Intro to Discrete Math & Functional Programming— CSCI 54— Fall 2023
Instructor: Chen
Homework - Week 15 (4.5 points)
Due: 10:00PM on Friday, December 8

- This problem set is intentionally worth very few points, not because it is unusually short but in the hopes of keeping it low stress since it's due after the end of classes and I don't know what other obligations you may have.
- You may work in groups of size up to 3. Note that you are all responsible for what you turn in and when you turn it in.

1. **[1.5 points] Counting** You must briefly justify each of your answers. Just giving the correct answer is worth minimal partial credit.

   (a) How many 6-digit binary sequences contain exactly 2 1s?

   SOLUTION: For a 6-digit binary sequence, we can say that the order doesn't matter and that there is no repetition. Since there are two $1's$, we can decide to start with any of them and place them in any of the positions therefore order does not matter. Since we cannot place the two $1's$ in the same position, there is no repetition. Therefore, to find this we say that $\binom{6}{2}$. This is equal to 15.
   $\binom{6}{2} = \frac{6!}{(6-2)!2!}$

   (b) How many n-digit binary sequences contain exactly k 1s?

   SOLUTION:
   Order doesn't matter since the first 1 can take any of the positions as well as the second 1 until the kth 1 as long as the position has not been occupied by an other 1. There is no repetition since a position once occupied cannot be occupied by another 1. Therefore;
   $\binom{n}{k} = \frac{n!}{(n-k)!k!}$

   (c) 30 runners compete in a 10k qualifying race. Only the 10 fastest runners will advance to the finals. How many different outcomes (set of 10 fastest runners) are there?

   SOLUTION:

For this question, out of 30 runners only the 10 fastest runners will advance. Since we are looking only for the 10 fastest, without caring about which place they came in, we can say that order doesn't matter. Since 1 runner cannot compete 2 twice, there is no repetition. Therefore;
$\binom{30}{10} = \frac{30!}{(30-10)!10!}$.

2. **[1 points] Probability**

   (a) A pair of 6-sided dice are rolled. What is the probability that both die show an even number? Do this problem by defining a sample space of equally likely outcomes and calculating the probability as a ratio of the cardinality of the event and the cardinality of the sample space

   SOLUTION:
   Sample space can be defined by $S = \{(1,1),(1,2),(1,3),...\}$, where the first digit in the first tuple represents the results of the first die and the second digit in that same tuple represents the results of the second die, and so on.

   $\frac{|E|}{|S|} = \frac{3\times3}{6\times6} = \frac{9}{36} = \frac{1}{4}$

   (b) A pair of 6-sided dice are rolled. What is the probability that exactly one die shows an even number? Do this problem by defining a sample space of equally likely outcomes and calculating the probability as a ratio of the cardinality of the event and the cardinality of the sample space.

   SOLUTION:
   Sample space can be defined by $S = \{(1,1),(1,2),(1,3),...\}$, where the first digit in the first tuple represents the results of the first die and the second digit in that same tuple represents the results of the second die, and so on.

   $\frac{|E|}{|S|} = \frac{(3\times3)+(3\times3)}{6\times6} = \frac{18}{36} = \frac{1}{2}$

3. **[2 points] Working with Error Correcting Codes**

   Assume you have a binary string of length 3 that you will encode by tripling every digit (e.g. `001` to `000000111`).

   (a) How many bit strings are there of length 9?

SOLUTION: Each digit in the string can take on one of two values: 0 or 1. Given the length of the bit string as 9, the total number of possible combinations is 2^9, resulting in 2^9 distinct bit strings of length 9.

(b) Given the 3-bit message `001`, how many bit strings of length 9 decode to the correct message? (For example, both `010010111` and `000100101` decode correctly to `001`. However, `110000111` does not.)

SOLUTION: For each bit in the message 001, there are 4 different combinations which will result in the correct message being encoded. There are 3 bits in the message, which means that $4^3 = 64$ bit strings of length 9 decode to the correct message.

(c) Is the answer to the previous question the same for any 3-bit message? Briefly explain your answer.

SOLUTION: Yes, it is the same for any 3-bit message. Because for each bit in a 3-bit message, there are 4 combinations which will lead to it being decoded to the correct message. For example, for the bit "1", the 4 combinations that will lead to the correct message are 110, 011, 101, and 111. The same logic will apply to all bits in a 3 bit message, ending up with 4^3 or 64 different bit strings of length 9 which decode to the correct message.

(d) Now consider the general case where messages have length k. How many bit strings are there of length $3k$? How many bit-strings of length $3k$ decode to the correct message?

SOLUTION: For the general case:
There are 2^{3k} bit strings of length $3k$.
There are 4^k bit strings of length 3k that decode to the correct message.