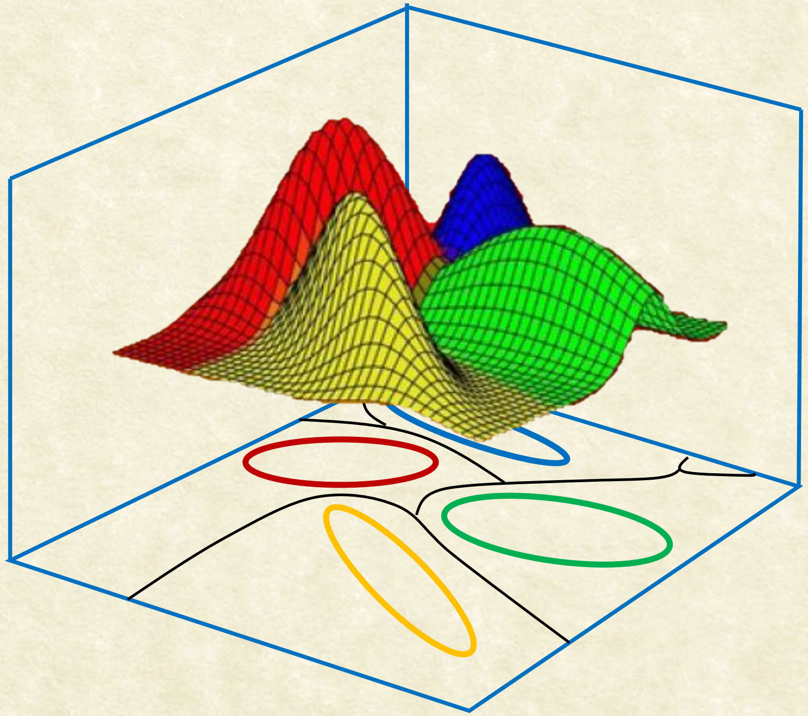
CS7.404: Digital Image Processing

Monsoon 2023: Fourier Transform: Recap



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Fourier Transform (FT): Recap

- The Fourier Transform of a function $f(t)$ is defined by:

$$F(\omega) = \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt$$

- The Inverse Fourier Transform (IFT) is given by:

$$f(t) = \mathcal{F}^{-1}[F(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t}\, d\omega$$

FT of the Impulse Train: Recap

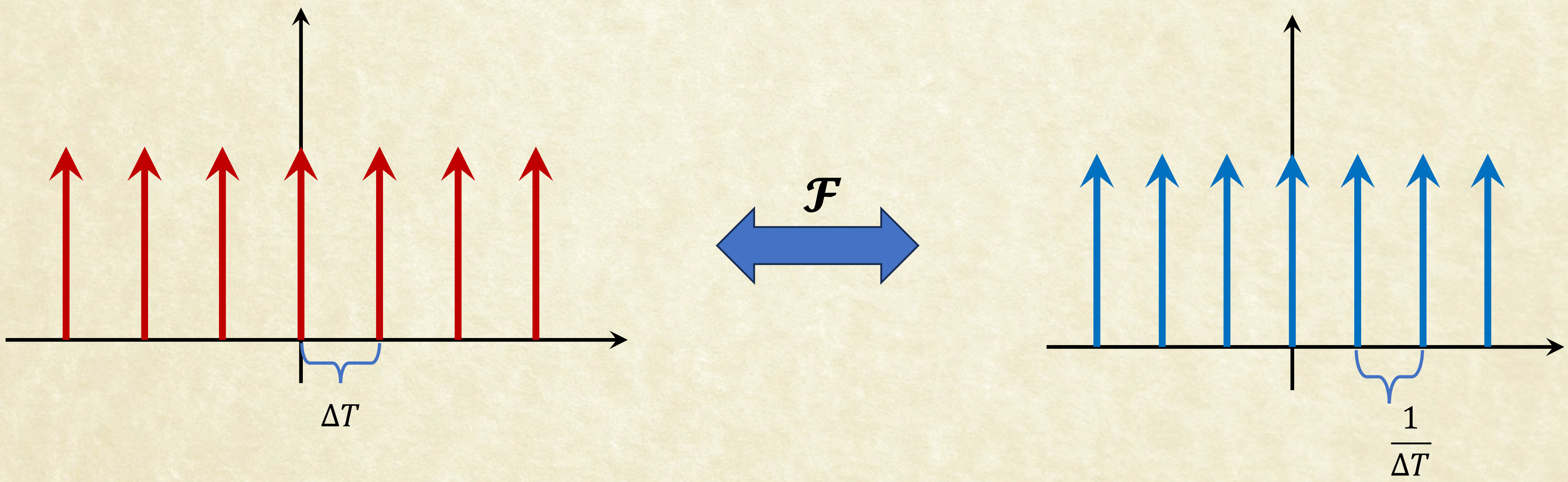
$$\mathcal{F}\big(s_{\Delta T}(t)\big) = \mathcal{F}\left(\frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} e^{i\frac{2\pi n}{\Delta T}t}\right)$$

$$= \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} \mathcal{F}\left(e^{i\frac{2\pi n}{\Delta T}t}\right)$$

$$\boxed{\mathcal{F}\big(s_{\Delta T}(t)\big) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} \delta\left(\mu - \frac{n}{\Delta T}\right)}$$

Fourier Transform of an Impulse Train: Recap

- The FT of an Impulse train with period ΔT is an Impulse train with period $\frac{1}{\Delta T}$.



FT of Convolution

$$(f \star h)(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau$$

$$\mathcal{F}\{(f \star h)(t)\} = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau\right] e^{-i2\pi\mu t}\, dt$$

$$= \int_{-\infty}^{\infty} f(\tau)\left[H(\mu)e^{-i2\pi\mu\tau}\right] d\tau$$

$$= H(\mu)\int_{-\infty}^{\infty} f(\tau)e^{-i2\pi\mu\tau}d\tau = H(\mu)F(\mu) = (H \cdot F)(\mu)$$

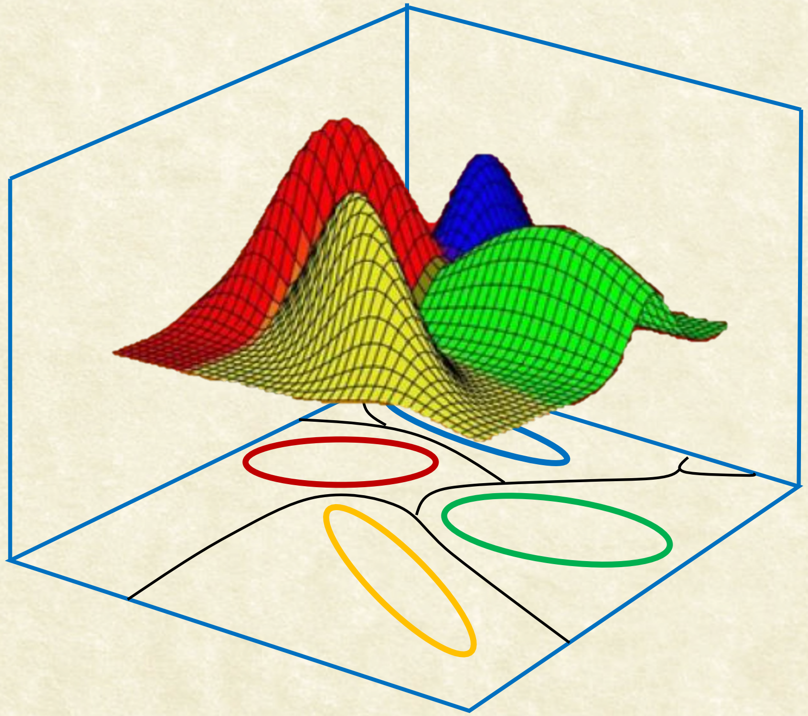
$$\boxed{(f \star h)(t) \iff (H \cdot F)(\mu)}$$

Questions?

CS7.404: Digital Image Processing

Monsoon 2023: Discrete Fourier Transform

Sampling and FT



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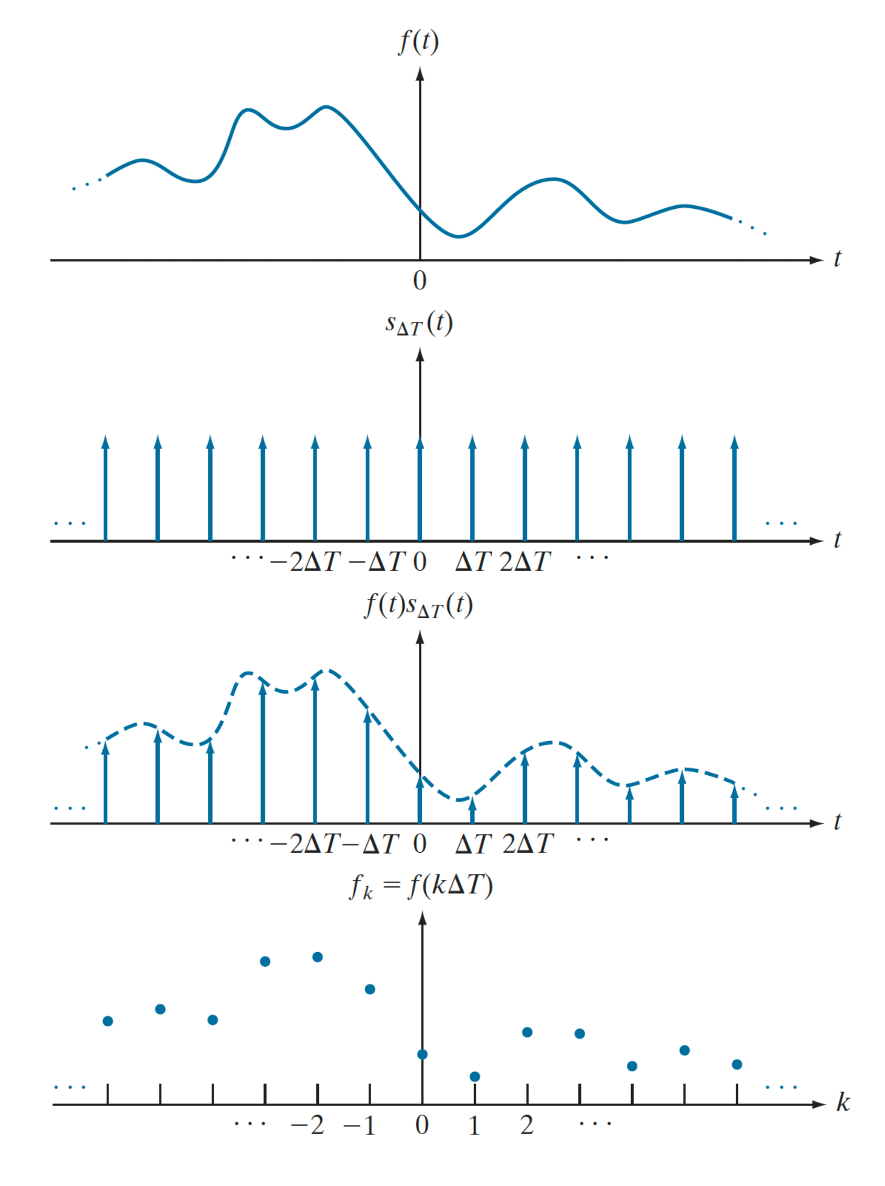
Sampling: $f(t)$ x Impulse Train



$$s_{\Delta T}(t) = \sum_{n=-\infty}^{n=\infty} \delta(t - n\Delta T)$$

$$\tilde{f}(t) = \sum_{n=-\infty}^{n=\infty} f_n\, \delta(t - n\Delta T)$$

FT of Sampled Function

$$\boxed{\mathcal{F}\left(s_{\Delta T}(t)\right) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty}\delta\left(\mu - \frac{n}{\Delta T}\right)}$$

$$\tilde{F}(\mu) = (F \star S)(\mu) = \int_{-\infty}^{\infty} F(\tau)S(\mu - \tau)\,d\tau$$

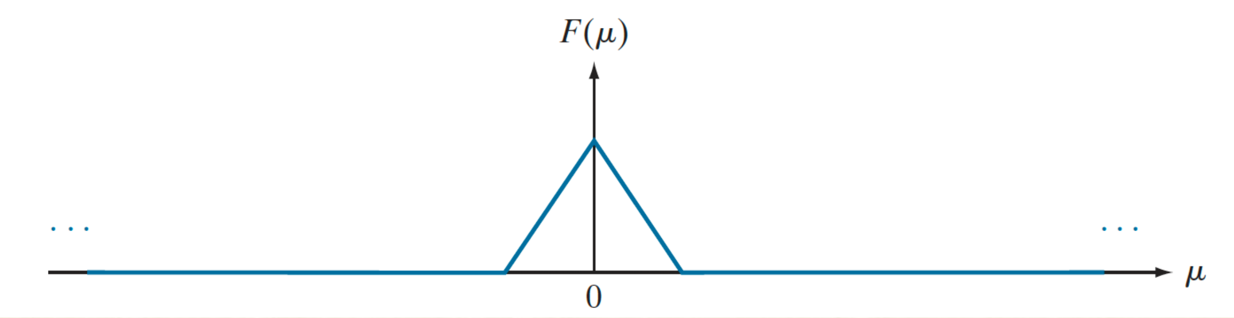
$$= \frac{1}{\Delta T}\int_{-\infty}^{\infty} F(\tau)\sum_{n=-\infty}^{\infty}\delta\left(\mu - \tau - \frac{n}{\Delta T}\right)d\tau$$

$$= \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} F(\tau)\delta\left(\mu - \tau - \frac{n}{\Delta T}\right)d\tau$$

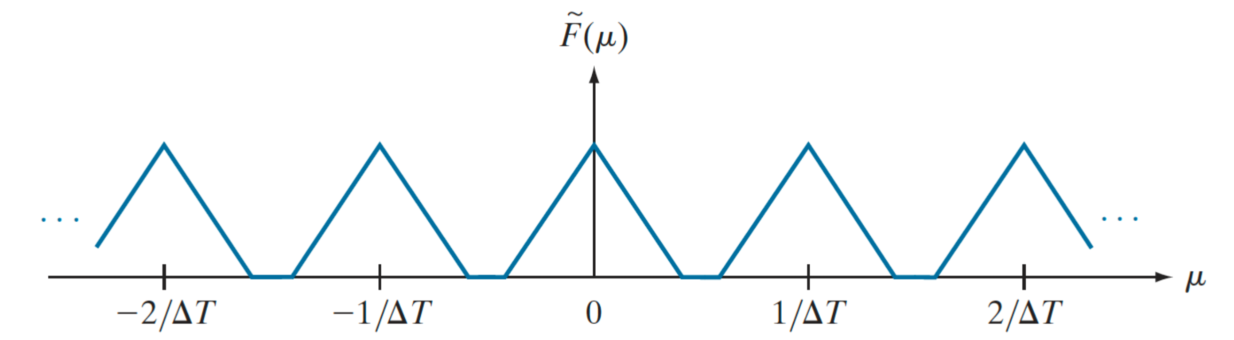
$$\boxed{\mathcal{F}\left(\tilde{f}(t)\right) = \tilde{F}(\mu) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} F\left(\mu - \frac{n}{\Delta T}\right)}$$

FT & Sampling

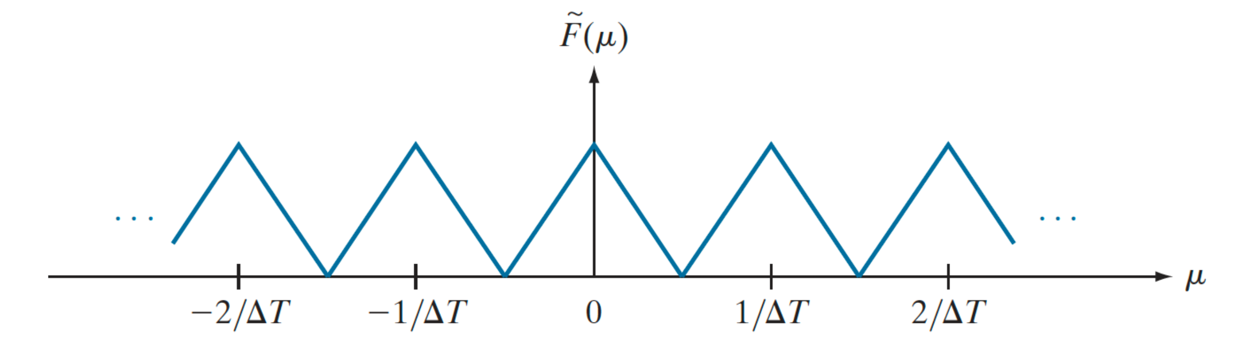
$$\mathcal{F}\left(\tilde{f}(t)\right) = \tilde{F}(\mu) = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} F\left(\mu - \frac{n}{\Delta T}\right)$$



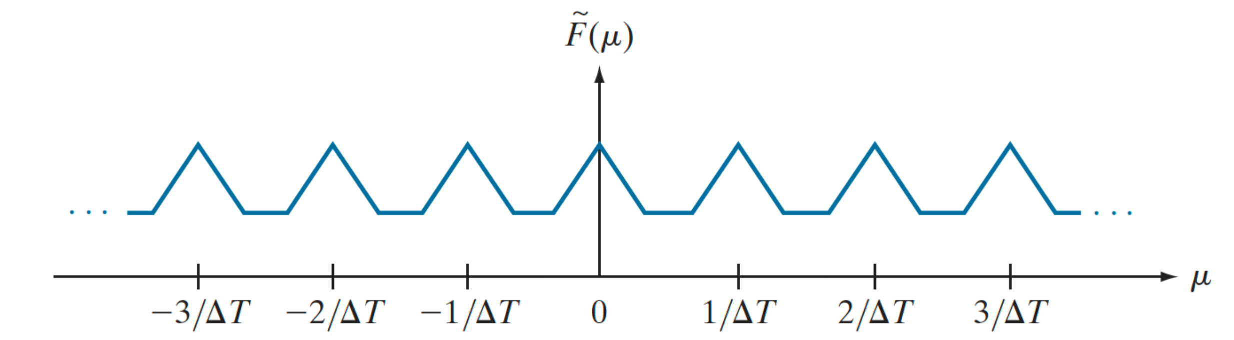
Over sampling



Critical Sampling



Under sampling



Sampling Theorem

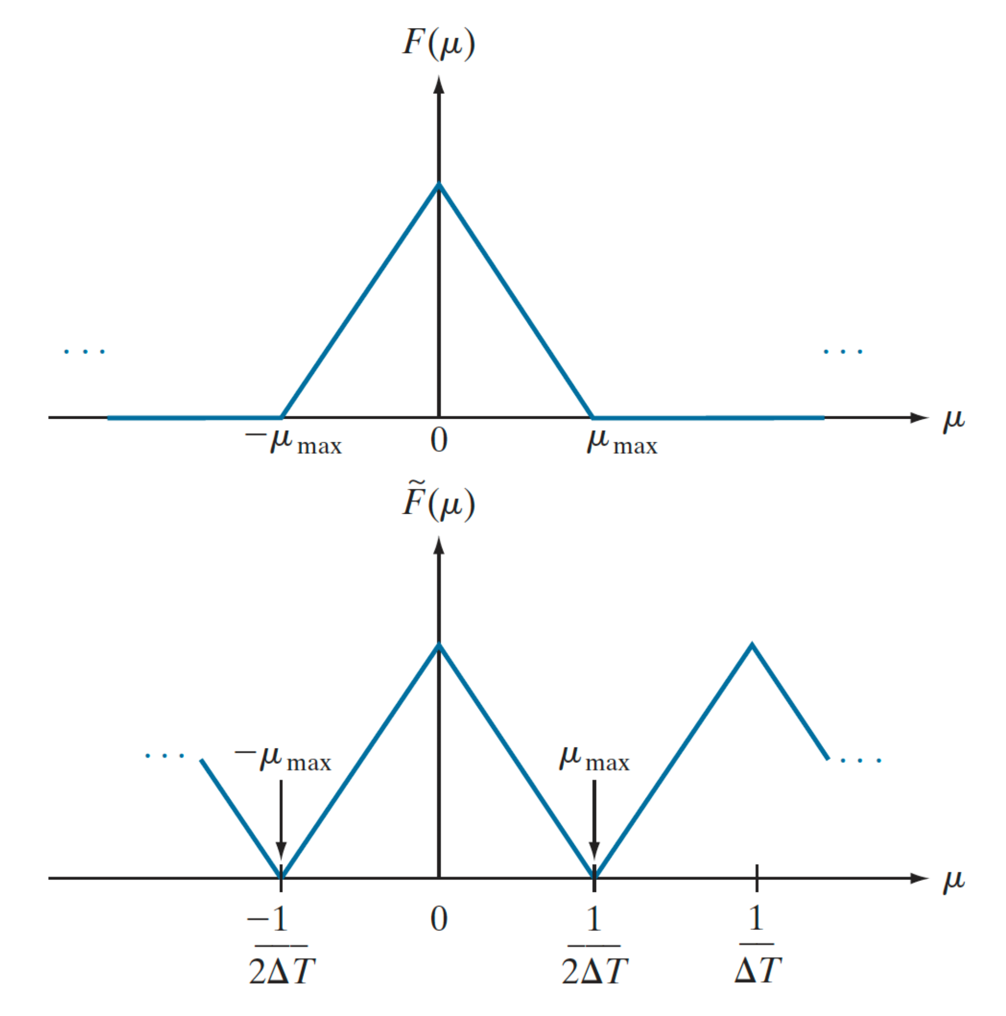
- Signal can be reconstructed perfectly if ***F*(μ)** is not corrupted
- Separation is guaranteed if:

$$\tfrac{1}{2}\Delta T > \mu_{max}$$

or

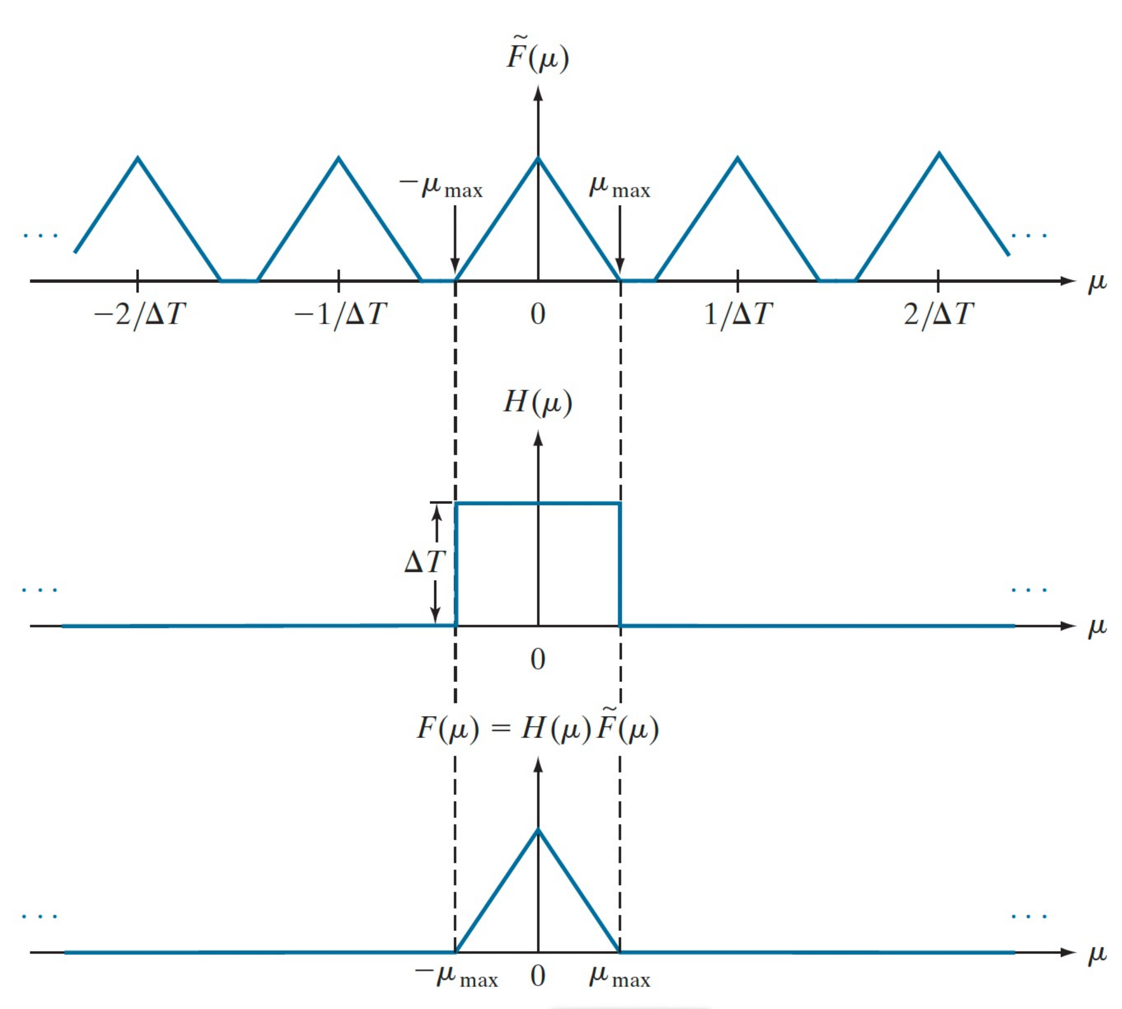
$$\boxed{\frac{1}{\Delta T} > 2\mu_{max}}$$

Nyquist Theorem



Recovering $F(\mu)$:

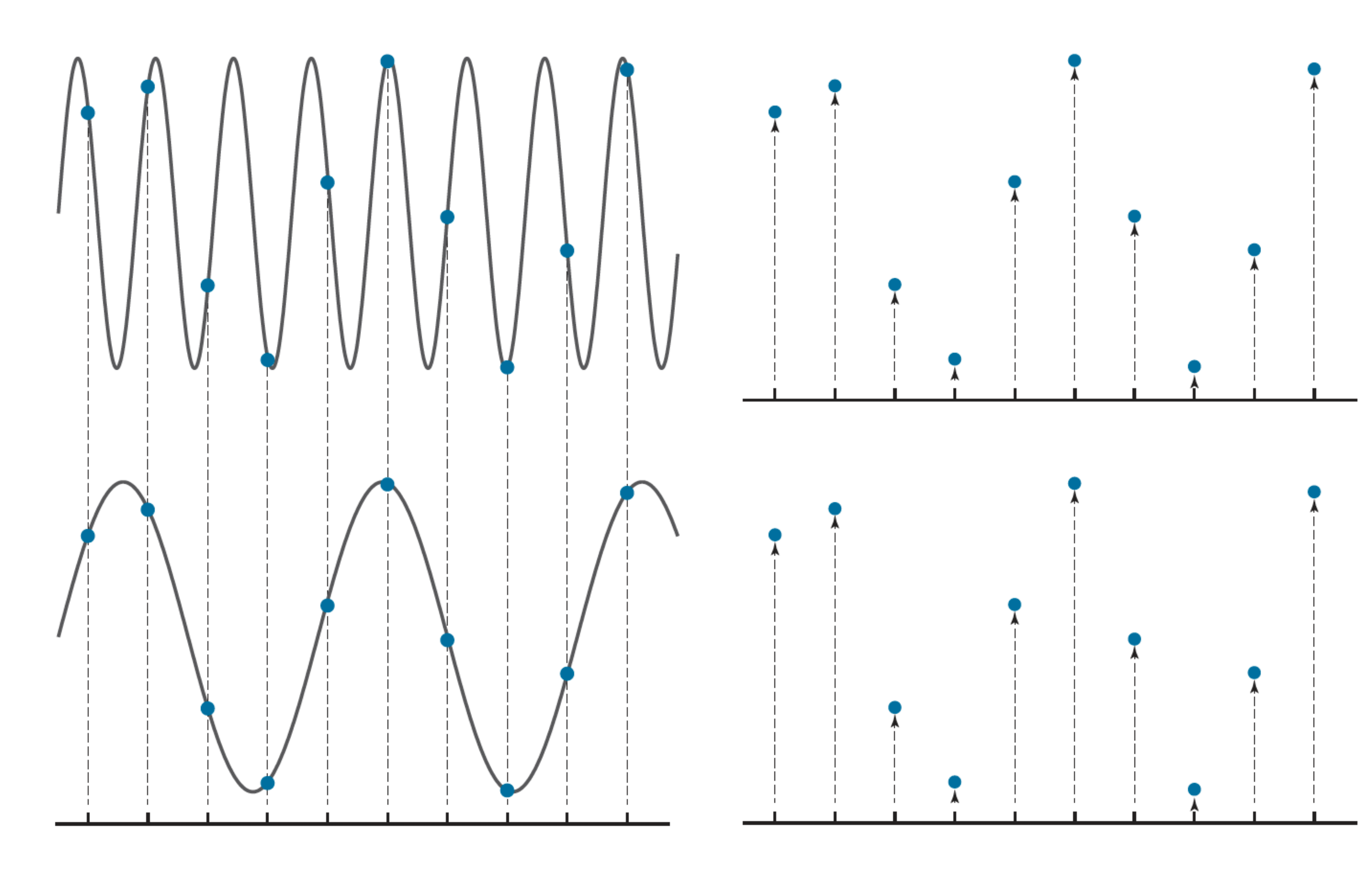
- If the sampling frequency is higher than Nyquist Rate:
  - An Ideal Low-pass Filter can recover $F(\mu)$ from $\tilde{F}(\mu)$.
- What is the implication in spatial domain?
- Band-Limited Signal



Aliasing

The two signals are different, but their sampled versions are identical !!

Sampling rate is low for signal-1



Problems with higher frequencies

- Consider the original signal, f(t), that is band limited and sampled above Nyquist Rate.
- What happens when we spatially limit the function?
  - Multiplication with a square pulse
  - Convolving with Sync function in Frequency domain
- What does this mean?

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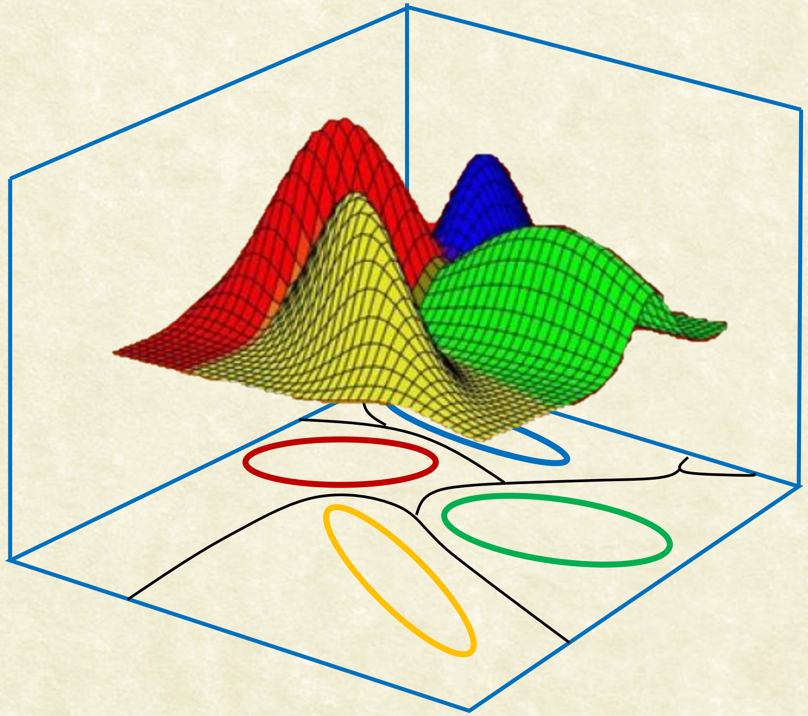
Monsoon 2023: Sampling the Frequencies

DFT of Finite Sample Sets



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Sampling the Frequency

- Fourier Transform of Sampled Signal

$$\tilde{F}(\mu) = \int_{-\infty}^{\infty} \tilde{f}(t) e^{-j2\pi\mu t} dt = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T) e^{-j2\pi\mu t} dt$$

$$= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\delta(t - n\Delta T) e^{-j2\pi\mu t} dt$$

$$= \sum_{n=-\infty}^{\infty} f_n e^{-j2\pi\mu n\Delta T}$$

- Sampling μ at M points:

$$F_m = \sum_{n=0}^{M-1} f_n e^{-j2\pi mn/M} \qquad m = 0, 1, 2, \ldots, M-1$$

Discrete Fourier Transform

- The forward DFT is:

$$F_m = \sum_{n=0}^{M-1} f_n e^{-j2\pi mn/M} \qquad m = 0, 1, 2, \ldots, M-1$$

- The inverse DFT is:

$$f_n = \frac{1}{M} \sum_{m=0}^{M-1} F_m e^{j2\pi mn/M} \qquad n = 0, 1, 2, \ldots, M-1$$

Computing DFT

- How do we compute the DFT?
- How many operations?

$$F_m = \sum_{n=0}^{M-1} f_n e^{-j2\pi mn/M} \qquad m = 0, 1, 2, \ldots, M-1$$

- Simplifying using Matrix Multiplication.

DFT as Matrix Multiplication

$$F_m = \sum_{n=0}^{M-1} f_n e^{-j2\pi mn/M} \qquad m = 0, 1, 2, \ldots, M-1$$

$$Let\ \omega = e^{-j2\pi/M}$$

$$\begin{bmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \\ \vdots \\ F_{M-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \cdots & \omega^{M-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(M-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(M-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{M-1} & \omega^{2(M-1)} & \omega^{3(M-1)} & & \omega^{(M-1)^2} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{M-1} \end{bmatrix}$$

The FFT Algorithm

- Assume the data vector, $\boldsymbol{f}$, is of length 2^{k+1}, and let $\boldsymbol{\mathcal{F}}_d$ represent the complete DFT matrix of dimensions: $d \times d$.
- Reorder the terms of the data vector $\boldsymbol{f}$ into even and odd groups
- The DFT matrix can be split into two matrices:

$$\begin{bmatrix} F_1 \\ \vdots \\ F_{2^{k+1}} \end{bmatrix} = \left[\boldsymbol{\mathcal{F}}_{2^{k+1}}\right] \begin{bmatrix} f_1 \\ \vdots \\ f_{2^{k+1}} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{D}_\omega \\ \mathbf{I} & \mathbf{D}_\omega \end{bmatrix} \begin{bmatrix} \boldsymbol{\mathcal{F}}_{2^k} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\mathcal{F}}_{2^k} \end{bmatrix} \begin{bmatrix} \boldsymbol{f}_{2^k}^{\boldsymbol{evn}} \\ \boldsymbol{f}_{2^k}^{\boldsymbol{odd}} \end{bmatrix},$$

where $\mathbf{D}_\omega$ is a diagonal matrix of powers of ω, and $\mathbf{I}$ is the identity matrix.

- Apply the split Recursively to get the FFT algorithm.
- DFT vs FFT: $4N^2$ vs. $2\ \mathrm{N}\log_2 N$.

(Some) Properties of FT

	Name:	Condition:	Property:
1	Amplitude scaling	$f(t) \leftrightarrow F(\omega)$, constant K	$Kf(t) \leftrightarrow KF(\omega)$
2	Addition	$f(t) \leftrightarrow F(\omega)$, $g(t) \leftrightarrow G(\omega)$, $\cdots$	$f(t) + g(t) + \cdots \leftrightarrow F(\omega) + G(\omega) + \cdots$
3	Hermitian	Real $f(t) \leftrightarrow F(\omega)$	$F(-\omega) = F^*(\omega)$
4	Even	Real and even $f(t)$	Real and even $F(\omega)$
5	Odd	Real and odd $f(t)$	Imaginary and odd $F(\omega)$
6	Symmetry	$f(t) \leftrightarrow F(\omega)$	$F(t) \leftrightarrow 2\pi f(-\omega)$
7	Time scaling	$f(t) \leftrightarrow F(\omega)$, real s	$f(st) \leftrightarrow \frac{1}{\lvert s\rvert} F(\frac{\omega}{s})$
8	Time shift	$f(t) \leftrightarrow F(\omega)$	$f(t - t_o) \leftrightarrow F(\omega)e^{-j\omega t_o}$
9	Frequency shift	$f(t) \leftrightarrow F(\omega)$	$f(t)e^{j\omega_o t} \leftrightarrow F(\omega - \omega_o)$
10	Modulation	$f(t) \leftrightarrow F(\omega)$	$f(t)\cos(\omega_o t) \leftrightarrow \frac{1}{2}F(\omega - \omega_o) + \frac{1}{2}F(\omega + \omega_o)$
11	Time derivative	Differentiable $f(t) \leftrightarrow F(\omega)$	$\frac{df}{dt} \leftrightarrow j\omega F(\omega)$
12	Freq derivative	$f(t) \leftrightarrow F(\omega)$	$-jtf(t) \leftrightarrow \frac{d}{d\omega}F(\omega)$
13	Time convolution	$f(t) \leftrightarrow F(\omega)$, $g(t) \leftrightarrow G(\omega)$	$f(t) * g(t) \leftrightarrow F(\omega)G(\omega)$
14	Freq convolution	$f(t) \leftrightarrow F(\omega)$, $g(t) \leftrightarrow G(\omega)$	$f(t)g(t) \leftrightarrow \frac{1}{2\pi}F(\omega) * G(\omega)$
15	Compact form	Real $f(t)$	$f(t) = \frac{1}{2\pi}\int_0^\infty 2\lvert F(\omega)\rvert \cos(\omega t + \angle F(\omega))d\omega$
16	Parseval, Energy W	$f(t) \leftrightarrow F(\omega)$	$W \equiv \int_{-\infty}^{\infty} \lvert f(t)\rvert^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert F(\omega)\rvert^2 d\omega$

https://hkn.illinois.edu/wiki/_media/sidebar:screen_shot_2016-04-11_at_12.35.36_pm.png?w=800&tok=1f0b2e

References & Fun Reading/Viewing

- GW DIP textbook, 3rd Ed.
  - 4.1 to 4.2
  - 4.2.4, 4.2.5,4.3.1,4.3.2, 4.4.1

- https://betterexplained.com/articles/intuitive-understanding-of-sine-waves/
- A visual introduction to Fourier Transform: https://www.youtube.com/watch?v=spUNpyF58BY
- Fourier Transform, Fourier Series and Frequency Spectrum: https://www.youtube.com/watch?v=r18Gi8lSkfM

Questions?