1. 证明下面这个双自旋态是纠缠态。

$|\Phi\rangle = \frac{3}{5}|uu\rangle + i\frac{4}{5}|dd\rangle$ (1).

证明：假设$|\Phi\rangle$是一个直积态，比较直积态$|\psi_{12}\rangle = a_1a_2|uu\rangle + a_1b_2|ud\rangle + b_1a_2|du\rangle + b_1b_2|dd\rangle$和(1)中的系数，有

$$a_1a_2 = \frac{3}{5}, b_1b_2 = i\frac{4}{5}, \quad a_1b_2 = b_1a_2 = 0$$

从前两个等式可推出$a_1a_2b_1b_2 = \frac{12}{25}i$，而从后两个等式可推出$a_1a_2b_1b_2 = 0$。相互矛盾，$|\Phi\rangle$不是直积态，而是纠缠态。

2. 针对上面这个纠缠态$|\Phi\rangle$，计算$\langle\Phi|\hat{\sigma}_y|\Phi\rangle$和$\langle\Phi|\hat{\sigma}_y \otimes \hat{\tau}_x|\Phi\rangle$。

解：$\langle\Phi|\hat{\sigma}_y|\Phi\rangle = \left(\frac{3}{5}\langle uu| - i\frac{4}{5}\langle dd|\right)\hat{\sigma}_y\left(\frac{3}{5}|u\rangle \otimes |u\rangle + i\frac{4}{5}|d\rangle \otimes |d\rangle\right)$

$= \left(\frac{3}{5}\langle uu| - i\frac{4}{5}\langle dd|\right)\left(\frac{3}{5}\left(\hat{\sigma}_y|u\rangle\right) \otimes |u\rangle + i\frac{4}{5}\left(\hat{\sigma}_y|d\rangle\right) \otimes |d\rangle\right).$

$= \left(\frac{3}{5}\langle uu| - i\frac{4}{5}\langle dd|\right)(\frac{3}{5}\mathrm{i}|du\rangle + \frac{4}{5}|ud\rangle) = 0.$

$\langle\Phi|\hat{\sigma}_y \otimes \hat{\tau}_x|\Phi\rangle = \left(\frac{3}{5}\langle uu| - i\frac{4}{5}\langle dd|\right)\hat{\sigma}_y \otimes \hat{\tau}_x\left(\frac{3}{5}|u\rangle \otimes |u\rangle + i\frac{4}{5}|d\rangle \otimes |d\rangle\right).$

$= \left(\frac{3}{5}\langle uu| - i\frac{4}{5}\langle dd|\right)\left(\frac{3}{5}\left(\hat{\sigma}_y|u\rangle\right) \otimes (\hat{\tau}_x|u\rangle) + i\frac{4}{5}\left(\hat{\sigma}_y|d\rangle\right) \otimes (\hat{\tau}_x|d\rangle)\right).$

$= \left(\frac{3}{5}\langle uu| - i\frac{4}{5}\langle dd|\right)\left(\frac{3}{5}i|d\rangle \otimes |d\rangle + i\frac{4}{5}(-i|u\rangle) \otimes |u\rangle\right).$

$= \left(\frac{3}{5}\langle uu| - i\frac{4}{5}\langle dd|\right)\left(\frac{3}{5}i|dd\rangle + \frac{4}{5}|uu\rangle\right) = \frac{12}{25} + \frac{12}{25} = \frac{24}{25}.$

3. 验证$|\mathrm{S}\rangle = -\frac{e^{-i\varphi}}{\sqrt{2}}(|n_+n_-\rangle - |n_-n_+\rangle)$，　($|\mathrm{S}\rangle = \frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle)$)

即需证明 $-\frac{e^{-i\varphi}}{\sqrt{2}}(|n_+n_-\rangle - |n_-n_+\rangle) = \frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle)$。

$|n_+\rangle = \begin{pmatrix} cos\frac{\theta}{2} \\ e^{i\varphi}sin\frac{\theta}{2} \end{pmatrix}$，$|n_-\rangle = \begin{pmatrix} sin\frac{\theta}{2} \\ -e^{i\varphi}cos\frac{\theta}{2} \end{pmatrix}$

左边$= -\frac{e^{-i\varphi}}{\sqrt{2}}\left(\left(cos\frac{\theta}{2}|u\rangle + e^{i\varphi}sin\frac{\theta}{2}|d\rangle\right) \otimes \left(sin\frac{\theta}{2}|u\rangle - e^{i\varphi}cos\frac{\theta}{2}|d\rangle\right) - \left(sin\frac{\theta}{2}|u\rangle - e^{i\varphi}cos \frac{\theta}{2}|d\rangle\right) \otimes \left(\cos\frac{\theta}{2}|u\rangle + e^{i\varphi}sin\frac{\theta}{2}|d\rangle\right)\right).$

$= -\frac{e^{-i\varphi}}{\sqrt{2}}\left(cos\frac{\theta}{2}sin\frac{\theta}{2}|uu\rangle - e^{i\varphi}cos^2\frac{\theta}{2}|ud\rangle + e^{i\varphi}sin^2\frac{\theta}{2}|du\rangle - e^{2i\varphi}cos\frac{\theta}{2}sin\frac{\theta}{2}|dd\rangle - \cos\frac{\theta}{2}sin \frac{\theta}{2}|uu\rangle - e^{i\varphi}sin^2\frac{\theta}{2}|ud\rangle + e^{i\varphi}cos^2\frac{\theta}{2}|du\rangle + e^{2i\varphi}cos\frac{\theta}{2}sin\frac{\theta}{2}|dd\rangle\right).$

$= -\frac{e^{-i\varphi}}{\sqrt{2}}(-e^{i\varphi}\left(sin^2\frac{\theta}{2} + cos^2\frac{\theta}{2}\right)|ud\rangle + e^{i\varphi}\left(sin^2\frac{\theta}{2} + cos^2\frac{\theta}{2}\right)|du\rangle).$

$= -\frac{e^{-i\varphi}}{\sqrt{2}}(e^{i\varphi}|du\rangle - e^{i\varphi}|ud\rangle) = \frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle).$

由于左边=右边，等式成立。

4. 对于双自旋态$|S_3\rangle$，选择处于xz平面内的两个方向$\overrightarrow{e_1}=\left(sin\frac{\Pi}{6},0,cos\frac{\Pi}{6}\right)$，$\overrightarrow{e_2}=\left(sin\frac{\Pi}{4},0,cos\frac{\Pi}{4}\right)$。计算自旋1沿$\overrightarrow{e_1}$正方向同时自旋2沿$\overrightarrow{e_2}$负方向的概率$p(\mathrm{e}_1^+,\mathrm{e}_2^-)$。

$\left|\mathrm{e}_1^+\right\rangle=cos\frac{\Pi}{12}|u\rangle+sin\frac{\Pi}{12}|d\rangle$，$\left|\mathrm{e}_2^-\right\rangle=sin\frac{\Pi}{8}|u\rangle-cos\frac{\Pi}{8}|d\rangle$

$\left|\mathrm{e}_1^+\mathrm{e}_2^-\right\rangle=\left(cos\frac{\Pi}{12}|u\rangle+sin\frac{\Pi}{12}|d\rangle\right)\otimes\left(sin\frac{\Pi}{8}|u\rangle-cos\frac{\Pi}{8}|d\rangle\right).$

$=cos\frac{\Pi}{12}sin\frac{\Pi}{8}\left|uu\right\rangle-cos\frac{\Pi}{12}cos\frac{\Pi}{8}\left|ud\right\rangle+sin\frac{\Pi}{12}sin\frac{\Pi}{8}\left|du\right\rangle-sin\frac{\Pi}{12}cos\frac{\Pi}{8}\left|dd\right\rangle.$

$p\left(\mathrm{e}_1^+,\mathrm{e}_2^-\right).$

$=|(cos\frac{\Pi}{12}sin\frac{\Pi}{8}\langle uu|-cos\frac{\Pi}{12}cos\frac{\Pi}{8}\langle ud|+sin\frac{\Pi}{12}sin\frac{\Pi}{8}\langle du|-sin\frac{\Pi}{12}cos\frac{\Pi}{8}\langle dd|)|S_3\rangle|^2.$

$=\frac{1}{2}\left|-cos\frac{\Pi}{12}cos\frac{\Pi}{8}\langle ud|ud\rangle-sin\frac{\Pi}{12}sin\frac{\Pi}{8}\langle du|du\rangle\right|^2$。

$=\frac{1}{2}(cos\frac{\Pi}{12}cos\frac{\Pi}{8}+sin\frac{\Pi}{12}sin\frac{\Pi}{8})^2=\frac{1}{2}cos^2\frac{\Pi}{24}.$

5. 巧克力版贝尔不等式：总共24块巧克力，正好12块是黑色的，12块酒心的，12块圆形的，请按如下的规则成对放入下面的12个长方盒中：同一长方盒子中的两块巧克力不能都是黑色的，不能都是酒心的，不能都是圆形的。小娟拿走了上面的12块巧克力，请数一下小娟手中下面三种巧克力的个数：（1）黑色但不是酒心M_1.；（2）酒心但不是圆形M_2.；（3）黑色但不是圆形M_3.。看看是否满足$\mathrm{M}_1+\mathrm{M}_2\geq\mathrm{M}_3$.

黑色	O	O	O	O	O	O	O	O	O	O	O	O
酒心	X	O	X	O	X	O	X	O	X	O	X	O
圆形	O	X	O	X	O	X	O	X	O	X	O	X
黑色	X	X	X	X	X	X	X	X	X	X	X	X
酒心	O	X	O	X	O	X	O	X	O	X	O	X
圆形	X	O	X	O	X	O	X	O	X	O	X	O

$\mathrm{M}_1=6$，$\mathrm{M}_2=6$，$\mathrm{M}_3=6$

满足$\mathrm{M}_1+\mathrm{M}_2\geq\mathrm{M}_3$.