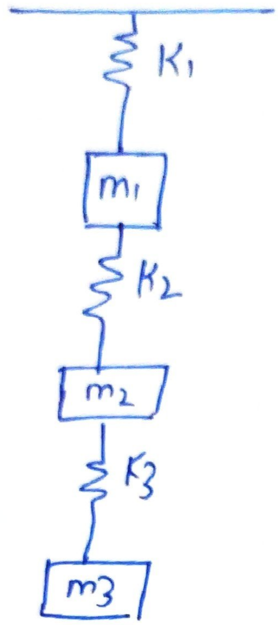
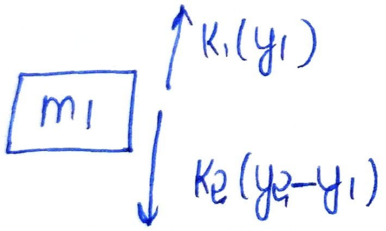
Q2: Consider the system of springs and masses as shown in the figure below with three mass m_1, m_2, m_3 at vertical displacement y_1, y_2, y_3 connected with three spring constant K_1, K_2, K_3



Write equation of motion of this system.

assuming displacement are taken from equilibrium then, $\forall (y_3 > y_2 > y_1)$

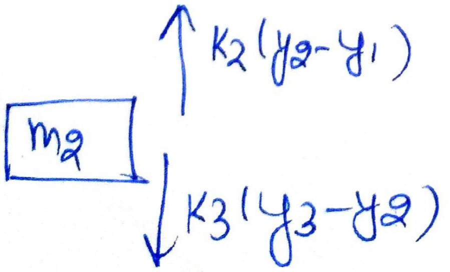
for m_1



equation of motion assuming downwards displacement

$$m_1 a_1 = K_2(y_2 - y_1) - K_1 y_1 \quad \text{——} \quad (1)$$

for m_2



equation of motion assuming downward displacement

$$m_2 a_2 = K_3(y_3 - y_2) - K_2(y_2 - y_1) \quad \text{— (II)}$$

for m_3

$$\uparrow K_3(y_3 - y_2)$$

$$\boxed{m_3}$$

$$m_3 a_3 = -K_3(y_3 - y_2) \quad \text{— (III)}$$

above three equations can rewrited to have

$$-m_1 a_1 = (K_1 + K_2) y_1 - K_2 y_2$$

$$-m_2 a_2 = -K_2 y_1 + (K_2 + K_3) y_2 - K_3 y_3$$

$$-m_3 a_3 = -K_3 y_2 + K_3 y_3$$

$$M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}$$ is called mass matrix

$$A = \begin{bmatrix} -a_1 \\ -a_2 \\ -a_3 \end{bmatrix}, \quad K = \begin{bmatrix} (K_1 + K_2) & -K_2 & 0 \\ -K_2 & (K_2 + K_3) & -K_3 \\ 0 & -K_3 & K_3 \end{bmatrix}$$ stiffness matrix

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$ we hence have $MA = Ky$ —(i*)

Assuming A_1, A_2, A_3 are amplitude of motion of m_1, m_2, m_3.
Then position of $m_1 m_2 m_3$ can be written as,

$$\left.\begin{aligned} y_1(t) &= A_1 e^{i\omega t} \\ y_2(t) &= A_2 e^{i\omega t} \\ y_3(t) &= A_3 e^{i\omega t} \end{aligned}\right\}$$

$$a_1 = \frac{d^2}{dt^2}(y_1(t)) = A_1 \omega^2 e^{i\omega t}$$

$$\boxed{a_1 = -A_1 \omega^2 e^{i\omega t}}$$

$$a_2 = \frac{d^2}{dt^2}(y_2(t)) = A_2 \omega^2 e^{i\omega t}$$

$$\boxed{a_2 = -A_2 \omega^2 e^{i\omega t}}$$

$$a_3 = \frac{d^2}{dt^2}(y_3(t)) = +A_3 \omega^2 e^{i\omega t}$$

$$\boxed{a_3 = -A_3 \omega^2 e^{i\omega t}}$$

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} A_1 e^{i\omega t} \\ A_2 e^{i\omega t} \\ A_3 e^{i\omega t} \end{bmatrix} \qquad A = \begin{bmatrix} -a_1 \\ -a_2 \\ -a_3 \end{bmatrix} = \begin{bmatrix} A_1 \omega^2 e^{i\omega t} \\ A_2 \omega^2 e^{i\omega t} \\ A_3 \omega^2 e^{i\omega t} \end{bmatrix}$$

System of Equation can be written as

$$MA = Ky$$

$$M(\omega^2 y) = Ky \quad (\because A = \omega^2 y)$$

$$(K - \omega^2 M) y = 0 \text{ ——— } (*)$$

where
$$K = \begin{bmatrix} (K_1+K_2) & -K_2 & 0 \\ -K_2 & (K_2+K_3) & -K_3 \\ 0 & -K_3 & K_3 \end{bmatrix} \qquad M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}$$

$$y = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$$

To find frequency we use the method which we used to find eigen values,

$$\det(K - \alpha I) = 0$$

here $\alpha = m\omega^2$

$$\begin{vmatrix} 2-2\omega^2 & -1 & 0 \\ -1 & 2-2\omega^2 & -1 \\ 0 & -1 & 1-4\omega^2 \end{vmatrix} = 0$$

$$\rightarrow (2-2\omega^2)\left((2-2\omega^2)(1-4\omega^2) - 1\right) + 1(-1+4\omega^2)$$

$$\rightarrow (2-2\omega^2)^2(1-4\omega^2) - (2-2\omega^2) - 1 + 4\omega^2$$

$$\rightarrow 4(1-\omega^2)^2(1-4\omega^2) - 2(1-\omega^2) - 1 + 4\omega^2 = 0$$

So we get

$$16\omega^6 - 36\omega^4 + 20\omega^2 - 3 = 0$$

solving this equation we get value as

$\omega = 0.2517, -0.2517, 0.7977, -0.7977, -1.2451, 1.2451$

Since we need natural frequency so we will ignore -ve values

$\omega = 0.2517, 0.7977, 1.2451$

$\omega_1 = 0.2517, \quad \omega_2 = 0.7977, \quad \omega_3 = 1.2451$