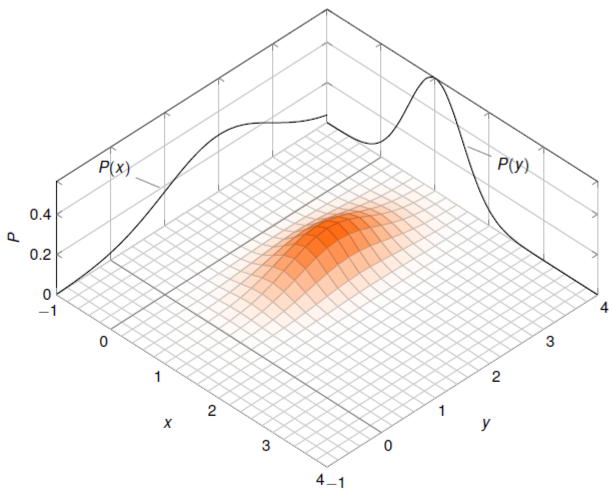
We say that random variables X and Y are **jointly continuous** if there exists a function $f(x,y)$ defined for all $x, y \in \mathbb{R}$, such that

$$P((X,Y) \in A) = \iint_{(x,y)\in A} f(x,y)\; dx\; dy$$

for any region A in the xy-plane.



The function $f(x,y)$ is called the **joint probability density function of X and Y.**

A bivariate function f is a **joint probability density function** of a pair of continuous random variables X and Y if its values satisfy

1. $f(x,y) \geq 0$ for all $x, y \in \mathbb{R}$.
2. $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\; dx\; dy = 1.$

Example 3.5.6 *Let*

$$f(x,y) = \begin{cases} \frac{3}{5}x(y+x) & \text{for } 0 < x < 1, 0 < y < 2 \\ 0 & \text{elsewhere} \end{cases}$$

1. *Verify that f can serve as a probability density function for two jointly continuous random variables X and Y.*
2. *For $A = \{(x,y) | 0 < X < \frac{1}{2}, 1 < Y < 2\}$ find $P((X,Y) \in A)$.*

1) We see that $f(x,y) \geq 0$ for all $0 < x < 1, 0 < y < 2$.

Next we integrate over the entire plane.

We see that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, dx\, dy = \int_0^2 \int_0^1 \frac{3}{5} x(y+x)\, dx\, dy$$

since $f(x,y) = 0$ for all other regions.

We proceed by integrating first with respect to x, treating y as constant:

$$\int_0^2 \left(\int_0^1 \frac{3}{5}(yx + x^2)\, dx \right)\, dy =$$

Finally integrate with respect to y.

$$\frac{3}{5} \int_0^2 \frac{y}{2} + \frac{1}{3}\, dy =$$

Therefore $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1$ as required.

2) For $A = \{(x,y) | 0 < X < \frac{1}{2}, 1 < Y < 2\}$, therefore we have

$$P((X,Y) \in A) = \int_1^2 \int_0^{\frac{1}{2}} \frac{3}{5} x(y+x)\, dx\, dy$$

$$=$$

If X and Y are jointly continuous random variables, with joint probability density f, the function given by

$$F(x, y) = P(X \leq x, Y \leq y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(s, t)\, ds\, dt$$

for $x, y \in \mathbb{R}$, is called the **joint cumulative distribution function of X and Y** (or simply the **joint distribution function**).

As with the discrete case we have that

1. $\lim_{x,y\to-\infty} F(x, y) = 0$,

2. $\lim_{x,y\to\infty} F(x, y) = 1$, and

3. If $a \leq c$ and $b \leq d$ then $F(a, b) \leq F(c, d)$.

It also follows that $f(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y)$ is the joint probability density function.

Example 3.5.7 *The joint probability density function of X and Y is given by*

$$f(x,y) = \begin{cases} 4xy & \text{for } 0 < x < 3, 0 < y < \frac{1}{3} \\ 0 & \text{elsewhere} \end{cases}.$$

Find $F(x,y)$.

To find $F(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(s,t)\, ds\, dt$ we must consider different regions in the plane where $f(x,y)$ is defined.

If either $x < 0$ or $y < 0$ then $f(x,y) = 0$ and so $F(x,y) = 0$.

If $0 < x < 3$ and $0 < y < \frac{1}{3}$ then

$$F(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(s,t)\, ds\, dt = \int_{0}^{y} \int_{0}^{x} 4st\, ds\, dt = x^2 y^2.$$

If $x \geq 3$ and $0 < y < \frac{1}{3}$ then

$$F(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(s,t)\, ds\, dt = \int_{0}^{y} \int_{0}^{3} 4st\, ds\, dt = 9y^2.$$

If $0 < x < 3$ and $y \geq \frac{1}{3}$ then

$$F(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(s,t)\, ds\, dt = \int_{0}^{\frac{1}{3}} \int_{0}^{x} 4st\, ds\, dt = \frac{x^2}{9}.$$

Finally if $x \geq 3$ and $y \geq \frac{1}{3}$ then

$$F(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(s,t)\, ds\, dt = \int_{0}^{\frac{1}{3}} \int_{0}^{3} 4st\, ds\, dt = 1.$$

In summary

$$F(x,y) = \begin{cases} 0 & \text{for } x < 0 \text{ or } y < 0 \\ x^2y^2 & \text{for } 0 < x < 3, 0 < y < \frac{1}{3} \\ 9y^2 & \text{for } x \geq 3, 0 < y < \frac{1}{3} \\ \frac{x^2}{9} & \text{for } 0 < x < 3, y \geq \frac{1}{3} \\ 1 & \text{for } x \geq 3, y \geq \frac{1}{3} \end{cases} .$$

Note that joint probability distributions/densities can be defined similarly for three or more random variables, but that is beyond the scope of this course.

3.5.1 Marginal Distributions

$$
\begin{array}{cc|ccc|c}
 & & & x & & \\
 & & 0 & 1 & 2 & \\
\hline
 & 0 & \frac{6}{36} & \frac{12}{36} & \frac{3}{36} & \frac{21}{36} \\
\text{y} & 1 & \frac{8}{36} & \frac{6}{36} & & \frac{14}{36} \\
 & 2 & \frac{1}{36} & & & \frac{1}{36} \\
\hline
 & & \frac{15}{36} & \frac{18}{36} & \frac{3}{36} &
\end{array}
$$

Returning to the caplet example, sum the rows and columns of the table of probabilities.

The column sums are the probabilities that $X = 0, 1, 2$ respectively, and the row sums are probabilities that $Y = 0, 1, 2$ respectively.

Therefore, the column totals are the probability distribution for X: for $x = 0, 1, 2$

$$g(x) = P(X = x) = \sum_{y=0}^{2} f(x, y),$$

and the row totals are the probability distribution for Y: for $y = 0, 1, 2$

$$h(y) = P(Y = y) = \sum_{x=0}^{2} f(x, y).$$

If X and Y are discrete random variables, and $f(x, y)$ is their joint probability distribution, then the function

$$g(x) = \sum_y f(x, y)$$

is called the **marginal distribution of** X and the function

$$h(y) = \sum_x f(x, y)$$

is called the **marginal distribution of** Y. The sums are over all values of either y or x respectively.

If X and Y are jointly continuous random variables, and $f(x, y)$ is their joint probability density function, then

$$g(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$$

is called the **marginal density of** X and the function

$$h(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$$

is called the **marginal density of** Y. These functions are defined for all $x \in \mathbb{R}$,$y \in \mathbb{R}$ respectively.

Example 3.5.8 *Find the marginal densities of X and Y given their joint probability density*

$$f(x,y) = \begin{cases} \frac{2}{3}(x+2y) & \textit{for } 0 < x < 1, 0 < y < 1 \\ 0 & \textit{elsewhere} \end{cases}$$

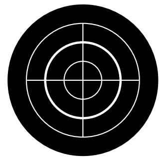
Marginal density of X:

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\, dy = \int_0^1 \frac{2}{3}(x+2y)\, dy = \dots$$

Marginal density of Y:

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\, dx = \int_0^1 \frac{2}{3}(x+2y)\, dx = \dots$$

Note that joint marginal distributions can be defined in the case of 3 or more random variables, and that is beyond the scope of this course too.



Example 3.5.9 *A circular biathlon target for prone position has a diameter of 45mm. Suppose that each point on the target has equally likely probability of being hit by a shot.*

Let $(0,0)$ *be the centre of the target, and define random variables* X *and* Y*, so that* (X,Y) *denotes the coordinates (in millimetres) of the shot fired.*

The joint density function for X *and* Y *is then, for some constant* k*,*

$$f(x,y) = \begin{cases} k & \textit{for } x^2+y^2 \leq (22.5)^2 \\ 0 & \textit{elsewhere} \end{cases}$$

It follows that $k = \frac{1}{(22.5)^2\pi}$, so the integral of the joint density function equals 1 over the area of the circle.

To find the marginal density for X, integrate over all y values:

$$x^2 + y^2 \leq (22.5)^2 \Rightarrow \quad y^2 \leq (22.5)^2 - x^2$$

$$\Rightarrow \quad -\sqrt{(22.5)^2 - x^2} \leq y \leq \sqrt{(22.5)^2 - x^2}$$

Thus

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\, dy = \int_{-\sqrt{(22.5)^2 - x^2}}^{\sqrt{(22.5)^2 - x^2}} \frac{1}{(22.5)^2 \pi}\, dy$$

$$=$$

The marginal density of X can be used to find the probability the shot will land any horizontal distance x from the centre, regardless of its vertical position.

Notice that $g(x)$ is largest when $x = 0$ and gets smaller as x gets near the boundary of the target.

3.5.2 Conditional Distributions

Recall: Conditional probability of event A given event B:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \qquad (P(B) \neq 0)$$

In terms of random variables: If A is the event $X = x$ and B is the event $Y = y$ then

$$P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}.$$

For discrete random variables with joint probability distribution $f(x, y)$ we have

$$P(X = x, Y = y) = \frac{f(x, y)}{h(y)},$$

where $h(y) \neq 0$ is the marginal distribution of Y.

If X and Y are discrete random variables with joint probability distribution $f(x, y)$, and respective marginal distributions $g(x)$ and $h(y)$, the function

$$f(x|y) = \frac{f(x, y)}{h(y)}$$

is called the **conditional distribution of X given $Y = y$**, provided $h(y) \neq 0$. The function

$$f(y|x) = \frac{f(x, y)}{g(x)}$$

is called the **conditional distribution of Y given $X = x$**, provided $g(x) \neq 0$.

Example 3.5.10

$$
\begin{array}{cc|ccc|c}
 & & & x & & \\
 & & \mathit{0} & \mathit{1} & \mathit{2} & \\
\hline
 & \mathit{0} & \frac{6}{36} & \frac{12}{36} & \frac{3}{36} & \frac{21}{36} \\
y & \mathit{1} & \frac{8}{36} & \frac{6}{36} & & \frac{14}{36} \\
 & \mathit{2} & \frac{1}{36} & & & \frac{1}{36} \\
\hline
 & & \frac{15}{36} & \frac{18}{36} & \frac{3}{36} &
\end{array}
$$

Caplet example: The conditional distribution of X given $Y = 1$ is, $f(x|1) = \frac{f(x,1)}{h(1)}$.

Its values are:

$$f(0|1) = \frac{f(0,1)}{h(1)} = \dots$$

$$f(1|1) = \frac{f(1,1)}{h(1)} = \dots$$

$$f(2|1) = \dots$$

Conditional distribution for discrete random variables is extended to the idea of conditional density for jointly continuous random variables:

For jointly continuous random variables X and Y with joint density $f(x, y)$, and marginal densities $g(x)$ and $h(y)$:

The function
$$f(x|y) = \frac{f(x, y)}{h(y)}$$
is called the **conditional density of** X **given** $Y = y$, provided $h(y) \neq 0$.

The function
$$f(y|x) = \frac{f(x, y)}{g(x)}$$
is called the **conditional density of** Y **given** $X = x$, provided $g(x) \neq 0$.

Example 3.5.11 *Let X and Y be jointly continuous random variables with joint probability density given by*

$$f(x,y) = \begin{cases} \frac{3}{5}x(y+x) & \text{for } 0 < x < 1, 0 < y < 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the conditional probability of Y given $X = x$.

(b) Find $P(0 < Y < 1|X = 0.75)$.

First we need the marginal density function for X:

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^2 \frac{3}{5}x(y+x)\,dy = \ldots$$

Thus $g(x) = \frac{6}{5}(x + x^2)$.

The conditional density function for $0 < x < 1, 0 < y < 2$ is then

$$f(y|x) = \frac{f(x,y)}{g(x)} = \ldots$$

Finally

$$P(0 < Y < 1|X = 0.75) = \int_0^1 f(y|0.75)\, dy =$$

Example 3.5.12 *Let X and Y be jointly continuous random variables with joint probability density given by*

$$f(x,y) = \begin{cases} 4xy & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $P(0 < X < 1 = 0.5|Y = 0.5)$.

Just as we defined the concept of independent events, we may speak of independent random variables.

If random variables X and Y have joint probability distribution (or density) $f(x, y)$ and marginal distributions (resp. densities) $g(x)$ and $h(y)$, then we say X and Y are **independent** if and only if

$$f(x, y) = g(x) \cdot h(y).$$

Example 3.5.13 *Let X and Y be jointly continuous random variables with joint probability density function*

$$f(x, y) = \begin{cases} 6e^{-2x}e^{-3y} & \text{for } 0 < x < \infty, 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Show that X and Y are independent random variables.