Using rational numbers to key nested sets

Dan Hazel*

Abstract

This report details the generation and use of tree node ordering keys in a single relational database table. The keys for each node are calculated from the keys of its parent, in such a way that the sort order places every node in the tree before all of its descendants and after all siblings having a lower index. The calculation from parent keys to child keys is simple, and reversible in the sense that the keys of every ancestor of a node can be calculated from that node's keys without having to consult the database.

Proofs of the above properties of the key encoding process and of its correspondence to a finite continued fraction form are provided.

1 Introduction: Nested Sets

The database of interest uses an encoding of nested sets or nested intervals to maintain the hierarchical structure of its data.

Nested Sets: LVs and RVs

Earlier revisions of the database used the left values and right values described by Celko [Cel04] to key tree nodes.

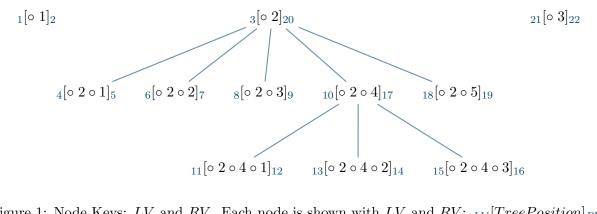


Figure 1: Node Keys: LV and RV. Each node is shown with LV and RV: $_{LV}[TreePosition]_{RV}$.

The nodes of trees keyed in this way are amenable to hierarchy painting predicates that are simple enough to be expressed in SQL.

^{*}Technology One. Email: dan_hazel@technologyonecorp.com

For each of my ancestor nodes,

$$LV_{anc} < LV_{me} < RV_{me} < RV_{anc} \tag{1.1}$$

And so of course for each of my descendant nodes,

$$LV_{me} < LV_{desc} < RV_{desc} < RV_{me} \tag{1.2}$$

These predicates are useful in the database for determining the ancestor nodes of a given node, for determining the descendant nodes of a given node, and most importantly, for imposing an order of display of a result set that relates directly to the tree.

The immediate problem with this approach is that node insertion eventually requires subtrees to the right to be re-encoded.

Consider for example in the tree shown in Figure 1, inserting another node $[\circ 2 \circ 2 \circ 1]$, under $[\circ 2 \circ 2]$. To make room for the keys of the new node that must satisfy the above predicates, all LVs and RVs in the tree having values greater than or equal to 7 must be incremented.

1.2 Rational numbers as nodes keys

Using rationals as keys obviates the problem with insertion into nested sets keyed on integer LVs and RVs. Within data representation limits, there will always be an arbitrary number of rational values between the rational key of any given node and the rational key of its next closest sibling. These rational values are available as keys for the descendants of that node.

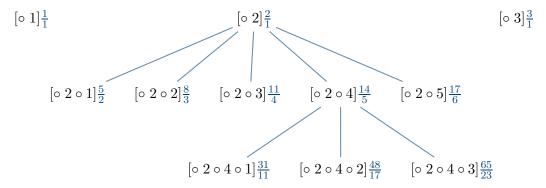


Figure 2: Node Keys: Numerators (nv) and Denominators (dv). Each node is shown as: $[TreePosition]\frac{nv}{dv}$.

Notice in Figure 2 for example, the rational key for each node under $[\circ 2]$, falls strictly between $\frac{2}{1}$ and $\frac{3}{1}$. Similarly, the rational key for each node under $[\circ 2 \circ 4]$, falls strictly between $\frac{14}{5}$ and $\frac{17}{6}$.

1.3 Determining nv and dv

A finite continued fraction encoding of tree position provides a unique rational for each position in the tree. For example, the 3rd child of the 4th child of the 2nd top level position, is encoded as

$$\mathbb{Q}_{[\circ 2\circ 4\circ 3]} = 2 + \frac{1}{1 + \frac{1}{4 + \frac{1}{1 + \frac{1}{3}}}} = \frac{65}{23} \tag{1.3}$$

This is a simple or regular finite continued fraction with an odd number of terms, where every even numbered term is unity. A definition of this $\mathbb{Q}_{[seq]}$ notation is provided in Section 4 on page 8.

This encoding is very similar to the encoding presented by Tropashko [Tro04]. The Tropashko encoding of the above tree position is

$$\operatorname{Trop}_{[\circ 2 \circ 4 \circ 3]} = 2 + \frac{1}{4 + \frac{1}{3}} = \frac{29}{13} \tag{1.4}$$

One problem with the Tropashko encoding, when taken as a finite continued fraction, is that the rational result for any first child is the same as that for any next sibling. For example

$$\operatorname{Trop}_{[\circ 2\circ 4\circ 3\circ 1]} = 2 + \frac{1}{4 + \frac{1}{3 + \frac{1}{1}}} = 2 + \frac{1}{4 + \frac{1}{3 + 1}} = 2 + \frac{1}{4 + \frac{1}{4}} = \operatorname{Trop}_{[\circ 2\circ 4\circ 4]}$$
(1.5)

Tropashko recognises this and ensures that the encodings are always nicely expressed as their associated continued fraction descendant functions. The constraint placed on the form of our encoding allows us to use simple finite continued fractions instead.

The more difficult problem with the Tropashko encoding is that although the rational key of each descendant lies strictly between the keys of its parent and its parent's next sibling, they do not order monotonically. Every second row reverses the ordering.

For example, on level 3 of the Tropashko tree

$$2 + \frac{1}{4 + \frac{1}{3}} < 2 + \frac{1}{4 + \frac{1}{4}} < 2 + \frac{1}{4 + \frac{1}{5}} \tag{1.6}$$

that is,

$$\frac{29}{18} < \frac{38}{17} < \frac{47}{21}$$

However, on level 4 of the tree

$$2 + \frac{1}{4 + \frac{1}{3 + \frac{1}{3}}} > 2 + \frac{1}{4 + \frac{1}{3 + \frac{1}{4}}} > 2 + \frac{1}{4 + \frac{1}{3 + \frac{1}{5}}}$$

$$(1.7)$$

that is,

$$\frac{96}{43} > \frac{125}{56} > \frac{154}{69}$$

Notice the difference in direction between the inequalities in (1.7) and those in (1.6). This makes the order by clause over a database result set extremely difficult to phrase. It could be said that our encoding has reestablished monotonicity by leaving out every second row.

Tropashko recognises the monotonicity problem in his later paper [Tro05]. A workaround using reverse continued fractions, different from our encoding, is suggested there. However imposing an order where

a child key is greater than a parent key, corresponds to keys for earlier siblings being greater than more recent siblings.

2 Sibling quadruples

2.1 Next sibling numerator and denominator: snv and sdv

We find it expedient to store with each node, not only the nv and dv that define its rational key, but also the numerator and denominator of the node's next sibling, snv and sdv. These values will be used when searching for a nodes descendants as is hinted in Section 1.2. They are also seminally useful when determining the nv and dv of an inserted child node and, as explained in Section 3, when relocating subtrees.

The choice to keep snv and sdv with nv and dv diverges from common practice in the continued fractions literature of keeping the parent keys. It is our constraint on the form of continued fractions we employ that makes snv and sdv the preferred associated pair. The matrices we associate with tree nodes should not be confused with those often used in reasoning about continued fractions.

The next sibling of $[\circ 2 \circ 4 \circ 3]$ is $[\circ 2 \circ 4 \circ 4]$.

$$\mathbb{Q}_{[\circ 2\circ 4\circ 3]} = \frac{65}{23}$$

$$\mathbb{Q}_{[\circ 2\circ 4\circ 4]} = 2 + \frac{1}{1 + \frac{1}{4 + \frac{1}{4}}} = \frac{82}{29}$$

Tree position	$\mathbf{n}\mathbf{v}$	$d\mathbf{v}$	\mathbf{snv}	sdv
$[\circ 2]$	2	1	3	1
$[\circ\ 2\circ 1]$	5	2	8	3
$[\circ\ 2\circ 2]$	8	3	11	4
$[\circ\ 2\circ 3]$	11	4	14	5
$[\circ\ 2\circ 4]$	14	5	17	6
$[\circ\ 2\circ 4\circ 1]$	31	11	48	17
$[\circ\ 2\circ 4\circ 2]$	48	17	65	23
$[\circ\ 2\circ 4\circ 3]$	65	23	82	29

Figure 3: Some example keys

Figure 3 shows the nv, dv, snv and sdv of some of the nodes in our example tree. Notice that when determining the nv and dv of the node $[\circ 2 \circ 4 \circ 3]$, we could either perform the continued fraction calculation shown at (1.3), or we could use the values on the parent row, $[\circ 2 \circ 4]$. Adding nv to snv gives the numerator of the first child. Adding dv to sdv gives the denominator of the first child. Adding

nv to $3 \times snv$ gives the numerator of the third child. Adding dv to $3 \times sdv$ gives the denominator of the third child.

In general, given the nv, dv, snv and sdv of a parent node p, we can determine the nv and dv of its c^{th} child as follows:

$$nv_c = nv_p + c \times snv_p \tag{2.1}$$

$$dv_c = dv_p + c \times s dv_p \tag{2.2}$$

A proof of (2.1) and (2.2) is provided in Section 4.

Since the next sibling of the c^{th} child of node p, is the $(c+1)^{th}$ child of node p, it follows that

$$snv_c = nv_p + (c+1) \times snv_p \tag{2.3}$$

$$sdv_c = dv_p + (c+1) \times sdv_p \tag{2.4}$$

A concrete example from values in Figure 3 is

$$65 = 14 + 3 \times 17$$

$$23 = 5 + 3 \times 6$$

$$82 = 14 + (3 + 1) \times 17$$

$$29 = 5 + (3 + 1) \times 6$$

That is:

$$\begin{split} nv_{[\circ 2\circ 4\circ 3]} &= nv_{[\circ 2\circ 4]} + 3\times snv_{[\circ 2\circ 4]} \\ dv_{[\circ 2\circ 4\circ 3]} &= dv_{[\circ 2\circ 4]} + 3\times sdv_{[\circ 2\circ 4]} \\ snv_{[\circ 2\circ 4\circ 3]} &= nv_{[\circ 2\circ 4]} + (3+1)\times snv_{[\circ 2\circ 4]} \\ sdv_{[\circ 2\circ 4\circ 3]} &= dv_{[\circ 2\circ 4]} + (3+1)\times sdv_{[\circ 2\circ 4]} \end{split}$$

2.2 Tree hierarchy predicates

The predicates that can be used to filter ancestors of a given node or descendants of a given node are not quite as simple as those available when using LVs and RVs to key nodes. See Predicates (1.1) and (1.2).

For the encoding presented here, if a node, me, has keys, $(nv_{me}, dv_{me}, snv_{me}, sdv_{me})$, then a node, anc, with keys, $(nv_{anc}, dv_{anc}, snv_{anc}, sdv_{anc})$, is an ancestor of me iff:

$$\frac{nv_{anc}}{dv_{anc}} < \frac{nv_{me}}{dv_{me}} < \frac{snv_{anc}}{sdv_{anc}} \tag{2.5}$$

and a node, desc, with keys, $(nv_{desc}, dv_{desc}, snv_{desc}, sdv_{desc})$, is a descendant of me iff:

$$\frac{nv_{me}}{dv_{me}} < \frac{nv_{desc}}{dv_{desc}} < \frac{snv_{me}}{sdv_{me}} \tag{2.6}$$

In practice, the predicate to filter ancestors is not used. This is because with the continued fractions encoding, the keys of all ancestors of a given node, η , can be calculated from the nv and dv keys of η .

There is rarely a need to use the inequalities of (2.5) to test whether a node is an ancestor of another node.

The source code for a SQL Server 2005 function to return the ancestors of a node indicated by argument numerator and denominator is provided in Figure 4 on page 7. This algorithm performs a simple root to leaf walk through the continued fraction encoding.

On the other hand, while in principle, calculation of descendants is possible, the (2.6) inequalities are used to filter descendant subtree searches since we would have to go to the database anyway to ask how many children each descendant has.

3 Transformations

3.1 Offspring transformation

If we draw the quadruple: (nv, dv, snv, sdv), as a 2×2 matrix:

$$\begin{bmatrix} nv & snv \\ dv & sdv \end{bmatrix}$$
 (3.1)

Then

$$\mathbb{M}_{[\circ 2\circ 4]} = \begin{bmatrix} 14 & 17\\ 5 & 6 \end{bmatrix} \tag{3.2}$$

And

$$\mathbb{M}_{[\circ 2 \circ 4 \circ 3]} = \begin{bmatrix} 65 & 82 \\ 23 & 29 \end{bmatrix} \tag{3.3}$$

And

$$\mathbb{M}_{[\circ 2\circ 4\circ 3]} = \mathbb{M}_{[\circ 2\circ 4]} \begin{bmatrix} 1 & 1 \\ 3 & (3+1) \end{bmatrix}$$
 (3.4)

The equality shown in (3.4) is just an application of equations (2.1) through (2.4).

Because of equations (2.1) through (2.4), the matrix corresponding to each node in our tree is built of a product of transformations that lead back to the root of the tree. For example:

$$\mathbb{M}_{[\circ 2 \circ 4 \circ 3]} = \begin{bmatrix} 65 & 82 \\ 23 & 29 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & (2+1) \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 4 & (4+1) \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & (3+1) \end{bmatrix}$$
(3.5)

An important observation in regard to performing calculations within the database is that the determinant of each of the factor matrices is either -1 or 1.

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1 \tag{3.6}$$

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1$$

$$\begin{vmatrix} 1 & 1 \\ c & (c+1) \end{vmatrix} = 1 \times (c+1) - 1 \times c = 1$$
(3.6)

(3.8)

And so, the determinant of the product is

$$\det\left(\mathbb{M}_{[\circ 2\circ 4\circ 3]}\right) = -1 \times 1 \times 1 \times 1 = -1 \tag{3.9}$$

This property is used in Section 4 to show that each pair of nv and dv are relatively prime. This is as good a normal form as any.

```
create function getAncestorsAsTable(
    Onumerator bigint,
    @denominator bigint
) returns @ancestortable table (nv bigint not null, dv bigint not null)
as
begin
    declare @ancnv bigint
    declare @ancdv bigint
    declare @ancsnv bigint
    declare @ancsdv bigint
    declare @div bigint
    declare @mod bigint
    set @ancnv = 0
    set @ancdv = 1
    set @ancsnv = 1
    set @ancsdv = 0
    while @numerator > 0 and @denominator > 0
    begin
        set @div = @numerator / @denominator
        set @mod = @numerator % @denominator
        set @ancnv = @ancnv + @div * @ancsnv
        set @ancdv = @ancdv + @div * @ancsdv
        set @ancsnv = @ancnv + @ancsnv
        set @ancsdv = @ancdv + @ancsdv
        insert into @ancestortable (nv, dv) values (@ancnv, @ancdv)
        set @numerator = @mod
        if @numerator <> 0
        begin
            set @denominator = @denominator % @mod
            if @denominator = 0
            begin
                set @denominator = 1
            end
        end
    end
    return
```

Figure 4: SQL Server 2005 function to return a table of ancestor keys when passed a numerator and denominator

Also, since the determinant of the matrix of each node in our tree is -1, the inverse of any matrix, $\begin{bmatrix} nv & snv \\ dv & sdv \end{bmatrix}$ is given by

$$\begin{bmatrix} nv & snv \\ dv & sdv \end{bmatrix}^{-1} = \frac{1}{\begin{vmatrix} nv & snv \\ dv & sdv \end{vmatrix}} \times \begin{bmatrix} sdv & -snv \\ -dv & nv \end{bmatrix}$$
(3.10)

$$= \begin{bmatrix} -sdv & snv \\ dv & -nv \end{bmatrix}$$
 (3.11)

Which means that inverse transformations can be calculated in the database without the need to leave integer arithmetic. Inverse transformations are important to the process of moving subtrees.

3.2 Moving subtrees

If it is required to move a subtree from under the nth child of the node with matrix p_0 to under the mth child of the node with matrix p_1 , this can be achieved using the relatively immediate availability of the inverses of the matrices. Say an arbitrary node in that subtree is given by matrix M_0 , then there must be a φ such that

$$p_0 \times \left[\begin{smallmatrix} 1 & 1 \\ n & n+1 \end{smallmatrix} \right] \times \varphi = M_0 \tag{3.12}$$

$$\begin{bmatrix} 1 & 1 \\ n & n+1 \end{bmatrix} \times \varphi = p_0^{-1} \times M_0 \tag{3.13}$$

$$\varphi = \begin{bmatrix} 1 & 1 \\ n & n+1 \end{bmatrix}^{-1} \times p_0^{-1} \times M_0 \tag{3.14}$$

$$\begin{bmatrix} 1 & 1 \\ m & m+1 \end{bmatrix} \times \varphi = \begin{bmatrix} 1 & 1 \\ m & m+1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ n & n+1 \end{bmatrix}^{-1} \times p_0^{-1} \times M_0$$
(3.15)

$$p_1 \times \begin{bmatrix} 1 & 1 \\ m & m+1 \end{bmatrix} \times \varphi = p_1 \times \begin{bmatrix} 1 & 1 \\ m & m+1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ n & n+1 \end{bmatrix}^{-1} \times p_0^{-1} \times M_0$$
(3.16)

The left hand side of the equality (3.16) expresses the relocation of the subtree to the m^{th} child of p_1 . Simplifying:

$$\begin{bmatrix} 1 & 1 \\ m & m+1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ n & n+1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ (m-n) & 1 \end{bmatrix}$$
 (3.17)

Restating, when a subtree identified as the descendants of the *n*th child of the node with matrix p_0 is relocated to the subtree identified as the descendants of the *m*th child of the node with matrix p_1 , any descendant node having matrix M_0 before the relocation, will have matrix M_1 after, where M_1 is given by

$$M_1 = p_1 \times \begin{bmatrix} 1 & 0 \\ (m-n) & 1 \end{bmatrix} \times p_0^{-1} \times M_0$$
 (3.18)

4 Properties of the encoding

Definition 4.1. Our encoding uses a simple or regular finite continued fraction with an odd number of terms, where every even numbered term is unity. It allows the terms to range over positive real

numbers, \mathbb{R}^+ , to make the proofs easier, though in general use, the terms are strictly in \mathbb{N} .

$$\forall N_1, N_2, \cdots, N_n \in \mathbb{R}^+$$

It is accepted and used without proof that:

$$\forall N_1, \cdots, N_n, \delta, \varepsilon \in \mathbb{R}^+$$

$$\mathbb{Q}_{[\circ N_1 \circ \cdots \circ N_n \circ \delta \circ \varepsilon]} = \mathbb{Q}_{\left[\circ N_1 \circ \cdots \circ N_n \circ \left(\delta + \frac{1}{1 + \frac{1}{\varepsilon}}\right)\right]}$$

$$(4.2)$$

Below, $\prod_{k=1}^{m} M_k$ denotes the product of a sequence of 2×2 matrices in the order $M_1 M_2 \cdots M_m$.

Theorem 4.2. The generated keys of tree nodes are in their lowest terms. That is, each numerator and denominator pair has no common divisors.

FOR ALL

 $m \in \mathbb{N}$

AND

 $nv_m, dv_m, snv_m, sdv_m, N_1, ..., N_m \in \mathbb{N}$

PROVIDED

$$\begin{bmatrix} nv_m & snv_m \\ dv_m & sdv_m \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \prod_{k=1}^m \begin{bmatrix} 1 & 1 \\ N_k & (N_k+1) \end{bmatrix}$$

$$(4.3)$$

HOLDS

$$\gcd(nv_m, dv_m) = 1 \tag{4.4}$$

$$\gcd(snv_m, sdv_m) = 1 \tag{4.5}$$

$$\gcd(nv_m, snv_m) = 1 \tag{4.6}$$

$$\gcd(dv_m, sdv_m) = 1 \tag{4.7}$$

Proof of Theorem 4.2

Consider first, for $nv_m, dv_m, snv_m, sdv_m, N_1, ..., N_m \in \mathbb{N}$:

$$nv_{m} \times sdv_{m} - dv_{m} \times snv_{m} = \begin{vmatrix} nv_{m} & snv_{m} \\ dv_{m} & sdv_{m} \end{vmatrix}$$
 by definition of determinant
$$= \det \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \prod_{k=1}^{m} \begin{bmatrix} 1 & 1 \\ N_{k} & (N_{k}+1) \end{bmatrix} \right)$$
 using hypothesis (4.3)
$$= \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \times \prod_{k=1}^{m} \begin{vmatrix} 1 & 1 \\ N_{k} & (N_{k}+1) \end{vmatrix}$$
 (scalar \prod now)
$$= -1 \times \prod_{k=1}^{m} 1$$
 by calculation
$$= -1$$

For all $a \in \mathbb{N}$ such that a is a divisor of both nv_m and dv_m there must be $b, c \in \mathbb{N}$ such that $nv_m = a \times b$ and $dv_m = a \times c$.

In which case,

$$-1 = nv_m \times sdv_m - dv_m \times snv_m \quad \text{using (4.8)}$$

$$= a \times b \times sdv_m - a \times c \times snv_m$$

$$= a \times (b \times sdv_m - c \times snv_m)$$
(4.9)

It follows that for all $a \in \mathbb{N}$ such that a is a divisor of both nv_m and dv_m , a = 1, as required for (4.4). The other gcd results (4.5), (4.6) and (4.7) are proven similarly, also using (4.8).

 \square Theorem 4.2.

Theorem 4.3. The generated key pair $\frac{nv}{dv}$ of each tree node, is unique to the tree. Uniqueness rests on well known uniqueness properties of (carefully constrained) continued fractions.

FOR ALL

 $m \in \mathbb{N}$

AND

 $nv_m, dv_m, snv_m, sdv_m, N_1, ..., N_m \in \mathbb{N}$

PROVIDED

$$\begin{bmatrix} nv_m & snv_m \\ dv_m & sdv_m \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \prod_{k=1}^m \begin{bmatrix} 1 & 1 \\ N_k & (N_k+1) \end{bmatrix}$$

$$(4.10)$$

HOLDS

$$\frac{nv_m}{dv_m} = \mathbb{Q}_{[\circ N_1 \circ \dots \circ N_m]} \tag{4.11}$$

$$\frac{snv_m}{sdv_m} = \mathbb{Q}_{[\circ N_1 \circ \cdots \circ N_{m-1} \circ (N_m + 1)]} \tag{4.12}$$

$$\forall \delta \in \mathbb{R}^+ \bullet \frac{nv_m + \delta \times snv_m}{dv_m + \delta \times sdv_m} = \mathbb{Q}_{[\circ N_1 \circ \dots \circ N_m \circ \delta]}$$

$$\tag{4.13}$$

Proof of Theorem 4.3

This proof is by induction over m, the depth of the tree.

Basis: m=1

Using Definition 4.1:

$$\mathbb{Q}_{[\circ N_1]} = N_1 \tag{4.14}$$

$$\mathbb{Q}_{[\circ(N_1+1)]} = N_1 + 1 \tag{4.15}$$

and for all $\delta \in \mathbb{R}^+$,

$$\mathbb{Q}_{[\circ N_1 \circ \delta]} = N_1 + \frac{1}{1 + \frac{1}{\delta}} \\
= \frac{N_1 + \delta \times N_1 + \delta}{\delta + 1}$$
(4.16)

It is also useful to expand hypothesis (4.10), for m = 1:

$$\begin{bmatrix}
 nv_1 & snv_1 \\
 dv_1 & sdv_1
\end{bmatrix} = \begin{bmatrix}
 0 & 1 \\
 1 & 0
\end{bmatrix} \begin{bmatrix}
 1 & 1 \\
 N_1 & (N_1+1)
\end{bmatrix}
= \begin{bmatrix}
 N_1 & (N_1+1) \\
 1 & 1
\end{bmatrix}$$
(4.17)

Then,

$$\frac{nv_1}{dv_1} = \frac{N_1}{1} \quad \text{using (4.17)}$$
$$= \mathbb{Q}_{[\circ N_1]} \quad \text{using (4.14)}$$

as required to show (4.11) for n = 1.

And,

$$\frac{snv_1}{sdv_1} = \frac{N_1 + 1}{1} \quad \text{using (4.17)}$$
$$= \mathbb{Q}_{[\circ(N_1 + 1)]} \quad \text{using (4.15)}$$

as required to show (4.12) for m = 1.

And,

$$\frac{nv_1 + \delta \times snv_1}{dv_1 + \delta \times sdv_1} = \frac{N_1 + \delta \times (N_1 + 1)}{1 + \delta \times 1} \quad \text{using (4.17)}$$

$$= \frac{N_1 + \delta \times N_1 + \delta}{\delta + 1} \quad \text{simplifying}$$

$$= \mathbb{Q}_{[\circ N_1 \circ \delta]} \quad \text{using (4.16)}$$

as required to show (4.13) for m = 1.

Inductive step

It is enough to show that for

$$m \in \mathbb{N}$$

and

 $nv_m, dv_m, snv_m, sdv_m, nv_{m+1}, dv_{m+1}, snv_{m+1}, sdv_{m+1}, N_1, ..., N_{m+1} \in \mathbb{N}$

THAT

$$\frac{nv_{m+1}}{dv_{m+1}} = \mathbb{Q}_{[\circ N_1 \circ \dots \circ N_{m+1}]} \tag{4.18}$$

and

$$\frac{snv_{m+1}}{sdv_{m+1}} = \mathbb{Q}_{[\circ N_1 \circ \cdots \circ N_m \circ (N_{m+1} + 1)]} \tag{4.19}$$

and

$$\forall \gamma \in \mathbb{R}^+ \bullet \frac{nv_{m+1} + \gamma \times snv_{m+1}}{dv_{m+1} + \gamma \times sdv_{m+1}} = \mathbb{Q}_{[\circ N_1 \circ \dots \circ N_{m+1} \circ \gamma]} \tag{4.20}$$

PROVIDED

$$\begin{bmatrix} nv_m & snv_m \\ dv_m & sdv_m \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \prod_{k=1}^m \begin{bmatrix} 1 & 1 \\ N_k & (N_k+1) \end{bmatrix}$$

$$(4.21)$$

and

$$\frac{nv_m}{dv_m} = \mathbb{Q}_{[\circ N_1 \circ \dots \circ N_m]} \tag{4.22}$$

and

$$\frac{snv_m}{sdv_m} = \mathbb{Q}_{[\circ N_1 \circ \dots \circ N_{m-1} \circ (N_m + 1)]} \tag{4.23}$$

and

$$\forall \xi \in \mathbb{R}^+ \bullet \frac{nv_m + \xi \times snv_m}{dv_m + \xi \times sdv_m} = \mathbb{Q}_{[\circ N_1 \circ \dots \circ N_m \circ \xi]}$$

$$\tag{4.24}$$

and

$$\begin{bmatrix} nv_{m+1} & snv_{m+1} \\ dv_{m+1} & sdv_{m+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \prod_{k=1}^{m+1} \begin{bmatrix} 1 & 1 \\ N_k & (N_k+1) \end{bmatrix}$$
(4.25)

Proof of Inductive step

It is required to prove (4.18), (4.19) and (4.20) given the hypotheses (4.21), (4.22), (4.23), (4.24) and (4.25).

It is useful to first calculate, using (4.25) and (4.21):

$$\begin{bmatrix}
nv_{m+1} & snv_{m+1} \\
dv_{m+1} & sdv_{m+1}
\end{bmatrix} = \begin{bmatrix}
nv_m & snv_m \\
dv_m & sdv_m
\end{bmatrix} \begin{bmatrix}
1 & 1 \\
N_{m+1} & (N_{m+1}+1)
\end{bmatrix}
= \begin{bmatrix}
(nv_m + N_{m+1} \times snv_m) & (nv_m + (N_{m+1}+1) \times snv_m) \\
(dv_m + N_{m+1} \times sdv_m) & (dv_m + (N_{m+1}+1) \times sdv_m)
\end{bmatrix}$$
(4.26)

Consider then hypothesis (4.24), choosing N_{n+1} for ξ :

$$\mathbb{Q}_{[\circ N_1 \circ \cdots \circ N_m \circ N_{n+1}]} = \frac{nv_m + N_{n+1} \times snv_m}{dv_m + N_{n+1} \times sdv_m}$$

$$= \frac{nv_{m+1}}{dv_{m+1}} \qquad \text{using the final equality at (4.26)}$$

as required to show (4.18).

Again using the hypothesis (4.24), but this time choosing $N_{n+1} + 1$ for ξ :

$$\mathbb{Q}_{[\circ N_1 \circ \cdots \circ N_m \circ (N_{n+1}+1)]} = \frac{nv_m + (N_{n+1}+1) \times snv_m}{dv_m + (N_{n+1}+1) \times sdv_m}$$

$$= \frac{snv_{m+1}}{sdv_{m+1}} \qquad \text{using the final equality at (4.26)}$$

as required to show (4.19).

Finally, again using the hypothesis (4.24), but now choosing $N_{n+1} + \frac{1}{1 + \frac{1}{\delta}}$ for ξ , where $\delta \in \mathbb{R}^+$,

$$\begin{split} &\mathbb{Q}_{[\circ N_1 \circ \cdots \circ N_m \circ N_{m+1} \circ \delta]} \\ &= \mathbb{Q}_{\left[\begin{array}{c} \circ N_1 \circ \cdots \circ N_m \circ \left(N_{n+1} + \frac{1}{1 + \frac{1}{\delta}}\right) \\ \end{array}\right]} \\ &= \frac{nv_m + \left(N_{n+1} + \frac{1}{1 + \frac{1}{\delta}}\right) \times snv_m}{dv_m + \left(N_{n+1} + \frac{1}{1 + \frac{1}{\delta}}\right) \times sdv_m} \\ &= \frac{nv_m + \left(N_{n+1} + \frac{1}{1 + \frac{1}{\delta}}\right) \times snv_m}{dv_m + \left(N_{n+1} + \frac{\delta}{1 + \delta}\right) \times sdv_m} \\ &= \frac{nv_m + \left(N_{n+1} + \frac{\delta}{1 + \delta}\right) \times sdv_m}{dv_m + \frac{N_{n+1} \times \delta + N_{n+1} + \delta}{1 + \delta} \times snv_m} \\ &= \frac{nv_m + \frac{N_{n+1} \times \delta + N_{n+1} + \delta}{1 + \delta} \times sdv_m}{dv_m + \frac{N_{n+1} \times \delta + N_{n+1} + \delta}{1 + \delta} \times sdv_m} \\ &= \frac{nv_m \times (1 + \delta) + (N_{n+1} \times \delta + N_{n+1} + \delta) \times snv_m}{dv_m \times (1 + \delta) + (N_{n+1} \times \delta + N_{n+1} + \delta) \times sdv_m} \\ &= \frac{nv_m + N_{n+1} \times snv_m + \delta \times (nv_m + (N_{n+1} + 1) \times snv_m)}{dv_m + N_{n+1} \times sdv_m + \delta \times (dv_m + (N_{n+1} + 1) \times sdv_m)} \\ &= \frac{nv_{m+1} + \delta \times snv_{m+1}}{dv_{m+1} + \delta \times sdv_{m+1}} \end{aligned} \qquad \text{using the final equality at (4.26)}$$

which, generalizing δ to γ , is enough to show (4.20).

 \square Theorem 4.3.

Theorem 4.4. The rational key corresponding to a node is less than the rational key corresponding to its next sibling.

 $FOR\ ALL$

 $m \in \mathbb{N}$

AND

 $nv_m, dv_m, snv_m, sdv_m, N_1, ..., N_m \in \mathbb{N}$

PROVIDED

$$\begin{bmatrix} nv_m & snv_m \\ dv_m & sdv_m \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \prod_{k=1}^m \begin{bmatrix} 1 & 1 \\ N_k & (N_k+1) \end{bmatrix}$$

$$(4.27)$$

HOLDS

$$\frac{nv_m}{dv_m} < \frac{snv_m}{sdv_m} \tag{4.28}$$

Proof of Theorem 4.4

Recalling calculation (4.8), given (4.27), it follows that:

$$nv_m \times sdv_m - dv_m \times snv_m = -1 \tag{4.29}$$

Since each of dv_m and sdv_m is strictly positive, it follows that:

$$\frac{nv_m}{dv_m} - \frac{snv_m}{sdv_m} = \frac{-1}{dv_m \times sdv_m} \tag{4.30}$$

Or:

$$\frac{nv_m}{dv_m} + \frac{1}{dv_m \times sdv_m} = \frac{snv_m}{sdv_m} \tag{4.31}$$

From which (4.28) is immediate.

 \square Theorem 4.4.

Theorem 4.5. The rational key corresponding to a child node lies strictly between the rational key of its parent and the rational key of its parents next sibling.

FOR ALL

 $m \in \mathbb{N}$

AND

 $nv_m, dv_m, snv_m, sdv_m, nv_{m+1}, dv_{m+1}, snv_{m+1}, sdv_{m+1}, N_1, ..., N_{m+1} \in \mathbb{N}$

PROVIDED

$$\begin{bmatrix} nv_m & snv_m \\ dv_m & sdv_m \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \prod_{k=1}^m \begin{bmatrix} 1 & 1 \\ N_k & (N_k+1) \end{bmatrix}$$

$$(4.32)$$

$$\begin{bmatrix} nv_{m+1} & snv_{m+1} \\ dv_{m+1} & sdv_{m+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \prod_{k=1}^{m+1} \begin{bmatrix} 1 & 1 \\ N_k & (N_k+1) \end{bmatrix}$$
(4.33)

HOLDS

$$\frac{nv_m}{dv_m} < \frac{nv_{m+1}}{dv_{m+1}} < \frac{snv_m}{sdv_m} \tag{4.34}$$

Proof of Theorem 4.5

To prove (4.34) it is required to prove

$$\frac{nv_m}{dv_m} < \frac{nv_{m+1}}{dv_{m+1}} \tag{4.35}$$

and to prove

$$\frac{nv_{m+1}}{dv_{m+1}} < \frac{snv_m}{sdv_m} \tag{4.36}$$

Again it is useful to first calculate, using (4.33) and (4.32):

$$\begin{bmatrix}
nv_{m+1} & snv_{m+1} \\
dv_{m+1} & sdv_{m+1}
\end{bmatrix} = \begin{bmatrix}
nv_m & snv_m \\
dv_m & sdv_m
\end{bmatrix} \begin{bmatrix}
1 & 1 \\
N_{m+1} & (N_{m+1}+1)
\end{bmatrix}
= \begin{bmatrix}
(nv_m + N_{m+1} \times snv_m) & (nv_m + (N_{m+1}+1) \times snv_m) \\
(dv_m + N_{m+1} \times sdv_m) & (dv_m + (N_{m+1}+1) \times sdv_m)
\end{bmatrix}$$
(4.37)

The requirement (4.35):

$$\begin{aligned} &\frac{nv_m}{dv_m} < \frac{nv_{m+1}}{dv_{m+1}} \\ &\iff &\frac{nv_m}{dv_m} < \frac{nv_m + N_{m+1} \times snv_m}{dv_m + N_{m+1} \times sdv_m} & \text{using (4.37)} \\ &\iff &nv_m \times (dv_m + N_{m+1} \times sdv_m) < (nv_m + N_{m+1} \times snv_m) \times dv_m & \text{all terms} \in \mathbb{N} \\ &\iff &N_{m+1} \times nv_m \times sdv_m < N_{m+1} \times snv_m \times dv_m & \text{simplifying} \\ &\iff &nv_m \times sdv_m < snv_m \times dv_m & \text{since } N_{m+1} \in \mathbb{N} \end{aligned}$$

which follows as a result of Theorem 4.4.

The requirement (4.36):

$$\frac{nv_{m+1}}{dv_{m+1}} < \frac{snv_m}{sdv_m}$$

$$\iff \frac{nv_m + N_{m+1} \times snv_m}{dv_m + N_{m+1} \times sdv_m} < \frac{snv_m}{sdv_m}$$
 using (4.37)
$$\iff (nv_m + N_{m+1} \times snv_m) \times sdv_m < snv_m \times (dv_m + N_{m+1} \times sdv_m)$$
 all terms $\in \mathbb{N}$

$$\iff nv_m \times sdv_m + N_{m+1} \times snv_m \times sdv_m < snv_m \times dv_m + N_{m+1} \times snv_m \times sdv_m$$
 simplifying
$$\iff nv_m \times sdv_m < snv_m \times dv_m$$

which follows as a result of Theorem 4.4.

 \square Theorem 4.5.

5 Acknowledgements

Thanks to Evan Jones for his review and suggestions.

References

- [Cel04] Joe Celko. Joe Celko's SQL for Smarties: Trees and Hierarchies. Morgan Kaufmann Publishers Inc., San Francisco, CA, USA, 2004.
- [Tro04] Vadim Tropashko. Nested Intervals Tree Encoding with Continued Fractions. CoRR, cs.DB/0402051, 2004.
- [Tro05] Vadim Tropashko. Nested intervals tree encoding in SQL. SIGMOD Record, 34(2):47–52, 2005.