MATH1550
Practice Set 4 - Solutions

These exercises are suited to Chapter 3 up to Cumulative Distribution for discrete random variables.
Topics Covered:

- Random variables
- Discrete random variables
- Probability distributions for random variables
- Cumulative distributions for random variables

---

1. (a) Describe what a *random variable* is for a probability experiment.
   (b) What is the *range* of a random variable?
   (c) Give two examples of random variables that can be defined on the experiment of rolling two dice.
   (d) How do we differentiate between a *discrete* random variable and a *continuous* random variable?
   (e) What conditions must a probability distribution satisfy for a discrete random variable?
   (f) How is a *cumulative distribution* for a random variable defined?

   Solution. (a) A random variable is a function which assigns a real number to each outcome of the sample space.
   (b) The range of a random variable is the set of values that it takes (or those potentially having non-zero probability).
   (c) We could take for example, the sum of the two dice or the number of fives that appear.
   (d) A discrete random variable has a countable range, whereas a continuous random variable has a range containing an interval (or made up of intervals) in $\mathbb{R}$.
   (e) For a function f to represent a probability distribution for a random variable X with respect to some probability measure, we must have that $f(x) \geq 0$ for all $x \in R(X)$ in the range of and $\sum_{x\in R(X)} f(x) = 1$ where $R(X)$ is the range of X
   (f) The cumulative distribution for a random variable X is a function F such that $F(x) = P(X \leq x)$.

   □

2. In each case write the range of the given random variable X.
   (a) Three apples are drawn from a bushel of 120 apples of which 9 are rotten. Let X be the the number of rotten apples drawn.
   (b) A coin is tossed until heads appears. Let X be the number of tails that appear.
   (c) A regular 6-sided die is thrown and then a coin is tossed. If the coin toss is heads take the half the number shown on the roll of the die and if the coin toss is tails, multiply the die roll by 2. Let X be the number that you get.

   Solution. (a) The range of X is $\{0, 1, 2, 3\}$.
   (b) The range of X is $\{0, 1, 2, 3, \dots\}$.
   (c) The range of X is $\{\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, 4, 6, 8, 10, 12\}$.

   □

3. A bag has 3 billiard balls numbered 1, 2 and 3. Two balls are drawn from the bag with replacement. Let X be the sum of the two numbers appearing on the balls that are drawn from the bag. Find the probability distribution of X.

Solution. This experiment has sample space $\{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$ and each of these outcomes is equally likely. The range of X is therefore $\{2,3,4,5,6\}$ and its probability distribution is

x	2	3	4	5	6
$P(X=x)$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{3}{9}$	$\frac{2}{9}$	$\frac{1}{9}$

(multiply $\frac{1}{9}$ times the number of pairs that sum to the given number.) We could also express this as

$$f(x) = \frac{3-|x-4|}{9}$$

for $x = 2,3,4,5,6$. □

4. A balanced coin is tossed 6 times. Let X be the number of heads that appear.

   (a) Write the range of X.

   (b) Find the probability distribution for X.

   (c) Write the cumulative distribution for X.

Solution. (a) The range of X is $\{0,1,2,3,4,5,6\}$.

(b) The probability distribution is given by

$$f(x) = \frac{\binom{6}{x}}{2^6}$$

for $x = 0,1,2,3,4,5,6$; choosing x coins out of the 6 to show heads counts the number of different ways x heads may be obtained in 6 tosses, and there are 2^6 different outcomes for the 6 tosses. The explicit values are

x	0	1	2	3	4	5	6
$P(X=x)$	$\frac{1}{64}$	$\frac{6}{64}$	$\frac{15}{64}$	$\frac{20}{64}$	$\frac{15}{64}$	$\frac{6}{64}$	$\frac{1}{64}$

(c) The cumulative distribution for X is given by

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{64} & 0 \le x < 1 \\ \frac{7}{64} & 1 \le x < 2 \\ \frac{22}{64} & 2 \le x < 3 \\ \frac{42}{64} & 3 \le x < 4 \\ \frac{57}{64} & 4 \le x < 5 \\ \frac{63}{64} & 5 \le x < 6 \\ 1 & x \ge 6 \end{cases}$$

□

5. Suppose a random variable X has the probability distribution given below.

x	-3	-1	2	5
$P(X = x)$	$\frac{2k-3}{10}$	$\frac{k-2}{10}$	$\frac{k-1}{10}$	$\frac{k+1}{10}$

(a) Find the value for $k \in \mathbb{R}$.

(b) Write the cumulative distribution for X.

(c) Draw the probability histogram for X.

Solution. (a) Since probabilities must sum to 1 we have

$$1 = \frac{2k-3}{10} + \frac{k-2}{10} + \frac{k-1}{10} + \frac{k+1}{10} = \frac{5k-5}{10} \qquad \Rightarrow \qquad k = 3.$$

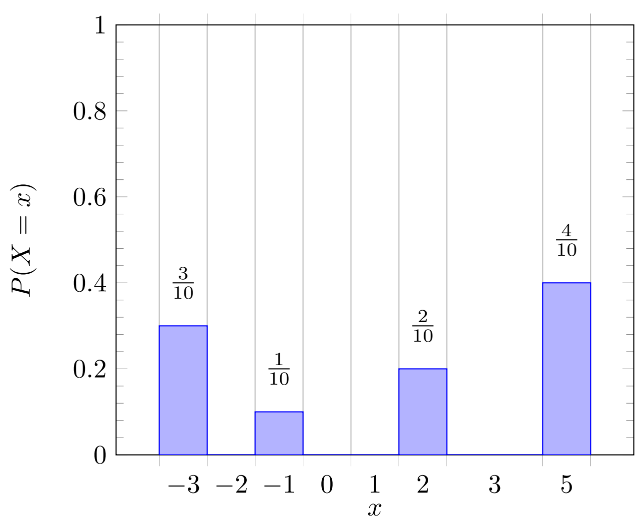
(b) Since $k = 3$, the probability distribution for X is

x	-3	-1	2	5
$P(X = x)$	$\frac{3}{10}$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{4}{10}$

and so the cumulative distribution is

$$F(x) = \begin{cases} 0 & x < -3 \\ \frac{3}{10} & -3 \le x < -1 \\ \frac{4}{10} & -1 \le x < 2 \\ \frac{6}{10} & 2 \le x < 5 \\ 1 & x \ge 5 \end{cases}$$

(c) The histogram for X is below



$\square$

6. A balanced coin is tossed until heads or five tails occurs. Let X be the number of tosses. Find the probability distribution for X.

Solution. The sample space for this experiment is $\{H, TH, TTH, TTTH, TTTTH, TTTTT\}$ and so the range of X is $\{1, 2, 3, 4, 5\}$. Using the multiplication rule, and the fact that coin tosses are independent events, we have

$$
\begin{aligned}
P(X = 1) &= P(\{H\}) = \frac{1}{2} \\
P(X = 2) &= P(\{TH\}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \\
P(X = 3) &= P(\{TTH\}) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8} \\
P(X = 4) &= P(\{TTTH\}) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{16} \\
P(X = 5) &= P(\{TTTTH\}) + P(\{TTTTT\}) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{16}
\end{aligned}
$$

□

7. In a certain coin toss game, the player tosses a balanced coin twice. The player wins \$20 if 2 heads appear, and \$10 if only 1 head appears. The player losses \$ if no heads appear. Let X be the amount of money won on this game. Give the probability distribution of X.

Solution. The range of X is $\{20, 10, -30\}$. Since all outcomes of the sample space $\{HH, HT, TH, TT\}$ are equally likely, the probability distribution of X is

$$
\begin{aligned}
P(X = 20) &= P(\{HH\}) = \frac{1}{4} \\
P(X = 10) &= P(\{HT, TH\}) = \frac{1}{2} \\
P(X = -30) &= P(\{TT\}) = \frac{1}{4}
\end{aligned}
$$

□

8. The cumulative distribution function for a discrete random variable X is given below:

$$
F(x) = \begin{cases} 0 & x < -2 \\ \frac{1}{4} & -2 \leq x < 1 \\ \frac{3}{8} & 1 \leq x < 2 \\ \frac{7}{8} & 2 \leq x < 4 \\ 1 & x \geq 4 \end{cases}
$$

(a) Find $F(2.5)$

(b) Find $P(X = 1)$

(c) Find $P(X = 1.5)$

(d) Find $P(0.5 \leq X < 4)$

(e) Find the probability distribution of X.

Solution. (a)

$$F(2.5) = \frac{7}{8}$$

(b)

$$P(X = 1) = F(1) - F(-2) = \frac{3}{8} - \frac{1}{4} = \frac{1}{8}.$$

(c) Looking at the domain of $F(X)$, we see that the range of X is $\{-2, 1, 2, 4\}$ and thus

$$P(X = 1.5) = 0.$$

(d)

$$P(0.5 \leq X < 4) = P(\{1, 2\}) = F(2) - F(-2) = \frac{7}{8} - \frac{1}{4} = \frac{5}{8}.$$

(e) Using the fact that $P(X = x_i) = F(x_i) - F(x_{i-1})$ (for each x_i in the range of X), we obtain the distribution

$$f(x) = \begin{cases} \frac{1}{4} & x = -2 \\ \frac{1}{8} & x = 1 \\ \frac{1}{2} & x = 2 \\ \frac{1}{8} & x = 4 \end{cases}$$

□

9. let X be a discrete random variable with range $\{0, 1, 2, 3\}$ and probability distribution given by

$$f(x) = \frac{\binom{3}{x}\binom{4}{4-x}}{\binom{7}{4}}$$

(a) Verify that this is a valid probability distribution.

(b) Write the cumulative distribution function.

Solution. (a) We have

$$f(0) + f(1) + f(2) + f(3) = \frac{\binom{3}{0}\binom{4}{4}}{\binom{7}{4}} + \frac{\binom{3}{1}\binom{4}{3}}{\binom{7}{4}} + \frac{\binom{3}{2}\binom{4}{2}}{\binom{7}{4}} + \frac{\binom{3}{3}\binom{4}{1}}{\binom{7}{4}} = \frac{1 + 12 + 18 + 4}{35} = 1$$

and see that $f(x) \geq 0$ for each x, thus f is a valid probability distribution.

(b) The cumulative distribution function is

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{35} & 0 \leq x < 1 \\ \frac{13}{35} & 1 \leq x < 2 \\ \frac{31}{35} & 2 \leq x < 3 \\ 1 & x \geq 3 \end{cases}$$

□