Example 3.4.6 *Find the cumulative distribution function $F(x)$ for*

$$f(x) = \begin{cases} 3e^{-3x} & \textit{for } x > 0 \\ 0 & \textit{elsewhere} \end{cases}$$

and use it to evaluate $P(0.5 \leq X \leq 1)$.

For $x > 0$ we have

$$F(x) = \int_{-\infty}^{\infty} f(t)\, dt = \int_0^x 3e^{-3t}\, dt = -e^{-3t}\Big|_0^x = -e^{-3x} + 1.$$

For $x \leq 0$, $f(x) = 0$.

So,

$$F(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ 1 - e^{-3x} & \text{for } x > 0 \end{cases}$$

Then using the theorem,

$$P(0.5 \leq X \leq 1) = \dots$$

3.5 Multivariate Distributions

We now consider the case when two or more random variables are defined on the same (joint) sample space. We start with the **bivariate** case, that is when two random variables X and Y are defined for a common sample space.

For example X could be the sum of rolling two dice, and Y could be the product.

Write $P(X = x, Y = y)$ for the probability of the intersection of events $X = x$ and $Y = y$.

Example 3.5.1 *Two caplets are randomly selected from a bottle containing 3 aspirin, 2 sedative, and 4 laxative.*

Let X be the number of aspirin selected, and Y be the number of sedative selected.

Find the probabilities associated to each possible pair of values for X and Y.

The possible pairs for X, Y are: $(0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (0, 2)$.

There are $\binom{9}{2} = 36$ different possible two-pill selections that can be drawn.

The number of different ways to draw x aspirin, y sedative, and therefore $2 = x - y$ laxative (where $0 \leq x + y \leq 2$) is

$$\binom{3}{x}\binom{2}{y}\binom{4}{2-x-y}.$$

Thus

$$P(X = x, Y = y) = \frac{\binom{3}{x}\binom{2}{y}\binom{4}{2-x-y}}{36}$$

We summarize the probabilities in a table:

		x		
		0	1	2
	0			
y	1			
	2			

If X and Y are discrete random variables, then the function

$$f(x, y) = P(X = x, Y = x)$$

for each pair (x, y) in the range of X and Y is called the **joint probability distribution of X and Y.**

Theorem 3.5.2 *A bivariate function f can serve as a joint probability distribution for discrete random variables X and Y if and only if*

1. $f(x, y) \geq 0$.
2. $\sum_x \sum_y f(x, y) = 1$, *where the double sum is taken over all possible pairs* (x, y).

		x		
		0	1	2
	0	$\frac{6}{36}$	$\frac{12}{36}$	$\frac{3}{36}$
y	1	$\frac{8}{36}$	$\frac{6}{36}$	
	2	$\frac{1}{36}$		

To verify the theorem for the caplet example, note that all values are non-negative and

$$\sum_x \sum_y f(x, y) = f(0, 0) + f(1, 0) + f(2, 0) + f(0, 1) + f(1, 1) + f(0, 2)$$

$$= \frac{6}{36} + \frac{12}{36} + \frac{3}{36} + \frac{8}{36} + \frac{6}{36} + \frac{1}{36} = 1.$$

Example 3.5.3 *Suppose the joint probability distribution of X and Y is given by*

$$f(x, y) = c(x^2 + y^2)$$

for all pairs (x, y) with $x = -1, 0, 1, 3$ and $y = -1, 2, 3$. Find the value of c

If X and Y are discrete random variables, with joint probability distribution f, then the function

$$F(x, y) = P(X \leq x, Y \leq y) = \sum_{s \leq x} \sum_{t \leq y} f(s, t)$$

defined for all $x, y \in \mathbb{R}$, is called the **joint cumulative distribution of X and Y**, or the **joint distribution function**.

		x		
		0	1	2
	0	$\frac{6}{36}$	$\frac{12}{36}$	$\frac{3}{36}$
y	1	$\frac{8}{36}$	$\frac{6}{36}$	
	2	$\frac{1}{36}$		

Example 3.5.4 *Let $F(x, y)$ be the joint cumulative distribution of the caplet example. Find $F(2, 1.3)$.*

To find $F(2, 1.3) = P(X \leq 2, Y \leq 1.3)$ we must sum the probabilities $f(x, y)$ over all pairs (x, y) in the range of X and Y with $x \leq 2$ and $y \leq 1.3$.

The pairs included here are ...

Therefore

$$F(2, 1.3) = f(0, 0) + \dots$$

As with the single variable case we have the following properties.

Theorem 3.5.5 *If $F(x, y)$ is the joint cumulative distribution for discrete random variables X and Y then*

1. $\lim_{x,y\to-\infty} F(x, y) = 0$,

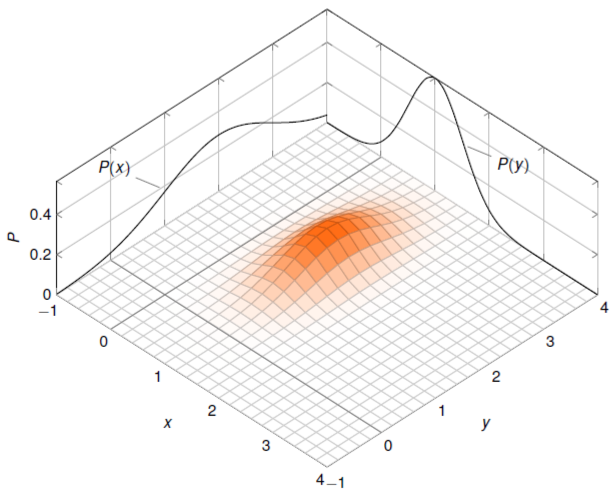
2. $\lim_{x,y\to\infty} F(x, y) = 1$, *and*

3. *If $a \leq c$ and $b \leq d$ then $F(a, b) \leq F(c, d)$.*

We say that random variables X and Y are **jointly continuous** if there exists a function $f(x, y)$ defined for all $x, y \in \mathbb{R}$, such that

$$P((X, Y) \in A) = \iint_{(x,y)\in A} f(x, y)\, dx\, dy$$

for any region A in the xy-plane.



The function $f(x, y)$ is called the **joint probability density function of X and Y.**

A bivariate function f is a **joint probability density function** of a pair of continuous random variables X and Y if its values satisfy

1. $f(x, y) \geq 0$ for all $x, y \in \mathbb{R}$.
2. $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1.$

Example 3.5.6 *Let*

$$f(x,y) = \begin{cases} \frac{3}{5}x(y+x) & \textit{for } 0 < x < 1, 0 < y < 2 \\ 0 & \textit{elsewhere} \end{cases}$$

1. *Verify that f can serve as a probability density function for two jointly continuous random variables X and Y.*
2. *For $A = \{(x,y)|0 < X < \frac{1}{2}, 1 < Y < 2\}$ find $P((X,Y) \in A)$.*

1) We see that $f(x,y) \geq 0$ for all $0 < x < 1, 0 < y < 2$.

Next we integrate over the entire plane.

We see that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_0^2\int_0^1 \frac{3}{5}x(y+x)\,dx\,dy$$

since $f(x,y) = 0$ for all other regions.