UNIVERSITY OF CALIFORNIA, BERKELEY

EE-120: Lecture 11

Oscar Ortega

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1 2D FOURIER TRANSFORMS

We can arbitrarily extend fourier transforms to any number of dimensions, we will stick to 2d today because of its relevance to image processing.

1.1 CTFT

In two dimensions we will define the following function fourier transform pair as

$$x(t_1,t_2) \leftrightarrow X(w_1,w_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t_1,t_2)e^{-j\omega_1 t_1}e^{-j\omega_2 t_2}dt_1 dt_2 \tag{1.1}$$

Which will have the following analysis equation:

$$x(t_1,t_2) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t_1,t_2)e^{j\omega_1 t_1}e^{j\omega_2 t_2}d\omega_1 d\omega_2 \tag{1.2}$$

1.2 DTFT

For the discrete time fourier transform we have the following form for the fourier transform $X(e^{j\omega_1}, e^{j\omega_2})$

$$\sum_{n_2=-\infty}^{\infty}\sum_{n_1=-\infty}^{\infty} x[n_1,x_2]e^{-j\omega_1 n_1}e^{-j\omega_2 n_2} \tag{1.3}$$

Note, that we know have periodicity in both arguments for the DTFT.

$$X(e^{j(\omega_1+2\pi)}, e^{j\omega_2}) = X(e^{j\omega_1}, e^{j\omega_2}) = X(e^{j\omega_1}, e^{j(\omega_2+2\pi)}) \tag{1.4}$$

We can define the analysis equation as follows:

$$x[n_1, n_2] = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(e^{j\omega_1}, e^{j\omega_2}) e^{j\omega_1} e^{j\omega_2} d\omega_1 d\omega_2 \tag{1.5}$$

Absolute integrability/summability guarantee convergence:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |x(t_1, t_2)| dt_1 dt_2 < \infty \tag{1.6}$$

$$\sum_{n_2=-\infty}^{\infty} \sum_{n_1=-\infty}^{\infty} |x[n_1, n_2]| < \infty \tag{1.7}$$

Eg: $x[n_1, n_2] = \delta[n_1, n_2]$ You could think of the two dimensional delta as just a spike that occurs in the origin of a plane.

$$X(e^{j\omega_1}, e^{j\omega_2}) = e^{-j\omega_1 n_1} e^{-j\omega_2 n_2}|_{(n_1,n_2)=(0,0)} = 1 \tag{1.8}$$

We can define $u[n_1, n_2] = u[n_1]u[n_2]$ Therefore, because the functions are homomorphic, we can also separate this double sum form of the fourier transform into a product of sums over the two domains of the fourier transforms. ie:

$$x[n_1, n_2] = x_1[n_1]x_2[n_2] \leftrightarrow X_1(e^{j\omega_1})X_2(e^{j\omega_2}) \tag{1.9}$$

2 2D Systems

Consider a system that takes in $x[n_1, n_2]$ and outputs a $y[n_1, n_2]$, we will still consider the response of the system to a delta as the impulse response $h[n_1, n_2]$s

2.1 2D moving avg filter

$$y[n_1, n_2] = \frac{1}{9} \sum_{k_2=-1}^{1} \sum_{k_1=1}^{1} x[n_1 - k1, n_2 - k_2] \tag{2.1}$$

What is the impulse response of this system? Easy a three by three square over the origin each spike with height $\frac{1}{9}$

We can think of time invariance as just shift invariance because its weird to think about the reliance on time when dealing with images. LTI is equivalent to a linear shift invariant system.

Shift invarance: $x[n_1, n_2]$ produces $y[n_1, n_2]$ and for all shifts k_1, k_2, $x[n_1 - k_1, n_2 - k_2]$ produces $y[n_1 - k_1, n_2 - k_2]$

Property:

$$h[n_1, n_2] * x[n_1, n_2] \leftrightarrow H(e^{j\omega_1}, e^{j\omega_2})X(e^{j\omega_1}, e^{j\omega_2}) \tag{2.2}$$

2d Separable ideal low pass filter would be $H(e^{j\omega 1}, e^{j\omega_2}) = 1$ in the rectangle about the origin with width and height ω_1, ω_2. If the frequency response is separable recall that this would imply the frequency response is separable and we can compute the fourier transforms with respect to each direction and multiply the two.

3 2D DFT

We define the 2d DFT as follows:

$$X[k_1, k_2] = \sum_{n_2=0}^{N_2-1} \sum_{n_1=0}^{N_1-1} x[n_1, n_2] e^{-j\frac{2\pi}{N_1}k_1 n_1} e^{-j\frac{2\pi}{N_2}k_2 n_2} \tag{3.1}$$

Similarly, we can define the synthesis equation as follows:

$$x[n_1, n_2] = \frac{1}{N_1 N_2} \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} X[k_1, k_2] e^{j\frac{2\pi}{N_1}k_1 n_2} e^{j\frac{2\pi}{N_2}k_2 n_2} \tag{3.2}$$