Consider a random variable Y that denotes the proportion of successes in n trials.

So, $Y = \frac{X}{n}$, where X is the binomial random variable.
Then, the following holds.

Theorem 5.3.6 *Let X be a binomial random variable and let $Y = \frac{X}{n}$. Then*

Proof omitted

mean of Y

variance of Y

$$E(Y) = \theta, \quad \sigma_Y^2 = \frac{\theta(1-\theta)}{n}.$$

Chebyshev: $P(|X-\mu| < k\sigma) \geq 1 - \frac{1}{k^2}$ for all $k > 0$

mean of X

By Chebyshev's Theorem, with $C = k\sigma$ $(k = \frac{C}{\sigma})$ we have

mean of Y

$$P(|Y-\theta| < C) \geq 1 - \frac{1}{k^2} = 1 - \frac{1}{(\frac{C^2}{\sigma^2})} = 1 - \frac{\theta(1-\theta)}{C^2 n}$$

$k\sigma$

$$1 - \frac{1}{\frac{C^2}{\sigma^2}} = 1 - \frac{\sigma^2}{C^2} = 1 - \frac{\frac{\theta(1-\theta)}{n}}{C^2} = 1 - \frac{\theta(1-\theta)}{C^2 n}$$

Thus for any value of $C > 0$ we have

$$1 \geq P\left(\left|\frac{X}{n} - \theta\right| < C\right) \geq 1 - \frac{\theta(1-\theta)}{C^2 n}.$$

Y

as $n \to \infty$ → approaches 0

When n is large, the fraction on the right side gets small, and so

(proportion of successes)

$$\lim_{n\to\infty} P\left(\left|\frac{X}{n} - \theta\right| < C\right) = 1.$$

success probability

This holds for any $C > 0$, no matter how small.

Explanation: The more trials we perform, the more likely it is that the proportion of successes will be close to the probability of a success θ.

Example: In repeatedly flipping a balanced coin, the more flips we perform (n), the more likely that the proportion of heads obtained ($\frac{X}{n}$) will be 0.5 (θ).

The Binomial distribution gives the probability of getting x successes in n trials.

Suppose we want to know the probability that the kth success occurs precisely on trial n.

For the kth success to occur on the nth trial, there must be exactly $k-1$ successes on the first $n-1$ trials, and consequently $n-1-(k-1) = n-k$ failures.

If θ is the probability of a success on a given trial, then the probability of getting $k-1$ successes in $n-1$ trails is

$$\underbrace{b(k-1;}_{\#\text{successes}}\underbrace{n-1}_{\#\text{trials}},\theta) = \binom{n-1}{k-1}\theta^{k-1}(1-\theta)^{n-k}.$$

$(n-1)-(k-1) = n-k$ # failures

Then the probability that the kth success is on trial n is

$$b^*(k;n,\theta) = \theta \cdot b(k-1;n-1,\theta) = \binom{n-1}{k-1}\theta^k(1-\theta)^{n-k}.$$

prob. that k^{th} success is precisely on trial n

prob. that the n^{th} trial is a success

ex: Find the probability that when tossing a coin the third T occurs precisely on the 6^{th} trial. Suppose that prob. of T on each trial is 0.4.

$$b^*(3;6,0.4) = 0.4\ b(2;5,0.4)$$
$$= \binom{5}{2}\cdot(0.4)^3\cdot(1-0.4)^{6-3}$$
$$= \binom{5}{2}\cdot(0.4)^3\cdot(0.6)^3$$

$\longrightarrow$ use a calculator

$b^*(k;n,\theta) = \binom{n-1}{k-1} \cdot \theta^k \cdot (1-\theta)^{n-k}$

for $k=1$: $b^*(1;n,\theta) = \binom{n-1}{0} \cdot \theta^1 \cdot (1-\theta)^{n-1} = \theta \cdot (1-\theta)^{n-1}$

The special case when $k = 1$ (first success appears in trial n) is called the **geometric distribution**:

$$g(n;\theta) = b^*(1;n,\theta) = \theta(1-\theta)^{n-1}$$

For a geometric distribution, we have the following:

mean variance

$$\mu = \frac{1}{\theta}, \quad \sigma^2 = \frac{1}{\theta}\left(\frac{1}{\theta} - 1\right) = \frac{1-\theta}{\theta^2}$$

Proof is omitted

Chapter 6

Special Probability Densities

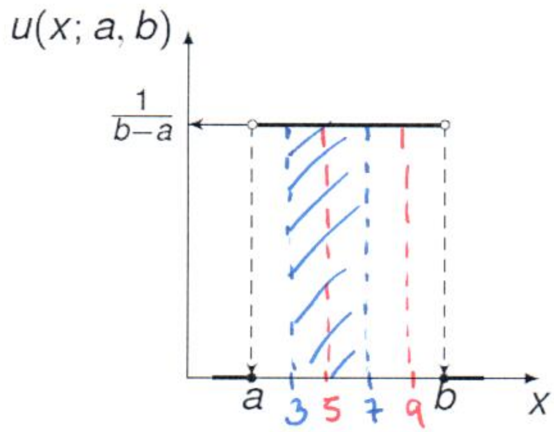
Just as was done in Chapter 5, we now present some common probability densities in the case of a continuous random variable.

6.1 Uniform Distribution

A continuous random variable X is said to have **uniform distribution** if and only if its probability density function is given by

$$u(x; a, b) = \begin{cases} \frac{1}{b-a} & \text{for } a < x < b \\ 0 & \text{otherwise} \end{cases}$$

This means that if (a_1, b_1) and (a_2, b_2) are two intervals of equal length inside of (a, b), then $P(a_1 < X < b_2) = P(a_2 < X < b_2)$.



$(a_1, b_1) = (3, 7)$ $7-3=4$

$(a_2, b_2) = (5, 9)$ $9-5=4$

6.2 Normal Distribution

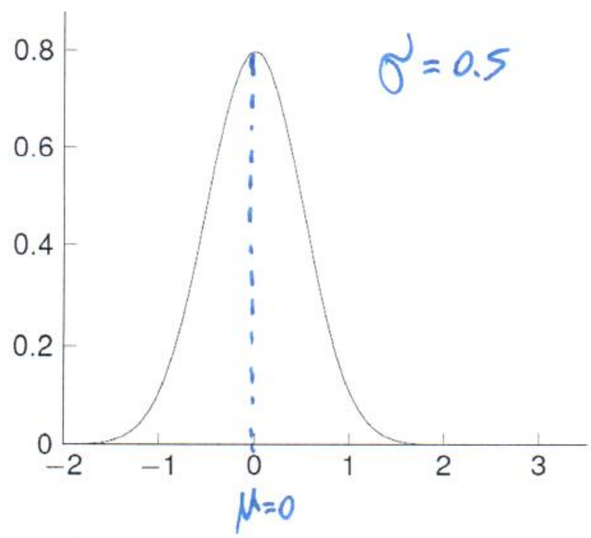
A continuous random variable X has **normal distribution**, and is called a **normal random variable** if and only if its probability density is given by

$$n(x;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

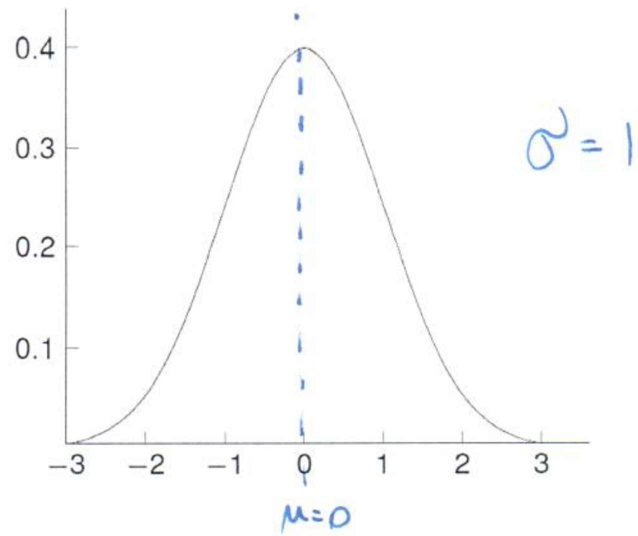
for all $x \in \mathbb{R}$, where $\sigma > 0$.

Showing that this function integrates to 1 over $\mathbb{R}$ requires a trick involving a change of variables to polar coordinates (found in a multivariable calculus course); we will omit this here and accept that this is a valid probability density for any μ and any $\sigma > 0$.

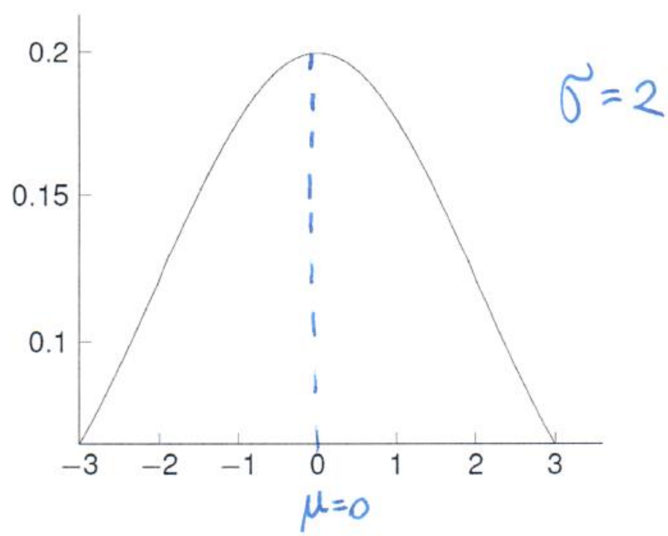
Plot of the Normal Distribution when $\mu = 0$, $\sigma = 0.5$



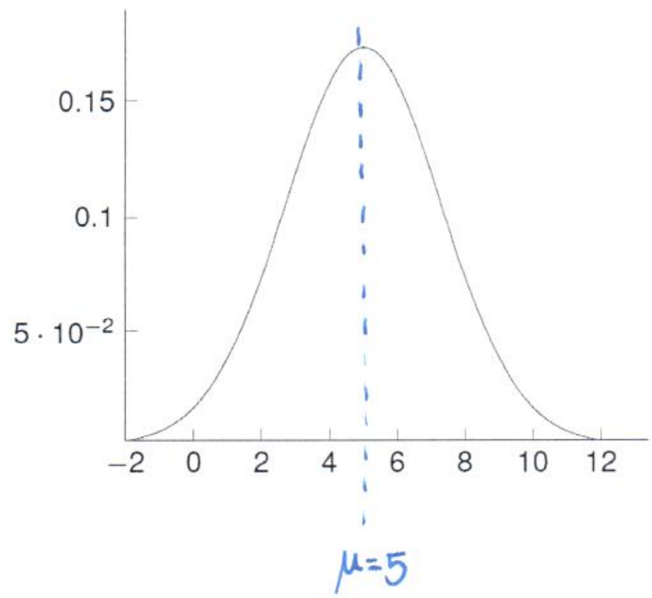
Plot of the Normal Distribution when $\mu = 0$, $\sigma = 1$



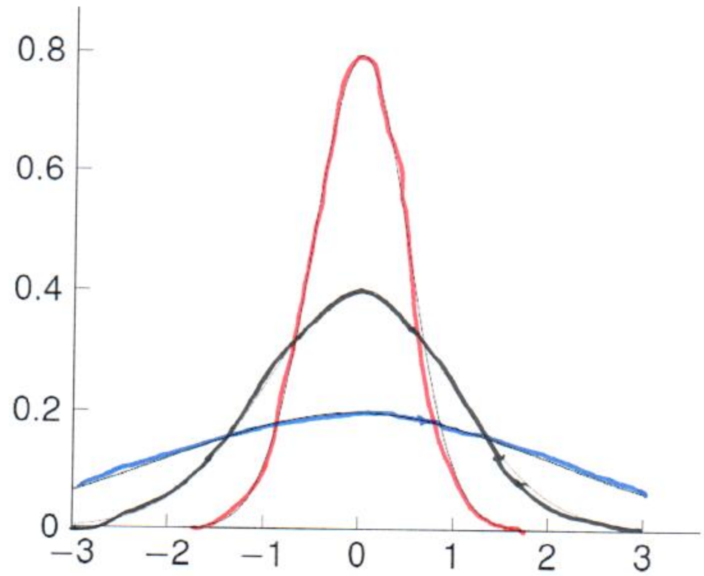
Plot of the Normal Distribution when $\mu = 0$, $\sigma = 2$



Plot of the Normal Distribution when $\mu = 5$, $\sigma = 2.3$



Plot of Normal Distributions with $\mu = 0$



Red - $\sigma = 0.5$, ~~Green~~ Black - $\sigma = 1$, Blue - $\sigma = 2$

One thing to notice about these graphs is their **"bell" shape**. There is a higher probability density in the middle, which rapidly decreases as we move outward.

The next thing to notice is that these graphs have symmetry about the value μ, which is due to the $\left(\frac{x-\mu}{\sigma}\right)^2$ in the exponent.

It also appears that the σ parameter controls the dispersion of the probability.

Indeed, μ and σ are the **mean** and **standard deviation** of a normally distributed random variable X, as we will see.