3.5.1 Marginal Distributions

		$x=0$	$x=1$	$x=2$		
y	0	$\frac{6}{36}$	$\frac{12}{36}$	$\frac{3}{36}$	$\frac{21}{36}$	$P(Y=0)$
	1	$\frac{8}{36}$	$\frac{6}{36}$	·	$\frac{14}{36}$	$P(Y=1)$
	2	$\frac{1}{36}$			$\frac{1}{36}$	$P(Y=2)$
		$\frac{15}{36}$	$\frac{18}{36}$	$\frac{3}{36}$		
		$P(X=0)$	$P(X=1)$	$P(X=2)$		

3 aspirin
2 sedatives
4 laxatives

Choose 2 caplets

x : # aspirin chosen

y : # sedative chosen

$f(x,y) \rightsquigarrow$ probability distribution

Returning to the caplet example, sum the rows and columns of the table of probabilities.

The column sums are the probabilities that $X = 0, 1, 2$ respectively, and the row sums are probabilities that $Y = 0, 1, 2$ respectively.

Therefore, the column totals are the probability distribution for X: for $x = 0, 1, 2$

$$g(x) = P(X = x) = \sum_{y=0}^{2} f(x, y),$$

$g(0) = P(X=0) = f(0,0) + f(0,1) + f(0,2)$

$= \sum_{y=0}^{2} f(0,y) = \frac{6}{36} + \frac{8}{36} + \frac{1}{36} = \frac{15}{36}$

and the row totals are the probability distribution for Y: for $y = 0, 1, 2$

$$h(y) = P(Y = y) = \sum_{x=0}^{2} f(x, y).$$

$h(1) = P(Y=1) = \sum_{x=0}^{2} f(x,1)$

$= f(0,1) + f(1,1) + f(2,1)$

$= \frac{8}{36} + \frac{6}{36} + 0 = \frac{14}{36} = \frac{7}{18}$

If X and Y are discrete random variables, and $f(x, y)$ is their joint probability distribution, then the function

fixed

$$g(x) = \sum_y f(x, y)$$

is called the **marginal distribution of X** and the function

$$h(y) = \sum_x f(x, y) \text{ — fixed}$$

is called the **marginal distribution of Y.** The sums are over all values of either y or x respectively.

If X and Y are jointly continuous random variables, and $f(x, y)$ is their joint probability density function, then

$$g(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$$

is called the **marginal density of X** and the function

$$h(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$$

is called the **marginal density of Y.** These functions are defined for all $x \in \mathbb{R}$,$y \in \mathbb{R}$ respectively.

Example 3.5.8 *Find the marginal densities of X and Y given their joint probability density*

$$f(x,y) = \begin{cases} \frac{2}{3}(x+2y) & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Marginal density of X:

because $0<y<1$

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\, dy = \int_0^1 \frac{2}{3}(x+2y)\, dy = \ldots \frac{2}{3}\int_0^1 x+2y\, dy$$

$$= \frac{2}{3}\left(\left(xy+y^2\right)\Big|_{y=0}^{y=1}\right) = \frac{2}{3}\left((x+1)-(0+0)\right) = \frac{2}{3}(x+1)$$

Marginal density of Y:

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\, dx = \int_0^1 \frac{2}{3}(x+2y)\, dx = \ldots \frac{2}{3}\int_0^1 x+2y\, dx$$

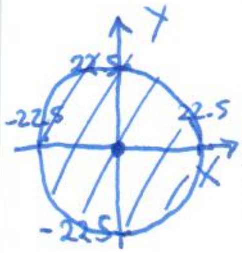
$$= \frac{2}{3}\left(\left(\frac{x^2}{2}+2yx\right)\Big|_{x=0}^{x=1}\right) = \frac{2}{3}\left(\left(\frac{1}{2}+2y\right)-(0+0)\right)$$

$$= \frac{2}{3}\left(\frac{1}{2}+2y\right) = \frac{1}{3}+\frac{4y}{3}$$

Note that joint marginal distributions can be defined in the case of 3 or more random variables, and that is beyond the scope of this course too.

radius: $\frac{45}{2} = 22.5$

Example 3.5.9 *A circular biathlon target for prone position has a diameter of 45mm. Suppose that each point on the target has equally likely probability of being hit by a shot.* Also, suppose that we are not missing the target.



Let $(0,0)$ be the centre of the target, and define random variables X and Y, so that (X,Y) denotes the coordinates (in millimetres) of the shot fired.

The joint density function for X and Y is then, for some constant k,

$$f(x,y) = \begin{cases} k & \textit{for } x^2+y^2 \le (22.5)^2 \\ 0 & \textit{elsewhere} \end{cases}$$

equation for the disk at the origin with radius 22.5

It follows that $k = \frac{1}{(22.5)^2\pi}$, so the integral of the joint density function equals 1 over the area of the circle.

(Using the joint density function and making use of the rule that the probability over the entire domain is 1, we can calculate k. It turns out that $k = \frac{1}{(22.5)^2 \cdot \pi}$

joint density function

$$f(x,y) = \begin{cases} \dfrac{1}{(22.5)^2\pi} & \text{for } x^2+y^2 \le (22.5)^2 \\ 0 & \text{elsewhere} \end{cases}$$

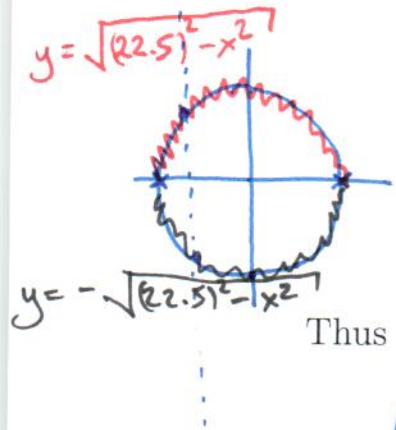
To find the marginal density for X, integrate over all y values:

First:

$$x^2 + y^2 \le (22.5)^2 \Rightarrow \quad y^2 \le (22.5)^2 - x^2$$

Take the square root of both sides

$$\Rightarrow \quad -\sqrt{(22.5)^2 - x^2} \le y \le \sqrt{(22.5)^2 - x^2}$$



Thus

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{-\sqrt{(22.5)^2-x^2}}^{\sqrt{(22.5)^2-x^2}} \frac{1}{(22.5)^2\pi}\,dy = \frac{1}{(22.5)^2\pi}\int_{-\sqrt{\cdots}}^{\sqrt{\cdots}} 1\,dy$$

$$= \frac{1}{(22.5)^2\pi}\cdot\left(y \Big|_{-\sqrt{\cdots}}^{\sqrt{\cdots}} \right) = \frac{1}{(22.5)^2\pi}\left(\sqrt{(22.5)^2-x^2} - \left(-\sqrt{(22.5)^2-x^2}\right)\right)$$

$$= \frac{1}{(22.5)^2\pi}\cdot 2\sqrt{(22.5)^2-x^2} = \frac{2\sqrt{(22.5)^2-x^2}}{(22.5)^2\cdot\pi}$$

The marginal density of X can be used to find the probability the shot will land any horizontal distance x from the centre, regardless of its vertical position.

Notice that $g(x)$ is largest when $x = 0$ and gets smaller as x gets near the boundary of the target.

3.5.2 Conditional Distributions

Recall: Conditional probability of event A given event B:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \qquad (P(B) \neq 0)$$

A given B

In terms of random variables: If A is the event $X = x$ and B is the event $Y = y$ then

$$P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}.$$

f(x,y)

h(y)

For discrete random variables with joint probability distribution $f(x, y)$ we have

$$P(X = x|Y = y) = \frac{f(x, y)}{h(y)},$$

where $h(y) \neq 0$ is the marginal distribution of Y.

If X and Y are discrete random variables with joint probability distribution $f(x, y)$, and respective marginal distributions $g(x)$ and $h(y)$, the function

$$f(x|y) = \frac{f(x, y)}{h(y)}$$

is called the **conditional distribution of X given $Y = y$**, provided $h(y) \neq 0$. The function

$$f(y|x) = \frac{f(x, y)}{g(x)}$$

is called the **conditional distribution of Y given $X = x$**, provided $g(x) \neq 0$.