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Lecture 20: EE-127

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1 CONES

A set C is called a cone, if for every $x \in C$ and $\lambda \geq 0, \lambda x \in C$. A set is known as a convex cone if it is convex and it is a cone. We can also define the **conic hull** of a set c as the set of all conic combinations of points in C, i.e:

$$\{\sum_{i=1}^{k} \lambda_i x_i : x_i \in C, \lambda_i \geq 0, i = 1, ..., k\}$$

Note that this is in fact also the smallest convex cone that contains C.
We can also define the **second-order cone** as follows:

$$\mathcal{K}_n := \{(x, t) | \|x\|_2^2 \leq t\} \subseteq \mathbf{R}^{n+1}$$

Note that the second order cone is convex, as it can be expressed as an infinite number of half-spaces.

$$\mathcal{K}_n := \bigcap_{u:\|u\|_2 \leq 1} \{(x, t) | x^T y \leq t\}$$

Lemma: For $y, z \in \mathbf{R}, x \in \mathbf{R}^n$

$$\| \begin{bmatrix} x \\ \frac{1}{\sqrt{2}(y-z)} \end{bmatrix} \|_2^2 \leq (\frac{1}{\sqrt{2}}(y+z))^2 \to \|x\|_2^2 \leq 2yz$$

Proof:

$$\|x\|_2^2 + \frac{1}{2}(y^2 - 2yz + z^2) \leq (\frac{1}{\sqrt{2}}(y+z))^2 \tag{1.1}$$

$$\|x\|_2^2 + \frac{1}{2}(y^2 - 2yz + z^2) \leq (\frac{1}{2}(y + 2yz + z^2)) \tag{1.2}$$

$$\|x\|_2^2 - yz \leq yz \tag{1.3}$$

$$\|x\|_2^2 \leq 2yz \tag{1.4}$$

From this, we can now define the **rotated second-order cone** as follows:

$$\mathcal{K} = \{(x, y, z) | x \in \mathbf{R}^n, y, z \in \mathbf{R} \text{ s.t } \|x\|_2^2 \leq 2yz, y \geq 0, z \geq 0\}$$

Which is also equivalent to the following:

$$\mathcal{K} = \{(x, y, z) | x \in \mathbf{R}^n, y, z, \in \mathbf{R}, \text{ s.t } \| \begin{bmatrix} x \\ \frac{1}{\sqrt{2}(y-z)} \end{bmatrix} \|_2^2 \leq (\frac{1}{\sqrt{2}}(y+z))^2\}$$

Note, that this defines a second order cone, such that $w = (x, \frac{(y-z)}{\sqrt{2}})$, and $t = \frac{(y+z)}{2}$
There two sets of variables are related by a rotation matrix:

$$R = \begin{bmatrix} I_n & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

s.t

$$\begin{bmatrix} w \\ t \end{bmatrix} = R \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

2 Standard SOC Constraints

In general, a second order constraint will an inequality constraint of the norm of an affine function and an affine function. i.e:

$$\|Ax + b\|_2 \leq c^T x + d$$

And in general, a standard **Second Order Cone Program**, or SOCP will be of the following form:

$$\min_x : c^T x$$

$$\text{s.t } \|A_i x + b_i\|_2 \leq c_i^T x + d_i, i = 1, \ldots, m$$

2.1 Linear Programs as SOCPs

We can cast a Linear Program in the following form:

$$\min_x c^T x$$

$$\text{s.t } a_i^T x \leq b_i : i = 1, \ldots, m$$

Into SOCP form as follows:

$$\min_x c^T x$$

$$\text{s.t } \|C_i^T x + d_i\|_2 \leq b_i - a_i^T x : i = 1, \ldots, m$$

Where the matrices, C_i, and the vectors, d_i, are 0.

2.2 Quadratic Programs as SOCPs

Consider the following QP:

$$\min_x x^T Q x + c^T x$$

$$\text{s.t } a_i^T x \leq b_i : i = 1, \ldots, m$$

where Q is PSD.
We can cast this as an SOCP by defining $w = Q^{\frac{1}{2}} x$ and defining slack variables y, z such that $z = 1$, which allows us to define the following region as a rotated cone.
This would result in the following QP:

$$\min_x y + c^T x$$

$$\text{s.t } w^T w \leq w$$

$$w = Q^{\frac{1}{2}} x$$

$$a_i^T x \leq b_i$$

2.3 QCQPs as SOCPs

Consider a QCQP in the following form:

$$\min_x x^T Q_0 x + a_0^T x$$

$$\text{s.t } x^T Q_i x + a_i^T x \leq b_i : i = 1, \ldots, m$$

We can first begin to reformulate this using the epigraph reformulation of the problem

$$\min_x t + a_0^T x$$

$$\text{s.t } x^T Q_i x + a_i^T x \leq b_i : i = i, \ldots, m$$

$$x^T Q_0 x \leq t$$

We would then apply the same transformation we performed on the QP to every inequality constraint. Leaving us with the following form:

$$\min_{(x,t)} a_0^T x + t$$

$$\text{s.t } w_0^T w_0 \leq t, w_0 = Q_0^{\frac{1}{2}} x$$
$$w_i^T Q_i w_i \leq b_i - q_i^T x, i = 1, \ldots, m$$