MORAWETZ ON MINKOWSKI

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We compute the deformation tensor of X for a vector field $X = f(r)\partial_r$, then compute the corresponding divergence term K^X.

1. Current formalism

Let φ be a function. The energy-momentum tensor is a symmetric $(0,2)$-tensor defined by

$$T_{\alpha\beta}[\varphi] = \partial_\alpha\varphi\partial_\beta\varphi - \frac{1}{2}g_{\alpha\beta}|\nabla\varphi|^2, \tag{1}$$

where we write $|\nabla\varphi|^2 = \nabla^\mu\varphi\nabla_\mu\varphi = g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi$. This tensor satisfies $\nabla^\alpha T_{\alpha\beta} = \Box\varphi\nabla_\beta\varphi$, so the energy momentum tensor associated to the solution of the wave equation is divergence free. For a vector field X, we introduce the associated current (a one-form)

$$J^X_\alpha[\varphi] = T_{\alpha\beta}[\varphi]X^\beta \tag{2}$$

and the deformation tensor

$${}^{(X)}\pi_{\alpha\beta} = \frac{1}{2}(\mathcal{L}_X g)_{\alpha\beta} = \frac{1}{2}(\nabla_\alpha X_\beta + \nabla_\beta X_\alpha). \tag{3}$$

The divergence of the one-form J^X satisfies

$$\nabla^\alpha J^X_\alpha[\varphi] = K^X[\varphi] + \mathcal{E}^X[\varphi]. \tag{4}$$

where

$$K^X[\varphi] = {}^{(X)}\pi^{\alpha\beta}T_{\alpha\beta}[\varphi] = {}^{(X)}\pi(\mathrm{d}\varphi, \mathrm{d}\varphi) - \frac{1}{2}\operatorname{tr}{}^{(X)}\pi|\nabla\varphi|^2 \tag{5}$$

and

$$\mathcal{E}^X := X\varphi\Box\varphi. \tag{6}$$

Moreover, if w is a weight function, we introduce the auxiliary current

$$J^{\mathrm{aux},w}_\alpha[\varphi] = w\varphi\nabla_\alpha\varphi - \frac{1}{2}\varphi^2\nabla_\alpha w. \tag{7}$$

The divergence of this one form is given by

$$\nabla^\alpha J^{\mathrm{aux},w}_\alpha[\varphi] = K^{\mathrm{aux},w}[\varphi] + \mathcal{E}^{\mathrm{aux},w}[\varphi] \tag{8}$$

for

$$K^{\mathrm{aux},w}[\varphi] := w|\nabla\varphi|^2 - \frac{1}{2}\Box w\varphi^2 \tag{9}$$

and

$$\mathcal{E}^{\mathrm{aux},w}[\varphi] := w\varphi\Box\varphi \tag{10}$$

Notice that

$$K^X[\varphi] + K^{\mathrm{aux},w}[\varphi] = {}^{(X)}\pi(\mathrm{d}\varphi, \mathrm{d}\varphi) + \Big(w - \frac{1}{2}\operatorname{tr}{}^{(X)}\pi\Big)|\nabla\psi|^2 - \frac{1}{2}\Box w\psi^2. \tag{11}$$

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2. Deformation tensor computations

Recall that the Minkowski metric in polar coordinates is

$$g = -\,\mathrm{d}t^2 + \mathrm{d}r^2 + \not{g}, \tag{12}$$

where $\not{g} = r^2 g_{\mathbf{S}^2}$ for $g_{\mathbf{S}^2}$ the round metric on the unit sphere.

Lemma 2.1. *Let Y be a vector field and set $X = f(r)Y$. Then*

$$^{(X)}\pi_{\alpha\beta} = f\,^{(Y)}\pi_{\alpha\beta} + \frac{1}{2}(\nabla_\alpha f Y_\beta + \nabla_\beta f Y_\alpha). \tag{13}$$

and

$$\mathrm{tr}\,^{(X)}\pi = f\,\mathrm{tr}\,^{(Y)}\pi + Yf. \tag{14}$$

If Y is a coordinate vector field, then

$$^{(Y)}\pi_{\alpha\beta} = \frac{1}{2} Y g_{\alpha\beta}. \tag{15}$$

Proof. Using $X = fY$, compute

$$^{(X)}\pi_{\alpha\beta} = f\,^{(Y)}\pi_{\alpha\beta} + \frac{1}{2}(\nabla_\alpha f Y_\beta + \nabla_\beta f Y_\alpha). \tag{16}$$

From (13) we obtain

$$\mathrm{tr}\,^{(X)}\pi = f\,\mathrm{tr}\,^{(Y)}\pi + Yf. \tag{17}$$

Using the definition $^{(Y)}\pi = \frac{1}{2}\mathcal{L}_Y g$, we compute (assuming Y is a coordinate vector field so that $\mathcal{L}_Y \partial_\alpha = 0$)

$$2\,^{(Y)}\pi_{\alpha\beta} = (\mathcal{L}_Y g)(\partial_\alpha, \partial_\beta) = Y g(\partial_\alpha, \partial_\beta) - g(\mathcal{L}_Y \partial_\alpha, \partial_\beta) - g(\partial_\alpha, \mathcal{L}_Y \partial_\beta) = Y g_{\alpha\beta} \tag{18}$$

□

Lemma 2.2. *Write $Y = \partial_r$ and set $X = f(r)\partial_r = f(r)Y$. Write $f' = \partial_r f$. We have*

$$\mathrm{tr}\,^{(X)}\pi = \frac{2}{r} f + f'. \tag{19}$$

and

$$^{(X)}\pi(\mathrm{d}\varphi, \mathrm{d}\varphi) = f'(\partial\varphi)^2 + \frac{f}{r}|\not\nabla\varphi|^2. \tag{20}$$

Proof. Using (15) and the diagonal form of the metric, compute that the non-zero components of $^{(Y)}\pi$ are

$$^{(Y)}\pi_{ab} = \frac{1}{r}\not{g}_{ab}. \tag{21}$$

It follows immediately that

$$\mathrm{tr}\,^{(Y)}\pi = \frac{2}{r} \tag{22}$$

and hence

$$\mathrm{tr}\,^{(X)}\pi = \frac{2}{r} f + f'. \tag{23}$$

Since the only non-zero component of Y with down indices is $Y_r = g_{rr}Y^r = g_{rr} = 1$, and f is a function only of r, the second term in (13) contributes only when $\alpha = \beta = r$. It follows that

$$^{(X)}\pi_{rr} = f' \quad {}^{(X)}\pi_{ab} = \frac{f}{r}\not{g}_{ab}. \tag{24}$$

We now compute

$$^{(X)}\pi(\mathrm{d}\varphi, \mathrm{d}\varphi) = f'(\partial_r\varphi)^2 + \frac{f}{r}|\not\nabla\varphi|^2. \tag{25}$$

□

3. Computing the bulk divergence term

Define

$$K^f[\varphi] := K^{f(r)\partial_r}[\varphi] + K^{\mathrm{aux},r^{-1}f(r)}[\varphi]. \tag{26}$$

Lemma 3.1. *We have*

$$K^f[\varphi] = \frac{1}{2}f'[(\partial_t\varphi)^2 + (\partial_r\varphi)^2] + \left(\frac{f}{r} - \frac{f'}{2}\right)|\nabla\!\!\!/\varphi|^2 - \frac{1}{2}r^{-1}f''\varphi^2. \tag{27}$$

Proof. From (11) and lemma 2.2, we have

$$K^X[\varphi] + K^{\mathrm{aux},w}[\varphi] = f'(\partial_r\varphi)^2 + \frac{f}{r}|\nabla\!\!\!/\varphi|^2 + \left(w - \frac{f}{r} - \frac{1}{2}f'\right)|\nabla\varphi|^2 - \frac{1}{2}\Box w\varphi^2. \tag{28}$$

Take $w = r^{-1}f$. To obtain the desired result, compute

$$\Box w = r^{-2}\partial_r(r^2\partial_r(r^{-1}f)) = r^{-1}f'' \tag{29}$$

and recall that $|\nabla\varphi| = -(\partial_t\varphi)^2 + (\partial_r\varphi)^2 + |\nabla\!\!\!/\varphi|^2$. $\Box$