Quantum Scribed Notes

February 9, 2023

Submitted by

- Prisha 2021101075
- Vrinda Agarwal 2021101110
- Khushi Wadhwa 2021101104
- Vanshita Mahajan 2021101102
- Apoorva Thirupathi 2019121012
- Rhythm Aggarwal 2021101081
- Nipun Tulsian 2021101055

1 Introduction to Quantum Information

Classical theories are a subset/special case of Quantum theory. Quantum information refers to the fastest and the most efficient way to communicate whereas Quantum Computing is about how this communication takes place.
Cloning – so easy to accomplish with classical information,turns out not to be possible in general in quantum mechanics. This no-cloning theorem, discovered in the early 1980s, is one of the earliest results of quantum computation and quntum information.
Shannon came up with the idea of how likely we are gonna look at a particular piece of information? His theory is about surprise.
He concluded that if an event has high probability of occurring then the information gained is small and vice versa.
Consider this,
If ρ is the probability of an event to occur then, information passed is proportional to $1/\rho$ As the value of $1/\rho$ can be very large so,we take log of the value for convenience.
Entropy is the expectation value of the degree of surprise.Shannon's entropy is defined as :
Entropy: (H(x))= -$\sum_i p_i \log_2 p_i$

2 The Stern-Gerlach Experiment

In the original Stern–Gerlach experiment, hot atoms were ‘beamed’ from an oven through a magnetic field which caused the atoms to be deflected, and then the position of each atom was recorded. The original experiment was done with silver atoms.
Hydrogen atoms contain a proton and an orbiting electron,this electron can be imagined as a little ‘electric current’ around the proton. This electric current causes the atom to have a magnetic field; each atom has what physicists call a ‘magnetic dipole moment’. As a result each atom behaves like a little bar magnet with an axis corresponding to the axis the electron is spinning around. Throwing little bar magnets through a magnetic field causes the magnets to be deflected by the field, and we expect to see a similar deflection of atoms in the Stern–Gerlach experiment. How the atom is deflected depends upon both the atom’s magnetic dipole moment – the axis the electron is spinning around – and the magnetic field generated by the Stern– Gerlach device which can cause atom to be deflected by an amount that depends upon the zˆ component of the atom’s magnetic dipole moment, where zˆ is some fixed external axis.
Observations made are:-
First, since the hot atoms exiting the oven would naturally be expected to have their dipoles oriented randomly in every direction, it would follow that there would be a continuous distribution of atoms seen at all angles exiting from the Stern–Gerlach device. Instead, what is seen is atoms emerging from a discrete set of angles. Physicists were able to explain this by assuming that the magnetic dipole moment of the atoms is quantized, that is, comes in discrete multiples of some fundamental amount.

Various experiments were performed,However two beams of equal intensity were observed every single time.

3 Postulates Of Quantum Theory

Close systems are a subset of open systems therefore,the properties applying to open systems are equally applicable on closed systems but vice-versa is not true. A complex vector space known as the Hilbert space is the one over which inner products are defined.
1) Inner product of a vector with itself is non-negative. $\langle x, x\rangle \geq 0$ and $\langle x, y\rangle =$ 0 ,iff x=0.
2)$\langle x, \alpha y_1 + y_2\rangle = \alpha\langle x, y_1\rangle + \langle x, y_2\rangle$
3)$\langle x, y\rangle = \overline{\langle y, x\rangle}$

Note-The complete description of a physical system is given by its state represented by $|\psi\rangle$ which belongs to Hilbert space,χ.
$|\psi\rangle\langle\psi|$ is defined as the dual of Hilbert space.

$\langle\psi|=[|\psi\rangle]^+$
d: degrees of freedom can be countably infinite For a given state having some vector representation measuring them in identical system may or may not yield same results.

4 Properties of Closed Quantum System

A closed quantum system is a system in which energy is preserved over time.

1. Let χ be a Hilbert Space. For any quantum state $|\psi> \ \epsilon \ \chi$, $<\psi|\psi> = 1$
And the state $|\psi>$ represents the state of the system in mathematical form, from which the probabilities of various outcomes of measurements can be calculated.
2. Observables are given by Hermitian Operators which take only real eigenvalues.
Say, for a complex vector space χ dim (χ) = d.

Let Orthonormal bases be denoted by $\{\ |\alpha_1>, |\alpha_2>, ..., |\alpha_d>\ \}$.
Then any vector state $|x>$ in the Hilbert Space can be represented in terms of basis vectors as

$$|x> = \ \Sigma c_i|\alpha_i>$$

Where c_i s are complex scalars with the constraint that $\Sigma|c_i|^2 = 1$
$|\alpha_1>, |\alpha_2>, ..., |\alpha_d>$ are also often represented as $|0>, |1>, ..., |d-1>$
So bases can now be represented as $\{|0>, |1>, ..., |d-1>\}$.

If $|x>$ is an entry in a Hilbert Space then $<x|$ is an entry in dual Hilbert Space and

$$(|x>)^\dagger = (<x|)$$

Example if we take the 0^{th} state $|0> = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$(|0>)^\dagger = <0| = [1 \ 0 \ 0 \ldots 0]$

$|x> = \begin{bmatrix} e^{i\theta} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ $\quad <x| = [e^{-i\theta} \ 0 \ldots . \ 0]$

Let us demonstrate how operators act on a quantum state in one Hilbert space χ_1 and change it to a quantum state in another Hilbert space χ_2

$\hat{O} : \chi_1 \rightarrow \chi_2$
$\hat{O} = \hat{O}^\dagger$ [Hermitian Operator]
$<$ x, $\hat{O}$y $>$ = $<$ $\hat{O}^\dagger$x, y $>$ where y $\in \chi_1$ and x $\in \chi_2$
Therefore, $\hat{O}^\dagger$y $\in \chi_2$ since inner product can only be taken with vectors in the same vector space.

3. Measurement M corresponding to some Observable O for any state $|\psi>$ then M $|\psi> \rightarrow |a_i>$
That is, the result of the measurement will be one of the eigenstates of the Observable with outcome λ_i i.e. the corresponding eigenvalue.

Similarly, if outcome is λ_i then we can conclude that our state has changed to a_i.
We can represent our measurable O as

$$O = \Sigma \lambda_i |a_i >< a_i|$$

Multiplying both sides by the ket $|a_3>$

$$O|a_3> = \Sigma \lambda_i |a_i >< a_i||a_3>$$

Now since all a_is form an orthonormal basis ,

$$< a_i||a_3> = \delta_{i_3} = 1 \text{ when i = 3, 0 otherwise}$$

$$O|a_3> = \Sigma \lambda_i |a_i > \delta_{i_3}$$
$$= \lambda_3 |a_3>$$

For Hermitian Operator since A†A = AA† we will always have spectral decomposition.
Note - If once upon measurement we receive the eigenstate a_i, then no matter how many times we measure it again, the outcome does not change i.e. we will keep getting a_i.
Takeaways :

- $\psi >$ can be written in terms of a_is since they form the orthonormal bases.
- Measurement always leads to evolution to a new state that is part of the bases.

4. Evolution of quantum states is given by Unitary Transformations. Unitary is a linear operator. This is because unitary transformations preserve the inner

product structure of the Hilbert space. This means that the norm of a state (the square root of the probability) remains constant under unitary evolution. Furthermore, unitary transformations are reversible

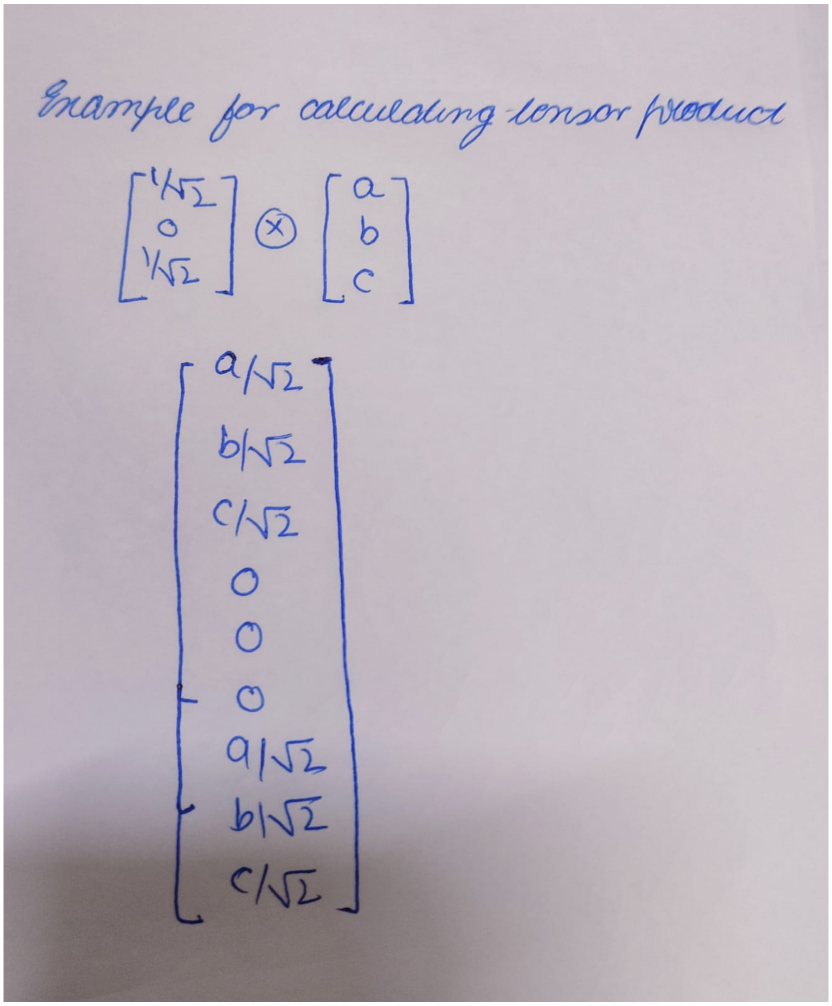
5 Properties of Quantum State

Let χ be a Hilbert Space. For any quantum state $|\psi>\ \epsilon\, \chi$, $<\psi|\psi> = 1$.
Consider an operator A that acts on $|\psi>$ such that
A$|\psi> = |\psi'>$
for $|\psi'>$ to be valid quantum state $<\psi'|\ \psi'>$ =1

Since we said that $|\psi'>$ =A$|\psi>$
hence $|\psi'>= \bar{A}|\psi>$,
taking dagger on both sides we get
$|\psi'>^{\dagger}= \bar{A}|\psi>)^{\dagger}$
Hence, we obtain $\langle\bar{\psi}| = \langle\psi|A^{\dagger}$
Combining the results, we get $\langle\psi|A^{\dagger}A\bar{\psi}\rangle = 1$,
since $\langle\psi|$ is a state, hence $\langle\psi|\psi\rangle = 1$.
Thus, we obtain $A\bar{A}^{\dagger} = 1$.

6 Noisy Quantum Theory

Assuming that $|\psi\rangle$ belongs to Hilbert space X such that $\langle\psi|\psi\rangle = 1$.
Let us consider X_a and X_b as Hilbert spaces, then the Hilbert space of the composite system will be $X_a \otimes X_b$.
Let the dimension of X_a be d_a and the dimension of X_b be d_b, then the dimension of the composite system will be $d_a \times d_b$.
Here is an example of calculating the tensor product of two matrices



7 Expectation

Now we will see expectation of observable A, $\langle A\rangle = \langle\psi|A^\dagger|\psi\rangle$. This is expectation with respect to state $\langle\psi|$. Say $e^{i\theta}|\psi\rangle$ be the state we are considering, then the expectation is given by
$\langle\psi|A^\dagger|\psi\rangle$. substituting the values we get:

$$\langle e^{i\theta}|\psi\rangle|A^\dagger|e^{i\theta}|\psi\rangle = \langle\psi|e^{-i\theta}A^\dagger e^{i\theta}|\psi\rangle = \langle\psi|A^\dagger|\psi\rangle,$$

which is the same as $\langle\psi|A^\dagger|\psi\rangle$, hence they would give the same measurement.

This also shows that the expectation of A does not change under unitary transformation.

8 Density Operators

A quantum state is represented by a density operator defined over Hilbert space $\mathcal{H}$. Naturally, operators map elements of one Hilbert space to another, but this one maps to the same Hilbert space.

The properties of density operators are:

1. $e \geq 0$ (non-negative)
2. $e = e^{\dagger}$ (Hermitian)
3. Trace(e) $= 1$ (normalization)

A density operator can be represented as the outer product of a state vector, $|\psi\rangle$, with its conjugate, $\langle\psi|$, such that: $\rho = |\psi\rangle\langle\psi|$

It is also true that $\rho = \langle\psi|\psi\rangle\ |\psi\rangle\langle\psi| = |\psi\rangle\langle\psi|$

The outer product $|\psi\rangle\langle\psi|$ is a Hermitian operator, meaning that it is equal to its own adjoint/conjugate transpose:

$(|\psi\rangle\langle\psi|)^{\dagger} = |\psi\rangle\langle\psi|$

This is a direct result of the inner product of two state vectors being a scalar and therefore it is a hermitian operator.

Any Hermitian operator can be written in terms of its eigenvectors and eigenvalues. This is known as the spectral decomposition of the operator.

For a given Hermitian operator ρ, its spectral decomposition can be written as: $\rho = \sum_i p_i |i\rangle\langle i|$

Where $|i\rangle$ are the eigenvectors of the operator and p_i are the corresponding eigenvalues. The eigenvalues are non-negative and add up to 1.

It is possible for different density operators to represent the same quantum state. This is because density operators are defined up to a constant factor. For example,

suppose we have two density operators, $\rho_1 = \frac{1}{2}|\psi\rangle\langle\psi|$ and $\rho_2 = |\psi\rangle\langle\psi|$. Both of these operators represent the same quantum state $|\psi\rangle$, but they are defined up to a constant factor of $\frac{1}{2}$.

Let χ be a Hilbert Space. For any quantum state $|\psi >\ \epsilon\ \chi$, $< \psi|\psi > = 1$. Consider an operator A that acts on $|\psi >$ such that

A$|\psi > = |\psi' >$

A is generally written as

$$A = \Sigma_{i,j}(a_i j|i >_B < j|_A) \text{ where A } :\mathcal{H}_A \rightarrow \mathcal{H}_B$$

Note that A and B can have different dimensions

Also , If ψ^2 is equal to ψ then ψ is pure , otherwise it is mixed . If it is mixed, then it is a classical mixture of different pure states . i.e,

$$\text{P(getting } |0 >< 0|) = P(getting|1 >< 1|) = \tfrac{1}{2}$$

$$\sigma = \frac{1}{2}(|0><0|) + \frac{1}{2}(|1><1|)$$

$$\text{Moreover },\ \sigma = \Sigma_{(}p_i)|\psi_i><\psi_i|$$

Here,$|\psi >= \Sigma(C_i|\phi_i >)$
This is basically the superposition of states. Here rank will be 1 , and the state is pure

9 Observables

Observables are represented by Hermitian operators. There exists a Hermitian operator for each observable..This is because when we measure something in the real world , we get only real values . This guarantees that eigenvalues are real

10 Measurement Operators

Measurement Operators can be represented by 'Positive Operator Valued Measurements'(POVMs) denoted by $(\lambda^x)_x$, such that

11 Projective Measurement

We Find spectral decomposition with respect to the observable.

Projective Operators $\mathbb{P}_x$ satisfy the following :

1. $\mathbb{P}_x^2 = \mathbb{P}_x$ for each x
2. $\mathbb{P}x\mathbb{P}_y = \delta xy\mathbb{P}_x$

12 Properties of POVMs $\{\Lambda^x\}_x$

1. $\Lambda^x \geq 0 \forall x$ (non-negative)
2. $\Sigma_x \Lambda^x = \mathbb{1}$ (probability is preserved)
3. $p(x) = Tr[\Lambda^x \rho]$ (Born's Rule)

We can construct Λ^x with a hermitian. In a way density operators
are observables (hermitian) Measurement Operators
Making ρ from $\Theta = \Theta^\dagger$ (make Trace = 1)
First exponentiate it then make trace = 1

$$\frac{\Theta}{Tr[\Theta]} \rightarrow \frac{e^\Theta}{Tr[e^\Theta]}$$

Let the current state be $|\phi >$ then after measurement the state of the system will be $\frac{\mathbb{P}\rho\mathbb{P}^\dagger}{Tr[\mathbb{P}\rho\mathbb{P}^\dagger]}$ where $\mathbb{P}$ is the Projective measurement
Note that $\mathbb{P} = \mathbb{P}^\dagger$ and $\mathbb{P} = \mathbb{P}^2$
p(i) = Tr[$\mathbb{P}_i\rho$]. Also if $\hat{O} = \Sigma_i O_i |i >< i|$
then the set of projectors can be $|i >< i|$

13 Transformations/Evolution of Quantum States

State ρ after evolution is $U\rho U^\dagger$
In pure state the evolution is as follows $|\Phi > \rightarrow U|\Phi >$ (Unitary evolution is noiseless)
In General Evolution $\rightarrow$ Quantum Channel (Anything that transforms a quantum state)

13.1 Quantum Channel

Quantum channel are completely Positive and trace preserving($Tr[\rho] = \mathbb{1}$) maps (PTP maps)

$$N_{A\rightarrow B} : B(H_A) \rightarrow B(H_B)$$

where B is a set of operators. This equation can be interpreted as from 1 point in time to another
$N_{A\rightarrow B}$ is a map that maps operators from $B(H_A)$ to $B(H_B)$

14 Quantum Channel

$$\mathcal{N}A \longrightarrow B : \beta(\mathcal{H}A) \longrightarrow \beta(\mathcal{H}_B)$$

where,

1. $\mathcal{N}$ represents a Quantum Channel
2. $\mathcal{N}$ is a a super operator, i.e. it acts on other operators.
3. $\beta(\mathcal{H}_A)$ denotes a set of operators
4. β is bounded, trace class operators i.e. trace is finite

Therfore, $\mathcal{N}_{A\longrightarrow B}$ is trace preserving

$\mathcal{N}_{A\longrightarrow B}$ is positive operator i.e. if

$$\text{X} \geq 0 \Longrightarrow \mathcal{N}(\text{X}) \geq 0$$

where, A ≥ 0 means eignevalues of A are all positive

Note: $\{|i\rangle\, A\}_i \otimes \{|j\rangle\, B\}j$ can be written as $\{|i\rangle\, A \otimes |j\rangle\, B\}i, j$

15 Tensor Product of Hilbert Space

$$\mathcal{H}_A \otimes \mathcal{H}_B$$
$$= \{|i\rangle A\}_i \otimes \{|j\rangle B\}_j$$
$$= \{|i\rangle A \otimes |j\rangle B\}_{i,j}$$

where,
$|i\rangle_A$ is the standard basis of $\mathcal{H}_A$
$|j\rangle_B$ is the standard basis of $\mathcal{H}_B$

16 Composite Systems

16.1 Product State

If:

$$\rho_{AB} = \rho_A \otimes \rho_B$$

$\Longrightarrow$ A and B are in product state
$\Longrightarrow$ A and B are mutually independent

16.2 Seperable State

$$\rho_{AB} = \sum_x p_x(\rho_A^x \otimes \rho_B^x)$$

If such a decomposition exists, A and B are in a seperable state.
Finding such a decomposition is a NP-hard problem.

16.3 Entangled State

If a decomposition does not exist, A and B are in a entangled state.

$$\Phi_{AB} = \frac{1}{d} \sum_{i,j} |i\rangle_A \otimes |i\rangle_B \langle j|_A \otimes \langle j|_B$$

where,

$$\text{d=min}\{\text{dim(A)},\text{dim(B)}\}$$

This state cannot be decomposed into Seperable form.
Example of such a Φ_{AB}:

$$\text{let } |\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B)$$

then $|\psi\rangle\langle\psi|$ is a Φ of d=2

$\mathcal{N}_{A \longrightarrow B}$ is completely positive operator if

$$(id)B \otimes \mathcal{N}A \longrightarrow B(\phi_{AB}) \geq 0 \forall \phi_{AB} \geq 0$$

Where $(id)B$ is called a super operator and applied on $\mathcal{N}A \longrightarrow B$ to nullify affect of $\mathcal{N}_{A \longrightarrow B}$ on state of B

$$\Phi = |\psi\rangle \langle\psi|$$

where,

$$|\psi\rangle = \sum \frac{1}{\sqrt{d}} |i\rangle_A \otimes |i\rangle_B$$

$\implies \phi$ is a special entangled state, checking for this is enough to satisfy the conditions for a valid quantum channel :

1. $Tr[\rho_A] = Tr[\mathcal{N}A \longrightarrow B(\phi A)]$
2. $(id)B \otimes \mathcal{N}A \longrightarrow B(\phi_{AB}) \geq 0$

$$\phi_{RA} = |\phi\rangle \langle\phi| RA = \frac{1}{d} \sum i, j\, |i\rangle_R \otimes |i\rangle_A \langle j|_R \otimes \langle j|_A$$

$$|\phi\rangle RA = \frac{1}{\sqrt{d}} \sum i^{d-1} |i\rangle R \otimes |i\rangle A$$

$$|\psi\rangle RA = \sum i = 0^{d-1} \sqrt{p_i}\, |\psi\rangle R \otimes |\phi\rangle A$$

where,
$\{|\psi\rangle_R\}$ denotes orthonormal basis in R
$\{|\phi\rangle_A\}$ denotes orthonormal basis in A

17 Kraus Operators

$$\mathcal{N}A \longrightarrow B(X) = \sum iK_i(X)K_i^+$$

such that

$$\sum_i K_i^+ K_i = I$$

Where the set of operators K_i are called **Kraus Operators**
We could call projection operators $(\{\mathcal{P}_i\})$ as a special case of Kraus Operators

18 Introduction to Quantum Computation

Recall

Given the properties of quantum states and transformations, noticing the similarities is inevitable. There is, in fact, a correspondence between quantum channels and quantum states. The Choi-Jamiolkowski isomorphism establishes a connection between linear maps from Hilbert space 1 to Hilbert space 2 and operators in their tensor product space.

A quantum channel is a completely positive, trace-preserving linear map.

$$\Phi : \mathscr{L}(\mathscr{H}_1) \mapsto \mathscr{L}(\mathscr{H}_2).$$

Here equation (1) is the space of linear operators on the Hilbert space. The map takes a density matrix acting on the system in the Hilbert space 1 to a density matrix acting on the system in the Hilbert space 2.

Since density matrices are positive, the map must preserve positivity, hence the requirement for a positive map. Furthermore, if an ancilla of some finite dimension n is coupled to the system, then the induced map, where one is the identity map on the ancilla, must be positive—that is, the cross product is positive for all n. Such maps are called completely positive.

The last constraint, the requirement to preserve the trace, derives from the density matrices having trace 1.

For example, the unitary time evolution of a system is a quantum channel. It maps states in the same Hilbert space, and it is trace-preserving because it is unitary. More generic quantum channels between two Hilbert spaces act as communication channels which transmit quantum information. In this regard, quantum channels generalize unitary transformations.

We define a matrix for a completely positive, trace-preserving map in the following way:

This is called the Choi matrix of the map. By Choi's theorem on completely positive maps, the map is completely positive if and only if the density matrix is positive. The density matrix is the state dual to the quantum channel.

This map is known as the Choi-Jamiołkowski isomorphism. This duality is convenient, as instead of studying linear maps, we can study the associated linear operators, which underlines the analogue that these maps are the generalization of unitary operators.

THANK YOU