Provably stable and high-order accurate finite difference schemes on staggered grids and their potential application for seismic hazard analysis

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Objectives

We investigate the accuracy of the free-surface boundary condition (FS1) used in many of SCEC's high-performance computing efforts.

- Proof of stability for FS1 in one dimension
- Develop new high-order free-surface boundary conditions (SBP4W, SBP4S, SBP6W, SBP6S)
- Compare FS1 vs SBP4W, SBP4S, SBP6W, SBP6S in one dimension

Introduction

When it comes to wave propagation over large distances, high-order and centered finite difference approximations on staggered grids are highly effective. For this reason, the fourth-order staggered grid scheme is used for many of SCEC's high performance computing efforts, ranging from large-scale earthquake hazard simulations like ShakeOut to full-waveform tomography to reciprocity-based CyberShake calculations. In these simulations, it is essential to accurately model surface waves due to their dominant role in ground motion at all but the closest distances.

To model a traction-free surface, SCEC implements the free-surface boundary condition explained in [**?**, **?**, **?**] (also known as *FS1* [**?**]). FS1 is known to be stable in practice, but no stability proofs exist. We prove that FS1 is stable in 1D and propose new provably stable implementations of the free-surface boundary condition that are second-order accurate (fourth-order interior accuracy) and third-order accurate (sixth-order interior accuracy) using either a strong or weak enforcement of the boundary condition. These difference approximations can also be used for solving interface-driven problems.

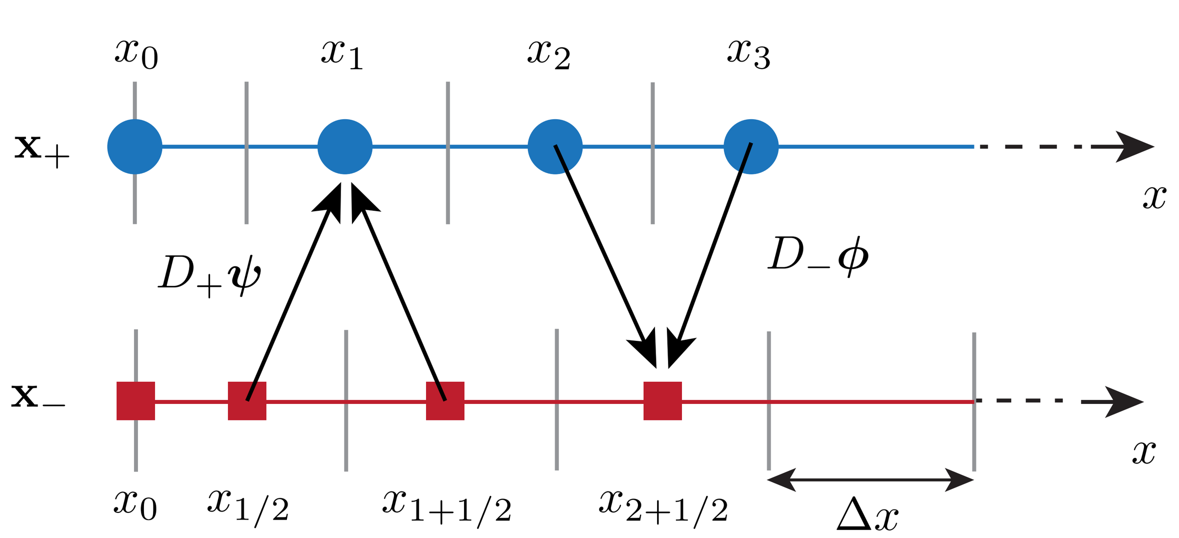
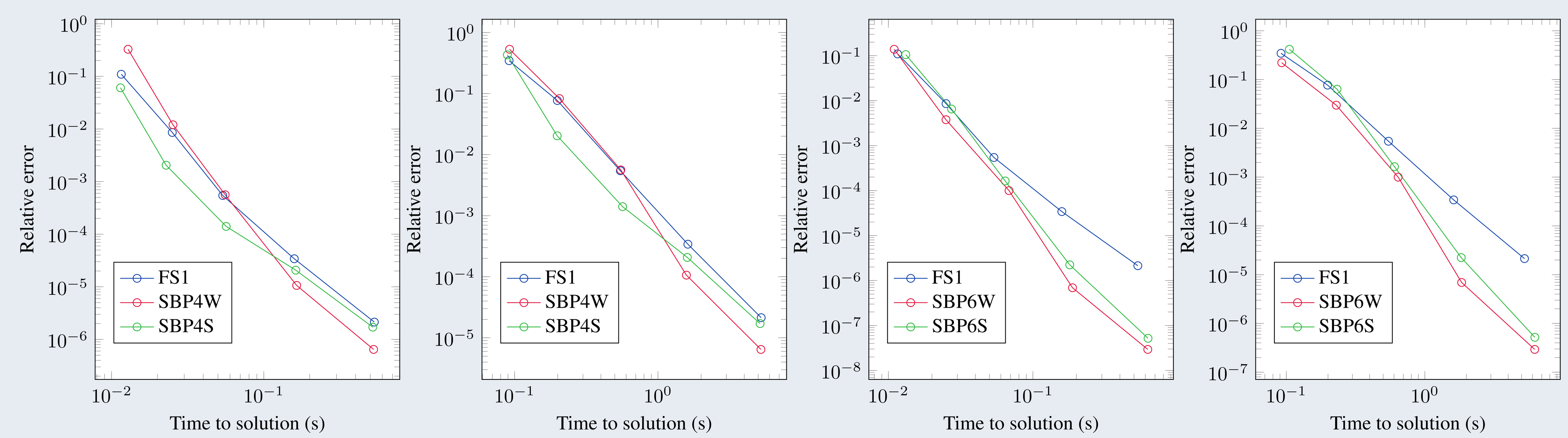


Figure 1: Example of second order difference approximations on staggered grids in one dimension.

Accuracy of free-surface boundary condition implementations in 1D



(a) time $T = 10$ (b) time $T = 100$ (c) time $T = 10$ (d) time $T = 100$

Figure 2: Error as a function of simulation time at time T (number of reflections) (a),(b) compares FS1 vs new fourth-order boundary implementations, (c), (d) compares FS1 vs new sixth-order boundary implementations

Test using initial condition $\sigma(x,0) = v(x,0) = e^{-a(x-0.5)^2}$ for $0 \leq x \leq 1$. Boundary conditions are: $\sigma(0,t) = \sigma(1,t) = 0$. We use $a = 100$, $\rho = \mu = 1$, and let the solution reflect against each boundary T times per boundary. A fourth-order Runge-Kutta method with time step $\Delta t = 0.4\Delta x$ is used in time. SBP**XY**, **X** = interior order of accuracy, **Y** = **W**eak or **S**trong enforcement of boundary conditions.

The elastic wave equation with a weak enforcement of the free-surface boundary condition

Elastic wave equation (anti-plane) with velocity v and shear stresses $\sigma = (\sigma_{13}, \sigma_{23})^T$:

$$\rho v_t = \nabla \cdot \sigma,\ \sigma_t = \mu \nabla v,\ (x,y) \in \Omega,$$
$$T = \hat{n}^T \sigma = 0 \text{ on } \Gamma.$$

Variational formulation with weak enforcement of boundary conditions:

$$\int_\Omega \rho \phi v_t d\Omega = \int_\Omega \phi \nabla \cdot \sigma d\Omega - \int_\Gamma \phi (T - \hat{T}) ds,$$
$$\int_\Omega \frac{1}{\mu} \varphi^T \sigma_t d\Omega = \int_\Omega \varphi^T \nabla v d\Omega - \int_\Gamma \varphi^T (v - \hat{v}) \hat{n} ds.$$

Fluxes chosen as

$$\hat{T} = 0,\ \hat{v} = v - ZT, Z = \rho\mu. \tag{1}$$

The choice $\phi = v$ and $\varphi = \sigma$ yields energy rate:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho v^2 + \frac{1}{\mu}\sigma^T \sigma d\Omega = -\int_\Gamma Z T^2 ds \leq 0. \tag{2}$$

Provably stable and high-order accurate finite difference approximations

Variational formulation of elastic wave equation (1) approximated by

$$\phi^T \rho H_+ \frac{dv}{dt} = \phi^T Q_+ \sigma - (r_+^T \phi)^T r_+^T (T - \hat{T}),$$
$$\varphi^T \mu H_- \frac{d\sigma}{dt} = \varphi^T Q_- v - (r_-^T \varphi)^T r_+^T \hat{n} (v - \hat{v}),$$

where $D_- = P^{-1} Q_-$, $D_+ = P^{-1} Q_+$ (difference approximations, see Figure 1), $H = diag(..)$ (quadrature rule), and vectors r_-, r_+ restricting solution to the boundary.

The choice $\phi = v$, $\varphi = \sigma$ yields energy rate

$$\frac{1}{2}\frac{d}{dt}(v^T \rho H_- v + \sigma^T \mu^{-1} H_+ \sigma) = v^T (Q_+ + Q_-^T + \hat{n} r_+ r_-^T)\sigma + \text{Diss},$$

where Diss ≤ 0. The scheme is a summation-by-parts (SBP) scheme if:

$$Q_+ + Q_-^T + \hat{n} r_+ r_-^T = 0. \tag{3}$$

In 1-D FS1 satisfies (3) with Diss $= 0$. Thus $dE_h/dt = 0$.

Construction of free-surface boundary conditions

We construct new difference approximations satisfying (3). These approximations are the same as FS1 in the interior, but uses different boundary closures. These boundary closures incur a small increase in computational cost because of having more non-zero coefficients in the difference stencils near the boundary. On the other hand, we gain free parameters which are tuned to minimize the truncation error.

Conclusions

The free-surface boundary condition at the Earth's surface give rise to surface waves which play a crucial role in ground motion estimation. In this work we aim for improving the accuracy of the free-surface boundary condition by proposing new and provably stable implementations. Since our analysis is restricted to one dimension, we do not a priori know how the implementations perform in practical applications. However, our preliminary work in one dimension indicates that they can outperform the commonly used implementation (FS1) in most cases. The way these new implementations are designed also opens up the possibility for handling interface problems, which is essential in e.g., dynamic earthquake rupture simulations.

Additional Information

Our MATLAB demo implementation that was used to produce the results in this work is available here: `https://github.com/ooreilly/scec2016/`

References

[1] Alan R Levander.
Fourth-order finite-difference p-sv seismograms.
Geophysics, 53(11):1425–1436, 1988.

[2] Robert W Graves.
Simulating seismic wave propagation in 3d elastic media using staggered-grid finite differences.
Bulletin of the Seismological Society of America, 86(4):1091–1106, 1996.

[3] E Gottschämmer and KB Olsen.
Accuracy of the explicit planar free-surface boundary condition implemented in a fourth-order staggered-grid velocity-stress finite-difference scheme.
Bulletin of the Seismological Society of America, 91(3):617–623, 2001.