MATH1550
Practice Set 10

These exercises are suited to Chapter 4, from Bivariate Moments to Conditional Expectations.
Topics Covered:

- Product moments about the origin
- Means in a joint distribution
- Product moments about the means
- Covariance
- Covariance and indpendent random variables
- Conditional expectation

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1. Let X and Y be jointly distributed random variables (either discrete or continuous).
   (a) Write the definition for the *rth and sth product moment about the origin* for X and Y where $r, s \in \{0, 1, 2, 3, \dots\}$.
   (b) What do the symbols μ_X and μ_Y mean and how to we compute these?
   (c) Write the definition for the *rth and sth product moment about the means* for X and Y where $r, s \in \{0, 1, 2, 3, \dots\}$.
   (d) What is the *covariance* of X and Y?
   (e) What does a positive covariance imply versus a negative covariance?
   (f) What is the relationship between covariance and independent random variables?
   (g) Write the definition for the *conditional mean of X given $Y = y$*.

   Solution. (a) The *rth and sth product moment about the origin* for X and Y is defined as $E(X^rY^s)$. In the case of discrete random variables this is computed as

   $$E(X^rY^s) = \sum_x \sum_y x^r y^s f(x, y),$$

   where $f(x, y)$ is the joint distribution of X and Y, and in the case of continuous random variables this is computed as

   $$E(X^rY^s) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^r y^s f(x, y)\, dx\, dy$$

   where $f(x, y)$ is the joint density of X and Y.

   (b) The symbols μ_X and μ_Y denote the *mean of X* and the *mean of Y* respectively in the context of jointly distributed random variables X and Y. They are defined by $E(X)$ and $E(Y)$ respectively.

   (c) The *rth and sth product moment about the means* for X and Y is defined as $E((X-\mu_X)^r(Y-\mu_Y)^s)$. In the case of discrete random variables this is computed as

   $$E((X - \mu_X)^r(Y - \mu_Y)^s) = \sum_x \sum_y (x - \mu_X)^r(y - \mu_Y)^s f(x, y)$$

   where $f(x, y)$ is the joint distribution of X and Y, and in the case of continuous random variables this is computed as

   $$E((X - \mu_X)^r(Y - \mu_Y)^s) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \mu_X)^r(y - \mu_Y)^s f(x, y)\, dx\, dy$$

   where $f(x, y)$ is the joint density of X and Y.

(d) The *covariance* of X and Y, denoted $\text{cov}(X,Y)$ or σ_{XY} is defined as $E((X-\mu_X)(Y-\mu_Y))$. This can be computed with the short cut formula $E((X-\mu_X)(Y-\mu_Y)) = E(XY) - \mu_X\mu_Y$.

(e) A positive covariance implies that large/small values of X and Y (values greater/less than than their respective means) occur together in higher probability than do those where large/small values for X occur with small/large values of Y. A negative covariance implies the opposite, that large/small values for X occur with small/large values of Y in higher probability than large/small values of X and Y occurring together.

(f) If X and Y are independent then $\text{cov}(X,Y) = 0$. However, $\text{cov}(X,Y) = 0$ does not mean that X and Y are independent.

(g) The *conditional mean of* X *given* $Y = y$, denoted $E(X|Y=y)$, is defined as the expected value of X with conditional distribution/density $f(x|y)$. In the case of discrete random variables this is

$$E(X|Y=y) = sum_x x f(x|y)$$

and in the case of continuous random variables this is

$$E(X|Y=y) = int_{-\infty}^{\infty} x f(x|y)\, dx.$$

Recall that

$$f(x|y) = \frac{f(x,y)}{h(y)}$$

where $h(y)$ is the marginal distribution/density of Y.

□

2. Two professional taste-testers are rating various cookie recipes. They give a rating of 1, 2 or 3 for each recipe. Let X and Y represent the ratings given by each person. The joint distribution of ratings is given below (i.e. entries are the percentage of ratings that the first person has given x while the second person has given y).

		x = 1	2	3
	1	0.10	0.08	0.06
y	2	0.12	0.12	0.06
	3	0.16	0.10	0.20

(a) Find $E(X)$ and $E(Y)$.

(b) Find $\text{var}(X)$ and $\text{var}(Y)$.

(c) Find $\text{cov}(X,Y)$.

(d) The *correlation* between X and Y is defined as $\rho(X,Y) = \frac{\text{cov}(X,Y)}{\sqrt{\text{var}(X)}\sqrt{\text{var}(Y)}}$, provided $\text{var}(X), \text{var}(Y) \neq 0$. It provides a measure of the degree of linearity between X and Y (this would appear in course on statistics). Find the correlation between X and Y.

(e) Find the expected rating of person Y given that person X has rated a 3.

Solution. (a)

$$E(X) = \sum_x \sum_y x f(x,y) = 1\cdot(0.10+0.12+0.16)+2\cdot(0.08+0.12+0.10)+3\cdot(0.06+0.06+0.20) = 1.94$$

$$E(Y) = \sum_x \sum_y y f(x,y) = 1\cdot(0.10+0.08+0.06)+2\cdot(0.12+0.12+0.06)+3\cdot(0.16+0.10+0.20) = 2.22$$

(b)

$$E(X^2) = \sum_x \sum_y x f(x,y) = 1^2\cdot(0.10+0.12+0.16)+2^2\cdot(0.08+0.12+0.10)+3^2\cdot(0.06+0.06+0.20) = 4.46$$

$$\mathrm{var}(X) = E(X^2) - (E(X))^2 = 4.46 - (1.94)^2 = 0.6964.$$

$$E(Y^2) = \sum_x \sum_y y f(x,y) = 1^2\cdot(0.10+0.08+0.06)+2^2\cdot(0.12+0.12+0.06)+3^2\cdot(0.16+0.10+0.20) = 5.58$$

$$\mathrm{var}(Y) = E(Y^2) - (E(Y))^2 = 5.58 - (2.22)^2 = 0.6516.$$

(c)

$$\begin{aligned} E(XY) &= \sum_x \sum_y xy f(x,y) \\ &= 1\cdot(0.10) + 2\cdot(0.08) + 3\cdot(0.06) + 2\cdot(0.12) + 4\cdot(0.12) \\ &\quad + 6\cdot(0.06) + 3\cdot(0.16) + 6\cdot(0.10) + 9\cdot(0.20) \\ &= 4.4 \end{aligned}$$

$$\mathrm{cov}(X,Y) = E(XY) - E(X)E(Y) = 4.4 - (1.94)(2.22) = 0.0932.$$

(d)

$$\rho(X,Y) = \frac{\mathrm{cov}(X,Y)}{\sqrt{\mathrm{var}(X)}\sqrt{\mathrm{var}(Y)}} = \frac{0.0932}{\sqrt{0.6964}\sqrt{0.6516}} \approx 0.1384.$$

(e) Let $g(x)$ be the marginal distribution for X. The conditional distribution for Y given $X = 3$ (denoted $f(y|3)$) is

$$\begin{aligned} f(1|3) &= \frac{f(3,1)}{g(3)} = \frac{0.06}{0.32} = \frac{3}{16}, \\ f(2|3) &= \frac{f(3,1)}{g(3)} = \frac{0.06}{0.32} = \frac{3}{16}, \\ f(2|3) &= \frac{f(3,1)}{g(3)} = \frac{0.20}{0.32} = \frac{10}{16}. \end{aligned}$$

The conditional expectation of Y given $X = 3$ is then

$$E(Y|X=3) = \sum_y y f(y|3) = 1\cdot\frac{3}{16} + 2\cdot\frac{3}{16} + 3\cdot\frac{10}{16} = \frac{39}{16} = 2.4375,$$

i.e. we expect person Y to give (on average) a rating of 2.437 for those cookies that person X has rated 3.

□

3. The joint density function for continuous random variables X and Y is given by

$$f(x,y) = \begin{cases} \frac{1}{3}(x+y) & 0 < x < 1, 0 < y < 2 \\ 0 & \text{otherwise} \end{cases}.$$

Find $\mathrm{cov}(X,Y)$. Are X and Y independent?

Solution.

$$\begin{aligned}
E(X) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xf(x,y)\;dx\;dy \\
&= \int_0^2\int_0^1 \frac{x}{3}(x+y)\;dx\;dy \\
&= \int_0^2 \frac{x^3}{9}+\frac{x^2y}{6}\Big|_0^1\;dy \\
&= \int_0^2 \frac{1}{9}+\frac{y}{6}\;dy \\
&= \frac{y}{9}+\frac{y^2}{12}\Big|_0^2 \\
&= \frac{5}{9}
\end{aligned}$$

$$\begin{aligned}
E(Y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yf(x,y)\;dx\;dy \\
&= \int_0^2\int_0^1 \frac{y}{3}(x+y)\;dx\;dy \\
&= \int_0^2 \frac{x^2y}{6}+\frac{xy^2}{3}\Big|_0^1\;dy \\
&= \int_0^2 \frac{y}{6}+\frac{y^2}{3}\;dy \\
&= \frac{y^2}{12}+\frac{y^3}{9}\Big|_0^2 \\
&= \frac{11}{9}
\end{aligned}$$

$$\begin{aligned}
E(XY) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)\;dx\;dy \\
&= \int_0^2\int_0^1 \frac{xy}{3}(x+y)\;dx\;dy \\
&= \int_0^2 \frac{x^3y}{9}\frac{x^2y^2}{6}\Big|_0^1\;dy \\
&= \int_0^2 \frac{y}{9}\frac{y^2}{6}\;dy \\
&= \frac{y^2}{18}\frac{y^3}{18}\Big|_0^2 \\
&= \frac{2}{3}
\end{aligned}$$

$$\operatorname{cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{2}{3} - \left(\frac{5}{9}\right)\left(\frac{11}{9}\right) = -\frac{1}{81}.$$

Since $\operatorname{cov}(X,Y) \neq 0$, it follows that X and Y are not independent. □

4. Let X be a continuous random variable with probability density given by

$$f(x) = \begin{cases} 1+x & -1 < x \leq 0 \\ 1-x & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}.$$

Let $U = X$ and $V = X^2$. Show that $\text{cov}(U, V) = 0$.

Solution.

$$\begin{aligned} E(U) = E(X) &= \int_{-\infty}^{\infty} x f(x)\, dx \\ &= \int_{-1}^{0} x + x^2\, dx + \int_{0}^{1} x - x^2\, dx \\ &= \frac{x^2}{2} + \frac{x^3}{3}\Big|_{-1}^{0} + \frac{x^2}{2} - \frac{x^3}{3}\Big|_{0}^{1} \\ &= -\frac{1}{2} + \frac{1}{3} + \frac{1}{2} - \frac{1}{3} \\ &= 0 \end{aligned}$$

$$\begin{aligned} E(V) = E(X^2) &= \int_{-\infty}^{\infty} x^2 f(x)\, dx \\ &= \int_{-1}^{0} x^2 + x^3\, dx + \int_{0}^{1} x^2 - x^3\, dx \\ &= \frac{x^3}{3} + \frac{x^4}{4}\Big|_{-1}^{0} + \frac{x^3}{3} - \frac{x^4}{4}\Big|_{0}^{1} \\ &= \frac{1}{3} - \frac{1}{4} + \frac{1}{3} - \frac{1}{4} \\ &= \frac{1}{6} \end{aligned}$$

$$\begin{aligned} E(UV) = E(X^3) &= \int_{-\infty}^{\infty} x^3 f(x)\, dx \\ &= \int_{-1}^{0} x^3 + x^4\, dx + \int_{0}^{1} x^3 - x^4\, dx \\ &= \frac{x^4}{4} + \frac{x^5}{5}\Big|_{-1}^{0} + \frac{x^4}{4} - \frac{x^5}{5}\Big|_{0}^{1} \\ &= -\frac{1}{4} + \frac{1}{5} + \frac{1}{4} - \frac{1}{5} \\ &= 0 \end{aligned}$$

$$\text{cov}(U, V) = E(UV) - E(U)E(V) = 0 - 0\left(\frac{1}{6}\right) = 0.$$

□

5. Suppose 2 balls are removed (without replacement) from an urn containing n red balls and m blue balls, with $n, m \geq 2$. For $i = 1, 2$, let $X_i = 1$ if the ith ball removed is red and $X_i = 0$ if it is blue (i.e. not red).

   (a) Do you think $\text{cov}(X_1, X_2)$ is positive, negative or zero?

(b) Compute $\text{cov}(X_1, X_2)$ to justify your answer to (a).

(c) Suppose the red balls are numbered 1 through n. Let $Y_i = 1$ if red ball number i is removed, and $Y_i = 0$ otherwise. Do you think $\text{cov}(Y_1, Y_2)$ is positive, negative or zero?

(d) Compute $\text{cov}(Y_1, Y_2)$ to justify your answer to (c).

Solution. (a) We might guess negative here. If the first ball is red, there is a greater chance the second one will not be red, and vice versa. i.e. there is a greater probability that high values for X_1 occur with low values for X_2 and vice versa.

(b)

$$\begin{aligned}
E(X_1) &= \sum_{x_1}\sum_{x_2} x_1 f(x_1, x_2) \\
&= 0 \cdot \left(\frac{m(m-1)}{(n+m)(n+m-1)} + \frac{nm}{(n+m)(n+m-1)}\right) \\
&\quad + 1 \cdot \left(\frac{nm}{(n+m)(n+m-1)} + \frac{n(n-1)}{(n+m)(n+m-1)}\right) \\
&= \frac{n}{n+m}
\end{aligned}$$

$$\begin{aligned}
E(X_2) &= \sum_{x_1}\sum_{x_2} x_2 f(x_1, x_2) \\
&= 0 \cdot \left(\frac{m(m-1)}{(n+m)(n+m-1)} + \frac{nm}{(n+m)(n+m-1)}\right) \\
&\quad + 1 \cdot \left(\frac{nm}{(n+m)(n+m-1)} + \frac{n(n-1)}{(n+m)(n+m-1)}\right) \\
&= \frac{n}{n+m}
\end{aligned}$$

$$\begin{aligned}
E(X_1X_2) &= \sum_{x_1}\sum_{x_2} x_1 x_2 f(x_1, x_2) \\
&= 0 \cdot \left(\frac{m(m-1)}{(n+m)(n+m-1)} + \frac{nm}{(n+m)(n+m-1)} + \frac{nm}{(n+m)(n+m-1)}\right) \\
&\quad + 1 \cdot \left(\frac{n(n-1)}{(n+m)(n+m-1)}\right) \\
&= \frac{n(n-1)}{(n+m)(n+m-1)}
\end{aligned}$$

$$\begin{aligned}
\text{cov}(X_1, X_2) = E(X_1X_2) - E(X_1)E(X_2) &= \frac{n(n-1)}{(n+m)(n+m-1)} - \left(\frac{n}{n+m}\right)^2 \\
&= \frac{n(n-1)(n+m) - n^2(m+n-1)}{(n+m)^2(n+m-1)} \\
&= \frac{-nm}{(n+m)^2(n+m-1)}
\end{aligned}$$

Therefore $\text{cov}(X_1, X_2) < 0$.

(c) A tough call? The likelihood of drawing ball 1 and/or 2 is low, and so the expected value for Y_1 and Y_2 should be closer to zero. There is a low probability that both Y_1 and Y_2 are 1 simultaneously, however there is a high probability that they are both 0 simultaneously. We will calculate directly to find out.

(d)

$$E(Y_1) = \sum_{y_1} y_1 f(y_1) = 0 \cdot \left(\frac{(n+m-1)(n+m-2)}{(n+m)(n+m-1)} \right) + 1 \cdot \left(\frac{2(n+m-1)}{(n+m)(n+m-1)} \right)$$
$$= \frac{2}{(n+m)}$$

Similarly $E(Y_2) = \frac{2}{(n+m)}$.

$$\begin{aligned} E(Y_1 Y_2) &= \sum_{y_1} \sum_{y_2} y_1 y_2 f(y_1, y_2) \\ &= 0 \cdot \left(\frac{(n+m-2)(n+m-3)}{(n+m)(n+m-1)} + \frac{2(n+m-2)}{(n+m)(n+m-1)} + \frac{2(n+m-2)}{(n+m)(n+m-1)} \right) \\ &\quad + 1 \cdot \left(\frac{2}{(n+m)(n+m-1)} \right) \\ &= \frac{2}{(n+m)(n+m-1)} \end{aligned}$$

$$\begin{aligned} \operatorname{cov}(Y_1, Y_2) = E(Y_1 Y_2) - E(Y_1)E(Y_2) &= \frac{2}{(n+m)(n+m-1)} - \left(\frac{2}{n+m} \right)^2 \\ &= \frac{2(n+m) - 4(n+m-1)}{(n+m)^2 (n+m-1)} \\ &= \frac{4 - 2n - 2m}{(n+m)^2 (n+m-1)} \end{aligned}$$

Since $m, n \geq 2$, it follows that $\operatorname{cov}(Y_1, Y_2) < 0$.

$\square$