MATH1550
Practice Set 5 - Solutions

These exercises are suited to Chapter 3, Continuous Random Variables to Cumulative Distributions (for continuous random variables).
Topics Covered:

- Continuous random variables
- Probability density functions
- Cumulative distributions for continuous random variables

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1. (a) What is a probability density function and how is it used?
   (b) What conditions must a probability density function satisfy for a continuous random variable?
   (c) How is a *cumulative distribution* for a continuous random variable defined?
   (d) Give an example of a probability experiment with a continuous random variables defined on it.

   Solution. (a) For a continuous randoms X, a function f is a probability density function if

$$P(a \leq X \leq b) = \int_a^b f(x)\, dx.$$

   (b) A probability density function f must satisfy $f(x) \geq 0$ for all $x \in \mathbb{R}$ and

$$\int_{-\infty}^{\infty} f(x)\, dx = 1.$$

   (c) A cumulative distribution for a continuous random variable X is a function F such that $F(x) = P(X \leq x)$. (The definition is the same for discrete random variables.)
   (d) An LED light bulb is chosen at random from a factory that produces them. Let X be lifespan, in hours, of the selected bulb. Here we would allow for fractions of hours (e.g. 43576.286 hours) not just whole numbers of hours.

$\square$

2. Determine whether the following functions can serve as a valid probability density function.

   (a)
$$f(x) = \begin{cases} \frac{1}{2}x & 0 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

   (b)
$$f(x) = \begin{cases} \frac{1}{2}x & 0 \leq x \leq 2 \\ 0 & \text{elsewhere} \end{cases}$$

   (c)
$$f(x) = \begin{cases} \frac{1}{4}x & -1 \leq x \leq 3 \\ 0 & \text{elsewhere} \end{cases}$$

   (d)
$$f(x) = \begin{cases} 1 & 0 \leq x \leq 0.5 \\ 2 & 0.5 \leq x \leq 0.75 \\ 0 & \text{elsewhere} \end{cases}$$

(e)

$$f(x) = \begin{cases} \sin(x) & 0 \leq x \leq \pi \\ 0 & \text{elsewhere} \end{cases}$$

Solution. (a) Since

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_0^1 \frac{1}{2}x\, dx = \frac{x^2}{4}\Big|_0^1 = \frac{1}{4} \neq 1$$

this is not a valid probability density function.

(b) Since $f(x) \geq 0$ for all $x \in \mathbb{R}$ and

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_0^2 \frac{1}{2}x\, dx = \frac{x^2}{4}\Big|_0^2 = 1,$$

this is a valid probability density function.

(c) Since, for example, $f(-1) = -\frac{1}{4} < 0$, this is not a valid probability density function.

(d) Since $f(x) \geq 0$ for all $x \in \mathbb{R}$ and

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_0^{0.5} 1\, dx + \int_{0.5}^{0.75} 2\, dx = x\Big|_0^{0.5} + 2x\Big|_{0.5}^{0.75} = 0.5 + (2(0.75) - 2(0.5)) = 1,$$

this is a valid probability density function.

(e) Since

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_0^{\pi} \sin(x)\, dx = -\cos(x)\Big|_0^{\pi} = -(-1) - (-1) = 2 \neq 1$$

this is not a valid probability density function.

□

3. Let X be a continuous random variable with probability density function given by

$$f(x) = \begin{cases} \frac{2x+k}{8} & 0 \leq x \leq k \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find a suitable value for $k \in \mathbb{R}$.

(b) Evaluate the following probabilities

i. $P(0 \leq X \leq 1)$.

ii. $P(0.5 \leq X \leq 3)$.

iii. $P(X \leq 2)$.

iv. $P(X > 1)$

(c) Find the cumulative distribution function for X.

Solution. (a) Note that $k \geq 0$. We must have that

$$1 = \int_{-\infty}^{\infty} f(x)\, dx = \int_0^k \frac{2x+k}{8}\, dx = \frac{x^2+kx}{8}\Big|_0^k = \frac{k^2}{4}$$

which implies that $k = 2$.

(b) Evaluate the following probabilities

i.

$$P(0 \leq X \leq 1) = \int_0^1 \frac{2x+2}{8}\, dx = \frac{x^2+2x}{8}\Big|_0^1 = \frac{3}{8}.$$

ii.
$$P(0.5 \le X \le 3) = \int_{0.5}^{2} \frac{2x+2}{8}\,dx = \frac{x^2+2x}{8}\Big|_{0.5}^{2} = 1 - \frac{5}{32} = \frac{27}{32} = 0.84375.$$

iii.
$$P(X \le 2) = \int_{-\infty}^{2} f(x)\,dx = \int_{0}^{2} \frac{2x+2}{8}\,dx = 1.$$

iv.
$$P(X > 1) = \int_{1}^{\infty} f(x)\,dx = \int_{1}^{2} \frac{2x+2}{8}\,dx = \frac{x^2+2x}{8}\Big|_{0.5}^{2} = 1 - \frac{3}{8} = \frac{5}{8},$$

or using the value for $P(0 \le X \le 1)$ above,
$$P(X > 1) = 1 - P(X \le 1) = 1 - \int_{-\infty}^{1} f(x)\,dx = 1 - \int_{0}^{1} f(x)\,dx = 1 - \frac{3}{8} = \frac{5}{8}.$$

(c) For $x < 0$,
$$P(X \le x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{x} 0\,dt = 0.$$

For $0 \le x \le 2$,
$$P(X \le x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{0} 0\,dt + \int_{0}^{x} \frac{2t+2}{8}\,dt = 0 + \frac{t^2+2t}{8}\Big|_{0}^{x} = \frac{x^2+2x}{8}.$$

For $x > 2$,
$$P(X \le x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{0} 0\,dt + \int_{0}^{2} \frac{2t+2}{8}\,dt + \int_{2}^{x} 0\,dt = 0 + \frac{t^2+2t}{8}\Big|_{0}^{2} + 0 = 1.$$

In summary, the cumulative distribution for X is
$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x^2+2x}{8} & 0 \le x \le 2 \\ 1 & x > 2 \end{cases}$$

□

4. Let X be a continuous random variable with cumulative distribution given by the function
$$F(x) = \begin{cases} 0 & x \le 10 \\ 1 - \frac{10}{x} & 10 < x \end{cases}$$

(a) Evaluate the following probabilities
   i. $P(X \le 20)$.
   ii. $P(X \ge 20)$.
   iii. $P(15 < X < 35)$.

(b) Find the density function for X.

Solution. (a) i.
$$P(X \le 20) = F(20) = 1 - \frac{10}{20} = 0.5.$$

ii.
$$P(X \ge 20) = 1 - P(X \le 20) = 1 - 0.5 = 0.5.$$

iii.
$$P(15 < X < 35) = F(35) - F(15) = \left(1 - \frac{10}{35}\right) - \left(1 - \frac{10}{15}\right) = \frac{8}{21} \approx 0.3810.$$

(b) Since $f(x) = \frac{d}{dx}F(x)$, we have
$$f(x) = \begin{cases} 0 & x \leq 10 \\ \frac{10}{x^2} & 10 < x \end{cases}$$

$\square$

5. Let X be the volume of water, in millions of litres, that a certain city uses per day. From past experience, the probability density for X has been found to be
$$f(x) = \begin{cases} \frac{1}{9}xe^{-\frac{x}{3}} & x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

(a) What is the probability that the daily water consumption does not exceed 6 million litres?

(b) What is the probability that the city's water supply will be inadequate if the daily capacity for this city is 9 million litres?

Solution. (a) Note that $\int xe^x \, dx = xe^x - e^x + C$.
$$P(X \leq 6) = \int_{-\infty}^{6} f(x)\, dx = \int_{0}^{6} \frac{1}{9}xe^{-\frac{x}{3}}\, dx = -\frac{1}{3}xe^{-\frac{x}{3}} - e^{-\frac{x}{3}}\Big|_0^6 = -2e^{-2} - e^{-2} - (-1) = 1 - \frac{3}{e^2}$$
$$\approx 0.5940$$

(b)
$$P(X > 9) = \int_{9}^{\infty} f(x)\, dx = \int_{9}^{\infty} \frac{1}{9}xe^{-\frac{x}{3}}\, dx = -\frac{1}{3}xe^{-\frac{x}{3}} - e^{-\frac{x}{3}}\Big|_9^\infty = 3e^{-3} + e^{-3} = \frac{4}{e^3} \approx 0.1991$$

or
$$P(X > 9) = 1 - P(X \leq 9) = 1 - \int_{-\infty}^{9} f(x)\, dx = 1 - \int_{0}^{9} \frac{1}{9}xe^{-\frac{x}{3}}\, dx = 1 - \left(-\frac{1}{3}xe^{-\frac{x}{3}} - e^{-\frac{x}{3}}\Big|_0^9\right)$$
$$= 1 + 3e^{-3} + e^{-3} - 1 = \frac{4}{e^3} \approx 0.1991$$

$\square$

6. Let X be a continuous random variable with probability density
$$f(x) = \begin{cases} x & 0 < x < 1 \\ 2 - x & 1 \leq x < 2 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Evaluate the following probabilities

i. $P(0.2 \leq X \leq 0.5)$.

ii. $P(0.75 \leq X \leq 1.25)$.

iii. $P(X \leq 1.5)$.

(b) Find the cumulative distribution function for X.

(c) Use the cumulative distribution function to compute the following probabilities

i. $P(X \leq 0.5)$.

ii. $P(0.5 \leq X \leq 1.5)$.

iii. $P(X \geq 1.5)$.

Solution. (a) i.

$$P(0.2 \leq X \leq 0.5) = \int_{0.2}^{0.5} f(x)\,dx = \int_{0.2}^{0.5} x\,dx = \frac{x^2}{2}\Big|_{0.2}^{0.5} = \frac{21}{200} = 0.105$$

ii.

$$P(0.75 \leq X \leq 1.25) = \int_{0.75}^{1.25} f(x)\,dx = \int_{0.75}^{1} x\,dx + \int_{1}^{1.25} 2-x\,dx = \left[\frac{x^2}{2}\right]_{0.75}^{1} + \left[2x - \frac{x^2}{2}\right]_{1}^{1.25}$$

$$= \frac{7}{32} + \frac{7}{32} = \frac{7}{16} = 0.4375$$

iii.

$$P(X \leq 1.5) = \int_{-\infty}^{1.5} f(x)\,dx = \int_{0}^{1} x\,dx + \int_{1}^{1.5} 2-x\,dx = \left[\frac{x^2}{2}\right]_{0}^{1} + \left[2x - \frac{x^2}{2}\right]_{1}^{1.5}$$

$$= \frac{1}{2} + \frac{3}{8} = \frac{7}{8} = 0.875$$

(b) For $x \leq 0$, $P(X \leq x) = 0$ and for $x \geq 2$, $P(X \leq x) = 1$. For $0 < x < 1$,

$$P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt = \int_{0}^{x} t\,dt = \frac{t^2}{2}\Big|_{0}^{x} = \frac{x^2}{2}.$$

For $1 \leq x < 2$,

$$P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt = \int_{0}^{1} t\,dt + \int_{1}^{x} 2-t\,dt = \frac{t^2}{2}\Big|_{0}^{1} + \left[2t - \frac{t^2}{2}\right]_{1}^{x} = 2x - \frac{x^2}{2} - 1.$$

In summary, the cumulative distribution function is

$$F(x) = \begin{cases} 0 & x \leq 0 \\ \frac{x^2}{2} & 0 < x < 1 \\ 2x - \frac{x^2}{2} - 1 & 1 \leq x < 2 \\ 1 & x \geq 2 \end{cases}$$

(c) i.

$$P(X \leq 0.5) = F(0.5) = \frac{(0.5)^2}{2} = \frac{1}{8}.$$

ii.

$$P(0.5 \leq X \leq 1.5) = F(1.5) - F(0.5) = \left(2(1.5) - \frac{(1.5)^2}{2} - 1\right) - \frac{1}{8} = \frac{7}{8} - \frac{1}{8} = \frac{3}{4}$$

iii.

$$P(X \geq 1.5) = 1 - P(X \leq 1.5) = 1 - F(1.5) = 1 - \frac{7}{8} = \frac{1}{8}$$

$\square$