UNIVERSITY OF CALIFORNIA, BERKELEY

Lecture 19: EE127

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1 Quadratic Optimization Problems

An optimization problem is called a **Quadratic Program** is the objective function is quadratic and the constraint functions are affine: ie:

$$\min x^T H_0 x + 2c_0^T x + d$$

$$\text{s.t } Gx \le h$$

$$Ax = b$$

Note that the quadratic program is convex when it is the case H_0 is PSD.
If the inequality constraints are also convex and quadratic, the program is known as a **quadratically constrained quadratic program**, or a QCQP. ie:

$$\min x^T Hx + 2c_0^T x + d$$

$$\text{s.t } \frac{1}{2} x_i^T H_i X + 2c_i^T x + d_i \le 0$$

$$\frac{1}{2} x_i^T H_i X + 2c_i^T x + d_i = 0$$

In a QCQP, we minimize a convex quadratic function over a feasible region that is the intersection of ellipsoids.Note, that quadratic programs are a superset of linear programs.

2 EXAMPLES

2.1 UNCONSTRAINED MINIMIZATION OF QUADRATIC FUNCTIONS

Consider the general quadratic case:

$$p^* = \min_{x \in \mathbf{R}^n} \frac{1}{2} x^T hX + c^T x + d$$

You will encounter one of two cases:

1. H has a negative eigenvalue $\lambda < 0$. the let u be the corresponding eigenvector and take $x = au$ with $a \neq 0$. Since $Hu =$

$$f_0(x) = \frac{1}{2} x^T Hx + c^T x + d$$

$$= \frac{1}{2} \lambda a^2 \| u \|_2^2 + a(c^T u) + d$$

   Which tends to $-\infty$ as a approaches infinity. Hence, the objective is unbounded from below.

2. If all the eigenvalues of H are non-negative, then we know the objective is convex. and that the minimum is characterized by the condition that the gradient of the function is zero:

$$\nabla f_0(x) = Hx + c = 0$$

$$\rightarrow Hx = -c$$

   - $c \notin \mathscr{R}(H)$, then there is no minimizer, and also implies that H is a singular matrix, since the matrix has an eigenvalue of 0. Thus taking $x = au$ we have that, along direction u, $f_0(x) = a(c^T u) + d$. But because $c^T u \neq 0$ since $u \in \mathscr{N}(H)$ and $c \in \mathscr{R}(H)$ must have a nonzero component along $\mathscr{N}(H)$, therefore, $f_0(x)$ is unbounded below.
   - If $c \in \mathscr{R}(H)$, then f_0 has a finite global minimum value

$$x^* = -H^{\dagger} c$$

$$p^* = -\frac{1}{2} C^T H^{\dagger} c + d$$

     Note that if the matrix H is positive definite, then the pseudo-inverse of the transformation H is just the inverse of the matrix.

$$f(n) = \begin{cases} \frac{-1}{2} H^{\dagger} c + d : \text{if } H \geq 0 \text{ and } c \in \mathbb{R} \\ -\infty \text{ else} \end{cases}$$

2.2 QUADRATIC MINIMIZATION UNDER LINEAR EQUALITY CONSTRAINTS

Consider, the linear equality-constrained problem:

$$\min f_0(x)$$

$$\text{subject to} : Ax = b$$

We reparametrize all $x : Ax = b$ as $x = \tilde{x} + Nz$, where $\tilde{x}$ is a specific solution to $Ax = b$, and N is a matrix containing a basis for the null-space of A, and z is a vector of free variables. We can then substitute for x in the objective to obtain the following unconstrained problem in the variable z:

$$\mathbf{min}_z \phi(z) = \frac{1}{2} z^T N^T H N + N^T (c + H\tilde{x}) + (d + c^T \tilde{x} + \frac{1}{2} \tilde{x}^T H \tilde{x}$$

3 FINITE-HORIZON, DISCRETE-TIME LQR

LQR stands for **Linear Quadratic Regulator** and for a discrete time linear system is described by

$$x_{k+1} = Ax_k + Bu_k$$

$$x_0 = x_{\text{init}}$$

with a performance index (cost function) defined as follows:

$$J(U) = \sum_{z=0}^{n-1} x_z^T Q x_z + u_z^T R u_z) + x_n^T Q x_n$$

, where we can interpret the terms with the Q matrix as a cumulative penalty to the target, the terms with the R matrix as a cumulative running cost, and the final term as a terminal cost. Note that we can our dynamical system can be purely expressed in terms of the the A, B, u_i and the initial vector x_{init}

$$x_0 = x_0 \tag{3.1}$$

$$x_1 = Ax_0 + Bu_0 \tag{3.2}$$

$$\tag{3.3}$$

This will allow us to derive a solution to this dynamical system in the next lecture.