第九次作业参考答案

1. 假设$|\Phi\rangle$是直积态,则可设

$$
\begin{aligned}
|\Phi\rangle &= \Big(a_1|u\rangle + b_1|d\rangle\Big)\otimes\Big(a_2|u\rangle + b_2|d\rangle\Big)\\
&= a_1a_2|u\rangle\otimes|u\rangle + a_1b_2|u\rangle\otimes|d\rangle + b_1a_2|d\rangle\otimes|u\rangle + b_1b_2|d\rangle\otimes|d\rangle.
\end{aligned}
$$

又由题意,有$|\Phi\rangle = \frac{3}{5}|u\rangle\otimes|u\rangle + \frac{4}{5}|d\rangle\otimes|d\rangle$,因此有

$$a_1a_2 = \frac{3}{5}, \tag{1}$$

$$b_1a_2 = 0, \tag{2}$$

$$a_1b_2 = 0, \tag{3}$$

$$b_1b_2 = \frac{4}{5}i. \tag{4}$$

(1)式乘(4)式得到$a_1a_2b_1b_2 = \frac{12}{25}i$,而(2)式乘(3)式得到$a_1a_2b_1b_2 = 0$,矛盾.因此$|\Phi\rangle$是纠缠态.

2. 我们有

$$
\begin{aligned}
\hat{\sigma}_y|\Phi\rangle &= \frac{3}{5}\left(\hat{\sigma}_y|u\rangle\right)\otimes|u\rangle + \frac{4}{5}i\left(\hat{\sigma}_y|d\rangle\right)\otimes|d\rangle\\
&= \frac{3}{5}i|d\rangle\otimes|u\rangle + \frac{4}{5}|u\rangle\otimes|d\rangle,
\end{aligned}
$$

因此

$$\langle\Phi|\hat{\sigma}_y|\Phi\rangle = \Big(\frac{3}{5}\langle uu| + \frac{4}{5}i\langle dd|\Big)\Big(\frac{3}{5}i|du\rangle + \frac{4}{5}|ud\rangle\Big) = 0.$$

又有

$$
\begin{aligned}
\hat{\sigma}_y\otimes\hat{\tau}_x|\Phi\rangle &= \frac{3}{5}(\hat{\sigma}_y|u\rangle)\otimes(\hat{\tau}_x|u\rangle) + \frac{4}{5}i(\hat{\sigma}_y|d\rangle)\otimes(\hat{\tau}_x|d\rangle)\\
&= \frac{3}{5}i|d\rangle\otimes|d\rangle + \frac{4}{5}|u\rangle\otimes|u\rangle,
\end{aligned}
$$

因此

$$\langle\Phi|\hat{\sigma}_y\otimes\hat{\tau}_x|\Phi\rangle = \Big(\frac{3}{5}\langle uu| + \frac{4}{5}i\langle dd|\Big)\Big(\frac{3}{5}i|dd\rangle + \frac{4}{5}|uu\rangle\Big) = \frac{3}{5}\cdot\frac{4}{5} + \frac{4}{5}\cdot\frac{3}{5} = \frac{24}{25}.$$

3. 由公式

$$
\begin{aligned}
|n_+\rangle &= \cos\frac{\theta}{2}|u\rangle + e^{i\varphi}\sin\frac{\theta}{2}|d\rangle\,, \\
|n_-\rangle &= \sin\frac{\theta}{2}|u\rangle - e^{i\varphi}\cos\frac{\theta}{2}|d\rangle
\end{aligned}
$$

可得

$$
\begin{aligned}
&-\frac{e^{-i\varphi}}{\sqrt{2}}|n_+n_-\rangle + \frac{e^{-i\varphi}}{\sqrt{2}}|n_-n_+\rangle \\
=&-\frac{e^{-i\varphi}}{\sqrt{2}}\Big(\cos\frac{\theta}{2}|u\rangle + e^{i\varphi}\sin\frac{\theta}{2}|d\rangle\Big)\otimes\Big(\sin\frac{\theta}{2}|u\rangle - e^{i\varphi}\cos\frac{\theta}{2}|d\rangle\Big) \\
&+\frac{e^{-i\varphi}}{\sqrt{2}}\Big(\sin\frac{\theta}{2}|u\rangle - e^{i\varphi}\cos\frac{\theta}{2}|d\rangle\Big)\otimes\Big(\cos\frac{\theta}{2}|u\rangle + e^{i\varphi}\sin\frac{\theta}{2}|d\rangle\Big) \\
=&\Big(-\frac{e^{-i\varphi}}{\sqrt{2}}\cos\frac{\theta}{2}\sin\frac{\theta}{2} + \frac{e^{-i\varphi}}{\sqrt{2}}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\Big)|uu\rangle \\
&+\Big(-\frac{1}{\sqrt{2}}\sin\frac{\theta}{2}\sin\frac{\theta}{2} - \frac{1}{\sqrt{2}}\cos\frac{\theta}{2}\cos\frac{\theta}{2}\Big)|du\rangle \\
&+\Big(\frac{1}{\sqrt{2}}\cos\frac{\theta}{2}\cos\frac{\theta}{2} + \frac{1}{\sqrt{2}}\sin\frac{\theta}{2}\sin\frac{\theta}{2}\Big)|du\rangle \\
&+\Big(\frac{e^{i\varphi}}{\sqrt{2}}\sin\frac{\theta}{2}\cos\frac{\theta}{2} - \frac{e^{-i\varphi}}{\sqrt{2}}\cos\frac{\theta}{2}\sin\frac{\theta}{2}\Big)|dd\rangle \\
=&\frac{1}{\sqrt{2}}\Big(|ud\rangle - |du\rangle\Big) = |S\rangle\,.
\end{aligned}
$$

4. 算子$\vec{e_1}\cdot\hat{\sigma}$的本征值+1对应的本征态是

$$
\left|e_1^+\right\rangle = \cos\frac{\pi}{12}|u\rangle + \sin\frac{\pi}{12}|d\rangle\,,
$$

算子$\vec{e_2}\cdot\hat{\sigma}$的本征值$-1$对应的本征态是

$$
\left|e_2^-\right\rangle = \sin\frac{\pi}{8}|u\rangle - \cos\frac{\pi}{8}|d\rangle\,.
$$

它们的直积态为

$$
\left|e_1^+e_2^-\right\rangle = \Big(\cos\frac{\pi}{12}|u\rangle + \sin\frac{\pi}{12}|d\rangle\Big)\otimes\Big(\sin\frac{\pi}{8}|u\rangle - \cos\frac{\pi}{8}|d\rangle\Big),
$$

从而

$$
\begin{aligned}
\langle e_1^+ e_2^- | S_3 \rangle &= \langle e_1^+ e_2^- | \left(\frac{1}{\sqrt{2}} |ud\rangle + \frac{1}{\sqrt{2}} |du\rangle \right) \\
&= \frac{1}{\sqrt{2}} \left(\sin\frac{\pi}{12} \sin\frac{\pi}{8} - \cos\frac{\pi}{12} \cos\frac{\pi}{8} \right) \\
&= -\frac{1}{\sqrt{2}} \cos\left(\frac{\pi}{8} + \frac{\pi}{12} \right) = -\frac{1}{\sqrt{2}} \cos\frac{5\pi}{24}.
\end{aligned}
$$

因此测得自旋1沿$\vec{e_1}$正方向,自旋2沿$\vec{e_2}$负方向的概率为

$$
p\left(e_1^+, e_2^-\right) = \left|\langle e_1^+ e_2^- | S \rangle\right|^2 = \frac{1}{2} \cos^2 \frac{5\pi}{24}
$$

5. 如下表,每一格上的0/1串表示该格中的巧克力的性质,第1位是1代表是黑色,第2位是1代表是酒心,第3位是1代表是圆形.

001	001	110	001	011	011	010	110	110	110	001	101
110	110	001	110	100	100	101	001	001	001	110	010

统计得,$M_1 = 1$,$M_2 = 5$,$M_3 = 4$.因此$M_1 + M_2 \geq M_3$成立.