MATH1550
Practice Set 1

Do these exercises after Chapter 1.
Topics Covered:

- Basic counting principles
- Permutations
- Combinations
- Partitions
- Binomial and multinomial coefficients.

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1. (a) What does the counting rule for compound events say?
   (b) In how many ways can we order n distinct objects?
   (c) What does the number ${}_nP_r = \frac{n!}{(n-r)!}$ count? Give an example.
   (d) How many sets of size r can be made from a set of size n? Give an example.
   (e) What is the summation formula used to expand $(x+y)^n$, where n is a positive integer.

   Solution.

   (a) If a process consists of k steps, where step i can be done in n_i ways, for each $i \in \{1, \dots, k\}$, then the entire process can be done in $n_1 \cdot \dots \cdot n_k$ ways.
   (b) There are $n!$ ways to order n distinct objects.
   (c) The number ${}_nP_r = \frac{n!}{(n-r)!}$ is the number of permutations of n objects taken r at a time. For example,
   $${}_{26}P_9 = \frac{26!}{17!} = 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 19 \cdot 18 = 1133836704000$$
   is the number of batting lineups of 9 players chosen from a team of 26 players (note that the order matters).
   (d) The number of sets of size r that can be made from a set of size n
   $$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$
   For example, the number of ways to choose a 5-player basketball rotation (ignoring positions) from a team of 13 players is
   $$\binom{13}{5} = \frac{13!}{5! \cdot 8!} = \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{5!} = 1287.$$
   (e) The binomial expansion formula is
   $$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$$
   or since $\binom{n}{r} = \binom{n}{n-r}$ we can write
   $$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}.$$

   $\square$

2. Suppose we are drawing items at random from a certain set; for example drawing cards from a deck of cards. Items are said to be drawn *with replacement* if, after an item is drawn, it is returned to the set and can potentially be drawn again. If the item is not returned to the set after being drawn, it is drawn *without replacement* and cannot be drawn again.

   How many ways can 5 cards be drawn from a deck of 52 cards:

   (a) With replacement?
   (b) Without replacement?
   (c) Without replacement and not counting the order they came out?
   (d) With replacement, but not counting order they came out?

   Solution.

   (a) Thinking of this as a 5 step process, with 52 choices at each step, the number is
   $$(52)^5 = 380204032.$$
   (b) If cards are chosen without replacement we lose one choice of card at each of the 5 steps. Thus the number is
   $${}_{52}P_5 = 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 = 311875200.$$
   (c) This is given by
   $$\binom{52}{5} = 2598960.$$
   (d) There are:
   - 52 hands where all 5 cards are the same,
   - $52 \cdot 51 = 2652$ hands where exactly 4 cards are the same,
   - $\frac{52 \cdot 51 \cdot 50}{2} = 66300$ hands where 3 cards are the same and the other two do not match,
   - $52 \cdot 51 = 2652$ hands where 3 cards are the same and the other 2 are the same,
   - $\frac{52 \cdot 51 \cdot 50 \cdot 49}{3!} = 1082900$ hands where 2 cards are the same and the other 3 are all different,
   - $\frac{52 \cdot 51 \cdot 50}{2} = 66300$ hands where exactly 2 sets of 2 cards are the same (but sets different),
   - $\frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5!} = 2598960$ hand where all cards are different.

   In total this is 3819816 different hands.

   $\square$

3. Compute the following:

   (a) ${}_7P_3$
   (b) $\binom{9}{4}$
   (c) The number of 3 digit pass codes using digits 0 through 9.
   (d) The number of different ways to seat 6 people at a round table.
   (e) The number of different 8 digit numbers that can be made using the digits 2, 2, 2, 3, 3, 4, 4, 4.
   (f) The expansion of $(x + 1)^5$.
   (g) The coefficient on $x_1^3 x_2 x_4^4$ in the expansion of $(x_1 + x_2 + x_3 + x_4)^8$.

   Solution.

   (a)
   $${}_7P_3 = 210.$$

(b)
$$\binom{9}{4} = 126.$$

(c)
$$10^3 = 1000.$$

(d)
$$5! = 120.$$

(e)
$$\binom{8}{3,2,3} = \frac{8!}{3! \cdot 2! \cdot 3!} = 560.$$

(f)
$$(x+1)^5 = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1.$$

(g)
$$\binom{8}{3,1,4} = \frac{8!}{3! \cdot 1! \cdot 4!} = 280.$$

□

4. Find the number of ways that 9 toys can be distributed to between 4 children if the youngest receives 3 toys and the others each receive 2.

   Solution. We consider this as a partition of the set of 9 toys into sets of size 3, 2, 2 and 2 respectively. The number of ways is given by
$$\binom{9}{3,2,2,2} = \frac{9!}{3! \cdot 2! \cdot 2! \cdot 2!} = 7560.$$
   We could also count this more directly as
$$\binom{9}{3}\binom{6}{2}\binom{4}{2}\binom{2}{2} = 84 \cdot 15 \cdot 6 \cdot 1 = 7560.$$
□

5. Aras and John are playing tennis. The first person to win 2 games in a row, or win 3 games in total, wins the tournament. How many ways can the tournament play out?

   Solution. Writing A if Aras wins, and J if John wins, a possible tournament (for example) is $AJAA$, which means Aras wins the first game, John the second game, and Aras the third and fourth games. The possible tournaments that Aras wins are
$$AA, AJAA, AJAJA, JAA, JAJAA.$$
   By symmetry, there is the same number of tournaments where John is the winner. Therefore there are 10 possible tournaments all together.

□

6. How many 4 letter words can be made from the 26 letters of the alphabet:

   (a) If letters are not allowed to repeat.

(b) If we allow repeated letters.

(c) If we allow only one letter to be repeated once (i.e. a double but not a triple).

(d) If we allow only one letter to be repeated any number of times.

Solution.

(a) If letter are not allowed to repeat there are

$$26 \cdot 25 \cdot 24 \cdot 23 = 358800$$

words that can be made.

(b) If we allow repeated letters there are

$$26^4 = 456976$$

words that can be made.

(c) If exactly 2 letters are the same, there are $\frac{26 \cdot 25 \cdot 24}{2} = 7800$ different sets of letters that can be chosen (e.g. AABC) and $\frac{4!}{2! \cdot 1! \cdot 1!} = 12$ ways to arrange them. So there are

$$7800 \cdot 12 = 93600$$

words where exactly 2 letters are the same. Add this to the number of words where all letters are different and we get

$$358800 + 93600 = 452400$$

4 letter words where only one letter may be a double.

(d) If one letter is repeated 3 times (i.e. a triple), there are $26 \cdot 25$ different sets of letters that can be chosen (e.g. AAAB) and $\frac{4!}{3! \cdot 1!} = 4$ ways to arrange them. So there are

$$650 \cdot 4 = 2600$$

words where exactly 3 letters are the same. There are 26 4 letter words where all letters are the same. Adding these to the possibilities in (c) we get

$$452400 + 2600 + 26 = 455026$$

where we allow only one letter to be repeated any number of times.

□