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Spectral Theorom and The Multivariate Gaussian

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1 THE GENERAL GAUSSIAN DISTRIBUTION

Let X be a random vector $\in \mathbb{R}^n$ X is a Multivariate gaussian if the following holds:

$$X = \left| x_1, x_2, ..., x_n \right|^T = \mathcal{N}(\mu, \sigma) := p(x : \mu, \Sigma) =$$

$$= \frac{1}{(2\pi)^{n/2} |\Sigma|^{1/2}} exp(-\frac{1}{2}(x-\mu)^T \sigma^{-1}(x-\mu))$$

2 RELATING TO THE UNIVARIATE GAUSSIAN

Recall the density for the normal distribution:

$$p(x, \sigma^2, \mu) = \frac{1}{\sqrt{2\pi\sigma}} exp(-\frac{1}{2\sigma^2}(x-\mu)^2)$$

In both cases:

$$\frac{1}{\sqrt{2\pi\sigma}} \text{ and } \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}}$$

are normalization terms that ensure that the integrating the support of the density yields a valid probability distribution.

also note:

$$-\frac{1}{2\sigma^2}(x-\mu)^2 \text{ and } -\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)$$

are both quadratic forms on x and you can think of the argument of the exponential as being a downward opening quadratic bowl.

3 THE COVARIANCE MATRIX

Recall the definition of Covariance for random variables X,Y:

$$Cov(X,Y) = \mathbb{E}(X - \mathbb{E}(X))(Y - \mathbb{E}(Y)) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

. Definition:

$$\text{Covariance Matrix} := (n \times n)\text{matrix s.t. entry}_{i,j} = Cov(X_i, X_j)$$

$$\text{usually denoted by } \Sigma$$

Furthermore: $\forall$ random vector X with mean μ and covariance matrix Σ.

$$\Sigma = \mathbb{E}((X-\mu)(X-\mu)^T) = \mathbb{E}(XX^T) - \mu\mu^T$$

It will be useful to know that Σ is always a symettric matrix as $Cov(i,j) = Cov(j,i)$ and that the covariance of ANY random vector must always be positive semidefinite.

4 SPECIAL CASES: THE DIAGONAL COVARIANCE MATRIX CASE

Note how when the covariance matrix of a random gaussian vector is diagonal, this implies that the covariances of differing components $i \neq j = 0$ and that this random vector behaves identically to a collection of independent gaussians where $x_i = \mathcal{N}(\mu_i, \Sigma_{(i,i)})$

5 ISOCONTOURS

Recall:

$$\text{For a function } f : \mathbb{R}^2 \to \mathbb{R} \text{ isocontour}_f := \{x \in \mathbb{R}^2 : f(x) = c\} \text{ for some c} \in \mathbb{R}$$

Let

$$X = \begin{vmatrix} x_1 x_2 \end{vmatrix}^T \mu = \begin{vmatrix} \mu_1 \mu_2 \end{vmatrix}^T, \text{and } \Sigma = \begin{vmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{vmatrix}$$

and if we wanted to solve for the level set c: $c = p(x : \mu, \Sigma)$: it turns out the level sets define an axis - aligned elipse! with center (μ_1, μ_2) and where the length of axis $x_i = 2r_i$

$$1 = (\frac{x_1 - \mu_1}{r_1})^2 + (\frac{x_2 - \mu_2}{r_2})^2$$

$$r_i = \sqrt{2\sigma_i^2 log(\frac{1}{2\pi\sigma_1\sigma_2})}$$

It turns out in the non-diagonal case: the isocontours turn out to simply be rotated ellipses and in higher dimensions, the isocountours form ellipsoids in $\mathbb{R}^n$

6 LINEAR TRANSFORMATION INTERPRETATION

Theorom:

$$\text{Let } X = \mathcal{N}(\mu, \Sigma) \text{ for some } \mu \in \mathbb{R}^n and \Sigma \in \mathbf{S}^n_{++}.$$

$$\text{Then. there exists a matrix } B \in \mathbb{R}^{n,n} \text{s.t } Z = B^{-1}(X - \mu), \text{then } Z = \mathcal{N}(0, I)$$

Corollary:

$$\text{Given this } B \to X = BZ + \mu$$

Interpretation: This theorom implies that any gaussian vector in $\mathbb{R}^n$ is simply a linear transformation of n independent standard normal random variables, and that the linear combination of these gaussian vectors will also be a gaussian vector.