The Probability of an Event:

A **probability**, or **probability measure**, is a function P which maps events in the sample space S to real numbers.

In order to assign probabilities in a meaningful way, P must satisfy the following called the **postulates** (or **axioms**) **of probability**.

P1: The probability of any event A in S is a non-negative real number, i.e. $P(A) \geq 0$.

P2: $P(S) = 1$.

P3: If $A_1, A_2, A_3, \ldots,$ is a finite or infinite sequence of (pairwise) mutually exclusive events in S then

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots) = P(A_1) + P(A_2) + P(A_3) + \cdots$$

(P is **countably additive**)

2.1.5 Postulates of Probability

- Interpreting a probability as a frequency, or a proportion of time, it makes sense that $P(A) \geq 0$; in fact we will show that $0 \leq P(A) \leq 1$ for any event A.
- P2 says that the probability that outcome of the experiment lies in S must be assigned value 1. Since this is certain to happen, we interpret $P(A) = 1$ as "A happens 100 percent of the time."
- P3 is for consistency. For example, if events A_1 and A_2 share no common outcomes, then the probability that either event occurs, $P(A_1 \cup A_2)$, is the sum of their individual probabilities.

A technical detail has been overlooked in the postulates of probability presented above. In a discrete sample space S, an "event" can be any subset of S, however in the continuous case one has to be more careful about which subsets of S are allowed as events. A precise definition for these allowable events comes in a course on **measure theory**. In this course we won't require that level of detail; i.e. the subsets we assign probabilities to will be allowable events.

Single Die Roll:
Let S be the sample space for rolling a die once.

Example 2.1.8 *Each outcome in S is its own event, call these $A_1, \ldots, A_6$.*

Events $A_1, \ldots, A_6$ are mutually exclusive, and any event E in S is a union of these, for example let $E = A_2 \cup A_4 \cup A_5$.

By the classical probability concept, $P(E) = \frac{3}{6}$ (successes/number of outcomes), and $P(A_i) = \frac{1}{6}$ for each i.

It follows that this satisfies the postulates of probability:

- $P(B) \geq 0$ *for any* $B \subset S$.
- $P(S) = \frac{6}{6} = 1$.
- *P3 is satisfied: for example* $P(E) = \frac{3}{6} = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = P(A_2) + P(A_4) + P(A_5)$.

Example 2.1.9 *Suppose we assigned probabilities in this experiment in a different way. Using the same notation as before say for any event B we specify that*

$$P(B) = \sum_{A_i \in B} P(A_i), \text{ and}$$

$$P(A_1) = \tfrac{1}{2}, P(A_2) = \tfrac{1}{4}, P(A_3) = \tfrac{1}{8},$$
$$P(A_4) = 0, P(A_5) = \tfrac{1}{16}, P(A_6) = \tfrac{1}{16}$$

Are the postulates of probability still satisfied?

Example 2.1.10 *An experiment has four possible outcomes A, B, C, D that are mutually exclusive. Explain why the following assignments of probabilities are not permissible.*

(a) $P(A) = 0.12, P(B) = 0.63, P(C) = 0.45, P(D) = -0.20$

(b) $P(A) = \frac{9}{120}, P(B) = \frac{45}{120}, P(C) = \frac{27}{120}, P(D) = \frac{46}{120}$

2.1.6 The Probability of an Event

Theorem 2.1.11 *If A is an event in a discrete sample space S, then $P(A)$ is the sum of the probabilities of the individual outcomes (elements) of A.*

(Note that the theorem assumes that P is a probability measure, and hence satisfies the postulates.)

Example 2.1.12 *Experiment: Tossing a coin three times.*
Sample space:

Event A: Getting at least two heads $\{HHH, HHT, HTH, THH\}$

Event B: Getting exactly two tails

Event C: Getting two consecutive heads $\{HHH, HHT, THH\}$

*Assuming this is a **balanced (fair)** coin, i.e. equal likely heads or tails, what are the probabilities of the events above?*

Example 2.1.13 *Suppose our six sided die is weighted so that each odd number is twice as likely to occur than each even number.*

What is the probability of rolling a number greater than 3?

What if instead each even number is four times as likely to occur than each odd number?

Infinite Discrete Sample Space: When a sample space has countably infinite outcomes, probabilities must be assigned via some rule/formula as opposed to listing them individually.

Example 2.1.14 *Tossing a coin until heads is reached:*
$S = \{H, TH, TTH, TTTH, TTTH, \ldots\}$.

If A_i *is the event of i flips, then* $P(A_i) = \frac{1}{2^i}$ *defines a probability on* S *(assuming countable additivity). From the geometric series formula we get P2:*

$$P(S) = \sum_{i=1}^{\infty} P(A_i) = -1 + \sum_{i=0}^{\infty} \frac{1}{2^i} = -1 + \frac{1}{1 - \frac{1}{2}} = -1 + 2 = 1.$$

Brief note on infinite series:

- Sequence: countably infinite list of real numbers; $r_1, r_2, r_3, \ldots$.

$$\text{e.g.} \quad 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots$$

- Geometric sequence: terms occur in a common ratio r; $a, ar, ar^2, ar^3 \ldots$

$$\text{e.g.} \quad 4, \tfrac{4}{3}, \tfrac{4}{9}, \tfrac{4}{27}, \tfrac{4}{81}, \ldots\ldots$$

- Partial sum of a sequence: $T_n = \displaystyle\sum_{i=1}^{n} r_i = r_1 + r_2 + \cdots + r_n$.

- Series: Limit of partial sums (if it exists); $\displaystyle\lim_{n\to\infty} T_n = \sum_{i=1}^{\infty} r_i$.

- Partial sum of a geometric sequence: $G_n = a + ar + \cdots + ar^n$.

$$\begin{aligned}
(1-r)G_n &= (1-r)(a + ar + ar^2 + \cdots + ar^n) \\
&= a + ar + ar^2 + \cdots + ar^n \\
&\quad - (ar + ar^2 + ar^3 + \cdots + ar^{n+1}) \\
&= a - ar^{n+1}
\end{aligned}$$

So $G_n = \frac{a - ar^{n+1}}{(1-r)}$ (for $r \neq 1$) .

- If $-1 < r < 1$ then $\displaystyle\lim_{n\to\infty} r^n = 0$, and so it follows that

$$\sum_{i=0}^{\infty} ar^i = \lim_{n\to\infty} G_n = \frac{a}{1-r}.$$

- In the coin flipping example above, $a = 1$ and $r = \frac{1}{2}$.

Theorem 2.1.15 ***(The Probability of an Event with equally likely outcomes)***

If an experiment has N equally likely outcomes and A is an event made up of k of those outcomes then

$$P(A) = \frac{k}{N}.$$

Example 2.1.16 *A five-card poker hand dealt from a deck of 52 playing cards is said to be a full house if it consists of three of a kind and a pair. If all the five-card hands are equally likely, what is the probability of being dealt a full house?*

$$8\spadesuit, J\spadesuit, 8\heartsuit, 8\clubsuit, J\diamondsuit$$