3.5.2 Conditional Distributions

Recall: Conditional probability of event A given event B:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \qquad (P(B) \neq 0)$$

In terms of random variables: If A is the event $X = x$ and B is the event $Y = y$ then

$$P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}.$$

For discrete random variables with joint probability distribution $f(x, y)$ we have

$$P(X = x, Y = y) = \frac{f(x, y)}{h(y)},$$

where $h(y) \neq 0$ is the marginal distribution of Y.

If X and Y are discrete random variables with joint probability distribution $f(x, y)$, and respective marginal distributions $g(x)$ and $h(y)$, the function

$$f(x|y) = \frac{f(x, y)}{h(y)}$$

is called the **conditional distribution of X given $Y = y$**, provided $h(y) \neq 0$. The function

$$f(y|x) = \frac{f(x, y)}{g(x)}$$

is called the **conditional distribution of Y given $X = x$**, provided $g(x) \neq 0$.

Example 3.5.10

		x = *0*	*1*	*2*	
y	*0*	$\frac{6}{36}$	$\frac{12}{36}$	$\frac{3}{36}$	$\frac{21}{36}$
	1	$\frac{8}{36}$	$\frac{6}{36}$		$\frac{14}{36}$
	2	$\frac{1}{36}$			$\frac{1}{36}$
		$\frac{15}{36}$	$\frac{18}{36}$	$\frac{3}{36}$	

Caplet example: The conditional distribution of X *given* $Y = 1$ *is,* $f(x|1) = \frac{f(x,1)}{h(1)}$.

Its values are:

$$f(0|1) = \frac{f(0,1)}{h(1)} = \dots$$

$$f(1|1) = \frac{f(1,1)}{h(1)} = \dots$$

$$f(2|1) = \dots$$

Conditional distribution for discrete random variables is extended to the idea of conditional density for jointly continuous random variables:

For jointly continuous random variables X and Y with joint density $f(x, y)$, and marginal densities $g(x)$ and $h(y)$:

The function

$$f(x|y) = \frac{f(x, y)}{h(y)}$$

is called the **conditional density of** X **given** $Y = y$, provided $h(y) \neq 0$.

The function

$$f(y|x) = \frac{f(x, y)}{g(x)}$$

is called the **conditional density of** Y **given** $X = x$, provided $g(x) \neq 0$.

Example 3.5.11 *Let X and Y be jointly continuous random variables with joint probability density given by*

$$f(x,y) = \begin{cases} \frac{3}{5}x(y+x) & \text{for } 0 < x < 1, 0 < y < 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the conditional probability of Y given $X = x$.

(b) Find $P(0 < Y < 1 | X = 0.75)$.

First we need the marginal density function for X:

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\, dy = \int_0^2 \frac{3}{5}x(y+x)\, dy = \dots$$

Thus $g(x) = \frac{6}{5}(x + x^2)$.

The conditional density function for $0 < x < 1, 0 < y < 2$ is then

$$f(y|x) = \frac{f(x,y)}{g(x)} = \dots$$

Finally

$$P(0 < Y < 1|X = 0.75) = \int_0^1 f(y|0.75)\, dy =$$

Example 3.5.12 *Let X and Y be jointly continuous random variables with joint probability density given by*

$$f(x,y) = \begin{cases} 4xy & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $P(0 < X < 1 = 0.5 | Y = 0.5)$.

Just as we defined the concept of independent events, we may speak of independent random variables.

If random variables X and Y have joint probability distribution (or density) $f(x,y)$ and marginal distributions (resp. densities) $g(x)$ and $h(y)$, then we say X and Y are **independent** if and only if

$$f(x,y) = g(x) \cdot h(y).$$

Example 3.5.13 *Let X and Y be jointly continuous random variables with joint probability density function*

$$f(x,y) = \begin{cases} 6e^{-2x}e^{-3y} & \textit{for } 0 < x < \infty, 0 < y < \infty \\ 0 & \textit{otherwise} \end{cases}$$

Show that X and Y are independent random variables.

Chapter 4

Mathematical Expectation

4.1 Expectation (Expected Value)

Example 4.1.1 *Suppose you are at a casino that has a dice game which costs \$1000 for a single roll of two 6-sided dice. You win \$5,555 by rolling a 7 and lose your money otherwise.*

Do you think it is worthwhile to play this game? Could you expect to come out ahead by repeatedly playing this game?

Example 4.1.2 *Suppose a university fundraiser sells 10,000 raffle tickets at a dollar apiece with a grand prize of \$5,000, a second prize of \$1,000 and two third place prizes of \$500 each.*

Do you think your ticket is worth \$1? How much do you think it is worth? In other words, how much can you "expect" to win in this raffle?

If X is a discrete random variable and $f(x)$ is the value of its probability distribution at x, the **expected value** of X (or **expectation of** X) is defined

$$E(X) = \sum_x x \cdot f(x).$$

where the sum is over all x in the range of X.

The sum must be defined in order for the expected value to have meaning.

In the first example of the dice game, the random variable X is the amount of money won on each roll. The range of X is $\{0, 5555\}$.

Since the probability of rolling a 7 is $\frac{1}{6}$ we have $P(X = 5555) = \frac{1}{6}$ and therefore $P(X = 0) = \frac{5}{6}$.

The expected value is

$E(X) =$

This analysis shows that this is a losing game, because our expected value is less than the cost to play.

In the long run, we can expect to lose money.

In the raffle ticket example we let X denote the possible winnings for our raffle ticket. Typically once a ticket is drawn it is not replaced to be drawn again, so the range of X is $\{0, 500, 1000, 5000\}$.

Four tickets will be drawn for the four prizes and there is an equally likely chance of $\frac{1}{10000}$ for each prize.

Therefore $P(X = 0) = \frac{9996}{10000}$, $P(X = 500) = \frac{2}{10000}$, $P(X = 1000) = \frac{1}{10000}$, $P(X = 5000) = \frac{1}{10000}$.

The expected value of X is

$E(X) =$

By playing the raffle repeatedly, we expect to win \$0.70 on average; therefore losing money with the \$1 cost. We could place a value of \$0.70 for our ticket.

Raffle with replacement:

Returning to the previous raffle example, let's compute the expected value of a single ticket with only three prize draws of \$5,000, \$1,000, \$500 each.

Now tickets are replaced each time to allow for multiple wins. (Again 10000 tickets sold)

Let X be the total prize money won.

The range of X is $\{0, 500, 1000, 1500, 5500, 5000, 6000, 6500\}$.

Then,

$P(X = 0) = \left(\frac{9999}{10000}\right)^3$,

$P(X = 500) = \left(\frac{9999}{10000}\right)^2 \left(\frac{1}{10000}\right)$,

$P(X = 1000) = \left(\frac{9999}{10000}\right) \left(\frac{1}{10000}\right) \left(\frac{9999}{10000}\right)$,

$P(X = 1500) = \left(\frac{9999}{10000}\right) \left(\frac{1}{10000}\right)^2$,

$P(X = 5000) = \left(\frac{1}{10000}\right) \left(\frac{9999}{10000}\right)^2$,

$P(X = 5500) = \left(\frac{1}{10000}\right) \left(\frac{9999}{10000}\right) \left(\frac{1}{10000}\right)$,

$P(X = 6000) = \left(\frac{1}{10000}\right)^2 \left(\frac{9999}{10000}\right)$,

$P(X = 6500) = \left(\frac{1}{10000}\right)^3$

The expected value of X is

$E(X) =$

Example 4.1.3 *A slot machine has three wheels with 20 symbols on each wheel.*

There is one 7*, two* $\overline{\underline{BAR}}$*, and three bell icons on each wheel.*

It costs $0.25 to play.

Payouts:

Wheel 1	Wheel 2	Wheel 3	
7	7	7	$500
$\overline{\underline{BAR}}$	$\overline{\underline{BAR}}$	$\overline{\underline{BAR}}$	$100
🔔	🔔	🔔	$20

(all other permutations lose).

What is the expected value of this game (ignore cost to play).

4.1.1 The Expected Value of a Continuous Random Variable

If X is a continuous random variable and $f(x)$ is its probability distribution function, then the expected value of X is defined as

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx.$$

The integral must exist in order for the expected value to have meaning.

Example 4.1.4 *A contractor's profit on a construction job can be considered as a continuous random variable having probability density*

$$f(x) = \begin{cases} \frac{1}{18}(x+1) & \textit{for } -1 < x < 5 \\ 0 & \textit{otherwise} \end{cases}$$

(where the units are in \$1,000). What is her expected profit?

The expected value of X, where X denotes the contractor's profit in \$1,000's, is

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx \qquad = \int_{-1}^{5} x \cdot \frac{1}{18}(x+1)\, dx$$

$$=$$

4.1.2 Expectation of a Function of a Random Variable

We are not limited to considering a random variable by itself.

We can as well consider a function $g(X)$ of a random variable, and evaluate its expected value.

Theorem 4.1.5 *If X is a discrete random variable with probability distribution $f(x)$, the expected value of $g(X)$ is given by*

$$E(g(X)) = \sum_x g(x) \cdot f(x).$$

If X is a continuous random variable with probability density function $f(x)$, the expected value of $g(X)$ is given by

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) \cdot f(x)\, dx.$$

Example 4.1.6 *Let X be a random variable that takes the values $-1, 0, 1$, and has probability distribution given by*

$$f(-1) = 0.2, \quad f(0) = 0.5, \quad f(1) = 0.3.$$

Find $E(X^2)$.

Before using the theorem, let's find $E(X^2)$ directly. We view X^2 as a new random variable which we'll call Y.

The range of Y is $\{0, 1\}$ and it has probability distribution

$$P(Y = 0) = P(X = 0) = f(0) = 0.5$$

$$P(Y = 1) = P(X = 1) + P(X = -1) = f(1) + f(-1) = 0.5$$

Then,

$$E(X^2) = E(Y) = 0 \cdot P(Y = 0) + 1 \cdot P(Y = 1) = 0 \cdot (0.5) + 1 \cdot (0.5) = 0.5.$$

We can find the same result using the theorem as well.

Let $g(X) = X^2$, and let $x_1 = -1, x_2 = 0, x_3 = 1$. Then according to the theorem

$$
\begin{aligned}
E(g(X)) &= \sum_{i=1}^{3} g(x_i) f(x_i) \\
&= g(x_1)f(x_1) + g(x_2)f(x_2) + g(x_3)f(x_3) \\
&= g(-1)f(-1) + g(0)f(0) + g(1)f(1) \\
&= (-1)^2 \cdot (0.2) + (0)^2 \cdot (0.5) + (1)^2 \cdot (0.3) \\
&= 0.5.
\end{aligned}
$$

Note that: $E(X^2) = 0.5 \neq (E(X))^2 = 0.01$

Example 4.1.7 *Suppose X has probability density*

$$f(x) = \begin{cases} e^{-x} & \textit{if } x > 0 \\ 0 & \textit{otherwise} \end{cases}$$

Find the expected value of $g(X) = e^{3X/4}$.

By our theorem

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) f(x)\, dx$$

$$= \int_{0}^{\infty} e^{3x/4} \cdot e^{-x}\, dx$$

$$=$$