QUANTUM ASSIGNMENT 3

PRISHA
2021101075

Q1: Consider the qubit states $\{|\psi_K\rangle\langle\psi_K|\}_{K=0}^{K=4}$ where $|\psi_K\rangle$ is defined as

$$|\psi_K\rangle = \cos\frac{2\pi K}{5}|0\rangle + \sin\frac{2\pi K}{5}|1\rangle \qquad K \in \{0,1,2,3,4\}$$

to check whether $\left\{\frac{2}{5}|\psi_K\rangle\langle\psi_K|\right\}_{K=0}^{K=4}$ is a POVM or not.

Check if PM.

Solution, $|\psi_0\rangle = \cos\frac{2\pi(0)}{5}|0\rangle + \sin\frac{2\pi(0)}{5}|1\rangle$

$$= 1\cdot|0\rangle + 0\cdot|1\rangle = \begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} \quad \text{—(i)}$$

so $|\psi_1\rangle = \cos\frac{2\pi}{5}|0\rangle + \sin\frac{2\pi}{5}|1\rangle = \cos\frac{2\pi}{5}\begin{bmatrix}1\\0\end{bmatrix} + \sin\frac{2\pi}{5}\begin{bmatrix}0\\1\end{bmatrix}$

$$= \begin{bmatrix}\cos 2\pi/5\\ \sin 2\pi/5\end{bmatrix} \quad \text{—(ii)}$$

$|\psi_2\rangle = \cos\frac{4\pi}{5}|0\rangle + \sin\frac{4\pi}{5}|1\rangle = \cos\frac{4\pi}{5}\begin{bmatrix}1\\0\end{bmatrix} + \sin\frac{4\pi}{5}\begin{bmatrix}0\\1\end{bmatrix}$

$$= \begin{bmatrix}\cos 4\pi/5\\ \sin 4\pi/5\end{bmatrix} = \begin{bmatrix}\cos(\pi-\pi/5)\\ \sin(\pi-\pi/5)\end{bmatrix}$$

$\left(\begin{matrix}\cos(180-\theta) = -\cos\theta\\ \sin(180-\theta) = \sin\theta\end{matrix}\right)$

$$= \begin{bmatrix}-\cos\pi/5\\ \sin\pi/5\end{bmatrix} \quad \text{—(iii)}$$

$$|\psi_3\rangle = \cos\frac{6\pi}{5}|0\rangle + \sin\frac{6\pi}{5}|1\rangle$$

$$|\psi_3\rangle = \cos\frac{6\pi}{5}\begin{bmatrix}1\\0\end{bmatrix} + \sin\frac{6\pi}{5}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}\cos 6\pi/5\\ \sin 6\pi/5\end{bmatrix} = \begin{bmatrix}\cos(\pi+\pi/5)\\ \sin(\pi+\pi/5)\end{bmatrix}$$

$$= \begin{bmatrix}-\cos\pi/5\\ -\sin\pi/5\end{bmatrix}$$

$$|\psi_4\rangle = \cos\frac{8\pi}{5}|0\rangle + \sin\frac{8\pi}{5}|1\rangle = \cos\frac{8\pi}{5}\begin{bmatrix}1\\0\end{bmatrix} + \sin\frac{8\pi}{5}\begin{bmatrix}0\\1\end{bmatrix}$$

$$= \begin{bmatrix}\cos 8\pi/5\\ \sin 8\pi/5\end{bmatrix} = \cancel{\cos}\begin{bmatrix}\cos(2\pi-\frac{2\pi}{5})\\ \sin(2\pi-\frac{2\pi}{5})\end{bmatrix}$$

$$= \begin{bmatrix}-\cos 2\pi/5\\ \sin 2\pi/5\end{bmatrix}$$

we evaluate for the expressions,

$\cancel{\langle\psi_0|\psi_0\rangle = /[/1\ /0]}$ $\quad|\psi_k\rangle\langle\psi_k|$ so,

$$|\psi_0\rangle\langle\psi_0| = \underset{2\times1}{\begin{bmatrix}1\\0\end{bmatrix}}\underset{1\times2}{\begin{bmatrix}1 & 0\end{bmatrix}} = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} \text{ --- } (i^*) \text{ (using } i\text{)}$$

$$|\psi_1\rangle\langle\psi_1| = \begin{bmatrix}\cos 2\pi/5\\ \sin 2\pi/5\end{bmatrix}\begin{bmatrix}\cos 2\pi/5 & \sin 2\pi/5\end{bmatrix} = \begin{bmatrix}\cos^2 2\pi/5 & \cos\frac{2\pi}{5}\sin\frac{2\pi}{5}\\ \sin\frac{2\pi}{5}\cos\frac{2\pi}{5} & \sin^2\frac{2\pi}{5}\end{bmatrix}$$

$$|\psi_2\rangle\langle\psi_2| = \begin{bmatrix}-\cos\pi/5\\ \sin\pi/5\end{bmatrix}\begin{bmatrix}-\cos\pi/5 & \sin\pi/5\end{bmatrix} = \begin{bmatrix}\cos^2\pi/5 & -\cos\frac{\pi}{5}\sin\frac{\pi}{5}\\ \sin\frac{\pi}{5}\cos\frac{\pi}{5} & \sin^2\frac{\pi}{5}\end{bmatrix}$$

$$|\psi_3\rangle\langle\psi_3| = \begin{bmatrix} -\cos\pi/5 \\ -\sin\pi/5 \end{bmatrix} \begin{bmatrix} -\cos\pi/5 & -\sin\pi/5 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\pi/5 & \cos\pi/5 \sin\pi/5 \\ \sin\frac{\pi}{5}\cos\frac{\pi}{5} & \sin^2\pi/5 \end{bmatrix}$$

$$|\psi_4\rangle\langle\psi_4| = \begin{bmatrix} -\cos(2\pi/5) \\ \sin(2\pi/5) \end{bmatrix} \begin{bmatrix} -\cos\left(\frac{2\pi}{5}\right) & \sin\frac{2\pi}{5} \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2 2\pi/5 & -\cos\frac{2\pi}{5}\sin\frac{2\pi}{5} \\ -\sin\left(\frac{2\pi}{5}\right)\cos\left(\frac{2\pi}{5}\right) & \sin^2\left(\frac{2\pi}{5}\right) \end{bmatrix}$$

We want to check if summation of elements yields identity

$$\sum_{K=0}^{K=4} \frac{2}{5} |\psi_K\rangle\langle\psi_K| = \frac{2}{5}\left(\sum_{K=0}^{K=4} |\psi_K\rangle\langle\psi_K|\right)$$

$$\frac{2}{5}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \cos^2 2\pi/5 & \cancel{\cos 2\pi/5 \sin 2\pi/5} \\ \cancel{+\sin\frac{2\pi}{5}\cos\frac{2\pi}{5}} & \sin^2\frac{2\pi}{5} \end{bmatrix} + \begin{bmatrix} \cos^2\pi/5 & \cancel{-\cos\frac{\pi}{5}\sin\frac{\pi}{5}} \\ \cancel{-\sin\frac{\pi}{5}\cos\frac{\pi}{5}} & \sin^2\frac{\pi}{5} \end{bmatrix}\right.$$

$$\left. + \begin{bmatrix} \cos^2\pi/5 & \cancel{\cos\pi/5 \sin\pi/5} \\ \cancel{\sin\pi/5 \cos\pi/5} & \sin^2\pi/5 \end{bmatrix} + \begin{bmatrix} \cos^2 2\pi/5 & \cancel{-\cos\frac{2\pi}{5}\sin\frac{2\pi}{5}} \\ \cancel{-\sin\frac{2\pi}{5}\cos\frac{2\pi}{5}} & \sin^2\frac{2\pi}{5} \end{bmatrix}\right)$$

$$\frac{2}{5}\left(\begin{bmatrix} 1 + 2\cos^2\pi/5 + 2\cos^2 2\pi/5 & 0 \\ 0 & 2\sin^2\pi/5 + 2\sin^2 2\pi/5 \end{bmatrix}\right)$$

$$\frac{2}{5}\left(\begin{bmatrix} 5/2 & 0 \\ 0 & 5/2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

now, as we see for each $|\psi\rangle$ we have normalized values

hence, $|\psi\rangle\langle\psi|$ would be a density matrix.

this is semi-definite.

Since, it satisfies both condition of POVM i.e

✓ measurement operator are +ve semidefinite

✓ sum of measurement operator equals identity

Since both condition are satisfied so it's POVM

but not projective measurement as for projective measurement, operators need to be mutually orthogonal which is not in given scenario.

Q2 Given we are doing walsh hadamard transform,

$$|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad |1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

(1) find matrix representation of H for basis $\{|0\rangle, |1\rangle\}$

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Solution: walsh hadamard transform is a single qubit operation denoted by H, performs following transformation

$$|0\rangle \longrightarrow \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|1\rangle \longrightarrow \frac{1}{\sqrt{2}}(|0\rangle \cancel{+} - |1\rangle)$$

$$= \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right](1 \;\; 0) + \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right](0 \;\; 1)$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}(1 \;\; 0) + \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}(0 \;\; 1)$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$$

$$\Rightarrow \boxed{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}$$

(iii) find matrix representation of H in basis $|+\rangle, |-\rangle$.

where $|+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $|-\rangle = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

now,

$$\Rightarrow \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right]\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix} + \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right]\frac{1}{\sqrt{2}}(1 \;$$

$$\Rightarrow \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\begin{pmatrix} 2 \\ 0 \end{pmatrix}\right]\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} + \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 2 \end{pmatrix}\right]\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$$

$$= \left[\frac{1}{2}\begin{pmatrix} 2 \\ 0 \end{pmatrix}\right]\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} + \left[\frac{1}{2}\begin{pmatrix} 0 \\ 2 \end{pmatrix}\right]\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

$$= \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}$$

$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

(iii) Given operator $H = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$, find inverse of this operator.

<u>Solution</u> so we have $H = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$ so inverse would be $\dfrac{adj(H)}{det(H)}$

$$det(H) = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{-1}{\sqrt{2}}\right) - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{-1}{2} - \frac{1}{2} = -1$$

adj(H):-

If matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow[\text{cofactor matrix}]{} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \Big\downarrow \text{adjoint matrix}$$

$$\begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \leftarrow \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}^T$$

So, $adj(H) = \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}}$

adj(A) :- if matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow[\text{cofactor matrix}]{} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \Big\downarrow \text{adjoint matrix}$$

$$\begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \longleftarrow \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}^T$$

so, $\text{adj}(A) = \underbrace{\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & +1/\sqrt{2} \end{bmatrix}}$ hence inverse $= \dfrac{\begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}{-1}$

$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$ Ans

Question: Consider a system in state $|\psi\rangle\langle\psi|$ ---

.... getting value 1 for B.

Solution: (i) Let $|\phi\rangle$ represent normalized state.

Eigen values for A will be, $|A-dI|=0$

$$\frac{1}{\sqrt{2}}\begin{vmatrix} (2-d) & 0 & 0 \\ 0 & 1-d & i \\ 0 & -i & 1-d \end{vmatrix} = 0$$

$$(2-d)\left[(1-d)(1-d)-(-i)(i)\right]=0$$

$$(2-d)\left(1+d^2-2d-(-i^2)\right)=0$$

$$(2-d)\left(1+d^2-2d-(-(-1))\right)=0$$

$$(2-d)(d)(d-2)=0$$

$d_1=0$

$d_2=2$

$d_3=2$

Now eigen vector corresponding to 0 for A,

$$(A-dI)V=0$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & i \\ 0 & -i & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0$$

$2x_1=0$

$\boxed{x_1=0}$

$x_2+ix_3=0 \quad x_2=-ix_3$

$-ix_2+x_3=0 \quad x_3=ix_2$

so, eigen vector we have is $\begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix}$

hence, normalized eigenvector $|\hat{x}\rangle = \dfrac{\begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix}}{\sqrt{2}}$

hence probability of getting zero :: $|\langle \hat{x}|\phi\rangle|^2$

$$\left| \frac{(0 \;\; +i \;\; 1)}{\sqrt{2}} \frac{\begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}}{\sqrt{17}} \right|^2$$

→ conventional dot product

(where $|\phi\rangle$: normalized state)

$$= \left|\frac{4}{\sqrt{34}}\right|^2 = \frac{16}{34} = \boxed{\frac{8}{17}}$$

(ii) Once we are already in the state corresponding to $d = 0$

let $B = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{vmatrix}$

so find B's eigen vector →

$$|B - dI| = 0$$

$$(1-d)(d^2 - (-i^2)) = 0$$

$$(1-d)(d^2 - 1) = 0$$

→ $d_1 = 1$, $d_2 = -1$, $d_3 = 1$

now, eigen vector corresponding to $\lambda = 1$ is

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -i \\ 0 & i & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\boxed{x_1 = 0}$

$-x_2 - ix_3 = 0$

$\boxed{x_2 = -ix_3}$ — (ii)

$ix_2 - x_3 = 0$

$\boxed{x_3 = ix_2}$ — (i)

$\downarrow$

$x_3 = i(-ix_3)$

$x_3 = i(-ix_3)$ ✓

$= -(-1)(x_3)$

hence, eigen vector : $\begin{bmatrix} 0 \\ 1 \\ i \end{bmatrix}$ normalize :

$\begin{bmatrix} 0 \\ 1/\sqrt{2} \\ i/\sqrt{2} \end{bmatrix}$ or $\begin{bmatrix} 0 \\ -i/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

This is same as state of vector before and after measurement.

hence state won't change and measurement outcome $= 1$ (eigen value corresponding to eigen state).