UNIVERSITY OF CALIFORNIA, BERKELEY

Lecture 18 - EE127

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July 16, 2021

When the objective and constraint are all affine, the problem is called a **Linear Program**, a general linear program will be of the following form:

$$\min\ c^T x + d$$

$$\text{subject to: } Gx \leq h$$
$$Ax = b$$

Linear Programs are, of course convex. It is common to omit the constant d as it does not affect the feasible set of the solutions.

0.1 STANDARD AND INEQUALITY FORM LINEAR PROGRAMS

Two special cases of LP are so widely encountered that they have been given separate names. In a **standard form LP** the only inequalities are component-wise non negativity constraints:

$$\min c^T x$$

$$\text{subject to } Ax = b$$
$$x \geq 0$$

If the LP has no inequality constraints, it is called an inequality form LP

$$\min\ c^T x$$

$$\text{subject to } Ax \leq b$$

0.2 Feasible Regions

- constraints will always be a polytope
- the solution not always just a vertex, but one can have an infinite set of solutions as well.

0.3 Motivation for the solution of an LP

Consider the taylor expansion:

$$f(x_0+v) \approx f(x_0) + \nabla f^T(v)$$
$$f(x_0) = f(x_0) + c^T v$$
$$0 = c^T v$$

Therefore, our solution is one such that we want our vector orthogonal to the gradient, which is why we want to go in the direction of -c!

1 Epigraphic Reformulation

Consider the following minimization:

$$\min_x f_0(x) = |x|$$
$$\text{s.t} -1 \le x \le 1$$

And recall the definition of the epigraph of a function f:

$$\text{Epi}(f) = \{(x,t) \in \mathbf{R}^{n+1} | x \in \text{Dom}(f), t \ge f_0(x)\}$$

$$\text{Epi}(f_0) = \{(x,t) \in \mathbf{R}^3 | x \in [-1,1], t \ge f_0\}$$

Because the information of the objective value is contained in the variable t, we can now reformulate the LP as follows:

$$\min\ t$$
$$\text{s.t}\ |x| \le t$$
$$-x-1 \le 0$$
$$x-1 \le 0$$

and because we can replace the absolute value with two linear constraints we can once again reformulate as follows:

$$\min\ t$$
$$\text{s.t}\ x-t \le 0$$
$$-x-t \le 0$$
$$-x-1 \le 0$$
$$x-1 \le 0$$

2 EXAMPLES

2.1 CHEBYSHEV CENTER OF A POLYHEDRON

Consider the problem of finding the largest Euclidean ball that lies in a polyhedron described by linear inequalities:

$$\mathscr{P} = \{x \in \mathbf{R}^n a_i^T x \leq b_i : i = 1, ..., m\}$$

and we want to find the center of the optimal ball within the polyhedron described:

$$\mathscr{B} = \{x_c + u | \|u\|_2 \leq r\}$$

So we can formulate this optimization problem as follows

$$\max r$$

$$\text{subject to } \mathscr{B} = \{x_c + u | \|u\|_2 \leq r\}$$

$$\mathscr{B} \subseteq \mathscr{P}$$

Consider the constraint that the ball lies in one halfspace of $a_i^T x \leq b_i$ i.e

$$\|u\|_2 \leq r \rightarrow a_i^T (x_c + u) \leq b_i$$

Since the maximum value that can be achieved is equal to $r\|a_i\|_2$ We can wrtie the previous equation as the following:

$$a_i^T x_c + r\|a_i\|_2 \leq b_i$$

Node that because the a_i variables are not over the terms in the optimization program, this is actually an affine constraint. Therefore, we can reformulate as the following:

$$\max{}_{x,r} r$$

$$\text{s.t } a_i^T x + \|a_i\|_2 \leq b_i : i = 1, ..., m$$

2.2 PIECEWISE LINEAR MINIMIZATION

consider the following optimization program:

$$\min f_0(x) = \min_x \max_{i=1,...,m} (a_i^T x + b_i)$$

Note here that using the epigraphic reformulation from above, we can reformulate this as such:

$$\min t$$

$$\text{s.t } \max_{i=1,...,m} (a_i^T x + b_i) \leq t$$

Furthermore, we note that if the pointwise-maximum of the affine functions is less than some value t, this implies all the affine functions are less than t, which means we can reformulate as the following:

$$\min t$$

$$\text{s.t } a_i^T x + b_i \leq t : i = 1, ..., m$$

2.3 l_∞ REGRESSION

Consider the following minimization:

$$\min_x \|Ax - b\|_\infty = \max_i |(a_i^T x - b_i)|$$

Here, we can use both the tricks used in epigraphic reformulation and the piecewise linear minimization to reduce the program to the following:

$$\min_{x,t} t$$

$$\text{s.t } a_i^T x - b_i \leq t : i = 1, \ldots, n$$
$$a_i^T x - b_i \geq -t : i = 1, \ldots, n$$