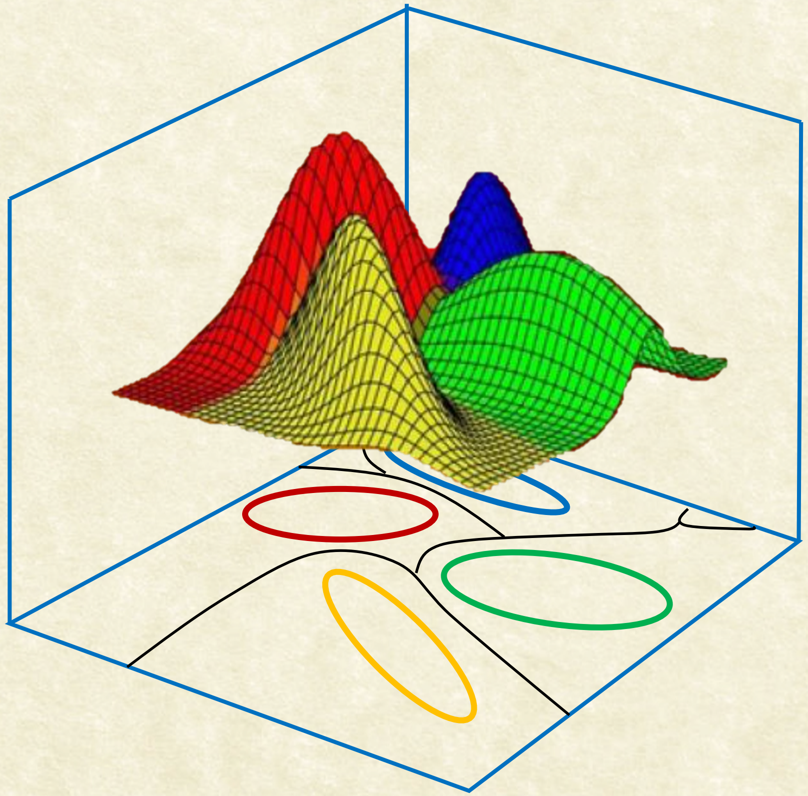
CS7.404: Digital Image Processing

Monsoon 2023: Fourier Series



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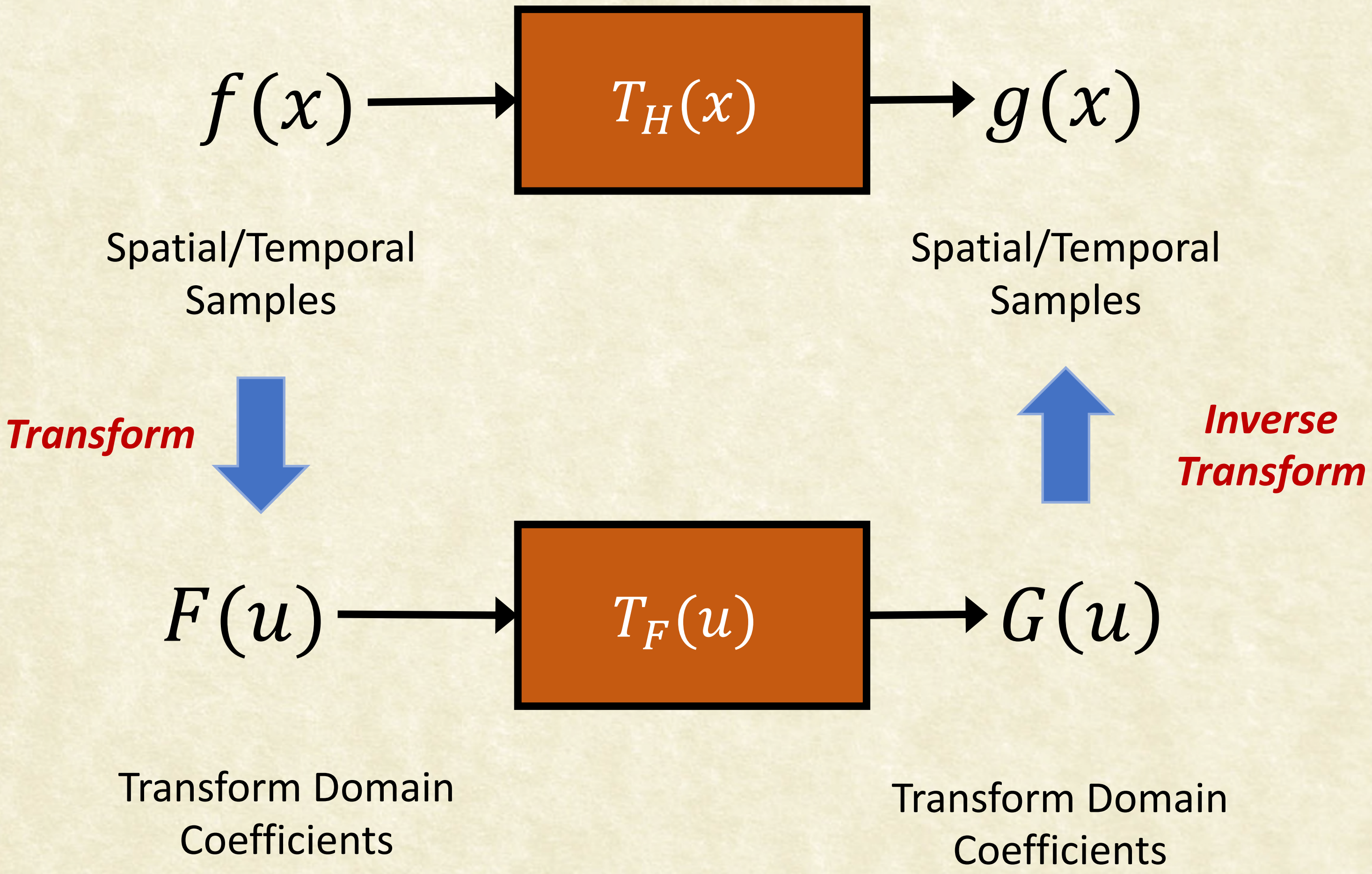
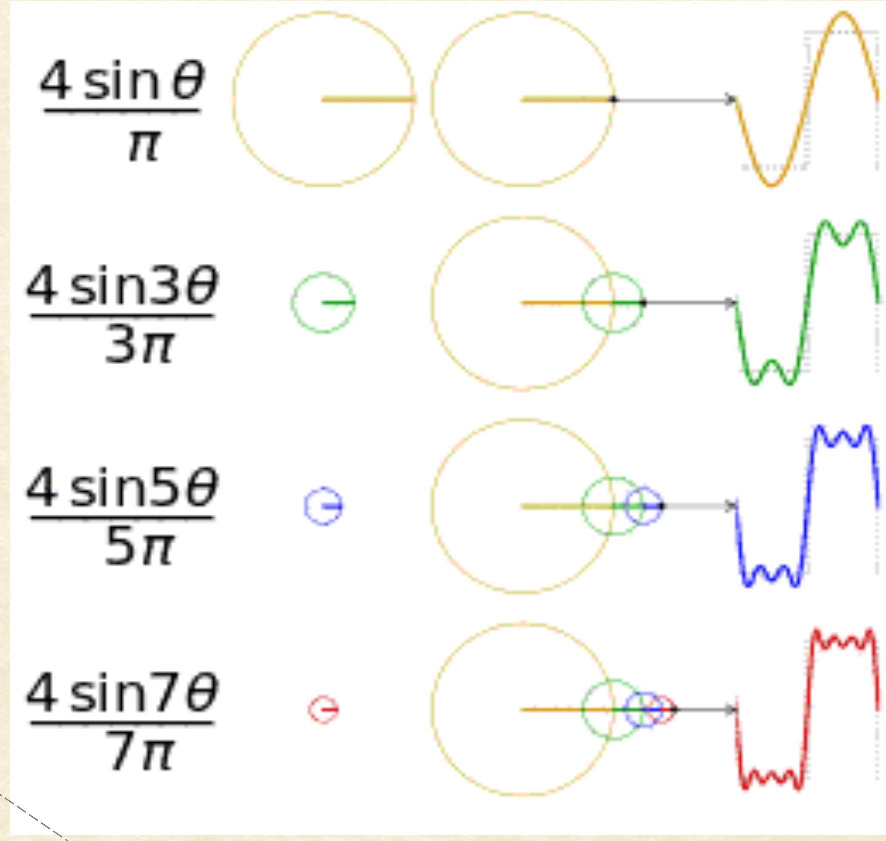
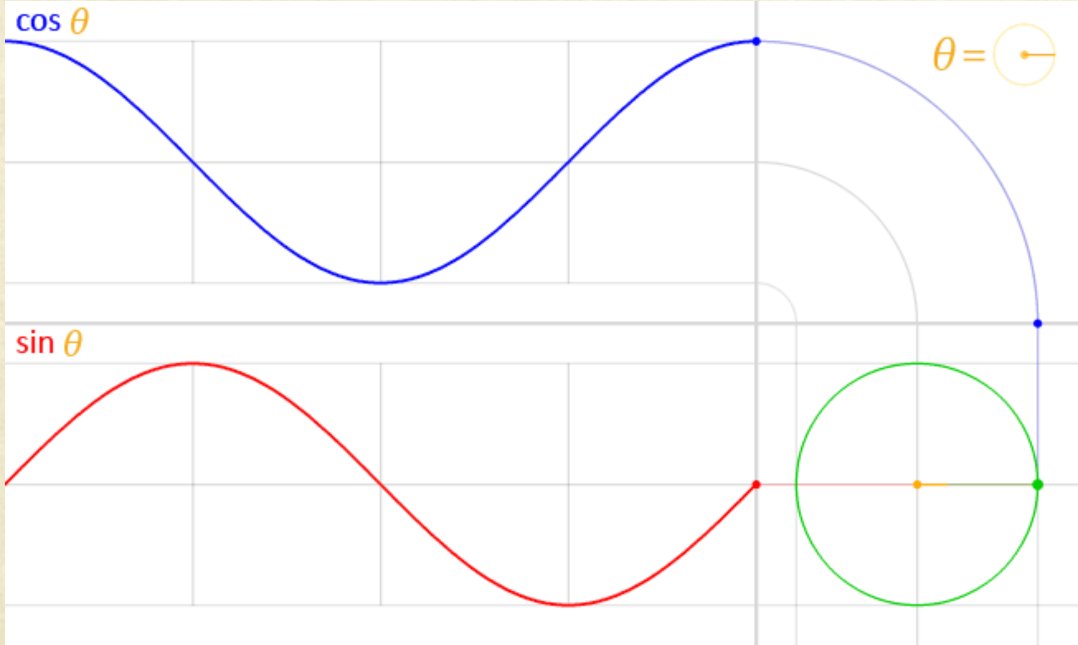
The Systems View



Periodic Signals

- Periodic → **Frequency** of occurrence
  - Repetitions/<Unit> (cycles/sec = Hz)

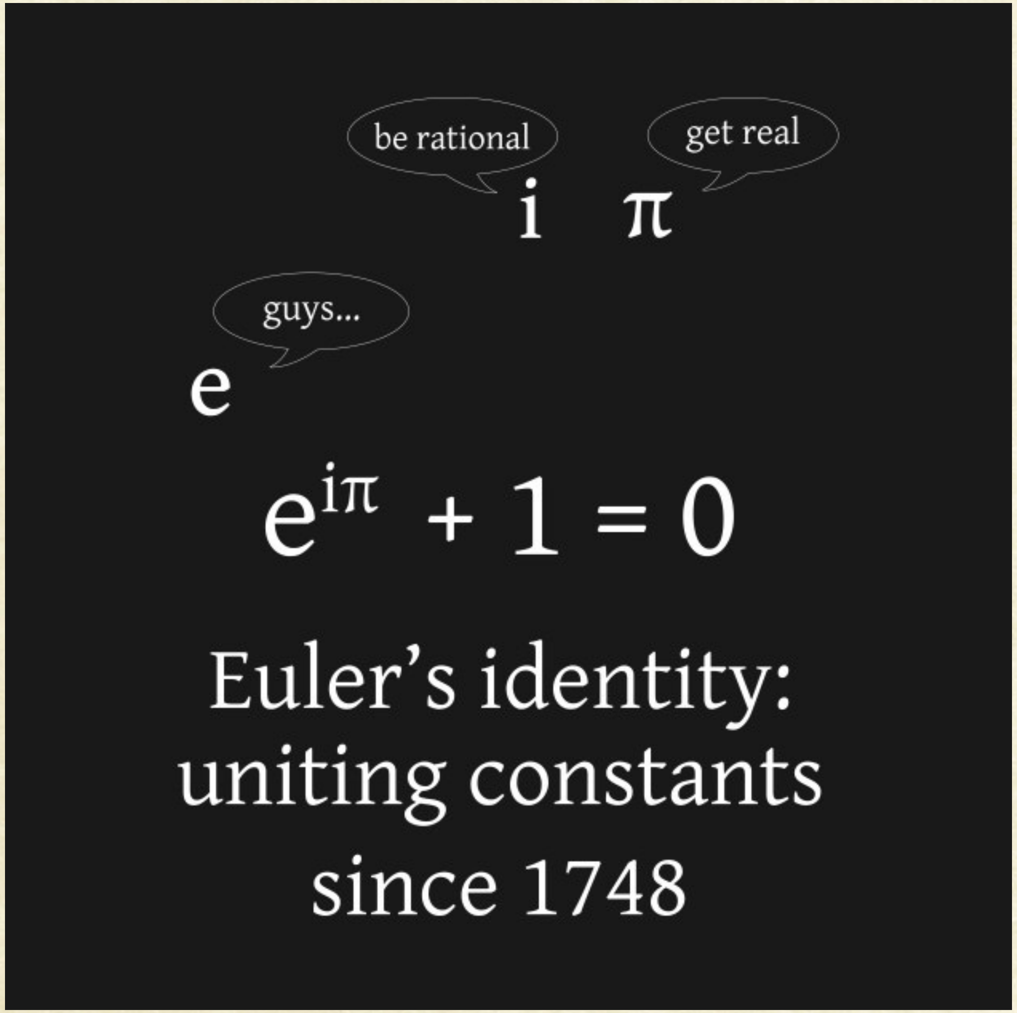




$$x(t) = A\cos(\omega t) = Acos(\boxed{2\pi} f t) = Acos(\frac{2\pi}{T}t)$$

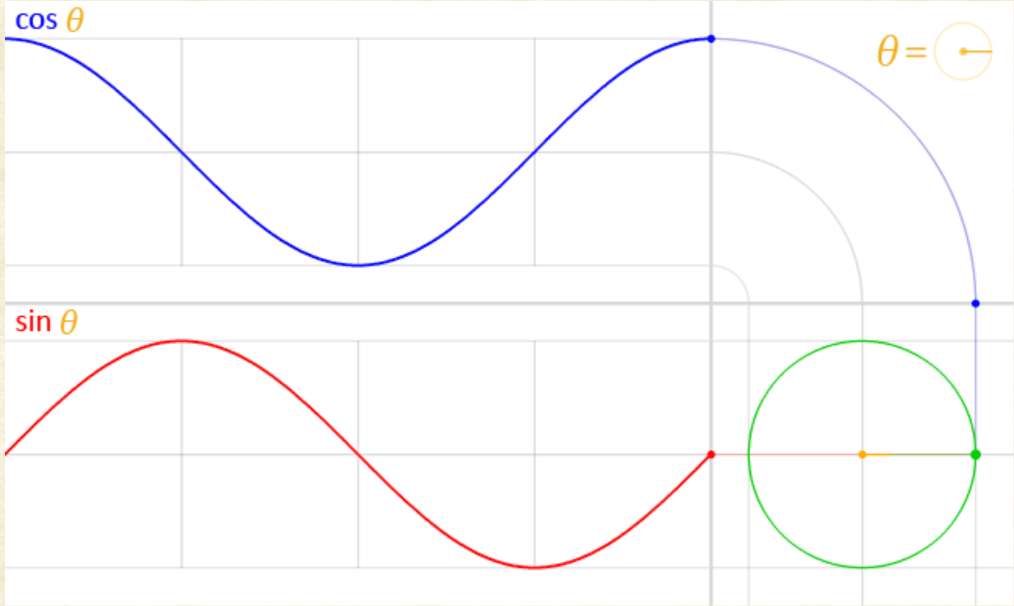
Angular frequency

Fundamental Period



$$e^{it} = \cos t + i \sin t$$

$$i = \sqrt{-1}$$

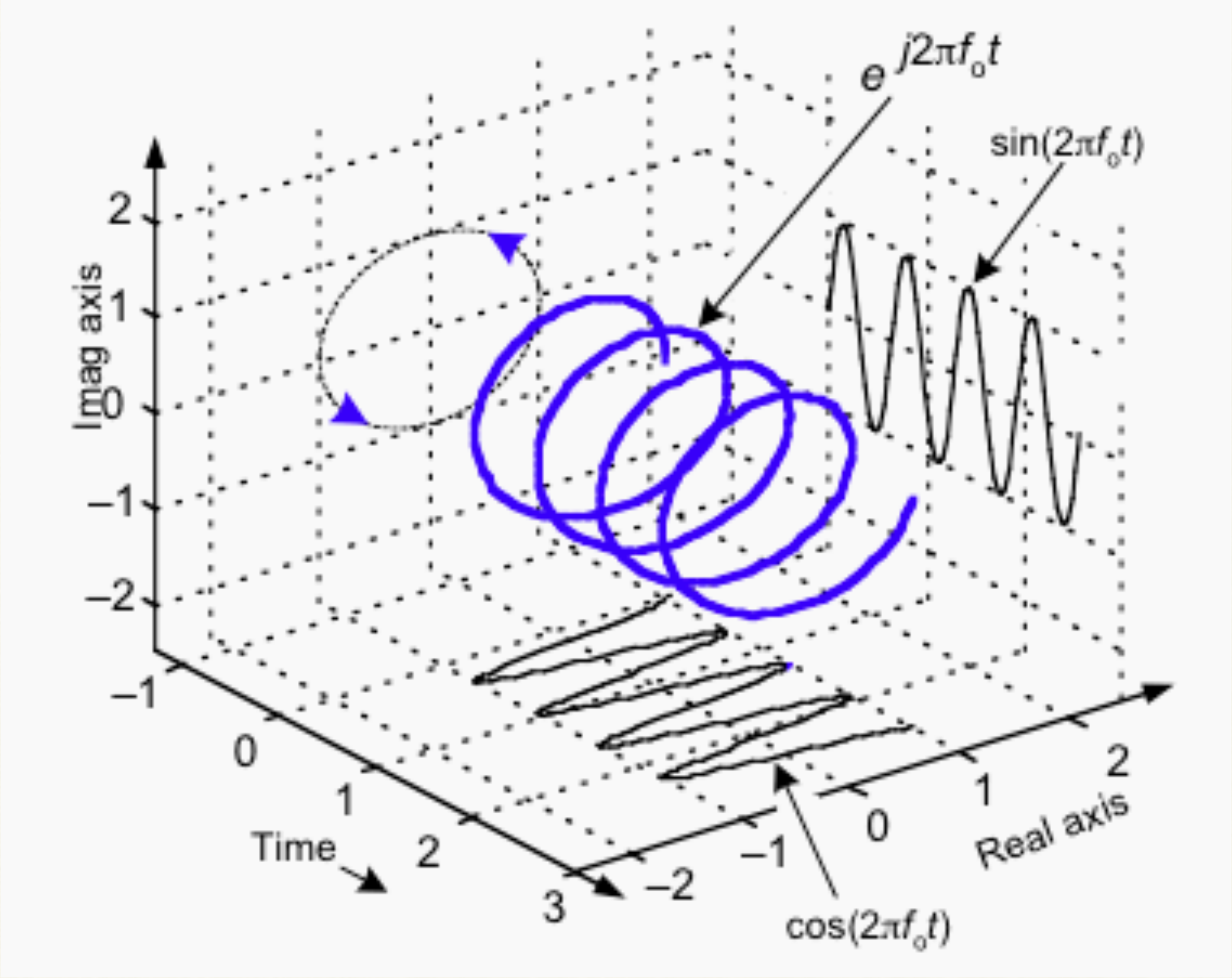


https://res.cloudinary.com/teepublic/image/private/s--s4GX0hG3--/t_Preview/b_rgb:191919,c_limit,f_jpg,h_630,q_90,w_630/v1486157785/production/designs/1177382_1.jpg

Complex Sinusoid

$$e^{it} = \cos t + i \sin t$$

$$i = \sqrt{-1}$$



Fourier Series in terms of complex coefficients

$$f(t) = a_0 + \sum_{m=1}^{\infty} a_m \cos\left(\frac{2\pi m t}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n t}{T}\right)$$
$$= \sum_{m=0}^{\infty} a_m \cos\left(\frac{2\pi m t}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n t}{T}\right)$$

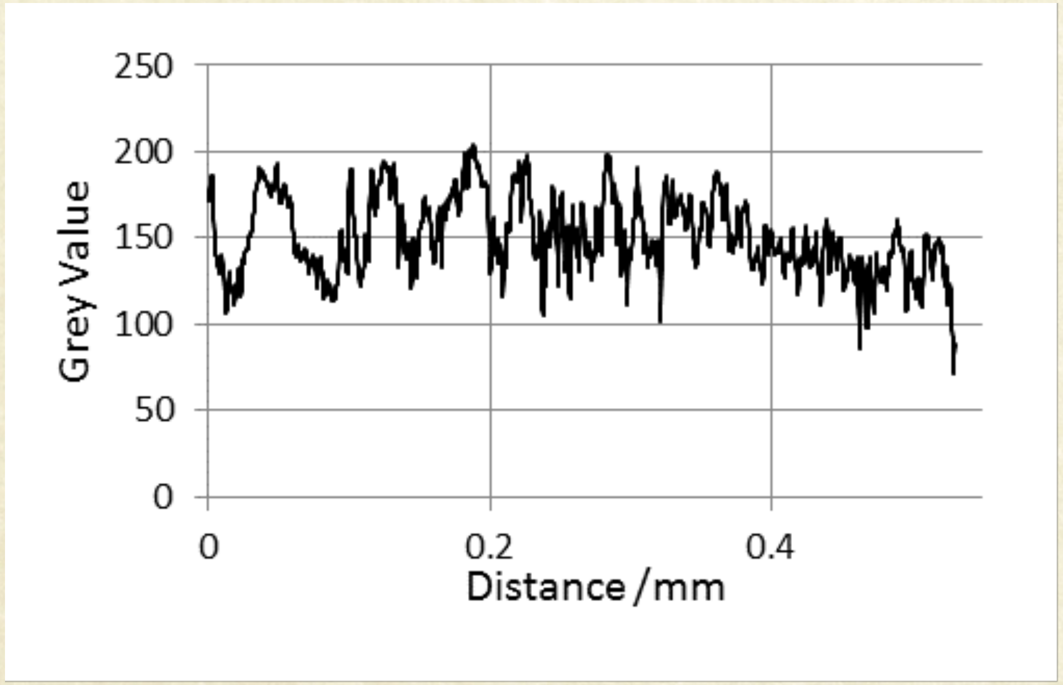
$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt$$
$$a_m = \frac{2}{T}\int_0^T f(t)\cos\left(\frac{2\pi m t}{T}\right)dt$$
$$b_n = \frac{2}{T}\int_0^T f(t)\sin\left(\frac{2\pi n t}{T}\right)dt$$

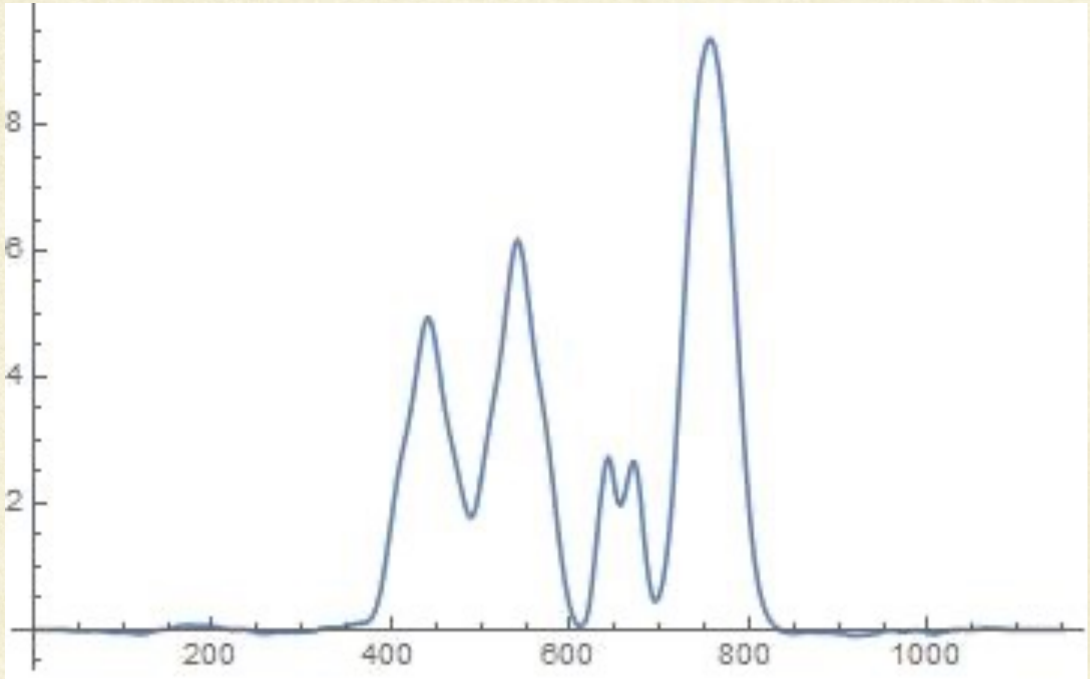
$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi n t}{T}}$$

$$c_n = \frac{1}{T}\int_0^T f(t) e^{-i\frac{2\pi n t}{T}}\,dt$$

http://www.thefouriertransform.com/series/complexcoefficients.php

What if f(t) is non-periodic ?





$$\cancel{f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi n t}{T}}}$$

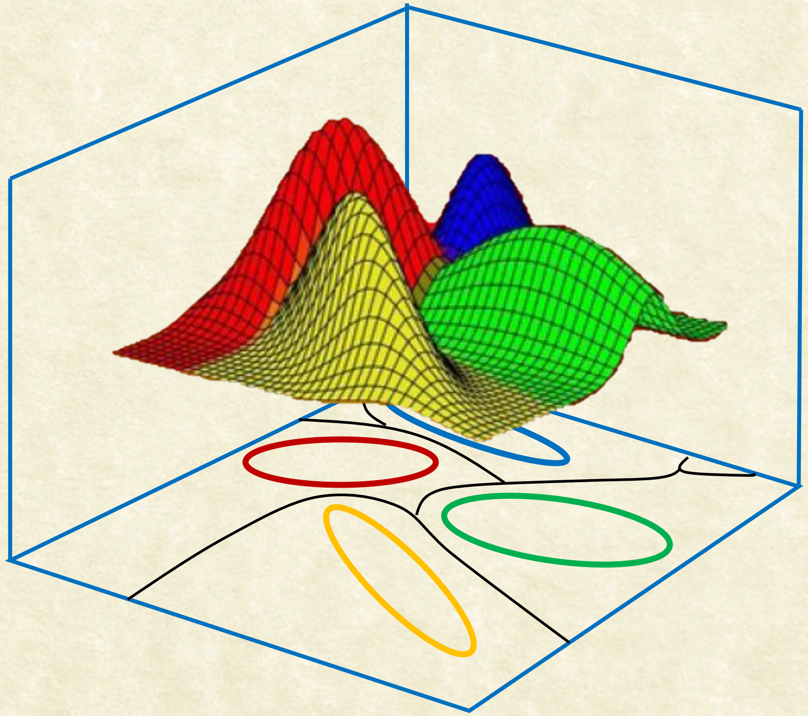
https://i.stack.imgur.com/fKznF.jpg

Questions?

CS7.404: Digital Image Processing

Monsoon 2023: Fourier Transform

Approximate **non-periodic signals** with complex sinusoids



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Definition: Fourier Transform (FT)

- The Fourier Transform of a function $f(t)$ is defined by:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt$$

- The result is a function of ω (frequency).
- Compared to Fourier Series, the Fourier coefficient of nth sinusoid is given by:

$$\boxed{c_n = \frac{1}{T}\int_0^T f(t)e^{-i\frac{2\pi n}{T}t}\,dt}$$

Intuition for FT

- $f(t)$: It is a single (real) number at each t.

- $F(\omega)$: How much of frequency ω is present for all values of t ?
  - Project $f(t)$ on to the complex sinusoid with frequency ω.

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt$$

$$F(\omega) = \mathcal{F}[f(t)]$$

Definition: Inverse Fourier Transform (IFT)

- The IFT of a function $F(\omega)$ is given by:

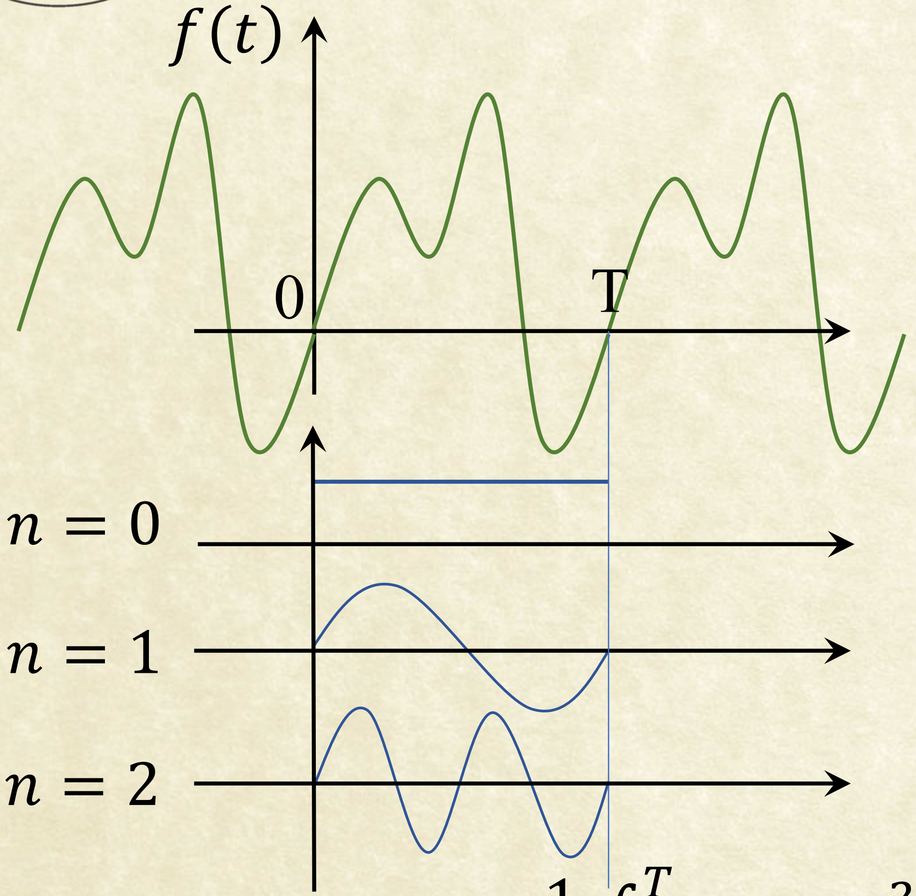
$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t}\, d\omega$$

$$f(t) = \mathcal{F}^{-1}[F(\omega)]$$

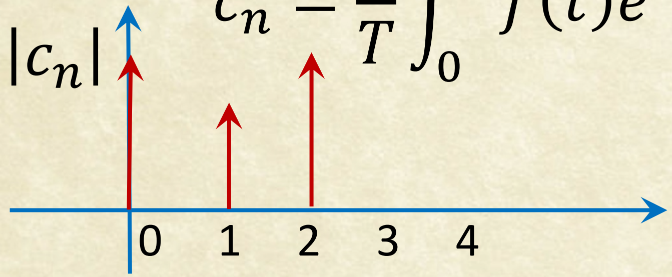
- Note: The corresponding equation in Fourier Series is:

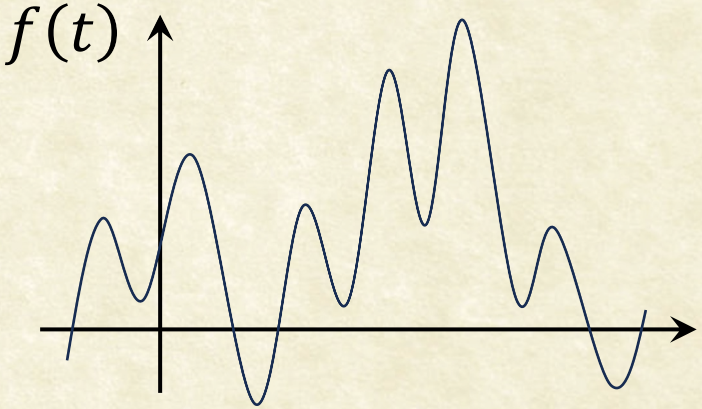
$$\boxed{f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi n}{T}t}}$$

Fourier Transform vs. Series

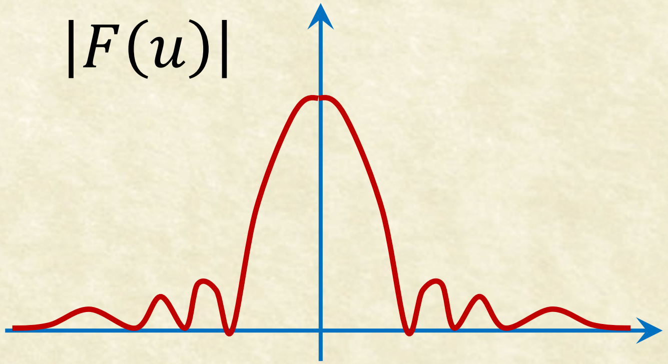


$$c_n = \frac{1}{T}\int_0^T f(t)e^{-i\frac{2\pi n}{T}t}\,dt$$





$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt$$



Processing in Spatial vs Frequency Domain

Spatial/Temporal Domain

$$f(x) \longrightarrow \boxed{T_H(x)} \longrightarrow g(x)$$

Transform

$\mathcal{F}[f(t)]$

Inverse Transform

$\mathcal{F}^{-1}[F(\omega)]$

Frequency Domain

$$F(u) \longrightarrow \boxed{T_F(u)} \longrightarrow G(u)$$

Questions?

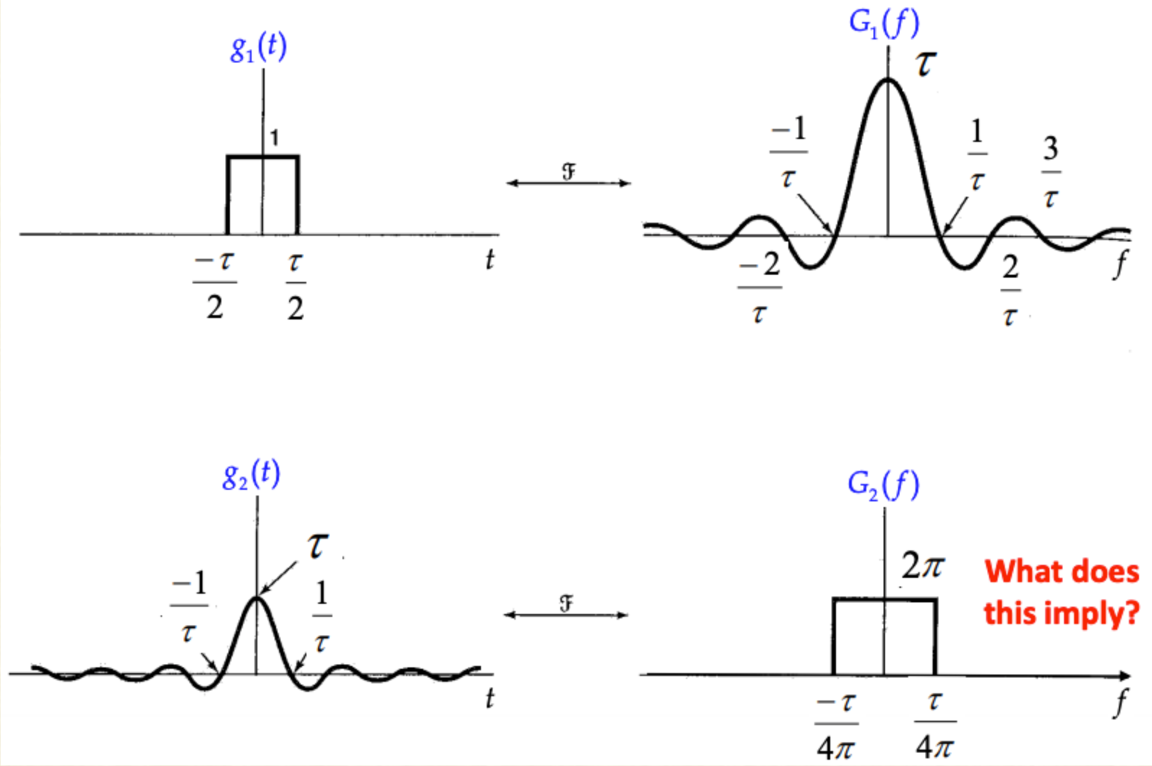
Symmetry property of FT

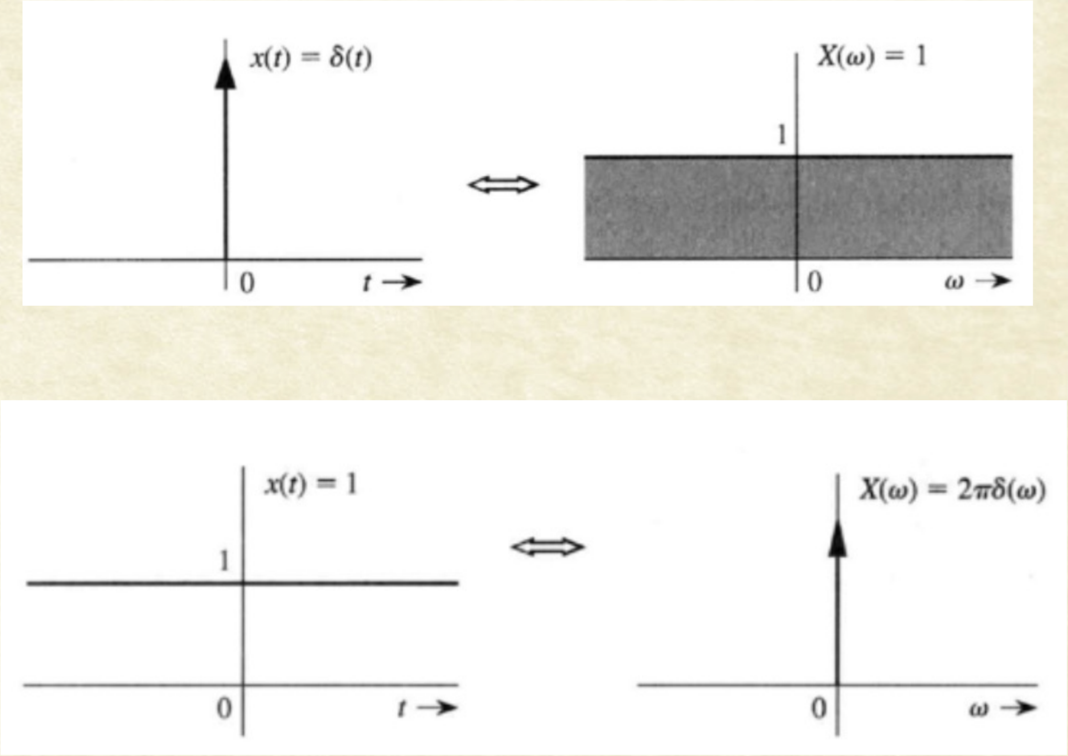
$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}\,d\omega$$

$$\mathcal{F}[f(t)] = F(\omega)$$

$$\Longrightarrow \mathcal{F}[F(t)] = 2\pi f(-\omega)$$





Symmetry property of FT

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

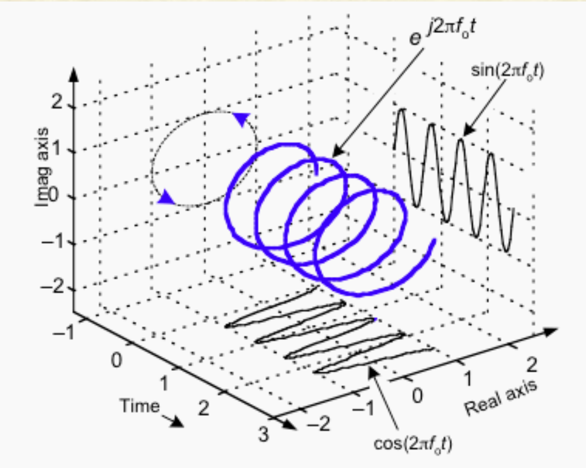
$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}\,d\omega$$

$$\mathcal{F}[f(t)] = F(\omega)$$

$$\Longrightarrow \mathcal{F}[F(t)] = 2\pi f(-\omega)$$

FT of complex exponential

$$e^{jn\omega_0 t} \xleftrightarrow{F} 2\pi\delta(\omega - n\omega_0)$$

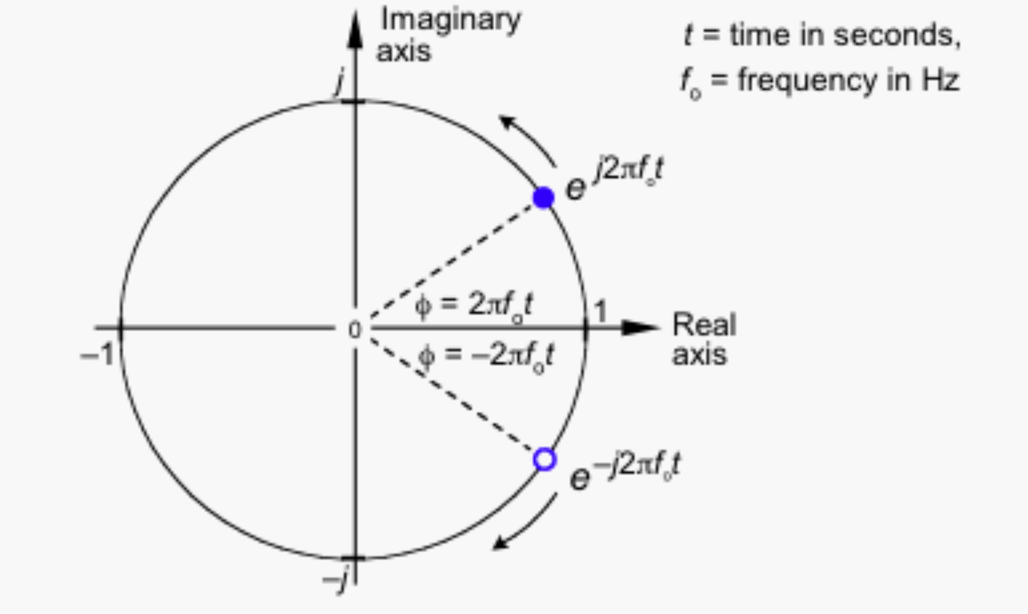


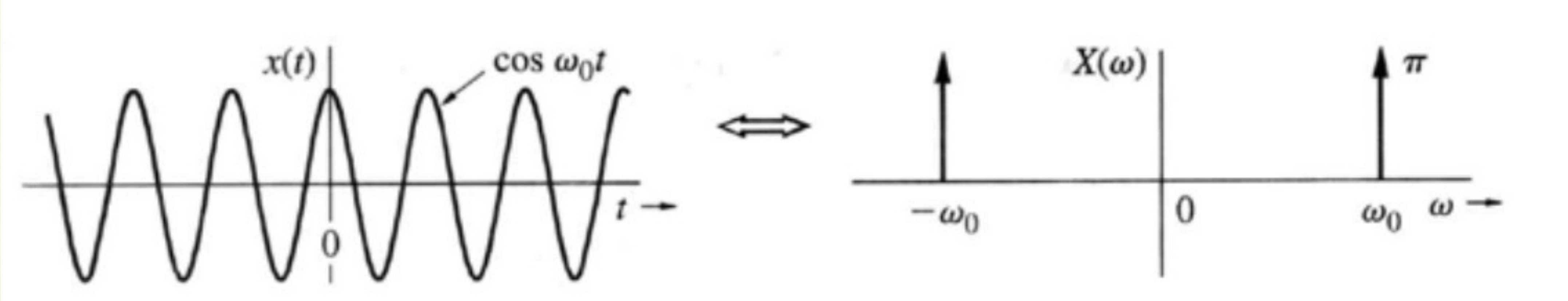
FT of cosine

$$e^{j\omega_0 t} \Leftrightarrow 2\pi\,\delta(\omega - \omega_0)$$

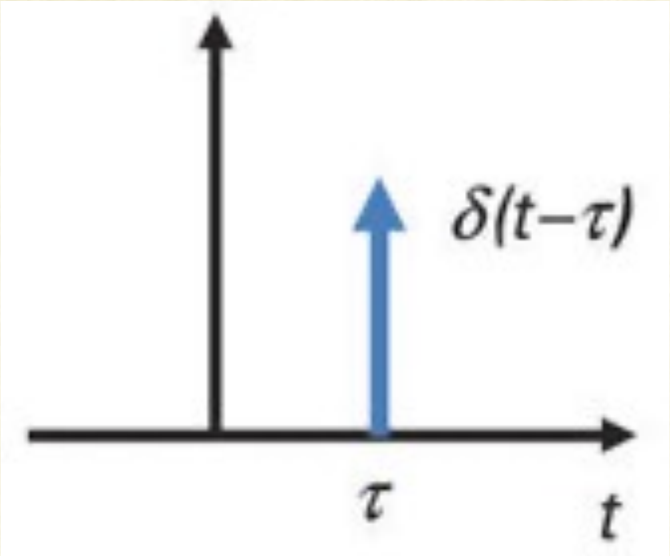
$$e^{-j\omega_0 t} \Leftrightarrow 2\pi\,\delta(\omega + \omega_0)$$

$$\cos\omega_0 t = \frac{1}{2}(e^{-j\omega_0 t} + e^{j\omega_0 t})$$





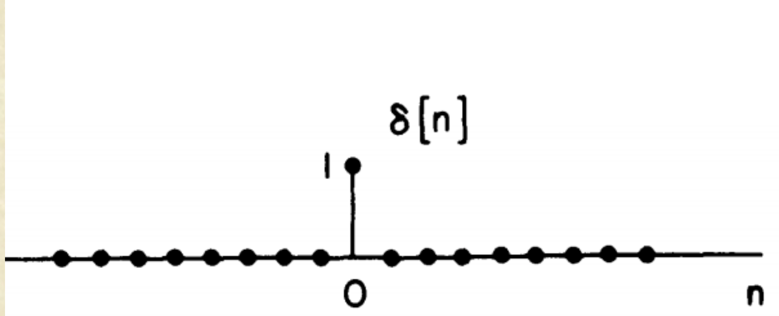
Impulse Function



$$\begin{aligned}\delta(t) &= 0, \quad for\ t \neq T \\ &= \infty, \quad for\ t = T\end{aligned}$$

$$\int_{-\infty}^{\infty} \delta(t-T)dt = 1$$

Discrete Impulse Function



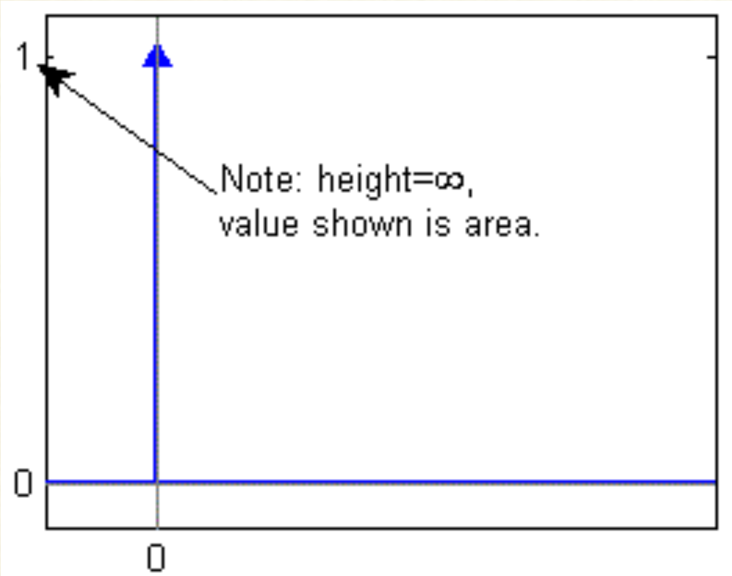
$$\begin{aligned}\delta[n] &= 1, \quad for\ n = 0 \\ &= 0, \quad for\ n \neq 0\end{aligned}$$

http://lpsa.swarthmore.edu/BackGround/ImpulseFunc/ImpFunc.html

Convolving with Unit Impulse Function

$$\begin{aligned}\delta(t) &= 0, \quad for\ t \neq 0 \\ &= \infty, \quad for\ t = 0\end{aligned}$$

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$



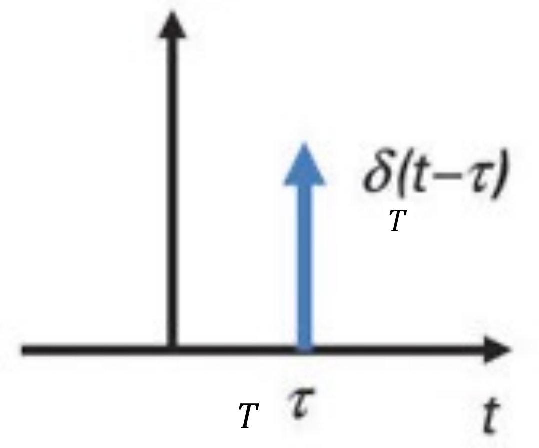
$$\int_a^b \delta(t)dt = \begin{cases} 1, & a < 0 < b \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned}\int_a^b \delta(t) \cdot f(t)dt &= \int_a^b \delta(t) \cdot f(0)dt \\ &= f(0) \cdot \int_a^b \delta(t)dt \\ &= \begin{cases} f(0), & a < 0 < b \\ 0, & \text{otherwise} \end{cases}\end{aligned}$$

http://lpsa.swarthmore.edu/BackGround/ImpulseFunc/ImpFunc.html

Convolving with Shifted Impulse Function

Shifted impulse



$$\begin{aligned}\delta(t) &= 0, \quad for\ t \neq T \\ &= \infty, \quad for\ t = T\end{aligned}$$

$$\int_{-\infty}^{\infty} \delta(t-T)dt = 1$$

Sifting Property

$$\int_a^b \delta(t-T)\,x(t)dt = x(T), \qquad a < T < b$$

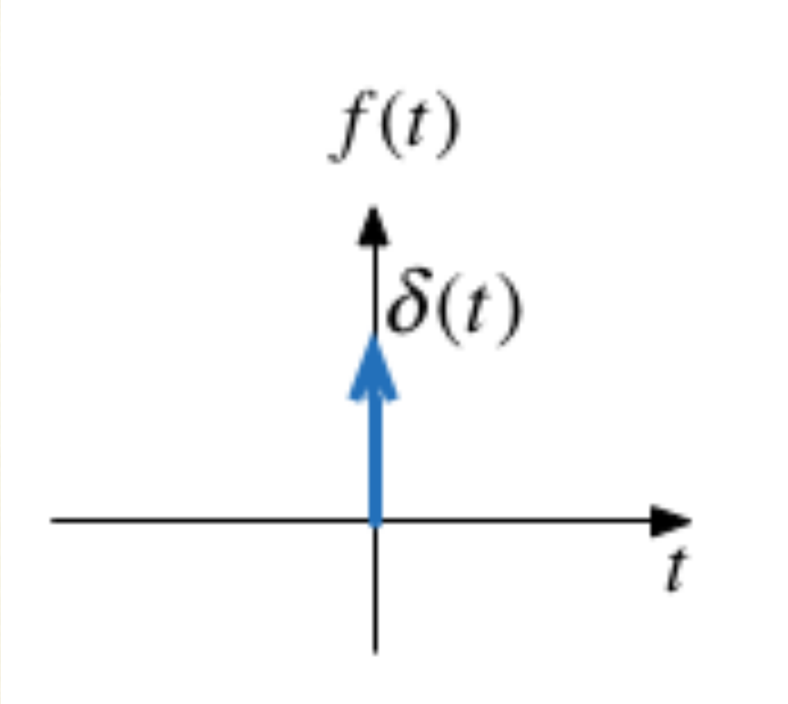
$$= 0\ otherwise$$

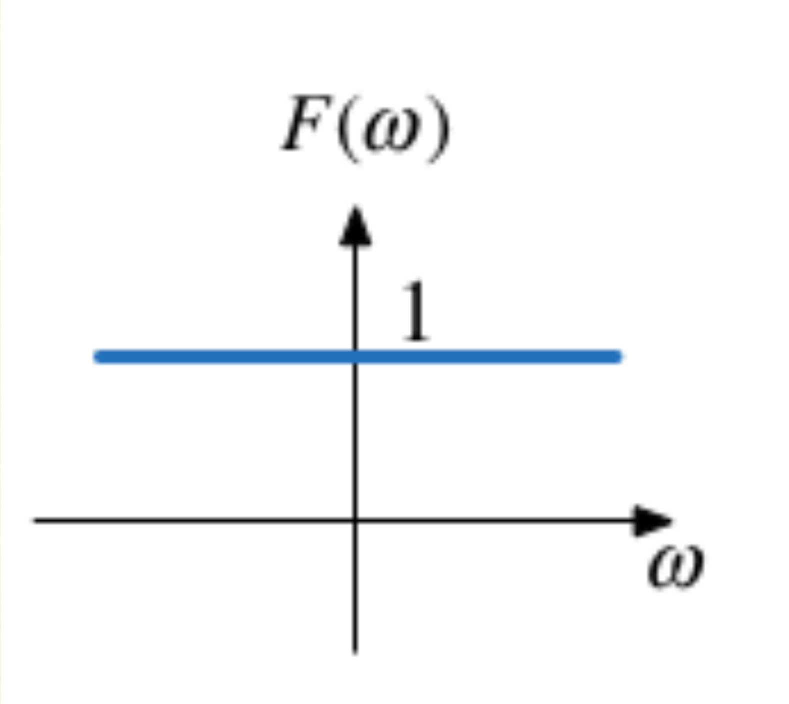
http://lpsa.swarthmore.edu/BackGround/ImpulseFunc/ImpFunc.html

FT of impulse function

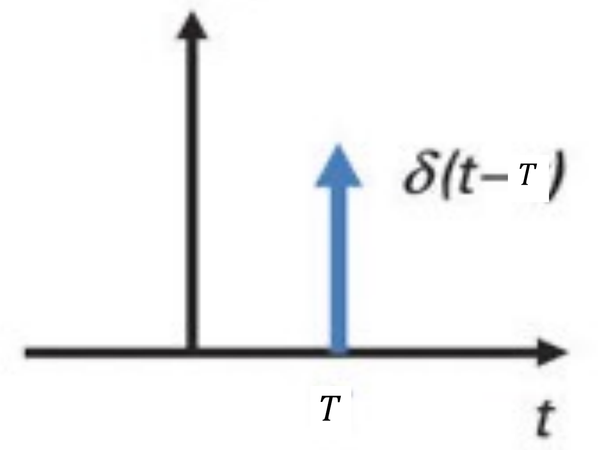
$$f(t) = \delta(t)$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$





FT of time-shifted impulse



$$\int_a^b \delta(t-T)\, x(t)dt = x(T), \qquad a < T < b$$

$$= 0 \; otherwise$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

$$= e^{-i\omega T}$$

Questions?