Theorem : $E(aX+b) = aE(X)+b$

Example 4.1.9 *Returning to ~~out~~ our slot machine example, we chose our random variable X to be the expected payout, and not the expected profit. Then we calculated the expected payout.* We found that $E(X) = 0.23$

Suppose this time that we want the expected profit.

calculate $E(Y)$ without using the theorem

If Y is our expected profit then Y has range $\{-0.25, 19.75, 99.75, 499.75\}$

X: 0 20 100 500

So then $P(Y = y) = P(X = y + 0.25)$*, and*

cost is 0.25

$$E(Y) = (-0.25) \cdot P(X = 0) + (19.75) \cdot P(X = 20) + (99.75) \cdot P(X = 100) + (499.75) \cdot P(X = 500)$$

$$= -0.25 \cdot \left(1 - \frac{27}{8000} - \frac{8}{8000} - \frac{1}{8000}\right) + 19.75 \cdot \frac{27}{8000} + 99.75 \cdot \frac{8}{8000} + 499.75 \cdot \frac{1}{8000} = -0.02$$

On the other hand since $Y = g(X) = X - 0.25$, we can compute

$$E(Y) = E(\underbrace{X - 0.25}_{Y}) = \underbrace{1 \cdot E(X) - 0.25}_{E(1 \cdot X - 0.25),\ a=1,\ b=-0.25} = 0.23 - 0.25 = -0.02$$

using the theorem (which, in this case is nicer calculation).

Theorem: $E(aX+b) = aE(X)+b$

We can extend the theorem above to more expressions:

Theorem 4.1.10 *If $c_1, c_2, \ldots, c_n$ are constants, then*

$$E\left(\underbrace{\sum_{i=1}^{n} c_i g_i(X)}_{h(x)}\right) = \sum_{i=1}^{n} c_i E(g_i(X)),$$

where the g_i are functions.

$n=2$

$$E\left(c_1 g_1(x) + c_2 g_2(x)\right)$$
$$= E(c_1 g_1(x)) + E(c_2 g_2(x))$$
$$= c_1 E(g_1(X)) + c_2 E(g_2(X))$$
$$= \sum_{i=1}^{n} c_i E(g_i(X))$$

Proof (continuous case): Suppose X has p.d.f. $f(x)$. Let $h(x) = \sum_{i=1}^{n} c_i g_i(X)$. Then

$$E(h(X)) = \int_{-\infty}^{\infty} h(x) \cdot f(x)\, dx$$

$$= \int_{-\infty}^{\infty} \left(\sum_{i=1}^{n} c_i g_i(X)\right) \cdot f(x)\, dx$$

$$= \int_{-\infty}^{\infty} c_1 g_1(x) f(x) + c_2 g_2(X) \cdot f(x) + c_3 g_3(X) \cdot f(x) \cdots + c_n g_n(X) \cdot f(x)\, dx$$

$$= \int_{-\infty}^{\infty} c_1 g_1(X) f(x)\, dx + \int_{-\infty}^{\infty} c_2 g_2(X) \cdot f(x)\, dx \cdots + \int_{-\infty}^{\infty} c_n g_n(X) \cdot f(x)\, dx$$

$$= c_1 \int_{-\infty}^{\infty} g_1(X) \cdot f(x)\, dx + \ldots + c_n \int_{-\infty}^{\infty} g_n(X) \cdot f(x)\, dx$$

$$= \sum_{i=1}^{n} c_i \int_{-\infty}^{\infty} g_i(X) \cdot f(x)\, dx$$

$$= \sum_{i=1}^{n} c_i E(g_i(X)).$$

$$= c_1 \underbrace{\int_{-\infty}^{\infty} g_1(X) \cdot f(x)\, dx}_{E(g_1(x))} + \dots + c_n \underbrace{\int_{-\infty}^{\infty} g_n(X) \cdot f(x)\, dx}_{E(g_n(x))}$$

$$= c_1 E(g_1(X)) + \dots + c_n E(g_n(X))$$

$$= \sum_{i=1}^{n} c_i E(g_i(x))$$

4.1.4 Multivariate Expected Value

Suppose X and Y are random variables with a joint probability distribution/density $f(x, y)$.

Then $Z = g(X, Y)$ is a random variable defined by the function g depending on X and Y.

The expected value of Z may be computed in the following way.

Theorem 4.1.11 *With notation as above if X and Y are discrete random variables, then*

$$E(\underbrace{g(X, Y)}_{Z}) = \sum_x \sum_y g(x, y) \cdot f(x, y)$$

joint probability distribution

where sums are taken over x and y in the ranges of X and Y respectively.

In the continuous case, we have the following result.

$$E(g(X, Y)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) \cdot f(x, y)\, dx\, dy$$

joint probability density

Back to the caplet example (yet again!):

Two caplets are randomly selected from a bottle containing 3 aspirin, 2 sedative, and 4 laxative.

Let X be the number of aspirin selected, and Y be the number of sedative selected.

$Z = X + Y$

		$x = 0$	$x = 1$	$x = 2$
y	0	$\frac{6}{36}$	$\frac{12}{36}$	$\frac{3}{36}$
	1	$\frac{8}{36}$	$\frac{6}{36}$	0
	2	$\frac{1}{36}$	0	0

Let $Z = X + Y$. Then Z is the random variable which gives the total number of aspirin or sedative when two caplets are drawn.
plus

What is the expected value of Z?

Theorem: $E(\underbrace{g(X,Y)}_{X+Y}) = \sum_x \sum_y g(x,y) \cdot f(x,y)$

$$E(X+Y) = \sum_{x=0}^{2} \sum_{y=0}^{2} (x+y) \cdot f(x,y)$$

$$= \cancel{(0+0) \cdot \frac{6}{36}} + (0+1) \cdot \frac{8}{36} + (0+2) \cdot \frac{1}{36}$$

$$+ (1+0) \cdot \frac{12}{36} + (1+1) \cdot \frac{6}{36} + \cancel{(1+2) \cdot 0}$$

$$+ (2+0) \cdot \frac{3}{36} + \cancel{(2+1) \cdot 0} + \cancel{(2+2) \cdot 0}$$

$$= \frac{8}{36} \cdot 1 + \frac{1}{36} \cdot 2 + \frac{12}{36} \cdot 1 + \frac{6}{36} \cdot 2 + \frac{3}{36} \cdot 2$$

$$= \frac{8}{36} + \frac{2}{36} + \frac{12}{36} + \frac{12}{36} + \frac{6}{36} = \frac{40}{36} \approx 1.1111$$

Example 4.1.12 *The joint probability density of X and Y is given by*

$$f(x,y) = \begin{cases} x+y & \text{for } 0<x<1, 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

$$Z = g(X,Y) = X \cdot Y$$

Find the expected value of XY.

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) \cdot f(x,y)\,dx\,dy$$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \cdot f(x,y)\,dx\,dy$$

$$= \int_0^1 \left(\int_0^1 xy \cdot (x+y)\,dx \right) dy$$

$$= \int_0^1 \left(\int_0^1 x^2y + xy^2\,dx \right) dy$$

$$= \int_0^1 \left(\frac{x^3}{3}y + \frac{x^2}{2}y^2 \right) \Bigg|_{x=0}^{x=1} dy$$

$$= \int_0^1 \left(\frac{y}{3} + \frac{y^2}{2} \right) - (0+0)\,dy = \int_0^1 \frac{2y}{6} + \frac{3y^2}{6}\,dy = \int_0^1 \frac{2y+3y^2}{6}\,dy$$

$$= \frac{1}{6}\int_0^1 2y + 3y^2\,dy = \frac{1}{6} \cdot (y^2 + y^3) \Bigg|_{y=0}^{y=1}$$

$$= \frac{1}{6}\left((1+1) - (0+0) \right)$$

$$= \frac{1}{6} \cdot 2 = \frac{1}{3}$$