap C) = 0.18, \quad P(B \cap C) = 0$

   Compute the following probabilities.

   (a) $P(A \cup B)$
   (b) $P(B \cup C)$
   (c) $P(B|A)$
   (d) $P(A \cup B \cup C)$

   Solution. (a)
   $$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.72 + 0.56 - 0.42 = 0.86$$

   (b)
   $$P(B \cup C) = P(B) + P(C) - P(B \cap C) = 0.56 + 0.25 - 0 = 0.81$$

   (c)
   $$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{0.42}{0.72} \approx 0.5833$$

   (d) Since $A \cap B \cap C \subset B \cap C$ we must have $P(A \cap B \cap C) \leq P(B \cap C)$, and hence $P(A \cap B \cap C) = 0$. Thus

   $$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) = 0.93$$

   □

2. A pair of regular, six-sided dice are tossed. The sample space for this experiment is the following set of 36 ordered pairs
   $$S = \{(d_1, d_2) | d_1, d_2 \in \{1, 2, 3, 3, 5, 6\}\}$$
   where each outcome is equally likely.

   (a) List the elements for the following events:
   - $A = \{$The sum of the dice is 6$\}$
   - $B = \{$The first die shows 2$\}$
   - $C = \{$2 appears on at least one die$\}$
   - $D = \{$6 appears on at least one die$\}$

   (b) List the elements in the following events:
   - $A \cap B$
   - $A \cap C$
   - $A \cap D$
   - $B \cap C$

- $B \cap D$
- $C \cap D$

(c) Compute the following conditional probabilities:

- The probability that the sum of the dice is 6, given that the first die shows 2; i.e. $P(A|B)$.
- The probability that the sum of the dice is 6, given that 2 appears on at least one die.
- Find $P(C|A)$ and describe this event in words.
- Find $P(A|D)$ and describe this event in words.
- Find $P(B|C)$ and describe this event in words.
- The probability that the first die shows 2 given that 6 appears on at least one die.
- Find $P(C|D)$ and describe this event in words.

Solution. (a)

- $A = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$
- $B = \{(2,1),(2,2),(2,3),(2,4),(2,5),(2,6)\}$
- $C = \{(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),(1,2),(3,2),(4,2),(5,2),(6,2)\}$
- $D = \{(6,1),(6,2),(6,3),(6,4),(6,5),(6,6),(1,6),(2,6),(3,6),(4,6),(5,6)\}$

(b)

- $A \cap B = \{(2,4)\}$
- $A \cap C = \{(2,4),(4,2)\}$
- $A \cap D = \emptyset$
- $B \cap C = B$
- $B \cap D = \{(2,6)\}$
- $C \cap D = \{(6,2),(2,6)\}$

(c) Compute the following conditional probabilities:

- Note that since all outcomes are equally likely we have that $P(X) = \frac{|X|}{36}$ for any event X, where $|X|$ is the number of elements in X. Thus

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{36}}{\frac{6}{36}} = \frac{1}{6}.$$

- The probability that the sum of the dice is 6, given that 2 appears on at least one die is

$$P(A|C) = \frac{A \cap C}{P(C)} = \frac{\frac{2}{36}}{\frac{11}{36}} = \frac{2}{11}.$$

The probability given by $P(C|A)$ is the probability that a 2 appears on at least one die given that the sum of dice is 6. This is

$$P(C|A) = \frac{A \cap C}{P(A)} = \frac{\frac{2}{36}}{\frac{5}{36}} = \frac{2}{5}.$$

- The probability given by $P(A|D)$ is the probability that the sum of dice is 6 given that a 6 appears on at least one die. This is

$$P(C|A) = \frac{A \cap D}{P(D)} = \frac{0}{\frac{5}{36}} = 0.$$

- The probability given by $P(B|C)$ is the probability that a 2 appears on the first die given that a 2 appears on at least one die. This is

$$P(B|C) = \frac{B \cap C}{P(C)} = \frac{\frac{6}{36}}{\frac{11}{36}} = \frac{6}{11}.$$

- The probability that the first die shows 2 given that 6 appears on at least one die is

$$P(B|D) = \frac{B \cap D}{P(D)} = \frac{\frac{1}{36}}{\frac{11}{36}} = \frac{1}{11}.$$

- The probability given by $P(C|D)$ is the probability a 2 appears on at least one die given that a 6 appears on at least one die. This is

$$P(C|D) = \frac{C \cap D}{P(D)} = \frac{\frac{2}{36}}{\frac{11}{36}} = \frac{2}{11}.$$

□

3. A coin is tossed 2 times. What is the probability that both flips are heads given that at least one of the flips is heads?

Solution.

$$S = \{HH, HT, TH, TT\}$$
$$A = \{HH\}, B = \{HH, HT, TH\}$$
$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(\{HH\})}{P(B)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3} = 0.33....$$

□

4. A coin is tossed 3 times. What is the probability that all three flips are heads given that the first flip is heads?

Solution.

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$
$$A = \{HHH\}, B = \{HHH, HHT, HTH, HTT\}$$
$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(\{HHH\})}{P(B)} = \frac{\frac{1}{8}}{\frac{4}{8}} = \frac{1}{4} = 0.25.$$

□

5. A coin is tossed 4 times. What is the probability that at least two consecutive flips are heads given that at least one flip is tails?

Solution.

$$S = \{HHHH, HHHT, HHTH, HTHH, THHH, HHTT, HTHT, HTTH,$$
$$THHT, THTH, TTHH, HTTT, THTT, TTHT, TTTH, TTTT\}$$

$$A = \{HHHH, HHHT, HHTH, HTHH, THHH, HHTT, THHT, TTHH\}$$
$$B = S \setminus \{HHHH\}$$

$$
\begin{aligned}
P(A|B) &= \frac{P(A\cap B)}{P(B)} \\
&= \frac{P(\{HHHT, HHTH, HTHH, THHH, HHTT, THHT, TTHH\})}{P(B)} \\
&= \frac{\frac{7}{16}}{\frac{15}{16}} = \frac{7}{15} = 0.4666....
\end{aligned}
$$

□

6. A coin is tossed 4 times. What is the probability that at least two consecutive flips are heads given that the third flip is heads?

Solution.

$$
\begin{aligned}
S = \{&HHHH, HHHT, HHTH, HTHH, THHH, HHTT, HTHT, HTTH, \\
&THHT, THTH, TTHH, HTTT, THTT, TTHT, TTTH, TTTT\}
\end{aligned}
$$

$$
\begin{aligned}
A &= \{HHHH, HHHT, HHTH, HTHH, THHH, HHTT, THHT, TTHH\} \\
B &= \{HHHH, HHHT, HTHH, THHH, HTHT, THHT, TTHH, TTHT\}
\end{aligned}
$$

$$
P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(\{HHHH, HHHT, HTHH, THHH, THHT, TTHH\})}{P(B)} = \frac{\frac{6}{16}}{\frac{8}{16}} = \frac{6}{8} = 0.75.
$$

□

7. Suppose you buy a lotto 649 ticket, and watch the drawing on t.v. What is the probability that you win the jackpot given that the first 3 numbers drawn match yours?

Solution. In lotto 649, six numbers are drawn from numbers 1-49 without replacement. For one play, you choose six numbers (without repetition) from 1-49 (or you can have the computer randomly choose six numbers for you) and if your six match the set of six drawn you win the jackpot. The order that the numbers are drawn/chosen does not matter.

There are $\binom{49}{6} = 13983816$ different outcomes, and so the probability that your six numbers wins is $\frac{1}{13983816} \approx 0.000000072$.

Given that three of our numbers have already been drawn, we need only three more to win. There is only one set of three out of the remaining 46 numbers that will do it. i.e. number of successes is $\binom{3}{3} = 1$. The total number of outcomes, now that three numbers have been drawn, is $\binom{46}{3} = 15180$. So the probability of winning the jackpot, given that we have already three numbers, is $\frac{1}{15180} \approx 0.000066$.

Alternative: Let W be the event that all six of our numbers match the numbers drawn (i.e. a winning lotto number) and let F_3 be the event that (at least) the first three numbers drawn match any three of ours. Then the probability of winning the jackpot, given that three of our numbers have already come up is

$$
P(W|F_3) = \frac{P(W\cap F_3)}{P(F_3)}.
$$

Since $W \cap F_3 = W$ we have

$$
P(W\cap F_3) = P(W) = \frac{1}{\binom{49}{6}}
$$

and

$$P(F_3) = \frac{\frac{1}{2}\binom{6}{3}\binom{46}{3}}{\frac{1}{2}\binom{49}{3}\binom{46}{3}} = \frac{\binom{6}{3}}{\binom{49}{3}}$$

by choosing 3 of the 6 winning numbers to draw first out of the 49 choose 3 possibilities. Thus

$$P(W|F_3) = \frac{\frac{1}{\binom{49}{6}}}{\frac{\binom{6}{3}}{\binom{49}{3}}} = \frac{1}{15180}.$$

Alternative: We will make use of the order in which the numbers are drawn. Let W be the event that all six of our numbers match the numbers drawn (i.e. a winning lotto number) and let F_3 be the event that (at least) the first three numbers drawn match any three of ours. Then the probability of winning the jackpot, given that three of our numbers have already come up is

$$P(W|F_3) = \frac{P(W \cap F_3)}{P(F_3)}.$$

First we note that $W \cap F_3 = W$, so $P(W \cap F_3) = P(W)$. There are 6! ways that the winning six lotto numbers can be drawn, and ${}_{49}P_6 = 49 \cdot 48 \cdot 47 \cdot 46 \cdot 45 \cdot 44$ total ways to draw six numbers including order. So $P(W) = \frac{6!}{49 \cdot 48 \cdot 47 \cdot 46 \cdot 45 \cdot 44} = \frac{1}{13983816}$ (the same was result as obtained above by neglecting order).

To compute $P(F_3)$ we need to count the number of ways that at least the first three numbers could match any 3 of ours; there are $6 \cdot 5 \cdot 4 \cdot 46 \cdot 45 \cdot 44$ ways this can happen (again including the order). Therefore $P(F_3) = \frac{6 \cdot 5 \cdot 4 \cdot 46 \cdot 45 \cdot 44}{49 \cdot 48 \cdot 47 \cdot 46 \cdot 45 \cdot 44}$. Now we can find the conditional probability:

$$P(W|F_3) = \frac{P(W \cap F_3)}{P(F_3)} = \frac{\left(\frac{6!}{49 \cdot 48 \cdot 47 \cdot 46 \cdot 45 \cdot 44}\right)}{\left(\frac{6 \cdot 5 \cdot 4 \cdot 46 \cdot 45 \cdot 44}{49 \cdot 48 \cdot 47 \cdot 46 \cdot 45 \cdot 44}\right)} = \frac{3!}{46 \cdot 45 \cdot 44} = \frac{1}{\binom{46}{3}} \approx 0.000066.$$

Alternative: Let N_1, N_2, N_3, N_4, N_5 and N_6 be the events that the 1$^{\text{st}}$, 2$^{\text{nd}}$, 3$^{\text{rd}}$, 4$^{\text{th}}$, 5$^{\text{th}}$, and 6$^{\text{th}}$, numbers drawn match ours. Winning the jackpot is the event $W = N_1 \cap N_2 \cap N_3 \cap N_4 \cap N_5 \cap N_6$, and the event that the first three numbers drawn matches ours is $N_1 \cap N_2 \cap N_3$. Thus

$$P(W|N_1 \cap N_2 \cap N_3) = \frac{P(W \cap (N_1 \cap N_2 \cap N_3))}{P(N_1 \cap N_2 \cap N_3)} = \frac{P(N_1 \cap N_2 \cap N_3 \cap N_4 \cap N_5 \cap N_6)}{P(N_1 \cap N_2 \cap N_3)}$$

since $W \cap (N_1 \cap N_2 \cap N_3)) = W$. Now

$$\begin{aligned} P(W) &= P(N_1 \cap N_2 \cap N_3 \cap N_4 \cap N_5 \cap N_6) \\ &= P(N_1) \cdot P(N_2|N_1) \cdot P(N_3|N_1 \cap N_2) \cdot P(N_4|N_1 \cap N_2 \cap N_3) \cdot P(N_5|N_1 \cap N_2 \cap N_3 \cap N_4) \\ &\quad \cdot P(N_6|N_1 \cap N_2 \cap N_3 \cap N_4 \cap N_5) \\ &= \frac{6}{49} \cdot \frac{5}{48} \cdot \frac{4}{47} \cdot \frac{3}{46} \cdot \frac{2}{45} \cdot \frac{1}{44} \\ &= \frac{1}{\binom{49}{6}} \end{aligned}$$

and

$$\begin{aligned} P(N_1 \cap N_2 \cap N_3) &= P(N_1) \cdot P(N_2|N_1) \cdot P(N_3|N_1 \cap N_2) \\ &= \frac{6}{49} \cdot \frac{5}{48} \cdot \frac{4}{47} \\ &= \frac{\binom{6}{3}}{\binom{49}{3}} \end{aligned}$$

So

$$P(W|N_1 \cap N_2 \cap N_3) = \frac{\frac{6}{49} \cdot \frac{5}{48} \cdot \frac{4}{47} \cdot \frac{3}{46} \cdot \frac{2}{45} \cdot \frac{1}{44}}{\frac{6}{49} \cdot \frac{5}{48} \cdot \frac{4}{47}} = \frac{1}{15180}$$

□

8. Two regular *fair* dice are rolled (i.e. 6-sided dice, where fair means that each side has an equally likely chance of coming up). What is the conditional probability that at least one lands on 6 given that the dice land on different numbers?

Solution. Let A be the event that at least one die lands on 6, and B be the event that the dice land on different numbers. The conditional probability we are interested in is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Here

$$A \cap B = \{(6, d_2)|d_2 = 1, 2, 3, 4, 5\} \cup \{(d_1, 6)|d_1 = 1, 2, 3, 4, 5\}$$

and since these sets are disjoint,

$$P(A \cap B) = P(\{(6, d_2)|d_2 = 1, 2, 3, 4, 5\}) + P(\{(d_1, 6)|d_1 = 1, 2, 3, 4, 5\}) = \frac{5}{36} + \frac{5}{36} = \frac{5}{18}.$$

The number of ways the two dice can show different values is $6 \cdot 5 = 30$, therefore

$$P(B) = \frac{30}{36} = \frac{5}{6}.$$

Therefore

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\left(\frac{5}{18}\right)}{\left(\frac{5}{6}\right)} = \frac{1}{3}.$$

□

9. In a certain community, 36 percent of the families own a dog and 22 percent of the families that own a dog also own a cat. It is also known that 30 percent of all families own a cat.

   (a) What is the probability that a randomly selected family owns both a dog and a cat?

   (b) What is the conditional probability that a randomly selected family owns a dog given that it owns a cat?

Solution. (a) Let C be the event that family owns a cat and D be the event that a family owns a dog. Then we are given that

$$P(D) = 0.36, \quad P(C) = 0.30, \quad P(C|D) = 0.22$$

Therefore

$$P(C \cap D) = P(D) \cdot P(C|D) = (0.36) \cdot (0.22) = 0.0792.$$

(b)

$$P(D|C) = \frac{P(C \cap D)}{P(C)} = \frac{0.0792}{0.3} = 0.264.$$

□

10. At a certain dance studio, 90% of the students take ballet, 70% of the students take jazz, and 40% of the students take tap. It is known that 75% of those who take ballet are also in jazz, while 55% of those in jazz are also in tap. If a student is selected at random, what is the probability that they take both tap and jazz?

Solution. Let B, J and T be the events that that a randomly selected student takes ballet, tap or jazz respectively. We are given

$$P(B) = 0.9, \quad P(J) = 0.7, \quad P(T) = 0.4), \quad P(J|B) = 0.75, \quad P(T|J) = 0.55,$$

and we want to know $P(J \cap T)$. By the multiplication rule,

$$P(J \cap T) = P(J) \cdot P(T|J) = (0.7) \cdot (0.55) = 0.385.$$

□

11. A total of 500 married working couples were polled about their annual salaries, with the following information resulting:

		husband less than \$125, 000	husband more than \$125, 000
wife	less than \$125, 000	212	198
wife	more than \$125, 000	36	54

For instance, in 36 of the couples the wife earned more and then husband earned less than \$ 125,000. If a couple is chosen at random:

(a) What is the probability that the husband earns less than \$125,000?

(b) What is the conditional probability that the wife earns more than \$125,000, given that the husband earns more than this amount?

(c) What is the conditional probability that the wife earns more than \$125,000, given that the husband earns less than this amount?

Solution. (a)

$$P(H < \$125,000) = \frac{212 + 36}{500} = \frac{248}{500} = 0.496.$$

(b)

$$P(W > \$125,000 | H > \$125,000) = \frac{54}{198 + 54} = \frac{54}{252} \approx 0.2143.$$

(c)

$$P(W > \$125,000 | H < \$125,000) = \frac{36}{212 + 36} = \frac{36}{248} \approx 0.1452.$$

□

12. All job applicants for a certain teaching position are organized into the following table.

	Master's degree	No master's degree
3 or more years experience	18	9
Less than 3 years experience	36	27

Let M be the event that a randomly selected applicant has a master's degree, and E be the event that that applicant has at least 3 years of experience. Find $P(E|M)$.

Solution.

$$P(E|M) = \frac{P(E \cap M)}{P(M)} = \frac{\left(\frac{18}{90}\right)}{\left(\frac{54}{90}\right)} = \frac{1}{3}.$$

□

13. If A and B are events from sample space S, with $P(B) \neq 0$, then the definition of conditional probability says that

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Suppose S is a finite set and that all outcomes of S are equally likely. Show that

$$P(A|B) = \frac{|A \cap B|}{|B|},$$

where $|X|$ denotes the number of elements in the sets X.

Solution. If S is a finite set such that all outcomes of S are equally likely, then the postulate of countable additivity says that $P(A) = \frac{|A|}{|S|}$. Thus

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{|A \cap B|}{|S|}}{\frac{|B|}{|S|}} = \frac{|A \cap B|}{|B|}.$$

□

14. A penny, nickle and dime are tossed. Find the probability that they all show heads given that:

(a) The penny is heads.

(b) At least one of the coins is heads.

(c) The dime is tails.

Assume all outcomes are equally likely.

Solution. (a) The event that the penny is heads is

$$A = \{HHH, HHT, HTH, HTT\}.$$

Since each outcome is equally likely we can compute this conditional probability as

$$P(\{HHH\}|A) = \frac{|\{HHH\} \cap A|}{|A|} = \frac{1}{4}.$$

(b) The event that at least one coin is heads is

$$B = \{HHH, HHT, HTH, THH, HTT, THT, TTH\}$$

so

$$P(\{HHH\}|B) = \frac{|\{HHH\} \cap B|}{|B|} = \frac{1}{7}.$$

(c) The event that the dime is tails is

$$C = \{HHT, HTT, THT, TTT\},$$

so

$$P(\{HHH\}|C) = \frac{|\{HHH\} \cap C|}{|C|} = \frac{0}{4} = 0.$$

□

15. A billiard ball is drawn at random from a bag of 15 balls numbered 1 to 15.

(a) What is the probability that ball drawn has number greater than 10?

(b) If we know that the ball drawn has an even number, what is the probability its number is greater than 10? Express this as a conditional probability $P(A|B)$ with suitable events A and B.

Solution. (a) The event of having a number greater than 10 is $A = \{11, 12, 13, 14, 15\}$ so

$$P(A) = \frac{5}{15} = \frac{1}{3}.$$

(With no other information given, we must assume each outcome is equally likely.)

(b) Event A has been defined in part (a), so we let B be the event that the ball drawn has an even number; $B = \{2, 4, 6, 8, 10, 12, 14\}$. The probability that the ball drawn has number greater than 10, given that the ball is even is

$$P(A|B) = \frac{|A \cap B|}{|B|} = \frac{|\{12, 14\}|}{|\{2, 4, 6, 8, 10, 12, 14\}|} = \frac{2}{7}.$$

□

16. A bin contains 25 red balls, 40 white balls and 35 black balls. What is the probability that one ball will be red and one ball will be white if the first ball is returned to the bin before drawing the second ball?

Solution. Let R_1 and W_1 be the events that the first ball drawn is red and white respectively, and R_2 and W_2 be the events that the second ball drawn is red and white respectively. We want the probability of the event

$$(R_1 \cap W_2) \cup (W_1 \cap R_2).$$

Then

$$\begin{aligned}
P(R_1 \cap W_2) \cup (W_1 \cap R_2) &= P(R_1 \cap W_2) + P(W_1 \cap R_2) \quad \text{(mutually exclusive events)} \\
&= P(R_1) \cdot P(W_2|R_1) + P(W_1) \cdot P(R_2|W_1) \quad \text{(multiplication rule)} \\
&= \frac{25}{100}\frac{40}{100} + \frac{40}{100}\frac{25}{100} \\
&= 0.2
\end{aligned}$$

Note that $P(W_2|R_1) = P(W_2)$ and $P(R_2|W_1) = P(R_2)$ since the ball is replaced.

□

17. In a certain science degree program, 25 percent of the students failed their mathematics exam, 15 percent of the students failed their physics exam and 10 percent of the student failed both exams. A student is selected at random.

(a) If the student failed physics, what is the probability that they failed mathematics?

(b) If the student failed mathematics, what is the probability that they failed physics?

(c) What is the probability that the student failed the mathematics or physics exam?

(d) What is the probability that the student failed neither of these exams?

Solution. (a) Let A be the event that a student fails the mathematics exam and B be the event that they fail the physics exam. The probability that a student fails mathematics, given that they failed physics is

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.1}{0.15} = \frac{2}{3} \approx 0.6667$$

(b) The probability that a student fails physics, given that they failed mathematics is

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{0.1}{0.25} = \frac{2}{5} = 0.4$$

(c) The probability that the student failed the mathematics or physics exam is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.25 + 0.15 - 0.1 = 0.3.$$

(d) The probability that the student failed neither exam is

$$P((A \cup B)') = 1 - P(A \cup B) = 1 - 0.3 = 0.7.$$

□

18. Let A and B be events in a common space with $P(A) = 0.6$, $P(B) = 0.3$ and $P(A \cap B) = 0.2$. Find the following probabilities:

(a) $P(A|B)$

(b) $P(B|A)$

(c) $P(A')$

(d) $P(B')$

(e) $P(A'|B')$

(f) $P(B'|A')$

Solution. (a)

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.2}{0.3} = \frac{2}{3} \approx 0.6667$$

(b)

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{0.2}{0.6} = \frac{1}{3} \approx 0.3333$$

(c)

$$P(A') = 1 - P(A) = 1 - 0.6 = 0.4.$$

(d)

$$P(B') = 1 - P(B) = 1 - 0.3 = 0.7.$$

(e)

$$P(A'|B') = \frac{P(A' \cap B')}{P(B')} = \frac{P((A \cup B)')}{0.7} = \frac{1 - P(A \cup B)}{0.7}$$
$$= \frac{1 - P(A) - P(B) + P(A \cap B)}{0.7} = \frac{0.3}{0.7} = \frac{3}{7} \approx 0.4286$$

(f)

$$P(B'|A') = \frac{P(A' \cap B')}{P(A')} = \frac{0.3}{0.4} = \frac{3}{4} = 0.75.$$

□

19. Two different digits are chosen at random from the digits 1 through 5.

(a) If the sum of the two digits is odd, what is the probability that 2 is one of the digits that was chosen?

(b) If 2 is one of the digits that was chosen, what is the probability that the sum of the two digits is odd?

Solution. (a) The sample space for this experiment is the set of 20 pairs

$$S = \{(x, y)|x, y \in \{1, 2, 3, 4, 5\}, x \neq y\}$$

where each outcome is equally likely.
Let A be the event that the sum of digits is odd, and B be the event that 2 is one of the digits. The probability that 2 is one of the digits given that the sum is odd is

$$P(B|A) = \frac{|A \cap B|}{|A|}.$$

To find $|A \cap B|$ we can simply list the pairs that contain a 2 and whose whose sum is odd,

$$A \cap B = \{(2, 1), (2, 3), (2, 5), (1, 2), (3, 2), (5, 2)\}.$$

In this case the set is small and easy to list, but we could also count the elements in this set as

$$|A \cap B| = |\text{odd sum with 2 first}| + |\text{odd sum with 2 second}| = 3 + 3 = 6$$

because if the first digit is 2, there are 3 digits it can pair with (namely 1,3,5) to make an odd sum, and similarly if the second digit is 2.
Next we count the elements in A. Again we could list them all, but if this set were larger it may be better to count in a different way. We see that

$$|A| = 2 \cdot 3 + 3 \cdot 2 = 12,$$

where $2 \cdot 3$ are the pairs where the first digit is even and the second digit is odd, and vice versa for $3 \cdot 2$. Thus

$$P(B|A) = \frac{6}{12} = \frac{1}{2}.$$

Alternative: The solution above took into consideration the order that the pair of digits was chosen, but since each outcome is equally likely, we can compute probabilities by ignoring order as follows. Let A be the event that the sum of digits is odd, and B be the event that 2 is one of the digits. Then

$$P(A) = \frac{\binom{2}{1}\binom{3}{1}}{\binom{5}{2}} = \frac{6}{10} = \frac{3}{5}$$

(choose 1 of 2 even digits and 1 of 3 odd digits to make an odd sum), and

$$P(A \cap B) = \frac{\binom{3}{1}}{\binom{5}{2}} = \frac{3}{10}$$

(choose 1 of 3 odd digits to pair with 2 for an odd sum). Thus

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{3}{10}}{\frac{3}{5}} = \frac{1}{2}.$$

(b) The probability we are interested in is

$$P(A|B) = \frac{|A \cap B|}{|B|}.$$

We see that

$$|B| = 4 + 4 = 8$$

by adding the number of pairs where 2 is the first digit, to the number where 2 is the second digit. Thus

$$P(A|B) = \frac{6}{8} = \frac{3}{4}.$$

Alternative: Ignoring order (as was done in part (a)) we have

$$P(B) = \frac{\binom{4}{1}}{\binom{5}{2}} = \frac{4}{10} = \frac{2}{5}$$

(choose 1 of the other 4 digits to pair with 2), so

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{3}{10}}{\frac{2}{5}} = \frac{3}{4}.$$

□

20. A sample of people are surveyed. It is found that 65 percent of those people buy Sparkle-Sparkle laundry detergent, 40 percent buy Sparkle-Sparkle dish detergent and 20 percent buy both products. If a person at random is chosen from this sample, find the probability that:

(a) The person buys Sparkle-Sparkle laundry detergent or Sparkle-Sparkle dish detergent.

(b) The person buys Sparkle-Sparkle laundry detergent if they also buy Sparkle-Sparkle dish detergent.

(c) The person buys neither product.

Solution. (a) Let A be the event that a person buys Sparkle-Sparkle laundry detergent and B the event they buy Sparkle-Sparkle dish detergent. The probability that they buy either product or both is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.65 + 0.4 - 0.2 = 0.85.$$

(b) The probability that the person buys Sparkle-Sparkle laundry detergent given that they also buy Sparkle-Sparkle dish detergent is

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.2}{0.4} = 0.5.$$

(c) The probability that the person buys neither product is

$$P((A \cup B)') = 1 - P(A \cup B) = 0.15.$$

$\square$

21. A blue die, a red die and a yellow die are rolled (assume these are regular 6-sided dice). Let B, R, Y denote the numbers appearing on the blue, red and yellow dice respectively.

    (a) What is the probability that no two dice land on the same number?

    (b) Given that no two dice land on the same number, what is the conditional probability that $B < R < Y$?

    (c) What is $P(B < R < Y)$?

    Solution. (a) The total number of outcomes in the same space is $6^3 = 216$. The total number of ways that all three dice show different numbers is $6 \cdot 5 \cdot 4 = 120$. If D is the event that no two dice show the same number then

    $$P(D) = \frac{120}{216} = \frac{10}{18} \approx 0.5556.$$

    (b) If no two dice show the same value, then we have either of the following $3! = 6$ types of outcomes:

    $$B < R < Y, \quad B < Y < R, \quad R < B < Y, \quad R < Y < B, \quad Y < B < R, \quad Y < R < B.$$

    Since these 6 types are equally likely, we have that

    $$P(B < R < Y | D) = \frac{1}{6} \approx 0.1667$$

    (c) Using the probabilities found above we have

    $$P(B < R < Y) = P(D) \cdot P(B < R < Y | D) = \frac{10}{18} \cdot \frac{1}{6} = \frac{10}{108} \approx 0.093.$$

    $\square$

22. An urn contains 4 white balls and 6 black balls. Suppose 3 balls are randomly drawn without replacement. What is the probability that the last ball drawn is white given the first two balls drawn are of the same colour?

    Solution. Let W_1, W_2, W_3 denote the events that the first, second, and third ball drawn is white, respectively; and B_1, B_2, B_3 the events that the first, second and third ball drawn is black, respectively. The probability we are interested in is

    $$\begin{aligned}
    &P(W_3 | (W_1 \cap W_2) \cup (B_1 \cap B_2)) \\
    &= \frac{P(W_3 \cap ((W_1 \cap W_2) \cup (B_1 \cap B_2)))}{P((W_1 \cap W_2) \cup (B_1 \cap B_2))} \\
    &= \frac{P(W_1 \cap W_2 \cap W_3) + P(B_1 \cap B_2 \cap W_3)}{P(W_1 \cap W_2) + P(B_1 \cap B_2)} \\
    &= \frac{P(W_1) \cdot P(W_2 | W_1) \cdot P(W_3 | W_1 \cap W_2) + P(B_1) \cdot P(B_2 | B_1) \cdot P(W_3 | B_1 \cap B_2)}{P(W_1) \cdot P(W_2 | W_1) + P(B_1) \cdot P(B_2 | B_1)} \\
    &= \frac{\frac{4}{10} \cdot \frac{3}{9} \cdot \frac{2}{8} + \frac{6}{10} \cdot \frac{5}{9} \cdot \frac{4}{8}}{\frac{4}{10} \cdot \frac{3}{9} + \frac{6}{10} \cdot \frac{5}{9}} \\
    &= \frac{3}{7} \approx 0.4286
    \end{aligned}$$

$\square$

23. An urn contains 6 white balls and 9 black balls. Suppose 4 balls are randomly drawn without replacement. What is the probability that the first two drawn are white, and the last two drawn are black?

Solution. Let W_1, W_2 be the respective events that the first and second balls drawn are white, and B_3, B_4 the respective events that the third and fourth balls drawn are black. By the multiplication rule of probability we have

$$\begin{aligned} P(W_1 \cap W_2 \cap B_3 \cap B_4) &= P(W_1) \cdot P(W_2|W_1) \cdot P(B_3|W_1 \cap W_2) \cdot P(B_4|W_1 \cap W_2 \cap B_3) \\ &= \frac{6}{15} \cdot \frac{5}{14} \cdot \frac{9}{13} \cdot \frac{8}{12} \\ &= \frac{6}{91} \\ &\approx 0.0659 \end{aligned}$$

$\square$

24. A roulette wheel has 38 spaces (all spaces are the same size); numbers 1-36, 0 and 00. Alice always bets that the ball will land on a number between 1-12. What is the probability that Alice will:

(a) lose 5 times in a row.

(b) lose twice and then win.

Solution. (a) If either of the spaces 1-12 are landed on, then Alice wins; otherwise she loses. Hence $P(WIN) = \frac{12}{38}$ and $P(LOSE) = 1 - P(WIN) = \frac{26}{38}$. Since spins are independent, the probability of losing 5 times in a row with this bet is

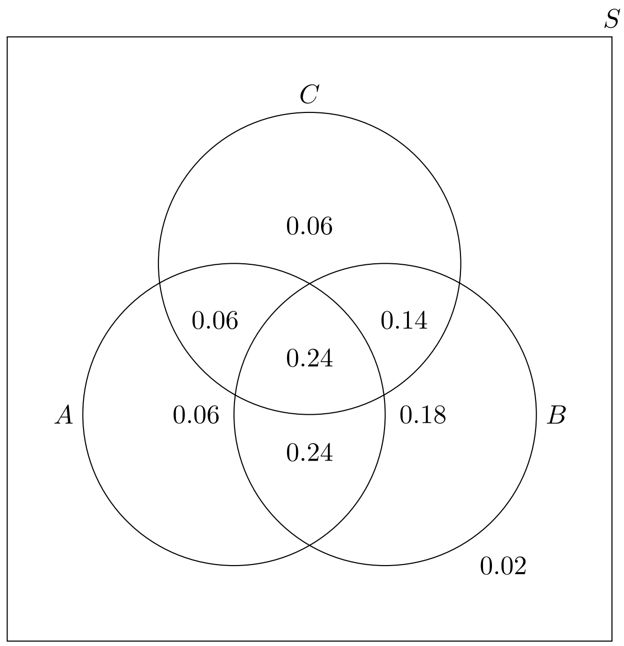
$$\left(\frac{26}{38}\right)^5 \approx 0.14995$$

(b) The probability of losing twice and then winning once is

$$\left(\frac{26}{38}\right)^2 \left(\frac{12}{38}\right) \approx 0.14783$$

$\square$

25. The following Venn diagram, shows events A, B and C (the three circles) in the sample space S. The values lying in each region are the probabilities of the subsets of S depicted by those regions.



Are events A, B and C independent?

Solution. Note that

$$P(B) = 0.18 + 0.24 + 0.24 + 0.14 = 0.8$$

and

$$P(C) = 0.06 + 0.06 + 0.24 + 0.14 = 0.5$$

but

$$P(B \cap C) = 0.24 + 0.14 = 0.38 \neq (0.5)(0.8) = P(B)P(C).$$

Since B and C are not independent (pairwise) it follows that A, B and C are not independent (taken together).

□

26. A coin is tossed twice, and heads or tails is observed on each toss. Let

$$S = \{HH, HT, TH, TT\}$$

be the sample space for this experiment and consider the events

$$\begin{aligned} H_1 &= \{\text{heads appears on the first toss}\}, \\ H_2 &= \{\text{heads appears on the second toss}\}, \\ T_1 &= \{\text{tails appears on the first toss}\}, \\ T_2 &= \{\text{tails appears on the second toss}\}. \end{aligned}$$

(a) Suppose each outcome in S is equally likely. Show that successive coin tosses are independent events; i.e. show that $P(A \cap B) = P(A)P(B)$ where A and B are any successive pair of events from H_1, H_2, T_1, or T_2.

(b) Suppose the probability of each outcome in S is as follows:

$$P(\{HH\}) = 0.1, \quad P(\{HT\}) = 0.2, \quad P(\{TH\}) = 0.4, \quad P(\{TT\}) = 0.3.$$

Show that successive coin tosses are not independent in this case; this means $P(A \cap B) \neq P(A)P(B)$ for some pair of successive events A and B.

(c) Suppose the probability of each outcome in S is as follows:

$$P(\{HH\}) = 0.16, \quad P(\{HT\}) = 0.24, \quad P(\{TH\}) = 0.24, \quad P(\{TT\}) = 0.36.$$

Show that successive coin tosses are independent.

(d) Suppose the probability of each outcome in S is as follows:

$$P(\{HH\}) = 0.1, \quad P(\{HT\}) = 0.25, \quad P(\{TH\}) = 0.25, \quad P(\{TT\}) = 0.4.$$

Determine whether coin tosses independent in this case?

(e) Suppose that the coin is constructed so that the probability of getting heads on any toss is 0.3 and the probability of getting tails on any toss is 0.7. This is the same as assuming

$$P(H_1) = 0.3, \quad P(H_2) = 0.3, \quad P(T_1) = 0.7, \quad P(T_2) = 0.7.$$

If we assume that successive coin tosses are independent event, find

- $P(\{HH\})$
- $P(\{HT\})$
- $P(\{TH\})$
- $P(\{TT\})$

(f) Suppose that $P(H_1) \neq P(H_2)$ for some valid probability P on S. Is it possible that successive coin tosses are independent events?

Solution. (a) We have

$$H_1 = \{HH, HT\}, H_2 = \{HH, TH\}, T_1 = \{TH, TT\}, T_2 = \{HT, TT\}.$$

Since all outcomes are equally likely, we have

$$P(H_1) = P(H_2) = P(T_1) = P(T_2) = \frac{2}{4} = \frac{1}{2}.$$

Then

$$\begin{aligned}
P(H_1 \cap H_2) = P(\{HH\}) &= \frac{1}{4} = P(H_1)P(H_2) \\
P(H_1 \cap T_2) = P(\{HT\}) &= \frac{1}{4} = P(H_1)P(T_2) \\
P(T_1 \cap H_2) = P(\{TH\}) &= \frac{1}{4} = P(T_1)P(H_2) \\
P(T_1 \cap T_2) = P(\{TT\}) &= \frac{1}{4} = P(T_1)P(T_2)
\end{aligned}$$

which shows that successive coin tosses are independent events.

(b) In this case $P(H_1) = P(\{HH\}) + P(\{HT\}) = 0.3$, and $P(H_2) = P(\{HH\}) + P(\{TH\}) = 0.5$, but

$$P(H_1 \cap H_2) = P(\{HH\}) = 0.1 \neq 0.15 = P(H_1)P(H_2)$$

and hence H_1 and H_2 are not independent events. We conclude that successive coin tosses are not independent for this distribution of probability.

Note that

$$P(H_2|H_1) = \frac{P(H_1 \cap H_2)}{P(H_1)} = \frac{0.1}{0.3} = \frac{1}{3} < 0.5 = P(H_2),$$

i.e. heads is less likely to appear on the second toss if it has already appeared on the first toss.

(c) In this case we have

$$\begin{aligned}
P(H_1) &= P(\{HH\}) + P(\{HT\}) = 0.4 \\
P(H_2) &= P(\{HH\}) + P(\{TH\}) = 0.4 \\
P(T_1) &= P(\{TH\}) + P(\{TT\}) = 0.6 \\
P(T_2) &= P(\{HT\}) + P(\{TT\}) = 0.6
\end{aligned}$$

and

$$\begin{aligned}
P(H_1 \cap H_2) &= P(\{HH\}) = 0.16 = P(H_1)P(H_2) \\
P(H_1 \cap T_2) &= P(\{HT\}) = 0.24 = P(H_1)P(T_2) \\
P(T_1 \cap H_2) &= P(\{TH\}) = 0.24 = P(T_1)P(H_2) \\
P(T_1 \cap T_2) &= P(\{TT\}) = 0.36 = P(T_1)P(T_2)
\end{aligned}$$

which shows that successive coin tosses are independent events.

(d) Note that

$$\begin{aligned}
P(H_1) &= P(\{HH\}) + P(\{HT\}) = 0.35 \\
P(H_2) &= P(\{HH\}) + P(\{TH\}) = 0.35 \\
P(T_1) &= P(\{TH\}) + P(\{TT\}) = 0.6 \\
P(T_2) &= P(\{HT\}) + P(\{TT\}) = 0.6
\end{aligned}$$

but

$$P(H_1 \cap H_2) = P(\{HH\}) = 0.1 \neq 0.1225 = P(H_1)P(H_2)$$

which shows that successive coin tosses not are independent events in this case.

(e) By independence we have

$$\begin{aligned}
P(\{HH\}) &= P(H_1 \cap H_2) = P(H_1)P(H_2) = (0.3)(0.3) = 0.09 \\
P(\{HT\}) &= P(H_1 \cap T_2) = P(H_1)P(T_2) = (0.3)(0.7) = 0.21 \\
P(\{TH\}) &= P(T_1 \cap H_2) = P(T_1)P(H_2) = (0.7)(0.3) = 0.21 \\
P(\{TT\}) &= P(T_1 \cap T_2) = P(T_1)P(T_2) = (0.7)(0.7) = 0.49
\end{aligned}$$

(f) This is possible, for example suppose

$$P(\{HH\}) = 0.02, \quad P(\{HT\}) = 0.08, \quad P(\{TH\}) = 0.18, \quad P(\{TT\}) = 0.72.$$

Then

$$\begin{aligned}
P(H_1) &= P(\{HH\}) + P(\{HT\}) = 0.1 \\
P(H_2) &= P(\{HH\}) + P(\{TH\}) = 0.2 \\
P(T_1) &= P(\{TH\}) + P(\{TT\}) = 0.9 \\
P(T_2) &= P(\{HT\}) + P(\{TT\}) = 0.8
\end{aligned}$$

(in particular $P(H_1) \neq P(H_2)$) and

$$\begin{aligned}
P(H_1 \cap H_2) &= P(\{HH\}) = 0.02 = P(H_1)P(H_2) \\
P(H_1 \cap T_2) &= P(\{HT\}) = 0.08 = P(H_1)P(T_2) \\
P(T_1 \cap H_2) &= P(\{TH\}) = 0.18 = P(T_1)P(H_2) \\
P(T_1 \cap T_2) &= P(\{TT\}) = 0.72 = P(T_1)P(T_2)
\end{aligned}$$

which shows that successive coin tosses are independent events.

(Note that while the various probability distributions given above are valid, they don't necessarily provide an appropriate model for an actual coin!) □

27. Three machines M_1, M_2 and M_3 produce respectively $25, 35$ and 40 percent of the items in a factory. It is known that

    - 3 percent of the items M_1 makes are defective,
    - 2 percent of the items M_2 makes are defective, and
    - 5 percent of the items M_3 makes are defective.

    (a) What is the probability that a randomly selected item came from M_2?

    (b) Find the probability that a randomly selected item is defective.

    (c) Find the probability that a randomly selected item came from M_2 given that it is defective.

    Solution. We will use M_1, M_2, M_3 to denote the events that the randomly selected part came from either of those machines, and D to denote the event that the part is defective.

    (a)
    $$P(M_2) = 0.35.$$

    (b) Using the rule of total probability:
    $$\begin{aligned} P(D) &= P(M_1) \cdot P(D|M_1) + P(M_2) \cdot P(D|M_2) + P(M_3) \cdot P(D|M_3) \\ &= (0.25)(0.03) + (0.35)(0.02) + (0.40)(0.05) \\ &= 0.0345. \end{aligned}$$

    (c) From the definition of conditional probability:
    $$\begin{aligned} P(M_2|D) &= \frac{P(M_2 \cap D)}{P(D)} \\ &= \frac{P(M_2) \cdot P(D|M_2)}{P(D)} \\ &= \frac{(0.35)(0.02)}{0.0345} \\ &\approx 0.2029. \end{aligned}$$

    (This is also Bayes' Theorem)

    □

28. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ be the sample space for an experiment with equally likely outcomes and define events
    $$A = \{1, 2, 3, 4\}, \quad B = \{2, 3, 4, 5\}, \quad C = \{4, 6, 7, 8\}.$$
    Is $P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$? Are A, B, C independent?

    Solution. Since $A \cap B \cap C = \{4\}$ and all outcomes out of the eight are equally likely, we have $P(A \cap B \cap C) = \frac{1}{8}$. On the other hand $P(A) = P(B) = P(C) = \frac{4}{8} = \frac{1}{2}$ so $P(A) \cdot P(B) \cdot P(C) = \frac{1}{8} = P(A \cap B \cap C)$.

To check that A, B, C are independent, we must show that they are pairwise independent, however

$$P(A \cap B) = P(\{2,3,4\}) = \frac{3}{8} \neq \frac{1}{4} = P(A) \cdot P(B).$$

Since one pair fails to be independent, the three A, B, C events cannot be independent. $\square$

29. A box contains 3 different types of disposable flashlights. Suppose that 20% of the box is type 1, 30% is type 2 and 50% is type 3. The probability that type 1, type 2 and type 3 will give over 100 hours of use is respectively 0.70, 0.40 and 0.30. What is the probability that a randomly chosen flashlight will give over 100 hours of use? If the selected flashlight lasted over 100 hours, what is the probability it was of type 2?

Solution. Let H be the event that a flashlight lasts over 100 hours, and T_1, T_2, T_3 be the events that a flashlight is of type 1, 2, 3, respectively. Using the rule for total probability we have

$$\begin{aligned} P(H) &= P(T_1)P(H|T_1) + P(T_2)P(H|T_2) + P(T_3)P(H|T_3) \\ &= (0.20)(0.70) + (0.30)(0.40) + (0.50)(0.30) \\ &= 0.41. \end{aligned}$$

By Bayes' formula,

$$P(T_2|H) = \frac{P(T_2)P(H|T_2)}{P(H)} = \frac{(0.30)(0.40)}{0.41} \approx 0.2927.$$

$\square$

30. A total of 48 percent of the women, and 37 percent of the men who took a certain "quit smoking" class remained nonsmokers for at least one year after completing the class. These people then attended a success party at the end of a year. Assuming that everyone in the class was either a male or female, if 62 percent of the original class was male:
    (a) What percentage of those attending the party were women?
    (b) What percentage of the original class attended that party?

Solution. (a) Let M denote the set of men in the class, W denote the set of women in the class, and S denote the set of people who were successful after one year (i.e. the people who attended the party). The information given the question is

$$P(S|W) = 0.48, \quad P(S|M) = 0.37, \quad P(M) = 0.62.$$

From this we immediately deduce that $P(W) = 1 - P(M) = 0.38$. The percentage of those attending the party that are women is

$$P(W|S) = \frac{P(W \cap S)}{P(S)}$$

by Bayes' Theorem this is

$$P(W|S) = \frac{P(W) \cdot P(S|W)}{P(W) \cdot P(S|W) + P(M) \cdot P(S|M)} = \frac{(0.38) \cdot (0.48)}{(0.38) \cdot (0.48) + (0.62) \cdot (0.37)} \approx 0.4429.$$

Thus about 0.44 or 44% of those attending the party were women.

(b) By the rule of total probability we have

$$P(S) = P(W) \cdot P(S|W) + P(M) \cdot P(S|M) = (0.38) \cdot (0.48) + (0.62) \cdot (0.37) = 0.4118.$$

Thus about 0.41 or 41% of the original class attended that party? □

31. Two countries C_1 and C_2 produce respectively 40 and 60 percent of humanoid robots worldwide. It is known that

- 2 percent of the humanoid robots that C_1 makes are not compatible with safety regulations, and
- 5 percent of the humanoid robots that C_2 makes are not compatible with safety regulations.

(a) Find the probability that a randomly selected humanoid robot was made in C_1.
(b) Find the probability that a randomly selected humanoid robot does not meet safety regulations.
(c) Find the probability that a randomly selected humanoid robot was made in C_1 given that it does not meet safety regulations.

Solution. Let A denote the event that a robot does not meet the safety regulations.

(a)
$$P(C_1) = 0.4.$$

(b) Using the rule of total probability:
$$\begin{aligned} P(A) &= P(C_1) \cdot P(A|C_1) + P(C_2) \cdot P(A|C_2) \\ &= (0.4)(0.02) + (0.6)(0.05) \\ &= 0.038. \end{aligned}$$

(c)] From the definition of conditional probability:
$$\begin{aligned} P(C_1|A) &= \frac{P(C_1 \cap A)}{P(A)} \\ &= \frac{P(C_1) \cdot P(A|C_1)}{P(A)} \\ &= \frac{(0.4)(0.02)}{0.038} \\ &\approx 0.2105. \end{aligned}$$

□

32. A fair die (i.e. 6-sided die, where fair means that each side has an equally likely chance of coming up) is rolled 1000 times. What is the conditional probability that the last roll is a 6 given that none of the first 999 rolls result is a 6?

Solution. Note that die rolls are independent events: Let $R_1(i)$ be the event of getting i on roll 1, and $R_2(j)$ the event of getting j on roll 2, where $i, j \in \{1, 2, 3, 4, 5, 6\}$. Since each outcome is equally likely we have $P(R_1(i) \cap R_2(j)) = \frac{1}{36} = \frac{1}{6} \cdot \frac{1}{6} = P(R_1(i)) \cdot P(R_2(j))$ (this verifies that die rolls are independent). This extends to any number of rolls.

Since rolls are independent, the probability of the 1000th roll is unaffected by the previous rolls. Consequently, we have
$$P(6 \text{ on last roll}|\text{first 999 rolls not 6}) = P(6 \text{ on last roll}) = \frac{1}{6}.$$

□

33. Let A and B be events with nonzero probability from a common sample space. If A and B are mutually exclusive, are they independent events?

Solution. Since

$$P(A \cap B) = P(\emptyset) = 0 \neq P(A)P(B)$$

we see that A and B are dependent. *(This makes intuitive sense, because if one event occurs, the other does not occur.)* □

34. Two probability experiments can be combined to form one experiment with a larger sample space; for example, the experiment of tossing a coin combined with rolling a die. If S_1 and S_2 are the sample spaces for the two probability experiments, we can form the new sample space with the Cartesian product

$$S_1 \times S_2 = \{(x, y) | x \in S_1, y \in S_2\}.$$

If A is an event in sample space S_1 (from the first experiment) we can identify the occurrence of A in the new experiment with the event $A \times S_2$ in $S_1 \times S_2$. Similarly, identify the event B from S_2 with $S_1 \times B$ in $S_1 \times S_2$. The event $A \times B = (A \times S_2) \cap (S_1 \times B)$ is when both A and B have occurred.

(a) A common way to assign probabilities to $S_1 \times S_2$ is to assume that the two experiments involved are independent of each other, and define

$$P(A \times B) = P(A)P(B),$$

where $P(A)$ and $P(B)$ are the probabilities from their individual experiments respectively. Then extend P to all of $S_1 \times S_2$ by assuming countable additivity. *(This way of assigning probabilities may be appropriate for the coin-toss die-roll experiment, where it is reasonable to assume that the two experiments don't interact.)*

Show that $(A \times S_2)$ and $(S_1 \times B)$ are independent events in the sample space $S_1 \times S_2$.

(b) Suppose S_1 and S_2 are finite, and all outcomes in S_1 and S_2 are equally likely (as individual experiments). If we specify that all outcomes of $S_1 \times S_2$ are equally likely, show that

$$P(A \times B) = P(A)P(B).$$

for any events $A \subseteq S_1$ and $B \subseteq S_2$.

(c) Suppose a fair coin is tossed, and then a pair of regular 6-sided dice are rolled. Assigning probabilities as in part (a), what is the probability that the coin shows tails and a sum of 8 is rolled on the dice.

(d) Bag 1 contains 5 red balls and 7 yellow balls. Bag 2 contains 12 balls numbered 1 through 12. Each ball has an equally likely chance of being drawn from its bag. Assuming that it is equally likely to choose any pair of balls, what is the probability of drawing a red ball along with a ball whose number is less than 5.

Solution. (a) By definition

$$P(A \times S_2) = P(A)P(S_2) = P(A),$$

and

$$P(S_1 \times B) = P(S_1)P(B) = P(B),$$

since both $P(S_2) = 1$ and $P(S_1) = 1$. Therefore

$$\begin{aligned} P((A \times S_2) \cap (S_1 \times B)) &= P(A \times B) \\ &= P(A)P(B) \quad \text{(by definition)} \\ &= P(A \times S_2)P(S_1 \times B) \end{aligned}$$

which shows that $A \times S_2$ and $S_1 \times B$ are independent events.

(b) By assumption,

$$P(A \times B) = \frac{|A \times B|}{|S_1 \times S_2|} = \frac{|A||B|}{|S_1||S_2|} = \frac{|A|}{|S_1|} \cdot \frac{|B|}{|S_2|} = P(A)P(B)$$

(c) The probability of getting tails is

$$P(\{T\}) = \frac{1}{2}.$$

The probability of rolling an sum of 8 is

$$P(\{\text{sum of 8}\}) = P(\{\text{⚁⚅}, \text{⚁⚄}... \})$$

Thus the probability of getting tails and rolling a sum of 8 is

$$P(\{T\} \times \{\text{sum of 8}\}) = \frac{1}{2} \cdot \frac{5}{36} = \frac{5}{72}$$

(d) Let A be the event of drawing a red ball from bag 1, and B be the event of drawing a number less than 5 from bag 2. Using what was shown in part (b) we have

$$P(A \times B) = P(A)P(B) = \frac{5}{12} \cdot \frac{4}{12} = \frac{5}{36}.$$

□

35. Suppose a coin is weighted so that $P(H) = \frac{2}{3}$ and $P(T) = \frac{1}{3}$. The coin is tossed 3 times. Assuming that coin flips are independent events, what is the probability of each outcome of the 3-flip experiment?

Solution. The sample space is $\{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$.

Since events are independent we multiply probabilities; e.g. $P(HHT) = P(H) \cdot P(H) \cdot P(T)$.

$$P(HHH) = \frac{8}{27}, \quad P(HHT) = P(HTH) = P(THH) = \frac{4}{27},$$

$$P(HTT) = P(THT) = P(TTH) = \frac{2}{27}, \quad P(TTT) = \frac{1}{27}.$$

□

36. In a certain geographical region during the month of April, it is known that the probability that a rainy day is followed by another rainy day is 0.8 and the probability that a sunny day is followed by a rainy day 0.6. If April 1st is a rainy day, find the probability that it will be rainy on April 3rd. *Assume it's sunny if it's not rainy.*

Solution. Let R_i be the event that it rains on the ith day of April, and S_i be the event that it is sunny on the ith day of April. The sample space can be partitioned into mutually exclusive events as $S = R_2 \cup R_2' = R_2 \cup S_2$. By the rule of total probability we have

$$P(R_3) = P(R_2) \cdot P(R_3|R_2) + P(S_2) \cdot P(R_3|S_2) = (0.8)(0.8) + (0.2)(0.6) = 0.76.$$

□

37. Persons A, B and C are responsible for shipping orders from a warehouse for a certain online retailer. Person A fills 30% of all orders, person B fills 40% of all orders and person C fills the remaining 30% of the orders. Based on past experience it is known that person A makes a mistake in an order 1% of the time, person B makes a mistake 5% of the time and person C makes a mistake 3% of the time.

An email complaint is received about a mistake in an order. What is the probability that mistake was made by person C?

Solution. Let A, B and C be the events that a randomly selected order was filled by person A, B or C respectively, and let M be the event that a randomly selected order was a mistake. By Bayes' Theorem,

$$\begin{aligned} P(C|M) &= \frac{P(C) \cdot P(M|C)}{P(A) \cdot P(M|A) + P(B) \cdot P(M|B) + P(C) \cdot P(M|C)} \\ &= \frac{(0.3)(0.03)}{(0.3)(0.01) + (0.4)(0.05) + (0.3)(0.03)} \\ &= 0.28125 \end{aligned}$$

□