NOTES ON "POLYNOMIAL TIME DECAY FOR SOLUTIONS TO THE KLEIN–GORDON EQUATION ON A SUBEXTREMAL REISSNER–NORDSTRÖM BLACK HOLE"

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These are notes on [SM23] (see also [PSM24]) for the Princeton–Cambridge GR seminar talks given on March 28 and April 4, 2025. For simplicity, we restrict to Schwarzschild (though there is no additional difficulty with subextremal Reissner–Nordström).

1. Introduction

We study the Klein–Gordon equation with an inhomogeneity on Schwarzschild:

$$\Box\varphi - m^2\varphi = F \tag{1}$$

Write $\psi = r\varphi$. We expand ψ into spherical harmonics $\psi = \sum_L \psi_L$. Then

$$-\partial_t^2\psi_L + \partial_s^2\psi_L = \underbrace{\Big(1-\frac{2M}{r}\Big)\Big(m^2 + \frac{L(L+1)}{r^2} + \frac{2M}{r^3}\Big)}_{:=V(r)}\psi_L + \Big(1-\frac{2M}{r}\Big)F_L. \tag{2}$$

where s is the tortoise coordinate (satisfying $\frac{\mathrm{d}s}{\mathrm{d}r} = (1-2M/r)^{-1}$).

1.1. The first result.

Theorem 1.1 ([PSM24])**.** *A solution φ_L arising from smooth compactly supported data supported on the L-th spherical mode satisfies*

$$r\varphi_L(t^*,r) = \mathcal{F}_L(t^*,r)\cdot(t^*)^{-5/6} + O_{R,L,\mathrm{data},\delta}((t^*)^{-1+\delta}) \quad \textit{in } \{r \le R\} \tag{3}$$

for an explicit oscillating profile

$$\mathcal{F}_L(t^*,r) \coloneqq O_{\mathrm{data}}(r^{1/4})\sum_{q=1}^{\infty}(-\Gamma_L)^{q-1}\cos\Big(mt^* - Cq^{2/3}(t^*)^{1/3} + O_q((t^*)^{-1/3}) + f(r)\Big), \tag{4}$$

where $0 < \Gamma_L < 1$.

Remark 1.2 (Comparison with massless waves)*.* This decay rate is slower than the rate for massless waves on Schwarzschild, which is $(t^*)^{-3-2L}$ (this is known as Price's law). In particular, each angular mode decays with the same rate. Note that on Minkowski, solutions to the Klein–Gordon equation decay faster than solutions to the massless wave equation.

Remark 1.3 (Comparison with Kerr)*.* On Kerr, the phenomenon of superradiance (lack of coercivity of the ∂_t-energy) gives rise to exponentially growing mode solutions, due to Yakov [Shl14], at least for an open set of Klein–Gordon masses. He also constructed time-periodic solutions. Note that solutions to the massless wave equation on Kerr decay, despite the presence of superradiance.

In contrast, on Schwarzschild one can use standard energy estimates (as for the wave equation) to show that $|r\varphi| \lesssim 1$ (this is for the full solution, not restricted to a single angular mode). The question that remains is decay.

Remark 1.4 (Removing the restriction to a compact-r region and obtaining decay without a rate)*.* By interpolating between $|\varphi_L| \lesssim r^{-1}$ (from energy estimates) and $\sup_{r\le R}|\varphi_L| \le C_{R,L}(t^*)^{-5/6}$, one sees that each angular mode decays without a rate, in the sense that $|\varphi_L|(t^*,r) \to 0$ as $t^* \to \infty$ for each $r > 2M$.

A soft argument (e.g. using angular commutation and the dominated convergence theorem) then shows that the full solution φ decays without a rate.[1]

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[1] I do not know how to make a statement uniform in r, i.e. $\lim_{t^*\to\infty}\sup_r|\varphi|(t^*,r) = 0$

Remark 1.5 (Absence of bound states). The decay shows that there are no bound states, even though this is an equation with a long range potential. The most famous example of such an equation is the hydrogen atom $\Box\varphi + \frac{1}{r}\varphi = 0$. Sussman [Sus23] used the bound states of the hydrogen atom to construct bound states for the Klein–Gordon equation on static asymptotically Schwarzschild backgrounds that *do not have a horizon*.

Heuristically, massive waves with insufficient kinetic energy will fall into the horizon, creating decay, and when there is no horizon there need not be decay. We will see this quantitatively in the form of a *positive energy flux* at the horizon, which has to do with the fact that the potential is real-valued.

Note that these linear bound states are related to boson stars, which are non-black hole stationary solutions of the Einstein–Klein–Gordon equations.

Remark 1.6 (Quasimodes). The absence of bound states is however consistent with the existence of quasimodes, which are approximate solutions corresponding to "almost bound states". These typically arise in the setting of stable trapping.[2]

On Schwarzschild–AdS there is stable trapping due to the boundary at infinity. Holzegel–Smulevici [HW14] exploited this to construct quasimodes for the appropriate analogue of Klein–Gordon (namely $(\Box_{g_{\mathrm{SAdS}}} + \alpha)\varphi$ for $-\Lambda^{-1}3\alpha \in (3/2, 9/4)$, i.e. satisfying the Breitenlohner–Freedman condition). This leads to $1/\log t^*$ decay, which is sharp [HW14; HS13]. These quasimodes have compact support in s. Note that for a single angular mode there is exponential decay [HS13; HS11].

The Schwarzschild black string is the product of the usual 4D Schwarzschild with a trivial S^1_ψ factor. In particular, $e^{im\psi}\varphi(t, r, \theta, \varphi)$ solves the wave equation on this spacetime if and only if φ solves the Klein–Gordon equation on Schwarzschild. Benomio [Ben21] constructed quasimodes for the wave equation on this spacetime (all supported in a fixed compact set), leading to logarithmic lower bounds.

Similar phenomena occur for the wave equation in the work of Keir [Kei16] on ultracompact neutron stars.

Remark 1.7 (Stable timelike trapping for large L). Of course, Schwarzschild does not have stable null trapping. However, it is *timelike* trapping that is relevant for the Klein–Gordon equation. When $L \gg 1$ and $0 \le m^2 - \omega^2 \ll L^{-2}$, the potential (which has an attractive $-2Mm^2/r$ Coulomb term due to the Klein–Gordon mass) admits a minimum in a classically allowed region, which is the qualitative source of stable timelike trapping.

1.2. **The new result.** The main theorem of [SM23] is that, despite the presence of quasimodes and stable timelike trapping, decay holds for solutions to the Klein–Gordon equation, with no restriction to finitely many angular modes.

Theorem 1.8 ([SM23]). *A solution φ arising from smooth compactly supported data satisfies*

$$\sup_{r\le R}|\varphi| \lesssim_{R,\delta} (t^*)^{-5/6+\delta}(\|\varphi_{\mathrm{data}}\|_{H^N\times H^{N-1}}), \tag{5}$$

where

(i) *$\delta > 0$ can be taken arbitrarily small if the exponent pair conjecture holds,*
(ii) *or $\delta = 1/23$ unconditionally (using the best known results on the conjecture).*

Remark 1.9. The compact support assumption can be relaxed to include data decaying at a sufficiently fast exponential rate ($\exp(-r^{1/2-\epsilon})$ is enough, and is essentially sharp).

1.3. **Consequences of stable timelike trapping.**

Theorem 1.10 (Unbounded Fourier transform, [SM23]). *There exist smooth and compactly supported data such that the solution has an unbounded Fourier transform.*

Proof. One first constructs a source so that the solution with zero data has an unbounded Fourier transform. The mechanism is the concentration of the source as $L \to \infty$ around a discrete set of "bad frequencies" (see the discussion in section 2.3.3, in particular (28) and remark 2.10). Then, one runs a contradiction argument that uses Duhamel's formula. ☐

A similar argument shows that the Morawetz estimate with a source fails.

[2] I think this idea goes back to Ralston.

Theorem 1.11 (Failure of Morawetz with a source, [SM23])**.** *Fix $p, q, N \in \mathbf{N}$ and $R > 2M$. For any $C > 0$, there exists a smooth and compactly supported source F_C such that the solution φ_C to $(\Box - m^2)\varphi_C = F_C$ with vanishing Cauchy data satisfies*

$$\int_0^\infty \int_{2M}^R |r^{-p}\partial^{\leq 1}\varphi_C|^2 \,\mathrm{d}r\,\mathrm{d}t \geq C \int_0^\infty \int_{2M}^R |r^q \partial^{\leq N}\varphi_C|^2 \,\mathrm{d}r\,\mathrm{d}t. \tag{6}$$

Remark 1.12 (Morawetz with a source without derivatives)*.* The Morawetz estimate with a source continues to fail if only a zeroth order term is present on the left side.

Remark 1.13 (Morawetz with a source for a single angular mode or for massless waves)*.* The Morawetz estimate with a source holds for solutions supported on a singular angular mode, with a constant that depends on the angular mode number. Moreover, the estimate with a source holds for massless wave equations.

Remark 1.14 (Morawetz for the free Klein–Gordon equation with data on the right side)*.* Note that the pointwise estimate in theorem 1.8 implies that a Morawetz estimate in terms of initial data holds for solutions without a source.

A key consequence of stable timelike trapping is the existence of quasimodes.

Theorem 1.15 (Quasimodes, [SM23])**.** *There are functions $\Phi_L = e^{i(m - O(L^{-p}))t^*}R(r)Y(\theta,\phi)$ supported on the L-th angular mode such that*

(i) $\operatorname{supp}\Phi_L \subset \{L^2 \lesssim r \lesssim L^p\}$ *and* $\|\Phi_L\|_{L^2(\Sigma_{t^*})} = 1$,
(ii) $|(\Box - m^2)\Phi_L| \lesssim e^{-CL}$.

Remark 1.16 (Logarithmic lower bounds for polynomially decaying data)*.* A standard argument (see for example [HW14, Sec. 7]) shows that theorem 1.15 implies that there cannot be a *uniform* pointwise decay statement with a rate faster than logarithmic for solutions arising from data of only polynomial decay (note that such a statement is consistent with any particular solution decaying faster than logarithmically).

Remark 1.17 (Support of the quasimodes)*.* These quasimodes do not preclude polynomial decay for compactly supported data (i.e., the statement of theorem 1.8), because their support moves to $r = \infty$ as $L \to \infty$.[3] It is this point that would perhaps be a saving grace for applications to nonlinear problems with sufficiently localized data.

2. Strategy of the proof

The rough strategy is to derive estimates for the Fourier transform of the solution, and then obtain decay via Fourier inversion and stationary/non-stationary phase arguments.

2.1. **Reduction to a problem in frequency space.** Let φ_{sol} be a solution of $(\Box - m^2)\varphi_{\mathrm{sol}} = 0$ with compactly supported data. Let $\chi(t)$ be a cutoff function that is 0 for $t \leq 0$ and 1 for $t \geq 1$. Then $\varphi \coloneqq \chi\varphi_{\mathrm{sol}}$ solves

$$(\Box - m^2)\varphi = F \tag{7}$$

for a source F satisfying

- $|F| \lesssim |\varphi_{\mathrm{sol}}| + |\partial\varphi_{\mathrm{sol}}|$,
- and $\operatorname{supp} F \subset \{0 \leq t \leq 1\} \cap \{r \leq R_0\}$ (by finite speed of propagation and the compact support of the data).

Remark 2.1. We fix once and for all the value of R_0, and allow all constants to depend on it.

We will work in frequency space. Expand φ into spherical harmonics

$$\varphi(t, r, \theta, \phi) = \sum_{\mathcal{M}, L} \varphi_{\mathcal{M},L}(t, r) e^{i\mathcal{M}\phi} Y_L(\theta). \tag{8}$$

Define the Fourier transformed quantities

$$u_{\mathcal{M},L}(\omega, r) \coloneqq r\int_{-\infty}^{\infty} e^{it\omega}\varphi_{\mathcal{M},L}(t,r)\,\mathrm{d}t, \qquad H_{\mathcal{M},L}(\omega, r) \coloneqq -r\Big(1 - \frac{2M}{r}\Big)\int_{-\infty}^{\infty} e^{it\omega}F_{\mathcal{M},L}(t,r)\,\mathrm{d}t. \tag{9}$$

[3] Recall that the quasimodes discussed in remark 1.6 are supported in L-independent compact-r regions.

Then $u_{\mathcal{M},L}$ solves

$$-\partial_s^2 u_{\mathcal{M},L} + (-\omega^2 + V)u_{\mathcal{M},L} = H_{\mathcal{M},L}. \tag{10}$$

An argument from the first paper [PSM24] shows that the Fourier transform is justified.

Proposition 2.2 ([PSM24])**.** *Let $\omega \in \mathbf{R} \setminus \{m\}$. Under the support assumptions of φ and the source H, the Fourier transform (a priori defined only as a distribution) is justified.*

Moreover, u satisfies "outgoing boundary conditions." That is, we have the asymptotics

$$u = e^{-i\omega s}(1 + h(\omega, s)), \quad \text{with } |h|, |\partial_s h| \lesssim_\omega r - 2M \text{ as } s \to -\infty, \tag{11}$$

and

$$u \sim \begin{cases} r^{Mm^2(\omega^2-m^2)^{-1/2}} \exp(-(\omega^2 - m^2)^{1/2}s) & \omega < m, \\ r^{iMm^2(\omega^2-m^2)^{-1/2}} \exp(i(\omega^2 - m^2)^{1/2}s) & \omega > m \end{cases} \qquad \text{as } s \to \infty. \tag{12}$$

Remark 2.3 (Behaviour of the source H towards the horizon)*.* In particular, F is regular near the horizon, so H has a $1 - 2M/r$ weight.

Proof. The Fourier transform is justified via an exponential damping argument in [PSM24, Prop. 4.1] (see [PSM24, Sec. 4.3]).

The proof relies on the solutions u_H and u_I to the unsourced problem regular near the horizon and infinity. For u_H, one analyzes an ODE with a regular singularity, and for u_I, one analyzes an ODE with an irregular singularity. In both cases, one then solves a Volterra integral equation to get the claimed asymptotics as $s \to -\infty$ for u_H (see [PSM24, Lem. 4.3]) and as $s \to \infty$ for u_I (see [PSM24, Lem. 4.4]). Note that the estimates for h in (11) are needed to establish the redshift estimate lemma 3.6 (not just $h = o(1)$ as in the statement of [PSM24, Lem. 4.3]), and in fact the proof of [PSM24, Lem. 4.3] implies such estimates (see the estimates for the Volterra equation in [PSM24, Thm. B.1]).

To get the claimed asymptotics for u, which solves a problem with a source, one could in principle redo the argument involving the Volterra integral equation used to derive the asymptotics of u_H and u_I, this time tracking the influence of the source, but there is an easier argument using the Green's formula in [PSM24, Eqn. (4.1)], which expresses u in terms of u_H and u_I and their Wronskian. One can compute the Wronskian of u_H and $\overline{u_H}$ explicitly, and then express u_I in terms of u_H and $\overline{u_H}$ (see the proof of [PSM24, Lem. 4.5]). Then the Green's formula implies that u solving an equation with source $H = (1 - 2M/r)\widehat{F}$ (note in particular the weight $1 - 2M/r$) the asymptotics of u_H as $s \to -\infty$. A similar argument shows that u inherits the asymptotics of u_I as $s \to \infty$. ◻

We now estimate

$$\begin{aligned} |r\varphi| &\underset{\text{(I)}}{\lesssim} \sum_L L^{3/2} |r\varphi_{\mathcal{M},L}| \lesssim \sup_{\mathcal{M},L} (1 + L^4)|r\varphi_{\mathcal{M},L}| \\ &\underset{\text{(II)}}{\lesssim} \sup_{\mathcal{M},L} (1 + L^4) \Big| \int_{-\infty}^{\infty} e^{-it\omega} u_{\mathcal{M},L} \, \mathrm{d}\omega \Big| \underset{\text{(III)}}{\lesssim} \sup_{\mathcal{M},L} (1 + L^4) \Big| \int_0^\infty e^{-iv\omega} e^{i\omega s} u_{m,L} \, \mathrm{d}\omega \Big|, \end{aligned} \tag{13}$$

We used

(I): $|Y_L| \lesssim L^{1/2}$, a classical fact about spherical harmonics[4], together with $|\mathcal{M}| \lesssim L$,

(II): Fourier inversion,

(III): φ real-valued implies $u(-\omega, r) = \overline{u(\omega, r)}$, so we can restrict to positive frequencies $\omega > 0$, and $t = v - s$ for the usual Eddington–Finkelstein outgoing coordinate v.

Introduce the energy norm (for some integer N which we won't track)

$$\mathcal{E}[H] = \int_{-\infty}^{\infty} (1 + |s|^N)[|\partial_\omega H|^2 + |\partial_s H|^2 + |H|^2]\Big(1 - \frac{2M}{r}\Big)^{-1} \mathrm{d}s. \tag{14}$$

We are done if we can (1) show that:

$$\sup_{r \le R} \Big| \int_0^\infty e^{-iv\omega} e^{i\omega s} u_{\mathcal{M},L}(\omega, r) \, \mathrm{d}\omega \Big| \lesssim_R v^{-5/6+\delta} \underbrace{(1 + L^N)(1 + \omega^N)\mathcal{E}[H_{\mathcal{M},L}]}_{(*)} \qquad \text{for } L \gg 1, \tag{15}$$

[4]In general, the L-th spherical harmonic on S^n is bounded pointwise by $L^{(n-1)/2}$ (see [Gar14, Sec. 7]).

since the low spherical harmonics are handled by [PSM24], and (2) show $\sup_{m,L,\omega}(*)$ is bounded by initial data in physical space.

Remark 2.4. In particular, we only ever work with functions supported on a single spherical harmonic, so the analysis of the first paper applies.

2.2. **Control of the energy norm by initial data.** Pulling out the $1-2M/r$ factor in H and using the finite-r support of H, we obtain

$$\mathcal{E}[H] \lesssim \int_{2M}^{R_0} (1+|s|^N)[|\widehat{tF}|^2 + |\widehat{\partial_r F}|^2 + |\widehat{F}|^2]\,\mathrm{d}r \lesssim \int_0^1 \int_{2M}^{R} (1+|s|^N)|\partial^{\le 2}\varphi_{\mathrm{sol}}|^2\,\mathrm{d}r\,\mathrm{d}t. \tag{16}$$

We passed the ∂_ω-derivative under the Fourier transform to get a factor of t. Then, since F has finite-t support, we used Cauchy–Schwarz in time (in the definition of the Fourier transform).

The s-weight is irrelevant away from the horizon since the integral is over a finite-r region. As $r \to 2M$, we have $s \sim \log(r-2M)$, so the s-weight is integrable. After using standard physical space estimates—namely energy estimates with redshift (recalling that the energy for Klein–Gordon includes a zeroth order term), angular commutation, Sobolev embedding, and elliptic estimates—to put the derivatives in L^∞ (controlled by a Sobolev norm of initial data), one is left with an integrable s-weight.

Remark 2.5. The same argument also controls

$$\sup_{\omega\in\mathbf{R}_{>0}\setminus\{m\},\mathcal{M},L} (1+L^N)(1+\omega^N)\mathcal{E}[H_{\mathcal{M},L}], \tag{17}$$

since the ω-weights become ∂_t-derivatives when passed through the Fourier transform, and the L-weights can be replaced with powers of Δ (since $H_{\mathcal{M},L}$ is supported on the L-th spherical mode).

2.3. **The three frequency regimes.** The goal is now to prove (15) using stationary/non-stationary phase arguments. We will split the integral over $\omega \in (0,\infty)$ into three regimes:

- $\omega > m$: the high frequencies,
- $m^2 - \omega^2 \ge L^{-p}$: the low frequencies bounded away from m, where $p \gg 1$ will be fixed in the course of the proof,
- and $0 < m^2 - \omega^2 \le L^{-p}$: the low frequencies near m.

Remark 2.6. From now on, we write V for the effective potential, i.e. $V(r) = -\omega^2 + V_{\mathrm{old}}(r)$, where V_{old} is the potential in (2).

2.3.1. *The high frequencies $\omega > m$.* In this regime it is in principle possible to establish smoothness of the Fourier transform, which leads to arbitrarily fast polynomial decay for the solution. The key is that in this region the potential has the form

$$V = -\Big(\omega^2 - m^2 + \frac{2Mm^2}{r}\Big) + O(L^2/r^2). \tag{18}$$

Since $\omega > m$, the main part is negative. This allows us to define a new WKB coordinate ζ by

$$\frac{\mathrm{d}\zeta}{\mathrm{d}s} = \sqrt{\omega^2 - m^2 + \frac{2Mm^2}{r}} \quad \text{for } \zeta \gg 1. \tag{19}$$

Then for $\zeta \gg 1$, the WKB solution $v = (\omega^2 - m^2 + 2Mm^2/r)^{1/4}u$ solves an equation with potential

$$W(\zeta) = -1 + c\frac{L(L+1)}{\zeta^2}(1+O(\zeta^{-1})) + O(\zeta^{-3}) \text{ for } \zeta \gg 1. \tag{20}$$

In particular, we no longer see the Klein–Gordon mass m or the time frequency ω, and the potential for large ζ looks like that of a wave equation on Schwarzschild for angular mode L and time frequency 1. This motivates the use of frequency-localized multipliers as in the work of Mihalis, Igor, and Yakov [DRS16] on the decay for massless waves on subextremal Kerr (see also [DR12]).

Remark 2.7. We will use the same notation for the multipliers, but not the archaic Greek notation used in that paper! Apparently it "proved unpopular."

2.3.2. *The low frequencies $m^2-\omega^2 \geq L^{-p}$.* In this region we get $(t^*)^{-1+\delta}$ decay. Note that, when $m^2-\omega^2 \ll L^{-2}$, the potential has three turning points:

$$r_{\mathrm{I}} \sim 1, \qquad r_{\mathrm{II}} \sim L^2, \qquad r_{\mathrm{III}} \sim (m^2-\omega^2)^{-1}. \tag{21}$$

Remark 2.8 (Coalescence of turning points). When $m^2-\omega^2 \sim L^{-2}$, the two turning points r_{II} and r_{III} coalesce, and when $m^2-\omega^2 \gg L^{-2}$ they disappear. This is why we do not construct a Green's function for the solution directly using the WKB method. Near a single turning point, we can use the Airy function approximation,[5] but when there are two turning points that can coalesce or disappear, this is essentially an open problem.

In between r_{II} and r_{III} is a classically allowed region, and to the left of r_{II} and to the right of r_{III} are classically forbidden regions.

- In particular, we expect exponential decay when $1 \ll r \ll L^2$ and when $r \gg L^p$. This is made rigorous by Agmon estimates.

Another key point is that the length of the first classically forbidden region (Agmon distance) is

$$\int_{r_{\mathrm{I}}}^{r_{\mathrm{II}}} \sqrt{V}\,\mathrm{d}r \sim \int_{2M}^{L^2} \sqrt{\frac{L(L+1)}{r^2}}\,\mathrm{d}r \sim 2L\log L. \tag{22}$$

In particular, we expect an estimate like

$$\|u\|_{L^2(\{r\sim 1\})} \lesssim e^{-2L\log L}\|u\|_{L^2(\text{allowed})} + \text{source}. \tag{23}$$

To estimate the solution in the classically allowed region, we approximate the Klein–Gordon operator in the region $r \gg 1$ by a hydrogen atom-type operator $m^2-\omega^2+Q$ for

$$Q = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{2Mm^2}{r} + \frac{L(L+1)}{r^2}(1+O(r^{-1})) + O(r^{-3}). \tag{24}$$

We can estimate the eigenvalues of Q by those of the exact hydrogen atom operator (where the big-O terms are not present), whose spectrum we know:

$$-\lambda_n \sim \frac{(Mm^2)^2}{(n+L)^2} \qquad \text{for } n \lesssim L^p, p \gg 1. \tag{25}$$

Away from these eigenvalues, say when

$$\inf_n |m^2-\omega^2+\lambda_n| \geq e^{-1/2L\log L}, \tag{26}$$

the operator $m^2-\omega^2+Q$ has an inverse bounded by $e^{1/2L\log L}$. This large loss is absorbed by the decay coming from the Agmon estimate (23), leading to an estimate $|u| \lesssim L^N$.

Near the eigenvalues, we can only prove

$$|u| \lesssim e^{O(L)}. \tag{27}$$

This is done using a Carleman-type estimate with an exponential microlocal multiplier. Since this exponential loss is only present in a small region of size $O(e^{-1/2L\log L})$, we still obtain improved integrated regularity.

2.3.3. *The low frequencies $m^2-\omega^2 < L^{-p}$.*

Remark 2.9. The arguments of the previous section can be extended to the range $m^2-\omega^2 \geq e^{-\epsilon L}$ (due to the exponential errors in the eigenvalue approximation done in lemma 4.19), but as $\omega \nearrow m$ we must do something different.

[5]Note that one cannot simply approximate by sines and cosines, since these do not interpolate between oscillating behaviour and exponential-type behaviour, as is needed to capture the transition between a classically allowed region and a classically forbidden region.

The goal is to show that, schematically,

$$\begin{aligned} u &\sim e^{Lf(r)}\epsilon_L^2 \frac{\cos(\pi k) + e^{O(L)}\sin(\pi k)}{\epsilon_L^2\cos(\pi k) + \sin(\pi k)}. \\ \epsilon_L &\sim e^{-2L\log L + O(L)} \\ k &\sim (m^2-\omega^2)^{-1/2} + O(1), \end{aligned} \tag{28}$$

where $f(r) = O(1)$ depends on data and is bounded away from 0 in any finite-r region.

Remark 2.10 (Exponential amplification near integers). When k is an integer (which happens infinitely often as $\omega \nearrow m$) the $\sin(\pi k)$ terms vanish, and we get an exponential amplification $|u| \gtrsim e^{CL}$.

Remark 2.11 (Lack of amplification away from integers). Away from $k \in \mathbf{N}$, there is no amplification, and in fact $|u| \lesssim e^{-4L\log L + O(L)}$. In particular, the exponentially bad behaviour only takes place on a very small k-interval (and hence ω-interval) of size $e^{-4L\log L+O(L)}$, so the stationary phase argument can still close.

The denominator can also be written as $1+\Gamma_L e^{2\pi i k}$ for $\Gamma_L \approx 1-\epsilon_L^2$. After expanding this into a geometric series, one sees a term like

$$\int_{m-L^{-p}}^{m} e^{-it\omega}\sum_{q=1}^{\infty}(-\Gamma_L)^q e^{2\pi i k q}\,\mathrm{d}\omega, \tag{29}$$

and the claim is that this is the slowest decaying term (in particular the terms coming from the numerator of the formula for u in (28) do not affect this analysis). Analyzing one of these terms gives

$$\int_{m-L^{-p}}^{m} \exp\Big(-it\Big(\omega - \frac{2q}{t}\pi k\Big)\Big)\,\mathrm{d}\omega. \tag{30}$$

We are now in the realm of stationary phase analysis.

Lemma 2.12. *For $A > 0$, the phase*

$$\Phi(t,A,\omega) = \omega - \frac{A}{t}\pi(m^2-\omega^2)^{-1/2} \tag{31}$$

has a unique critical point, at

$$\omega_c(t,A) = m - C_{M,m}(t/A)^{-2/3} + O_\delta((t/A)^{-4/3+\delta}), \tag{32}$$

and

$$\partial_\omega^2\Phi(t,A,\omega_c(t,A)) = -C'_{M,m}(t/A)^{2/3} + O_\delta((t/A)^\delta). \tag{33}$$

Corollary 2.13. *We get $t^{-5/6}$ decay for u.*

Remark 2.14. The point is that $t^{5/6} = t^{1/2}\cdot(t^{2/3})^{1/2}$.

Proof. By the stationary phase lemma, we have

$$\int_{m-L^{-p}}^{m} e^{-it\Phi(t,2q,\omega)}\,\mathrm{d}\omega = ce^{-it\Phi(t,2q,\omega_c(t,2q))}\sqrt{\frac{1}{t|\partial_\omega^2\Phi(t,2q,\omega_c(t,2q))|}} + \text{error} \sim t^{-5/6}. \tag{34}$$

□

Remark 2.15. In reality, k is not exactly $(m^2-\omega^2)^{-1/2}$, and, more seriously, one must track all the Γ_L-terms and $e^{-it\Phi(t,\omega_c)}$ coefficients, which produces a huge exponential sum. This is how the exponent pair conjecture enters.

The argument to prove the schematic formula (28) involves a WKB-type analysis of each turning point.

- The small constant ϵ_L comes from the transition between a compact region ($s \sim 1$) and the second turning point, which is a classically allowed region. Recall that the WKB length of this region is $2L\log L + O(L)$.
- The oscillating terms come from the transition between the second and third turning points (which are like $s = L^2$ and $s = k^{-2}$ respectively), which takes place in a classically allowed region.

- The $e^{Lf(r)}$ term comes from the fact that the fundamental solutions of the Fourier transformed equation behave like the Minkowskian solutions $f''(r) = L(L+1)/r^2 \cdot f(r)$, which are $r^{L+1} = e^{(L+1)\log r}$ and $r^{-L} = e^{-L\log r}$.

The goal is to study the so called Jost solutions regular at the horizon u_H and regular at infinity u_I. As justified in the first paper, their asymptotics are

$$u_H \sim e^{-i\omega s} \qquad \text{as } s \to -\infty, \tag{35}$$

and

$$u_I \sim Ck^{-2/3}V^{-1/4}(s)\exp\Big(\int_{s_{\mathrm{III}}}^{s} \sqrt{V(s)}\,\mathrm{d}s\Big) \qquad \text{as } s \to \infty. \tag{36}$$

Also in the first paper the following Green's formula was derived:

$$u(\omega, s) = \frac{1}{W(u_I, u_H)(\omega)}\Big[u_H(\omega, s)\int_s^\infty u_I(s')H(\omega, s')\,\mathrm{d}s' + u_I(\omega, s)\int_{-\infty}^{s} u_H(\sigma)H(\omega, s')\,\mathrm{d}s'\Big], \tag{37}$$

where W is the Wronskian

$$W(f, g) = \partial_s f \cdot g - f\partial_s g. \tag{38}$$

Remark 2.16 (Turning points in the large-L problem as opposed to in the fixed-L problem). The above strategy was also used in the first paper [PSM24]. Since L was fixed in that setting, there was only one large-r turning point, namely r_{III}. The compact-r turning point r_{I} stayed fixed, whereas for us it tends to $r = 2M$ as $L \to \infty$, and r_{II} was a compact-r turning point, whereas for us $r_{\mathrm{II}} \sim L^2$. In that paper, one uses a soft argument to control the solution near the first two compact-r turning points, and so one must only study the final turning point. In our setting we must contend with all three turning points.

The strategy is to do a sequence of approximations starting at a compact region and eventually getting to infinity, so that we can write u_H and u_I in terms of the same functions and use the Green's formula. In particular, this will allow us to compute the Wronskian and derive an estimate.

(i) (*u_H and the first turning point*) Analyze the first turning point $s_{\mathrm{I}} \sim -\log(L^{-1})$ and control u_H up to $e^{O(L)}$ errors. This is done following Donninger–Schlag–Soffer and Costin–Donninger–Schlag–Tanveer, exploiting the exponentially decaying character of the potential towards the horizon and finding the appropriate modified Bessel functions[6] to resolve the turning point. A change of variables is used that reduces to a WKB approximation away from the turning point.

(ii) (*Between the first and second turning points*) Between the first and second turning point, they use the WKB method to study the solutions $w_{1,\pm} \sim e^{\pm O(L)}$ and express u_H as a linear combination of $w_{1,\pm}$. In particular, $u_H \approx w_{1,+}$.

(iii) (*Airy near the second turning point*) For the second turning point, one uses the Airy function approximation in a region $s_{\mathrm{I}} \leq s \lesssim s_{\mathrm{II}}$, which interpolates between oscillating behaviour and exponential behaviour. Note that the range is L-dependent (since s_{II} is), but k-independent.

(iv) (*Connecting the first and second turning points*) Then we connect the first and second turning point by expressing the Airy functions in terms of the $w_{1,\pm}$, with coefficients that have exponential behaviour in $2L\log L + O(L)$. In particular, these coefficients depend only on L and ω, but not on s. Moreover, we have decay in L, but not in k.

(v) (*Bessel-like functions above the second turning point*) The Airy function approximation breaks down near the third turning point, in the sense that the ∂_ω derivatives blow up at a rate k^2. As we will see in the proof, the stationary phase argument can only handle $O(k^{-1/2})$ errors. In the previous work, one sets $\omega = m$ and neglects the terms depending on L to solve explicitly and obtain Bessel functions. Here one cannot ignore the dependence on L, but one can still do a WKB approximation to get something like a Bessel function, and then perturb to get exact solutions. This is valid in a region $s \lesssim k^{4/3}$ (which is still well below the third turning point). These solutions can be connected to the Airy functions near the second turning point, up to k-independent errors.

[6] Recall that the Bessel functions solve $x^2f''(x) + xf'(x) + (x^2 - \alpha^2)f(x) = 0$ and the modified ones solve the same equation with a $-$ sign in front of the zeroth order term.

(vi) (*WKB below the third turning point*) Another WKB approximation produces solutions $w_{2,\pm}$ in $\{s_{\mathrm{II}} \ll s \ll s_{\mathrm{III}}\}$. These can be connected to the "Bessel functions" of the previous step in a region $s \sim k$. The key point is that, because the behaviour is oscillatory, one can fix the errors to be zero at a certain point $s \sim k^2$. Moreover, the errors are $O(k^{-1})$ in $\{s \sim k^2\}$.

(vii) (u_I *above the third turning point*) Above the third turning point, one uses the function u_I. Since $\{s > s_{\mathrm{III}}\}$ is classically forbidden and $\{s < s_{\mathrm{III}}\}$ is classically allowed, it makes sense to use the Airy function approximation. This gives control in the region $\{\epsilon s_{\mathrm{III}} < s\}$.

(viii) (*Connecting* u_I *to* $w_{2,\pm}$) Finally, we can connect u_I to the WKB solutions $w_{2,\pm}$ in a region $\{\epsilon k^2 < s < s_{\mathrm{III}}\}$, with $O(k^{-1})$ errors.

Now we are done with all the approximations. We now use an energy identity which captures a *positive energy flux at the horizon*. Note that, for any solution of $-\partial_s^2 u + Vu = 0$, we have

$$\partial_s(\overline{u}\partial_s u) = |\partial_s u|^2 + V|u|^2 \implies \mathrm{Im}(\overline{u}\partial_s u) = \mathrm{const} \quad \text{if } V \text{ is real-valued.} \tag{39}$$

In particular,

$$\mathrm{Im}(\overline{u_H}\partial_s u_H) = -\omega, \tag{40}$$

since this is true at the horizon by the outgoing boundary conditions. Recall that u_H can be expressed in terms of the $w_{2,\pm}$, which are complex valued. By taking real and imaginary part, we can get solutions that are real-valued,

$$\text{"}w_{2,A} = \mathrm{Re}(w_{2,+})\text{"} \qquad \text{"}w_{2,B} = \mathrm{Im}(w_{2,-})\text{"} \tag{41}$$

Thus we can express u_H in terms of these (with complex coefficients)

$$u_H = \gamma_A(\omega)w_{2,A} + \gamma_B(\omega)w_{2,B} \tag{42}$$

Then we obtain the energy identity

$$-4\omega = |\gamma_A + i\gamma_B|^2 - |\gamma_A - i\gamma_B|^2. \tag{43}$$

In particular,

$$|\gamma_A + i\gamma_B|^2 < |\gamma_A - i\gamma_B|^2. \tag{44}$$

Thus

$$\Gamma(\omega) := \frac{\gamma_A + i\gamma_B}{\gamma_A - i\gamma_B} \text{ satisfies } |\Gamma| < 1. \tag{45}$$

The role of Γ is that by writing u_I and u_H in terms of $w_{2,\pm}$, we can express

$$W(u_I, u_H) = \text{prefactor} \cdot (1 + \Gamma(\omega) \cdot e^{-2\pi i\tilde{k}}) \tag{46}$$

for $\tilde{k}$ a slight modification of $k = (m^2 - \omega^2)^{-1/2}$. In particular, $W(u_I, u_H)$ is non-vanishing. In the first paper, this qualitative estimate was enough, but here we also need to quantify

$$|\Gamma(\omega)| = 1 - e^{-4L\log L + O(L)}. \tag{47}$$

In particular, we obtain the *lower bound* $\epsilon_L \gtrsim e^{-2L\log L + O(L)}$, which is crucial for the analysis of the schematic formula (28).

Finally, we can combine all the approximations to write u_I in terms of $w_{1,\pm}$:

$$u_I(\omega, s) = \alpha_+(\omega)w_{1,+}(\omega, s) + \alpha_-(w)w_{1,-}(\omega, s). \tag{48}$$

Substituting this into the Green's formula and using that $u_H \approx w_{1,+}$ is regular up to $e^{O(L)}$ errors, we get that the main terms in u are (up to acceptable $O(k^{-1/2})$ errors)

$$\frac{\alpha_+}{W(u_H, u_I)} \approx \epsilon_L^2 \frac{O(1)}{1 + \Gamma e^{2\pi ik}} \tag{49}$$

and

$$\frac{\alpha_-}{W(u_H, u_I)} \approx \frac{O(1)[e^{2\pi ik} - 1] + \epsilon_L^2 O(1)}{1 + \Gamma e^{2\pi ik}}. \tag{50}$$

Rewriting these in terms of sine and cosine gives the schematic formula (28).

3. The high frequencies $\omega > m$

Proposition 3.1. *Fix $R \gg 1$. For $\omega > m$, we have*

$$\sup_{r\leq R}|u|^2 \lesssim_R L^N\omega^N\mathcal{E}[H] \tag{51}$$

and

$$\sup_{r\leq R}|\partial_\omega(e^{i\omega s}u)|^2 \lesssim_R L^N(\omega^N + (\omega^2-m^2)^{-1})\mathcal{E}[H]. \tag{52}$$

3.1. Non-stationary phase analysis.

We use the following basic estimate, proven by integrating by parts:

$$\Big|\int_I e^{-iv\omega}f(\omega)\,\mathrm{d}\omega\Big| \lesssim v^{-1}[\|\partial_\omega f\|_{L^1(I)} + \|f\|_{L^\infty(\partial I)}]. \tag{53}$$

Lemma 3.2. *We have*

$$\sup_{r\leq R}\Big|\int_m^\infty e^{-iv\omega}e^{i\omega s}u\,\mathrm{d}\omega\Big| \lesssim v^{-1}\sup_{\omega>m,L}(L^N\omega^{N+2}\mathcal{E}[H])^{1/2}. \tag{54}$$

Proof. Immediate consequence of (53) and proposition 3.1, noting that the singular weight in (52) is integrable near $\omega = m$ (since it's like $(\omega^2-m^2)^{-1/2}$ after taking the square root). □

3.2. The WKB coordinate ζ.

Recall that the potential takes the form

$$V = -\Big(\omega^2 - m^2 + \frac{2Mm^2}{r}\Big) + \Big(1-\frac{2M}{r}\Big)\Big(\frac{L(L+1)}{r^2} + \frac{2M}{r^3}\Big). \tag{55}$$

For $S_\zeta \gg 1$, define $P(\omega, s)$ by

- $P = 1$ in $\{s \leq S_\zeta\}$,
- $P = \omega^2 - m^2 + 2Mm^2/r$ in $\{s \geq 2S_\zeta\}$,
- technical interpolation conditions in between.

Remark 3.3. We have the lower bound $P \gtrsim r^{-1}$ (due to the $2Mm^2/r$ term). In particular, P is bounded below in compact-r regions.

Now define a new coordinate (*note how $\omega > m$ is crucial*)

$$\frac{\mathrm{d}\zeta}{\mathrm{d}s} = \sqrt{P} \qquad \zeta|_{s=0} = 0 \implies \zeta \sim r\sqrt{P} + S_\zeta O(\omega). \tag{56}$$

The point of this new coordinate is that $v := P^{1/4}u$ has the nice asymptotics (*here h is as before, since $\zeta = s$ near the horizon*)

$$v = e^{-i\omega\zeta}(1+h) \text{ as } \zeta\to-\infty, \qquad v \sim e^{i\zeta} \text{ as } \zeta\to\infty. \tag{57}$$

and solves

$$-\partial_\zeta^2 v + Wv = P^{-3/4}H =: \tilde{H}, \tag{58}$$

for a new potential

$$W = P^{-1}V + \text{error} = -1 + c\frac{L(L+1)}{\zeta^2}(1+O(\zeta^{-1})) + O(\zeta^{-3}) \qquad \text{when } \zeta \gg 1. \tag{59}$$

3.3. Pointwise estimate from a Morawetz estimate.

Our goal is to prove the following Morawetz estimate:

$$\int_{-\infty}^\infty\Big[\Big(1-\frac{2M}{r}\Big)^{-1}|\partial_\zeta(e^{i\omega\zeta}v)|^2 + (1+|\zeta|)^{-1-\delta}|v|^2\Big]\,\mathrm{d}\zeta \lesssim \omega^4\int_{-\infty}^\infty\Big(1-\frac{2M}{r}\Big)^{-1}|H|^2\,\mathrm{d}\zeta. \tag{60}$$

Remark 3.4. Remember the form of the left side: it will appear frequently on the left side of estimates.

From (60) one can prove (51) using the following consequence of the fundamental theorem of calculus:

$$\sup_{\zeta\leq S_0}|f(\zeta)|^2 \lesssim_{S_0} \int_{-\infty}^{S_0}\Big(1-\frac{2M}{r}\Big)^{-1}|\partial_\zeta(e^{i\omega\zeta}f)|^2\,\mathrm{d}s + \int_0^1 f^2\,\mathrm{d}s \tag{61}$$

Proof. For $\zeta \leq S_0$, integrate from ζ to 0 to get

$$|f(\zeta)| \leq \int_{-\infty}^{S_0} |\partial_\zeta(e^{i\omega\zeta} f)|(\zeta)\,\mathrm{d}\zeta + |f(0)| \tag{62}$$

To conclude the estimate, average the second term on the right and use Cauchy–Schwarz (which produces an S_0-dependent constant, arising from integrating $1-2M/r$ up to $\zeta = S_0$). $\square$

Remark 3.5. This controls v in L^∞. To control u, use the lower bound on P in a compact-r region.

3.4. **The redshift estimate.** We first handle the terms near the horizon.

Lemma 3.6 (Redshift)**.** *Let $-s_{\text{red}} \gg 1$. Then*

$$\int_{-\infty}^{s_{\text{red}}-1} \left[\left(1-\frac{2M}{r}\right)^{-1} |\partial_s(e^{i\omega s} v)|^2 + L^2(1+|s|)^{-1-\delta}|v|^2\right] \mathrm{d}s \lesssim \int_{-\infty}^{s_{\text{red}}} \left(1-\frac{2M}{r}\right)^{-1} |H|^2\,\mathrm{d}s + \int_{s_{\text{red}}-1}^{s_{\text{red}}} L^2|v|^2\,\mathrm{d}s.$$

Remark 3.7. Recall that $s = \zeta$ in $-s \gg 1$, so it makes no difference whether we write the above estimate in terms of s or in terms of ζ.

Proof. We first note the following properties of the potential near the horizon:

- $0 < V + \omega^2 \lesssim L^2(1-2M/r)$ as $r \to 2M$,
- and $\frac{\mathrm{d}}{\mathrm{d}r}|_{r=2M}(V+\omega^2) \gtrsim L^2$.

Now define z by

$$z = \begin{cases} -L^2(V+\omega^2)^{-1}(1+(-s)^{-\delta}) & s \leq s_{\text{red}} - 1, \\ 0 & s \geq s_{\text{red}}. \end{cases} \tag{63}$$

The above properties of the potential imply that

$$|z| \lesssim (1-2M/r)^{-1} \qquad \partial_s z \gtrsim (1-2M/r)^{-1}. \tag{64}$$

Now multiply the equation by (i.e. take the complex L^2 inner product with) $z \cdot (\partial_s u + i\omega u)$ to get

$$\begin{aligned} &\int_{-\infty}^{\infty} [\partial_s z |\partial_s(e^{i\omega s} v)|^2 - \partial_s(z(V+\omega^2))|v|^2]\,\mathrm{d}s \\ &= -2\int_{-\infty}^{\infty} z\,\mathrm{Re}(H \cdot \overline{\partial_s u + i\omega u}) + \underbrace{|v(-\infty)|^2 \lim_{r\to 2M}(z(V+\omega^2))}_{\text{good sign}} + \underbrace{\lim_{r\to 2M} z|\partial_s(e^{i\omega s} v)|^2}_{\text{vanishes}}. \end{aligned} \tag{65}$$

There are no boundary terms at infinity because z is compactly supported in r. Note that the zeroth-order boundary term has a good sign, and the derivative boundary term vanishes due to the outgoing boundary conditions (specifically the decay rate of h in $e^{i\omega s}u = 1 + h$). $\square$

3.5. **The Morawetz estimate without redshift.** It remains to obtain a Morawetz estimate away from the horizon (in particular, the estimate is allowed to be degenerate towards the horizon).

Lemma 3.8. *We have*

$$\int_{-\infty}^{\infty} (1+|\zeta|)^{-1-\delta}[|\partial_s u|^2 + |u|^2]\,\mathrm{d}s \lesssim \omega^4 \int_{-\infty}^{\infty} (1+|\zeta|)^{1+\delta}|\tilde{H}|^2. \tag{66}$$

Remark 3.9. In a compact-r region, $P \gtrsim 1$, so $|\tilde{H}| \lesssim |H|$, and the right side is controlled by $\mathcal{E}[H]$.

Remark 3.10. When $\omega > 2m$, one does not even need to use the WKB technique, and can instead simply use microlocal multipliers, adapting the methods developed for the wave equation in [DRS16].[7] Indeed, the potential for the Klein–Gordon equation when $\omega > 2m$ looks like the potential for the wave equation away from zero frequency, with an additional $-2Mm^2/r$ Coulomb term. This term would break the argument near zero frequency, but away from zero frequency one can handle it perturbatively, in the sense that $1/r$ is not the dominant term in the potential: we have $2Mm^2 \ll L^2/r^2$, except when $r \gg L^2$, but in this region $2Mm^2/r \ll \omega^2 - m^2$. However, we will use the WKB coordinate for the whole frequency range $\omega > m$, for the sake of a unified treatment.

[7] Note in particular that one can handle $\omega \sim L$ (where there is unstable trapping).

Proof. Step 1: $\omega \gg L$. For this frequency range we can use a single multiplier $y\partial_\zeta u$ to get

$$\int_{-\infty}^{\infty} \big[\partial_\zeta y|\partial_\zeta v|^2 - \partial_\zeta(yW)|v|^2\big]\,\mathrm{d}\zeta = y(\infty)|v(\infty)|^2 - \omega^2|y(-\infty)|^2 + \int_{-\infty}^{\infty} 2\,\mathrm{Re}(\tilde{H}\overline{y\partial_\zeta v})\,\mathrm{d}s \tag{67}$$

If we use

$$y(\zeta) = P(\zeta)\exp\Big(A\int_{-\infty}^{\zeta}(1+|\zeta'|)^{-1-\delta}\,\mathrm{d}\zeta'\Big), \qquad A \gg 1, \tag{68}$$

then we have (*using the technical conditions on P and $\omega \gg L$, which in particular ensure $-W \gg |\partial_s W|$ and hence $-\partial_\zeta(yW) \gtrsim (-W)\partial_\zeta y$*)

$$\partial_\zeta y, -\partial_\zeta(yW) \gtrsim (1+|\zeta|)^{-1-\delta}. \tag{69}$$

The boundary term at the horizon has a good sign, and the boundary term at infinity is controlled by $\omega^2|v(\infty)|^2$ (since $P(\zeta) \to C\omega^2$ as $\zeta \to \infty$), which is controlled by the following "∂_t-estimate" (proven by multiplying by v and using the outgoing boundary conditions):

$$|v(\infty)|^2 + \omega|v(-\infty)|^2 = \int_{-\infty}^{\infty}\mathrm{Im}(\tilde{H}\overline{v})\,\mathrm{d}\zeta. \tag{70}$$

Step 2: $\omega \sim L$. I will skip this step. It is similar to $\omega \ll L$, but with some technical modifications.

Step 3: $\omega \ll L$. Use $2f\partial_\zeta v + \partial_\zeta f v$ as a multiplier to get

$$\begin{aligned}\int_{-\infty}^{\infty}[2\partial_\zeta f|\partial_\zeta v|^2 - f\partial_\zeta W|v|^2]\,\mathrm{d}s &\leq f(\infty)|v(\infty)|^2 - f(-\infty)\omega^2|v(-\infty)|^2 \\ &\quad + \int_{-\infty}^{\infty}|\partial_\zeta^3 f||v|^2\,\mathrm{d}s + \int_{-\infty}^{\infty}|H|[|f||\partial_\zeta v| + |\partial_\zeta f||v|].\end{aligned} \tag{71}$$

For $L \gg 1$, note that V has a unique (non-degenerate) critical point in a compact-r region, at $r = r_{\mathrm{crit}} = 3M + o_{L\to\infty}(1)$ (the other is at $r \sim L^2$). Let ζ_0 be sufficiently negative and let ζ_1 be sufficiently positive. Define f by

- $f = -1$ in $\zeta \leq \zeta_0$,
- $f = 1$ in $\zeta \geq \zeta_1$,
- $\partial_\zeta f \gtrsim 1$ in $[\zeta_0+1, \zeta_1-1]$,
- f vanishes at $r = r_{\mathrm{crit}}$.

Since $f \sim (r - r_{\mathrm{crit}})$ and $-\partial_\zeta W \sim L^2(r-r_{\mathrm{crit}})$ near r_{crit} and $-\partial_\zeta W \gtrsim_{\zeta_1} L^2$ in $\{r > 2r_{\mathrm{crit}}\} \cap \{\zeta \leq \zeta_1\}$,[8]

$$\int_{\zeta_0+1}^{\zeta_1-1}|\partial_\zeta v|^2\,\mathrm{d}\zeta + \int_{\zeta_0}^{\zeta_1} L^2(r-r_{\mathrm{crit}})^2|v|^2\,\mathrm{d}\zeta \lesssim_{\zeta_0,\zeta_1} \int_{\zeta_0}^{\zeta_1}|v|^2\,\mathrm{d}s + (1+\omega^2)\int_{-\infty}^{\infty}|H|[|\partial_\zeta v| + |v|]\,\mathrm{d}s. \tag{72}$$

We controlled the boundary terms with the following "∂_t-estimate" (proven by multiplying by v and using the outgoing boundary conditions):

$$|v(\infty)|^2 + \omega|v(-\infty)|^2 = \int_{-\infty}^{\infty}\mathrm{Im}(\tilde{H}\overline{v})\,\mathrm{d}\zeta. \tag{73}$$

At this point, we have good control of the derivative inside the interval, as well as of the zeroth order term in the interval away from trapping. To get control outside of the interval, we use $y\partial_\zeta v$ as a multiplier to get

$$\int_{-\infty}^{\infty} \big[\partial_\zeta y|\partial_\zeta v|^2 - \partial_\zeta(yW)|v|^2\big]\,\mathrm{d}\zeta = y(\infty)|v(\infty)|^2 - \omega^2|y(-\infty)|^2 + \int_{-\infty}^{\infty} 2\,\mathrm{Re}(\tilde{H}\overline{y\partial_\zeta v})\,\mathrm{d}s. \tag{74}$$

Define y by

- $y = -1 + \epsilon\zeta^{-\delta}$ for $\zeta \leq \zeta_0 + 2$,
- $y = 1 - \epsilon\zeta^{-\delta}$ for $\zeta \geq \zeta_1 - 2$,
- $y \equiv 0$ in a neighbourhood of $r = r_{\mathrm{crit}}$,
- $\partial_\zeta y > 0$ everywhere.

[8]It is crucial that $\omega \ll L$ for this positivity.

Then[9]

$$-\partial_\zeta(yW) \gtrsim \frac{\epsilon}{|\zeta|^{1+\delta}} \text{ outside } [\zeta_0+2,\zeta_1-2] \qquad |\partial_\zeta(yW)| = O(L^2) \text{ elsewhere.} \tag{75}$$

The first term provides the necessary control outside the interval, and inside the interval we treat the term as error; it can be absorbed because it vanishes near $r = r_{\text{crit}}$. If we add a small multiple of the y-multiplier estimate (with the smallness depending on ζ_0 and ζ_1), we can absorb the $|\partial_\zeta(yW)||v|^2$ error term in the interval (since it is supported away from r_{crit}), and arrive at

$$\begin{aligned}&\int_{-\infty}^{\infty}(1+|\zeta|)^{-1-\delta}|\partial_\zeta v|^2\,\mathrm{d}\zeta + \int_{\{|r-r_{\text{crit}}|\gtrsim 1\}}(1+|\zeta|)^{-1-\delta}|v|^2\,\mathrm{d}\zeta + \int_{\zeta_0}^{\zeta_1} L^2(r-r_{\text{crit}})^2|v|^2\,\mathrm{d}\zeta \\ &\lesssim \int_{\{|r-r_{\text{crit}}|\lesssim 1\}}|v|^2\,\mathrm{d}s + (1+\omega^2)\int_{-\infty}^{\infty}|H|[|\partial_\zeta v|+|v|]\,\mathrm{d}\zeta\end{aligned} \tag{76}$$

Note that we have controlled boundary terms with the "∂_t-estimate." Although the left side degenerates at r_{crit}, we can use the following lemma 3.11 and Cauchy–Schwarz to complete the proof. □

Lemma 3.11. *For a function h on I and a constant $A>0$, we have*

$$A\int_I |h|^2\,\mathrm{d}x \lesssim \int_I \left[|\partial_x h|^2 + A^2x^2|h|^2\right]\mathrm{d}x. \tag{77}$$

Proof. By an approximation argument, we can assume h vanishes at the endpoints. In this case

$$A\int_I |h|^2\,\mathrm{d}x = A\int_I \frac{\mathrm{d}}{\mathrm{d}x}(x)|h|^2\,\mathrm{d}x = -2A\int_I xh\partial_x h\,\mathrm{d}x \le \int_I [|\partial_x h|^2 + 4A^2x^2|h|^2]\,\mathrm{d}x. \tag{78}$$

□

Remark 3.12. The derivative estimate in (52) is proven similarly, but with an extra commutation with a suitable derivative operator. The singular weight comes from the fact that $|\partial_{\widehat{\omega}}W| \lesssim L^2(\omega^2-m^2)^{-1}$ near $\omega = m$, where $\partial_{\widehat{\omega}}$ is the coordinate derivative in the (ω,ζ) coordinate system.

4. The low frequencies away from m ($m^2-\omega^2 \ge L^{-p}$)

Proposition 4.1. *Fix $p \gg 1$, $\delta \ll 1$, and $R > 2M$. For $L \gg_p 1$, there exist critical frequencies $\{\lambda_n\}_{n=1}^{N(p)}$ (where $N(p) \sim L^{p/2}$) with the following properties. First,*

$$-\lambda_n \sim (n+L)^{-2}. \tag{79}$$

Next, if we define

$$\mathcal{I} \coloneqq \inf_n(m^2-\omega^2+\lambda_n) \tag{80}$$

and

$$\begin{aligned} I &\coloneqq \{\omega\in[0,m) : m^2-\omega^2 \ge L^{-p}\},\\ I_{\text{bad}} &\coloneqq I\cap\{\mathcal{I} < e^{-\frac{1}{2}L\log L}\},\\ I_{\text{good}} &\coloneqq I\cap\{\mathcal{I} \ge e^{-\frac{1}{2}L\log L}\},\end{aligned} \tag{81}$$

then

- (i) *for $\omega \in I_{\text{bad}}$, we have*
  - *(a)* $\sup_{r\le R}|u|^2 \lesssim e^{CL}\mathcal{E}[H]$,
  - *(b)* $\sup_{r\le R}|\partial_\omega(e^{i\omega s}u)|^2 \lesssim e^{(4+\delta)L\log L}\mathcal{E}[H]$
  - *(c)* $\sup_{r\le R}|\partial_\omega(e^{i\omega s}u)|^2 \lesssim \mathcal{I}^{-2}e^{\delta L\log L}\mathcal{E}[H]$ *when* $\mathcal{I} \ge e^{-2L\log L}$.
- (ii) *and for $\omega \in I_{\text{good}}$, we have* $\sup_{r\le R}(|\partial_\omega(e^{i\omega s}u)|^2+|u|^2) \lesssim L^4\mathcal{E}[H]$,

Remark 4.2. The estimate (1c) is an improvement over (1b).

[9] This is where we use $\omega \lesssim L$, in particular to ensure that the $-\epsilon\omega^2$ term in $(-\partial_\zeta y)W$ can be absorbed by $y(-\partial_\zeta W) \gtrsim L^2$ in the bound for $-\zeta \gg 1$.

Remark 4.3. Note an $e^{CL\log L}$ estimate is the best we can hope for in view of the possible growth of u to a size e^{CL} over an interval of size $e^{-1/2L\log L}$ (and such growth is indeed possible, by the analysis of the low frequencies near m (see (28))).

4.1. **Non-stationary phase analysis.**

Lemma 4.4. *For I as in proposition 4.1, we have*

$$\sup_{r\le R}\Big|\int_I e^{-iv\omega}e^{i\omega s}u\,\mathrm{d}\omega\Big|\lesssim v^{-1+\delta}L^{p/2}\mathcal{E}^{1/2}. \tag{82}$$

Proof. We split the integral using $I=I_{\text{good}}\cup I_{\text{bad}}$. On I_{good} we can integrate by parts:

$$\Big|\int_{I_{\text{good}}}e^{-iv\omega}e^{i\omega s}u\,\mathrm{d}\omega\Big|\lesssim v^{-1}L^2\mathcal{E}^{1/2}. \tag{83}$$

On I_{bad}, the naive estimate using the L^∞ bound for $|u|$ and the (superexponential) smallness of I_{bad} gives

$$\Big|\int_{I_{\text{bad}}}e^{-iv\omega}e^{i\omega s}u\,\mathrm{d}\omega\Big|\lesssim|I_{\text{bad}}|e^{CL}\mathcal{E}^{1/2}\lesssim e^{-1/4L\log L}\mathcal{E}^{1/2}\lesssim v^{-1}\mathcal{E}^{1/2}\qquad\text{when } v\le e^{1/4L\log L}. \tag{84}$$

For $v\ge e^{1/4L\log L}$, we have to work harder. Let J be one of then intervals in I_{bad}. By (79), there are $N(p)\sim L^{p/2}$ many intervals in I_{bad}, and each one contains a unique critical frequency λ_n (since the distance between the critical frequencies is much larger than $e^{-1/2L\log L}$).

Integrating by parts on J (noting that $\partial J\subset I_{\text{good}}$) gives

$$\Big|\int_J e^{-iv\omega}e^{i\omega s}u\,\mathrm{d}\omega\Big|\lesssim\underbrace{v^{-1}L^2\mathcal{E}^{1/2}}_{\text{boundary terms}}+v^{-1}\underbrace{\int_J|\partial_\omega(e^{i\omega s}u)|\,\mathrm{d}\omega}_{(*)} \tag{85}$$

Since there are $L^{p/2}$ many intervals J, it is enough to show that $(*)\lesssim\mathcal{E}^{1/2}v^\delta$. Use (1b) for $\{\mathcal{I}\le e^{-2L\log L}\}$ and (1c) for $\{\mathcal{I}\ge e^{-2L\log L}\}$ (with smallness constant $\epsilon>0$):

$$\begin{aligned}(*)&\le\int_{J\cap\{\mathcal{I}\le e^{-2L\log L}\}}|\partial_\omega(e^{i\omega s}u)|\,\mathrm{d}\omega+\int_{J\cap\{\mathcal{I}\ge e^{-2L\log L}\}}|\partial_\omega(e^{i\omega s}u)|\,\mathrm{d}\omega\\&\lesssim\mathcal{E}^{1/2}e^{(2+\epsilon)L\log L}\int_{J\cap\{\mathcal{I}\le e^{-2L\log L}\}}\mathrm{d}\omega+\mathcal{E}^{1/2}e^{\epsilon L\log L}\int_{J\cap\{\mathcal{I}\ge e^{-2L\log L}\}}\mathcal{I}^{-1}\,\mathrm{d}\omega\\&\lesssim\mathcal{E}^{1/2}(e^{\epsilon L\log L}+e^{\epsilon L\log L}\cdot L\log L)\\&\lesssim\mathcal{E}^{1/2}e^{2\epsilon L\log L}\\&\lesssim\mathcal{E}^{1/2}v^\delta\quad\text{when } v\ge e^{1/4L\log L}\text{ and }\epsilon\ll\delta.\end{aligned} \tag{86}$$

□

4.2. **Carleman-type exponential multipliers for all frequencies.** The estimate (1a) and (1b) in proposition 4.1 can be proved using global exponential multipliers.

Lemma 4.5. *For a constant $C=C(S_0)$, we have*

$$\int_{-\infty}^{S_0}\Big[\Big(1-\frac{2M}{r}\Big)^{-1}|\partial_s(e^{i\omega s}u)|^2+(1+|s|)^{-1-\delta}(|\partial_su|^2+|u|^2)\Big]\,\mathrm{d}s\lesssim_{S_0}e^{CL}\int_{-\infty}^{S_0}\Big(1-\frac{2M}{r}\Big)^{-1}|H|^2\,\mathrm{d}s. \tag{87}$$

Proof of lemma. We can easily handle the case $m^2-\omega^2\gg L^{-2}$ by using the redshift estimate and an estimate that uses the multiplier χu for a suitable cutoff χ (*in fact, there isn't even an exponential loss.*). The real challenge is $m^2-\omega^2\lesssim L^{-2}$. Since ω is bounded away from zero, we can use the "∂_t-estimate" to control $|u(-\infty)|^2$.

The challenge is to come up with a multiplier that is insensitive to turning point coalescence and works in all regions of s. Pick $A\gg1$ and set

$$x(s):=\sqrt{1-\frac{2M}{r}}\Big(2(1+\delta)\frac{L}{r}+A\frac{L}{r^{1+\delta}}+\frac{A}{r^{1/2}}\Big)\qquad y(s):=-\exp\Big(-\int_{-\infty}^{s}x(s')\,\mathrm{d}s'\Big). \tag{88}$$

- The weight in the front is to handle terms at the horizon.

- The L/r weight is related to the fact that even on Minkowski, $-\frac{\mathrm{d}^2}{\mathrm{d}r^2}+L(L+1)/r^2$ has solutions r^{L+1} and r^{-L}, which are roughly $|y|^{\pm 1}$.
- The middle term is useful away from the horizon but when $r\sim 1$. In this region the leading order term needs to be L/r, so this term has to decay faster. But since it is not leading order, we can put a large constant that helps us absorb certain terms.
- Finally, $Ar^{-1/2}$ is pretty much the largest possible weight that still lets you prove the estimate, because integrating it up to $O(L^2)$ gives you $O(L)$, which is less than $L\log L$ (which we need for later applications).

After using $y\partial_s u$ as a multiplier and doing a careful analysis of $\tilde{V}:=V+\chi\omega^2$ and $\partial_s(\tilde{V}/x^2)$, then integrating by parts in a clever way and using a Hardy inequality, one gets the desired estimate away from the horizon, with $(-y)$ weights on the left and an error term on the right supported in $r\gg L^2$:

$$\int_{-\infty}^{\infty}(-y)x[|\partial_s u|^2+|u|^2]\,\mathrm{d}s\lesssim\int_{s(cAL^2)}^{\infty}(-y)\frac{\tilde{V}^2}{x}|u|^2\,\mathrm{d}s+\int_{-\infty}^{\infty}|H|[(-y)|\partial_s u|+|u|]. \tag{89}$$

One can handle the error term using a couple more multipliers. The point is that

$$(-y)\geq e^{-C_R L}\text{ in }\{r\geq R\}. \tag{90}$$

This is where the e^{CL} term comes from on the right. The left side is degenerate towards the horizon, but we can use redshift. □

Remark 4.6. The derivative estimate is similar, but we cannot use the compact support of H in the last step. Instead, we contend with a bad $(-y)$-weight up to CL^2, and $(-y)\sim e^{-(4+\delta)L\log L+O(L)}$ there.

4.3. The reference operator at infinity.

Definition 4.7 (Reference operator at infinity)**.** Define an operator $Q:H^2_0((x_0,\infty))\to L^2((x_0,\infty))$ by

$$Q:=-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\tilde{V}, \tag{91}$$

where

$$\tilde{V}(x):=\Big(-\frac{2Mm^2}{x}+\frac{L(L+1)}{x^2}\big(1+O(x^{-1}\log x)\big)+O(x^{-2}\log x)\Big). \tag{92}$$

Remark 4.8. Observe that $-\partial_s^2+V(s)$ is of the form $Q+(m^2-\omega^2)$ when $x_0=S_0\gg 1$ (in a region where $s=r+O(\log r)$).

Remark 4.9. Observe that $\tilde{V}$ is bounded below uniformly in L and in x_0, for $x_0\gg 1$ (depending on the constants in the big-O terms). Moreover, this bound from below holds up to $x=0$ when the big-O terms are removed (i.e. for the hydrogen atom operator).

Lemma 4.10 (Properties of Q)**.** *The operator Q on $L^2((x_0,\infty))$ with domain $H^2((x_0,\infty))$ is*

- *self-adjoint,*
- *with essential spectrum $\sigma_{\mathrm{ess}}(Q)=[0,\infty)$,*
- *and has infinitely many eigenvalues $\lambda_n<0$ with $\lambda_n\to 0$ as $n\to\infty$.*

Definition 4.11 (Hydrogen atom operator)**.** Define $Q_q:H^2_0((0,\infty))\to L^2((0,\infty))$ by

$$Q_q:=-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\Big(-\frac{2Mm^2}{x}+\frac{q(q+1)}{x^2}\Big)\qquad(\beta>0\text{ and }q\gg 1). \tag{93}$$

Remark 4.12. Clearly anything we prove about Q will hold for Q_q if $q\sim L$.

Remark 4.13. Note that Q_q is defined on $L^2((0,\infty))$, while Q is only defined on $L^2((x_0,\infty))$.

The point is that we know the eigenvalues exactly.

Lemma 4.14 (Spectrum of the hydrogen atom)**.** *The operator Q_q has the properties of Q in lemma 4.10, and its discrete spectrum consists of the simple eigenvalues*

$$\Big\{-\frac{(Mm^2)^2}{(n+q)^2}\Big\}_{n=1}^{\infty}. \tag{94}$$

Proof. See [FY09, Sec. 32] or [Tak08, Sec. 3.5.1]. □

4.4. **Exponential decay in the classically forbidden region via an Agmon estimate.**

Lemma 4.15 (Agmon estimate)**.** *Suppose $v : [s_0, s_1] \to \mathbf{R}$ vanishes at the endpoints. Suppose v solves*

$$-\frac{\mathrm{d}^2}{\mathrm{d}s^2} v + Wv = P \tag{95}$$

for a real-valued W. Then for any Lipschitz function φ, we have

$$\int_{s_0}^{s_1} \Big[\big((e^{\varphi} v)'\big)^2 + (W - (\varphi')^2)(e^{\varphi} v)^2 \Big] \, \mathrm{d}s = \int_{s_0}^{s_1} e^{2\varphi} P v \, \mathrm{d}s, \tag{96}$$

where $'$ denotes differentiation by s.

Proof. Integration by parts. □

Remark 4.16. We will not use the first term on the left. This estimate is useful when the potential W is large and positive (a classically forbidden region), so that the left side is coercive even when φ is large. If P is small (or vanishes) where φ is large, then we get exponential smallness for v.

Now we show that eigenfunctions of Q (or of the hydrogen atom operator) must be exponentially small in L in the region $\{x \ll L^2\}$. The key point is that $\{x \ll L^2\}$ is well within the classically forbidden region of the potential $\tilde{V}$.

Lemma 4.17 (Exponential decay via an Agmon estinmate)**.** *There is $c > 0$ such that if ψ is an eigenfunction of Q, then*

$$\|\psi\|_{H^1(\{x \le cL^2\})} \le e^{-cL} \|\psi\|_{L^2}. \tag{97}$$

Proof. Suppose $Q\psi = \lambda\psi$ and $\|\psi\|_{L^2} = 1$. The equation ψ satisfies is

$$-\psi'' + (-\lambda + \tilde{V})\psi = 0. \tag{98}$$

Step 1: Elliptic estimate. For $X_0 \ge x_0$, we have

$$\|\psi\|_{H^1(\{x \le X_0\})} \lesssim \|\psi\|_{L^2(\{x \le X_0 + 1\})}. \tag{99}$$

To see this, multiply the equation by χ^2 for a cutoff χ adapted to $\{x \le X_0\}$, integrate by parts, and use $\lambda < 0$ and the bound from below for $\tilde{V}$ to control the zeroth order term. This gives

$$\int (\chi\psi')^2 = \int (\lambda - \tilde{V})(\chi\psi)^2 + \int 2|\chi\psi||\chi'\psi'|. \tag{100}$$

(*Pay attention to this elliptic estimate, since it will come up again later.*)

In particular:

(i) We only need to show decay for the L^2 norm,
(ii) and taking $X_0 \to \infty$ gives $\|\psi\|_{H^1} \lesssim 1$.

We can also commute the equation with a cutoff χ adapted to $\{x \le \delta L^2\}$. That is, $\tilde{\psi} := \chi\psi$ solves

$$-\tilde{\psi}'' + (-\lambda + \tilde{V})\tilde{\psi} = P \tag{101}$$

for $\|P\|_{L^2} \lesssim \|\psi\|_{H^1} \lesssim 1$ (by (2)) and $\operatorname{supp} P \subset \{x \ge \delta L^2\}$.

Step 2: Agmon estimate. The key observation is that for $x \le \delta L^2$, the potential is large:

$$x \le \delta L^2 \implies \tilde{V} \ge \frac{1}{2}\frac{L^2}{x^2}. \tag{102}$$

(*In fact, by choosing δ small enough, we can replace $1/2$ with $1 - \epsilon$. We will use this later.*) Now construct the weight function

$$\varphi(x) = \begin{cases} \frac{1}{2}\int_x^{\delta L^2} \sqrt{-\lambda + \tilde{V}} \, \mathrm{d}x & x \le \delta L^2, \\ 0 & x \ge \delta L^2. \end{cases} \tag{103}$$

The point is that

$$\{x \le \delta/2 \cdot L^2\} \implies \varphi(x) \gtrsim L \text{ and } (-\lambda + \tilde{V}) - (\varphi')^2 \gtrsim 1. \tag{104}$$

The Agmon estimate (and $\langle P, \psi \rangle \lesssim \|P\|_{L^2} \|\psi\|_{H^1} \lesssim 1$) implies

$$\int_{\{x \leq \delta/2 \cdot L^2\}} e^{2\varphi} \tilde{\psi}^2 \, \mathrm{d}x \leq \int_{\cancel{x_0} \delta L^2}^{\infty} \cancel{e^{2\varphi}} |P| |\psi| \, \mathrm{d}x \lesssim \|P\|_{L^2} \|\psi\|_{L^2} \lesssim 1 \tag{105}$$

(*Note the crucial use of the fact that P is supported away from the region where φ is large.*) Use that $\varphi \gtrsim L$ in the region of integration on the left. ☐

We can obtain a similar estimate in the region $\{x \gg L^p\}$, which is well within the forbidden region. This time, the estimate depends on the smallness of the eigenvalue.

Lemma 4.18. *Suppose $-\lambda \geq aL^{-p}$. Then for $Q\psi = \lambda\psi$ there is $c(a) > 0$ such that*

$$\|\psi\|_{H^1(\{x \geq c(a)^{-1} L^p\})} \leq e^{-c(a)L} \|\psi\|_{L^2}. \tag{106}$$

Proof. The proof is similar to that of lemma 4.17. The key observation this time is that

$$x \gg L^p \implies -\lambda + \tilde{V} \geq -\lambda - \frac{2Mm^2}{x} \geq \frac{1}{2}(-\lambda) \gtrsim aL^{-p}. \tag{107}$$

☐

4.5. **Comparison with the hydrogen atom.** We now approximate the eigenvalues of Q by those of Q_q.

Lemma 4.19. *Let $\{\lambda_n\}_{n=1}^{N(p)}$ be the eigenvalues of Q satisfying $-\lambda_n \geq L^{-p}$. There are $A \gg 1$ and $c \ll 1$ such that*

$$\frac{(Mm^2)^2}{(n + \underbrace{L + A \log L/L}_{L^+})^2} - e^{-cL} \leq -\lambda_n \leq \frac{(Mm^2)^2}{(n + \underbrace{L - A \log L/L}_{L^-})^2} + e^{-cL}. \tag{108}$$

In particular, $N(p) \sim L^{p/2}$ and $-\lambda_n \sim (Mm^2)^2/(n+L)^2$.

Proof. We first recall some functional analysis. The min-max principle ([RS78, Thm. XIII.1]) says that if T is a self-adjoint operator bounded from below with, say, infinitely many eigenvalues λ_n,[10] then

$$\lambda_n(T) = \sup_{\varphi_1, \ldots, \varphi_{n-1} \in \mathcal{H}} \inf_{\substack{\psi \in D(T), \|\psi\|=1 \\ \psi \in \{\varphi_i\}^\perp}} \langle \psi, T\psi \rangle. \tag{109}$$

As an easy consequence, we obtain the Rayleigh–Ritz technique ([RS78, Thm. XIII.3]): if $V \subset D(T)$ is a finite-dimensional subspace of the domain of T, then for $n \leq \dim V$ we have

$$\lambda_n(T) \leq \lambda_n(P_V T|_V) \underset{\text{by min-max}}{=} \sup_{\varphi_1, \ldots, \varphi_{n-1} \in V} \inf_{\substack{\psi \in V, \|\psi\|=1 \\ \psi \in \{\varphi_i\}^\perp}} \langle \psi, T\psi \rangle, \tag{110}$$

where P_V is the orthogonal projection onto V. That is, we can estimate the n-th eigenvalue of T by what is essentially its n-th eigenvalue on a finite-dimensional subspace.

We will apply these facts with T either Q or a hydrogen atom operator.[11] We let V be the span of the first $N = BL^{p/2}$ eigenfunctions of the hydrogen atom operator Q_{L^+} whose eigenvalues are larger than $B^{-1}L^{-p}$ in magnitude (where $B \gg 1$), cutoff to $\{x \geq x_0\}$ (so that they lie in the domain of Q). In symbols,

$$V = \{\chi\psi : Q_{L^+}\psi = E\psi \text{ for some } -E \geq \max(B^{-1}L^{-p}, -\lambda_N(Q_{L^+})) \text{ for } N = BL^{p/2}\}.$$

The point is that we want V to contain the eigenfunctions of Q_{L^+} with sufficiently negative eigenvalues, with the a priori constraint that the dimension of V not be too large,[12] and with the technical caveat that we must cut the eigenfunctions off so that they are in the domain of Q.

Define $I := [cL^2, c^{-1}L^p]$ for $c = c(B)$ as in the Agmon lemmas. The key observations are that we have exponential smallness outside I and $Q_{L^-} \leq Q \leq Q_{L^+}$ on I. More precisely:

---

[10] The theorem also works for operators with finitely many eigenvalues, but in this case λ_n will be the bottom of the essential spectrum of T for n sufficiently large.

[11] Recall that both of these operators have potentials that are bounded below, and are hence bounded below, so the min-max principle and the Rayleigh–Ritz technique apply.

[12] The argument shows that, a fortiori, this constraint is not necessary; that is, it is purely for technical purposes, because the estimates will depend on $\dim V$. This would cause a problem if Q_{L^+} somehow had exponentially many eigenvalues larger than L^{-p} in magnitude.

- outside I we have exponential smallness for eigenfunctions of Q by the Agmon lemmas, and hence for $\psi \in V$ satisfying $\|\psi\|_{L^2} = 1$, we have $\|\psi\|_{H^1((x_0,\infty)\setminus I)} \lesssim Ne^{-c(B)L}$ (where we have used the facts that the cut off eigenfunctions are orthogonal up to exponentially small error and that multiplying by a cutoff is bounded on H^1),
- it follows that $\langle Q\psi, \psi\rangle_{L^2((x_0,\infty)\setminus I)} \lesssim L^2 \|\psi\|^2_{H^1(I^c)} \le e^{-c(B)L}$ (for a different c) for ψ as before and $L \gg_c 1$ (integrate by parts and use the bound $\|\tilde{V}\|_{L^\infty((x_0,\infty))} \lesssim L^2$).
- and on I, we have

$$\int_I Q_{L^-}\psi \cdot \overline{\psi}\,\mathrm{d}x \le \int_I Q\psi \cdot \overline{\psi}\,\mathrm{d}x \le \int_I Q_{L^+}\psi \cdot \overline{\psi}\,\mathrm{d}x \tag{111}$$

for $A \gg_{c(B)} 1$, because of the corresponding inequality for the potentials on I.

Writing W for the vector space V without the cutoffs, and allowing c to change from line to line, we have

$$\begin{aligned}
&\lambda_n(Q) \\
&\le \sup_{\varphi_1,\dots,\varphi_{n-1}\in V} \Bigg(\inf_{\substack{\psi\in V, \|\psi\|_{L^2}=1 \\ \psi\in\{\varphi_i\}^\perp}} \int_{x_0}^\infty Q\psi\cdot\overline{\psi}\,\mathrm{d}x \Bigg) && \text{by Rayleigh–Ritz} \\
&\le e^{-cL} + \sup_{\varphi_1,\dots,\varphi_{n-1}\in V} \Bigg(\inf_{\substack{\psi\in V, \|\psi\|_{L^2}=1 \\ \psi\in\{\varphi_i\}^\perp}} \int_I Q_{L^+}\psi\cdot\overline{\psi}\,\mathrm{d}x \Bigg) && \text{by the second observation and (111)} \\
&\le e^{-cL} + \sup_{\varphi_1,\dots,\varphi_{n-1}\in V} \Bigg(\inf_{\substack{\psi\in V, \|\psi\|_{L^2}=1 \\ \psi\in\{\varphi_i\}^\perp}} \int_0^\infty Q_{L^+}\psi\cdot\overline{\psi}\,\mathrm{d}x \Bigg) && \text{by smallness outside } I \text{ for eigenfunctions of } Q_{L^+} \\
&\le e^{-cL} + \sup_{\varphi_1,\dots,\varphi_{n-1}\in W} \Bigg(\inf_{\substack{\psi\in W, \|\psi\|_{L^2}=1 \\ \psi\in\{\varphi_i\}^\perp}} \int_0^\infty Q_{L^+}\psi\cdot\overline{\psi}\,\mathrm{d}x \Bigg) && \text{by a finite-dimensional linear algebra argument[13]} \\
&= e^{-cL} + \lambda_n(Q_{L^+}) && \text{by the min-max principle}
\end{aligned}$$

In the last equality we used that W is spanned by genuine eigenfunctions of Q_{L^+}. This shows the desired lower bound on $-\lambda_n$. To get the upper bound, replace Q with Q_{L^-} and replace Q_{L^+} with Q. □

4.6. **Improved estimates away from the critical frequencies.**

Lemma 4.20 (Agmon + redshift). *For $S_0 \gg 1$, we have*

$$\begin{aligned}
&\int_{-\infty}^{S_0} \Big[\Big(1 - \frac{2M}{r}\Big)^{-1} |\partial_s(e^{i\omega s}u)|^2 + (1+|s|)^{-1-\delta}(|\partial_s u|^2 + |u|^2) \Big]\,\mathrm{d}s \\
&\lesssim e^{-(4-\delta)L\log L} \underbrace{\int_{S_0}^\infty |u|^2\,\mathrm{d}s}_{(*)} + L^2 \int_{-\infty}^\infty \Big(1 - \frac{2M}{r}\Big)^{-1} |H|^2\,\mathrm{d}s
\end{aligned} \tag{112}$$

Proof. As in the Agmon lemma, the key observation is that for $c \ll_\delta 1$, in $[s_{\mathrm{red}}, s(cL^2)]$ we have

$$(1-\delta)\Big(1 - \frac{2M}{r}\Big)\frac{L^2}{r^2} \le V \lesssim \Big(1 - \frac{2M}{r}\Big)\frac{L^2}{r^2} \tag{113}$$

In particular, for $S_0 \gg 1$ such that $1 - 2M/r$ is close to 1 and for $L \gg_\delta 1$, we have

$$\varphi_{\max} := (1-\delta)\int_{2S_0}^{s(cL^2)} \sqrt{V} \ge 2(1-2\delta)L\log L \tag{114}$$

[13] The finite dimensional linear algebra statement we are using is that the min-max quantity defined over V is exponentially close to the min-max quantity defined over W, since V and W are exponentially close.

This allows us to construct a weight function φ which satisfies

$$\begin{cases} \varphi \equiv \varphi_{\max} & \text{in } s \sim S_0, \\ \varphi \le \varphi_{\max} - aL & \text{in } s = s_{\mathrm{red}} + O(1), \\ \varphi = O(1) & \text{in } s = cL^2 + O(1), \\ V - (\varphi')^2 \gtrsim V & \text{in } [s_{\mathrm{red}}, cL^2]. \end{cases} \tag{115}$$

We can commute the equation with a cutoff χ adapted to $[s_{\mathrm{red}}, cL^2]$ to apply the Agmon estimates. The cutoff introduces an error term localized near s_{red} with a good exponential weight and a term localized near cL^2:

$$\begin{aligned} \int_{S_0/2}^{2S_0} [|\partial_s u|^2 + |u|^2] \le L^2 \int_{-\infty}^{\infty} \Big(1 - \frac{2M}{r}\Big)^{-1} |H|^2 \, \mathrm{d}s + e^{-4(1-3\delta)^2 L \log L} \int_{\{s \text{ near } cL^2\}} |u|^2 \, \mathrm{d}s \\ + \cancel{e^{-aL}} \int_{\{s \text{ near } s_{\mathrm{red}}\}} |u|^2 \, \mathrm{d}s. \end{aligned} \tag{116}$$

- The weight on the $|H|^2$ term comes from gaining a V^{-1} weight on the right after doing Cauchy–Schwarz on $|H||u|$.
- We get control of $|\partial_s u|^2$ on the left from elliptic estimates. The error terms on the right (namely the second and third terms) will include $|\partial_s u|^2$ terms, but we control these using elliptic estimates.
- In the region $[S_0/2, 2S_0]$, the left side of (116) controls the left side of (112). Moreover, the first two terms on the right of the above estimate are allowed on the right of (112).
- The intermediate region $[s_{\mathrm{red}}, S_0]$ can be controlled with the multiplier χu for χ adapted to that interval. This will create a good zeroth order term on the left with an L^2 weight (because of the size of the potential in this region), as well as an error term in the redshift region and an error term in the region covered by the left of the above estimate.
- The last term on the right is controlled with redshift (whose error term is controlled by the good term in the intermediate region), which completes the proof of (112) in the redshift region $(-\infty, s_{\mathrm{red}}]$.

□

Lemma 4.21. *Let $S_0 \gg 1$, and let λ_n be the eigenvalues of Q (with $x_0 = S_0$) that satisfy $-\lambda_n \ge L^{-p}$. Suppose*

$$\mathcal{I} = \inf_n (m^2 - \omega^2 + \lambda_n) \ge e^{1/2L \log L} \qquad (\textit{i.e. } \omega \in I_{\mathrm{good}}). \tag{117}$$

Then

$$\int_{-\infty}^{S_0} \Big[\Big(1 - \frac{2M}{r}\Big)^{-1} |\partial_s (e^{i\omega s} u)|^2 + (1+|s|)^{-1-\delta} (|\partial_s u|^2 + |u|^2) \Big] \, \mathrm{d}s \lesssim L^2 \int_{-\infty}^{\infty} \Big(1 - \frac{2M}{r}\Big)^{-1} |H^2| \, \mathrm{d}s. \tag{118}$$

Remark 4.22. By the fundamental theorem of calculus argument in section 3.3, this estimate implies the zeroth order bound in estimate (2) of proposition 4.1. The proof of the derivative bound is similar (commute the equation with a suitable derivative of u).

Proof. Recall from functional analysis that

$$T \text{ self-adjoint} \implies \|f\|^2 \le \operatorname{dist}(\lambda_0, \sigma(T))^{-2} \|(T - \lambda_0) f\|^2 \text{ by the spectral theorem.} \tag{119}$$

We apply this with

$$T = Q - (\omega^2 - m^2) = -\partial_s^2 + V(s) \text{ on } (S_0, \infty) \text{ for } S_0 \gg 1 \quad \text{and } f = \chi u \text{ for } \chi(S_0) = 0. \tag{120}$$

(*The cutoff introduces error terms supported near S_0 bounded by $\|u\|_{H^1}$, which we can control a priori by $\|u\|_{L^2\{\text{near } S_0\}}$ with elliptic estimates (as in the Agmon lemma).*) The assumption (together with the relative sizes of the eigenvalues) implies that the distance of $\omega^2 - m^2$ to the spectrum of T is at least $e^{-1/2L \log L}$. The upshot is

$$\int_{S_0+1}^{\infty} (|\partial_s u|^2 + |u|^2) \lesssim e^{L \log L} \Big[\int_{S_0}^{S_0+2} |u|^2 + \int_{S_0}^{\infty} |H|^2 \Big]. \tag{121}$$

(*This $e^{L \log L}$ weight looks bad, but we will absorb it with the good $e^{-(4-\delta) L \log L}$ weight in the Agmon/redshift estimate.*) Use the above estimate to control term $(*)$ of the Agmon/redshift estimate (lemma 4.20) applied with $S_0 + 2$ in place of S_0. □

Remark 4.23. The proof of (1c) is similar, except we also use the exponential multiplier to establish an estimate for u (with a loss of $e^{CL} \lesssim e^{\delta L \log L}$), then commute with a suitable ∂_ω-derivative of u and use the estimate leading to (1b) to establish the derivative bound.

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