MATH1550
Practice Set 6 - Solutions

These exercises are suited to Chapter 3, Multivariate Distributions to Joint Cumulative Distributions (for continuous random variables).
Topics Covered:

- Joint probability distributions for discrete random variables
- Joint cumulative distributions for discrete random variables
- Joint probability densities for continuous random variables
- Joint cumulative distributions for continuous random variables

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1. (a) Suppose X and Y are discrete random variables defined on the same sample space. What properties must a joint probability distribution $f(x, y)$ satisfy?
   (b) Give an example of probability experiment with two different discrete random variables defined on it, say X and Y, and give their joint probability distribution.
   (c) How is a *cumulative distribution* for jointly distributed discrete random variables X and Y defined?
   (d) Suppose X and Y are continuous random variables defined on the same sample space. What properties must a joint probability density $f(x, y)$ satisfy?
   (e) How is a *cumulative distribution* for jointly distributed continuous random variables X and Y defined?

Solution. (a) A bivariate function f can serve as a joint probability distribution for discrete random variables X and Y if and only if

i. $f(x, y) \geq 0$.
ii. $\displaystyle\sum_x \sum_y f(x, y) = 1$, where the sums are taken over all possible pairs (x, y).

(b) Example: Suppose a loonie ($1 coin) and a toonie ($2 coin) are tossed once each. Let X be the number of heads that appear. let $Y = 1$ if the loonie shows heads $Y = 2$ if the toonie shows heads, and $y = 0$ otherwise. Assuming fair coins (and that tosses are independent), the joint distribution of X and Y is

		$x=0$	$x=1$	$x=2$
y	0	$\frac{1}{4}$	0	0
	1	0	$\frac{1}{4}$	0
	2	0	$\frac{1}{4}$	0
	3	0	0	$\frac{1}{4}$

(c) The *cumulative distribution* for jointly distributed discrete random variables X and Y is a function F such that $F(x, y) = P(X \leq x, Y \leq y)$ for all $x, y \in \mathbb{R}$.

(d) A bivariate function f can serve as a joint probability density function of a pair of continuous random variables X and Y if it satisfies:

i. $f(x, y) \geq 0$ for all $x, y \in \mathbb{R}$.
ii. $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1$.

(e) The *cumulative distribution* for jointly distributed continuous random variables X and Y is a function F such that $F(x, y) = P(X \leq x, Y \leq y)$ for all $x, y \in \mathbb{R}$.

$\square$

2. (a) A fair coin is tossed twice. Let $X = 1$ if the first toss is heads and $X = 0$ if the first toss is tails. Let $Y = 1$ if both tosses are heads and $Y = 0$ otherwise. Give the joint distribution for X and Y.

   (b) A fair coin is tossed twice. Let $X = 1$ if the first toss is heads and $X = 0$ if the first toss is tails. Let $Y = 1$ if the second toss is heads and $Y = 0$ otherwise. Give the joint distribution for X and Y.

Solution. (a) The (X, Y) pairs $(0, 0)$, $(1, 0)$ and $(1, 1)$ correspond to the events $\{TT, TH\}$, $\{HT\}$ and $\{HH\}$ respectively. The pair $(0, 1)$ is impossible. The probability distribution is summarized in the table below.

		x	
		0	1
y	0	$\frac{1}{2}$	$\frac{1}{4}$
	1	0	$\frac{1}{4}$

(b) Note that the (X, Y) pairs $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$ correspond to the outcomes TT, TH, HT, and HH respectively. The probability distribution is summarized in the table below.

		x	
		0	1
y	0	$\frac{1}{4}$	$\frac{1}{4}$
	1	$\frac{1}{4}$	$\frac{1}{4}$

$\square$

3. Let X and Y be discrete random variables with joint probability distribution given by the following table:

			x	
		-5	0	10
y	1	k	0.3	0.18
	2	0.08	0.2	0.12

   (a) Determine the appropriate value for $k \in \mathbb{R}$ so that is is a valid joint probability distribution.

   (b) Find the following probabilities
   - $P(X = 0, Y = 2)$
   - $P(X \leq 2, Y = 1)$
   - $P(X < 10, Y = 2)$
   - $P(X > -2, Y \leq 3)$
   - $P(X = 0)$

- $P(Y \leq 3)$

Solution. (a) We require that

$$k + 0.3 + 0.18 + 0.08 + 0.2 + 0.12 = 1$$

and so it follows that $k = 0.12$.

(b)
- $P(X = 0, Y = 2) = 0.20$
- $P(X \leq 2, Y = 1) = 0.12 + 0.3 = 0.28$
- $P(X < 10, Y = 2) = 0.08 + 0.2 = 0.28$
- $P(X > -2, Y \leq 3) = 0.3 + 0.18 + 0.2 + 0.12 = 0.8$
- $P(X = 0) = 0.3 + 0.2 = 0.5$
- $P(Y \leq 3) = 1$

$\square$

4. Let X and Y be jointly distributed continuous random variables with joint density

$$f(x,y) = \begin{cases} k(3x^2 + 4y) & -1 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find an appropriate value for $k \in \mathbb{R}$ so that f is a valid probability density.

(b) Compute the following probabilities:

i. $P(-1 \leq X \leq 0, 0.5 \leq Y \leq 1)$

ii. $P(X \leq 0, Y \geq 0)$

iii. $P(Y \leq 0.5)$

iv. $P(0 \leq X \leq 0.25)$

(c) Find the joint cumulative distribution for X and Y.

(d) Make use the joint cumulative distribution to find the probabilities in part (b).

Solution. (a) We require that

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, dx\, dy \\ &= \int_0^1 \int_{-1}^1 k(3x^2 + 4y)\, dx\, dy \\ &= \int_0^1 kx^3 + 4kxy \Big|_{-1}^{1}\, dy \\ &= \int_0^1 (k + 4ky) - (-k - 4ky)\, dy \\ &= \int_0^1 2k + 8ky\, dy \\ &= 2ky + 4ky^2 \Big|_0^1 \\ &= 6k \end{aligned}$$

from which it follows that $k = \frac{1}{6}$.

(b) Compute the following probabilities:

i.

$$
\begin{aligned}
P(-1 \le X \le 0, 0.5 \le Y \le 1) &= \int_{0.5}^{1} \int_{-1}^{0} \frac{1}{6}(3x^2 + 4y)\ dx\ dy \\
&= \frac{1}{6} \int_{0.5}^{1} x^3 + 4xy \Big|_{-1}^{0}\ dy \\
&= \frac{1}{6} \int_{0.5}^{1} (0) - (-1 - 4y)\ dy \\
&= \frac{1}{6} \int_{0.5}^{1} 1 + 4y\ dy \\
&= \frac{1}{6} \left[y + 2y^2\right]_{0.5}^{1} \\
&= \frac{1}{6} \left[3 - 1\right]_{0.5}^{1} \\
&= \frac{1}{3}
\end{aligned}
$$

ii.

$$
\begin{aligned}
P(X \le 0, Y \ge 0) &= \int_{0}^{1} \int_{-1}^{0} \frac{1}{6}(3x^2 + 4y)\ dx\ dy \\
&= \frac{1}{6} \int_{0}^{1} x^3 + 4xy \Big|_{-1}^{0}\ dy \\
&= \frac{1}{6} \int_{0}^{1} (0) - (-1 - 4y)\ dy \\
&= \frac{1}{6} \int_{0}^{1} 1 + 4y\ dy \\
&= \frac{1}{6} \left[y + 2y^2\right]_{0}^{1} \\
&= \frac{1}{2}
\end{aligned}
$$

iii.

$$
\begin{aligned}
P(Y \le 0.5) &= \int_{0}^{0.5} \int_{-1}^{1} \frac{1}{6}(3x^2 + 4y)\ dx\ dy \\
&= \frac{1}{6} \int_{0}^{0.5} x^3 + 4xy \Big|_{-1}^{1}\ dy \\
&= \frac{1}{6} \int_{0}^{0.5} (1 + 4y) - (-1 - 4y)\ dy \\
&= \frac{1}{6} \int_{0}^{0.5} 2 + 8y\ dy \\
&= \frac{1}{6} \left[2y + 4y^2\right]_{0}^{0.5} \\
&= \frac{1}{3}
\end{aligned}
$$

iv.

$$
\begin{aligned}
P(0 \le X \le 0.25) &= \int_0^1 \int_0^{0.25} \frac{1}{6}(3x^2 + 4y)\; dx\; dy \\
&= \frac{1}{6}\int_0^1 x^3 + 4xy\Big|_0^{0.25}\; dy \\
&= \frac{1}{6}\int_0^1 \frac{1}{64} + y\; dy \\
&= \frac{1}{6}\left[\frac{y}{64} + \frac{y^2}{2}\right]_0^1 \\
&= \frac{11}{128}
\end{aligned}
$$

(c) For $x < -1$ or $y < 0$,

$$
\int_{-\infty}^{y}\int_{-\infty}^{x} f(s,t)\; ds\; dt = \int_{-\infty}^{y}\int_{-\infty}^{x} 0\; ds\; dt = 0.
$$

For $-1 \le x \le 1$ and $0 \le y \le 1$,

$$
\begin{aligned}
\int_{-\infty}^{y}\int_{-\infty}^{x} f(s,t)\; ds\; dt &= \int_0^y \int_{-1}^{x} \frac{1}{6}(3s^2 + 4t)\; ds\; dt \\
&= \frac{1}{6}\int_0^y s^3 + 4st\Big|_{-1}^{x}\; dt \\
&= \frac{1}{6}\int_0^y (x^3 + 4xt) - (-1 - 4t)\; dt \\
&= \frac{1}{6}\int_0^y x^3 + 4xt + 1 + 4t\; dt \\
&= \frac{1}{6}\left[x^3 t + 2xt^2 + t + 2t^2\right]_0^y \\
&= \frac{1}{6}\left[x^3 y + 2xy^2 + y + 2y^2\right]
\end{aligned}
$$

For $-1 \le x \le 1$ and $y > 1$,

$$
\begin{aligned}
\int_{-\infty}^{y}\int_{-\infty}^{x} f(s,t)\; ds\; dt &= \int_0^1 \int_{-1}^{x} \frac{1}{6}(3s^2 + 4t)\; ds\; dt \\
&= \frac{1}{6}\int_0^1 s^3 + 4st\Big|_{-1}^{x}\; dt \\
&= \frac{1}{6}\int_0^1 (x^3 + 4xt) - (-1 - 4t)\; dt \\
&= \frac{1}{6}\int_0^1 x^3 + 4xt + 1 + 4t\; dt \\
&= \frac{1}{6}\left[x^3 t + 2xt^2 + t + 2t^2\right]_0^1 \\
&= \frac{1}{6}\left[x^3 + 2x + 3\right]
\end{aligned}
$$

For $0 \leq y \leq 1$, and $x > 1$,

$$\begin{aligned}
\int_{-\infty}^{y} \int_{-\infty}^{x} f(s,t)\; ds\; dt &= \int_{0}^{y} \int_{-1}^{1} \frac{1}{6}(3s^2 + 4t)\; ds\; dt \\
&= \frac{1}{6} \int_{0}^{y} s^3 + 4st \Big|_{-1}^{1}\; dt \\
&= \frac{1}{6} \int_{0}^{y} (1 + 4t) - (-1 - 4t)\; dt \\
&= \frac{1}{6} \int_{0}^{y} 2 + 8t\; dt \\
&= \frac{1}{6} \left[2t + 4t^2\right]_0^y \\
&= \frac{1}{6} \left[2y + 2y^2\right]
\end{aligned}$$

For $x > 1$ and $y > 1$

$$\int_{-\infty}^{y} \int_{-\infty}^{x} f(s,t)\; ds\; dt = \int_{0}^{1} \int_{-1}^{1} \frac{1}{6}(3s^2 + 4t)\; ds\; dt = 1$$

In summary:

$$F(x,y) = \begin{cases} 0 & x < -1 \text{ or } y < 0 \\ \frac{1}{6}\left(x^3 y + 2xy^2 + y + 2y^2\right) & -1 \leq x \leq 1, 0 \leq y \leq 1 \\ \frac{1}{6}\left(x^3 + 2x + 3\right) & -1 \leq x \leq 1, y > 1 \\ \frac{1}{6}\left(2y + 2y^2\right) & 0 \leq y \leq 1, x > 1 \\ 1 & x > 1, y > 1 \end{cases}$$

(d) i. $P(-1 \leq X \leq 0, 0.5 \leq Y \leq 1) = F(0,1) - F(0,0.5) = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$

ii. $P(X \leq 0, Y \geq 0) = F(0,1) = \frac{1}{2}$

iii. $P(Y \leq 0.5) = F(\infty, 0.5) = \frac{1}{3}$

iv. $P(0 \leq X \leq 0.25) = F(0.25, \infty) - F(0, \infty) = \frac{75}{128} - \frac{1}{2} = \frac{11}{128}$

□

5. Consider the function

$$f(x,y) = \begin{cases} 6e^{-2x-3y} & x > 0, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Show that f is a valid joint probability density.

Solution. Note that $x, y >$ implies $6e^{-2x-3y} > 0$, and thus $f(x,y) \geq 0$ for all $x, y \in \mathbb{R}$. Also,

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy &= \int_0^{\infty}\int_0^{\infty} 6e^{-2x-3y}\,dx\,dy \\
&= \int_0^{\infty}\int_0^{\infty} -3e^{-2x-3y}\Big|_0^{\infty}\,dy \\
&= \int_0^{\infty}\int_0^{\infty} 0-(-3e^{-3y})\,dy \\
&= \int_0^{\infty}\int_0^{\infty} 3e^{-3y}\,dy \\
&= -e^{-3y}\Big|_0^{\infty} \\
&= 0-(-e^0) \\
&= 1
\end{aligned}$$

Thus f is a valid joint probability density. □

6. A group of university students attend a conference and where they are given a discount coupon for coffee and a discount coupon for cake at a local cafe. Let X be the percentage of students who make use of the coffee coupon and Y the percentage of students who make use of the cake coupon. The joint probability density of X is Y is given by

$$f(x,y) = \begin{cases} \frac{2}{5}(x+4y) & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) What is the probability that at most 20 percent of the students use the coffee coupon and less than 50 percent use the cake coupon?

   (b) What is the probability that at least 75 percent use the coffee coupon and between 25 and 75 percent use the cake coupon?

   (c) What is the probability that more than 50 percent of the students use the cake coupon?

   (d) Find the joint cumulative distribution for X and Y.

Solution. (a)

$$\begin{aligned}
P(X \leq 0.2, Y < 0.5) &= \int_0^{0.5}\int_0^{0.2} \frac{2}{5}(x+4y)\,dx\,dy \\
&= \int_0^{0.5} \frac{x^2+8xy}{5}\Big|_0^{0.2}\,dy \\
&= \int_0^{0.5} \frac{1}{125}+\frac{8y}{25}\,dy \\
&= \frac{y}{125}+\frac{4y^2}{25}\Big|_0^{0.5} \\
&= \frac{1}{250}+\frac{1}{25} \\
&= \frac{11}{250}
\end{aligned}$$

(b)

$$
\begin{aligned}
P(X \geq 0.75, 0.25 \leq Y \leq 0.75) &= \int_{0.25}^{0.75} \int_{0.75}^{1} \frac{2}{5}(x+4y)\, dx\, dy \\
&= \int_{0.25}^{0.75} \frac{x^2+8xy}{5}\Big|_{0.75}^{1}\, dy \\
&= \int_{0.25}^{0.75} \frac{1+8y}{5} - \frac{9}{80} - \frac{6y}{5}\, dy \\
&= \frac{y+4y^2}{5} - \frac{9y}{80} - \frac{3y^2}{5}\Big|_{0.25}^{0.75} \\
&= \frac{3}{5} - \frac{27}{320} - \frac{27}{80} - \frac{1}{10} + \frac{9}{320} + \frac{3}{80} \\
&= \frac{23}{160}
\end{aligned}
$$

(c)

$$
\begin{aligned}
P(Y \geq 0.75) &= \int_{0.5}^{1} \int_{0}^{1} \frac{2}{5}(x+4y)\, dx\, dy \\
&= \int_{0.5}^{1} \frac{x^2+8xy}{5}\Big|_{0}^{1}\, dy \\
&= \int_{0.5}^{1} \frac{1+8y}{5}\, dy \\
&= \frac{y+4y^2}{5}\Big|_{0.5}^{1} \\
&= 1 - \frac{3}{10} \\
&= \frac{7}{10}
\end{aligned}
$$

(d) For $x < 0$ or $y < 0$,

$$
\int_{-\infty}^{y} \int_{-\infty}^{x} f(s,t)\, ds\, dt = \int_{-\infty}^{y} \int_{-\infty}^{x} 0\, ds\, dt = 0.
$$

For $0 \leq x \leq 1$ and $0 \leq y \leq 1$,

$$
\begin{aligned}
\int_{-\infty}^{y} \int_{-\infty}^{x} f(s,t)\, ds\, dt &= \int_{0}^{y} \int_{0}^{x} \frac{2}{5}(s+4t)\, ds\, dt \\
&= \int_{0}^{y} \frac{s^2+8st}{5}\Big|_{0}^{x}\, dt \\
&= \int_{0}^{y} \frac{x^2+8xt}{5}\, dt \\
&= \frac{x^2t+4xt^2}{5}\Big|_{0}^{y} \\
&= \frac{x^2y+4xy^2}{5}
\end{aligned}
$$

For $0 \le x \le 1$ and $y > 1$,

$$\begin{aligned}
\int_{-\infty}^{y}\int_{-\infty}^{x} f(s,t)\,ds\,dt &= \int_0^1\int_0^x \frac{2}{5}(s+4t)\,ds\,dt \\
&= \int_0^1 \frac{s^2+8st}{5}\Big|_0^x\,dt \\
&= \int_0^1 \frac{x^2+8xt}{5}\,dt \\
&= \frac{x^2t+4xt^2}{5}\Big|_0^1 \\
&= \frac{x^2+4x}{5}
\end{aligned}$$

For $0 \le y \le 1$, and $x > 1$,

$$\begin{aligned}
\int_{-\infty}^{y}\int_{-\infty}^{x} f(s,t)\,ds\,dt &= \int_0^y\int_0^1 \frac{2}{5}(s+4t)\,ds\,dt \\
&= \int_0^y \frac{s^2+8st}{5}\Big|_0^1\,dt \\
&= \int_0^y \frac{1+8t}{5}\,dt \\
&= \frac{t+4t^2}{5}\Big|_0^y \\
&= \frac{y+4y^2}{5}
\end{aligned}$$

For $x > 1$ and $y > 1$

$$\int_{-\infty}^{y}\int_{-\infty}^{x} f(s,t)\,ds\,dt = \int_0^1\int_0^1 \frac{2}{5}(s+4t)\,ds\,dt = 1$$

In summary:

$$F(x,y) = \begin{cases} 0 & x<0 \text{ or } y<0 \\ \frac{x^2y+4xy^2}{5} & 0\le x\le 1, 0\le y\le 1 \\ \frac{x^2+4x}{5} & 0\le x\le 1, y>1 \\ \frac{y+4y^2}{5} & 0\le y\le 1, x>1 \\ 1 & x>1, y>1 \end{cases}$$

$\square$