The mean and the variance of a normal distribution:
First we find the moment generating function.

normal distr.

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left(-2xt\sigma^2+(x-\mu)^2\right)} dx$$

In the exponent we have:

$$-2xt\sigma^2 + (x-\mu)^2 = -2xt\sigma^2 + x^2 - 2x\mu + \mu^2 = x^2 - 2x(\mu + t\sigma^2) + \mu^2.$$

sketch of calculations to find the moment generating fnc.

Completing the square gives:

$$x^2 - 2x(\mu + t\sigma^2) + \mu^2 = (x - (\mu + t\sigma^2))^2 - 2\mu t\sigma^2 - t^2\sigma^4.$$

This allows us to write

$$M_X(t) = e^{\mu t + \frac{1}{2}t^2\sigma^2} \left(\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-(\mu+t\sigma^2)}{\sigma}\right)^2} dx \right) = e^{\mu t + \frac{1}{2}t^2\sigma^2}$$

try to follow it and fill in the gaps if you're interested

The moment generating function for normally distributed random variable X is:

$$M_X(t) = e^{\mu t + \frac{1}{2}t^2\sigma^2}$$

Now we will show that the mean and variance of X are indeed μ and σ^2.

First derivative of the moment generating function:

$$\frac{d}{dt}M_X(t) = \frac{d}{dt}e^{\mu t + \frac{1}{2}t^2\sigma^2} = e^{\mu t + \frac{1}{2}t^2\sigma^2} \cdot (\mu + \sigma^2 t)$$

mean :

then $\left.\frac{d}{dt}M_X(t)\right|_{t=0} = e^0 \cdot (\mu + 0) = \mu$ ← mean

chain rule

To find μ: we need $\left.\frac{d}{dt}M_X(t)\right|_{t=0}$

Second derivative of the moment generating function:

$$\frac{d}{dt}M_X(t) = e^{\mu t + \frac{1}{2}t^2\sigma^2} \cdot (\mu + \sigma^2 t)$$

$$\frac{d^2}{dt^2}M_X(t) = \frac{d}{dt}\left(\mu e^{\mu t + \frac{1}{2}t^2\sigma^2} + \sigma^2 t e^{\mu t + \frac{1}{2}t^2\sigma^2}\right)$$

chain rule

product & chain rule

$$= \mu e^{\mu t + \frac{1}{2}t^2\sigma^2} \cdot (\mu + \sigma^2 t) + \sigma^2 e^{\mu t + \frac{1}{2}t^2\sigma^2} + \sigma^2 t e^{\mu t + \frac{1}{2}t^2\sigma^2} \cdot (\mu + \sigma^2 t)$$

$= \mu \cdot e^{\mu t + \frac{1}{2}t^2\sigma^2} + \sigma^2 t \cdot e^{\mu t + \frac{1}{2}t^2\sigma^2}$

To find σ: We do $E(X^2) - \mu^2$ where to find $E(X^2)$ we need $\left.\frac{d^2}{dt^2}M_X(t)\right|_{t=0}$

Setting $t = 0$ in both derivatives:

$$\left.\frac{d^2}{dt^2}M_X(t)\right|_{t=0} = \mu^2 + \sigma^2$$

$E(X^2) = \left.\frac{d^2}{dt^2}M_X(t)\right|_{t=0} = \mu \cdot e^0 \cdot \mu + \sigma^2 e^0 + 0$

$= \mu^2 + \sigma^2$

So, variance $= E(X^2) - \mu^2$

$= \mu^2 + \sigma^2 - \mu^2$

Therefore **the mean, $E(X)$, is μ; and the variance is $E(X^2) - \mu^2 = \sigma^2$.**

normally distributed random variables:

mean = μ

and variance = σ^2

=> stand. deviation = σ

normal distribution μ, σ $\quad n(x;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

when $\mu = 0$ and $\sigma = 1$: $n(x;0,1) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-0}{1}\right)^2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$ ← standard normal distribution

The Standard Normal Distribution

The normal distribution with $\mu = 0$ and $\sigma = 1$ is called the *standard normal distribution*.

$$n(x; 0, 1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

Probabilities for the standard normal distribution may be found by way of a table of "pre-calculated" probabilities.

For example, Table III on the next page gives $P(0 \leq X \leq z)$ for various z values. Graphically this looks like,

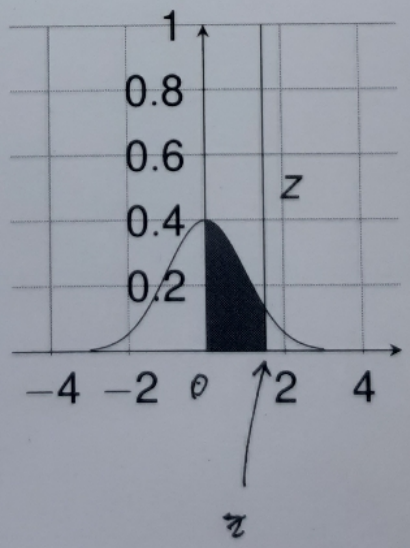


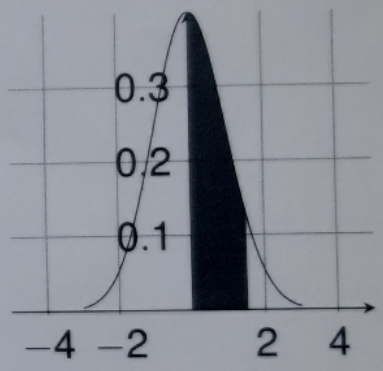
Table III: Standard Normal Distribution

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4988
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990

Standard Normal Table

Table III: Standard Normal Distribution

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4988
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990



$P(0 \leq X \leq 1.52) = 0.4357$

$P(0 \leq X \leq 1.52) = 0.4357$

$P(0 \leq X \leq 2.37) = 0.4911$

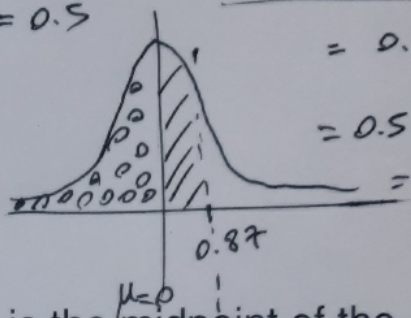
X is a random variable with standard normal distribution

$P(X \le \underbrace{0.87}_{z}) = P(X \le 0) + P(0 \le X \le 0.87)$

$= 0.5$ $\qquad = 0.3078$

$= 0.5 + 0.3078$

$= 0.8078$



Standard Normal Table

We noted earlier that μ, in this case 0, is the midpoint of the graph. Thus to find $P(X \le z)$, we look up our value of z in the table, then add 0.5.

Table III: Standard Normal Distribution

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
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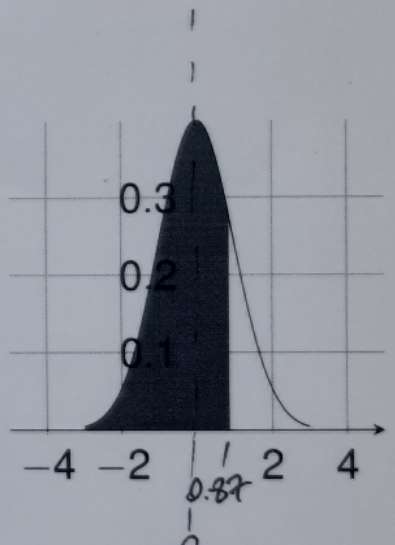
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4988
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990



$P(X \le 0.87)$

$= 0.3078 + 0.5$

$= 0.8078$

by symmetry

$= 0.5$

$P(X \leq 0)$

by table

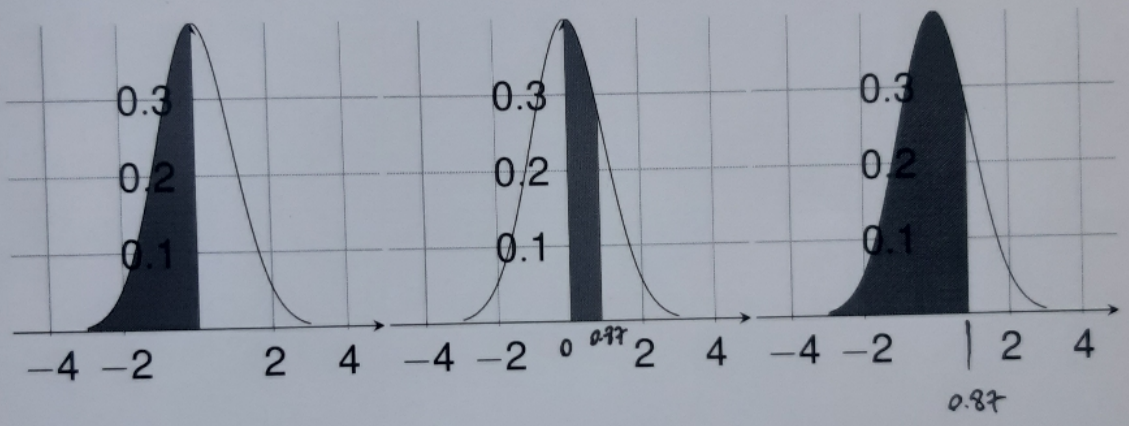
$= 0.3078$

$P(0 \leq X \leq 0.87)$

$P(X \leq 0.87)$

$= 0.5 + 0.3078$

$= 0.8078$



Adding the first two areas gives the third.