QUANTUM ASSIGNMENT-3

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(1) $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & \cancel{0} \end{pmatrix}$ $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

to check if matrix $\begin{pmatrix} \sigma_0 & \sigma_1 \\ -i\sigma_2 & \sigma_3 \end{pmatrix}$ is unitary.

for matrix to be unitary $UU^H = I$ where U^H is conjugate transpose of matrix.

$$U^H = \begin{pmatrix} \sigma_0 & \sigma_1 \\ i\sigma_2 & \sigma_3 \end{pmatrix}^T = \begin{pmatrix} \sigma_0 & i\sigma_2 \\ \sigma_1 & \sigma_3 \end{pmatrix}$$

(NOTE: conjugate of real matrix is matrix itself)

now, $UU^H = \begin{pmatrix} \sigma_0 & \sigma_{\cancel{0}} \\ -i\sigma_2 & \sigma_3 \end{pmatrix} \begin{pmatrix} \sigma_0 & i\sigma_2 \\ \sigma_1 & \sigma_3 \end{pmatrix}$

$$UU^H = \begin{pmatrix} \sigma_0^2 + \sigma_1^2 & i\sigma_0\sigma_2 + \sigma_1\sigma_3 \\ -i\sigma_0\sigma_2 + \sigma_{\cancel{1}}\sigma_3 & \sigma_2^2 + \sigma_3^2 \end{pmatrix}$$

(by property of pauli matrix than $\sigma^2 = I$ we have $\sigma_0^2 + \sigma_1^2 = 2I$)

now $\sigma_1\sigma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$i\sigma_0\sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -i^2 \\ i^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i^2 \\ i^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

hence $\sigma_1\sigma_3 + i\sigma_0\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 0$ (2x2 null matrix)

$UU^H = \begin{pmatrix} 2I & 0 \\ 0 & 2I \end{pmatrix} = 2\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \neq I$

hence, not unitary.

Q2 consider $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ where α and β are complex numbers

say $\alpha = r_\alpha e^{i\theta_\alpha}$ and $\beta = r_\beta e^{i\theta_\beta}$

hence $|\psi\rangle = r_\alpha e^{i\theta_\alpha}|0\rangle + r_\beta e^{i\theta_\beta}|1\rangle$

multiply by $e^{-i\theta_\alpha}$ then we have

$$|\psi\rangle = r_\alpha e^{i\theta_\alpha} \cdot e^{-i\theta_\alpha}|0\rangle + r_\beta e^{i\theta_\beta - i\theta_\alpha}|1\rangle$$

$$= r_\alpha|0\rangle + r_\beta e^{i(\phi_\beta - \phi_\alpha)}|1\rangle$$

now, have r_α and r_β essentially represent the probability of getting $|0\rangle$ & $|\phi\rangle$.

now $\quad r_\alpha^2 + r_\beta^2 = 1$

$$\cos\left(\frac{\theta}{2}\right) = r_\alpha \text{ and } \sin\frac{\theta}{2} = r_\beta$$

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\frac{\theta}{2} e^{(i(\phi_\beta - \phi_\alpha)}|1\rangle$$

$$\cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\frac{\theta}{2} e^{i\phi}|1\rangle$$

$$\phi = (\phi_\beta - \phi_\alpha) \qquad \theta = 2\cos^{-1} r_\alpha$$

$$= 2\sin^{-1} r_\beta$$

and point opposite to it on bloch sphere will be

$$|\psi'\rangle = \cos\left(\frac{\theta+\pi}{2}\right)|0\rangle + \sin\left(\frac{\theta+\pi}{2}\right) e^{i\phi}|1\rangle$$

$$= \cos\left(\frac{\theta}{2}+\frac{\pi}{2}\right)|0\rangle + \sin\left(\frac{\theta}{2}+\frac{\pi}{2}\right)e^{i\phi}|1\rangle$$

$$= -\sin\left(\frac{\theta}{2}\right)|0\rangle + \cos\left(\frac{\theta}{2}\right) e^{i\phi}|1\rangle$$

if they are orthogonal $\langle\psi|\psi'\rangle = 0$

$$\langle\psi, \psi'\rangle = \left(\cos\left(\frac{\theta}{2}\right)\langle 0| + \sin\frac{\theta}{2} e^{-i\phi}\langle 1|\right)\left(-\sin\frac{\theta}{2}|0\rangle + \cos\frac{\theta}{2} e^{i\phi}|1\rangle\right)$$

$$\Rightarrow = -\cos\frac{\theta}{2}\sin\frac{\theta}{2}\langle 0|0\rangle - \sin^2\frac{\theta}{2} e^{-i\phi}\langle 1|0\rangle + \cos^2\frac{\theta}{2}\langle 0|1\rangle e^{i\phi}$$

$$+ \sin\frac{\theta}{2}\cos\frac{\theta}{2} e^{-i\phi} e^{i\phi}\langle 1|1\rangle$$

now, $\langle 0|0\rangle = \langle 1|1\rangle = 1$, $\langle 1|0\rangle = 0 = \langle 0|1\rangle$

hence we have

we get

$$= -\cos\theta/_2 \cancel{\sin\theta/_2} - 0 - 0 + \sin\theta/_2 \cancel{\cos\theta/_2} = 0$$

hence orthogonal!

Q3 Consider the operator (4×4) matrix in Hilbert space C_4

$$\rho = \frac{1}{4}(1-\epsilon)I_4 + \epsilon\,(|0\rangle \otimes |0\rangle)(\langle 0| \otimes \langle 0|).$$

Is ρ ~~identity~~ density?

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Solution

$$|0\rangle \otimes |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ 0\begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{4\times1}$$

Similarly

$$\langle 0| \otimes \langle 0| = [1 \;\; 0] \otimes [1 \;\; 0] = [1[1 \;\; 0] \;\; 0[1 \;\; 0]]$$
$$= \underbrace{[1 \;\; 0 \;\; 0 \;\; 0]}_{1\times4}$$

now, finally we have

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}[1000] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rho = \frac{1}{4}I_4 - \epsilon\frac{I_4}{4} + \epsilon\cdot(|0\rangle \otimes |0\rangle)(\langle 0| \otimes \langle 0|)$$

$$\rho = \frac{1}{4}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \epsilon/4\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} \epsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1+3\epsilon}{4} & 0 & 0 & 0 \\ 0 & \frac{1-\epsilon}{4} & 0 & 0 \\ 0 & 0 & \frac{1-\epsilon}{4} & 0 \\ 0 & 0 & 0 & \frac{1-\epsilon}{4} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1+3\epsilon}{4} & 0 & 0 & 0 \\ 0 & \frac{1-\epsilon}{4} & 0 & 0 \\ 0 & 0 & \frac{1-\epsilon}{4} & 0 \\ 0 & 0 & 0 & \frac{1-\epsilon}{4} \end{bmatrix} = A$$

since matrix obtained is diagonal matrix, so its eigen values be diagonal elements

Since $\epsilon \in [0,1]$ hence all eigen values are positive.

therefore ρ is positive semidefinite — (1)

$$Tr(matrix) = \left(\frac{1+3\epsilon}{4}\right) + \left(\frac{1-\epsilon}{4}\right) + \left(\frac{1-\epsilon}{4}\right) + \left(\frac{1-\epsilon}{4}\right)$$

$$= \frac{4}{4} = 1 \quad — (2)$$

moreover we can see that above matrix is hermitian

as $(A^*)^T = A$

Hence ρ is valid density matrix !

(4) $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ hilbert space: C^2

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle\right)$$ hilbert space: C^4

$$|\alpha\rangle = \cos\phi\,|0\rangle + \sin\phi\,|1\rangle$$

$$|\beta\rangle = \cos\theta\,|0\rangle + \sin\theta\,|1\rangle$$

to find $p(\phi,\theta) = |(\langle\alpha| \otimes \langle\beta|)|\psi\rangle|^2$

$\llcorner$ as function of ϕ and θ.

<u>Solution</u> $|0\rangle \otimes |1\rangle$ (we know $u \otimes v = uv^T$)

$$\underset{2\times1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} \underset{1\times2}{\begin{bmatrix} 0 & 1 \end{bmatrix}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = |0\rangle \otimes |1\rangle$$

$|1\rangle \otimes |0\rangle$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = |1\rangle \otimes |0\rangle$$

$$|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$|\psi\rangle = \frac{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}{\sqrt{2}}$$

$$|\alpha\rangle = \cos\phi\,|0\rangle + \sin\phi\,|1\rangle$$

$$|\alpha\rangle = \cos\phi \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sin\phi \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix} \Longrightarrow \langle\alpha| = \begin{bmatrix} \cos\phi & \sin\phi \end{bmatrix}$$

$$|\beta\rangle = \cos\theta\,|0\rangle + \sin\theta\,|1\rangle$$

$$= \cos\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sin\theta \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\langle\beta| = \begin{bmatrix} \cos\theta & \sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

now say

$$\chi = (\langle\alpha| \otimes \langle\beta|)\,|\psi\rangle$$

$$= (\cos\phi\cos\theta)(0) + (\cos\phi\sin\theta)\left(\frac{1}{\sqrt{2}}\right) + (\sin\phi\cos\theta)\left(\frac{-1}{\sqrt{2}}\right)$$

$$+ \sin\phi\sin\theta\,(0)$$

$$= \frac{1}{\sqrt{2}}(\cos\phi\sin\theta - \sin\phi\cos\theta)$$

$$= \frac{1}{\sqrt{2}}(\sin(\theta-\phi)) = |\chi|$$

$$\underbrace{\chi^2 = \frac{1}{2}\sin^2(\theta-\phi)}_{\text{Answer}}$$