MATH1550
Exercise Set 9 - Solutions

- Moments
- Chebyshev's Theorem
- Moment Generating Functions

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1. Let X be a random variable with the following distribution

x	-3	-1	2	5
$P(X=x)$	0.3	0.1	0.2	0.4

(a) Find the expected value of X.
(b) Find the variance of X.
(c) Find the 3rd moment about the mean of X.

Solution. (a) $E(X) = (-3)\cdot(0.3) + (-1)\cdot(0.1) + (2)\cdot(0.2) + (5)\cdot(0.4) = 1.4$.

(b) We have our mean $\mu = E(X) = 1.4$ from part (a). The variance σ^2 is $E((X-\mu)^2)$.

$E((X-\mu)^2) = (-3-1.4)^2\cdot(0.3) + (-1-1.4)^2\cdot(0.1) + (2-1.4)^2\cdot(0.2) + (5-1.4)^2\cdot(0.4) = 11.64.$

We can also find the variance by the formula $E((X-\mu)^2) = E(X^2) - \mu^2$. We have

$$E(X^2) = (-3)^2\cdot(0.3) + (-1)^2\cdot(0.1) + (2)^2\cdot(0.2) + (5)^2\cdot(0.4) = 13.6,$$

and so

$$\sigma^2 = 13.6 - (1.4)^2 = 11.64.$$

(c) The third moment about the mean is

$E((X-\mu)^3) = (-3-1.4)^3\cdot(0.3) + (-1-1.4)^3\cdot(0.1) + (2-1.4)^3\cdot(0.2) + (5-1.4)^3\cdot(0.4) = -8.232.$

We can also find this with the formula

$$E((X-\mu)^3) = E(X^3) - 3\mu E(X^2) + 2\mu^3.$$

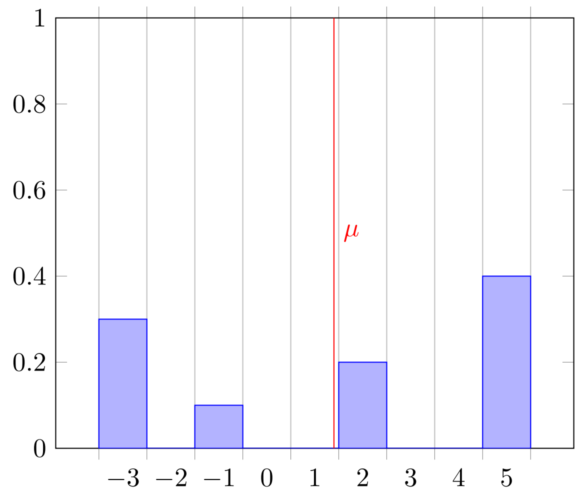
(you'll derive this formula in another problem). We find

$$E(X^3) = (-3)^3\cdot(0.3) + (-1)^3\cdot(0.1) + (2)^3\cdot(0.2) + (5)^3\cdot(0.4) = 43.4,$$

and using $E(X^2)$ from part (b) we have

$$E((X-\mu)^3) = 43.4 - 3(1.4)(13.6) + 2(1.4)^3 = -8.232.$$

Histogram representing the probability distribution of X:



□

2. Let Y be a random variable with the following distribution

y	2	3	4	5	6
$P(Y=y)$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{3}{9}$	$\frac{2}{9}$	$\frac{1}{9}$

   (a) Find the expected value of Y.
   (b) Find the variance of Y.
   (c) Find the 3rd moment about the mean of Y.

Solution. (a) $E(Y) = (2)\cdot\frac{1}{9} + (3)\cdot\frac{2}{9} + (4)\cdot\frac{3}{9} + (5)\cdot\frac{2}{9} + (6)\cdot\frac{1}{9} = 4.$

(b) $E((Y-\mu)^2)) = (2-4)^2\cdot\frac{1}{9} + (3-4)^2\cdot\frac{2}{9} + (4-4)^2\cdot\frac{3}{9} + (5-4)^2\cdot\frac{2}{9} + (6-4)^2\cdot\frac{1}{9} = \frac{12}{9} \approx 1.3333.$

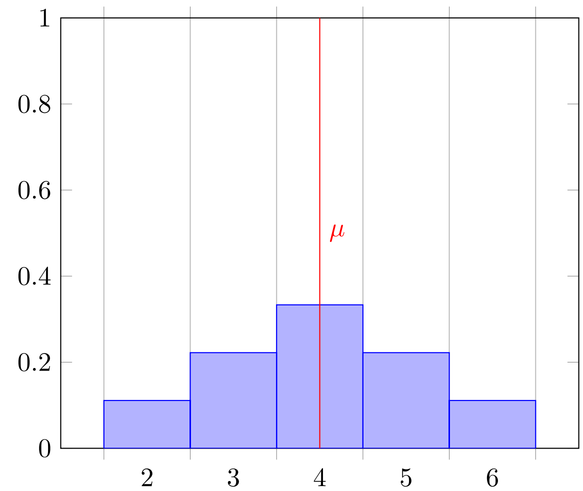
Applying the formula $\sigma^2 = E(Y^2) - \mu^2$, we have

$$E(Y^2) = (2)^2\cdot\frac{1}{9} + (3)^2\cdot\frac{2}{9} + (4)^2\cdot\frac{3}{9} + (5)^2\cdot\frac{2}{9} + (6)^2\cdot\frac{1}{9} = \frac{156}{9},$$

and so

$$\sigma^2 = \frac{156}{9} - (4)^2 = \frac{12}{9} \approx 1.3333.$$

(c) $E((Y-\mu)^3) = (2-4)^3\cdot\frac{1}{9} + (3-4)^3\cdot\frac{2}{9} + (4-4)^3\cdot\frac{3}{9} + (5-4)^3\cdot\frac{2}{9} + (6-4)^3\cdot\frac{1}{9} = 0.$
Histogram representing the probability distribution of Y:



□

3. Let X be a continuous random variable with probability density

$$f(x) = \begin{cases} \frac{1}{10}(3x^2 + 1) & \text{for } 0 \leq x \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

Find the mean and variance of X.

Solution. The mean is

$$\mu = \int_0^2 x \cdot \frac{1}{10}(3x^2 + 1)\, dx = \frac{1}{10}\left(\frac{3x^4}{4} + \frac{x^2}{2}\right)\Big|_0^2 = \frac{14}{10} = 1.4.$$

We have,

$$E(X^2) = \int_0^2 x^2 \cdot \frac{1}{10}(3x^2 + 1)\, dx = \frac{1}{10}\left(\frac{3x^5}{5} + \frac{x^3}{3}\right)\Big|_0^2 = \frac{164}{75},$$

thus the variance is

$$\sigma^2 = E(X^2) - \mu^2 = \frac{164}{75} - (1.4)^2 \approx 0.2267.$$

□

4. Write the definition for the 3rd moment about the mean, and then devise a “shortcut” formula in terms of the moments about the origin. Do this using properties of expected value as was done to obtain a formula for the second moment about the mean.

Solution. By definition:

$$E((X-\mu)^3) = \sum_x (x-\mu)^3 f(x) \quad \text{or} \quad E((X-\mu)^3) = \int_{-\infty}^{\infty} (x-\mu)^3 f(x)\, dx$$

Shortcut formula:

$$\begin{aligned} E((X-\mu)^3) &= E(X^3 - 3X^2\mu + 3X\mu^2 - \mu^3) \\ &= E(X^3) - 3\mu E(X^2) + 3\mu^2 E(X) - \mu^3 \\ &= E(X^3) - 3\mu E(X^2) + 2\mu^3 \end{aligned}$$

□

5. Derive an expression for $E((X-\mu)^4))$ which involves only terms $E(X^4), E(X^3), E(X^2), E(X)$. In other words, find a "shortcut" formula which allows us to compute $E((X-\mu)^4))$ from moments around the origin.

Solution.

$$\begin{aligned} E((X-\mu)^4)) &= E(X^4 - 4X^3\mu + 6X^2\mu^2 - 4X\mu^3 + \mu^4) \\ &= E(X^4) - 4\mu E(X^3) + 6\mu^2 E(X^2) - 4\mu^3 E(X) + \mu^4 \\ &= E(X^4) - 4\mu E(X^3) + 6\mu^2 E(X^2) - 3\mu^4 \end{aligned}$$

$$\text{or} \quad E(X^4) - 4E(X)E(X^3) + 6(E(X))^2 E(X^2) - 3(E(X))^4$$

□

6. Find $\mu = E(X)$, $E(X^2)$, σ^2 (variance) and σ (standard deviation) for the discrete random variable X that has the probability distribution $f(x) = \frac{1}{2}$ for $x = -2$ and $x = 2$.

Solution. The probability distribution for X is

x	$P(X = x)$
-2	$\frac{1}{2}$
2	$\frac{1}{2}$

The mean μ of X (or expected value $E(X)$) is

$$\mu = E(X) = (-2) \cdot \frac{1}{2} + (2) \cdot \frac{1}{2} = 0.$$

The second moment about the origin is

$$E(X^2) = (-2)^2 \cdot \frac{1}{2} + (2)^2 \cdot \frac{1}{2} = 4.$$

The variance for X is

$$\sigma^2 = E(X^2) - \mu^2 = 4 - 0^2 = 4.$$

The standard deviation is

$$\sigma = \sqrt{\sigma^2} = \sqrt{4} = 2.$$

□

7. If the probability density of X is given by

$$f(x) = \begin{cases} 630x^4(1-x)^4 & \text{for } 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

find the probability that X will take on a value within two standard deviations of the mean and compare this probability with the lower bound provided by Chebyshev's Theorem.

Solution. First we find the mean $\mu = E(X)$.

$$\begin{aligned}
\mu &= \int_{-\infty}^{\infty} xf(x)\,dx \\
&= \int_0^1 x(630x^4(1-x)^4)\,dx \\
&= 630\int_0^1 x^5(1-4x+6x^2-4x^3+x^4)\,dx \\
&= 630\left[\frac{x^6}{6}-\frac{4x^7}{7}+\frac{3x^8}{4}-\frac{4x^9}{9}+\frac{x^{10}}{10}\right]_0^1 \\
&= 630\left[\frac{1}{6}-\frac{4}{7}+\frac{3}{4}-\frac{4}{9}+\frac{1}{10}\right] \\
&= \frac{1}{2}
\end{aligned}$$

To find the variance we will first find $E(X^2)$

$$\begin{aligned}
E(X^2) &= \int_{-\infty}^{\infty} x^2 f(x)\,dx \\
&= \int_0^1 x^2(630x^4(1-x)^4)\,dx \\
&= 630\int_0^1 x^6(1-4x+6x^2-4x^3+x^4)\,dx \\
&= 630\left[\frac{x^7}{7}-\frac{x^8}{2}+\frac{2x^9}{3}-\frac{2x^{10}}{5}+\frac{x^{11}}{11}\right]_0^1 \\
&= 630\left[\frac{1}{7}-\frac{1}{2}+\frac{2}{3}-\frac{2}{5}+\frac{1}{11}\right] \\
&= \frac{3}{11}
\end{aligned}$$

Therefore the variance is

$$\sigma^2 = E(X^2)-\mu^2 = \frac{3}{11}-\left(\frac{1}{2}\right)^2 = \frac{1}{44}.$$

The standard deviation is

$$\sigma = \sqrt{\frac{1}{44}} \approx 0.1508.$$

Note that $\mu - 2\sigma \approx 0.1985$ and $\mu + 2\sigma \approx 0.8015$. Thus the probability that X will lie within two standard deviations of the mean is

$$\begin{aligned}
P(|X-\mu|<2\sigma) &= \int_{\mu-2\sigma}^{\mu+2\sigma} f(x)\,dx \\
&= \int_{\mu-2\sigma}^{\mu+2\sigma} 630x^4(1-x)^4\,dx \\
&= 630\int_{\mu-2\sigma}^{\mu+2\sigma} x^4(1-4x+6x^2-4x^3+x^4)\,dx \\
&= 630\left[\frac{x^5}{5}-\frac{2x^6}{3}+\frac{6x^7}{7}-\frac{x^8}{2}+\frac{x^9}{9}\right]_{\mu-2\sigma}^{\mu+2\sigma}
\end{aligned}$$

Rounding to $\mu - 2\sigma \approx 0.2$ and $\mu + 2\sigma \approx 0.8$ this expression yields

$$P(|X - \mu| < 2\sigma) \approx 0.96.$$

In this case Chebyshev's Theorem for $k = 2$ gives us the lower bound

$$P(|X - \mu| < 2\sigma) \geq 1 - \frac{1}{2^2} = \frac{3}{4}.$$

□

8. A study of the nutritional value of a certain kind of bread shows that the amount of thiamine (vitamin B_1) in a slice may be looked upon as a random variable X with $\mu = 0.260$ milligrams and $\sigma = 0.005$ milligrams. According to Chebyshev's Theorem, what interval of thiamine content values about μ must we consider, in order to include:

   (a) at least 35 of every 36 slices of bread?

   (b) at least 143 of every 144 slices of bread?

   Solution. Chebyshev's Theorem states

$$P(|X - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}.$$

   Thus we are solving for $\mu - k\sigma$ and $\mu + k\sigma$, and given μ, σ and $1 - \frac{1}{k^2}$.

   (a)

$$1 - \frac{1}{k^2} = \frac{35}{36} \quad \Rightarrow \quad \frac{1}{k^2} = \frac{1}{36} \quad \Rightarrow \quad k = 6.$$

   Thus Chebyshev's Theorem asserts that the thiamine content must be between

$$\mu - k\sigma = 0.260 - 6(0.005) = 0.23 \quad \text{and} \quad \mu + k\sigma = 0.260 + 6(0.005) = 0.29.$$

   Stated another way: The probability that the thiamine content is between 0.23 and 0.29 milligrams is at least $\frac{35}{36}$.

   (b)

$$1 - \frac{1}{k^2} = \frac{143}{144} \quad \Rightarrow \quad \frac{1}{k^2} = \frac{1}{144} \quad \Rightarrow \quad k = 12,$$

   and Chebyshev's Theorem asserts that the thiamine content must be between

$$\mu - k\sigma = 0.260 - 12(0.005) = 0.2 \quad \text{and} \quad \mu + k\sigma = 0.260 + 12(0.005) = 0.32.$$

□

9. Let X be a continuous random variable with probability density

$$f(x) = \begin{cases} \frac{1}{6}x + \frac{1}{12} & \text{for } 0 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find the mean μ of X.

   (b) Find the variance σ^2 of X.

   (c) Compute $P(1 \leq X \leq 2)$.

   (d) Find $P(|X - \mu| < \frac{3}{2}\sigma)$, and compare this value with what Chebyshev's Theorem tells us.

Solution. (a) $\mu = E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_0^3 x \cdot \left(\frac{1}{6}x + \frac{1}{12}\right)\,dx = \frac{x^3}{18} + \frac{x^2}{24}\Big|_0^3 = \frac{45}{24} = 1.875.$

(b) We will use the formula $\sigma^2 = E(X^2) - \mu^2$. First we have

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx = \int_0^3 x^2 \cdot \left(\frac{1}{6}x + \frac{1}{12}\right)\,dx = \frac{x^4}{24} + \frac{x^3}{36}\Big|_0^3 = \frac{33}{8} = 4.125.$$

Then

$$\sigma^2 = 4.125 - (1.875)^2 = \frac{33}{8} - \frac{2025}{576} = \frac{117}{192} = 0.609375.$$

(c) $P(1 \le X \le 2) = \int_1^2 \frac{1}{6}x + \frac{1}{12}\,dx = \frac{x^2}{12} + \frac{x}{12}\Big|_1^2 = \frac{1}{3}.$

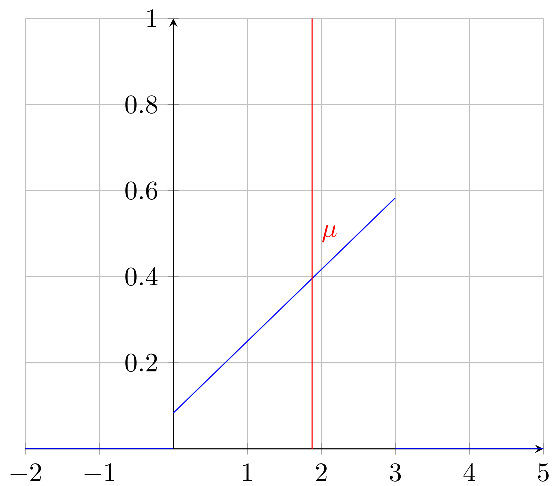
(d) To find $P(|X - \mu| < \frac{3}{2}\sigma)$ we integrate our density function from $\mu - \frac{3}{2}\sigma$ to $\mu + \frac{3}{2}\sigma$. Note that $\sigma \approx 0.7806$, $\mu - \frac{3}{2}\sigma \approx 0.7041$ and $\mu + \frac{3}{2}\sigma \approx 3.0459 > 3$. So we have

$$\begin{aligned}
P(|X - \mu| < \frac{3}{2}\sigma) &= \int_{\mu - \frac{3}{2}\sigma}^{\mu + \frac{3}{2}\sigma} f(x)\,dx \\
&= \int_{\mu - \frac{3}{2}\sigma}^{3} \frac{1}{6}x + \frac{1}{12}\,dx \\
&= \frac{x^2}{12} + \frac{x}{12}\Big|_{\mu - \frac{3}{2}\sigma}^{3} \\
&= \frac{3^2}{12} + \frac{3}{12} - \frac{(\mu - \frac{3}{2}\sigma)^2}{12} - \frac{\mu - \frac{3}{2}\sigma}{12} \\
&= 1 - \frac{\mu^2 - 3\sigma\mu + \frac{9}{4}\sigma^2}{12} - \frac{\mu - \frac{3}{2}\sigma}{12} \\
&\approx 0.90002.
\end{aligned}$$

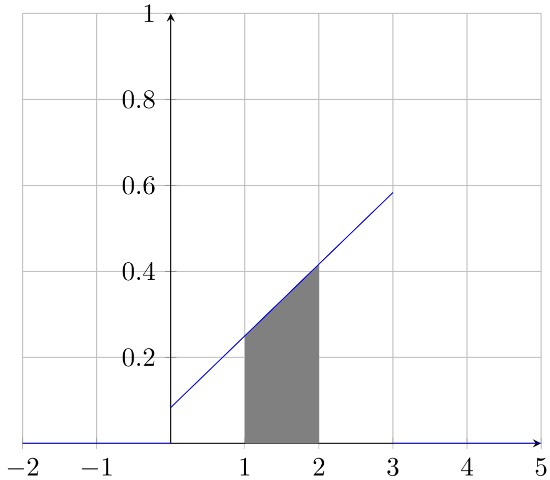
By Chebyshev's Theorem

$$P(|X - \mu| < \frac{3}{2}\sigma) \ge 1 - \frac{1}{\left(\frac{3}{2}\right)^2} = \frac{5}{9} \approx 0.55556$$

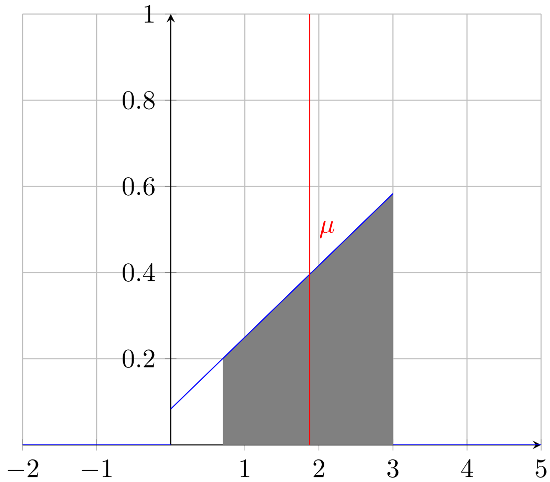
Plot of density function $f(x)$:



$P(1 \le X \le 2)$ represented by shaded region:



$P(|X-\mu| < \frac{3}{2}\sigma)$ represented by shaded region:



□

10. Let X be a continuous random variable with probability density given by

$$f(x) = \begin{cases} \frac{1}{8}(x+1) & \text{for } 2 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the mean μ of X.

(b) Find the variance of X.

(c) Find the 3rd moment about the mean for X.

(d) Find the standard deviation σ, and find $P(|X-\mu| < 2\sigma)$.

Solution. (a)

$$\begin{aligned}
\mu = E(X) &= \int_{-\infty}^{\infty} x \cdot f(x)\, dx \\
&= \int_2^4 x \cdot \frac{1}{8}(x+1)\, dx \\
&= \frac{1}{8}\int_2^4 x^2 + x\, dx \\
&= \frac{1}{8}\left(\frac{x^3}{3} + \frac{x^2}{2}\right)\Bigg|_2^4 \\
&= \frac{1}{8}\left(\frac{64}{3} + \frac{16}{2} - \frac{8}{3} - \frac{4}{2}\right) \\
&= \frac{37}{12}
\end{aligned}$$

(b) If we compute this directly using the definition:

$$\begin{aligned}
\sigma^2 = E((X-\mu)^2) &= \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f(x)\, dx \\
&= \int_2^4 (x-\mu)^2 \cdot \frac{1}{8}(x+1)\, dx \\
&= \frac{1}{8}\int_2^4 (x^2 - 2\mu x + \mu^2)(x+1)\, dx \\
&= \frac{1}{8}\int_2^4 x^3 + (1-2\mu)x^2 + (\mu^2 - 2\mu)x + \mu^2\, dx \\
&= \frac{1}{8}\left(\frac{x^4}{4} + \frac{(1-2\mu)x^3}{3} + \frac{(\mu^2-2\mu)x^2}{2} + \mu^2 x\right)\Bigg|_2^4 \\
&= \frac{1}{8}\left(64 + \frac{64(1-2\mu)}{3} + 8(\mu^2 - 2\mu) + 4\mu^2\right) - \frac{1}{8}\left(4 + \frac{8(1-2\mu)}{3} + 2(\mu^2-2\mu) + 2\mu^2\right) \\
&= \frac{1}{8}\left(60 + \frac{(1-2\mu)56}{3} + 6(\mu^2 - 2\mu) + 2\mu^2\right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{8}\left(\frac{236}{3} - \frac{148\mu}{3} + 8\mu^2\right) \\
&= \frac{59}{6} - \frac{37}{6}\left(\frac{37}{12}\right) + \left(\frac{37}{12}\right)^2 \\
&= \frac{59}{6} - \left(\frac{37}{12}\right)^2 \\
&= \frac{47}{144}
\end{aligned}$$

Using the theorem that says $\sigma^2 = E(X^2) - \mu^2$ we can save some time here.

$$\begin{aligned}
E(X^2) &= \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx \\
&= \int_2^4 x^2 \cdot \frac{1}{8}(x+1)\,dx \\
&= \frac{1}{8}\int_2^4 x^3 + x^2\,dx \\
&= \frac{1}{8}\left(\frac{x^4}{4} + \frac{x^3}{3}\right)\Big|_2^4 \\
&= \frac{1}{8}\left(64 + \frac{64}{3} - 4 - \frac{8}{3}\right) \\
&= \frac{1}{8}\left(60 + \frac{56}{3}\right) \\
&= \frac{59}{6}
\end{aligned}$$

Then

$$\sigma^2 = E(X^2) - \mu^2 = \frac{59}{6} - \left(\frac{37}{12}\right)^2 = \frac{47}{144}$$

(c) Instead of computing this directly let's devise a shortcut using the properties of expected value.

$$\begin{aligned}
E((X-\mu)^3) &= E(X^3 - 3X^2\mu + 3X\mu^2 - \mu^3) \\
&= E(X^3) - 3\mu E(X^2) + 3\mu^2 E(X) - \mu^3 \\
&= E(X^3) - 3\mu E(X^2) + 2\mu^3
\end{aligned}$$

we have $E(X^2)$ from a previous problem, now we just need $E(X^3)$.

$$E(X^3) = \int_{-\infty}^{\infty} x^3 \cdot f(x)\,dx = \int_2^4 x^3 \cdot \frac{1}{8}(x+1)\,dx = \frac{1}{8}\int_2^4 x^4 + x^3\,dx = \frac{1}{8}\left(\frac{x^5}{5} + \frac{x^4}{4}\right)\Big|_2^4 = \frac{323}{10}$$

Then

$$E((X-\mu)^3) = E(X^3) - 3\mu E(X^2) + 2\mu^3 = \frac{323}{10} - 3\left(\frac{37}{12}\right)\left(\frac{59}{6}\right) + 2\left(\frac{37}{12}\right)^3 = -\frac{139}{4320} \approx -0.0322$$

(d) Using the variance σ^2 from before, we see that the standard deviation $\sigma = \sqrt{\frac{47}{144}} \approx 0.5713$.

Recall that $|X-\mu| < 2\sigma$ implies $-2\sigma < X - \mu < 2\sigma$ and hence $\mu - 2\sigma < X < \mu + 2\sigma$. Also note that $\mu - 2\sigma \approx 1.941 < 2$ and $\mu + 2\sigma \approx 4.226 > 4$. Thus

$$\begin{aligned}
P(|X-\mu| < 2\sigma) &= \int_{\mu-2\sigma}^{\mu+2\sigma} f(x)\,dx \\
&= \int_2^4 f(x)\,dx \\
&= 1.
\end{aligned}$$

$\square$

11. Let X be a discrete random variable with the probability distribution given below. Find the variance

of X.

x	$f(x)$
-2	$\frac{1}{20}$
-1	$\frac{3}{20}$
0	$\frac{6}{20}$
1	$\frac{2}{20}$
2	$\frac{7}{20}$
3	$\frac{1}{20}$

Solution. The mean μ of X is

$$\mu = E(X) = (-2)\frac{1}{20} + (-1)\frac{3}{20} + (0)\frac{6}{20} + (1)\frac{2}{20} + (2)\frac{7}{20} + (3)\frac{1}{20} = \frac{14}{20}.$$

(This was found previously in Mini-Assignment 8). The variance of X is

$$\begin{aligned}
\sigma^2 &= E((X-\mu)^2) \\
&= \left(-2 - \frac{14}{20}\right)^2 \frac{1}{20} + \left(-1 - \frac{14}{20}\right)^2 \frac{3}{20} + \left(0 - \frac{14}{20}\right)^2 \frac{6}{20} + \left(1 - \frac{14}{20}\right)^2 \frac{2}{20} + \left(2 - \frac{14}{20}\right)^2 \frac{7}{20} \\
&\quad + \left(3 - \frac{14}{20}\right)^2 \frac{1}{20} \\
&= \left(-\frac{54}{20}\right)^2 \frac{1}{20} + \left(-\frac{34}{20}\right)^2 \frac{3}{20} + \left(-\frac{14}{20}\right)^2 \frac{6}{20} + \left(\frac{6}{20}\right)^2 \frac{2}{20} + \left(\frac{26}{20}\right)^2 \frac{7}{20} + \left(\frac{46}{20}\right)^2 \frac{1}{20} \\
&= \frac{14480}{8000} \\
&= \frac{181}{100}.
\end{aligned}$$

An easier way to compute this is using the formula $\sigma^2 = E(X^2) - \mu^2$, where

$$E(X^2) = (-2)^2\frac{1}{20} + (-1)^2\frac{3}{20} + (0)^2\frac{6}{20} + (1)^2\frac{2}{20} + (2)^2\frac{7}{20} + (3)^2\frac{1}{20} = \frac{46}{20}.$$

(also found in Mini-Assignment 8), so

$$\sigma^2 = \frac{45}{20} - \left(\frac{14}{20}\right)^2 = \frac{181}{100}.$$

□

12. Let X be a discrete random variable with the probability distribution given below. Find the third moment about the mean of X.

x	$f(x)$
-2	$\frac{1}{20}$
-1	$\frac{3}{20}$
0	$\frac{6}{20}$
1	$\frac{2}{20}$
2	$\frac{7}{20}$
3	$\frac{1}{20}$

Solution. The third moment about the mean is

$$
\begin{aligned}
\sigma^2 &= E((X-\mu)^3) \\
&= \left(-2-\frac{14}{20}\right)^3\frac{1}{20} + \left(-1-\frac{14}{20}\right)^3\frac{3}{20} + \left(0-\frac{14}{20}\right)^3\frac{6}{20} + \left(1-\frac{14}{20}\right)^3\frac{2}{20} + \left(2-\frac{14}{20}\right)^3\frac{7}{20} \\
&\quad + \left(3-\frac{14}{20}\right)^2\frac{1}{20} \\
&= \left(-\frac{54}{20}\right)^3\frac{1}{20} + \left(-\frac{34}{20}\right)^3\frac{3}{20} + \left(-\frac{14}{20}\right)^3\frac{6}{20} + \left(\frac{6}{20}\right)^3\frac{2}{20} + \left(\frac{26}{20}\right)^3\frac{7}{20} + \left(\frac{46}{20}\right)^3\frac{1}{20} \\
&= -\frac{71040}{160000} \\
&= -\frac{111}{250}.
\end{aligned}
$$

This can also be computed using the "short cut" formula

$$E((X-\mu)^3) = E(X^3) - 3\mu E(X^2) + 2\mu^3$$

where

$$E(X^3) = (-2)^3\frac{1}{20} + (-1)^3\frac{3}{20} + (0)^3\frac{6}{20} + (1)^3\frac{2}{20} + (2)^3\frac{7}{20} + (3)^3\frac{1}{20} = \frac{74}{20},$$

and $\mu = \frac{14}{20}$ and $E(X^2) = \frac{46}{20}$ (from the previous question), so

$$E((X-\mu)^3) = \frac{74}{20} - 3\left(\frac{14}{20}\right)\left(\frac{46}{20}\right) + 2\left(\frac{14}{20}\right)^3 = -\frac{111}{250}.$$

□

13. Let X be a continuous random variable with the probability density given below. Compute the variance of X.

$$f(x) = \begin{cases} \frac{x}{2} & \text{for } 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Solution. First we find the mean of X.

$$\mu = \int_{-\infty}^{\infty} xf(x)\,dx = \int_0^2 \frac{x^2}{2}\,dx = \frac{x^3}{6}\Big|_0^2 = \frac{4}{3}.$$

Then

$$\begin{aligned}
\sigma^2 &= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx \\
&= \int_0^2 \left(x - \frac{4}{3}\right)^2 \frac{x}{2}\, dx \\
&= \int_0^2 \left(x^2 - \frac{8}{3}x + \frac{16}{9}\right) \frac{x}{2}\, dx \\
&= \int_0^2 \frac{x^3}{2} - \frac{4x^2}{3} + \frac{8x}{9}\, dx \\
&= \frac{x^4}{8} - \frac{4x^3}{9} + \frac{4x^2}{9}\Big|_0^2 \\
&= 2 - \frac{32}{9} + \frac{16}{9} \\
&= \frac{2}{9}.
\end{aligned}$$

Alternatively,

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_0^2 \frac{x^3}{2}\, dx = \frac{x^4}{8}\Big|_0^2 = 2,$$

so

$$\sigma^2 = E(X^2) - \mu^2 = 2 - \left(\frac{4}{3}\right)^2 = \frac{2}{9}.$$

□

14. A random variable X has mean $\mu = 124$ and standard deviation $\sigma = 7.5$. According to Chebyshev's Theorem, what is the minimum probability that X lies between 64 and 184?

    Solution. Since

    $$\frac{\mu - 64}{\sigma} = \frac{124 - 64}{7.5} = 8$$

    and hence $\mu - 8\sigma = 64$; i.e. 64 is 8 standard deviations from the mean. Similarly we see $\mu + 8\sigma = 184$. Thus by Chebyshev's Theorem

    $$P(|X - \mu| < 8\sigma) \geq 1 - \frac{1}{8^2} = \frac{63}{64} = 0.984375.$$

    □

15. Let X be a discrete random variable with the probability distribution given below. What does Chebyshev's Theorem tell us is the minimum probability that X lies within 1.3 standard deviations of the mean?

x	$f(x)$
-2	$\frac{1}{20}$
-1	$\frac{3}{20}$
0	$\frac{6}{20}$
1	$\frac{2}{20}$
2	$\frac{7}{20}$
3	$\frac{1}{20}$

Solution. By Chebyshev's Theorem

$$P(|X-\mu| < (1.3)\sigma) \geq 1 - \frac{1}{(1.3)^2} = \frac{69}{169} \approx 0.4083.$$

$\square$

16. Let X be a discrete random variable with the probability distribution given below. What is the probability that X lies within 1.3 standard deviations of the mean?

x	$f(x)$
-2	$\frac{1}{20}$
-1	$\frac{3}{20}$
0	$\frac{6}{20}$
1	$\frac{2}{20}$
2	$\frac{7}{20}$
3	$\frac{1}{20}$

Solution. We have that $\mu = 0.7$ and $\sigma = \sqrt{\frac{181}{100}} \approx 1.3454$ (from previous question) so

$$\begin{aligned}
P(|X-\mu| < (1.3)\sigma) &= P(\mu - (1.3)\sigma < X < \mu + (1.3)\sigma) \\
&= P(-1.0490 < X < 2.4490) \\
&= P(-1 \leq X \leq 2) \\
&= P(X=-1) + P(X=0) + P(X=1) + P(X=2) \\
&= \frac{3}{20} + \frac{6}{20} + \frac{2}{20} + \frac{7}{20} \\
&= \frac{18}{20}
\end{aligned}$$

$\square$

17. Let X be a continuous random variable with probability density

$$f(x) = \begin{cases} 1 & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the moment generating function $M_X(t)$ of X.

Solution. $M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} \cdot f(x)\, dx = \int_0^1 e^{tx}\, dx = \left(\frac{1}{t} e^{tx}\right) \Big|_0^1 = \frac{1}{t}\left(e^t - 1\right)$ $\square$

18. Prove the following properties of moment generating functions:

   (a)
   $$M_{X+a}(t) = e^{at} \cdot M_X(t)$$

   (b)
   $$M_{bX}(t) = M_X(bt)$$

   (c)
   $$M_{\frac{X+a}{b}}(t) = e^{\frac{a}{b}t} \cdot M_X\left(\frac{t}{b}\right)$$

Solution. (a) $M_{X+a}(t) = E(e^{t(X+a)}) = \int_{-\infty}^{\infty} e^{t(x+a)} \cdot f(x)\,dx = \int_{-\infty}^{\infty} e^{tx} \cdot e^{at} \cdot f(x)\,dx = e^{at} \cdot \int_{-\infty}^{\infty} e^{tx} \cdot f(x)\,dx = e^{at} \cdot M_X(t)$

(b) $M_{bX}(t) = E(e^{tbX}) = \int_{-\infty}^{\infty} e^{tbx} \cdot f(x)\,dx = \int_{-\infty}^{\infty} e^{(bt)x} \cdot f(x)\,dx = M_X(bt)$

(c) $M_{\frac{X+a}{b}}(t) = E(e^{t((X+a)/b)}) = \int_{-\infty}^{\infty} e^{t((x+a)/b)} \cdot f(x)\,dx$

$= \int_{-\infty}^{\infty} e^{(t/b)x} \cdot e^{(a/b)t} \cdot f(x)\,dx = e^{(a/b)t} \cdot \int_{-\infty}^{\infty} e^{(t/b)x} \cdot f(x)\,dx$

$= e^{\frac{a}{b}t} \cdot M_X\left(\frac{t}{b}\right)$

□

19. Let X be a continuous random variable with probability density given by

$$f(x) = \begin{cases} 3e^{-3x} & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the moment generating function for X. (Hint: Express the integrand as $e^{-x(t-3)}$ and restrict $t < 3$.)

(b) Use the moment generating function to find the mean and variance of X.

Solution. (a)

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} \cdot f(x)\,dx = \int_0^{\infty} e^{tx}(3e^{-3x})\,dx = \int_0^{\infty} 3e^{-x(3-t)}\,dx.$$

Assume $t < 3$ or equivalently $3 - t > 0$ (so that the integral is finite), and hence

$$M_X(t) = \int_0^{\infty} 3e^{-x(3-t)}\,dx = 3\left.\frac{e^{-x(3-t)}}{-(3-t)}\right|_0^{\infty} = \frac{3}{3-t}.$$

(b)

$$\mu = E(X) = \left.\frac{d}{dt}M_X(t)\right|_{t=0} = \left.\frac{d}{dt}\left(\frac{3}{3-t}\right)\right|_{t=0} = \left.\frac{3}{(3-t)^2}\right|_{t=0} = \frac{1}{3}$$

$$E(X^2) = \left.\frac{d^2}{dt^2}M_X(t)\right|_{t=0} = \left.\frac{d}{dt}\left(\frac{3}{(3-t)^2}\right)\right|_{t=0} = \left.\frac{6}{(3-t)^3}\right|_{t=0} = \frac{2}{9}$$

$$\sigma^2 = E(X^2) - \mu^2 = \frac{1}{9}$$

□

20. Suppose the continuous random variable X has moment generating function given by

$$M_X(t) = 2(2-t)^{-1}$$

for $-2 < t < 2$. Find the mean and variance of X.

Solution.

$$\frac{d}{dt}M_X(t) = \frac{d}{dt}2(2-t)^{-1} = 2(2-t)^{-2}$$

$$\frac{d}{dt}M_X(t)\bigg|_{t=0} = \frac{1}{2}$$
$$\frac{d^2}{dt^2}M_X(t) = \frac{d}{dt}2(2-t)^{-2} = 4(2-t)^{-3}$$
$$\frac{d^2}{dt^2}M_X(t)\bigg|_{t=0} = \frac{1}{2}$$

Therefore
$$\mu = \frac{1}{2},$$
and
$$\sigma^2 = \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$

□

21. Find the moment generating function of the discrete random variable X that has probability distribution
$$f(x) = 2\left(\frac{1}{3}\right)^x, \quad \text{for } x \in \mathbb{N},$$
and use it to find the mean an variance of X. *Hint: Use the formula for sum of an infinite geometric series* $\sum_{i=0}^{\infty} ar^i = \frac{a}{1-r}$

Solution. We find the moment generating function as follows:

$$M_X(t) = E(e^{tX}) = \sum_{-\infty}^{\infty} e^{tx} \cdot f(x) \;= \sum_{x=1}^{\infty} e^{tx} \cdot 2 \cdot \left(\frac{1}{3}\right)^x$$
$$= 2 \cdot \sum_{x=1}^{\infty} e^{tx}\left(\frac{1}{3}\right)^x = 2 \cdot \sum_{x=1}^{\infty}\left(\frac{e^t}{3}\right)^x = 2 \cdot \left(\sum_{x=0}^{\infty}\left(\frac{e^t}{3}\right)^x - \left(\frac{e^t}{3}\right)^0\right)$$
$$= 2 \cdot \left(\sum_{x=0}^{\infty}\left(\frac{e^t}{3}\right)^x - 1\right) = 2 \cdot \left(\left(\frac{1}{1-\frac{e^t}{3}}\right) - 1\right) = 2 \cdot \left(\left(\frac{3}{3-e^t}\right) - 1\right)$$
$$= 2 \cdot \left(\frac{3-(3-e^t)}{3-e^t}\right) = \frac{2e^t}{3-e^t}.$$

To find the mean $\mu' = E(X)$ (the first moment about the origin) of X, we take the first derivative of the moment generating function with respect to t and evaluate it at $t = 0$.

$\dfrac{d}{dt}\dfrac{2e^t}{3-e^t} = \dfrac{6e^t}{(e^t-3)^2}$. Evaluating at $t = 0$, we find $\mu' = E(X) = \dfrac{3}{2}$.

To find the variance σ^2, we may use the formula $\sigma^2 = E(X^2) - \mu^2$. In order to find $E(X^2)$, we take the second derivative of the moment generating function and evaluate it at $t = 0$.

$$E(X^2) = \frac{d^2}{dt^2}M_X(t)\bigg|_{t=0} = \frac{d}{dt}\left(\frac{d}{dt}M_X(t)\right)\bigg|_{t=0} = \frac{d}{dt}\left(\frac{6e^t}{(e^t-3)^2}\right)\bigg|_{t=0}$$
$$= -\frac{6e^t(e^t+3)}{(e^t-3)^3}\bigg|_{t=0} = 3$$

Finally, $\sigma^2 = E(X^2) - \mu^2 = 3 - \left(\frac{3}{2}\right)^2 = \frac{3}{4}$.

$\square$

22. Suppose a random variable X has moment generating function

$$M_X(t) = e^{3t+8t^2}.$$

Find the mean and the variance of X.

Solution. The mean μ is the first moment about the origin. We can find it by taking the first derivative of the moment generating function and evaluating it at $t = 0$.

$$\frac{d}{dt} M_X(t) = (3 + 16t)\, e^{3t+8t^2}$$

Then,

$$(3 + 16t)\, e^{3t+8t^2} \bigg|_{t=0} = 3e^0 = 3.$$

The variance σ^2 can be found using the shortcut formula $\sigma^2 = E(X^2) - \mu^2$, where $E(X^2)$ is the second moment about the origin. To find the second moment about the origin we find the second derivative of the moment generating function and evaluate it at $t = 0$.

$$\frac{d^2}{dt^2} M_X(t) = \frac{d^2}{dt^2} e^{3t+8t^2} = \frac{d}{dt}(3 + 16t)\, e^{3t+8t^2} = 16e^{3t+8t^2} + (3 + 16t)^2 e^{3t+8t^2}$$

Then,

$$16e^{3t+8t^2} + (3 + 16t)^2 e^{3t+8t^2} \bigg|_{t=0} = 16e^0 + 3^2 e^0 = 16 + 9 = 25.$$

Therefore, from $\sigma^2 = E(X^2) - \mu^2$, we get $\sigma^2 = 25 - 3^2 = 16$.

$\square$

23. Let X be a random variable with moment generating function

$$M_X(t) = \frac{1}{1 - t^2}.$$

Find the mean of X.

Solution.

$$\begin{aligned}
\mu &= \frac{d}{dt} M_X(t) \bigg|_{t=0} \\
&= \frac{d}{dt} \frac{1}{1 - t^2} \bigg|_{t=0} \\
&= \frac{2t}{(1 - t^2)^2} \bigg|_{t=0} \\
&= 0
\end{aligned}$$

$\square$

24. Let X be a random variable with moment generating function

$$M_X(t) = \frac{1}{1 - t^2}.$$

Find the variance of X.

Solution.

$$\begin{aligned}
\sigma^2 &= \frac{d^2}{dt^2} M_X(t)\bigg|_{t=0} \\
&= \frac{d^2}{dt^2} \frac{1}{1-t^2}\bigg|_{t=0} \\
&= \frac{d}{dt} \frac{2t}{(1-t^2)^2}\bigg|_{t=0} \\
&= \frac{2(1-t^2)^2 - (2t)2(1-t^2)(-2t)}{(1-t^2)^4}\bigg|_{t=0} \\
&= 2
\end{aligned}$$

□