3. 2D Geometric Transformations

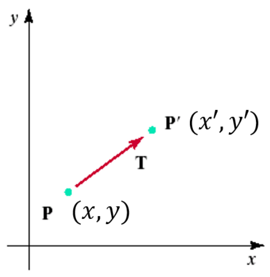
> 💡 **Basic 2D Transformations**
> - **Translation**
> - **Rotation**
> - **Scaling**
> - **Shear**
> - **Reflection**
>
> **Homogeneous Coordinates**
>
> **Matrix Representations**
> - **Inverse**
> - **Composite**

Basic Two-Dimensional Transformations

> 💡 Model 을 이동 → Model 은 Primitives (점선면) → 면은 선, 선은 점 → 결국 점을 이동한다는 의미

Translation

- 평행이동
- 점 (x,y) 를 x축으로 **tx**, y축으로 **ty**만큼 평행이동 했을 때의 점이 (x',y')



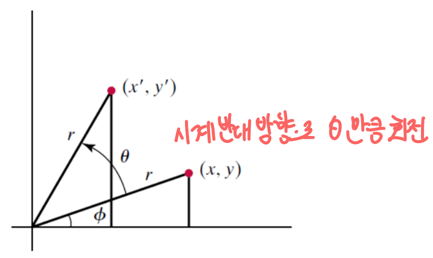
$$x' = x + t_x, \qquad y' = y + t_y$$

$$\mathbf{P} = \begin{bmatrix} x \\ y \end{bmatrix}, \qquad \mathbf{P}' = \begin{bmatrix} x' \\ y' \end{bmatrix}, \qquad \boxed{\mathbf{T} = \begin{bmatrix} t_x \\ t_y \end{bmatrix}}$$

$$\mathbf{P}' = \mathbf{P} + \mathbf{T}$$

Rotation

- 2차원 평면에서 회전
- 양의 회전 방향 = 시계 반대 방향
- 점 (x,y)를 원점 중심으로 Θ만큼 회전 이동했을 때의 점이 (x',y')



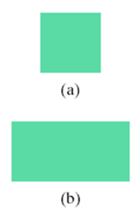
$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

$$\boxed{\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}$$

$$\mathbf{P}' = \mathbf{R} \cdot \mathbf{P}$$

Scaling

- x축, y축 방향으로 늘리거나 줄이는 것
- s_x : x축으로 얼마나 scaling할지 나타내는 수
- s_y : y축으로 얼마나 scaling할지 나타내는 수
- s_x, s_y가 1보다 크면 scale up, 1보다 작으면 scale down



$$x' = x \cdot s_x, \qquad y' = y \cdot s_y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\boxed{\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}}}_{=S} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{P}' = \mathbf{S} \cdot \mathbf{P}$$

> 💡 Translation 만 행렬 차원이 2x1, Translation 만 곱하기가 아닌 더하기 연산
> ⇒ consistent 하지 못하면 여러 문제가 발생
> ⇒ uniform 한 형태로 어떻게 통일할 수 있을까?
> ⇒ **Homogeneous Coordinates**

Homogeneous Coordinates

- **P' = MP** 로 모양을 통일

$$\textbf{Map } (x, y) \in \mathbb{R}^2 \textbf{ to } (x_h, y_h, h) \in \mathbb{RP}^3\textbf{, where } x = \frac{x_h}{h}, y = \frac{y_h}{h}$$

- h를 추가하여 1차원을 높임 (3차원으로)
- 일반적으로 h=1 **(i.e.** x=x_h,y=y_h**)** ⇒ (x,y) → (x, y, 1) 이므로 변환이 쉽기 때문

Matrix Representations

> 💡 Homogeneous coordinate로 변경해서 사용

Translation T(t_x, t_y)

결과는 1,2번째 값

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$x' = x + t_x$
$y' = y + t_y$

Rotation R(Θ)

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$x' = x\cos\theta - y\sin\theta$
$y' = x\sin\theta + y\cos\theta$

Scaling S(s_x, s_y)

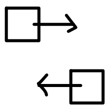
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$x' = x s_x$
$y' = y s_y$

Inverse Transformations

Translation

💡 t_x → -t_x, t_y → -t_y

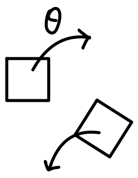


$$\mathbf{T}^{-1} = \begin{bmatrix} 1 & 0 & -t_x \\ 0 & 1 & -t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation

💡 Θ- → -Θ

$$\begin{matrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{matrix}$$



$$\mathbf{R}^{-1} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

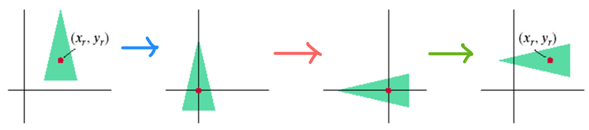
Scaling

💡 s_x → 1/s_x, s_y → 1/s_y

$$\mathbf{S}^{-1} = \begin{bmatrix} \frac{1}{s_x} & 0 & 0 \\ 0 & \frac{1}{s_y} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Composite Transformations

- 임의의 2D point(x_r, y_r) 에서 회전



(a) Original Position of Object and Pivot Point

(b) Translation of Object so that Pivot Point (x_r, y_r) is at Origin

(c) Rotation about Origin

(d) Translation of Object so that the Pivot Point is Returned to Position (x_r, y_r)

$$\mathbf{T}(x_r, y_r) \cdot \mathbf{R}(\theta) \cdot \mathbf{T}(-x_r, -y_r) = \mathbf{R}(x_r, y_r, \theta)$$

💡 진행순서 연산순서 반대

다른 Transformations

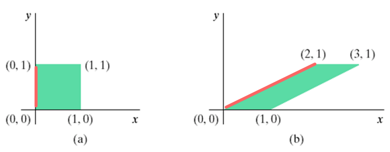
Shear

- 모양 왜곡
- sh_x, sh_y : 왜곡할 크기 결정

shear along x

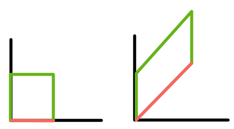
$$\begin{bmatrix} 1 & sh_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}$$

$$x' = x + sh_x \cdot y, \qquad y' = y$$



shear along y

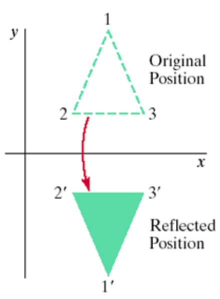
$$\begin{matrix} 1 & 0 & 0 \\ sh_y & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$



Reflection

- 대칭이동

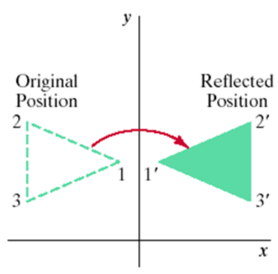
x축



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}$$

💡 x'=x, y'=-y

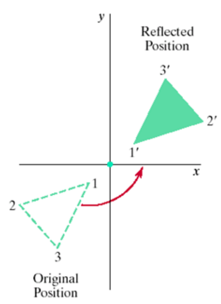
y축



$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

💡 x'=-x, y'=y

원점



$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

💡 x'=-x, y'=-y

문제1

Which of the following buffers is used for visual surface determination in 3D?

하나를 선택하세요.
- a. Double buffer
- b. Alpha channel
- c. Color buffer
- d. Z-buffer ✔

문제2

Rotate (1,0) by 120 degrees around the origin.

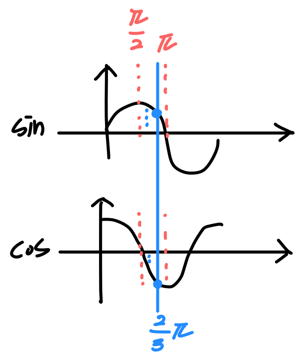
하나를 선택하세요.
- a. $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$ ✔
- b. $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$
- c. $(\frac{1}{2}, \frac{\sqrt{3}}{2})$
- d. $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$120 \times \frac{\pi}{180} = \frac{2}{3}\pi$$

$$\begin{bmatrix} \cos\frac{2}{3}\pi & -\sin\frac{2}{3}\pi & 0 \\ \sin\frac{2}{3}\pi & \cos\frac{2}{3}\pi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{2} & -(\frac{\sqrt{3}}{2}) & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 1 \end{bmatrix}$$



$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, $\cos\frac{\pi}{3} = \frac{1}{2}$

$\therefore (-\frac{1}{2}, \frac{\sqrt{3}}{2})$

문제3

Which of the following homogeneous coordinates in 3D projective space corresponds to $(1, 2)$ in 2D Euclidean coordinate?

하나를 선택하세요.
- a. (2, 4, 1)
- b. (2, 4, 0)
- c. (2, 4, 2) ✔
- d. $(2, 4, \frac{1}{2})$

Homogeneous Coordinates

Map $(x, y) \in \mathbb{R}^2$ to $(x_h, y_h, h) \in \mathbb{RP}^3$, where $x = \frac{x_h}{h}, y = \frac{y_h}{h}$

$(1, 2) \longrightarrow (1, 2, 1)$

배수도 동일한 유클리드 좌표 $(2, 4, 2)$
$(3, 6, 3)$
$(0.5, 1, 0.5)$
$\vdots$