P48

ex1、

(1) $\{1, x, x^2\}$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ 4 \end{pmatrix}$$

$$\begin{cases} a_0 = -\frac{7}{3} \\ a_1 = \frac{3}{2} \\ a_2 = \frac{5}{6} \end{cases} \Rightarrow P(x) = -\frac{7}{3} + \frac{3}{2}x + \frac{5}{6}x^2$$

(2) $w_{n+1}(x) = (x-1)(x+1)(x-2)$

$$l_k(x) = \frac{w_{n+1}}{(x-x_k)w_{n+1}'(x_k)}$$

$\{l_0, l_1, l_2\}$

$$\Rightarrow l_0 = \frac{(x+1)(x-2)}{2\times(-1)} = -\frac{1}{2}(x+1)(x-2)$$

$$l_1 = \frac{(x-1)(x-2)}{(-2)\times(-3)} = \frac{1}{6}(x-1)(x-2)$$

$$l_2 = \frac{(x-1)(x+1)}{1\times 3} = \frac{1}{3}(x-1)(x+1)$$

$$\Rightarrow P(x) = \sum_{i=0}^{2} f_i \cdot l_i = 0l_0 + (-3)\cdot l_1 + 4l_2$$

$$= -\frac{1}{2}(x-1)(x-2) + \frac{4}{3}(x^2-1)$$

$$= -\frac{1}{2}(x^2-3x+2) + \frac{4}{3}x^2 - \frac{4}{3}$$

$$= \frac{5}{6}x^2 + \frac{3}{2}x - \frac{7}{3}$$

(3) $\{1, x-x_0, (x-x_1)(x-x_0)\}$

$\Rightarrow P(x) = a_0 + (x-x_0)a_1 + (x-x_1)(x-x_0)a_2$

代入 $(x_0, f_0) \Rightarrow a_0 = 0$

代入 $(x_1, f_1) \Rightarrow (-2)a_1 = -3$ $\Rightarrow a_1 = \frac{3}{2}$

代入 $(x_2, f_2) \Rightarrow a_1 + 3\times 1\, a_2 = 4 \Rightarrow a_2 = \frac{5}{6}$

$\Rightarrow P(x) = \frac{5}{6}x^2 + \frac{3}{2}x$

ex4、

(1) 设 $f(x) = x^k$ $(k \le n)$

∴ 对 x_j 为 $f(x)$ 上的互异节点, $j = 0, \cdots, n$

$\Rightarrow P(x) = \sum_{i=0}^{n} x_i^k l_i(x)$

由误差估计 $\Rightarrow P(x) - f(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \cdot w_{n+1}(x) = 0$

$\Rightarrow P(x) = f(x) \Rightarrow \sum_{i=0}^{n} x_i^k \cdot l_i = x^k, k = 0, \cdots, n$

(2) 对 $(x_j - x)^k$，有 $(x_j - x)^k = \sum_{i=0}^{k} x_j^{k-i} \cdot x^i \cdot C_k^i$，展开有：

$$\sum_{j=0}^{n}(x_j - x)^k \cdot l_j(x) = \sum_{j=0}^{n} \cdot \sum_{i=0}^{k} x_j^{k-i}(-x)^i \cdot C_k^i \cdot l_j$$

$$\xlongequal[\text{可换序}]{\text{有限求和}} \sum_{i=0}^{k}(-x)^i C_k^i \cdot \sum_{j=0}^{n} x_j^{k-i} \cdot l_j$$

$$= \sum_{i=0}^{k} C_k^i (-x)^i \cdot x^{k-i}$$

$$= (x-x)^k \equiv 0^k \equiv 0$$

ex5、

$f \in C^2[a,b]$

$\therefore P(x) = f(a)\cdot\frac{x-b}{a-b} + f(b)\cdot\frac{x-a}{b-a}$

$\because f(a) = f(b) \equiv 0$

$\Rightarrow P(x) \equiv 0$

$\Rightarrow |f - P| < \frac{1}{2}|f''(\xi)\cdot(x-a)(x-b)|$

$\le \frac{1}{2}|f''(\xi)||(x-a)(x-b)|$

∵ 对 $\forall x \in [a,b]$,

$\max|(x-a)(x-b)| = \frac{1}{4}(b-a)^2$

$\Rightarrow |f - P| \le \frac{1}{8}(b-a)^2 \cdot \max_{a\le x\le b}|f''(x)|$

$\Rightarrow \max_{a\le x\le b}|f| \le \frac{1}{8}(b-a)^2 \cdot \max_{a\le x\le b}|f''(x)|$

证毕

ex6、

$$|R_n(x)| = \frac{|f^{(3)}(\xi)|_{max}}{3!} \cdot W_{n+1}(x)$$

$$= \frac{|f^{(3)}(\xi)|_{max}}{3!} \cdot |(x-x_0)(x-(x_0-h))(x-(x_0+h))|$$

其中 $x_0 \pm ah$ 为节点，x为插值点 则必有 $x = x_0 + bh$, $b \in Z$.

对 $f^{(3)}(\xi) = f^{(3)}(x) = e^x \le e^4$, $x \in [-4, 4]$

∴式 $\le \frac{e^4}{3!} \cdot |(b-1)(b+1)b| \cdot h^3$

$= |b(b-1)(b+1)| \cdot \frac{e^4 h^3}{6}$

设 $g(x) = x(x-1)(x+1)$

$g'(x) = 3x^2 - 1 \Rightarrow x = \pm\frac{1}{\sqrt{3}}$

而 $x = -\frac{1}{\sqrt{3}}$ 时取极大，$x = \frac{1}{\sqrt{3}}$ 时取极小.

$|g(x)|$ 在 $\pm\frac{1}{\sqrt{3}}$ 时取极大值 $\frac{2}{9}\sqrt{3}$

$\therefore \frac{e^4}{6} \cdot h^3 \cdot \frac{2}{9}\sqrt{3} < 10^{-6}$

$h^3 < \frac{9\sqrt{3}}{e^4} \times 10^{-6}$

$\Rightarrow h < \left(\frac{9\sqrt{3}}{e^4} \times 10^{-6}\right)^{\frac{1}{3}}$

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ex13、

由Hermite插值，设

$P(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + A(x-x_0)^3$

$P(x_1) = f(x_1)$

$$\Rightarrow A(x_1-x_0)^3 = \underbrace{f(x_0) - f(x_1) - f'(x_0)(x_1-x_0) + \frac{f''(x_0)}{2!}(x_1-x_0)^2}_{\Rightarrow K}$$

$A = \frac{K}{(x_1-x_0)^3}$

$\Rightarrow P(x) = \sum_{i=0}^{2} f^{(i)}(x_0)\cdot(x-x_0)^i + \frac{K}{(x_1-x_0)^3}\cdot(x-x_0)^3$

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ex15、

设余项函数 $R(x) = f(x) - P(x) = k(x)\cdot(x-x_k)^2(x-x_{k+1})^2$

设 $\varphi(t) = f(t) - P(t) - k(x)(t-x_k)^2(x-x_{k+1})^2$

则 $t = x_k, x_{k+1}, x_{k-1}$ 上至少五个零点

∴应用Rolle定理，知 $\exists\xi \in (x_k, x_{k+1})$ s.t. $\varphi^{(4)}(\xi) = 0$

$\because P(t) \in P^3[t]$ $\therefore P^{(4)} \equiv 0$

$\therefore f^{(4)}(\xi)/4! = k(x)$

$\therefore R_n = \frac{f^{(4)}(\xi)}{4!}\cdot(x-x_k)^2(x-x_{k+1})^2$

误差限估计：

$(x-x_k)^2(x-x_{k+1})^2$，设 $h = \frac{x_{k+1}-x_k}{2}$

当 $x = h$ 时取 max 值 $\Rightarrow \frac{h^4}{16}$

$\therefore R_n \le \frac{h^4}{384} \cdot \max_{x_k \le \xi \le x_{k+1}} f^{(4)}(\xi)$