Chapter 3 数值逼近

ex1、

(1) $f(x) = (x-1)^3$

$\|f\|_\infty = \max\limits_{[0,1]} f = 1$

$\|f\|_1 = \int_0^1 |f(x)|dx = -\frac{1}{4}(x-1)\Big|_0^1 = \frac{1}{4}$

$\|f\|_2 = \left(\int_0^1 f^2(x)dx\right)^{\frac{1}{2}} = \left[\frac{1}{7}(x-1)^7\Big|_0^1\right]^{\frac{1}{2}} = \frac{1}{\sqrt{7}}$

(2) $f(x) = x^m(1-x)^n$

$\|f\|_\infty = \max\limits_{[0,1]} f = \frac{m^m \cdot n^n}{(m+n)^{m+n}}$ $\rightarrow$ $f'(x) = m x^{m-1}(1-x)^n + (-1)\times n\, x^m(1-x)^{n-1}$

$= (m(1-x) - nx)\, x^{m-1}\cdot(1-x)^{n-1}$

$= (m-(m+n)x)\, x^{m-1}(1-x)^{n-1}$

当 $x = \frac{m}{m+n}$ 时，取极大值

ex2、

(1) $(f,g) = \int_a^b f'(x)\cdot g'(x)\,dx$

不构成内积：

$f'(x)$、$g'(x)$ 可以在 $[a,b]$ 上恒 <0

$\Rightarrow \int_a^b f'\cdot g'\,dx < 0 \Rightarrow$ 不满足正定性.

(2) 不构成内积：

同(1).

ex3、

Proof：

设 $T_n^*(x)$，$T_m^*(x)$，$\rho(x)$

有 $\int_a^b T_n^*\cdot T_m^*\cdot \rho\,dx = \int_0^1 T_m(2x-1)\,T_n(2x-1)\cdot\frac{1}{\sqrt{x-x^2}}\,dx$

$\overset{t=2x-1}{=\!=\!=} \int_{-1}^1 T_m(t)\cdot T_n(t)\cdot\frac{1}{\sqrt{\frac{1-t^2}{4}}}\cdot\frac{1}{2}\,dt$

$= \int_{-1}^1 T_m(t)\cdot T_n(t)\cdot\frac{1}{\sqrt{1-t^2}}\,dt$

即 Chebyshev 正交基数.

$\therefore T_0^*(x) = 1$

$T_1^*(x) = 2x-1$

$T_2^*(x) = 2(2x-1)^2-1$

$T_3^*(x) = 4(2x-1)^3 - 3(2x-1)$

ex4、

设 $\varphi_0(x) = 1$.

$\varphi_1(x) = (x-a_0)\varphi_0$

$\varphi_{k+1}(x) = (x-a_k)\cdot\varphi_k - b_k\cdot\varphi_{k-1}$

$\therefore a_0 = \frac{(x\varphi_0,\varphi_0)}{(\varphi_0,\varphi_0)} = \frac{\int_{-1}^1 x\,dx}{\int_{-1}^1 1\,dx} = 0$

$\Rightarrow \varphi_1(x) = x$

$a_1 = \frac{(x\varphi_1,\varphi_1)}{(\varphi_1,\varphi_1)} = 0$

$b_1 = \frac{(\varphi_1,\varphi_1)}{(\varphi_0,\varphi_0)} = \frac{2}{5}$

$\Rightarrow \varphi_2(x) = x^2 - \frac{2}{5}$

$a_2 = \frac{(x\varphi_2,\varphi_2)}{(\varphi_2,\varphi_2)} = 0$

$b_2 = \frac{(\varphi_2,\varphi_2)}{(\varphi_1,\varphi_1)} = \frac{17}{70}$

$\Rightarrow \varphi_3(x) = x^3 - \frac{9}{14}x$