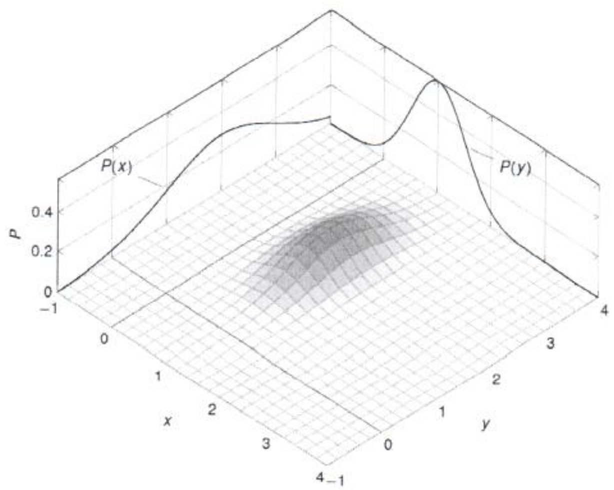
We say that random variables X and Y are **jointly continuous** if there exists a function $f(x, y)$ defined for all $x, y \in \mathbb{R}$, such that

$$P((X,Y) \in A) = \iint_{(x,y)\in A} f(x,y)\, dx\, dy$$

for any region A in the xy-plane.



The function $f(x, y)$ is called the **joint probability density function of X and Y.**

A bivariate function f that is a **joint probability density function** of a pair of continuous random variables X and Y ~~if its values satisfy~~ has the following properties:

1. $f(x, y) \geq 0$ for all $x, y \in \mathbb{R}$.
2. $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dx\, dy = 1.$

Example 3.5.6 *Let*

$$f(x,y) = \begin{cases} \frac{3}{5}x(y+x) & \text{for } 0 < x < 1, 0 < y < 2 \\ 0 & \text{elsewhere} \end{cases}$$

1. *Verify that f can serve as a probability density function for two jointly continuous random variables X and Y.*

2. *For $A = \{(x,y) | 0 < X < \frac{1}{2}, 1 < Y < 2\}$ find $P((X,Y) \in A)$.*

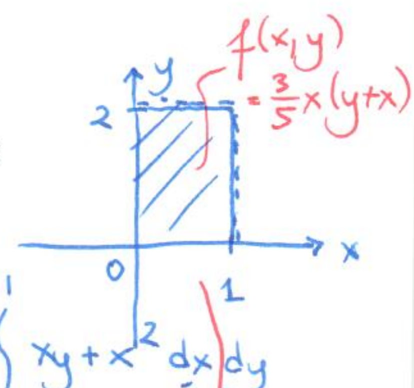
$$f(x,y) = \begin{cases} \frac{3}{5}x(y+x) & \text{for } 0<x<1,\ 0<y<2 \\ 0 & \text{elsewhere} \end{cases}$$

1) We see that $f(x,y) \geq 0$ for all $0 < x < 1, 0 < y < 2$, and $f(x,y)=0$ elsewhere; $f(x,y) \geq 0$ everywhere. Therefore property 1 of p.88 is satisfied.

Next we integrate over the entire plane.

We see that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, dx\, dy = \int_0^2 \left(\int_0^1 \frac{3}{5} x(y+x)\, dx \right) dy$$



since $f(x,y) = 0$ for all other regions.

$$\int_0^2 \left(\int_0^1 \frac{3}{5}x(y+x)\,dx \right) dy = \int_0^2 \left(\int_0^1 \frac{3}{5}(xy+x^2)\,dx \right) dy = \frac{3}{5} \int_0^2 \left(\int_0^1 xy + x^2\, dx \right) dy$$

iterated integrals

We proceed by integrating first with respect to x, treating y as constant:

$$\int_0^2 \left(\int_0^1 \frac{3}{5}(yx + x^2)\, dx \right)\, dy = \frac{3}{5}\int_0^2 \left(\int_0^1 xy + x^2\, dx \right) dy$$

$$= \frac{3}{5}\int_0^2 \left(\left(\frac{x^2}{2}\cdot y + \frac{x^3}{3} \right) \Bigg|_{x=0}^{x=1} \right) dy = \frac{3}{5}\int_0^2 \left(\left(\frac{y}{2} + \frac{1}{3} \right) - (0+0) \right) dy$$

$$= \frac{3}{5}\int_0^2 \frac{y}{2} + \frac{1}{3}\, dy$$

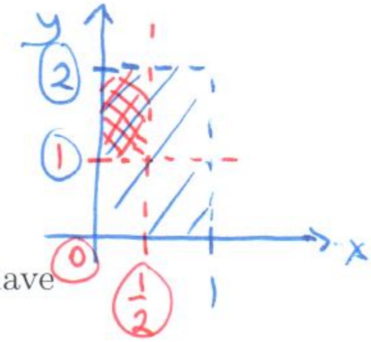
Finally integrate with respect to y.

$$\frac{3}{5}\int_0^2 \frac{y}{2} + \frac{1}{3}\, dy = \frac{3}{5}\left(\frac{y^2}{4} + \frac{y}{3} \right) \Bigg|_{y=0}^{y=2}$$

$$= \frac{3}{5}\left(\left(\frac{4}{4} + \frac{2}{3} \right) - (0+0) \right) = \frac{3}{5} \cdot \frac{5}{3} = 1$$

Therefore $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1$ as required.

$$f(x,y) = \begin{cases} \frac{3}{5}x(y+x) & \text{for } 0<x<1,\ 0<y<2 \\ 0 & \text{elsewhere} \end{cases}$$



2) For $A = \{(x,y)|0 < X < \frac{1}{2}, 1 < Y < 2\}$, therefore we have

$$P((X,Y) \in A) = \int_1^2 \int_0^{\frac{1}{2}} \frac{3}{5}x(y+x)\,dx\,dy$$

$$= \frac{3}{5}\int_1^2 \left(\int_0^{\frac{1}{2}} xy + x^2\,dx\right) dy = \frac{3}{5}\int_1^2 \left(\frac{x^2 y}{2} + \frac{x^3}{3}\right)\Bigg|_{x=0}^{x=\frac{1}{2}} dy$$

$$= \frac{3}{5}\int_1^2 \left(\frac{y}{8} + \frac{1}{24}\right) - (0+0)\,dy = \frac{3}{5}\int_1^2 \frac{3y+1}{24}\,dy = \frac{3}{5}\int_1^2 \frac{1}{24}\cdot(3y+1)\,dy$$

$$= \frac{3}{5}\cdot\frac{1}{24}\int_1^2 3y+1\,dy = \frac{1}{40}\cdot\int_1^2 3y+1\,dy = \frac{1}{40}\cdot\left(\frac{3y^2}{2} + y\right)\Bigg|_{y=1}^{y=2}$$

$$= \frac{1}{40}\cdot\left((6+2) - \left(\frac{3}{2}+1\right)\right) = \frac{1}{40}\left(\frac{11}{2}\right) = \frac{11}{80}$$

compare to p. 85
(discrete case)

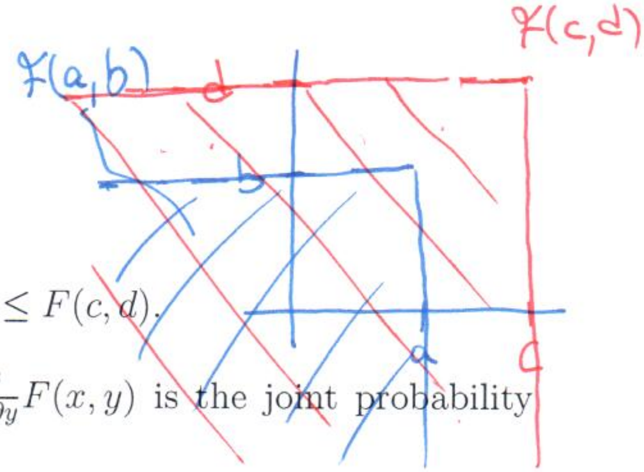
If X and Y are jointly continuous random variables, with joint probability density f, the function given by

$$F(x,y) = P(X \le x, Y \le y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(s,t)\, ds\, dt$$

for $x, y \in \mathbb{R}$, is called the **joint cumulative distribution function of X and Y** (or simply the **joint distribution function**).

As with the discrete case we have that

1. $\lim_{x,y \to -\infty} F(x,y) = 0$,
2. $\lim_{x,y \to \infty} F(x,y) = 1$, and
3. If $a \le c$ and $b \le d$ then $F(a,b) \le F(c,d)$.



It also follows that $f(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y)$ is the joint probability density function.

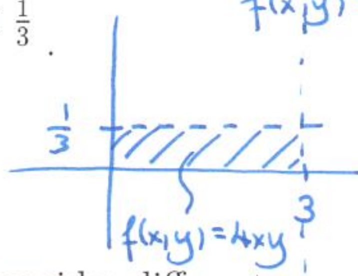
"take the partial derivative with respect to y and then take the partial derivative of the result with respect to x".

Example 3.5.7 *The joint probability density function of X and Y is given by*

outside: $f(x,y)=0$

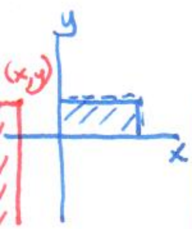
$$f(x,y)=\begin{cases} 4xy & \text{for } 0<x<3, 0<y<\frac{1}{3} \\ 0 & \text{elsewhere} \end{cases}.$$

Find $F(x,y)$.



To find $F(x,y)=\int_{-\infty}^{y}\int_{-\infty}^{x} f(s,t)\,ds\,dt$ we must consider different regions in the plane where $f(x,y)$ is defined.

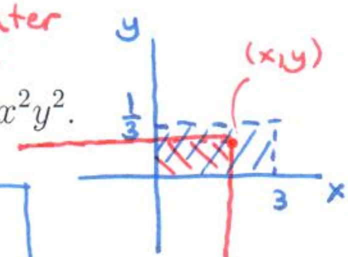
a) If either $x<0$ or $y<0$ then $f(x,y)=0$ and so $F(x,y)=0$.

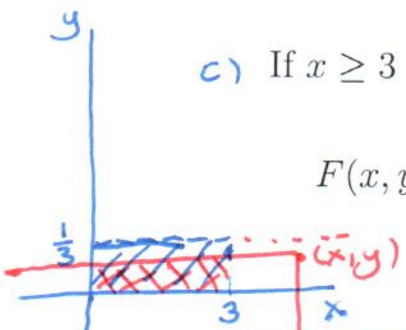


b) If $0<x<3$ and $0<y<\frac{1}{3}$ then

later

$$F(x,y)=\int_{-\infty}^{y}\int_{-\infty}^{x} f(s,t)\,ds\,dt=\int_{0}^{y}\int_{0}^{x} 4st\,ds\,dt=x^2y^2.$$





c) If $x\geq 3$ and $0<y<\frac{1}{3}$ then

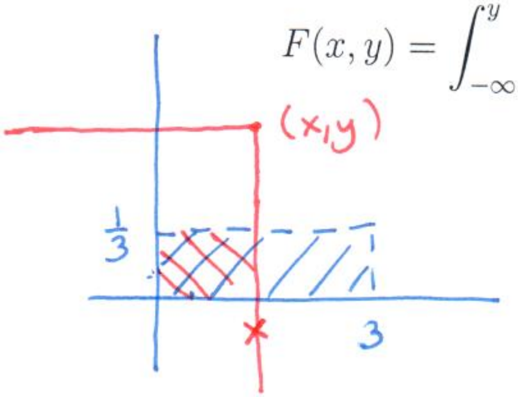
later

$$F(x,y)=\int_{-\infty}^{y}\int_{-\infty}^{x} f(s,t)\,ds\,dt=\int_{0}^{y}\int_{0}^{3} 4st\,ds\,dt=9y^2.$$

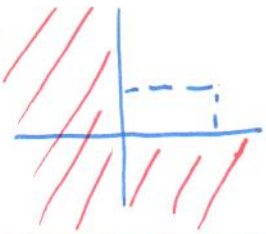
d) If $0<x<3$ and $y\geq\frac{1}{3}$ then

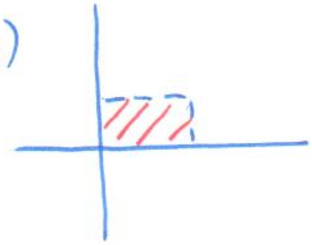
later

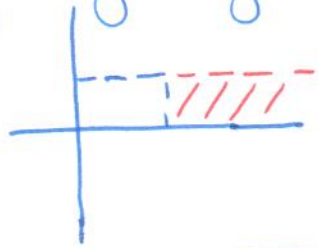
$$F(x,y)=\int_{-\infty}^{y}\int_{-\infty}^{x} f(s,t)\,ds\,dt=\int_{0}^{\frac{1}{3}}\int_{0}^{x} 4st\,ds\,dt=\frac{x^2}{9}.$$

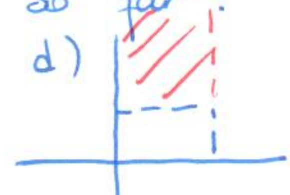


We considered points in the following regions so far:

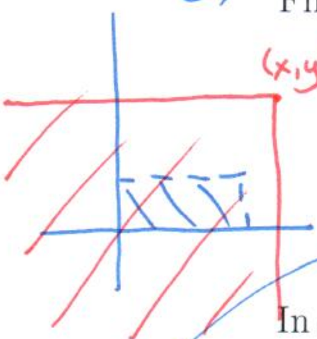
a) 

b) 

c) 

d) 

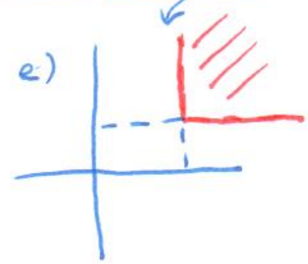
we need one more case

e) 

Finally if $x \geq 3$ and $y \geq \frac{1}{3}$ then

$$F(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(s, t)\, ds\, dt = \int_{0}^{\frac{1}{3}} \int_{0}^{3} 4st\, ds\, dt = 1.$$

by the properties of prob.

e) 

In summary

$$F(x, y) = \begin{cases} 0 & \text{for } x < 0 \text{ or } y < 0 & \text{from a)} \\ x^2y^2 & \text{for } 0 < x < 3, 0 < y < \frac{1}{3} & \text{from b)} \\ 9y^2 & \text{for } x \geq 3, 0 < y < \frac{1}{3} & \text{from c)} \\ \frac{x^2}{9} & \text{for } 0 < x < 3, y \geq \frac{1}{3} & \text{from d)} \\ 1 & \text{for } x \geq 3, y \geq \frac{1}{3} & \text{from e)} \end{cases} .$$

Note that joint probability distributions/densities can be defined similarly for three or more random variables, but that is beyond the scope of this course.

p.93 part b)

$$\int_0^y \left(\int_0^x 4st\, ds \right) dt = 4 \int_0^y \left(\int_0^x st\, ds \right) dt = 4 \int_0^y \left. \frac{s^2 t}{2} \right|_{s=0}^{s=x} dt$$

$$= 4 \int_0^y \frac{x^2 t}{2} - 0\, dt = 4 \int_0^y \frac{x^2 t}{2}\, dt = 4 \cdot \left. \left(\frac{x^2 t^2}{4} \right) \right|_{t=0}^{t=y}$$

$$= 4 \cdot \left(\frac{x^2 y^2}{4} - 0 \right) = x^2 y^2$$

p.93 part c)

$$\int_0^y \left(\int_0^3 4st\, ds \right) dt = 4 \int_0^y \left(\int_0^3 st\, ds \right) dt = 4 \int_0^y \left. \frac{s^2 t}{2} \right|_{s=0}^{s=3} dt$$

$$= 4 \int_0^y \left(\frac{9t}{2} - 0 \right) dt = 4 \int_0^y \frac{9t}{2}\, dt = 4 \cdot \frac{9}{2} \int_0^y t\, dt$$

$$= 4 \cdot \frac{9}{2} \left(\left. \frac{t^2}{2} \right|_{t=0}^{t=y} \right) = 4 \cdot \frac{9}{2} \left(\frac{y^2}{2} - 0 \right) = \cancel{4} \cdot \frac{9}{\cancel{2}} \cdot \frac{y^2}{\cancel{2}}$$

$$= 9y^2$$

p. 93 part d)

$$\int_0^{1/3}\left(\int_0^x 4st\, ds\right)dt = 4\int_0^{1/3}\left(\int_0^x st\, ds\right)dt$$

$$= 4\int_0^{1/3}\left(\frac{s^2t}{2}\Bigg|_{s=0}^{s=x}\right)dt = 4\int_0^{1/3}\frac{x^2t}{2} - 0\, dt$$

$$= 4\int_0^{1/3}\frac{x^2t}{2}\, dt = 4\cdot\frac{1}{2}\int_0^{1/3} x^2t\, dt$$

$$= 4\cdot\frac{1}{2}\left(\frac{x^2t^2}{2}\Bigg|_{t=0}^{t=\frac{1}{3}}\right) = 4\cdot\frac{1}{2}\left(\frac{x^2\cdot\frac{1}{9}}{2} - 0\right)$$

$$= 4\cdot\frac{1}{2}\cdot\frac{x^2}{18} = \frac{x^2}{9}$$