Random Walk and its extensions

Object Oriented Programming
2024 First Semester
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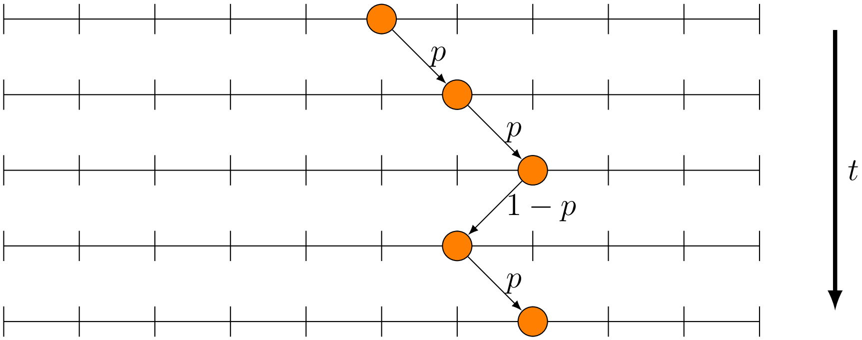
Introduction: What is *Random Walk* (酔歩)

- Random walk is a mathematical object that describes a path that consists of a succession of random steps.
- It is a simple but fundamental example of a stochastic process.
- It also gives us a good insight into the behavior of more complex systems.

One-dimensional Random Walk

- A one-dimensional random walk is a sequence of random steps on a line.
  - The walker starts at the origin and moves by one unit at each step.
  - At each step, the walker moves left with probability p and right with $1 - p$.
- The positions of the walker are not important, but the probability distribution of the position is.
- For estimating the probability distribution by simulations, we need to simulate motions of a large number of walkers.

Image of Random walk



Theoretical Analysis

- At t-th step, the position is x if the particle moves right $m = (t+x)/2$ times and left $n = t - m = (t-x)/2$ times.
- The probability arriving to the position x is given by the binomial distribution

$$P(x) = \binom{t}{\frac{t+x}{2}} p^{(t+x)/2}(1-p)^{(t-x)/2} \tag{2.1}$$

- The probability moving m steps to the right is given by

$$Q(m) = \binom{t}{m} p^{m}(1-p)^{t-m} \tag{2.2}$$

Generating Function (確率母関数) for $Q(m)$

- For obtaining averages, deviations and other moments, the generating function is a convenient tool.

$$G(z) = \sum_{m=0}^{t} Q(m) z^m \tag{2.3}$$

General formulae with generating functions

$$G(1) = \sum_{m=0}^{t} Q(m) = 1 \tag{2.4}$$

$$G'(z) = \sum_{m=1}^{t} mQ(m)z^{m-1} = \sum_{m=0}^{t} mQ(m)z^{m-1} \tag{2.5}$$

$$G'(1) = \sum_{m=0}^{t} mQ(m) = \langle m \rangle \tag{2.6}$$

$$G''(z) = \sum_{m=2}^{t} m(m-1)Q(m)z^{m-2} = \sum_{m=0}^{t} m(m-1)Q(m)z^{m-2} \tag{2.7}$$

$$G''(1) = \sum_{m=0}^{t} m(m-1)Q(m) = \langle m(m-1) \rangle \tag{2.8}$$

For binominal distribution

- For the binomial distribution, the generating function is given by a compact form.
- Those compact forms enable us to calculate the moments easily.

$$G(z) = \sum_{m=0}^{t} \binom{t}{m} p^m (1-p)^{t-m} z^m = (zp + 1 - p)^t \tag{2.9}$$

$$G(1) = 1 \tag{2.10}$$

$$G'(z) = tp\,(zp + 1 - p)^{t-1} \tag{2.11}$$

$$G'(1) = tp \tag{2.12}$$

$$\langle m \rangle = tp \tag{2.13}$$

$$\langle x \rangle = \langle 2m - 1 \rangle = 2tp - 1 = t\,(2p - 1) \tag{2.14}$$

$$G''(z) = t(t-1)p^2 \left(zp + 1 - p\right)^{t-2} \tag{2.15}$$

$$G''(1) = t(t-1)p^2 = \left\langle m^2 \right\rangle - \langle m \rangle \tag{2.16}$$

$$\sigma^2 = \left\langle x^2 \right\rangle - \langle x \rangle = 4tp(1-p) \tag{2.17}$$

Approximated distribution at the visinity of the mean

$$P(x) \propto \exp\left[-\frac{(x-\langle x\rangle)^2}{2\sigma^2}\right] \tag{2.18}$$

$$\langle x\rangle = t(2p-1) \tag{2.19}$$

$$\sigma^2 = 4tp(1-p) \tag{2.20}$$

- Normal (Gauss) distribution

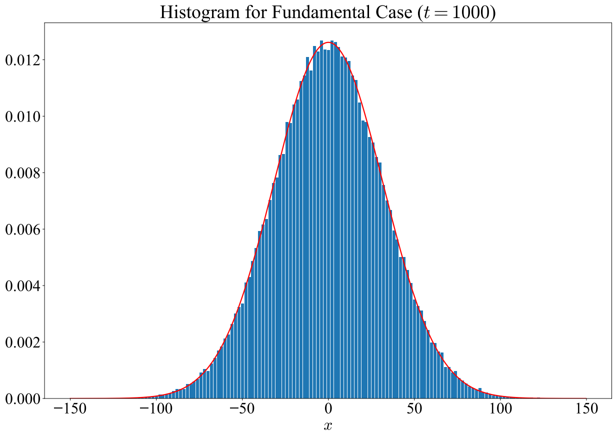
Class Planning in `model` package

- `Walker` class: A class for a single walker
- `Simulation` class: A class for simulating a large number of walkers
- `PositionHistogram` class: A class for creating the histogram of the positions

walk() method in Walker class

```java
public int walk() {
    double r = random.nextDouble();
    if (r < p) {//x++ with probability p
        x++;
    } else {//x-- with probability 1-p
        x--;
    }
    return x;
}
```

$p = 1/2$ case


Histogram for Fundamental Case ($t = 1000$)
0.000
0.002
0.004
0.006
0.008
0.010
0.012
−150
−100
−50
0
50
100
150
x

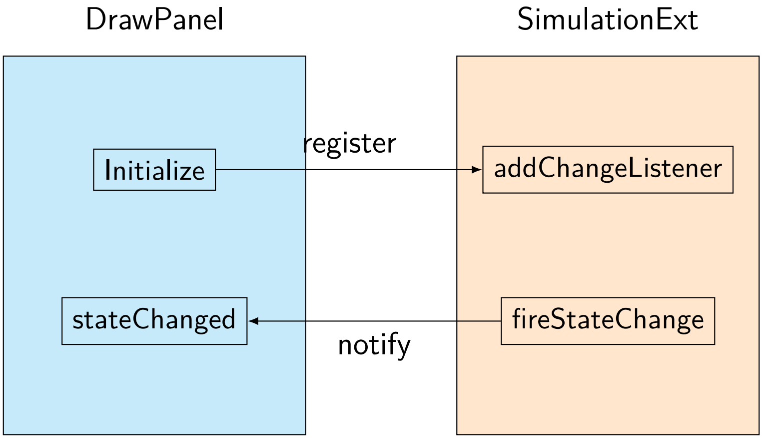
Observer Design Pattern

- You may want to monitor the progress of the simulation.
- However, you need not to monitor the every changes in the simulation.
- Moreover, you may not want to change the `Simulation` class.
- *Observer design pattern* is a solution for this case.

Random Walk Simulation with GUI

- `GuiMain` class: A class for the main frame extended from `JFrame`.
- `DrawPanel` class: A class for drawing the histogram extended from `JPanel`.
- `SimulationExt` class: A runnable extension of the `Simulation` class.
  - It observes the changes in the simulation every 10 steps.
  - State changes are notified to the `DrawPanel` class.

Notify State Changes



Notify State Changes

```java
public void run() {
    while (running) {
        this.oneStep();
        //Every 10 steps, notify the listeners
        if (t % 10 == 0) {
            fireStateChanged();
        }
        try {
            Thread.sleep(100);
        } catch (InterruptedException ex) {
        }
    }
}

private void fireStateChanged() {
    listeners.forEach(
            li -> li.stateChanged(new ChangeEvent(this)));
}
```

Continuous Random Walk

- In the continuous random walk, the displacement of a walker is given by a random variable with a probability distribution.
- The distribution of the positions of the walkers is given by a continuous function.

Class Planning

- `AbstractRandom` class: An abstract class for the random number generator
- `Walker` class: A class for a single walker, whose position is continuous.
- `Simulation` class: A class for simulating a large number of walkers
- `Histogram` class: A class for creating the histogram of the continuous positions

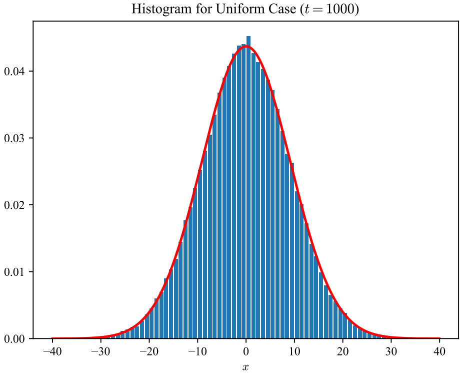
Walker class

```
public double walk() {
    double r = random.getNext();
    x += r;
    return x;
}
```

- `random` is an instance of `AbstractRandom` class.
- The distribution of the random variables is defined in the subclass of `AbstractRandom`.

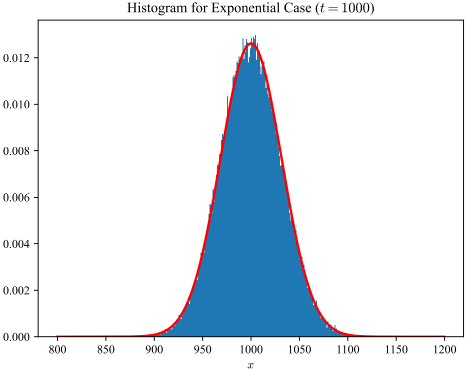
Uniform distribution

$$f(x) = \begin{cases} 1 & (-1/2 \le x \le 1/2) \\ 0 & (\text{otherwise}) \end{cases} \tag{5.1}$$



Exponential distribution

$$f(x) = e^{-x} \qquad x \in [0, \infty) \tag{5.2}$$



Central Limiting Theorem: 中心極限定理

- $\{X_k\}$: random variables obeying an identical distribution with the mean μ and deviation σ^2

$$S_n = \sum_{k=1}^{n} X_k \tag{5.3}$$

$$S_n^* = \frac{S_n - n\mu}{\sqrt{n}\sigma} \tag{5.4}$$

$$\lim_{n\to\infty} P(S_n^* \leq a^*) = \frac{1}{\sqrt{2}} \int_{-\infty}^{a*} \exp\left[-\frac{x^2}{2}\right] \mathrm{d}x \tag{5.5}$$

- In the limit $n \to \infty$, S_n^* obeys the standard normal distribution $N\left(0,1\right)$

Appendix: Generating nonuniform random numbers

- For generating nonuniform random numbers, we need to transform the uniform random numbers.
- Two types of methods
  - Transform method
  - Rejection method

Transform method

- For a probability density $f(x)$ $(x \in [a, b))$, the probability distribution F is given by

$$F(x) = \int_a^x f(t)\mathrm{d}t \tag{6.1}$$

- The transform method is available if the inverse function F^{-1} is obtained.
- procedure
  - Generate a uniform random number $r \in [0, 1)$.
  - $x = F^{-1}(r)$
  - $\{x\}$ is a random number obeying the probability density $f(x)$.

Generating random numbers obeying exponential distribution

- In `continuousModel/Main1.java`, the random numbers obeying the exponential distribution $f(x) = e^{-x}$ $(x \in [0, \infty])$ are generated.

$$F(x) = \int_0^x e^{-t}\mathrm{d}t = 1 - e^{-x} \tag{6.2}$$

$$F^{-1}(x) = -\ln(1-x) \tag{6.3}$$

```java
AbstractRandom random = new Transform(x->-Math.log(1-x), seed);
Simulation sys = new Simulation(n, random);
```

Appendix: Characteristic functions: 特性関数

- The characteristic function of a continuous random variable is a counterpart of the generating function of a discrete random variable.
- For the probability density function $f(x)$, the characteristic function is defined by

$$\varphi(z) = \int_{-\infty}^{\infty} f(x) e^{izx} \mathrm{d}x \tag{7.1}$$

General formulae with characteristic functions

$$\varphi(0) = \int_{-\infty}^{\infty} f(x)\mathrm{d}x = 1 \tag{7.2}$$

$$\varphi'(z) = i\int_{-\infty}^{\infty} xf(x)e^{izx}\mathrm{d}x \tag{7.3}$$

$$\varphi'(0) = i\int_{-\infty}^{\infty} xf(x)\mathrm{d}x = i\left\langle x\right\rangle \tag{7.4}$$

$$\varphi''(z) = -\int_{-\infty}^{\infty} x^2f(x)e^{izx}\mathrm{d}x \tag{7.5}$$

$$\varphi''(0) = -\int_{-\infty}^{\infty} x^2f(x)\mathrm{d}x = -\left\langle x^2\right\rangle \tag{7.6}$$

Characteristic function for uniform distribution

$$f(x) = \begin{cases} 1 & (-1/2 \le x \le 1/2) \\ 0 & (\text{otherwise}) \end{cases} \tag{7.7}$$

$$\varphi(z) = \int_{-1/2}^{1/2} e^{izx} \mathrm{d}x = \frac{2}{z} \sin(\frac{z}{2}) \tag{7.8}$$

$$= 1 - \frac{z^2}{24} + \frac{z^4}{1920} + O(z^6) \tag{7.9}$$

$$\langle x \rangle = 0 \tag{7.10}$$

$$\left\langle x^2 \right\rangle = \frac{1}{12} \tag{7.11}$$

Characteristic function for exponentioal distribution

$$f(x) = e^{-x} \qquad x \in [0, \infty) \tag{7.12}$$

$$\varphi(z) = \int_0^\infty e^{izx} e^{-x} \mathrm{d}x = \frac{1}{1 - iz} \qquad \varphi(0) = 1 \tag{7.13}$$

$$\varphi'(z) = \frac{i}{(1 - iz)^2} \qquad \varphi'(0) = i \tag{7.14}$$

$$\varphi''(z) = \frac{2}{(1 - iz)^3} \qquad \varphi''(0) = 2 \tag{7.15}$$

$$\langle x \rangle = 1 \qquad \left\langle x^2 \right\rangle = 2 \tag{7.16}$$

$$\sigma^2 = 1 \tag{7.17}$$