Background: Solving Linear Systems of Equations

Topic 2.5

Week 2 – Jan. 16th, 2019

Representing a Linear Eqn. in Matrix Form

- Suppose we have N unknowns $x_1, \ldots, x_N$, and an eqn. of the form:

$$a_1 x_1 + a_2 x_2 + \ \ldots + a_N x_N = b$$

(where $a_1, \ldots, a_N$ and b are known quantities)

- Represent coefficients as a **row vector** (1xN matrix): $[a_1 \ a_2 \ \ \ldots a_N]$

- Represent unknowns as a **column vector** (Nx1 matrix): $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$

Representing a Linear Eqn. in Matrix Form

- We can now represent the equation,

$$a_1 x_1 + a_2 x_2 + \ \ldots + a_N x_N = b$$

- In matrix form,

$$[a_1\ a_2\ \ \ldots a_N] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = b$$

$$\Updownarrow$$

- Recall, matrix multiplication ➔ $\sum_{j=1}^{N} a_j x_j = b$

Systems of Linear Equations

- Multiple equations
  - The variables are the same for all M equations, still use single column vector
  - Coefficients are different, must use different row vector for each of M equations

$$\begin{matrix} [a_{11}\ a_{12}\ \dots a_{1N}] \\ [a_{21}\ a_{22}\ \dots a_{2N}] \\ \vdots \\ [a_{M1}\ a_{M2}\ \dots a_{MN}] \end{matrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{matrix} b_1 \\ b_2 \\ \vdots \\ b_M \end{matrix}$$

$$\Updownarrow$$

$$\sum_{j=1}^{N} a_{ij} x_j = b_i \quad (i = 1, \dots, M)$$

Systems of Linear Equations in Matrix Form

- If we also represent b_i as the rows of a column vector,

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ & \vdots & & \\ a_{M1} & a_{M2} & \dots & a_{MN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_M \end{bmatrix}$$

$$\Updownarrow$$

$$\sum_{j=1}^{N} a_{ij} x_j = b_i \quad (i = 1, \dots, M)$$

$$\Updownarrow$$

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

Solving Systems of Linear Equations

- Now we have a nice matrix equation,
$$\boldsymbol{Ax} = \boldsymbol{b}$$
- We want to solve for the unknowns, so we can isolate $\boldsymbol{x}$:
$$\boldsymbol{x} = \boldsymbol{bA}^{-1}$$
- Where $\boldsymbol{A}^{-1}$ is the **matrix inverse** of $\boldsymbol{A}$
- Not all matrices $\boldsymbol{A}$ will have an inverse
  - Our system of equations must not be underdetermined or overdetermined, i.e. # equations must be same as # unknowns, $\boldsymbol{A}$ must be square (N = M)
  - Singular matrices ($\det(\boldsymbol{A}) = 0$) have no inverse

Next
Topic 3: HDR