MATH1550
Practice Set 5

These exercises are suited to Chapter 3, Continuous Random Variables to Cumulative Distributions (for continuous random variables).
Topics Covered:

- Continuous random variables
- Probability density functions
- Cumulative distributions for continuous random variables

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1. (a) What is a probability density function and how is it used?
   (b) What conditions must a probability density function satisfy for a continuous random variable?
   (c) How is a *cumulative distribution* for a continuous random variable defined?
   (d) Give an example of a probability experiment with a continuous random variables defined on it.

2. Determine whether the following functions can serve as a valid probability density function.

   (a)
   $$f(x) = \begin{cases} \frac{1}{2}x & 0 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

   (b)
   $$f(x) = \begin{cases} \frac{1}{2}x & 0 \leq x \leq 2 \\ 0 & \text{elsewhere} \end{cases}$$

   (c)
   $$f(x) = \begin{cases} \frac{1}{4}x & -1 \leq x \leq 3 \\ 0 & \text{elsewhere} \end{cases}$$

   (d)
   $$f(x) = \begin{cases} 1 & 0 \leq x \leq 0.5 \\ 2 & 0.5 \leq x \leq 0.75 \\ 0 & \text{elsewhere} \end{cases}$$

   (e)
   $$f(x) = \begin{cases} \sin(x) & 0 \leq x \leq \pi \\ 0 & \text{elsewhere} \end{cases}$$

3. Let X be a continuous random variable with probability density function given by
   $$f(x) = \begin{cases} \frac{2x+k}{8} & 0 \leq x \leq k \\ 0 & \text{elsewhere} \end{cases}$$

   (a) Find a suitable value for $k \in \mathbb{R}$.
   (b) Evaluate the following probabilities
      i. $P(0 \leq X \leq 1)$.
      ii. $P(0.5 \leq X \leq 3)$.
      iii. $P(X \leq 2)$.
      iv. $P(X > 1)$
   (c) Find the cumulative distribution function for X.

4. Let X be a continuous random variable with cumulative distribution given by the function

$$F(x) = \begin{cases} 0 & x \leq 10 \\ 1 - \frac{10}{x} & 10 < x \end{cases}$$

   (a) Evaluate the following probabilities
      i. $P(X \leq 20)$.
      ii. $P(X \geq 20)$.
      iii. $P(15 < X < 35)$.
   (b) Find the density function for X.

5. Let X be the volume of water, in millions of litres, that a certain city uses per day. From past experience, the probability density for X has been found to be

$$f(x) = \begin{cases} \frac{1}{9} x e^{-\frac{x}{3}} & x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

   (a) What is the probability that the daily water consumption does not exceed 6 million litres?
   (b) What is the probability that the city's water supply will be inadequate if the daily capacity for this city is 9 million litres?

6. Let X be a continuous random variable with probability density

$$f(x) = \begin{cases} x & 0 < x < 1 \\ 2 - x & 1 \leq x < 2 \\ 0 & \text{elsewhere} \end{cases}$$

   (a) Evaluate the following probabilities
      i. $P(0.2 \leq X \leq 0.5)$.
      ii. $P(0.75 \leq X \leq 1.25)$.
      iii. $P(X \leq 1.5)$.
   (b) Find the cumulative distribution function for X.
   (c) Use the cumulative distribution function to compute the following probabilities
      i. $P(X \leq 0.5)$.
      ii. $P(0.5 \leq X \leq 1.5)$.
      iii. $P(X \geq 1.5)$.