3.5.2 Conditional Distributions

Recall: Conditional probability of event A given event B:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \qquad (P(B) \neq 0)$$

A given B

In terms of random variables: If A is the event $X = x$ and B is the event $Y = y$ then

$$P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}.$$

f(x,y)
h(y)

For discrete random variables with joint probability distribution $f(x, y)$ we have

$$P(X = x|Y = y) = \frac{f(x, y)}{h(y)},$$

where $h(y) \neq 0$ is the marginal distribution of Y.

If X and Y are discrete random variables with joint probability distribution $f(x, y)$, and respective marginal distributions $g(x)$ and $h(y)$, the function

$$f(x|y) = \frac{f(x, y)}{h(y)}$$

is called the **conditional distribution of X given $Y = y$**, provided $h(y) \neq 0$. The function

$$f(y|x) = \frac{f(x, y)}{g(x)}$$

is called the **conditional distribution of Y given $X = x$**, provided $g(x) \neq 0$.

Example 3.5.10

$$
\begin{array}{cc|ccc|c}
 & & & x & & \\
 & & 0 & 1 & 2 & \\
\hline
 & 0 & \frac{6}{36} & \frac{12}{36} & \frac{3}{36} & \frac{21}{36} = h(0) \\
y & 1 & \frac{8}{36} & \frac{6}{36} & & \frac{14}{36} = h(1) \\
 & 2 & \frac{1}{36} & & & \frac{1}{36} = h(2) \\
\hline
 & & \frac{15}{36} = g(0) & \frac{18}{36} = g(1) & \frac{3}{36} = g(2) &
\end{array}
$$

Caplet example: The conditional distribution of X given $Y = 1$ is, $f(x|1) = \frac{f(x,1)}{h(1)}$.

Its values are:

$$f(0|1) = \frac{f(0,1)}{h(1)} = \ldots \frac{\frac{8}{\cancel{36}}}{\frac{14}{\cancel{36}}} = \frac{8}{14} = \frac{4}{7}$$

$$f(1|1) = \frac{f(1,1)}{h(1)} = \ldots \frac{\frac{6}{\cancel{36}}}{\frac{14}{\cancel{36}}} = \frac{6}{14} = \frac{3}{7}$$

$$f(2|1) = \ldots \frac{f(2,1)}{h(1)} = \frac{0}{\frac{14}{36}} = 0$$

Conditional distribution for discrete random variables is extended to the idea of conditional density for jointly continuous random variables:

For jointly continuous random variables X and Y with joint density $f(x, y)$, and marginal densities $g(x)$ and $h(y)$:

The function

$$f(x|y) = \frac{f(x, y)}{h(y)}$$

same formulas as on p. 100

is called the **conditional density of X given** $Y = y$, provided $h(y) \neq 0$.

The function

$$f(y|x) = \frac{f(x, y)}{g(x)}$$

is called the **conditional density of Y given** $X = x$, provided $g(x) \neq 0$.

Example 3.5.11 *Let X and Y be jointly continuous random variables with joint probability density given by*

$$f(x,y) = \begin{cases} \frac{3}{5}x(y+x) & \text{for } 0 < x < 1, 0 < y < 2 \\ 0 & \text{otherwise} \end{cases}$$

$$f(y|x) = \frac{f(x,y)}{g(x)}$$

conditional density of Y given $X = x$,

(a) Find the conditional ~~probability~~ density function of Y given $X = x$.

(b) Find $P(0 < Y < 1|X = 0.75)$.

First we need the marginal density function for X:

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^2 \underbrace{\frac{3}{5}x(y+x)}_{f(x,y)}\,dy = \ldots$$

$$= \frac{3}{5}\int_0^2 xy + x^2\,dy = \frac{3}{5}\left(\frac{xy^2}{2} + yx^2\right)\Bigg|_{y=0}^{y=2} = \frac{3}{5}\left(2x + 2x^2 - 0\right) = \frac{3}{5}\cdot 2(x+x^2)$$

$$= \frac{6}{5}(x+x^2)$$

Thus $g(x) = \frac{6}{5}(x + x^2)$.

The conditional density function for $0 < x < 1, 0 < y < 2$ is then

$$f(y|x) = \frac{f(x,y)}{g(x)} = \ldots \frac{\frac{3}{\cancel{5}}x(y+x)}{\frac{6}{\cancel{5}}(x+x^2)} = \frac{1}{2}\,\frac{\cancel{x}(y+x)}{\cancel{x}(1+x)}$$

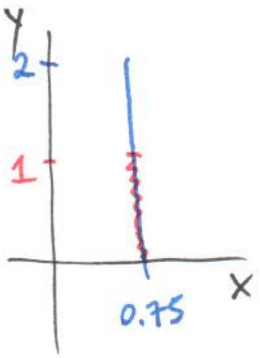
$$= \frac{y+x}{2(1+x)} = \frac{y+x}{2+2x}$$

$f(y|x) = 0$ elsewhere

In a) we found $f(y|x) = \dfrac{y+x}{2+2x}$ $\quad 0<x<1,\ 0<y<2.$

$$\underset{x=0.75}{f(y|0.75)} = \frac{y+0.75}{2+2\cdot 0.75} = \frac{y+0.75}{3.5}$$

Finally



$$P(0<Y<1|X=0.75) = \int_0^1 f(y|0.75)\,dy = \int_0^1 \frac{y+0.75}{3.5}\,dy$$

$$= \int_0^1 \frac{1}{3.5}(y+0.75)\,dy = \frac{1}{3.5}\int_0^1 y+0.75\,dy$$

$$= \frac{2}{7}\left(\frac{y^2}{2}+0.75y\right)\Bigg|_{y=0}^{y=1} = \frac{2}{7}\cdot\left(\frac{1}{2}+0.75-0\right)$$

$$= \frac{2}{7}\cdot\left(\frac{1}{2}+\frac{3}{4}\right) = \frac{\overset{1}{\cancel{2}}}{7}\cdot\frac{5}{\underset{2}{\cancel{4}}} = \frac{5}{14}$$

Example 3.5.12 *Let X and Y be jointly continuous random variables with joint probability density given by*

$$f(x,y) = \begin{cases} 4xy & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $P(0 < X < 1$ [redacted] $|Y = 0.5)$.

We need to find $f(x|y) = \dfrac{f(x,y)}{h(y)}$ → given → need to find → marginal density for y

Let's find $h(y)$. $h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^1 4xy\,dx$

$$= 2x^2y \Big|_{x=0}^{x=1} = 2y - 0 = \underline{\underline{2y}}$$

So, $f(x|y) = \dfrac{4xy}{2y} = 2x \quad 0<x<1,\ 0<y<1$

$(f(x|y) = 0 \text{ otherwise})$

Now, we can find $P(0<X<1 \mid Y=0.5)$.

$f(x|0.5) = 2x$

$$P(0<X<1 \mid Y=0.5) = \int_0^1 2x\,dx = x^2 \Big|_{x=0}^{x=1} = 1^2 - 0^2 = 1$$

Just as we defined the concept of independent events, we may speak of independent random variables.

If random variables X and Y have joint probability distribution (or density) $f(x, y)$ and marginal distributions (resp. densities) $g(x)$ and $h(y)$, then we say X and Y are **independent** if and only if

$$f(x, y) = g(x) \cdot h(y).$$

Example 3.5.13 *Let X and Y be jointly continuous random variables with joint probability density function*

$$f(x, y) = \begin{cases} 6e^{-2x}e^{-3y} & \text{for } 0 < x < \infty, 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Show that X and Y are independent random variables.

Follow the previous examples to find the marginal densities $\underset{\text{for } X}{g(x)}$ and $\underset{\text{for } Y}{h(y)}$. Then, check if $f(x,y) = g(x) \cdot h(y)$.

exercise

(consider two cases
1) $0 < x < \infty$, $0 < y < \infty$
2) elsewhere)

If equal, then X and Y are independent random variables.

Chapter 4

Mathematical Expectation

4.1 Expectation (Expected Value)

Example 4.1.1 *Suppose you are at a casino that has a dice game which costs \$1000 for a single roll of two 6-sided dice. You win \$5,555 by rolling a 7 and lose your money otherwise.*

Do you think it is worthwhile to play this game? Could you expect to come out ahead by repeatedly playing this game?

Let X be the sum of the two dice.

$$P(X=7) = \frac{6}{36} = \frac{1}{6}$$

$$P(X \neq 7) = 1 - \frac{1}{6} = \frac{5}{6}$$

Imagine we're playing this game n times for some really large n.

expected winnings (average)

$$\frac{\frac{n}{6} \cdot 5555 + \left(1 - \frac{n}{6}\right) \cdot 0}{n}$$

$$= \frac{\frac{n}{6} \cdot 5555 + \frac{5n}{6} \cdot 0}{n}$$

$$= \frac{n \cdot \left(\frac{5555}{6} + \frac{5}{6} \cdot 0 \right)}{n} = \frac{\cancel{n} \cdot \left(5555 \cdot \frac{1}{6} + 0 \cdot \frac{5}{6} \right)}{\cancel{n}}$$

$$= 5555 \cdot \frac{1}{6} + 0 \cdot \frac{5}{6} \approx 926$$

Let Y be the amount of money one can make in one game

range of Y is $\{0, 5555\}$

Example 4.1.2 *Suppose a university fundraiser sells 10,000 raffle tickets at a dollar apiece with a grand prize of \$5,000, a second prize of \$1,000 and two third place prizes of \$500 each.*

Do you think your ticket is worth \$1? How much do you think it is worth? In other words, how much can you "expect" to win in this raffle?

$$5000 \cdot \frac{1}{10000} + 1000 \cdot \frac{1}{10000} + 500 \cdot \frac{2}{10000}$$

$$= 0.5 + 0.1 + 0.1 = 0.7$$

X has values $5000, 1000, 500$ or 0

If X is a discrete random variable and $f(x)$ is the value of its probability distribution at x, the **expected value** of X (or **expectation of** X) is defined

$$E(X) = \sum_x x \cdot f(x).$$

where the sum is over all x in the range of X.

The sum must be defined in order for the expected value to have meaning.

In the first example of the dice game, let the random variable ~~X~~ Y ~~is~~ be the amount of money won on each roll. The range of ~~X~~ Y is $\{0, 5555\}$.

Since the probability of rolling a 7 is $\frac{1}{6}$ we have $P(Y = 5555) = \frac{1}{6}$ and therefore $P(Y = 0) = \frac{5}{6}$.

$$E(Y) = \sum_y y \cdot f(y) = 0 \cdot \frac{5}{6} + 5555 \cdot \frac{1}{6} \approx 926$$

[redacted]

[redacted]

This analysis shows that this is a losing game, because our expected value is less than the cost to play.

In the long run, we can expect to lose money.

In the raffle ticket example we let X denote the possible winnings for our raffle ticket. Typically once a ticket is drawn it is not replaced to be drawn again, so the range of X is $\{0, 500, 1000, 5000\}$.

Four tickets will be drawn for the four prizes and there is an equally likely chance of $\frac{1}{10000}$ for each ~~prize~~. ticket.

Therefore $P(X = 0) = \frac{9996}{10000}$, $P(X = 500) = \frac{2}{10000}$, $P(X = 1000) = \frac{1}{10000}$, $P(X = 5000) = \frac{1}{10000}$.

The expected value of X is

$$E(X) = \sum_{x} x \cdot f(x) = 0 \cdot \frac{9996}{10000} + 500 \cdot \frac{2}{10000} + 1000 \cdot \frac{1}{10000} + 5000 \cdot \frac{1}{10000}$$

$$= 0 + 0.1 + 0.1 + 0.5 = 0.7$$

By playing the raffle repeatedly, we expect to win \$0.70 on average; therefore losing money with the \$1 cost. We could place a value of \$0.70 for our ticket.

Raffle with replacement:

Returning to the previous raffle example, let's compute the expected value of a single ticket with only three prize draws of \$5,000, \$1,000, \$500 each.

Now tickets are replaced each time to allow for multiple wins. (Again 10000 tickets sold)

\$5000 1st draw for the 1st prize
\$1000 2nd draw for the 2nd prize
\$500 3rd draw for the 3rd prize

Let X be the total prize money won.

The range of X is $\{0, 500, 1000, 1500, 5500, 5000, 6000, 6500\}$.

Then,

$P(X = 0) = \left(\frac{9999}{10000}\right)^3,$

$P(X = 500) = \left(\frac{9999}{10000}\right)^2 \cdot \left(\frac{1}{10000}\right),$

$P(X = 1000) = \left(\frac{9999}{10000}\right) \cdot \left(\frac{1}{10000}\right) \cdot \left(\frac{9999}{10000}\right),$

$P(X = 1500) = \left(\frac{9999}{10000}\right) \cdot \left(\frac{1}{10000}\right)^2,$

$P(X = 5000) = \left(\frac{1}{10000}\right) \cdot \left(\frac{9999}{10000}\right)^2,$

$P(X = 5500) = \left(\frac{1}{10000}\right) \cdot \left(\frac{9999}{10000}\right) \cdot \left(\frac{1}{10000}\right),$

$P(X = 6000) = \left(\frac{1}{10000}\right)^2 \cdot \left(\frac{9999}{10000}\right),$

$P(X = 6500) = \left(\frac{1}{10000}\right)^3$

The expected value of X is

$$E(X) = \sum_{x} x \cdot f(x) = 0 \cdot \left(\frac{9999}{10000}\right)^3 + 500 \cdot \left(\frac{9999}{10000}\right)^2 \cdot \frac{1}{10000}$$

$$+ 1000 \cdot \frac{9999}{10000} \cdot \frac{1}{10000} \cdot \frac{9999}{10000} + 1500 \cdot \frac{9999}{10000} \cdot \left(\frac{1}{10000}\right)^2$$

$$+ 5500 \cdot \frac{1}{10000} \cdot \frac{9999}{10000} \cdot \frac{1}{10000} + 5000 \cdot \frac{1}{10000} \cdot \left(\frac{9999}{10000}\right)^2$$

$$+ 6000 \cdot \left(\frac{1}{10000}\right)^2 \cdot \frac{9999}{10000} + 6500 \cdot \left(\frac{1}{10000}\right)^3$$

$$= 0.65$$

(65 cents)

So, we can say that a fair value for a raffle ticket in this case is 65 cents.

Same problem without replacement

$$E(x) = \sum x \cdot f(x) = 0 \cdot \frac{9997}{10000}$$

$$+ 500 \cdot \frac{1}{10000} + 1000 \cdot \frac{1}{10000}$$

$$+ 5000 \cdot \frac{1}{10000}$$

$$= 0.05 + 0.1 + 0.5 = 0.65$$

(65 cents)

Example 4.1.3 *A slot machine has three wheels with 20 symbols on each wheel.*

There is one 7, two $\overline{BAR}$, and three bell icons on each wheel.

It costs \$0.25 to play.

1×7
$2 \times \overline{BAR}$
$3 \times$ 🔔
$14 \times$ other symbols

Payouts:

Wheel 1	Wheel 2	Wheel 3	
7	7	7	\$500
$\overline{BAR}$	$\overline{BAR}$	$\overline{BAR}$	\$100
🔔	🔔	🔔	\$20

(all other permutations lose).

What is the expected value of this game (ignore cost to play).

Let X be the random variable that denotes the amount of winnings. $\underbrace{\cancel{X}}_{\text{range of } X} = \{0, 20, 100, 500\}$

$$P(X=0) = 1 - \left(\frac{27}{8000} + \frac{8}{8000} + \frac{1}{8000}\right)$$

$$P(X=20) = \frac{3}{20} \cdot \frac{3}{20} \cdot \frac{3}{20} = \frac{27}{8000}$$

$$P(X=100) = \frac{2}{20} \cdot \frac{2}{20} \cdot \frac{2}{20} = \frac{8}{8000}$$

$$P(X=500) = \frac{1}{20} \cdot \frac{1}{20} \cdot \frac{1}{20} = \frac{1}{8000}$$

$$E(X) = 0 \cdot \left(1 - \left(\frac{27}{8000} + \frac{8}{8000} + \frac{1}{8000}\right)\right)$$

$$+ \; 20 \cdot \frac{27}{8000}$$

$$+ \; 100 \cdot \frac{8}{8000}$$

$$+ \; 500 \cdot \frac{1}{8000}$$

$$\text{So, } E(X) = 0 + \frac{540}{8000} + \frac{800}{8000} + \frac{500}{8000} = \frac{1840}{8000}$$

$$= \frac{184}{800} = \frac{\cancel{8} \cdot 23}{\cancel{8} \cdot 100} = \frac{23}{100} = 0.23$$

The expected value is 23 cents