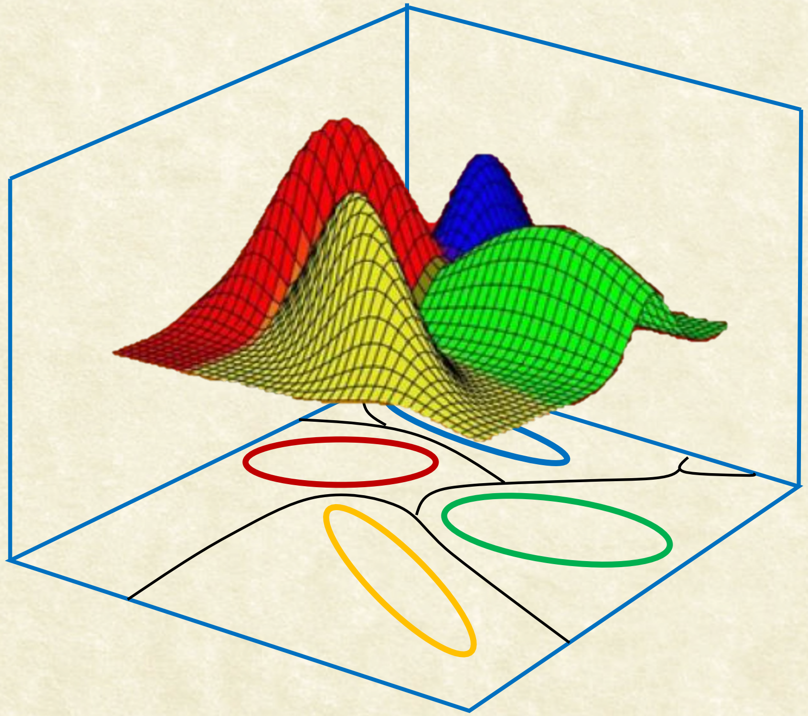
CS7.404: Digital Image Processing

Monsoon 2023: Fourier Transform - 2

A Closer Look



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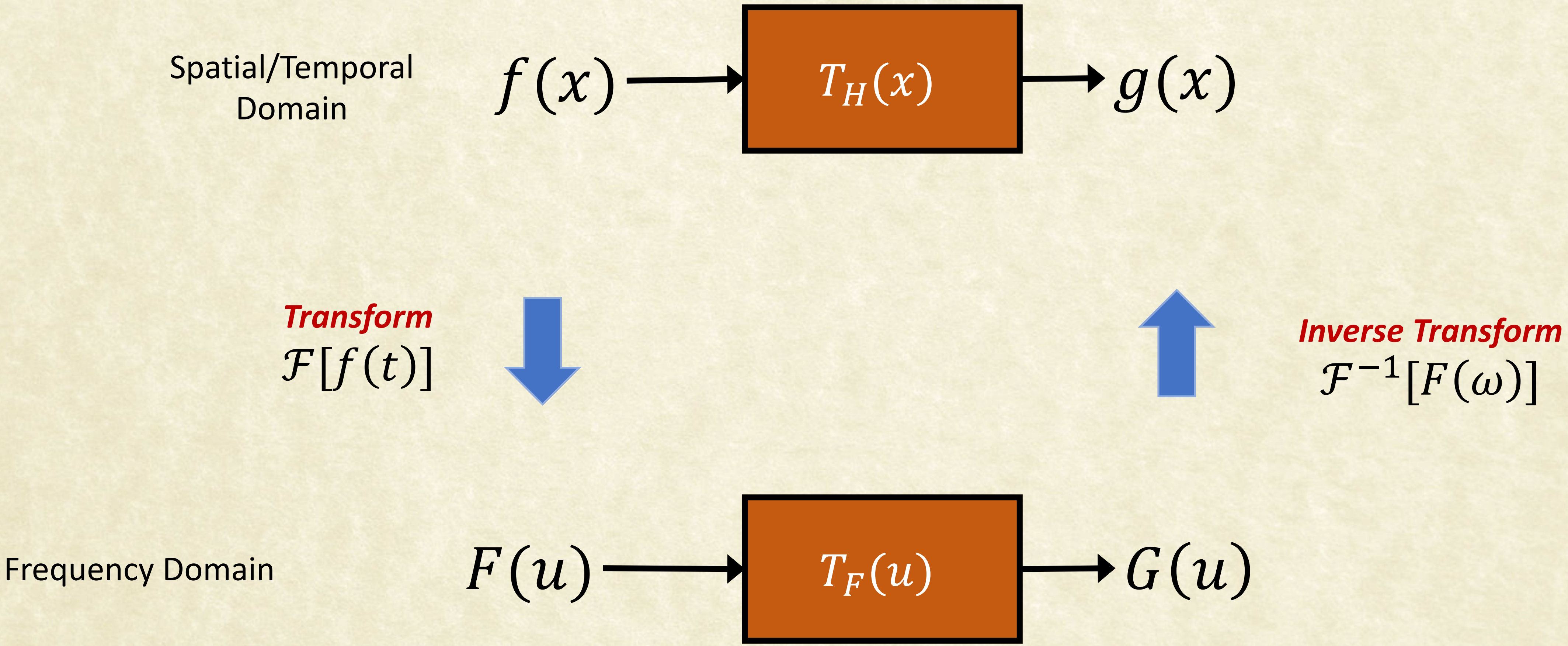
Processing in Spatial vs Frequency Domain



Fourier Transform (FT): Recap

- The Fourier Transform of a function $f(t)$ is defined by:

$$F(\omega) = \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt$$

- The Inverse Fourier Transform (IFT) is given by:

$$f(t) = \mathcal{F}^{-1}[F(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}\,d\omega$$

Fourier Series: Recap

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi n}{T}t}$$

$$c_n = \frac{1}{T}\int_0^T f(t)e^{-i\frac{2\pi n}{T}t}\, dt$$

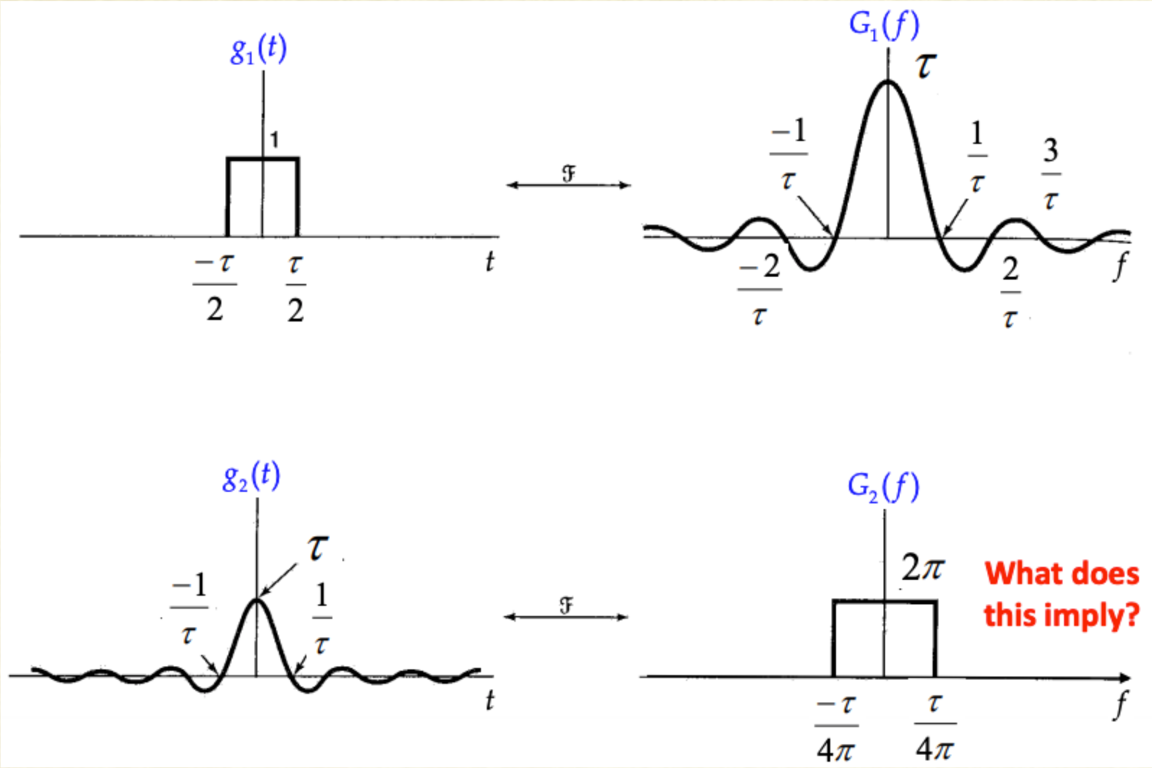
Symmetry property of FT

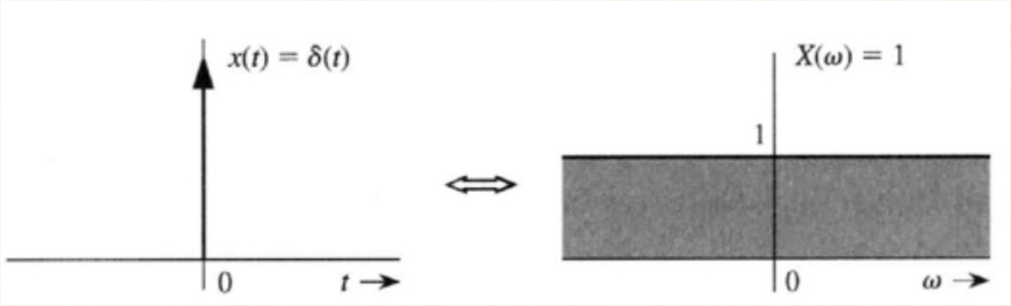
$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

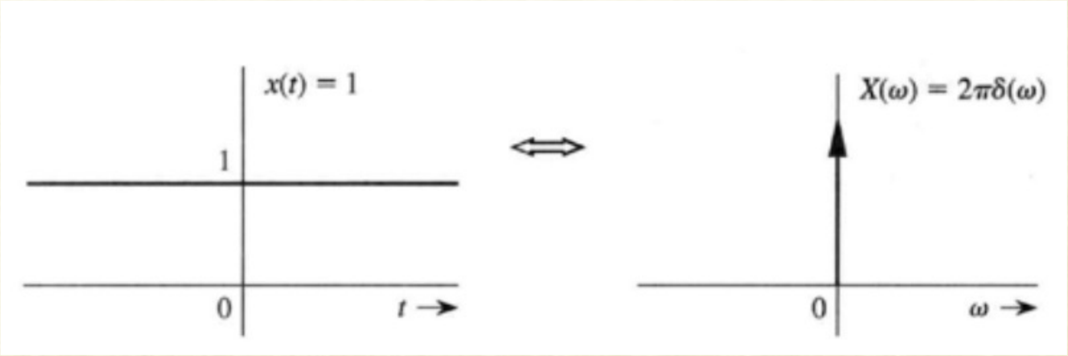
$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}\,d\omega$$

$$\mathcal{F}[f(t)] = F(\omega)$$

$$\Longrightarrow \mathcal{F}[F(t)] = 2\pi\, f(-\omega)$$







Fourier Transform of a Square Pulse

- $F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$; $\quad F(\mu) = \int_{-\frac{W}{2}}^{\frac{W}{2}} A e^{-i2\pi\mu t} dt$

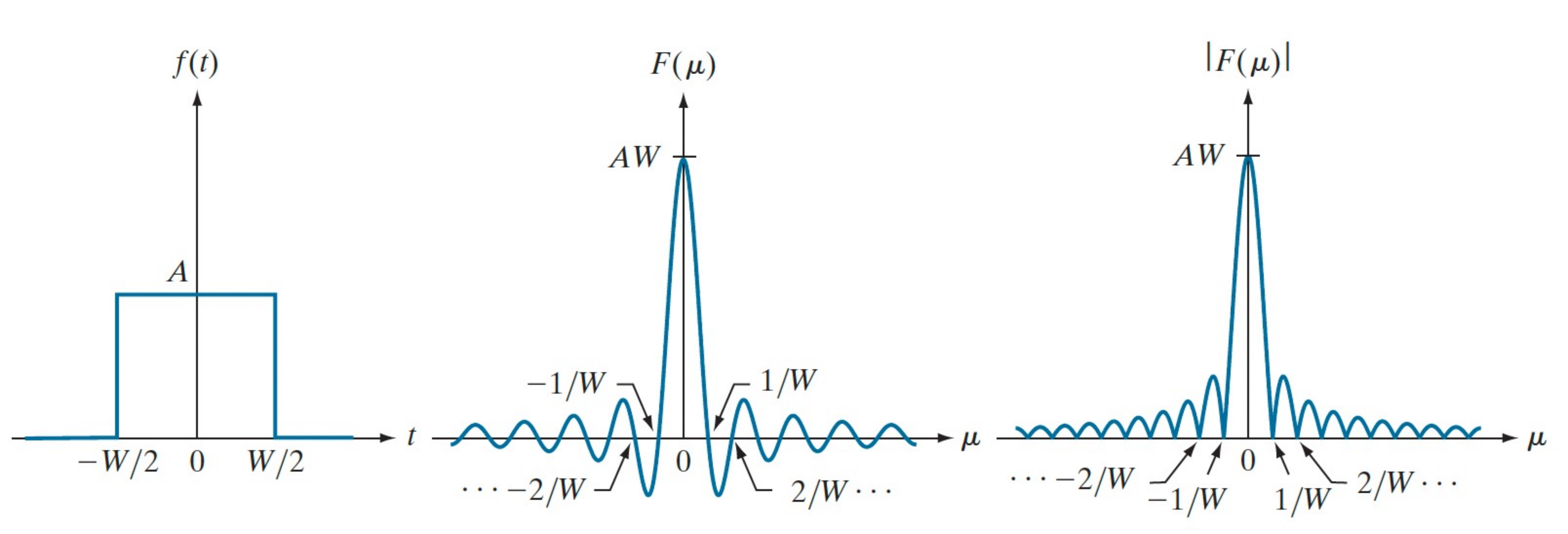


Fig: 4.4; Gonzalez and Woods

Fourier Transform of a Square Pulse

$$F(\mu) = \int_{-\frac{W}{2}}^{\frac{W}{2}} Ae^{-i2\pi\mu t}dt \qquad = \frac{-A}{i2\pi\mu}\left[e^{-i2\pi\mu t}\right]_{-W/2}^{W/2}$$

$$= \frac{-A}{i2\pi\mu}\left[e^{-i\pi\mu W} - e^{i\pi\mu W}\right] \qquad = \frac{A}{i2\pi\mu}\left[e^{i\pi\mu W} - e^{-i\pi\mu W}\right]$$

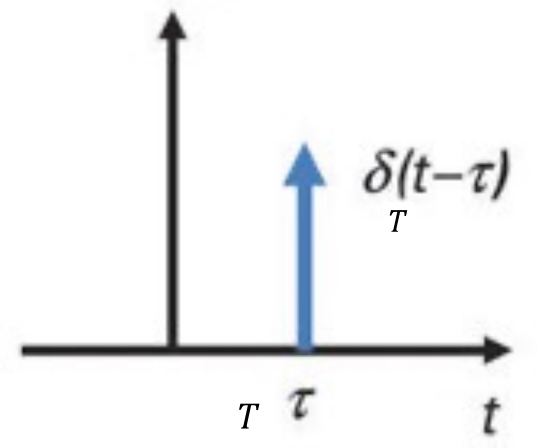
$$\boxed{F(\mu) = AW\frac{\sin(\pi\mu W)}{(\pi\mu W)}}$$

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

Questions?

Convolving with Shifted Impulse Function

Shifted impulse



$$\begin{aligned} \delta(t) &= 0, \qquad for\ t \neq T \\ &= \infty, \qquad for\ t = T \end{aligned}$$

$$\int_{-\infty}^{\infty} \delta(t-T)dt = 1$$

Sifting Property

$$\int_{a}^{b} \delta(t-T)\, f(t)dt = f(T), \qquad a < T < b$$

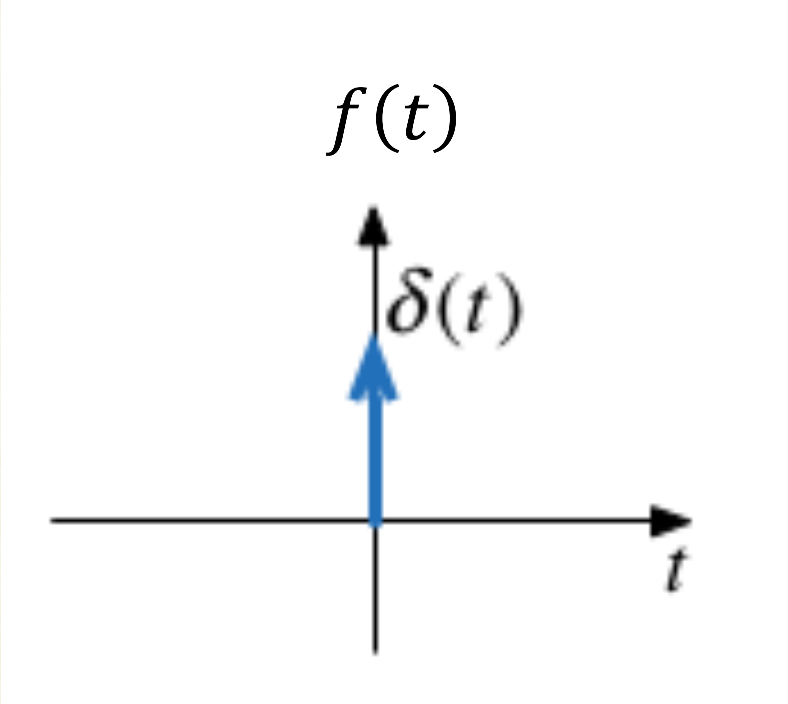
$$= 0\ otherwise$$

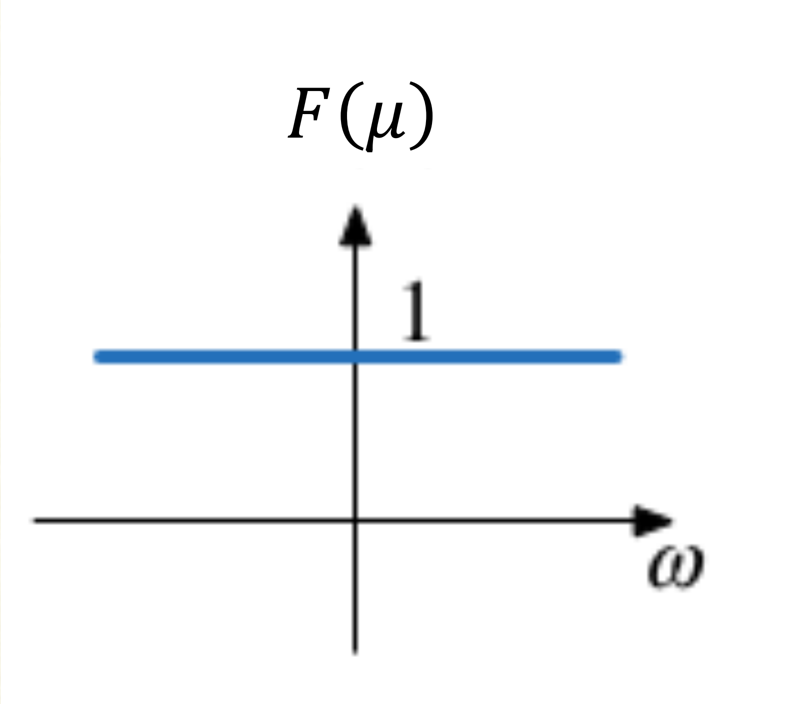
http://lpsa.swarthmore.edu/BackGround/ImpulseFunc/ImpFunc.html

FT of impulse function

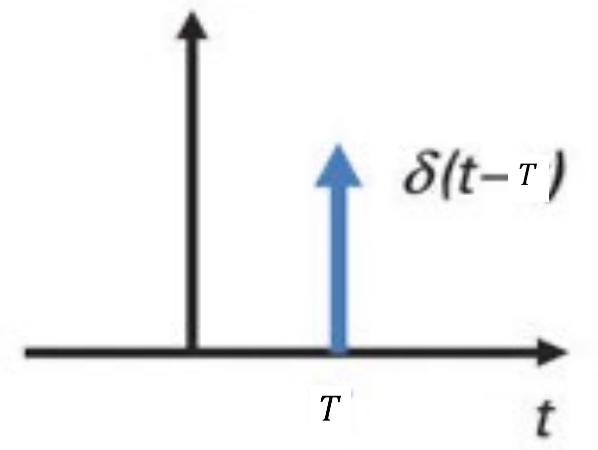
$$f(t) = \delta(t)$$

$$F(\mu) = \int_{-\infty}^{\infty} \delta(t) e^{-i2\pi\mu t} dt$$





FT of time-shifted impulse



$$\int_a^b \delta(t - t_0)\, f(t) dt = f(t_0), \qquad a < t_0 < b$$

$$= 0, \qquad otherwise$$

$$F(\mu) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-i2\pi\mu t} dt$$

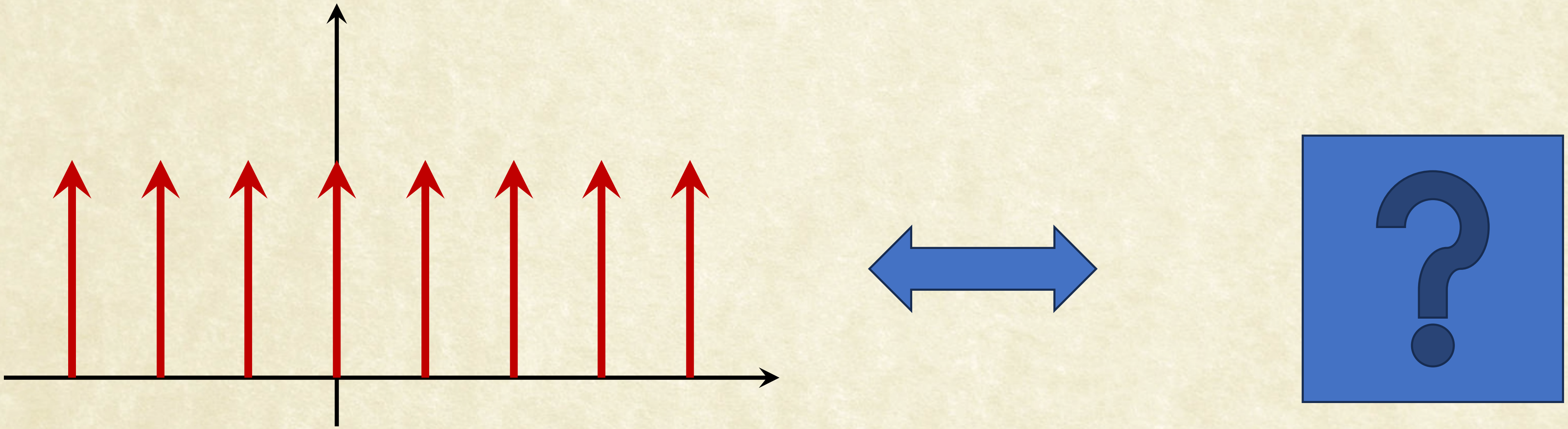
$$= e^{-i2\pi\mu t_0}$$

Questions?

FT of an Impulse Train

- We know that:

$$\mathcal{F}[\delta(t)] = e^{-i2\pi\mu t_0}$$



FT of a Periodic Function

Fourier Series:

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi n}{\Delta T}t}$$

$$c_n = \frac{1}{\Delta T}\int_{-\Delta T/2}^{\Delta T/2} s_{\Delta T}(t) e^{-i\frac{2\pi n}{\Delta T}t}\, dt$$

The Impulse Train is:

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi n}{\Delta T}t} \qquad = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} e^{i\frac{2\pi n}{\Delta T}t}$$

$$c_n = \frac{1}{\Delta T}\int_{-\Delta T/2}^{\Delta T/2} \delta(t) e^{-i\frac{2\pi n}{\Delta T}t}\, dt$$

$$= \frac{1}{\Delta T} e^0 = \frac{1}{\Delta T}$$

FT of the Impulse Train

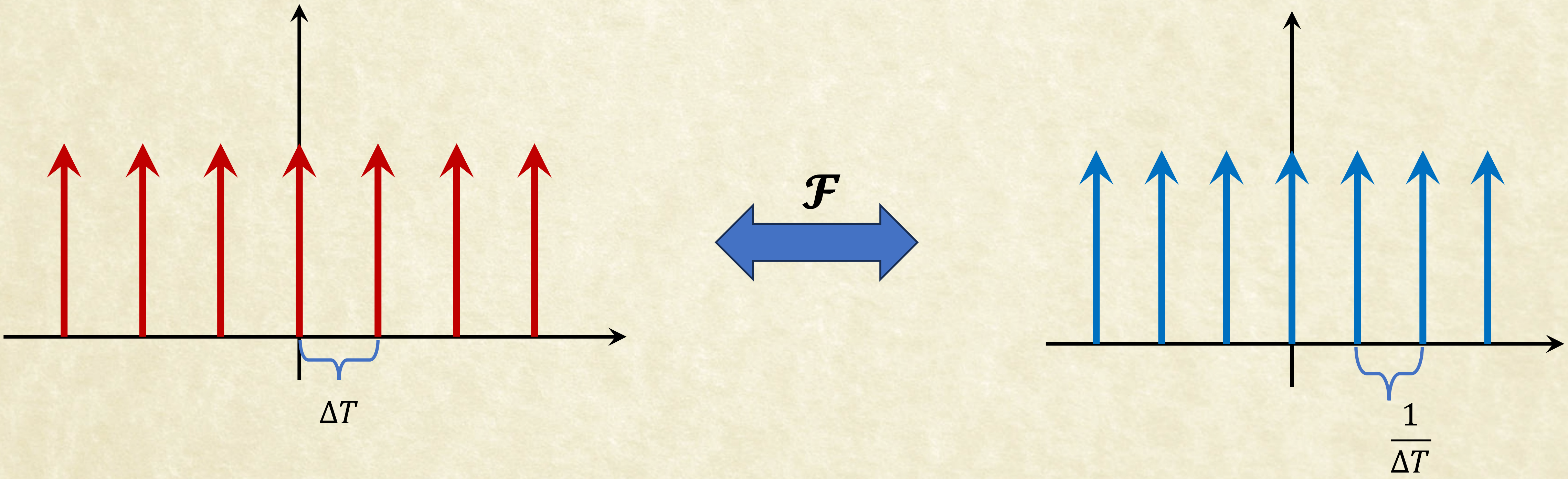
$$\mathcal{F}\left(s_{\Delta T}(t)\right) = \mathcal{F}\left(\frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} e^{i\frac{2\pi n}{\Delta T}t}\right)$$

$$= \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} \mathcal{F}\left(e^{i\frac{2\pi n}{\Delta T}t}\right)$$

$$= \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} \delta\left(\mu - \frac{n}{\Delta T}\right)$$

Fourier Transform of an Impulse Train

- The FT of an Impulse train with period ΔT is an Impulse train with period $\frac{1}{\Delta T}$.



Questions?

FT of Convolution

$$(f \star h)(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau$$

$$\mathcal{F}\{(f \star h)(t)\} = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau\right] e^{-i2\pi\mu t}\, dt$$

$$= \int_{-\infty}^{\infty} f(\tau) \left[\int_{-\infty}^{\infty} h(t-\tau)e^{-i2\pi\mu t}dt\right] d\tau$$

$$\mathcal{F}\{(f \star h)(t)\} = \int_{-\infty}^{\infty} f(\tau)\left[H(\mu)e^{-i2\pi\mu\tau}\right] d\tau$$

FT of Convolution

$$\mathcal{F}\{(f \star h)(t)\} = \int_{-\infty}^{\infty} f(\tau)\left[H(\mu)e^{-i2\pi\mu\tau}\right] d\tau$$

$$= H(\mu)\int_{-\infty}^{\infty} f(\tau)e^{-i2\pi\mu\tau} d\tau$$

$$= H(\mu)F(\mu) = H.F(\mu)$$

Questions?