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Spacetime Geometry and Gravity

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Abstract

This report has a general view of Einstein's theory of general relativity. It starts with geometry to understand curvature which consequence of matter and contains much information about general relativity and spacetime. Einstein field equation is derived with understanding curvature and energy-momentum tensor. The equation of motion is examined and compared with Newtonian gravity. At the final FRW universe is described by Friedmann equations which describe homogeneous, isotropic and expanding the universe with cosmological parameters.

Notations

Some fundemental constants

$c = 1$ Naturel unit system is used
$G = 6.672 \times 10^{-11}\, m^3\, kg^{-1}\, sec^{-2}$ Newton's gravitational constant

Some conversion factors

$1\, pc = 3.261\, light\, year = 3.086 \times 10^{16}\, m$
$1\, yr = 3.156 \times 10^{7} sec$
$1\, M_{\odot} = 1.989 \times 10^{30} kg$

Commonly-used symbols

H_0: Hubble constant
H : Hubble parameter
ρ: mass or enrgy density
$\rho_c = \rho_{crit}$: critical density
p: pressure
K: curvature
Ω: density parameter
Ω_0: present density parameter
Ω_K: curvature density parameter
q: deceleration parameter
Λ: cosmological constant
a: scale factor

1 Introduction

General relativity is Einstein's theory of space, time and gravitation. Main idea is that while most forces of nature are represented by fields defined on spacetime, gravity is inherent in spacetime itself. In particular, what we experience as "gravity" is curvature of spacetime.

So to understand spacetime, we need to understand curvature and how curvature becomes gravity.

1.1 The Equivelance Principle

"The possibility of explaning the numerical equality of inertia and gravitation by the unity of their nature gives to the general relativity, according to my conviction, such a superiority over the conceptions of classical mechanics, that all the diffulties encountered in development must be considered as small in comparison" A. Einstein[2]

The equivalence principle is probably the single most important ideo which underlies all of general relativity and it is also allows us to make transition from special relativity to general relativity. It is the basic principle that Einstein used in his formulation and discovery of the theory of general relativity.

The idea is that in small neigbourhood of every point in spacetime, the geometry of spacetime looks approximately like Minkowski space, $R^{3,1}$. For example, someone who is on the very large sphere, neigborhood of every point on that sphere looks approximately like flat plane.

In Newtonian gravity

$$\vec{F} = m_i \vec{a} \tag{1}$$

If object is moving in the presence of a gravitational field such as the planets in the solar system this force is determined by

$$\vec{F} = -m_g \vec{\nabla} \Phi \tag{2}$$

Inertial and gravitational masses are equal ($m_i = m_g$) for every known object. It turns out to be

$$\vec{a} = \vec{\nabla} \Phi \tag{3}$$

It means that equation of motion of particle independent of mass of that particle. It's only determined by the gravitational field and not by any particular properties of that obeject. In theory of gravity there exist a prefereed set of trajectories which all object follow (inertial trajectories).

Galileo proved this with different types of objects. And Eötvös established that all bodies have the same ratio of inertial to gravitational mass with high precision.

If we consider small region of spacetime where gravitational force is constant ($\nabla\Phi = c$)

$$\vec{\nabla}\Phi = -\vec{a_0} \tag{4}$$

The equivalence priciple is the statement that in this very small region of spacetime the effect of gravity identical to that of being in a constantly acceleratig to referance frame, there might be two boxes, one is moving with gravitational field, other one is moving with acceleration $a_0 = g$.

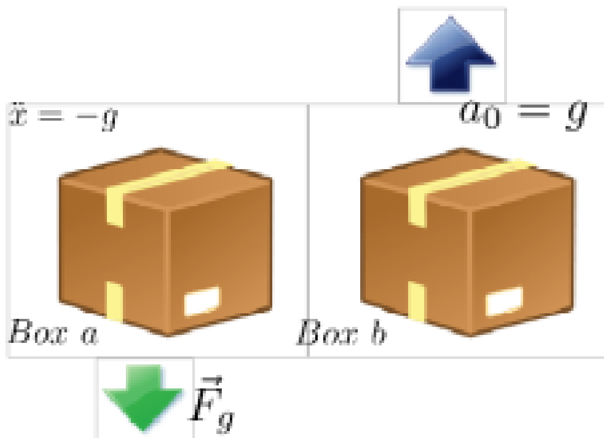

Figure 1: Equivalence Princple

In a sufficient box, gravity is indistinguisable from acceleration. For box b

$$x^{'} = x - \frac{1}{2} g\, t^2 \tag{5}$$

and

$$\ddot{x}^{'} = -g \tag{6}$$

This famous thought experiment is also known as Einstein's elevator.

Presence of a constant gravitational field is just like being in Minkowski space ($R^{3,1}$), but in accelerated coordinate system. Acceleratin x coordinate

$$x' = x - \frac{1}{2} g\, t^2 \qquad (t,\, x) \rightarrow (t,\, x')$$

The line element is in Minkowski space for one one spatial coordinate and time coordinate

$$ds^2 = -dt^2 + dx^2 \tag{7}$$

It represents infinitesimal displacements t and x coordinates. And we need to ask when $t \to t + dt$ and $x \to x + dx$ changes infinitesimaly how does $x^{'}$ change?

$$\begin{aligned} dx' =& dx - gt\,dt \\ dx =& dx' + gt\,dt \end{aligned}$$

and line elements become

$$\begin{aligned} ds^2 =& -dt^2 + (dx' + gt\,dt)^2 \\ =& (-1 + g^2t^2)dt^2 + 2gt\,dx'\,dt\, + dx'^2 \\ =& g_{\mu\nu}dx^{\mu}dx^{\nu} \end{aligned}$$

This is the formula for the line element in this coordinate system which accelerating with constant acceleration g and physics in a constant gravitationla field is by the equivalence principle completely equivalent to physics in Minkowski space in this coordinate system. As we see gravity change the metric. $g_{\mu\nu}dx^{\mu}dx^{\nu}$ is not constant.

If $\nabla\Phi$ is not approximately constant, we cannot think of gravity as equevalent to acceleration.

2 Geometry

2.1 Vectors and Tensors

Tensor means that physical quantities which conform transformation rules. It is generelization of notion of vector or matrix to object with more indices. If we consider a D-dimensional manifold M, a patch may be smoothly labelled by two coordinate system x^μ and $x^{\mu'}$. And we can say that $x^\mu = x^{\mu'}(x^\mu)$. If take a scaler, assigment of a number to each point in M in away which is independent of coordinate system. A scalar is a $(0,0)$ tensor. If function of f is invariant under coordinate transformation, we can say,

$$f(x^\mu) = f(x^\mu(x^{\mu'})) \tag{8}$$

We can define a vector which is set of D functions V^μ can transform under coordinate trasnformation($x^\mu \to x^{\mu'}$) as $V^\mu(x^\mu)$ in x^μ coordinate

$$V^{\mu'}(x^{\mu'}) = \frac{\partial x^{\mu'}}{\partial x^\mu} V^\mu(x^\mu(x^{\mu'})) \tag{9}$$

This known as a *covariant vector* or $(1,0)$ tensor. So a vector is a $(0,1)$ tensor. If we want to remark in vector calculus,

$$V' = J.V \tag{10}$$

We know how to use a vector in $\mathbf{R^3}$

$$\vec{V} = \begin{pmatrix} V^1 \\ V^2 \\ V^3 \end{pmatrix} \tag{11}$$

Under a rotation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to R \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ or $x^i \to x^{i'}$

$$x^{i'} = R^{i'}_j x^j \tag{12}$$

R^i_j is the jacobian matrix $\frac{\partial x^{i'}}{\partial x^j}$.
We can also show *Lorentz Vector* using same way in $\mathbf{R^{3,1}}$,

$$V^\mu = \begin{pmatrix} V^0 \\ V^1 \\ V^2 \\ V^3 \end{pmatrix} \tag{13}$$

and we can transform it as,

$$x^{\mu'} = \Lambda^{\mu'}_{\nu} x^{\nu} \tag{14}$$

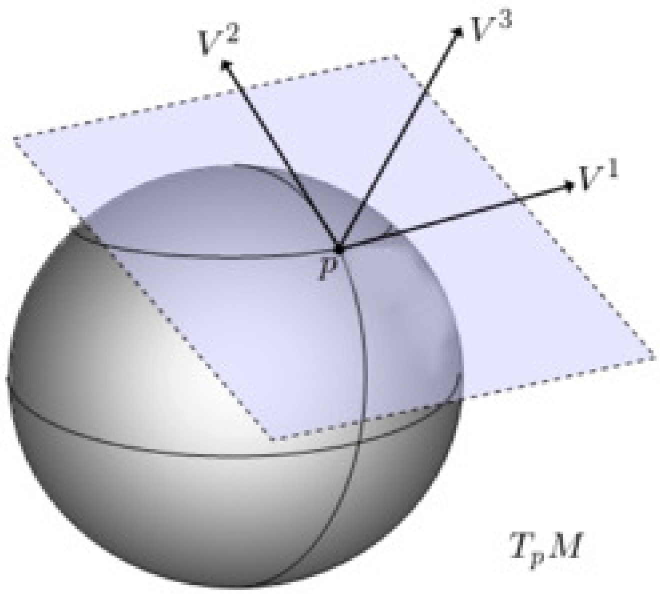

Figure 2: T_pM →all vectors attached to p

We can think of a vector as an arrow attached to each point in spacetime, we can call this T_pM(all vectors attached to p , in manifold M). It is same for a world line $x^{\mu}(\lambda)$,

$$V^{\mu} = \frac{\partial x^{\mu}}{\partial \lambda} \tag{15}$$

is tangent vector. If we want to redefine T_pM for world line, we say, T_pM is the tangent vectors of path through p.

A vector attached to each point in spacetime is a vector field. A vector is also known as a differential operator,

$$\mathbf{V} = V^{\mu}\frac{\partial}{\partial x^{\mu}} = V^{\mu}\partial_{\mu} \tag{16}$$

If $V^{\mu} = \frac{\partial x^{\mu}}{\partial \lambda}$ is the tangent vector along a curve,

$$\mathbf{V} = V^{\mu}\frac{\partial}{\partial x^{\mu}} = \frac{\partial x^{\mu}}{\partial \lambda}\frac{\partial}{\partial x^{\mu}} = \frac{d}{d\lambda} \tag{17}$$

If we look equation (16), ∂_μ can be seen as basis vector. Sometimes we can think of it as basis vector, because, as we know $\vec{V} = V^i \hat{e_i}$.

As I said at the beginning a one-form or covariant vector is set of D functions which can transform as,

$$V_{\mu'} = \left(\frac{\partial x^\mu}{\partial x^{\mu'}}\right) V_\mu \tag{18}$$

This is useful, because a vector of one-form can be combine to make scalar

$$\begin{aligned} V^{\mu'}\omega_{\mu'} &= V^\mu \left(\frac{\partial x^{\mu'}}{\partial x^\mu}\right) \left(\frac{\partial x^\nu}{\partial x^{\mu'}}\right) \omega_\nu \\ &= V^\mu \delta^\nu_\mu \omega_\nu \\ &= V^\mu \omega_\nu \end{aligned}$$

So it is a scalar.

In linear-algebra a vector is a cloumn vector, a one-form is a row vector.

$$V^\mu \omega_\mu = \begin{pmatrix} V^1 \\ \vdots \\ V^D \end{pmatrix} \begin{pmatrix} \omega_1 & \dots & \omega_D \end{pmatrix} = \vec{V}.\vec{\omega} \tag{19}$$

As a notation

$$\partial_\mu f \equiv f_{,\mu} \tag{20}$$

$A(k,l)$ tensor is defined as collection of D^{k+l} functions like $T^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l}$. It is transform as[1],

k is the contravariant(upper) indices and l is the covariant(lower) indicies. Order of indices matters.

And we got some tensor operations:

1. We can add together two tensors of same rank and we get another tensor of same rank

2. We can multiply tensors of different rank $T_1(k,l)$ and $T_2(m,n)$ tensors, after multiplication we get $(k+m, l+n)$ tensor.

3. We can contract an upper and a lower index of a(k,l) tensor to get a $(k-1, l-1)$ tensor.

4. We can also symmetrize and anti-symmetrize indices by

$$T_{[\mu_1 \ldots \mu_l]} = \frac{1}{l!} \sum_{\sigma} sgn(\sigma) T_{\sigma(\mu_1) \ldots \sigma(\mu_l)} \tag{21}$$

As an example for symmetrize,

$$T_{\nu}^{\mu_1 \mu_2} = \frac{1}{2}(T_{\nu}^{\mu_1 \mu_2} + T_{\nu}^{\mu_2 \mu_1}) \tag{22}$$

A symmetric tensor is unchanged under interchange of indices, on the hanf an anti-symmetric tensor pick up sign from $sgn(\sigma)$ under a permutation of indices[1]. An anti-symmetric tensor with equal indices is zero. For examle *kronecker-delta*,

$$\delta_{\nu}^{\mu} = \begin{cases} 1 & \mu = \nu \\ 0 & \mu \neq \nu \end{cases}$$

2.2 Metric Tensor

Spacetime ise psuedo-Riemmanian manifold, and its line element is

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} \tag{23}$$

Here, $g_{\mu\nu}$ is $(0, 2)$ tensor and it is called metric tensor. It is very important for general relativity. It is one of the quantities that tells curvature of space time. Properities of metric are[1]:

1. Symmetry: $g_{\mu\nu} = g_{\nu\mu}$
2. $g_{\mu\nu}$ is invertible and also $g^{\mu\nu}$ is symmetric: $det|g_{\mu\nu}| \neq 0$
3. $g_{\mu\nu}$ is diagonelizable wity eigenvalue $(-1, 1, 1, 1)$.

Third properities tells us metric is like signature of spacetime. And when we look second identities, we can see $g^{\mu\nu}$ is $(2, 0)$ tensor. As we saw kronecker-delta, in here $g^{\mu\nu} g_{\nu\rho} = \delta_{\nu}^{\mu}$.

Besides, We can use metric to raise and lower indices. As an example $\omega_{\nu} = g_{\mu\nu} \omega^{\mu}$ or $V^{\nu} = g^{\mu\nu} V_{\mu}$.

2.3 Geometry and Gravitation

"I was sitting in a chair at patent office at Bern when all of a sudden a thought occured to me: 'If a person falls freely he will not feel his own weight.' I was started. This simple thought had a deep impression on me. It impelled me toward a theory of gravitation"[4]

Massive bodies curve the spacetime. Because motion of stars, galaxies and groups of galaxies, curvature of spacetime is not same everywhere. So we cannot take static referance frame, changing of spacetime, changing of motion and relation mass-energy unseperable of this physical process. This is different from space Newton defined.

If we look at relation between arbitrary reference frame and freely falling particle which is under influence of arbitrary gravitational field by choosing reference frame ξ^α, the equation of motion are[6],

$$\frac{d^2\xi^\alpha}{d\tau^2} = 0 \tag{24}$$

Proper time is defined as

$$d\tau^2 = \eta_{\alpha\beta} d\xi^\alpha d\xi^\beta \tag{25}$$

The freely falling coordinates may be regarded as function of the coordinates $x^\mu$of any arbitrary reference frame curvilinear, accelerated or rotating.

$$\begin{aligned}
0 =& \frac{d}{d\tau}\left(\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{dx^\mu}{d\tau}\right) \\
=& \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{d\tau^2} + \frac{\partial^2\xi^\alpha}{\partial x^\mu\partial x^\nu}\frac{dx^\nu}{d\tau}\frac{dx^\nu}{d\tau}
\end{aligned}$$

We multiply by $\partial x^\lambda/\partial\xi^\alpha$, after that we use chain rule again,

$$\frac{dx^\lambda}{dx^\mu} = \frac{dx^\lambda}{d\xi^\alpha}\frac{d\xi^\alpha}{dx^\mu} = \delta^\lambda_\mu \tag{26}$$

We obtain this to equation of motion

$$\begin{aligned}
0 =& \frac{\partial x^\lambda}{\partial\xi^\alpha}\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{d\tau^2} + \frac{\partial x^\lambda}{\partial\xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial x^\mu\partial x^\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \\
=& \frac{d^2x^\mu}{d\tau^2} + \frac{\partial x^\lambda}{\partial\xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial x^\mu\partial x^\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}
\end{aligned}$$

Here,

$$\Gamma^{\lambda}_{\mu\nu} = \frac{\partial x^{\lambda}}{\partial \xi^{\alpha}} \frac{\partial^2 \xi^{\alpha}}{\partial x^{\mu} \partial x^{\nu}} \tag{27}$$

And we can say, equation of motion is

$$\frac{d^2 x^{\mu}}{d\tau^2} + \Gamma^{\lambda}_{\mu\nu} \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau} = 0 \tag{28}$$

$$\ddot{x}^{\mu} + \Gamma^{\lambda}_{\mu\nu} \dot{x}^{\mu} x^{\nu} = 0 \tag{29}$$

Here $\Gamma^{\lambda}_{\mu\nu}$is Christoffel symbol equation (27), also we can call it as *affine connection.* And result equation of motion(29) is named *geodesic*, the extremal path in the spacetime of an arbitrary gravitational field[6]. We can think of geodesic as path which is as straight as possible given the curvature of spacetime. We can see it as generilization of Newton law $\ddot{x}^{\mu} = 0$ to curved geometry or to accelerating coordinate system.

Christoffel symbol has got second derivative of metric. Also we write it as

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa} (g_{\kappa\mu,\nu} + g_{\kappa\nu,\mu} - g_{\mu\nu,\kappa}) \tag{30}$$

This equation known as Christoffel symbol of the second kind. Comma script notation denote differentiation. It is symmetric at lower indices $\Gamma^{\mu}_{\nu\rho} = \Gamma^{\mu}_{\rho\nu}$. And it is not a tensor.

3 Curvature

3.1 Covariant Derivative

We can think, notion of curvature just depends on metric. As we said metric is the signature of spacetime. Partial derivative is not a good operator in curved spacetime. In here we should define connection.

Connections means in geometry, transporting data along to curve or curves in a parallel and consistent. In subsection (2.3) we show affine connection as equation(27), it is one of example as connections. It transports tangent vector to a manifold from another manifold along a curve.

We need covariant derivative, that is, an operator that reduces to the partial derivative in flat space with inertial coordinates.

We need a new derivative $\nabla_\mu V^\rho$ which is a $(1,1)$ tensor. It obeys,

1. Linearity: $\nabla_\mu(v^\rho + \omega^\rho) = \nabla_\mu V^\rho + \nabla_\mu \omega^\rho$
2. Leibniz Rule: $\nabla_\mu(f V^\rho) = \partial_\mu f.V + f.\nabla_\mu V^\rho$

We write first $\nabla_\mu V^\rho = \partial_\mu V^\rho + G^\rho_{\mu\nu} V^\nu$ for some $G^\rho_{\mu\nu}$ and to make $\nabla_\mu V^\rho$ a tensor, we require:

$$\begin{aligned}
\nabla_{\mu'} V^{\rho'} =& \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\rho'}}{\partial x^\rho} \nabla_\mu V^\rho \\
=& \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\rho'}}{\partial x^\rho} (\partial_\mu V^\rho + G^\rho_{\mu\nu} V^\nu) \\
=& \partial_{\mu'} \left(\frac{\partial x^{\rho'}}{\partial x^\rho} V^\rho \right) + G^{\rho'}_{\mu'\nu'} V^{\nu'}
\end{aligned}$$

In here, $G^{\rho'}_{\mu'\nu'}$ must be,

$$G^{\rho'}_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\rho'}}{\partial x^\rho} G^\rho_{\mu\nu} V^\nu - \frac{\partial^2 x^{\rho'}}{\partial x^\rho \partial x^\mu} \frac{\partial x^\mu}{\partial x^{\mu'}} V^\nu \tag{31}$$

In this equation there is a correction term which is $\frac{\partial^2 x^{\rho'}}{\partial x^\rho \partial x^\mu}$. So $G^\rho_{\mu\nu}$ must transform exactly like $\Gamma^\rho_{\mu\nu}$. So now, we can define covariant derivative of vector V^ρ to be

$$\nabla_\mu V^\rho = \partial_\mu V^\rho + \Gamma^\rho_{\mu\nu} V^\nu \tag{32}$$

We can write general form of covariant derivative as

$$
\begin{aligned}
\nabla_\alpha T^{\mu_1 \ldots \mu_k}_{\nu_1 \ldots \nu_l} =& \partial_\alpha T^{\mu_1 \ldots \mu_k}_{\nu_1 \ldots \nu_l} \\
& + \Gamma^{\mu_1}_{a\rho} T^{\rho \ldots \mu_k}_{\nu_1 \ldots \nu_l} + \ldots \\
& - \Gamma^{\rho}_{a\nu_1} T^{\mu_1 \ldots \mu_k}_{\rho \ldots \nu_l} - \ldots
\end{aligned}
$$

It is important that covariant derivativedo not commute: $\nabla_\alpha \nabla_\beta V^\rho \neq \nabla_\beta \nabla_\alpha V^\rho$

Aas I said at the beggining, it must obey Leibniz Rule:

$$
\nabla_\alpha (T^{\mu\nu} \omega_\rho) = (\nabla_\alpha T^{\mu\nu}) \omega_\rho + T^{\mu\nu} (\nabla_\alpha \omega_\rho) \tag{33}
$$

And metric is important for us, hence we should look at covariant derivative of metric:

$$
\begin{aligned}
\nabla_\mu g_{\rho\sigma} =& g_{\rho\sigma,\mu} - \Gamma^\alpha_{\mu\rho} g_{\alpha\sigma} - \Gamma^\alpha_{\mu\sigma} g_{\alpha\rho} \\
=& g_{\rho\sigma,\mu} - \frac{1}{2}(g_{\mu\nu,\rho} + g_{\rho\sigma,\mu} - g_{\mu\rho,\sigma}) - \frac{1}{2}(g_{\mu\rho,\alpha} + g_{\sigma\rho,\mu} - g_{\mu\sigma,\rho}) \\
=& 0
\end{aligned}
$$

We can say for this covariant derivative is "*metric compatible*".

3.2 Riemann Curvature Tensor

Riemann Curvature Tensor($R^\rho_{\sigma\mu\nu}$) is a parameterization of local curvarture. $R^\rho_{\sigma\mu\nu}$ is a$(1,3)$ tensor which captures the notion of "curvature at point point as spacetime".

$$
\begin{aligned}
R^\rho_{\sigma\mu\nu} V^\rho =& [\nabla_\mu, \nabla_\nu] V^\rho \\
=& \nabla_\mu (\nabla_\nu V^\rho) - \nabla_\nu (\grave{\nabla}_\mu V^\rho) \\
=& \partial_\mu (\nabla_\nu V^\rho) - \Gamma^\lambda_{\mu\nu} \nabla_\lambda V^\rho + \Gamma^\rho_{\mu\sigma} \nabla_\nu V^\sigma - \partial_\nu (\nabla_\mu V^\rho) + \Gamma^\lambda_{\mu\nu} \nabla_\lambda V^\rho - \Gamma^\rho_{\nu\sigma} \nabla_\mu V^\sigma \\
=& \partial_\mu \partial_\nu V^\rho + \partial_\mu (\Gamma^\rho_{\nu\sigma} V^\sigma) + \Gamma^\rho_{\mu\sigma} \partial_\nu V^\sigma + \Gamma^\rho_{\mu\sigma} \Gamma^\sigma_{\nu\lambda} V^\lambda - (\mu \leftrightarrow \nu) \\
=& \partial_\mu \Gamma^\rho_{\nu\sigma} V^\sigma + \Gamma^\rho_{\mu\sigma} \partial_\nu V^\sigma + \Gamma^\rho_{\mu\sigma} \partial_\nu V^\sigma + \Gamma^\rho_{\mu\sigma} \Gamma^\sigma_{\nu\lambda} V^\lambda - (\mu \leftrightarrow \nu)
\end{aligned}
$$

So we can write $R^\rho_{\sigma\mu\nu}$

$$
R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma} \tag{34}
$$

$R^\rho_{\sigma\mu\nu}$ involves information ofsecond derivative of $g_{\mu\nu}$. If $R^\rho_{\sigma\mu\nu} = 0$, we say that spacetime is flat. Flat spacetime means that there is a coordinate system where metric is constant. The Riemann tensor tells us how spacetime deviates from being flat.

Properities of Riemann Curvature tensor:

1. $R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu}$ anti-symmetric in second pair of indices
2. $R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$ anti-symmetric in first pair of indices
3. $R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$ there is symmery between first pair and second pait
4. $R_{[\rho\sigma\mu\nu]} = 0$
5. $\nabla_{[\lambda}R_{\rho\sigma]\mu\nu} = 0$ is **Bianchi Idenity**[1]

3.3 Ricci and Einstein Tensor

Ricci tensor contains all of the trace informatin of Riemann tensor.

$$R_{\mu\nu} = R^\rho_{\mu\rho\nu} \tag{35}$$

It is symmetric, $R_{\mu\nu} = R_{\nu\mu}$.

And the Ricci scalar is

$$R = g^{\mu\nu}R_{\mu\nu} \tag{36}$$

It is unique scalar one can form by tracing the Riemann tensor. Bianchi Identity for Ricci tensor is

$$\nabla^\mu R_{\mu\nu} = \frac{1}{2}\nabla_\nu R \tag{37}$$

Now we can define Einstein tensor $G_{\mu\nu}$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \tag{38}$$

It is symmetric $(0,2)$ tensor and unique built out of Riemann tensor.

$$\nabla^\mu G_{\mu\nu} = 0 \tag{39}$$

This equation shows conservation of Einstein tensor.[1]

3.4 Einstein's Field Equations

3.4.1 Einstein's Field Equation Absence of Matter

"The geometry of spacetime is not given; it is determined by matter and its motion". W. Pauli, 1919

Massive bodies generate gravity and other bodies feels it near massive bodies. Mass is the source of gravity and we know mass-energy equivalence from *Special Relativity*. So we can say that mass-energy curve spacetime.[6] If we look at force law of Newtonian gravity.

$$\ddot{\vec{x}} = -\vec{\nabla}\phi \tag{40}$$

and

$$\nabla^2\phi = 4\pi G\rho \tag{41}$$

This should be replaced by an equation for second derivative of metric:

1. It must be a tensor equation.
2. It should be reproduce Newton's gravity in the right limits.

We can think first empty space, no matter($\rho = 0$). The only tensor which is second order derivative of metric is Riemann tensor and it should be zero at empty space, because it would be flat space.

$$R^{\rho}_{\mu\rho\nu} = 0 \tag{42}$$

So Einstein's equation must be zero in vacuum: $R_{\mu\nu} = 0$ and $R = 0$.

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0 \tag{43}$$

We can write Einstein's equation for vacuum

$$G_{\mu\nu} = 0 \tag{44}$$

3.4.2 Energy-Momentum Tensor for Perfect Fluid

Now we must define energy-momentum tensor $T_{\mu\nu}$. And four momentum$(E, \vec{p})$ is

$$p^{\mu} = m\frac{dx^{\mu}}{d\tau} \tag{45}$$

So $p^0 \equiv E = m\gamma$ and $\vec{p} = m\gamma\vec{v}$.
Energy-momentum tensor is

$$T_{\mu\nu} = \begin{pmatrix} E & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{46}$$

- T_{00} : Energy density
- T_{oi} : Energy shear
- T_{i0} : Momentum density
- T_{ij} : Momentum shear

$T_{\mu\nu}$is $(0, 2)$ tensor, so it obeys,

1. $T_{\mu\nu} = T_{\nu\mu}$
2. $\nabla^{\mu} T_{\mu\nu} = 0$, conservation equation

A perfect fluid is a medium in which the pressure is isotropic in the rest frame of each fluid element, and shear stresses and heat transport are absent[6]. A perfect fluid is characterized energy density(ρ) and pressure density(p). For a perfect fluid which is at rest in $R^{3,1}$, the stress tensor is,

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{47}$$

If fluid is moving with velocity $v^{\mu} = (\gamma, \gamma\vec{v})$[1], the stress tensor is

$$T^{\mu\nu} = -pg^{\mu\nu} + (p + \rho)v^{\mu}v^{\nu} \tag{48}$$

For many perfect fluid ρ and p are related by the equation of state, as simplest type

$$p = \rho\omega \tag{49}$$

ω is a constant which known as equation of state parameter.

1. $\omega = 0$, pressureless fluid, with any ρ, we call it as “dust”

[1] $c = 1$, $\gamma = \frac{1}{\sqrt{1-v^2}}$

2. $\omega = \frac{1}{3} \rightarrow p = \frac{1}{3}\rho$, gas of photon or other massles particle, "radiation"

3. $\omega = -1 \rightarrow p = -\rho$

If we look at three, we get stress tensor $T_{\mu\nu} \propto g^{\mu\nu}$ and we call a constant,

$$T_{\mu\nu} = \Lambda g_{\mu\nu} \tag{50}$$

And we name lambda as "**cosmological constant**" or vacuum energy.

The stress tensor of a perfect fluid is conserved:

$$\nabla^{\mu} T_{\mu\nu} = 0 \tag{51}$$

3.4.3 Einstein's Field Equation Presence of Matter

Now we should look at equation of motion of general relativity with source. We wrote equation of motion absence of matter($T_{\mu\nu} = 0$) as $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$.

What if $T_{\mu\nu} \neq 0$? We can write at the begining $R_{\mu\nu} = \kappa T_{\mu\nu}$, in here, there is problem in here, this cannot be right. $T_{\mu\nu}$ obeys conservation law, $R_{\mu\nu}$ does not. We should write a tensor which has information of curvature and it must obey conservation law. It is Einstein tensor, so the correct equation must be

$$G_{\mu\nu} = \kappa T_{\mu\nu} \tag{52}$$

This must reproduce $\nabla^2 \phi = 4\pi G \rho$ in the approximation where the metric is close to flat

$$ds^2 = -(1 + 2\phi) dt^2 + d\vec{x}^2 \tag{53}$$

when stress tensor is,

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{54}$$

the equation of motion

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu}$$

If we take trace both side by multiplying by $g^{\mu\nu}$,

$$R - 2R = \kappa T \tag{55}$$

where $T = g^{\mu\nu} T_{\mu\nu} = -\rho + \vartheta(\phi)$

So $R = \kappa\rho$ and $T_{00} = \rho$

The non-trivial equation of motion is

$$R_{00} = \kappa T_{00} + \frac{1}{2}g_{00}R = \kappa\rho + \frac{1}{2}(-1)\kappa\rho \tag{56}$$

So

$$R_{00} = \frac{1}{2}\kappa\rho \tag{57}$$

And we can compute R_{00} to get $\vartheta(\phi)$.

$$R_{00} = R^{\lambda}_{0\lambda 0} \tag{58}$$

We know metric $ds^2 = -(1+2\phi)dt^2 + d\vec{x}^2$ and $\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\kappa}(g_{\kappa\mu,\nu} + g_{\kappa\nu,\mu} - g_{\mu\nu,\kappa})$,

$$\begin{aligned}\Gamma^{i}_{00} =& \delta^{ij}\phi_{,j}\\ \Gamma^{0}_{0i} =& \phi_{,i}\end{aligned}$$

to get $\vartheta(\phi)$. And we compute Riemann tensor,

$$R^{\lambda}_{0\lambda 0} = R^{0}_{000} + R^{i}_{0i0}$$

So

$$\begin{aligned}R^{\lambda}_{0\lambda 0} =& 0\\ R^{i}_{0i0} =& \partial_i\Gamma^{i}_{00} = \delta^{ij}\partial_i\partial_j\phi = \nabla^2\phi\end{aligned}$$

To leading to order, $\nabla^2\phi = \frac{1}{2}\kappa\rho$ which is identical to $\nabla^2\phi = 4\pi G\rho$ where $\kappa = 8\pi G$

And finally we get, equation of motion of general relativity in the presence of matter is *Einstein's Equation* is

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} \tag{59}$$

If we get trace bot side $-R = 8\pi G T$. So, when we put the equation(59),

$$R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T) \tag{60}$$

This is known as *"Trace subtracted Einstein's Equation"*.

Unlike, Maxwell or Newtonian gravity, Einstein's equation s are non-linear: $R_{..} \approx \partial^2(g) + (\partial g)^2$. It means that we cannot add two solutions together. Maxwell equations are linear because electromagnetic fields are source of charges, and EM fields do not carry charges. Einstein's Equation are not linear because curvature is sourced by energy, gravitational fields have energy, just like everything else.

To solve Einstein's equation is hard and there are some ways to solve it:

1. Exactly if we have symmetry

2. Approximation

3. Numerically

4 Test of General Relativity

4.1 Schwarzschild Geometry

In 1916, Karl Schwarzschild solved Einstein's equation for empty space outside a static and spherical symmetric of curvature,for example spherical star and this is called **Schwarschild Geometry**. [3]

I would like to emphasize that we are looking at outside the object, it means we are working on vacuum. So it must be source free, in Newtonian gravity $\nabla^2\phi = 0$, $\phi = \frac{GM}{r}$. [1]Source free means,

$$R_{\mu\nu} = 0 \tag{61}$$

If we study spherical symmetric we use in Minkowski space, spherically symmetric metric,

$$ds^2_{Minkowski} = -dt^2 + dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\phi^2 \tag{62}$$

After this point, I'll write $d\Omega$ instead of $d\vartheta + \sin\vartheta d\phi$.

There will be two symmetries:

1. Rotational symmetries
2. $R_{\mu\nu} = 0$, time translation symmetries, metric will be independent of time coordinate.

So metric will take the form

$$ds^2 = -e^{2\alpha(\rho)} d\hat{t}^2 + e^{2\beta(\rho)} d\rho^2 + e^{2\gamma(\rho)} d\Omega^2 \tag{63}$$

We will also assume the metric is invariant under time translation $t \rightarrow -t$. This means, object has mass, but does not have angular momentum. A term $d\vartheta d\rho$ is not allowed by rotational symmetry and the term $dtd\rho$ is not time invariant under translation $t \rightarrow -t$.

We choose a radial coordinate $r(\rho) = e^{\gamma}$, to liken spherical coordinate, then metric looks like

$$ds^2 = -e^{2\alpha} d\hat{t}^2 + e^{2\hat{\beta}} dr^2 + r^2 d\Omega^2 \tag{64}$$

$$r = e^{\gamma} \tag{65}$$

we take derivative of this, we get this by using chain rule,

$$dr = \frac{d\gamma}{d\rho} e^{\gamma} d\rho \tag{66}$$

We multiply both side by e^{β} and take square power of both side

$$e^{2\beta}dr^2 = \left(\frac{d\gamma}{d\rho}e^{\gamma}\right)^2 e^{2\beta}d\rho^2 \tag{67}$$

And we put in order,

$$e^{2\beta}\left(\frac{d\gamma}{d\rho}e^{\gamma}\right)^{-2} dr^2 = e^{2\beta}d\rho^2 \tag{68}$$

We can easily see, $e^{2\beta}\left(\frac{d\gamma}{d\rho}e^{\gamma}\right)^{-2}$ is equal to $e^{2\hat{\beta}}$, when we look at equation(63). Now we change radial coordinate.

$$ds^2 = -e^{2\alpha}d\hat{t}^2 + e^{2\beta}dr^2 + r^2 d\Omega^2 \tag{69}$$

We've just took metric, and wen can finally evaluate Einstein'e equation. We start with computing Christoffel symbols.

$$\begin{aligned}
\Gamma^t_{tr} =& \frac{1}{2}g^{tt}(g_{tt,r} + g_{rt,t} - g_{tr,t}) \\
=& \frac{1}{2}e^{2\alpha}(\partial_r(e^{-2\alpha}) + 0 + 0) \\
\Gamma^t_{tr} =& \partial_r\alpha
\end{aligned}$$

Like this way we get other Christoffel's symbols

$$\begin{array}{rrr}
\Gamma^t_{tr} = \partial_r\alpha & \Gamma^r_{tt} = e^{2(\alpha-\beta)}\partial_r\alpha & \Gamma^r_{rr} = \partial_r\beta \\
\Gamma^{\vartheta}_{r\vartheta} = \dfrac{1}{r} & \Gamma^r_{\vartheta\vartheta} = -re^{-2\beta} & \Gamma^r_{rr} = \partial_r\beta \\
\Gamma^r_{\phi\phi} = -re^{-2\beta}\sin^2\vartheta & \Gamma^{\vartheta}_{\phi\phi} = -\sin\vartheta\cos\vartheta & \Gamma^{\phi}_{\vartheta\phi} = \dfrac{\cos\vartheta}{\sin\vartheta}
\end{array}$$

We calculated all Christoffel's symbols and so we can get components of Riemann tensor(34):

$$\begin{aligned}
R^t_{rtr} =& \partial_r\alpha\partial_r\beta - \partial_r^2\alpha - (\partial_r\alpha)^2 \\
R^t_{\vartheta t\vartheta} =& -re^{-2\beta}\partial_r\alpha \\
R^t_{\phi t\phi} =& -re^{-2\beta}\sin^2\vartheta\partial_r\alpha \\
R^r_{\vartheta r\vartheta} =& re^{-2\beta}\partial_r\beta \\
R^r_{\phi r\phi} =& re^{-2\beta}\sin^2\vartheta\partial_r\beta \\
R^{\vartheta}_{\phi\vartheta\phi} =& (1 - e^{-2\beta})\sin^2\vartheta
\end{aligned}$$

At final we should calculate Ricci by using Riemann tensor:

$$\begin{aligned}
R_{tt} =& e^{2(\alpha-\beta)}[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\alpha] \\
R_{rr} =& -\partial_r^2\alpha - (\partial_r\alpha)^2 + \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\beta \\
R_{\vartheta\vartheta} =& e^{-2\beta}[r(\partial_r\beta - \partial_r\alpha) - 1] + 1 \\
R_{\phi\phi} =& \sin^2\vartheta R_{\vartheta\vartheta}
\end{aligned}$$

As I said at the begining, it is source free, $R_{\mu\nu} = 0$. So we can write

$$0 = e^{2(\beta-\alpha)}R_{tt} + R_{rr} = \frac{2}{r}(\partial_r\alpha + \partial_r\beta) \tag{70}$$

And we get $\alpha = -\beta + c$, c is the contant of integration. And we can write metric again by this information.

$$ds^2 = -e^{2\beta+2c}d\hat{t}^2 + e^{2\beta}dr^2 + r^2d\Omega^2 \tag{71}$$

We use a new time coordinate,

$$\begin{aligned}
t =& e^c\hat{t} \\
dt^2 =& e^{2c}d\hat{t}^2
\end{aligned}$$

and

$$ds^2 = e^{-2\beta}dt^2 + e^{2\beta}dr^2 + r^2d\Omega^2 \tag{72}$$

In the final step, we figure out parameter β. We use $R_{\vartheta\vartheta}$ for this

$$\begin{aligned}
R_{\vartheta\vartheta} =& e^{-2\beta}[r(\partial_r\beta - \partial_r\alpha) - 1] + 1 = 0 \\
\rightarrow & e^{-2\beta}(2r\partial_r\beta - 1) + 1 = 0 \\
\rightarrow & e^{-2\beta}(1 - 2r\partial_r\beta) = 1 \\
\rightarrow & e^{-2\beta} + r\frac{\partial}{\partial r}e^{-2\beta} = 1 \\
\rightarrow & \frac{\partial}{\partial r}(re^{-2\beta}) = 1 \\
\rightarrow & re^{-2\beta} = r - R
\end{aligned}$$

R is the integration constant with dimension lenght. So,

$$e^{-2\beta} = 1 - \frac{R}{r} \tag{73}$$

And final form of the metric is,

$$ds^2 = -\left(1 - \frac{R}{r}\right) dt^2 + \left(1 - \frac{R}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{74}$$

Constant R is called **Schwarzschild radius** and we write it R_s instead of R.

$$ds^2 = -\left(1 - \frac{R_s}{r}\right) dt^2 + \left(1 - \frac{R_s}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{75}$$

When we look at Newtonian gravity $\phi = \frac{GM}{r}$ and $g_{tt} = -(1 - \frac{2GM}{r})$, so we can easily see that[2]

$$R_s = 2GM \tag{76}$$

This is called "*gravitational lenght scale of massive object*".

I think we should examine this metric in some approximation:

1. As $r \to \infty$, metric approaches $R^{3,1}$.
2. As $r \to R_s$, "*Event Horizon*" $r \to 2GM$
3. AS $r \to 0$, this is called "*singularity*"

For time being, studied objects's physical radius is much larger thang R_s. For example

- $R_s(Earth) \approx 1cm$
- $R_s(Sun) \approx 3km$

For "normal objects" the horizon and singularity are irrelevant.

4.2 Geodesic of Schwarzschild

We find Schwarzschild metric as

$$ds^2 = -\left(1 - \frac{R_s}{r}\right) dt^2 + \left(1 - \frac{R_s}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{77}$$

Now we should calculate non-zero Christoffel symbols for this metric,

[2] I'd like to say that R_s equals $\frac{2GM}{c^2}$, we take $c = 1$

$$\Gamma^r_{tt} = \frac{GM}{r^3}(r-2GM) \qquad \Gamma^r_{rr} = \tfrac{-GM}{r(r-2GM)} \qquad \Gamma^t_{tr} = \frac{GM}{r(r-2GM)}$$
$$\Gamma^\vartheta_{r\vartheta} = \frac{1}{r} \quad \Gamma^r_{\vartheta\vartheta} = -(r-2GM) \quad \Gamma^\phi_{r\phi} = \frac{1}{r}$$
$$\Gamma^r_{\phi\phi} = -(r-2GM)\sin^2\vartheta \quad \Gamma^\vartheta_{\phi\phi} = -\sin\vartheta\cos\vartheta \quad \Gamma^\phi_{\vartheta\phi} = \frac{\cos\vartheta}{\sin\vartheta}$$

And we use geodesic equation(29), we get geodesic equation, λ is affine parameter. So equation is

$$\begin{aligned}
\frac{d^2t}{d\lambda^2} + \frac{2GM}{r(r-2GM)}\frac{dr}{d\lambda}\frac{dt}{d\lambda} &= 0 \\
\frac{d^2r}{d\lambda^2} + \frac{GM}{r^3}(r-2GM)\left(\frac{dt}{d\lambda}\right)^2 - \frac{GM}{r(r-2GM)}\left(\frac{dr}{d\lambda}\right)^2 & \\
-(r-2GM)\left[\left(\frac{d\vartheta}{d\lambda}\right)^2 + \sin^2\vartheta\left(\frac{d\phi}{d\lambda}\right)\right] &= 0 \\
\frac{d^2\vartheta}{d\lambda^2} + \frac{2}{r}\frac{d\vartheta}{d\lambda}\frac{dr}{d\lambda} - \sin\vartheta\cos\vartheta\left(\frac{d\phi}{d\lambda}\right)^2 &= 0 \\
\frac{d^2\vartheta}{d\lambda^2} + \frac{2}{r}\frac{d\vartheta}{d\lambda}\frac{dr}{d\lambda} + 2\frac{\cos\vartheta}{\sin\vartheta}\frac{d\vartheta}{d\lambda}\frac{d\phi}{d\lambda} &= 0
\end{aligned}$$

We know there are four Killing vectors(K^μ), three of them for spherical symmetry and one is for time translation. These all lead to a constatn of the motion for a free particle. We know that,

$$K_\mu \frac{dx^\mu}{d\lambda} = constant \tag{78}$$

And also, ϵ is constant,

$$\epsilon = -g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \tag{79}$$

is constant along the path, for any trajectory.

$$\left(\frac{ds^2}{d\lambda}\right)^2 = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu < 0 \tag{80}$$

For massive body we chose $\lambda = \tau$ because geodesic will be timelike. In here λ is proper time as measured by an observer on this geodesic.

Our metric has time translation and rotation symmetry which are generated by killing vectors $\partial/\partial t$ and $\partial/\partial\phi$, independent of t and ϕ.

We should notice that, invariance under time translation leads to energy conservation, while invariance udner spatial rotations leads to conservation of the three components of angular momentum[1].

There are two quantities E and L which are independent of λ.

We have, given a geodesic $x^{\mu}(\lambda)$ and a killing vector $K^{\mu}$the quantitiy and the quantity $\dot{x}^{\mu}$. Also $K^{\mu}$is independent of λ.

$$\begin{aligned} K^{\mu}\partial_{\mu} =& \partial_t \\ K^{\mu} =& (1,0,0,0) \\ K_{\mu} =& (-(1-R/r),0,0,0) \end{aligned}$$

Conserved energy is $E = -K_{\mu}\dot{x}^{\mu} = (1-\frac{R}{r})\dot{t}$

$$\dot{t} = \frac{dt}{d\lambda} = (1-\frac{R}{r})E \tag{81}$$

And we use killing vector other symmetries, $K^{\mu}\partial_{\mu} = \partial_{\phi}$ and $K^{\mu} = (0,0,0,1)$, $K_{\mu} = (0,0,0,r^2)$. Angular momentum is

$$L = r^2\dot{\phi} \tag{82}$$

We turn back to metric and affine parameter we use:

$$\left(\frac{ds^2}{d\lambda}\right)^2 = g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = -1 \tag{83}$$

$$\begin{aligned} -1 =& -\left(1-\frac{R}{r}\right)\dot{t}^2 + \left(1-\frac{R}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\phi}^2 \\ -1 =& -\left(1-\frac{R}{r}\right)^{-1}E + \left(1-\frac{R}{r}\right)^{-1}\dot{r}^2 + \frac{L^2}{r^2} \end{aligned}$$

We know conservation of energy of a particle moving on potential

$$\frac{1}{2}\dot{r}^2 + V_{eff}(r) = \hat{E} = \frac{1}{2}E^2 \tag{84}$$

where

$$\begin{aligned} V_{eff} =& \left[(\left(1-\frac{R}{r}\right) + \frac{L^2}{r^2}(\left(1-\frac{R}{r}\right)\right] \\ V_{eff} =& \frac{1}{2} - \frac{R}{2r} + \frac{L^2}{2r^2} - \frac{RL^2}{2r^3} \end{aligned}$$

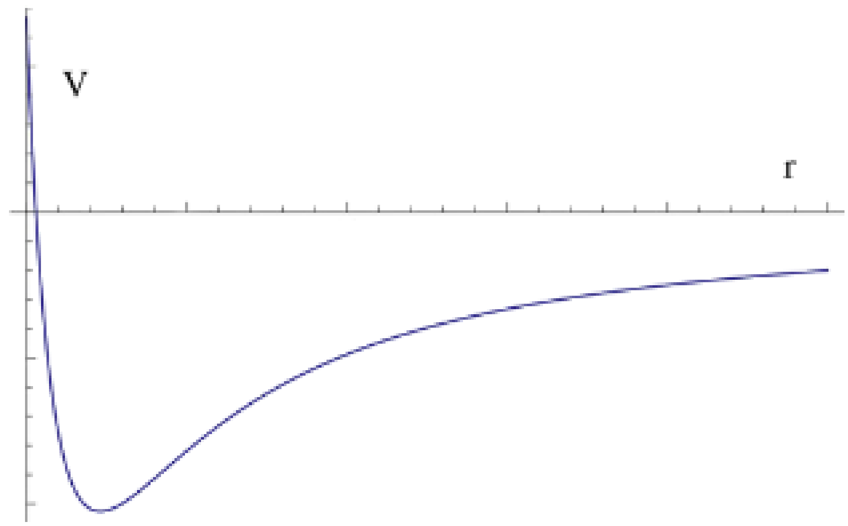

Figure 3: Newtonian potentian graph to distance

In here $\frac{RL^2}{2r^3}$ is corrention term for general relativity. And we can neglect first term of right hand side,

$$V_{eff} = -\frac{R}{2r} + \frac{L^2}{2r^2} - \frac{RL^2}{2r^3} \tag{85}$$

To understan this equation, we can look first Newtonian gravity for effective mass

$$V_{eff} = -\frac{R}{2r} + \frac{L^2}{2r^2} \tag{86}$$

1. For all L there is stable circular orbit $r_c = \frac{2L^2}{R}$
2. Orbits are conic sections
3. Angular momentum barier which repels us from $r = 0$
4. For all values of r there is a stable orbit.

Now we can see difference from Einstein gravity

$$V_{eff} = -\frac{R}{2r} + \frac{L^2}{2r^2} - \frac{RL^2}{2r^3} \tag{87}$$

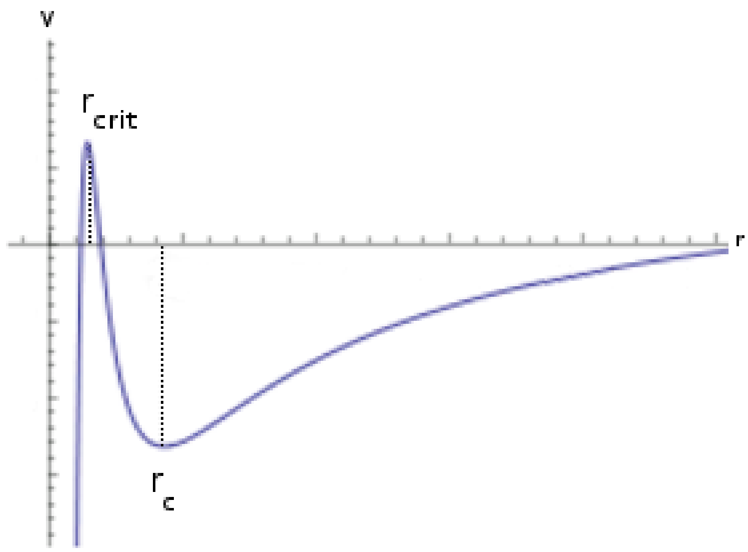

Figure 4: V(r)-r graph with GR term

$$
\begin{aligned}
r_c =& \frac{L^2 + \sqrt{L^4 - 3R^2L^2}}{R} < \frac{2L^2}{R} \\
r_{crit} =& \frac{L^2 + \sqrt{L^4 - 3R^2L^2}}{R}
\end{aligned}
$$

1. As in Newtonian case, we have stable circular orbits at r_c but one thing that you can see by looking at this equation for r_c is that this always less then $\frac{2L}{R^2}$ which was location of circular orbit in the case of Newtonian gravity. So these circular orbit will slightly closer to the center of Schwarschild solution R equals to zero.
   $r_c > 3R$ stable circular orbits only far from sufficiently large r.

2. Unstable circular orbits,

$$\frac{3}{2}R < r_{crit} < 3R \tag{88}$$

as L goes infinity $r_{crit} \approx \frac{L^2}{R}(1-\sqrt{1-\frac{3R^2}{L^2}}) \approx \frac{3}{2}R$ (Taylor expansion).

3. The orbits are not conic sections. The orbits will not close on themselves

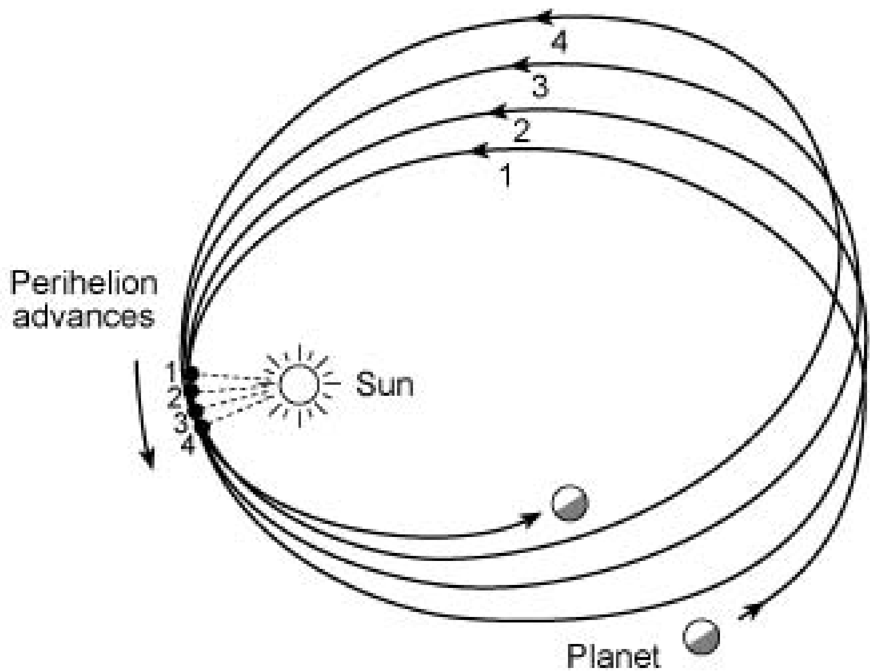

Figure 5: Perihelion

when $L < \sqrt{3}R$, there is no stable orbits.

4.2.1 The Precession of Perihelion

Our goal is to solve

$$\frac{1}{2}\dot{r}^2 + V_{eff} = \hat{E} \tag{89}$$

As we see, in general relativity orbits are not perfectly closed, with good approximation they are ellipses that flower pattern as shown in Figure(5)

First we should study $r(\phi)$,and we know $\dot{\phi} = \frac{L}{r^2}$, if we say

$$\frac{dr}{d\phi} = r' = \dot{r}\frac{r^2}{L^2} \tag{90}$$

So

$$\hat{E} = \frac{1}{2}\frac{L^2}{r^4}r'^2 - \frac{R}{2r} + \frac{L^2}{2r^2} - \frac{RL^2}{2r^3}$$
$$\frac{2r^2}{L^2}\hat{E} = \left(\frac{r'}{r}\right)^2 - \frac{Rr}{L^2} + 1 - \frac{R}{r}$$

We do simplification in here we call,

$$u = \frac{2L^2}{Rr} \tag{91}$$

and

$$\frac{r'}{r} = -\frac{u'}{u} \tag{92}$$

So we get

$$\frac{1}{u^2}\frac{8\hat{E}L^2}{R^2} = \frac{u'^2}{u'} - \frac{2}{u} + 1 - \frac{R^2}{2L^2}u \tag{93}$$

$$u' - 2u + u^2 - \frac{R^2}{2L^2}u^3 = constant \tag{94}$$

We take the derivative of this equation and diveded by 2

$$u'' - 1 + u - \frac{R^2}{2L^2}\frac{3}{2}u^2 = 0 \tag{95}$$

We organize this

$$u'' + u = 1 + \frac{3R^2}{4L^2}u^2 \tag{96}$$

Left hand side of this equation is simple harmonic oscillator, right hand side ise driving term. We can also split this equation as, $u'' + u = 1$ is Newtonian term, and $\frac{3R^2}{4L^2}u^2$ is the term which come from general relativity. We should solve this differential equation.

$$u = u_0 + u_1 + \ldots \tag{97}$$

u_0 is a solution to $u''+u = 1$. The solution for the zeroth-order equation is written as,

$$u_0 = 1 + e\cos\phi \tag{98}$$

This is result of Kepler solution, it describes a perfect ellipse with e the eccentricity. The eccentricity satisfies $e^2 = 1 - \frac{b^2}{a^2}$

And u_1

$$u_1'' + u_1 = \frac{3R^2}{4L^2} u_0^2 \tag{99}$$

simple harmonic oscillator with driving force. We use method of Green functions,

$$\begin{aligned} u_1(\phi) =& \frac{3R^2}{4L^2} \int d\phi \sin(\phi - \phi')(1 + e\cos\phi')^2 \\ \approx& \frac{3R^2}{4L^2} e\phi \sin\phi + terms\, periodic\, in\, \phi \end{aligned}$$

So we got u

$$\begin{aligned} u =& 1 + e\cos\phi + e\frac{3R^2}{4L^2}\phi. \sin\phi + \dots \\ \approx& 1 + e\cos(1-\alpha)\phi\, \alpha = \frac{3R^2}{4L^2} \end{aligned}$$

since $\cos(1-\alpha)\phi \approx \cos\phi + \alpha\phi\sin\phi + \dots$

So we see that, u is periodic but it is periodic that is slightly less than 2π. So the orbit is not closed. We pick up an angle

$$\Delta\phi = 2\pi\alpha \approx \frac{3\pi}{2}\frac{R^2}{L^2} = \frac{6\pi G^2 M^2}{L^2} \tag{100}$$

for per orbit.

It looks like not complete, we must write L more conventional form. We may use expressions valid for Newtonian orbits, since the quantity we are looking at is already a small perturbation. An ordinary ellipse satisfies,

$$r = \frac{(1-e^2)a}{1 + e\cos\phi} \tag{101}$$

here parameter a is distance from the massive object. So angular momentum is

$$L^2 \approx GM(1-e^2)a \tag{102}$$

W can write $\Delta\phi$ again,

$$\Delta\phi = \frac{6\pi GM}{c^2(1-e^2)a} \tag{103}$$

At the end of this section, we can finally calcute $\Delta\phi$ for Mercury. Parameters are for Sun and Mercury

$$\begin{aligned}\frac{GM_\odot}{c^2} =& 1.48 \times 10^5 cm\\ a =& 5.79 \times 10^{12} cm\\ e =& 0.2056\end{aligned}$$

So

$$\Delta\phi = 5.01 \times 10^{-7} radians/orbits = 0.103"/orbit \tag{104}$$

where " stands for arcsecond[3]. It is more conventional to express this in terms of precession per century. Mercury orbits once every 88 days,

$$\Delta\phi_{Mercury} = 43.0"/century. \tag{105}$$

[3] $arcsecond = \frac{1}{60} arcminute = \frac{1}{3600} degree$

5 Cosmology

5.1 Robertson-Walker Metric

"*To describe the real world, we are forced to give up the "perfect" Copernican principle, which implies symmetry throughout space and time, and postulate something more forgiving.* Sean Carrol[1]"

The Copernican principle which is the principle that the laws of physics are the same everywhere. That does not mean that the state of matter or the configuration of the universe is the same everywhere. We are interested in doing is looking at the universe one very large scales where we can approximate the matter and energy density of universe being a roughly constant. What we like to do is come up with a model for the evolution of the universe which treats every point in spacetime same. Then we need model for universe. There are two possibilities spacetime is

1. **Homogeneous:** For any two points in spacetime there is an isometry which connects them. Statemant that the universe is homogeneous is the statement roughly speaking that no point is different from any other point.

2. **Isotropic:** Statement of the universe is the same everywhere would be the statement that the universe is isotropic which is not the statement that every two points look the same bu rather that there is no prefed direction. More precisely that would be the statement that there would be an isometry connectiong any two vectors at each point.

These are two different notions. Universe is both homogeneous and isotropic. Geometry which is homogeneous and isotropic are symmetric. We know geometries which are homogeneous and isotropic which are Minkowski space, de Sitter and anti-de Sitter space. de Sitter space has positive curvature($R > 0$) and anti-de Sitter space is negative. In Minkowski space curvature is zero($R = 0$) respectively. And we know

$$R \begin{cases} = 0 & flat\,space,\,there\,is\,no\,curvature \\ < 0 & hyperbola \\ > 0 & sphere \end{cases}$$

Plane, sphere and hyperbola are Euclidean metric in the sense of a metric has only positive eigenvalues and Lorenzian metrics, slightly different geometries.

It is not true to say our universe appear to be homogeneous and isotropic. It must be sait as *"Our universe does appear to spatialy homogeneous and isotropic only on large scales"*.

At long distances

$$ds^2 = -dt^2 + a^2(t)d\sigma_K^2 \tag{106}$$

There are three possible Euclidean symmetric spaces, labeled by an index $K = -1,\, 0,\, +1$. $a(t)$ is a scale factor, $d\sigma_K$ is spatial part of the metric 3-dimensional symmetric space.

1. $K = 0$ flat space R^3, $d\sigma_0^2 = d\vec{x}^2$
2. $K = 1$ sphere S^3, $d\sigma_1^2 = d\vartheta^2 + \sin^2\vartheta d\Omega^2$(symmetric space with $R > 0$)
3. $K = -1$ hyperboloid H^3, $d\sigma_{-1}^2 = d\vartheta^2 + \sinh^2\vartheta d\Omega^2$

It is convenient a coordinate system where the metric on,

$$d\sigma_0^2 = dr^2 + r^2 d\Omega^2 \tag{107}$$

and we do $r = \sin\vartheta$ translation for sphere,

$$\begin{aligned} d\sigma_1^2 =& d\vartheta^2 + \sin^2\vartheta d\Omega^2 \\ =& \frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2 \end{aligned}$$

and same for hyperbola

$$\begin{aligned} d\sigma_{-1}^2 =& d\vartheta^2 + \sinh^2\vartheta d\Omega^2 \\ =& \frac{dr^2}{1 + r^2} + r^2 d\Omega^2 \end{aligned}$$

Now we can write generilize form

$$d\sigma_K^2 = \frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2 \tag{108}$$

$$K \begin{cases} = 0 & flat\, space,\, there\, is\, no\, curvature \\ = -1 & hyperbola \\ = +1 & sphere \end{cases}$$

These spaces are symmetric spaces. And spatial metric which is homogeneous and isotropic,

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2\right] \tag{109}$$

This known as "*Robertson-Walker metric*" and universe model which described by this metric is known as "*FRW Universe*" or *Friedmann-Lemaître-Robertson-Walker Universe*".

- $K = 1$: spatial slices are closed
- $K = -1$: spatial slices are open
- $K = 0$: spatial slices are flat(are also open)

This metric is an exact solution of Einstein's field equations of general relativity, it describes a homogeneous, isotropic and expanding universe. And it describes our universe at long distances.

In order to determine $a(t)$, we need to write equation of motion. We also need to decide on the matter contenct; Friedmann Model is with "perfect fluid". As we know energy-stress tensor for perfect fluid is

$$T_{\mu\nu} = (p + \rho)v_\mu v_\nu + pg_{\mu\nu} \tag{110}$$

becomes

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & & & \\ 0 & & g_{ij}p & \\ 0 & & & \end{pmatrix} \tag{111}$$

and $T^\mu_\nu = diag(-\rho, p, p, p,)$. Trace is given by

$$T = T^\mu_\mu = -\rho + 3p \tag{112}$$

For perfect fluid, four-velocity is $v^\mu = (1, 0, 0, 0)$, there is no special direction because of isotropy.

Before using Einstein equation to determine equation of motion and scale factor $a(t)$, we should consider conservation of energy

$$\begin{aligned} 0 =& \nabla_\mu T^\mu_0 \\ =& \partial_\mu T^\mu_0 + \Gamma^\mu_{\mu\lambda} T^\lambda_0 - \Gamma^\lambda_{\mu 0} T^\mu_\lambda \\ =& -\partial_0 \rho - 3\frac{\dot{a}}{a}(\rho + p) \\ \frac{\dot{\rho}}{\rho + p} =& -3\frac{\dot{a}}{a} \end{aligned}$$

While introducing energy-stress tensor for perfect fluid, as I said, ω is a constant which known as equation of state parameter.

1. $\omega = 0$, pressureless fluid, with any ρ, we call it as "dust"
2. $\omega = \frac{1}{3} \rightarrow p = \frac{1}{3}\rho$, gas of photon or other massles particle, "radiation"
3. $\omega = -1 \rightarrow p = -\rho$

We first look general statement, $p = \omega\rho$. We put this equation of conservation energy, it becomes

$$\frac{\dot{\rho}}{\rho}\frac{1}{1+\omega} = -3\frac{\dot{a}}{a} \tag{113}$$

We take the intergral

$$\rho = \rho_0 a^{-3(1+\omega)} \; or \; \rho \approx a^{-3(1+\omega)} \tag{114}$$

This tells us how the matter density evolves in time

- Dust: $\omega = 0 \; \rightarrow \; \rho = \rho_0 a^{-3}$
- Radiation: $\omega = \frac{1}{3} \; \rightarrow \; p = \rho_0 a^{-4}$
- Vacuum energy: $\omega = -1 \; \rightarrow \; \rho_0$, constant

In our universe, we don't just have dust, we don't just have radiation, we don't just have curvature to cosmological constant, we have all of these things.

5.2 Friedmann Equations

Friedman equations are dynamical equations of homogeneous and isotropic universe. It describes the expansion of the Universe. It maybe most important equations in cosmology.

Classical mechanics is a global theory which contains gravitational potential which diverges in a homogeneous and isotropic universe. Also Einstein had a problem with General Relativity. General Relativity is local theory as it uses differential geometry, instead of differential calculus.

In those days, static universe was the concept, also Einstein agreed. Eventhough he found dynamical equations that involved acceleration terms. And he did not accept it and add a constant term called the *cosmological constant*. After Friedman reached the conclusion, Einstein said that cosmological constant was the biggest mistake of my life. But cosmological constant is been defining as dark energy. We can write Einstein equations again with cosmological constant

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{115}$$

Robertson-Walker metric describes an isotropic universe, because it does not have crossed terms between time and space so there is not any privileged direction[7]. And it also describes homogeneous universe because of the spherical symmetry. We said before, universe is homogeneous and isotropic at large scales.

$$ds^2 = dt^2 - a^2(t)\left[\frac{1}{1-Kr^2}dr^2 + r^2 d\vartheta^2 + r^2\sin^2\vartheta d\phi^2\right] \tag{116}$$

K^{-2} is directly related to the curvature of the spatial hypersurface. If we take $a^2(t) = 1$

$$K^{-2} = \begin{cases} = 0 & euclidean \\ > 0 & close\, universe(volume\, integral\, converges) \\ < 0 & open\, universe(volume\, integral\, diverges) \end{cases}$$

Now we need to calculate Ricci tensor and Ricci scalar to seperate Einstein equations for a homogeneous and isotropic universe. Firs we calculate none-zero Christoffel symbols(30) of Robertson-Walker metric(116)

$$\begin{array}{rcc} \Gamma^t_{rr} = \dfrac{a\dot{a}}{1-Kr^2} & \Gamma^t_{\vartheta\vartheta} = r^2 a\dot{a} & \Gamma^t_{\phi\phi} = r^2 a\dot{a}\sin^2\vartheta \\ \Gamma^r_{tr} == \Gamma^\vartheta_{t\vartheta} = \Gamma^\phi_{t\phi} = \dfrac{\dot{a}}{a} & \Gamma^r_{rr} = \frac{r}{K(1-Kr^2)} & \Gamma^r_{\vartheta\vartheta} = -r(1-Kr^2) \\ \Gamma^r_{\phi\phi} = -r(1-Kr^2)\sin^2\vartheta & \Gamma^\vartheta_{r\vartheta} = \Gamma^\phi_{r\phi} = \frac{1}{r} & \Gamma^\vartheta_{\phi\phi} = -\sin\vartheta\cos\vartheta \\ \Gamma^\phi_{\vartheta\phi} = \dfrac{1}{\tan\vartheta} = \cot\vartheta & & \end{array}$$

As we did in Schwarzschild solution, we need Ricci Tensors.. We are only interested in Riemann tensor components that have the same top index as the middle bottom one.

$$\begin{aligned} R_{tt} = R^m_{tmt} &= R^r_{trt} + R^\vartheta_{t\vartheta t} + R^\phi_{t\phi t} = -3\frac{\ddot{a}}{a} \\ R_{rr} = R^m_{rmr} &= \frac{a\ddot{a}}{1-Kr^2} + \frac{2\dot{a}^2}{1-Kr^2} + \frac{2K}{1-Kr^2} \\ R_{\vartheta\vartheta} = R^m_{\vartheta m\vartheta} &= r^2(a\ddot{a} + 2\dot{a}^2 + 2K) \\ R_{\phi\phi} = R^m_{\phi m\phi} &= r^2(a\ddot{a} + 2\dot{a}^2 + 2K)\sin^2\vartheta \end{aligned}$$

If we look closely these Ricci tensors, we see, we can generalize,

$$R_{tt} = -3\frac{\ddot{a}}{a} \tag{117}$$

$$R_{ii} = \frac{-g_{ii}}{a^2}(a\ddot{a} + 2\dot{a}^2 + 2K) \tag{118}$$

Now we can get Ricci scalar

$$R = g^{ik}R_{ik} = -6\frac{\ddot{a}}{a} - 6\left(\frac{\dot{a}}{a}\right) - 6\frac{K}{a^2} \tag{119}$$

We can right right hand side of Einstein equation(59) by these components, we must think of right hand side of Einstein equation, I mean energy-momentum tensor for perfect fluid. As discuss later energy-stress tensor is $T^{\mu\nu} = -pg^{\mu\nu} + (p+\rho)v^{\mu}v^{\nu}$. v^{α} is macroscopic speed of medium, it has just time component $v^{\alpha} = (1,0,0,0)$, so $v^t = 1$. We should see T_{tt} and T_{ii}.

$$T_{tt} = \rho g_{tt} \tag{120}$$

$$T_{ii} = -pg_{ii} \tag{121}$$

We can think of universe being filled by a perfect fluid.

Finally we can write Friedmann equations, we calculate all components. We consider first time part

$$\begin{aligned} R_{tt} - \frac{1}{2}Rg_{tt} - \Lambda g_{tt} =& 8\pi G\rho v_t v_t \\ -3\frac{\ddot{a}}{a} + 3\frac{\ddot{a}}{a} + 3\left(\frac{\dot{a}}{a}\right)^2 + 3\frac{K}{a^2} - \Lambda =& 8\pi G\rho \end{aligned}$$

$$\left(\frac{\dot{a(t)}}{a(t)}\right)^2 = \frac{8\pi G}{3}\rho(t) + \frac{\Lambda}{3} - \frac{K}{a^2(t)} \tag{122}$$

Now we should work on spatial part.

$$\frac{g_{ii}}{a^2(t)}\left(a\ddot{a} + 2\dot{a}^2 + 2K\right) - \frac{1}{2}Rg_{ii} - \Lambda g_{ii} = 8\pi G(-p)g_{ii} \tag{123}$$

We can remove metric from both sides,

$$-\frac{\ddot{a}}{a} - 2\left(\frac{\dot{a}}{a}\right)^2 - \frac{2K}{a^2} + 3\frac{\ddot{a}}{a} + 3\left(\frac{\dot{a}}{a}\right)^2 + \frac{3K}{a^2} - \Lambda = -8\pi Gp \tag{124}$$

$$\frac{\ddot{a}(t)}{a(t)} + \frac{1}{2}\left(\frac{\dot{a}(t)}{a(t)}\right)^2 = -4\pi Gp + \frac{\Lambda}{2} - \frac{1}{2}\frac{K}{a^2(t)} \tag{125}$$

We should realize that if we make linear combination equation (122) and (125), we get an equation without term $\left(\frac{\dot{a}}{a}\right)$. I mean $2 \times (125) - (122)$. So we get

$$\frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} \tag{126}$$

These are **Friedmann Equations**, which describes dynamics of universe:

$$\left(\frac{\dot{a(t)}}{a(t)}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{K}{a^2(t)} \tag{127}$$

$$\frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} \tag{128}$$

First one is governing the expansion of universe. Second one known as "*Acceleration Equation*".If we want to write without cosmological constant

$$\left(\frac{\dot{a(t)}}{a(t)}\right)^2 = \frac{8\pi G}{3}\rho + -\frac{K}{a^2(t)}$$
$$\frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi G}{3}(\rho + 3p)$$

5.3 Evolution of the Universe

To understand how Universe might evolve, we need information of what it has in it. In cosmology, this is done by specifying the mass density ρ and pressure p . As we saw, this known as equation of state.

Matter: Matter is used in cosmology for non-relativistic matter and refers to any type of material which exerts negligible pressure, $p = 0$ A pressureless universe is the simplest assumption that can be made.

Radiation: Particles of light move, naturally enough, at speed of light[5]. Their kinetic energy leads to a pressure force, we can call it radiation pressure,

$$p = \frac{pc^2}{3} \tag{129}$$

More generally, any particles moving at highly-relativistic speeds have this equation ofstate, neutrinos being an obvious example.

5.3.1 Matter

We looked at equation of state for three conditions, dust, radiation and vacuum. And we see

$$\rho \propto \frac{1}{a^3} \tag{130}$$

for dust and we said $\rho = \rho_0 a^{-3}$. Dust is also known as matter, and universes whose energy density is mostly due to matter are known as **matter-dominated**, $\rho_M \propto a^{-3}$.

It means that density falls off in proportion to the volume of the Universe. It is natural.

If we solve Friedmann equation for $K = 0$, flat space, we see

$$a = \frac{8\dot{\pi} G \rho_0}{3} \frac{1}{a} \tag{131}$$

5.3.2 Radiation

A universe in which most of the energy density is in the form of radiation is known as **radiation-dominated**. The energy density falls off as

$$\rho_R \propto a^{-4} \tag{132}$$

The energy density in radiation falls off slightly faster than that in matter. Because the number density of photons decreases in the same way as the number density of non-relativistic particles, but photons alsı lose energy as a^{-1} as they redshift.

Novadays, it is believed that radiation density is much less than that of matter with

$$\frac{\rho_M}{\rho_R} \approx 10^3 \tag{133}$$

However, in the past of universe was much smaller, and the energy density in radiation would have dominated at very early times.

5.3.3 Vacuum

Since the energy density in matter and radiation decreases as the universe expands if there is a nonzero vacuum energy it tends to win out over the long term, as long as the universe does not start contracting. It this happens, we say that the universe becomes **vacuum-dominated**.

As we discussed,

$$\rho_\Lambda \propto a^0 \ or \ p_\Lambda = -\rho_\Lambda \tag{134}$$

5.4 Cosmological Parameters

5.4.1 The Expansion Rate H_0

These equations define Friedmann-Robertson-Walker (FRW) universes. The rate of expansions is characterized by the **Hubble paramter**,

$$H = \frac{\dot{a}}{a} \tag{135}$$

The Friedmann equation allows us to explain Hubble's discovery that velocity is proportional to the distance. The value of the Hubble parameter at he present epoch is the Hubble constant, H_0. Curren measurements lead us to belive that the Hubble constant is $70 \pm 10\, km/sec/Mpc$.[4]

It also ought to be the easiest to measure, since all galaxies are supposed to obey $\vec{v} = H_0 \vec{r}$. So all we have to do is measure the velocities and distances of as many galaxies as we can and get an answer.

We often parameterize the Hubble constatn as

$$H_0 = 100h\, km/sec/Mpc \tag{136}$$

$h \approx 0.7$. Typical cosmological scales are set by the **Hubble lenght**

$$\begin{aligned} d_H =& H_0^{-1} c \\ =& 9.25 \times 10^{27} h^{-1} cm \\ =& 3.00 \times 10^{3} h^{-1} Mpc \end{aligned}$$

and **Hubble time**

$$\begin{aligned} t_H =& H_0^{-1} \\ =& 3.09 \times 10^{17} h^{-1} sec \\ =& 9.78 \times 10^{9} h^{-1} yr \end{aligned}$$

I'd like remind, we use naturel units, light speed is $c = 1$.

We can expand Taylor series of scale factor $a(t)$,

$$a(t) = a(t_0) + \dot{a}[t - t_0] + \frac{1}{2}\ddot{a}(t_0)[t - t_0]^2 + \ldots \tag{137}$$

[4] Mpc stands for megaparsec, which is $3.09 \times 10^{24} cm$

If we divide through by $a(t_0)$

$$\frac{a(t)}{a(t_0)} = 1 + H_0[t - t_0] - \frac{q}{2}H_0^2[t - t_0]^2 + \dots \tag{138}$$

we defien q as **deceleration parameter**,

$$q = -\frac{a\ddot{a}}{\dot{a}^2} \tag{139}$$

5.4.2 The Density Parameter Ω

It is usefull way using density parameter(Ω_0) describe density of Universe. We have just said that $H = \dot{a}/a$. And we recall Friedmann equation by using Hubble parameter

$$H^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2} \tag{140}$$

For given value of H, there is a special value of the density. For flat space, **critical density** ρ_{crit} is

$$\rho_c = \frac{3H^2}{8\pi G} \tag{141}$$

We must recognize that critical density changes as long as H does. We calculate present critical density by unit changing, $G = 6.67 \times 10^{-11}\, m^3 kg^{-1} sec^{-2}$, but H is in unit megaparsec, megaparsex should be converted to meter. So present critical density is

$$\rho_c(t_0) = 1.88\, h^2 \times 10^{-26}\, kg\, m^{-3} \tag{142}$$

It is very small number ,if we compare any matter, like water($10^3 kg\, m^{-3}$). But is not, we can rewrite using masses in solar masses and distance in megaparsec and it becomes

$$\rho_{crit}(t_0) = 2.78\, h^{-1} \times\, 10^{11}\, M_\odot\, /(h^{-1} Mpc)^3 \tag{143}$$

Now it does not seem so small. In fact 10^{11} and 10^{12} solar masses is about the mass of a typical galaxies.

Another useful quantity is the **density parameter**,

$$\Omega = \frac{8\pi G}{3H^2}\rho = \frac{\rho}{\rho_c} \tag{144}$$

We can rearrange Friedmann equation and we can express it as Ω,

$$\Omega - 1 = \frac{K}{H^2 a^2} \tag{145}$$

The sign of K is therefore determined by whether Ω is greater thant or equal to, or less than

$$\begin{aligned}
\rho <&\rho_{crit} \iff \Omega < 1 \iff K < 0 \iff open \\
\rho =&\rho_{crit} \iff \Omega = 1 \iff K = 0 \iff flat \\
\rho >&\rho_{crit} \iff \Omega > 1 \iff K > 0 \iff closed
\end{aligned}$$

The density parameter tells us which of the three Robertson-Walker geometries describes our universe. Determining it observationally is of crucial importance. Recent measurements of the cosmic microwave background anisotropy lead us to believe that Ω is very close to unity.

We also link to disacceleration parameter

$$q = \frac{4\pi G}{3}\rho\frac{3}{8\pi G\rho_{crit}} = \frac{\Omega}{2} \tag{146}$$

6 Result and Discussion

In this work, general theor of reletivity is viewed from idea of the equivalence principle which leads to Einstein to discover this theory, to large scale of universe. And also whether gravitation is discussed geometry originated or not.

We started to look gravitayion as geometry. Metric is the most important term of geometry, we need to understant metric very well to understan gravity. Metric changes as gravity does. We derived geodesic equation(29), and discussed shortes lenght on the curvature and why straight line is not. Because spacetime is effectin from mass and it's structure is been changing by massive bodies. we have different geometry from Euclidean geometry.

In the third section, we looked curvature as Riemannian geometry to understant geometry. Partial derivative is not a good operator, we need connections to transport data, as an example we discussed affine connection. And we defined covariant derivative.

$$\nabla_\mu V^\rho = \partial_\mu V^\rho + \Gamma^\rho_{\mu\nu} V^\nu \tag{147}$$

Christoffel symbol has all information about first derivative of metric.

To parameterize of local curvature we defined Riemann curvature tensor. Riemann curvature tensor has all information about third derivative of metric. And it's trace is called Ricci tensor.Ricci tensor contains all of the trace information of Riemann tensor.

$$R_{\mu\nu} = R^\rho_{\mu\rho\nu} \tag{148}$$

And we defined Einstein tensor, and said Einstein tensor is conserved.

$$\nabla^\mu G_{\mu\nu} = 0 \tag{149}$$

When this part was finished, we need to write right hand side of Einstein's equation to able to write curvature. First we derived Einstein's equation absence of matter. If we say matter curves spacetime, curvature must be zore with absence of matter. After that, to obtain Einstein equation presence of matter, wee needed to define energy-momentum and energy stress tensor and we studied energy-momentum tensor for perfect fluid, because we assume spacetime is like a perfect fluid. We were ready to write field equations. To derive the equation, we always know that weak gravitational field Newtonian gravity works well. We used it to derive Einstein's equation, we did assumption and used conservation of Einstein tensor. Wen could not use other because energy momentum tensor is conserved. And finally we get right hand side of Einstein's equation too

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} \tag{150}$$

And we need to test this equation. We use Schwarzschild solution for test. It has rotational symmetries and source free which means there is no curvature, metric will be independent of time coordinate. Under this assumption we found out coefficient of metric

$$ds^2 = -\left(1 - \frac{R_s}{r}\right)dt^2 + \left(1 - \frac{R_s}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{151}$$

This is known as Schwarzschild metric or Schwarzschild geometry. Than we can use it our solar system, especially for Mecury that it's orbits can not be defined by Newtonian gravity, it was a good example. And use geodesic of Schwarzschild geometry and looked at differences between Newtonian gravity and General relativity. And we use it for to find the precession of perihelion of Mercury.

And we looked up our universe at large scales, and discussed possible geometries

$$R \begin{cases} = 0 & flat\,space,\,there\,is\,no\,curvature \\ < 0 & hyperbola \\ > 0 & sphere \end{cases}$$

and we generelize these possible geometries and find Robertson-Walker metric which describes geomety of universe.

$$ds^2 = dt^2 - a^2(t)\left[\frac{1}{1 - Kr^2}dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\phi^2\right] \tag{152}$$

$$K^{-2} = \begin{cases} = 0 & euclidean \\ > 0 & close\,universe(volume\,integral\,converges) \\ < 0 & open\,universe(volume\,integral\,diverges) \end{cases} \tag{153}$$

To find equation of motion we used Einstein equation and we find maybe the most important equations in cosmology, Friedmann equations. They are dynamical equations of homogeneous and isotropic universe.

$$\left(\frac{a(\dot{t})}{a(t)}\right)^2 = \frac{8\pi G}{3}\rho + -\frac{K}{a^2(t)} \tag{154}$$

$$\frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi G}{3}(\rho + 3p) \tag{155}$$

We use these equations to understan parameters of universe like expansion rate, density. Also energy density of universe was discussed as matter-dominated, radiation-dominated and vacuum-dominated.

A Differential Forms

A differantial form of rank n(an n-form) is a completely antismmetric $(0, n)$tensor.For example: f is $0-form$, ω_μ is $1-form$ and $F_{\mu\nu}$ is $2-form$. If we want to generalize this, we can say $A_{\mu_1\mu_2\ldots\mu_p}$ is a $p-form$.

We can use $p-form$ $A_{\mu_1\mu_2\ldots\mu_p}$ and $q-form$ $B_{\mu_1\mu_2\ldots\mu_q}$ to get $(p+q)-form$ tensor by **wedge product** $A \wedge B$. $\wedge$ is just generilization of cross product.

$$(A \wedge B)_{\mu_1\ldots\mu_{p+q}} = \frac{(p+q)!}{p!q!} A_{[\mu_1\ldots\mu_p} B_{\mu_{p+1}\ldots\mu_{p+q}]} \tag{156}$$

For example, wedge product of two $1-form$

$$(A \wedge B)_{\mu\nu} = 2A_{[\mu}B_{\nu]} = A_\mu B_\nu - A_\nu B_\nu \tag{157}$$

We can get $(p+1)-form$ tensor from $p-form$ by using **exterior derivative**. It is defined as an appropriately normalized, antisymmetric partial derivative:

$$(dA)_{\mu_1\ldots\mu_{p+1}} = (p+1)\partial_{[\mu_1} A_{\mu_2\ldots\mu_{p+1}]} \tag{158}$$

We can do an example for exterior derivative:

$$(d\omega)_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu \tag{159}$$

- Operator "d" on $0-form$, change scaler to vector: $\vec{\nabla} f$
- Operator "d" on $1-form$, change vector to vector: $\vec{\nabla} \times \vec{V}$
- Operator "d" on $2-form$, change vector to scalar: $\vec{\nabla} \cdot \vec{V}$

B Newtonian Derivation of Friedmann Equation

To derive the equation, we need to calculate gravitational potential energy, kinetic energy of test particle and then we use conservation of energy.

$$F = \frac{GMm}{r^2} \tag{160}$$

As we know $F = -\nabla V$. So that,

$$V = -\frac{GMm}{r} \tag{161}$$

We can say that total mass is $M = 4\pi r^3 \rho/3$. When put this to the (161), we get,

$$V = -\frac{4\pi G\rho r^2 m}{3} \tag{162}$$

and we can call energy conservation for total energy(U=T+V),

$$U = \frac{1}{2}m\dot{r}^2 - \frac{4\pi}{3}G\rho r^2 m \tag{163}$$

We will use **comoving coordinates**, because it is carrying along with expansion. Expansion is uniform, the relationship between real distance $\vec{r}$ and the comoving distance, which we can call $\vec{x}$, can be written

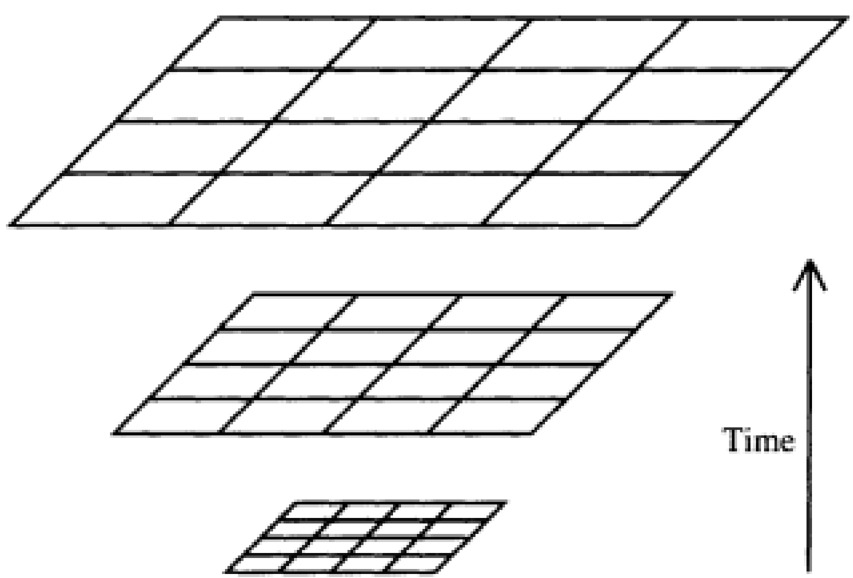

Figure 6: Comoving Coordinates

$$\vec{r} = a(t)\vec{x} \tag{164}$$

where the homogeneity property has been used to ensure that a is function of time alone

$$\dot{\vec{r}} = \dot{a}x \tag{165}$$

. We put these to the equation (163)

$$U = \frac{1}{2}m\dot{a}^2x^2 - \frac{4\pi}{3}G\rho a^2x^2m \tag{166}$$

Multiplying each side by $2/ma^2x^2$ and rearranging the terms gives

$$\frac{2U}{mx^2} = \dot{a}^2 - \frac{8\pi G}{3}a^2\rho \tag{167}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} \tag{168}$$

where $k = -2U/mc^2x^2$. The equation (168) is the standard form of **Friedmann Equation** .

Friedmann equation is of no use without an equation to describe how the density ρ of material in the Universe is evolving with time. This involves the pressure p of the material, and is called the fluid equation.

We can derive the fluid equation by considering the first law of thermodynamics[5]

$$dE + pdV = TdS \tag{169}$$

applied to an expanding volume V of unit *comoving radius.*

$$E = mc^2 \tag{170}$$

$$E = \frac{4\pi}{3}a^3\rho c^2 \tag{171}$$

The change of energy in a time dt

$$\frac{dE}{dt} = 4\pi a^2\rho c^2\frac{da}{dt} + \frac{4\pi}{3}a^3\frac{d\rho}{dt}c^2 = 0 \tag{172}$$

If we calculate chang in volume over $V = \frac{4\pi}{3}a^3x^3$

$$\frac{dV}{dt} = 4\pi a^2x^3\frac{da}{dt} \tag{173}$$

Assuming a reversible expansion $dS = 0$, now we can put the equation (172) and (173) into the equation (169) and rearranging

$$4\pi a^2 x^3 [\rho c^2 \dot{a} + \frac{1}{3} a c^2 \dot{\rho} + p\dot{a}] = 0 \tag{174}$$

$$\rho c^2 \dot{a} + \frac{1}{3} a c^2 \dot{\rho} + p\dot{a} = 0 \tag{175}$$

$$\dot{a}\left(\rho + \frac{p}{c^2}\right) + \frac{1}{3} a\dot{\rho} = 0 \tag{176}$$

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0 \tag{177}$$

This eqution is called **fluid equation**.

To derive *acceleration equation* we differentiate the equation (168) with respect to time, we obtain

$$2\frac{\dot{a}}{a}\frac{a\ddot{a} - \dot{a}^2}{a^2} = \frac{8\pi G}{3}\dot{\rho} + 2\frac{kc^2\dot{a}}{a^3} \tag{178}$$

Substituting in for $\dot{\rho}$ from equation (177) and cancelling the factor $a\dot{a}/a$ in each term gives

$$\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = -4\pi G\left(\rho + \frac{p}{c^2}\right) + \frac{kc^2}{a^2} \tag{179}$$

by using equation (168) again, we arrive at the **acceleration equation**

$$\frac{\ddot{a}}{a} = -\frac{3\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) \tag{180}$$

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