UNIVERSITY OF CALIFORNIA, BERKELEY

Lecture 12: EE127 Convexity 2

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1 OVERVIEW

In general, an optimization problem will be of the following form:

$$p* = min_x f_0(x)$$

$$\text{subject to:} f_0(x) \leq 0, i = 1, ..., m$$

$$h_i(x) = 0 : i = 1, ..., q$$

with $f_i's$ and $g_i's$ are arbitrary nonlinear functions, however in this class, we will focus on models that are 'reliable'.

For the optmization problem to be convex it must satisfy the following:

- the objective function is convex
- the functions defining the inequality constraints are convex
- the functions defining the equality constraints are afine.

We note here that there is a implicit constraint on x, that it lie in the domain of the problem, the intersection of the domains of the functions with inequality constraints.
We know that since the $f_i's$ are convex, the domain is convex.

1.1 Implicit Constraints

if it is the case the objective function is undefined at certain points or tends to ∞ it is the case there is implied constraint that x is not part of the undefined domain or the portions of the domain that tend towards infinity

1.2 Example of Convex Program: SOCP

Consider the following model of an **second order cone program**, known as an SOCP:

$$min_{x \in \mathbf{R}^n} c^T x$$

$$s.t : \|A_i x + b_i\|_2 \le c_i^T x + d_i, i = 1, ..., m$$

If we recall the point-wise maximum rule, the maximum of convex functions is a convex function, and the implicit maximum by computing the l_2 norm,

$$\|y_2\|_2 = max_{\|u\|=1} u^T y$$

and use the fact that affine transformations of functions that are convex remain convex we can immediately see this optimization program is convex.

1.3 Other standard forms

We can equivalently define a convex optimization program as minimizing a convex function subject to the decision variable lying in a **convex set**

$$p* = min_{x \in \mathscr{X}} f_0(x)$$

Notes:

- that if the problem is unconstrained, then $\mathscr{X} = \mathbb{R}^n$
- If $\mathscr{X}$ is the empty set, we say the problem is infeasible, in such a case it is customary to set $p^* = \infty$
- When $\mathscr{X}$ is nonempty, we say the problem is feasible
- If we do not now in advance if the feasible set $\mathscr{X}$ is empty or not; the task of determining if this is the case or not is referred to as a feasibility problem
- If the problem is **feasible** and $p* = -\infty$ we say that the problem in unbounded below.
- If $x^* \in \mathscr{X}_{\text{opt}}$ is such that $f_i(x^*) < 0$, we say that the ith constraint is inactive, or **slack**.
- The **optimal set** is defined as follows: $\mathscr{X}_{opt} = \{x \in \mathscr{X} : f_0(x) = p*\} = \text{argmin}_{\mathscr{X}} f_0(x)$

2 LOCAL AND GLOBAL OPTIMA

Theorem:
If f_0 is a convex function and $\mathcal{X}$ is a convex set, then any locally optimal solution is also globally optimal. Moreover, the set $\mathcal{X}_{opt}$ of optimal points in convex.
Furthermore, we know that the optimal set is convex, since it can be expressed as the p^* sublevel set of a convex function:

$$\mathcal{X}_{\text{opt}} = \{x \in \mathcal{X} : f_0(x) \leq p^*\}$$

3 PROBLEM TRANSFORMATIONS

An optimization problem can be transformed, or reformulated, into an equivalent one by the use of several useful 'tricks'.

We define two optimization problems as **equivalent** informally if from one optimization problem we can find the solution of the other optimization problem from the solution of the original.

3.1 MONOTONE OBJECTIVE TRANSFORMATION

consider an optimization problem in standard form:

$$p* = min_x f_0(x)$$

$$\text{subject to:} f_(x) \leq 0, i = 1, ..., m$$

$$h_i(x) = 0 : i = 1, ..., q$$

and let $\varphi : \mathbf{R} \to \mathbf{R}$ be a continuous a monotonically increasing function of $\mathcal{X}$, and consider the transformed problem.
Here are some examples of 'tricks':

- monotone objective transformation
- change of variables
- addition of slack variables
- epigraphic reformulation
- replacement of equality constraints with inequalities
- elimination of inactive constraints

$$g^* = \min_{x \in \mathbf{R}^n} \varphi(f_0(x))$$

$$\text{subject to} : f_i(x) \leq 0, i = 1, ..., m$$

$$h_i(x) = 0 : i = 1, ..., q$$

Then the original and transformed problems have the same set of optimal solutions. Note however that we cannot make any statements on the convexity of the function.

3.2 ADDITION OF SLACK VARIABLES

Equivalent problem formulations are also obtained by introduction of slack variables: As an example, consider the following:

$$p^* = \min_x f_0(x) + \sum_{i=1}^{x} \varphi_i(x)$$

$$\text{s.t } x \in \mathscr{X}$$

with the introduction of slack variables $t_1, ..., t_m$, we can reformulate this as the following:

$$q* = \min_{x,t} \sum_{i=1}^{n} t_i$$

$$\text{s.t } x \in \mathscr{X}$$

$$\varphi_i(x) \leq t_i : i = 1, ..., m$$

Here we say the problems are equivalent because the following holds:

- if x if feasible for the first problem, then x, $t_i = \varphi(x), i = 1, ..., r$ is feasible for the second.
- if x, t is feasible for the second then, x is feasible for the first
- $p^* = q^*$

Note that the statements also imply that the there is an equivalence relation between the optimal sets of these optimization problems.

4 RELAXATION

In some cases, we can substitute an equality constraint of the form $b(x) = u$ to an inequality constraint of the form $b(x) \leq u$.

This is useful in some cases, for gaining convexity. Specifically, if $b(x)$ is a convex function, then the set described by the equality constraint $\{x : b(x) = u\}$ is non-convex in general. However, the set described by the inequality constraint, $\{x : b(x) \leq u\}$ is the sub-level set of a convex function, and is therefore convex.

Consider a problem of the following form:

$$p^* = \min_{x \in \mathscr{X}} f_0(x)$$

$$b(x) = u$$

and consider the following relaxation:

$$q^* = \min_{x \in \mathscr{X}} f_0(x)$$

$$b(x) \leq u$$

We know that because the feasible set of the first problem is a subset of the second, then $p^* \geq q^*$

We know equality holds in the following conditions:

- f_0 is non-increasing over $\mathscr{X}$
- b is non-decreasing over $\mathscr{X}$
- the optimal value p^* is attained at some optimal point x^*, and the optimal value g^*, and the optimal value g^* is attained at some optimal point