Infinite Discrete Sample Space: When a sample space has countably infinite outcomes, probabilities must be assigned via some rule/formula as opposed to listing them individually.

$P(A_1) = \frac{1}{2}$

$P(A_2) = \frac{1}{2^2} = \frac{1}{4}$

Example 2.1.14 *Tossing a coin until heads is reached:*
$S = \{\underset{A_1}{H}, TH, TTH, TTTH, TTTTH, \ldots\}$.

If A_i is the event of i flips, then $P(A_i) = \frac{1}{2^i}$ defines a probability on S (assuming countable additivity). From the geometric series formula we get P2:

$$P(S) = \sum_{i=1}^{\infty} P(A_i) = -1 + \sum_{i=0}^{\infty} \frac{1}{2^i} = -1 + \frac{1}{1-\frac{1}{2}} = -1 + 2 = 1.$$

$$= P(A_1) + P(A_2) + P(A_3) + \ldots = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$$

$$= \sum_{i=1}^{\infty} \frac{1}{2^i} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

see: geometric series

So, assigning probabilities using the rule $P(A_i) = \frac{1}{2^i}$ is allowed because the sample has probability 1. (Also, note that all probabilities are non-negative.)

Brief note on infinite series:

- Sequence: countably infinite list of real numbers; $r_1, r_2, r_3, \ldots$.

  e.g. $\quad 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots$ Fibonacci sequence

- Geometric sequence: terms occur in a common ratio r; $a, ar, ar^2, ar^3 \ldots$ $\cdot r \quad \cdot r \quad \cdot r$

  $r = \frac{1}{3}$

  e.g. $\quad 4, \frac{4}{3}, \frac{4}{9}, \frac{4}{27}, \frac{4}{81}, \ldots\ldots$

- Partial sum of a sequence: $T_n = \sum_{i=1}^{n} r_i = r_1 + r_2 + \cdots + r_n$.

- Series: Limit of partial sums (if it exists); $\lim_{n\to\infty} T_n = \sum_{i=1}^{\infty} r_i$.

- Partial sum of a geometric sequence: $G_n = a + ar + \cdots + ar^n$.

$$
\begin{aligned}
(1-r)G_n &= (1-r)(a + ar + ar^2 + \cdots + ar^n) \\
&= a + \cancel{ar} + \cancel{ar^2} + \cdots + \cancel{ar^n} \\
&\quad - (\cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \cdots + ar^{n+1}) \\
&= a - ar^{n+1}
\end{aligned}
$$

$(1-r)\, G_n = a - ar^{n+1}$

So $G_n = \frac{a - ar^{n+1}}{(1-r)}$ (for $r \neq 1$) .

$\Rightarrow G_n = \frac{a - ar^{n+1}}{1-r}$

- If $-1 < r < 1$ then $\lim_{n\to\infty} r^n = 0$, and so it follows that

$$\boxed{\sum_{i=0}^{\infty} ar^i = \lim_{n\to\infty} G_n = \frac{a}{1-r}.}$$

$\lim_{n\to\infty} G_n = \lim_{n\to\infty} \frac{a - \cancel{ar^{n+1}}}{1-r}$

$\lim_{n\to\infty} G_n = \frac{a}{1-r}$

- In the coin flipping example above, $a = 1$ and $r = \frac{1}{2}$.

Theorem 2.1.15 *(The Probability of an Event with equally likely outcomes)*

If an experiment has N equally likely outcomes and A is an event made up of k of those outcomes then

$$P(A) = \frac{k}{N}.$$

Example 2.1.16 *A five-card poker hand dealt from a deck of 52 playing cards is said to be a full house if it consists of three of a kind and a pair. If all the five-card hands are equally likely, what is the probability of being dealt a full house?*

three of a kind

$$8\spadesuit, J\spadesuit, 8\heartsuit, 8\clubsuit, J\diamondsuit$$

pair

$P(A) = \frac{k}{N}$

A: "being dealt a full house"

k = # of different full house configurations

N = # of 5-card hands

So, $N = \binom{52}{5}$ (There are $N = \binom{52}{5}$ five-card hands, and they are all equally likely)

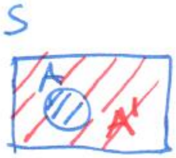
We need to find k.

For the three of a kind, we have 13 options (rank) ($A, 2, 3, \dots, 10, J, Q, K$). Once the rank is determined, we choose 3 of the 4 cards of that rank ($\binom{4}{3}$ ways)

12 options left for the rank of the pair. Once the rank of the pair is determined, we choose 2 of the 4 cards of that rank ($\binom{4}{2}$ ways). So, $k = 13 \cdot \binom{4}{3} \cdot 12 \cdot \binom{4}{2}$

So, $P(A) = \frac{k}{N} = \frac{13 \cdot \binom{4}{3} \cdot 12 \cdot \binom{4}{2}}{\binom{52}{5}} \approx \dots$ (compute this)

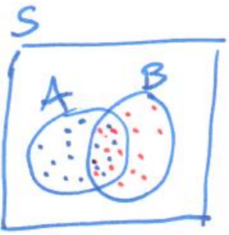
2.1.7 Rules of Probability



Theorem 2.1.17 *Let S be a sample space with probability measure P, and let A and B be events in S. Then*

1. $P(A) + P(A') = 1$, *or equivalently* $P(A') = 1 - P(A)$. → complement $P(A) = 1 - P(A')$
2. $P(\emptyset) = 0$.
3. *If* $A \subseteq B$ *then* $P(A) \leq P(B)$.

4. $0 \leq P(A) \leq 1$.



5. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. *(think of the Venn diagrams)* Inclusion-Exclusion Principle for two events/sets

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Example 2.1.18 *What is the probability that among a group of r people at least two have the same birthday (ignoring leap years)?*

year: 365 days

Let A be the event that at least two people have the same birthday.

What event does A' denote? A' is the event that all r people have different birthdays.

$r = 1: \; P(A') = \frac{365}{365} = 1$

$r = 2: \; P(A') = \frac{365}{365} \cdot \frac{364}{365}$

$r = 3: \; P(A') = \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365}$

so, for r people, $P(A') = \frac{365 \cdot 364 \cdot \ldots \cdot (365-r+1)}{365^r}$

Then, since $P(A) = 1 - P(A') = 1 - \dfrac{365 \cdot 364 \cdot \ldots \cdot (365-r+1)}{365^r}$

(when $r = 23$, from this calculation we find that $P(A) > 0.5$. In a room with 23 or more people, the probability of having two people share the same birthday is more than 50%.)

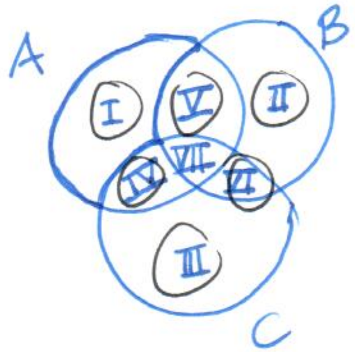
Theorem 2.1.19 *(The Inclusion-Exclusion Principle - two events)* *If A and B are any two events in sample space S, then*

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B).}$$

Theorem 2.1.20 *(The Inclusion-Exclusion Principle - three events)* *If A, B and C are any three events in sample space S, then*

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) \\ &\quad - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C). \end{aligned}$$

(Note that the Inclusion-Exclusion Principle can be generalized to many more sets.)



$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) \\ &\quad - P(A \cap B) - P(A \cap C) \\ &\quad - P(B \cap C) \\ &\quad + P(A \cap B \cap C) \end{aligned}$$

$$\begin{aligned} &\mathrm{I} + \cancel{\mathrm{V}} + \cancel{\mathrm{IV}} + \cancel{\mathrm{VII}} \\ &+ \mathrm{II} + \mathrm{V} + \cancel{\mathrm{VI}} + \cancel{\mathrm{VII}} \\ &+ \mathrm{III} + \mathrm{IV} + \mathrm{VI} + \cancel{\mathrm{VII}} \\ &\quad\quad \mathrm{VII} \end{aligned}$$

$P(A)$ $P(B)$ $P(A \cap B)$

Example 2.1.21 *Suppose the probabilities are* 0.86, 0.35, *and* 0.29, *respectively, that a family owns a laptop computer, a desktop computer, or both kinds. What is the probability that a family owns either ~~or both kinds of computer~~ and what is the probability that a family owns neither?*

Let A be the event that family owns a laptop and B the event that a family owns a desktop. Then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.86 + 0.35 - 0.29 = \underline{0.92}.$$

The probability that a family owns neither is $P((A \cup B)') = 1 - \underbrace{P(A \cup B)}_{= 0.92}$

$$P\left((A \cup B)'\right) = 1 - P(A \cup B) = 1 - 0.92 = 0.08.$$

$= 1 - 0.92 = 0.08$

2.2 Conditional Probability

Example 2.2.1 *Mayoral candidate Alice receives 56 percent of the entire vote, but only 47 percent of the female vote.*

Let $P(A)$ be the probability that a randomly selected person has voted for Alice, and let $P(A|F)$ denote the probability that a randomly selected female has voted for Alice.

"probability of A given F"

So,

$P(A) = 0.56$

$P(A|F) = 0.47$

Value $P(A|F)$ is called the ***conditional probability of*** A ***relative to*** F, *or the* ***conditional probability of*** A ***given*** F.

Example 2.2.2 *Let A be the event of rolling 8 with two dice; Then $P(A) = \frac{5}{36} \approx 0.1389$.*

as the sum of two dice

$A = \{(2,6), (3,5), (4,4), (5,3), (6,2)\}$

Suppose we are given that the roll of die 1 is 3. Knowing this (i.e. given that this event has occurred) what is the probability of rolling an 8?

Let $B = \{(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)\}$; the event that die 1 is 3. Since these outcomes are equally likely prior to knowing die 1

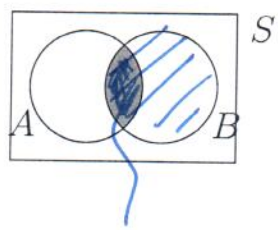
only outcome that results in 8 given the first die is a 3.

is 3, they are still equally likely given that B has occurred.

So given B has occurred, each have probability $\frac{1}{6}$ (the other 30 outcomes have probability 0).

Therefore the probability of rolling an 8 given that die 1 is a 3 is $P(A|B) = \frac{1}{6} \approx 0.1667$

In the example above, if B occurs, then in order for A to occur, the outcome must lie in both A and B. Thus $A \cap B$ becomes the event of interest, and B is considered the new sample space.



A: "rolling an 8"
B: "die 1 is a 3"

$A \cap B$

The conditional probability of A given B is the probability of $A \cap B$ relative to the probability of B.

Definition 2.2.3 *If A and B are events in S and $P(B) \neq 0$, then the* ***conditional probability*** *of A given B is*

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Example 2.2.4 *Results of a survey of 50 car dealerships:*

	Good service under warranty	*Poor service under warranty*
In business 10 years or more	*16*	*4*
In business less than 10 years	*10*	*20*

If a person randomly chooses one dealership, what is the probability that

(a) they get one who provides good service under warranty?

(b) they get one who provides good service under warranty if they select from dealers in business 10 years or more?

Let G be the event of getting good service and T the event that a dealer has been in business 10 year or more. Let $n(A)$ denote the number of element in event A.

G: event that you get good service under warranty

T: event that dealership in business 10 years or more.

a) We want $P(G)$. $n(G) = 16 + 10 = 26$ $\quad$ total $= 50$ $\quad P(G) = \frac{26}{50} = 0.52$

b) $P(G|T) = \frac{P(G \cap T)}{P(T)} = \frac{\frac{16}{50}}{\frac{20}{50}} = \frac{16}{20} = 0.8$

Example 2.2.5 *A coin is tossed twice. Assuming all* ~~*points*~~ outcomes *in the sample space*

$$S = \{HH, HT, TH, TT\}$$

are equally likely, what is the probability that both flips land on heads given that

$A := \{HH\}$

(a) the first flip is heads? $B = \{HH, HT\}$

(b) at least one flip is heads? $\{HH, HT, TH\} = C$

$S = \{HH, HT, TH, TT\}$

$A = \{HH\}$ - event that both flips are heads,

$B = \{HH, HT\}$ event that first flip is heads,

$C = \{HH, HT, TH\}$ - event that at least one flip is heads.

a) $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{2/4} = \frac{1}{2} = 0.5$

b) $P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3} \approx 0.33$

A = event that order will be ready for shipment on time

B = event that order is delivered on time

Example 2.2.6 *A manufacturer of airplane parts knows from past experience that the probability is* 0.80 *that an order will be ready for shipment on time, and it is* 0.72 *that an order will be ready for shipment on time and will also be delivered on time.*

$P(A) = 0.8$

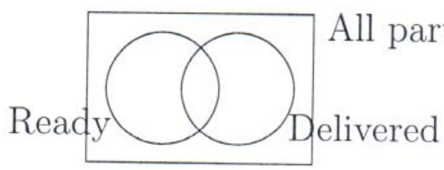
$P(A \cap B) = 0.72$

What is the probability that such an order will be delivered on time given that it was ready for shipment on time?

Q: $P(B|A) = \dfrac{P(B \cap A)}{P(A)}$

$= \dfrac{0.72}{0.8}$

$= 0.9$



this is the probability that the order is delivered on time given it was ready for shipment on time.

Conditional Probability Multiplication Rule:

Theorem 2.2.7 *If A and B are events in S and $P(A) \neq 0$, then*

$$P(A \cap B) = P(A) \cdot P(B|A).$$

This means that the probability that both A and B will occur is the product of the probability of A and the probability of B given A.

~~Now back to the problem:~~

$$P(B|A) = \frac{P(B \cap A)}{P(A)} \quad P(A) \neq 0$$

$$\Rightarrow P(A) \cdot P(B|A) = \underbrace{P(B \cap A)}_{= P(A \cap B)}$$

$$P(A) \cdot P(B|A) = P(A \cap B)$$

Example 2.2.8 *A pot contains 8 red balls and 4 green balls. We draw 2 balls without replacement. If each ball has an equally likely chance of being chosen, what is the probability that both balls are red?*

(*without replacement means that the first ball is not returned to the pot before the second ball is drawn*)

Let R_1 be the event that ball 1 is red, and R_2 be the event that ball 2 is red. Then $R_1 \cap R_2$ is the event that both are red.

We want $P(R_1 \cap R_2)$.

From p.41 (multiplication rule): $P(R_1 \cap R_2) = P(R_1) \cdot P(R_2 | R_1)$

$$P(R_1) = \frac{8}{8+4} = \frac{8}{12} = \frac{2}{3}$$

$$P(R_2 | R_1) = \frac{7}{7+4} = \frac{7}{11}$$

So, $P(R_1 \cap R_2) = \frac{2}{3} \cdot \frac{7}{11} = \frac{14}{33}$
≈ 0.4242

Since the the outcomes are equally likely, we could also have computed the probability as the number of successful outcomes over total number of outcomes:

$$P(R_1 \cap R_2) = \frac{\binom{8}{2}}{\binom{12}{2}} = \frac{28}{66} = \frac{14}{33} \approx 0.4242.$$