MATH1550
Exercise Set 1 - Solutions

- Counting
- Permutations
- Combinations
- Partitions
- Binomial Coefficients

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1. You have the following options when purchasing a new car: automatic or manual transmission, with or without A/C, with or without power windows, a choice of 4 stereo systems, and a choice of black, white or red, exterior paint. How many different vehicle setups are possible?

   Solution.
   $$2 \cdot 2 \cdot 2 \cdot 4 \cdot 3 = 96$$
   □

2. A test consists of 12 true-false questions. In how many different ways can a student mark the test paper with one answer to each question?

   Solution. Since each question can be answered in two ways, there are
   $$2^{12} = 4096$$
   different ways to mark the test paper. □

3. The menu at certain restaurant gives you the following options:

Course:	**Options:**
Appetizers	Mozzarella sticks, onion rings, or wings
Main	Spaghetti, burger, pizza, chicken wrap or fish
Sides	Salad, fries, soup or veggies
Dessert	Cheesecake, ice cream, pie, or chocolate cake

   How many different meal options are possible?

   Solution. There are
   $$3 \cdot 5 \cdot 4 \cdot 4 = 240$$
   different meal options. □

4. How many different permutations of the word *addressee* are there?

   Solution.
   $$\frac{9!}{2! \cdot 3! \cdot 2!} = 15120$$
   □

5. How many different permutations of the word *rearrange* are there?

Solution. There are

$$\frac{9!}{3! \cdot 2! \cdot 2!} = 15120$$

different permutations. □

6. In a certain election, voters rank 4 candidates from a selection of 12; i.e. they will choose their first, second, third and fourth preferred candidate from the list of 12. How many different selections of this type are possible?

   Solution. Note that the order of the selection does matter in this case. We can count this as

   $${}_{12}P_4 = 12 \cdot 11 \cdot 10 \cdot 9 = 11880,$$

   or using the formula

   $${}_{12}P_4 = \frac{12!}{(12-4)!} = \frac{12!}{8!} = \frac{479001600}{40320} = 11880.$$

   □

7. In how many ways can 3 of 20 laboratory assistants be chosen to assist with an experiment?

   Solution. The order that the 3 people are chosen does not matter, here we are only concerned with the set of 3. The number of possible sets of 3 lab assistants is

   $${}_{20}C_3 = \binom{20}{3} = \frac{20!}{3!(20-3)!} = \frac{20!}{3!17!} = \frac{20 \cdot 19 \cdot 18}{3 \cdot 2 \cdot 1} = 1140.$$

   □

8. A research team composed of 2 chemists and 3 physicists is to be made. If there are 7 chemists and 9 physicists to choose from, how many different ways can this research team be made?

   Solution. The set of 2 chemists can be chosen in

   $$\binom{7}{2} = \frac{7!}{2!5!} = \frac{7 \cdot 6}{2 \cdot 1} = 21$$

   ways, and the 3 physicists can be chosen in

   $$\binom{9}{3} = \frac{9!}{3!6!} = \frac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1} = 84.$$

   Therefore there are

   $$\binom{7}{2} \cdot \binom{9}{3} = 21 \cdot 84 = 1764$$

   different ways to select the research team.

   □

9. You need to hire 2 electricians, 1 plumber and 3 carpenters for a construction job. There are 4, 6, and 8 skilled tradespeople, respectively, from each of these three trades. How many different construction teams can be formed for this job?

   Solution. There are

   $$\binom{4}{2}\binom{6}{1}\binom{8}{3} = 2016$$

   possible teams. □

10. There are 16 graduate students and four offices labeled A, B, C and D. Offices A and B hold 5 people, office C holds 4 and office D holds 2. How many different office assignments are possible?

Solution.

$$\binom{16}{5,5,4,2} = \frac{16!}{5!\cdot 5!\cdot 4!\cdot 2!} = 30270240$$

□

11. A child has 12 blocks, of which 6 are black, 4 are red, 1 is white and 1 is blue. If the child puts the blocks in a line, how many arrangements are possible?

Solution.

$$\binom{12}{6,4,1,1} = \frac{12!}{6!\cdot 4!\cdot 1!\cdot 1!} = 27720$$

□

12. In how many ways can 8 people be seated in a row if
    (a) There are no restrictions on the seating arrangement?
    (b) Persons A and B must sit next to each other?
    (c) There are 5 men and they must sit next to one another?
    (d) There are 4 married couples and each couple must sit together?

Solution. (a)

$$8! = 40320$$

(b) A and B can be next to each other as AB or BA in 2 different ways. Other than the block containing A and B, we have 6 people. That is 7 items in total. Permute these 7 items.

$$2\cdot 7! = 10080$$

(c) 5 men can be arranged among themselves in 5! ways. Consider the block of 5 men as one item. Together with the remaining 3 people this makes 4 items. Permute the 4 items.

$$5!4! = 2880$$

(d) Each couple can be arranged in itself in two ways (depending on who is on the right/left). This yields the factor 2^4 (a factor of 2 for each couple). The 4 couples then can be permuted in 4! ways.

$$4!2^4 = 384$$

□

13. In how many ways can 8 people be seated at a round table?

Solution.

$$(8-1)! = 5040$$

□

14. How many different 6-card hands be made from a deck of 52 cards so that there is at least one card from each of the 4 suits ($\heartsuit, \diamondsuit, \spadesuit, \clubsuit$)? (Do not count the order in which the cards are drawn from the deck.)

Solution. If 3 of the 6 cards have the same suit, and hence 1 of each suit for the remaining cards, there are

$$\binom{4}{1}\binom{13}{3}\binom{13}{1}\binom{13}{1}\binom{13}{1} = 2513368$$

possible 6 card hands. Otherwise, there are 2 different suits which have 2 cards each of that suit, and hence 1 card each for the other 2 suits, which total

$$\binom{4}{2}\binom{13}{2}\binom{13}{2}\binom{13}{1}\binom{13}{1} = 6169176$$

different hands. Adding these we get

$$2513368 + 6169176 = 8682544$$

possible 6 card hands where each suit appears on at least one card. □

15. A dance class consist of 22 students, of which 10 are women and 12 are men. If 5 men and 5 women are to be chosen and then paired off, how many results are possible?

    Solution. There are $\binom{10}{5}\binom{12}{5}$ possible choices of the 5 women and 5 men. They can then be paired up in 5! ways, since if we arbitrarily order the men then the first man can be paired with any of the 5 women, the next with any of the remaining 4, and so on. Hence, there are $5!\binom{10}{5}\binom{12}{5}$ possible results. □

16. Seven different gifts are to be distributed among 10 children. How many distinct results are possible if no child is to receive more than one gift?

    Solution. The first gift can go to any of the 10 children, the second to any of the remaining 9 children, and so on. Hence, there are $10 \cdot 9 \cdot 8 \cdots 5 \cdot 4 = 604800$ possibilities. □

17. How many ways can 14 students be placed into 3 offices, A, B and C, where Office A holds 5 people, Office B holds 6 people and Office C holds 3 people?

    Solution.
    $$\binom{14}{5,6,3} = \frac{14!}{5! \cdot 6! \cdot 3!} = 168168$$
    □

18. Use the Binomial Theorem to expand $(4x-2)^3$.

    Solution.
    $$(4x-2)^3 = \binom{3}{0}(4x)^3(-2)^0 + \binom{3}{1}(4x)^2(-2)^1 + \binom{3}{2}(4x)^1(-2)^2 + \binom{3}{3}(4x)^0(-2)^3$$
    $$= 64x^3 - 96x^2 + 48x - 8$$
    □

19. Expand $(3x^2+y)^5$.

Solution. Let $A = 3x^2$ and $B = y$, and apply the binomial expansion formula to $(A+B)^5$.

$$\begin{aligned}
(3x^2+y)^5 &= (A+B)^5 \\
&= \sum_{i=0}^{5} \binom{5}{i} A^{5-i} B^i \\
&= \binom{5}{0} A^5 B^0 + \binom{5}{1} A^4 B^1 + \binom{5}{2} A^3 B^2 + \binom{5}{3} A^2 B^3 + \binom{5}{4} A^1 B^4 + \binom{5}{5} A^0 B^5 \\
&= A^5 + 5A^4 B + 10A^3B^2 + 10A^2B^3 + 5AB^4 + B^5 \\
&= (3x^2)^5 + 5(3x^2)^4 y + 10(3x^2)^3 y^2 + 10(3x^2)^2 y^3 + 5(3x^2) y^4 + y^5 \\
&= 243x^{10} + 405x^8 y + 270x^6y^2 + 90x^4y^3 + 15x^2y^4 + y^5
\end{aligned}$$

□

20. What is the coefficient on $x^{12}y^4$ in the expansion of $(4x^3 + 3y^2)^6$?

Solution. Letting $a = 4x^3$ and $b = 3y^2$ we have

$$(4x^3 + 3y^2)^6 = (a+b)^6 = \sum_{r=0}^{6} \binom{6}{r} a^r b^{6-r} = \sum_{r=0}^{6} \binom{6}{r} (4x^3)^r (3y^2)^{6-r}.$$

The $x^{12}y^4$ term occurs when $r = 4$, where

$$\binom{6}{4} (4x^3)^4 (3y^2)^{6-4} = 15 \cdot 4^4 \cdot 3^2 \cdot x^{12}y^4 = 34560x^{12}y^4.$$

So the coefficient on $x^{12}y^4$ is 34560. □

21. What is the coefficient of $x_2^2 x_3^3 x_4^2$ in the expansion of $(x_1 + x_2 + x_3 + x_4)^7$?

Solution.

$$\binom{7}{0,2,3,2} = \frac{7!}{0! \cdot 2! \cdot 3! \cdot 2!} = 210$$

□

22. Expand $(x_1 + 2x_2 + 3x_3)^4$.

Solution. Let $A = x_1, B = 2x_2, C = 3x_3$. Start by listing all possible monomials $A^{r_1}B^{r_2}C^{r_3}$ where $r_1 + r_2 + r_3 = 4$; actually we only need to list the non-negative integer triples (r_1, r_2, r_3) where

$r_1 + r_2 + r_3 = 4$. They are

$$\begin{gathered}(4,0,0)\\(3,1,0)\\(3,0,1)\\(2,2,0)\\(2,1,1)\\(2,0,2)\\(1,3,0)\\(1,2,1)\\(1,1,2)\\(1,0,3)\\(0,4,0)\\(0,3,1)\\(0,2,2)\\(0,1,3)\\(0,0,4)\end{gathered}$$

The coefficient on the monomial $A^{r_1}B^{r_2}C^{r_3}$ in the expansion of $(A+B+C)^4$ is given by

$$\binom{4}{r_1, r_2, r_3} = \frac{4!}{r_1!r_2!r_3!}.$$

Therefore

$$\begin{aligned}(x_1 + 2x_2 + 3x_3)^4 &= (A+B+C)^4\\
&= \binom{4}{4,0,0}A^4B^0C^0 + \binom{4}{3,1,0}A^3B^1C^0 + \binom{4}{3,0,1}A^3B^0C^1 + \binom{4}{2,2,0}A^2B^2C^0\\
&+ \binom{4}{2,1,1}A^2B^1C^1 + \binom{4}{2,0,2}A^2B^0C^2 + \binom{4}{1,3,0}A^1B^2C^0 + \binom{4}{1,2,1}A^1B^2C^1\\
&+ \binom{4}{1,1,2}A^1B^1C^2 + \binom{4}{1,0,3}A^1B^0C^3 + \binom{4}{0,4,0}A^0B^4C^0 + \binom{4}{0,3,1}A^0B^3C^1\\
&+ \binom{4}{0,2,2}A^0B^2C^2 + \binom{4}{0,1,3}A^0B^1C^3 + \binom{4}{0,0,4}A^0B^0C^4\\
&= (x_1)^4 + 4(x_1)^3(2x_2)^1 + 4(x_1)^3(3x_3)^1 + 6(x_1)^2(2x_2)^2\\
&+ 12(x_1)^2(2x_2)^1(3x_3)^1 + 6(x_1)^2(3x_3)^2 + 4(x_1)^1(2x_2)^2 + 12(x_1)^1(2x_2)^2(3x_3)^1\\
&+ 12(x_1)^1(2x_2)^1(3x_3)^2 + 4(x_1)^1(3x_3)^3 + (2x_2)^4 + 4(2x_2)^3(3x_3)^1\\
&+ 6(2x_2)^2(3x_3)^2 + 4(2x_2)^1(3x_3)^3 + (3x_3)^4\\
&= x_1^4 + 8x_1^3x_2 + 12x_1^3x_3 + 24x_1^2x_2^2\\
&+ 72x_1^2x_2x_3 + 54x_1^2x_3^2 + 16x_1x_2^2 + 144x_1x_2^2x_3\\
&+ 216x_1x_2x_3^2 + 108x_1x_3^3 + 16x_2^4 + 96x_2^3x_3\\
&+ 216x_2x_3^2 + 216x_2x_3^3 + 81x_3^4\end{aligned}$$

$\square$