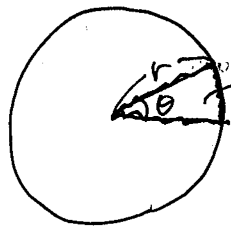
Polar curve

Recall: Find the area of a "sector" of a circle.



→ This shape is called "sector"

$$A = \frac{1}{2} r^2 \theta$$

15.4 Double Integrals in Polar coordinates

Recall: Given $z = f(x,y)$ over D → Rectangle $\{(x,y) \mid a \le x \le b, c \le y \le d\}$

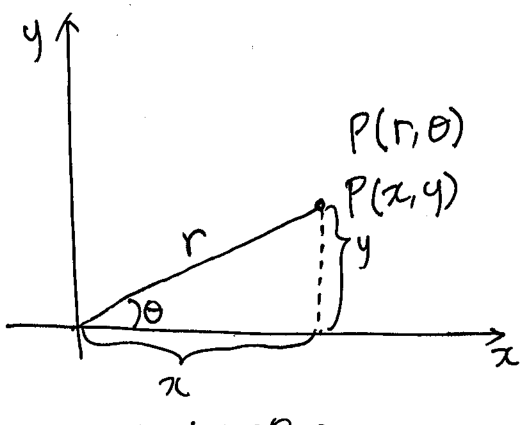
→ General → I, II

$$\iint_D f(x,y)\, \boxed{dA} \to dy\,dx \text{ (I)}, \quad dx\,dy \text{ (II)}$$

New Method: Convert to Polar coordinates

$$z = f(x,y) = f(r\cos\theta, r\sin\theta)$$

(polar curve → textbook chapter... maybe 10)



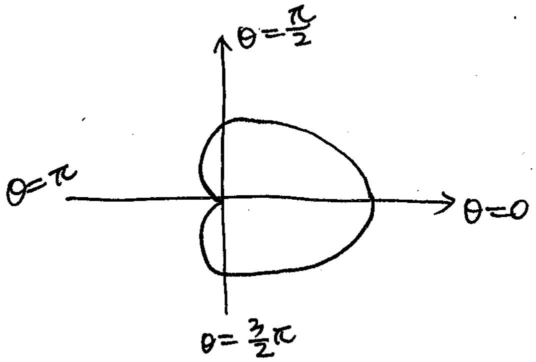
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ x^2 + y^2 = r^2 \end{cases}$$

"cardroid"

$r = 1 + \cos\theta$

r	2	1	0	1	2
θ	0	$\frac{\pi}{2}$	π	$\frac{3}{2}\pi$	2π

(Learn how to graph polar curves with a graphing calculator.)

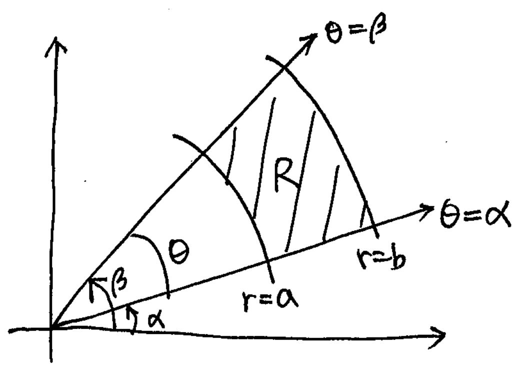


$$\left\{ \begin{array}{l} r = \cos^2 5\theta + \sin 3\theta + 0.3 \\ \quad [-2.5, 2.5] \times [-2, 1.5] \end{array} \right\}$$

try it graphing with a calculator just for fun

Polar Rectangle

$$R = \{ (r, \theta) \mid a \le r \le b,\ \alpha \le \theta \le \beta \}$$



$\Delta\theta = \beta - \alpha$

Find area of R

$$A(R) = \frac{1}{2}(b)^2(\Delta\theta) - \frac{1}{2}(a)^2(\Delta\theta)$$

$$= \frac{1}{2}(b^2 - a^2)\Delta\theta$$

$$= \frac{1}{2}(b+a)(b-a)\Delta\theta$$

$$= \frac{b+a}{2}(b-a)\Delta\theta$$

$$= \underbrace{\bar{r}}_{\text{average of "r"}} \Delta r \underbrace{\Delta\theta}_{\text{central angle}}$$

Find $\iint_R f(x,y)\,dA$ where $R =$ polar rectangle
$= \{(r,\theta) \mid a \le r \le b,\ \alpha \le \theta \le \beta\}$

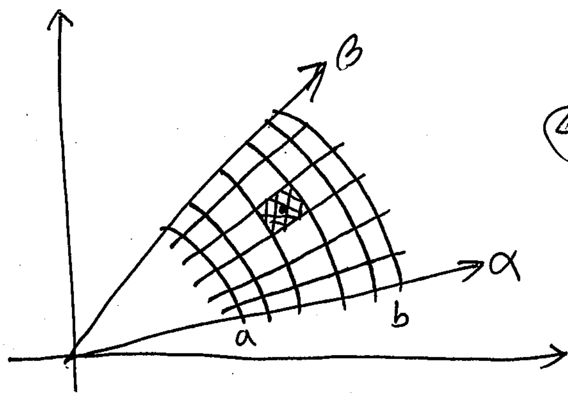
① Partition $[a, b]$ into m subintervals of width $\Delta r = \frac{b-a}{m}$

② Partition $[\alpha, \beta]$ into n subintervals of width $\Delta\theta = \frac{\beta-\alpha}{n}$

∴) we have mn sub-polar rectangles R_{ij}

③ Pick any point (r_i^*, θ_j^*) in each rectangle. (i.e. pick center)

∴) $(x,y) = (r_i^* \cos\theta_j^*,\ r_i^* \sin\theta_j^*)$



④ double Riemann Sum

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i^*, y_j^*)\,\Delta A$$

$\Delta x \Delta y$ or $\Delta y \Delta x$

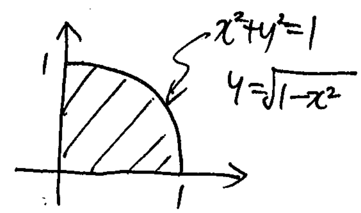
$r_i^* = \bar{r}_i$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*,\ r_i^* \sin\theta_j^*)\, r_i^* \Delta r \Delta\theta$$

$$∴)\ \iint_R f(x,y)\,dA = \int_\alpha^\beta\int_a^b f(r\cos\theta,\ r\sin\theta)\,\overbrace{r\,dr\,d\theta}^{dA}$$

ex) Evaluate $\iint_D \cos(x^2+y^2)\,dA$

when $D =$ bounded by $x=0,\ y=0,\ x^2+y^2=1$



(a) set up Type I rectangular Integral

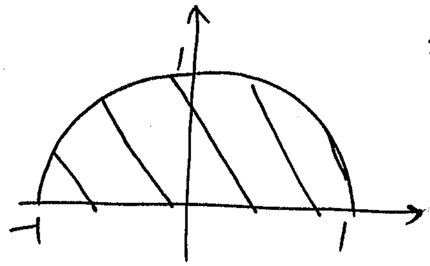
$$\int_0^1\int_0^{\sqrt{1-x^2}} \cos(x^2+y^2)\,dy\,dx$$

$R: \{(r,\theta) \mid 0 \le r \le 1,\ 0 \le \theta \le \frac{\pi}{2}\}$

(b) convert to polar!

$$\int_0^{\frac{\pi}{2}}\int_0^1 \cos(r^2)\, r\,dr\,d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin(r^2)\Big|_0^1\,d\theta$$

ex) Evaluate $\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (x^2+y^2)^{\frac{3}{2}}\,dy\,dx$



$$= \int_0^{\pi}\int_0^{1} (r^2)^{\frac{3}{2}}\, r\,dr\,d\theta$$

$$= \int_0^{\pi}\int_0^{1} r^4\,dr\,d\theta$$

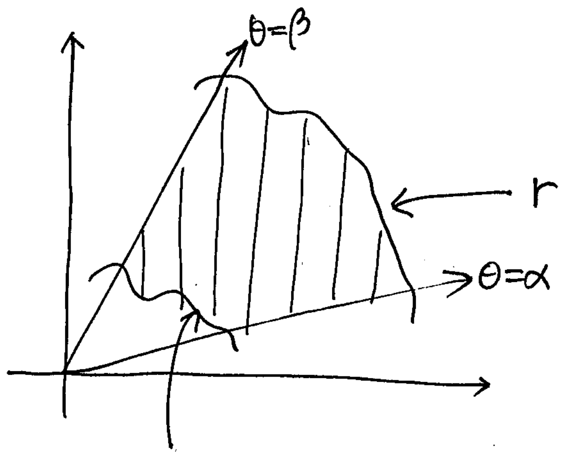
$$= \int_0^{\pi} \frac{1}{5}r^5\Big|_0^1\,d\theta = \int_0^{\pi}\frac{1}{5}\,d\theta$$

$$= \frac{\pi}{5}$$

General Polar Region

[polar function $r=f(\theta)$]

$$D=\{(r,\theta)\,|\,\alpha\le\theta\le\beta,\quad h_1(\theta)\le r\le h_2(\theta)\}$$

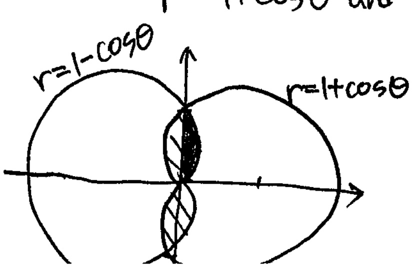


$$\int_{\alpha}^{\beta}\int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta,\, r\sin\theta)\, r\,dr\,d\theta$$

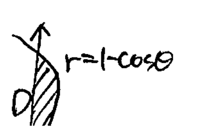
(Type I & II not interchangeable)

P.1002 Use a double integral to

(16) Find the area of region enclosed by both of the cardioids $r=1+\cos\theta$ and $r=1-\cos\theta$



$$A = 4(1^{st}\text{ quadrant}) = 4\iint_D dA$$



$$=4\int_0^{\frac{\pi}{2}}\int_0^{1-\cos\theta} r\,dr\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} (1-\cos\theta)^2 \, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} (1-2\cos\theta+\cos^2\theta) \, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \left(1-2\cos\theta+\tfrac{1}{2}[1+\cos 2\theta]\right) d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \left(\tfrac{3}{2}-2\cos\theta+\tfrac{1}{2}\cos 2\theta\right) d\theta$$

$$= 2\left[\tfrac{3}{2}\theta - 2\sin\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= 2\left[\left(\tfrac{3}{4}\pi - 2 + 0\right) - (0)\right]$$

$$= \boxed{\frac{3\pi}{2} - 4}$$

(22) Use polar coordinates to find volume of solid inside the sphere $x^2+y^2+z^2=16$ and outside the cylinder $x^2+y^2=4$

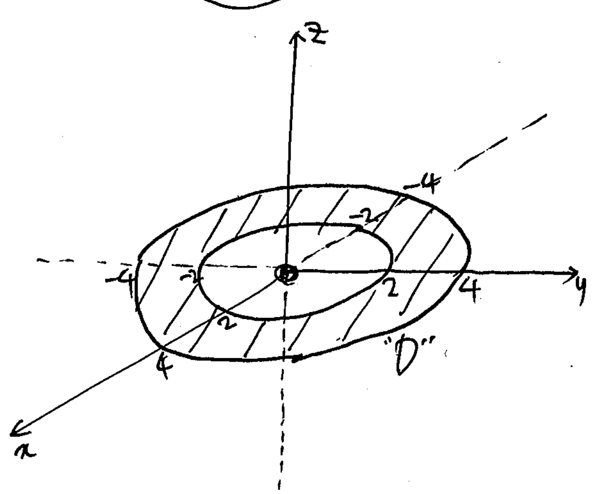
$z = \pm\sqrt{16-x^2-y^2}$

top

bottom

Recall: If $z=f(x,y)\geq 0$ over D, then $V=\iint_D f(x,y)\,dA$

$$= \iint_D z \, dA$$



$$V = 2\iint_D \sqrt{16-x^2-y^2}\, dA$$

$$= \boxed{2}\int_0^{2\pi}\int_0^4 \sqrt{16-r^2}\,\boxed{}\, r\,dr\,d\theta$$

$$= (-)\int_0^{2\pi}\int_0^4 \sqrt{16-r^2}\,(-2r)\,dr\,d\theta$$

$$= -\int_0^{2\pi} \frac{2}{3}(16-r^2)^{\frac{3}{2}}\Big|_2^4 \, d\theta$$

$$= \frac{2}{3} \int_0^{2\pi} (12)^{\frac{3}{2}} \, d\theta$$

$$= \frac{2}{3} \cdot (12)^{\frac{3}{2}} \cdot \int_0^{2\pi} d\theta$$

$$= \frac{2}{3} \cdot (12\sqrt{2}) \cdot (2\pi)$$

$$= \boxed{32\sqrt{3}\,\pi}$$