Example 3.4.6 *Find the cumulative distribution function $F(x)$ for*

$$f(x) = \begin{cases} 3e^{-3x} & \textit{for } x > 0 \\ 0 & \textit{elsewhere} \end{cases}$$

$x \leq 0$

and use it to evaluate $P(0.5 \leq X \leq 1)$.

probability density fnc.

an antiderivative for $3e^{-3t}$ is $-e^{-3t}$

For $x > 0$ we have

$$F(x) = \int_{-\infty}^{\infty} f(t)\,dt = \int_0^x 3e^{-3t}\,dt = -e^{-3t}\Big|_0^x = -e^{-3x} + 1.$$

$-e^{-3x} - (-e^{-3\cdot 0}) = -e^{-3x} - (-e^{0})$

$= -e^{-3x} + 1$

For $x \leq 0$, $f(x) = 0$. Therefore $F(x) = 0$

So,

$$F(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ 1 - e^{-3x} & \text{for } x > 0 \end{cases}$$

$e \approx 2.71\ldots$

Then using the theorem,

$$P(0.5 \leq X \leq 1) = \ldots F(1) - F(0.5)$$

$= (1 - e^{-3\cdot 1}) - (1 - e^{-3\cdot 0.5})$

$= \cancel{1} - e^{-3} - \cancel{1} + e^{-1.5} = e^{-1.5} - e^{-3}$

≈ 0.1733

3.5 Multivariate Distributions

We now consider the case when two or more random variables are defined on the same (joint) sample space. We start with the **bivariate** case, that is when two random variables X and Y are defined for a common sample space.

For example X could be the sum of rolling two dice, and Y could be the product.

Write $P(X = x, Y = y)$ for the probability of the intersection of events $X = x$ and $Y = y$.

Example 3.5.1 *Two caplets are randomly selected from a bottle containing 3 aspirin, 2 sedative, and 4 laxative.*

Let X be the number of aspirin selected, and Y be the number of sedative selected.

$3+2+4=9$ caplets totals

Find the probabilities associated to each possible pair of values for X and Y.

X : # aspirin selected

Y : # sedatives selected

3 aspirin

2 sedatives

4 laxatives

The possible pairs for X, Y are: $(0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (0, 2)$.

x y x y x y x y x y x y

There are $\binom{9}{2} = 36$ different possible two-pill selections that can be drawn.

The number of different ways to draw x aspirin, y sedative, and therefore $2 - x - y$ laxative (where $0 \leq x + y \leq 2$) is

$$\binom{3}{x} \cdot \binom{2}{y} \cdot \binom{4}{2-x-y}.$$

3 aspirin
2 ~~laxatives~~ sedatives
4 laxatives

9 caplets total

aspirin | sedatives | laxatives

Thus

$$P(X = x, Y = y) = \boxed{\frac{\binom{3}{x}\binom{2}{y}\binom{4}{2-x-y}}{36}}$$

We summarize the probabilities in a table:

$\frac{\binom{3}{0}\cdot\binom{2}{0}\cdot\binom{4}{2}}{36} = P(X=0, Y=0)$

$P(X=1, Y=0)$

$P(X=2, Y=0)$

		x 0	1	2
	0	$\frac{6}{36}$	$\frac{12}{36}$	$\frac{3}{36}$
y	1	$\frac{8}{36}$	$\frac{6}{36}$	0
	2	$\frac{1}{36}$	0	0

$\frac{\binom{3}{0}\cdot\binom{2}{1}\cdot\binom{4}{1}}{36} = P(X=0, Y=1)$

$P(X=x, Y=y) = \frac{\binom{3}{x}\cdot\binom{2}{y}\cdot\binom{4}{2-x-y}}{36}$

$P(X=1, Y=1)$
$= \frac{\binom{3}{1}\cdot\binom{2}{1}\cdot\binom{4}{0}}{36}$

$P(X=0, Y=2)$

If X and Y are discrete random variables, then the function

$$f(x,y) = P(X = x, Y = y)$$

for each pair (x, y) in the range of X and Y is called the **joint probability distribution of X and Y.**

Theorem 3.5.2 *A bivariate function f can serve as a joint probability distribution for discrete random variables X and Y if and only if*

1. $f(x,y) \geq 0$.
2. $\sum_x \sum_y f(x,y) = 1$, *where the double sum is taken over all possible pairs (x, y).*

		x = 0	1	2
y	0	$\frac{6}{36}$	$\frac{12}{36}$	$\frac{3}{36}$
	1	$\frac{8}{36}$	$\frac{6}{36}$	
	2	$\frac{1}{36}$		

To verify the theorem for the caplet example, note that all values are non-negative and

$$\sum_x \sum_y f(x,y) = f(0,0) + f(1,0) + f(2,0) + f(0,1) + f(1,1) + f(0,2)$$

$$= \frac{6}{36} + \frac{12}{36} + \frac{3}{36} + \frac{8}{36} + \frac{6}{36} + \frac{1}{36} = 1.$$

discrete random variables

Example 3.5.3 *Suppose the joint probability distribution of X and Y is given by*

$$f(x, y) = c(x^2 + y^2)$$

for all pairs (x, y) *with* $x = -1, 0, 1, 3$ *and* $y = -1, 2, 3$. *Find the value of* c.

all possible pairs: $(-1,-1)$, $(-1,2)$, $(-1,3)$
$(0,-1)$, $(0,2)$, $(0,3)$
$(1,-1)$, $(1,2)$, $(1,3)$
$(3,-1)$, $(3,2)$, $(3,3)$

Since $f(x,y)$ is a joint probability distribution, by Theorem 3.5.2 (part 2), we must have

$$f(-1,-1) + f(-1,2) + f(-1,3) + f(0,-1) + f(0,2) + f(0,3) + f(1,-1) + f(1,2) + f(1,3) + f(3,-1) + f(3,2) + f(3,3) = 1$$

So, $c((-1)^2+(-1)^2) + c((-1)^2+2^2) + \ldots = 1$.

Therefore, $c(2+5+10+1+4+9+2+5+10+10+13+18) = 1$
$\Rightarrow 89c = 1 \Rightarrow c = \frac{1}{89}$.

Observe that $f(x,y) = \underset{\frac{1}{89}}{\underset{\shortparallel}{c}}(x^2+y^2)$ is non-negative for all (x,y).

If X and Y are discrete random variables, with joint probability distribution f, then the function

$$F(x,y) = P(X \leq x, Y \leq y) = \sum_{s \leq x} \sum_{t \leq y} f(s,t)$$

defined for all $x, y \in \mathbb{R}$, is called the **joint cumulative distribution of X and Y**. ~~**or the joint distribution function**~~.

		x = 0	1	2	
	0	$\frac{6}{36}$	$\frac{12}{36}$	$\frac{3}{36}$	$-P(X=2, Y=0)$
y	1	$\frac{8}{36}$	$\frac{6}{36}$		$P(X=1, Y=1)$
	2	$\frac{1}{36}$			

Example 3.5.4 *Let $F(x,y)$ be the joint cumulative distribution of the caplet example. Find $F(2, 1.3)$.*

$$\sum_{s \leq 2} \sum_{t \leq 1.3} f(s,t)$$

To find $F(2,1.3) = P(X \leq 2, Y \leq 1.3)$ we must sum the probabilities $f(x,y)$ over all pairs (x,y) in the range of X and Y with $x \leq 2$ and $y \leq 1.3$.

$x=0, \ y=0 \qquad P(X=0, Y=0) = \frac{6}{36}$

$x=1, \ y=0 \qquad P(X=1, Y=0) = \frac{12}{36}$

$x=2, \ y=0 \qquad P(X=2, Y=0) = \frac{3}{36}$

$x=0, \ y=1 \qquad P(X=0, Y=1) = \frac{8}{36}$

$x=1, \ y=1 \qquad P(X=1, Y=1) = \frac{6}{36}$

$$F(2, 1.3) = \frac{6}{36} + \frac{12}{36} + \frac{3}{36} + \frac{8}{36} + \frac{6}{36} = \frac{35}{36}$$

$(0,0), (1,0), (2,0), (0,1), (1,1)$

The pairs included here are ...

done on the previous page

Therefore

$$F(2, 1.3) = f(0,0) + f(1,0) + f(2,0)$$
$$+ f(0,1) + f(1,1)$$
$$= \frac{6}{36} + \frac{12}{36} + \frac{3}{36} + \frac{8}{36} + \frac{6}{36} = \frac{35}{36}$$

As with the single variable case we have the following properties.

Theorem 3.5.5 *If $F(x, y)$ is the joint cumulative distribution for discrete random variables X and Y then*

1. $\lim_{x,y\to-\infty} F(x, y) = 0,$

2. $\lim_{x,y\to\infty} F(x, y) = 1,$ *and*

3. *If $a \le c$ and $b \le d$ then $F(a, b) \le F(c, d)$.*