QUANTUM ASSIGNMENT-1

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(1) $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

(a) say let's assume λ be the scalar quantity, representing eigen value.

$AV = \lambda V$ (V is a vector)

$AV - \lambda V = 0$

$AV - \lambda I \cdot V = 0$

$(A - \lambda I)V = 0 \Rightarrow A - \lambda I = 0$ hence, $|A - \lambda I| = 0$ — (1)

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

$-\lambda(\lambda^2 - 1) - 1(-\lambda) = 0$

$-\lambda^3 + \lambda + \lambda = 0$

$2\lambda = \lambda^3$

$\lambda_1 = 0$, $2 = \lambda^2$

$\lambda_2 = \sqrt{2}$ $\lambda_3 = -\sqrt{2}$

So, three eigen values

$\lambda_1 = 0$ $\lambda_2 = \sqrt{2}$ $\lambda_3 = -\sqrt{2}$

- finding eigen vector V say $V = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ now,

using (1)

$$\begin{bmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$\boxed{\lambda = 0}$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \qquad \begin{aligned} y &= 0 \\ x + z &= 0 \\ x &= -z \end{aligned} \equiv \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

normalize $|v| = \sqrt{1+1} = \sqrt{2}$

$$a_1 \leftarrow \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

$\boxed{\lambda = \sqrt{2}}$

$$\begin{bmatrix} -\sqrt{2} & 1 & 0 \\ 1 & -\sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \qquad \begin{aligned} -\sqrt{2}x + y &= 0 \\ z + x - \sqrt{2}y &= 0 \\ y - \sqrt{2}z &= 0 \end{aligned} \qquad \begin{aligned} z + x - 2\cancel{y}^{\,x} &= 0 \\ \boxed{x = z} \\ y &= \sqrt{2}z \end{aligned}$$

$$\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} \xrightarrow[|v| = \sqrt{1+1+2} = 2]{\text{normalize}} \begin{bmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{bmatrix} \rightarrow a_2$$

$\boxed{\lambda = -\sqrt{2}}$

$$\begin{bmatrix} \sqrt{2} & 1 & 0 \\ 1 & \sqrt{2} & 1 \\ 0 & 1 & \sqrt{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} \sqrt{2}x + y \\ x + \sqrt{2}y + z \\ y + \sqrt{2}z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{aligned} x &= z \\ y &= -\sqrt{2}z \end{aligned} \equiv \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$$

normalize $1/\sqrt{1+2+1}$

$$a_3 \leftarrow \begin{bmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{bmatrix}$$

(b) $|a_1\rangle = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$ $|a_2\rangle = \begin{bmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{bmatrix}$ $|a_3\rangle = \begin{bmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{bmatrix}$

so, $\langle a_1| = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{bmatrix}$ $\langle a_2| = \begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix}$ $\langle a_3| = \begin{bmatrix} \frac{1}{2} & \frac{-1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix}$

so, $\boxed{\langle a_j | a_k \rangle = \delta_{jk}}$

If we notice except when $j=k$ we have $\langle a_j | a_k \rangle = 0$ else Identity op

say $j=1$ $k=1$

$$\langle a_1 | a_1 \rangle = \underset{1\times 3}{\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}} = \begin{bmatrix} \frac{1}{2} + 0 + \frac{1}{2} \end{bmatrix} = [1] = I$$

$j=1$ $k=2$

$$\langle a_1 | a_2 \rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2\sqrt{2}} & \frac{-1}{2\sqrt{2}} \end{bmatrix} = [0] = 0$$

Similarly for others

$$\sum_{j=1}^{3} |a_j\rangle\langle a_j| = I$$

$$\underset{3\times 1}{\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}} \underset{1\times 3}{\begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}} + \begin{bmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{bmatrix} \begin{bmatrix} 1/2 & -1/\sqrt{2} & 1/2 \end{bmatrix} + \begin{bmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{bmatrix} \begin{bmatrix} 1/2 & -1/\sqrt{2} & 1/2 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix} + \begin{bmatrix} 1/4 & -\frac{1}{2\sqrt{2}} & 1/4 \\ \frac{1}{2\sqrt{2}} & -1/2 & 1/2\sqrt{2} \\ \frac{1}{4} & \frac{-1}{2\sqrt{2}} & 1/4 \end{bmatrix} + \begin{bmatrix} 1/4 & \frac{-1}{2\sqrt{2}} & 1/4 \\ \frac{-1}{2\sqrt{2}} & 1/2 & -1/2\sqrt{2} \\ 1/4 & +1/2\sqrt{2} & 1/4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

(c) so we have, matrix A as,

$$A = \begin{bmatrix} 1/\sqrt{2} & 1/2 & 1/2 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & 1/2 & 1/2 \end{bmatrix}$$

and final basis B is

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

inorder to find transformation matrix U.

$$U \cdot A = B$$

$$U = B \cdot A^{-1}$$

now, $A^{-1} = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/2 & 1/\sqrt{2} & 1/2 \\ 1/2 & -1/\sqrt{2} & 1/2 \end{pmatrix}$

since $B = I$
hence $B \cdot A^{-1} = A^{-1}$

hence $U = \underbrace{\begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/2 & 1/\sqrt{2} & 1/2 \\ 1/2 & -1/\sqrt{2} & 1/2 \end{pmatrix}}$

(2) $|\psi\rangle = |\phi_1\rangle + |\phi_2\rangle$

$|\chi\rangle = |\phi_1\rangle - |\phi_2\rangle$

show $\langle\psi|\psi\rangle + \langle\chi|\chi\rangle = 2\langle\phi_1|\phi_1\rangle + 2\langle\phi_2|\phi_2\rangle$

now as $|\psi\rangle = |\phi_1\rangle + |\phi_2\rangle$

$\langle\psi| = \langle\phi_1| + \langle\phi_2|$

similarly $\langle\chi| = \langle\phi_1| - \langle\phi_2|$

$$LHS = \langle\psi|\psi\rangle + \langle\chi|\chi\rangle$$

$$= (\langle\phi_1| + \langle\phi_2|)(|\phi_1\rangle + |\phi_2\rangle) + (\langle\phi_1| - \langle\phi_2|)(|\phi_1\rangle - |\phi_2\rangle)$$

$$= \langle\phi_1,\phi_1\rangle + \cancel{\langle\phi_1|\phi_2\rangle} + \cancel{\langle\phi_2|\phi_1\rangle} + \langle\phi_2|\phi_2\rangle + \langle\phi_1|\phi_1\rangle - \cancel{\langle\phi_1|\phi_2\rangle}$$

$$- \cancel{\langle\phi_2,\phi_1\rangle} + \langle\phi_2|\phi_2\rangle$$

$$= \underbrace{2\langle\phi_1,\phi_1\rangle + 2\langle\phi_2|\phi_2\rangle}_{RHS}$$

hence proved LHS = RHS

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(3) $|\psi\rangle = \frac{1}{\sqrt{2}}|\phi_1\rangle + \frac{1}{\sqrt{5}}|\phi_2\rangle + \frac{1}{\sqrt{10}}|\phi_3\rangle$ ($|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle$ orthonormal)

$\hat{B}|\phi_n\rangle = n^2|\phi_n\rangle$

(a) is $|\psi\rangle$ normalized?

normalized state. magnitude = 1.

$$\langle\psi| = \frac{1}{\sqrt{2}}\langle\phi_1| + \frac{1}{\sqrt{5}}\langle\phi_2| + \frac{1}{\sqrt{10}}\langle\phi_3|$$

$$\langle\psi|\psi\rangle = \left(\frac{1}{\sqrt{2}}\langle\phi_1| + \frac{1}{\sqrt{5}}\langle\phi_2| + \frac{1}{\sqrt{10}}\langle\phi_3|\right)\left(\frac{1}{\sqrt{2}}|\phi_1\rangle + \frac{1}{\sqrt{5}}|\phi_2\rangle + \frac{1}{\sqrt{10}}|\phi_3\rangle\right)$$

(note $\langle\phi_i|\phi_j\rangle = 0$ when $i \neq j$ as orthonormal states) — ①
if $i = j$ then $\langle\phi_i|\phi_j\rangle = 1$

$$\langle\psi|\psi\rangle = \left(\frac{1}{2}\langle\phi_1|\phi_1\rangle + \frac{1}{\sqrt{10}}(\langle\phi_1|\phi_2\rangle) + \frac{1}{\sqrt{20}}\langle\phi_1|\phi_3\rangle + \frac{1}{\sqrt{10}}\langle\phi_2|\phi_1\rangle + \frac{1}{5}\langle\phi_2|\phi_2\rangle + \frac{1}{10}\langle\phi_3|\phi_3\rangle\right) \quad (\text{using } 1)$$

$$\langle\psi|\psi\rangle = \left(\frac{1}{2} + \frac{1}{\sqrt{10}}(0) + \frac{1}{\sqrt{20}}(0) + \frac{1}{\sqrt{10}}(0) + \frac{1}{5} + \frac{1}{10}\right)$$

$$\langle\psi|\psi\rangle = \left(\frac{8}{10}\right)$$

$\longrightarrow \neq 1$ (hence not normalized)

(b) $$\langle\hat{B}\rangle = \frac{\langle\psi|\hat{B}|\psi\rangle}{\langle\psi|\psi\rangle}$$

we know, it doesnot matter on whom we act operator first say

$$= \frac{\langle\psi|(\hat{B}|\psi)\rangle}{\langle\psi|\psi\rangle}$$

$$\hat{B}|\psi\rangle = \hat{B}\left(\frac{1}{\sqrt{2}}|\phi_1\rangle + \frac{1}{\sqrt{5}}|\phi_2\rangle + \frac{1}{\sqrt{10}}|\phi_3\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\hat{B}|\phi_1\rangle + \frac{1}{\sqrt{5}}\hat{B}|\phi_2\rangle + \frac{1}{\sqrt{10}}\hat{B}|\phi_3\rangle$$

$$= \frac{1}{\sqrt{2}}\cdot 1|\phi_1\rangle + \frac{1}{\sqrt{5}}\cdot(2)^2|\phi_2\rangle + \frac{1}{\sqrt{10}}\cdot 9\cdot|\phi_3\rangle$$

$$\langle\psi|B|\psi\rangle = \left(\frac{1}{\sqrt{2}}\langle\phi_1| + \frac{1}{\sqrt{5}}\langle\phi_2| + \frac{1}{\sqrt{10}}\langle\phi_3|\right)\left(\frac{1}{\sqrt{2}}|\phi_1\rangle + \frac{4}{\sqrt{5}}|\phi_2\rangle + \frac{9}{\sqrt{10}}|\phi_3\rangle\right)$$

$$\langle\psi|B|\psi\rangle = \frac{1}{\sqrt{2}\times\sqrt{2}}\langle\phi_1|\phi_1\rangle + \frac{1\cdot 4}{5}\langle\phi_2|\phi_2\rangle + \frac{1\cdot 9}{10}\langle\phi_3|\phi_3\rangle$$

$$= \frac{1}{2} + \frac{4}{5} + \frac{9}{10} = \frac{5+8+9}{10} = \frac{22}{10}$$

$$\langle\psi|\psi\rangle = \underbrace{\left(\frac{1}{\sqrt{2}}|\phi_1\rangle + \frac{1}{\sqrt{5}}|\phi_2\rangle + \frac{1}{\sqrt{10}}|\phi_3\rangle\right)}_{|\psi\rangle}\left(\frac{1}{\sqrt{2}}\langle\phi_1| + \frac{1}{\sqrt{5}}\langle\phi_2| + \frac{1}{\sqrt{10}}\langle\phi_3|\right)$$

$$= \frac{1}{2} + \frac{1}{5} + \frac{1}{10} = \frac{5+2+1}{10} = 8/10 \quad \text{— (11)}$$

$$\frac{\langle\psi|B|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{22/\cancel{10}}{8/\cancel{10}} = \frac{11}{4} \quad \underline{\underline{\text{Ans}}}$$

(c) expectation value of $\bar{B}^2$.

$$\hat{B}|\theta_n\rangle = n^2|\theta_n\rangle$$

$$\hat{B}(\hat{B}|\theta_n\rangle) = \hat{B}n^2|\theta_n\rangle \quad (\text{now } \hat{B}|\theta_n\rangle = n^2|\theta_n\rangle)$$

$$\boxed{\hat{B}^2|\theta_n\rangle = n^2|\theta_n\rangle n^2}$$

now

$$\langle\psi|\bar{B}^2|\psi\rangle = \langle\psi|(\bar{B}^2|\psi)\rangle$$

$$\langle\bar{B}^2|\psi\rangle = \bar{B}^2\left(\frac{1}{\sqrt{2}}|\theta_1\rangle + \frac{1}{\sqrt{5}}|\theta_2\rangle + \frac{1}{\sqrt{10}}|\theta_3\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\bar{B}^2|\theta_1\rangle + \frac{1}{\sqrt{5}}\bar{B}^2|\theta_2\rangle + \frac{1}{\sqrt{10}}\bar{B}^2|\theta_3\rangle$$

$$= \frac{1}{\sqrt{2}}\cdot 1|\theta_1\rangle + \frac{1}{\sqrt{5}}(2)^4|\theta_2\rangle + \frac{1}{\sqrt{10}}\cdot(3)^4|\theta_3\rangle$$

$$= \frac{|\theta_1\rangle}{\sqrt{2}} + \frac{16}{\sqrt{5}}|\theta_2\rangle + \frac{81}{\sqrt{10}}|\theta_3\rangle \quad \text{--- (i)}$$

now $\langle\psi|\bar{B}^2|\psi\rangle = \left(\frac{1}{\sqrt{2}}\langle\theta_1| + \frac{16}{\sqrt{5}}\langle\theta_2| + \frac{1}{\sqrt{10}}\langle\theta_3|\right)\left((i)\right)$

$$= \left(\frac{1}{\sqrt{2}}\langle\theta_1| + \frac{\langle\theta_2|}{\sqrt{5}} + \frac{\langle\theta_3|}{\sqrt{10}}\right)\left(\frac{|\theta_1\rangle}{\sqrt{2}} + \frac{16}{\sqrt{5}}|\theta_2\rangle + \frac{81}{\sqrt{10}}|\theta_3\rangle\right)$$

$$\frac{1}{2} + \frac{16}{5} + \frac{81}{10} = \frac{5+32+81}{10} = \frac{128}{10}$$

$$\langle\psi|\psi\rangle = \frac{8}{10} \quad (\text{from (ii)}) \qquad = \frac{128/10}{8/10} = \frac{59}{4}$$

(4) Prove that for hermitian ($\hat{A}^{+}=\hat{A}$), all eigenvalue are real and eigen vector corresponding to diff eigenvalue are orthogonal.

<u>Solution</u>: So we will first prove that eigenvalues are real

say $\hat{H}$ be hermitian operator on inner product space V with complex number $\mathbb{C}$

Since hermitian $\hat{H}=\hat{H}^{+}$ then for eigen vector $|x\rangle \in V$ and eigen value d.

$$\hat{H}|x\rangle = d|x\rangle \quad \text{—— (i)}$$

complex conjugate

now, $\langle x|\hat{A}|y\rangle = \langle y|\hat{A}^{+}|x\rangle^{*}$

$$\langle x|\hat{H}|x\rangle = \langle x|d x\rangle$$
$$= d\langle x|x\rangle \quad \text{—— (1)}$$

Since $\underbrace{\langle x|\hat{H}|x\rangle}_{} = \langle x|\hat{H}|x\rangle^{*}$ (as $\hat{H}^{+}=\hat{H}$)

from (1)

$$d\langle x|x\rangle = \langle x|d|x\rangle^{*}$$
$$= (d\langle x|x\rangle)^{*}$$
$$d\langle x|x\rangle = d^{*}\langle x|x\rangle$$

hence $\boxed{d=d^{*}}$ hence eigenvalue are real

To prove ψ_1, ψ_2 are orthogonal we need $\psi_1^{*}\psi_2 = 0$

as, $\hat{A}\psi_1 = d_1\psi_1$ — (1)

$\hat{A}\psi_2 = d_2\psi_2$ — (2)

using (1) taking complex conjugate on both sides,

$$(\hat{A}\psi_1)^* = (d_1\psi_1)^*$$

$$\psi_1^* \hat{A}^* = d_1^* \psi_1^*$$

$$\psi_1^* \hat{A}^* = d_1 \psi_1^* \quad \text{— (III)}$$

using (III) taking inner product with ψ_2.

$$\psi_1^* A^* \psi_2 = d\psi_1^* \psi_2$$

since A is hermitian $A^* = A$

(hence $A\psi_2 = d_2\psi_2$)

$$\psi_1^* A \psi_2 = d_1 \psi_1^* \psi_2$$

$$\psi_1^* d_2 \psi_2 = d_1 \psi_1^* \psi_2$$

(since $d_1 \neq d_2$, so $\psi_1^* \psi_2 = 0$)

$$(d_1 - d_2)\psi_1^* \psi_2 = 0$$

since $\psi_1^* \psi_2 = 0$ (hence orthogonal)

Thanks!