1. (10分）在二维希尔伯特空间里有两个向量

$$|\psi_1\rangle = \frac{1}{2}\begin{pmatrix} 1+i \\ 1-i \end{pmatrix}, \qquad |\psi_2\rangle = \begin{pmatrix} \frac{8}{17}i \\ \frac{15}{17} \end{pmatrix}$$

计算 $\langle\psi_1|\psi_2\rangle$和$\langle\psi_2|\psi_1\rangle$.

$$\langle\psi_1|\psi_2\rangle = \begin{pmatrix} \frac{1-i}{2} & \frac{1+i}{2} \end{pmatrix}\begin{pmatrix} \frac{8i}{17} \\ \frac{15}{17} \end{pmatrix} = \frac{23}{34} + \frac{23}{34}i$$

$$\langle\psi_2|\psi_1\rangle = \begin{pmatrix} \frac{8i}{17} & \frac{15}{17} \end{pmatrix}\begin{pmatrix} \frac{1+i}{2} \\ \frac{1-i}{2} \end{pmatrix} = \frac{23}{34} - \frac{23}{34}i$$

2. (30分）在二维希尔伯特空间里定义两个向量

$$|\bar{e}_1\rangle = \frac{3}{5}|e_1\rangle + \frac{4}{5}|e_2\rangle) = \frac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix}, \qquad |\bar{e}_2\rangle = \frac{4}{5}|e_1\rangle - \frac{3}{5}|e_2\rangle) = \frac{1}{5}\begin{pmatrix} 4 \\ -3 \end{pmatrix}$$

(1) (15分) 证明这两个向量正交归一;

(2) (15分) 由于它们正交归一，所以可以用它们作正交基矢。 在这组正交基下，题1中的$|\psi_1\rangle$具有如下形式

$$|\psi_1\rangle = a|\bar{e}_1\rangle + b|\bar{e}_2\rangle$$

求a和b.

(1)

$$\langle\bar{e}_1|\bar{e}_2\rangle = \frac{1}{5}(3 \quad 4)\cdot\frac{1}{5}\begin{pmatrix} 4 \\ -3 \end{pmatrix} = 0$$

$$\langle\bar{e}_1|\bar{e}_1\rangle = \frac{1}{5}(3 \quad 4)\cdot\frac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = 1$$

$$\langle\bar{e}_2|\bar{e}_2\rangle = \frac{1}{5}(4 \quad -3)\cdot\frac{1}{5}\begin{pmatrix} 4 \\ -3 \end{pmatrix} = 1$$

所以，这两个向量正交归一；

(2)

$$\langle\bar{e}_1|\psi_1\rangle = \langle\bar{e}_1|\bar{e}_1\rangle a + \langle\bar{e}_1|\bar{e}_2\rangle b = a$$
$$\langle\bar{e}_2|\psi_1\rangle = \langle\bar{e}_2|\bar{e}_1\rangle a + \langle\bar{e}_2|\bar{e}_2\rangle b = b$$

所以

$$a = \langle\bar{e}_1|\psi_1\rangle = \frac{1}{5}(3 \quad 4)\cdot\frac{1}{2}\begin{pmatrix} 1+i \\ 1-i \end{pmatrix} = \frac{7-i}{10}$$

$$b = \langle\bar{e}_2|\psi_1\rangle = \frac{1}{5}(4 \quad -3)\cdot\frac{1}{2}\begin{pmatrix} 1+i \\ 1-i \end{pmatrix} = \frac{7i+1}{10}$$