UNIVERSITY OF CALIFORNIA, BERKELEY

EE-120: Homework 1

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1 Convolution Proofs

Commutativity:

$$(x * h)(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \tag{1.1}$$

Let $\tau' = t - \tau$
$\rightarrow \tau = t - \tau'$
$\rightarrow d\tau = -d\tau$'
Furthermore, we know the following is true.
$\tau \rightarrow \infty \equiv \tau' \rightarrow -\infty$
$\tau \rightarrow -\infty \equiv \tau' \rightarrow \infty$

$$(x * h)(t) = \int_{\infty}^{-\infty} -x(t-\tau')h(\tau')d\tau' \tag{1.2}$$

$$= \int_{-\infty}^{\infty} x(t-\tau')h(\tau')d\tau' \tag{1.3}$$

$$= (h * x)(t) \tag{1.4}$$

Distributive Property:

$$x * (h_1 + h_2)(t) = \int_{-\infty}^{\infty} x(\tau)(h_1 + h_2)(t-\tau)d\tau \tag{1.5}$$

$$= \int_{-\infty}^{\infty} x(\tau)h_1(t-\tau) + x(\tau)h_2(t-\tau)d\tau \tag{1.6}$$

$$= \int_{-\infty}^{\infty} x(\tau)h_1(t-\tau)d\tau + \int_{-\infty}^{\infty} x(\tau)h_2(t-\tau)d\tau \tag{1.7}$$

$$= (x * h_1)(t) + (x * h_2)(t) \tag{1.8}$$

Associative Property:

$$x * (h_1 * h_2)(t) = \int_{k=-\infty}^{\infty} x(t-k) \int_{r=-\infty}^{\infty} h_1(t-r)h_2(r)drdk \tag{1.9}$$

$$= \int_{k=-\infty}^{\infty} \int_{r=-\infty}^{\infty} x(t-k)h_1(r)h_2(t-r)drdk \tag{1.10}$$

Let $m = t-k, s = t+r-k \to r = t-s, k-r = t-s, dm = -dk, ds = dr$

$$= \int_{s=-\infty}^{\infty} \int_{m=\infty}^{-\infty} -x(m)h_1(s-m)h_2(t-s)dmds \tag{1.11}$$

$$= \int_{s=-\infty}^{\infty} \int_{m=-\infty}^{\infty} x(m)h_1(s-m)h_2(t-s)dmds \tag{1.12}$$

$$= \int_{s=-\infty}^{\infty} (x * h_1)(s)h_2(t-s)ds \tag{1.13}$$

$$= (x * h_1) * h_2(t) \tag{1.14}$$

2 Impulse Responses of LTI systems

- h[n] = u[n] - u[n-10]

$$u[n] - u[n-10] = \sum_{i=0}^{\infty} \delta[n-i] - \sum_{i=10}^{\infty} \delta[n-i] = \sum_{i=0}^{9} \delta[n-i] \tag{2.1}$$

$$h[n] = \begin{cases} 0 : n < 0 \\ 1 : n \in 0, 1, ...9 \end{cases} \tag{2.2}$$

Therefore, the system is causal as it is 0 if $n < 0$. It is FIR because there is a finite number of non-zero entries to the impulse response.

$$\sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=0}^{9} |h[k]| = \sum_{k=0}^{9} |\delta[n-k]| = \sum_{k=0}^{9} 1 < \infty \tag{2.3}$$

Therefore, the system is also bounded.

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$$h[n] = 2^n u[n] \tag{2.4}$$

$$\begin{cases} 0 : n < 0 \\ 2^n : n \geq 0 \end{cases} \tag{2.5}$$

Because $n < 0 \to$ the signal is 0, then we can conclude the system is causal. Furthermore, there are an infinite number of points where the signal is IIR.

$$\sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=0}^{\infty} |h[k]| = \sum_{k=0}^{\infty} 2^n \to \infty \tag{2.6}$$

Therefore, the system is unbounded.

- $$h[n] = 2^n u[-n] \tag{2.7}$$

$$\begin{cases} 2^{-n} : n < 0 \\ 0 : n \geq 0 \end{cases} \tag{2.8}$$

Because there are an infinite number of points where the signal is nonzero, this is an IIR signal. Furthermore, because there are indices $n < 0$ where the signal is nonzero, this is also not a causal system.

$$\sum_{k=\infty}^{\infty} |h[k]| = \sum_{k=-\infty}^{0} 2^{-k} = \sum_{k=0}^{\infty} 2^k \to \infty \tag{2.9}$$

Therefore, the system is also unbounded.

- $$h[n] = n(0.8)^n u[n] \tag{2.10}$$

$$\begin{cases} 0 : n < 0 \\ n(0.8)^n : n \geq 0 \end{cases} \tag{2.11}$$

Because $n < 0$ implies that $h[n] = 0$ for this impulse response, this means the system is causal. Because there are an infinite number of nonzero signals, this system is IIR. Because this signal is positive for all indices in the domain, we know the following:

$$\sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=\infty}^{\infty} h[k] = \sum_{k=0}^{\infty} h[k]$$

. By the ratio test, this sum converges to $B < \infty$.

$$\begin{aligned} \lim_{k \to \infty} \left| \frac{h[k+1]}{h[k]} \right| &= \left| \frac{0.8^{n+1} n + 1}{0.8^n n} \right| \\ &= \lim_{k \to \infty} \left| 0.8 + \frac{0.8}{n} \right| \\ &= 0.8 \leq 1 \end{aligned}$$

3 Constant-Coefficient Difference Equations

- Let $x[n], y[n]$ s.t

$$y[n] + \sum_{i=1}^{N} a_i y[n-i] = \sum_{j=0}^{M} b_j x[n-j]$$

$$\alpha\left(y[n] + \sum_{i=1}^{N} a_i y[n-i]\right) = \alpha\left(\sum_{j=0}^{M} b_j x[n-j]\right)$$

$$\alpha y[n] + \sum_{i=1}^{N} a_i \alpha y[n-i] = \sum_{j=0}^{M} b_j \alpha x[n-j]$$

$$\hat{y}[n] + \sum_{i=1}^{N} \hat{y}[n-i] = \sum_{j=0}^{M} b_j \hat{x}[n-j]$$

  Which also satisfies the constant-coefficient difference equation.

- Let $x_1[n], x_2[n], y_1[n], y_2[n]$ satisfy the following difference equation.

$$y_1[n] + \sum_{i=1}^{N} a_i y_1[n-i] = \sum_{j=0}^{M} b_j x_1[n-j] \text{ and } y_2[n] + \sum_{i=1}^{N} a_i y_2[n-i] = \sum_{j=0}^{M} b_j x_2[n-j]$$

$$y_1[n] + \sum_{i=1}^{N} a_i y_1[n-i] + y_2[n] + \sum_{i=1}^{N} a_i y_2[n-i] = \sum_{j=0}^{M} b_j x_1[n-j] + \sum_{j=0}^{M} b_j x_2[n-j]$$

$$(y_1 + y_2)[n] + \sum_{i=1}^{N} a_i (y_1 + y_2)[n-i] = \sum_{j=0}^{M} b_j (x_1 + x_2)[n-j]$$

$$\hat{y}[n] + \sum_{i=1}^{N} a_i \hat{y}[n-i] = \sum_{j=0}^{M} b_j \hat{x}[n-j]$$

  Which also satisfies the constant-coefficient difference equation.

- Let $n' = n + L$. This implies $n = n' - L$

$$y[n] + \sum_{i=1}^{N} a_i y[n-i] = \sum_{j=0}^{M} b_j x[n-j]$$

$$y[n'-L] + \sum_{i=1}^{N} a_i y[n'-L-i] = \sum_{j=0}^{M} b_j x[n'-L-j]$$

$$\hat{y}[n'] + \sum_{i=1}^{N} a_i \hat{y}[n'-i] = \sum_{j=0}^{M} b_j \hat{x}[n'-j]$$

  Which also satisfies the constant-coefficient difference equation. It is important in this case that the coefficients $\{a_i\}_{i=1}^{M}\{b_j\}_{j=0}^{N}$ remain constant because when we perform the shift in time, we would no longer be able to state that a_i at a time k is equal to a_i at time $k - L$.

- Let $k < 0$ be such that the $y[k] = 1$ and let $x[n] = \delta[n]$

$$y[n] = \begin{cases} 0 : n < k \\ 1 : n = k \\ 1 + \sum_{i=k+1}^{n} \delta[i] : n > k \end{cases} \tag{3.1}$$

  Now let k be the same as above and let $x[n] = \alpha\delta[n]$

$$\begin{aligned} y[n] &= \begin{cases} 0 : n < k \\ 1 : n = k \\ 1 + \alpha\sum_{i=k+1}^{n} \delta[i] : n > k \end{cases} \\ &\neq \alpha y[n] \end{aligned}$$

  So no, the response did not scale by the same constant.

- Consider y defined as follows:

$$y[n] = \begin{cases} 0 : n < 0 \\ y[n-1] + x[n] : n \geq 0 \end{cases}$$

  Note that this is equivalent to the following:

$$y[n] = \begin{cases} 0 : n < 0 \\ \sum_{i=0}^{n} x[i] : n \geq 0 \end{cases}$$

  Consider an accumulator y with inputs αx

$$\begin{aligned} y[n] &= \begin{cases} 0 : n < 0 \\ \sum_{i=0}^{n} \alpha x[i] : n \geq 0 \end{cases} \\ &= \alpha y[n] \end{aligned}$$

  Consider two accumulators y_1, y_2 with inputs x_1, x_2 and let $\hat{y} = y_1 + y_2$

$$y_1[n] + y_2[n] = \begin{cases} 0 : n < 0 \\ \sum_{i=0}^{n} x_1[i] + \sum_{j=0}^{n} x_2[j] : n \geq 0 \end{cases} \tag{3.2}$$

$$= \begin{cases} 0 : n < 0 \\ \sum_{i=0}^{n} x_1[i] + x_2[i] : n \geq 0 \end{cases} \tag{3.3}$$

$$= y_1 + y_2[n] = \hat{y}[n] \tag{3.4}$$

Therefore the system is linear.
Let $\hat{y}[n] = y[n-L]$ and let $n' = n+L$
This implies $y[n] = y[n'-L]$

$$
\begin{aligned}
y[n'-L] &= \begin{cases} 0 : n'-L < 0 \\ \sum_{i=0}^{n'-L} x[i] : n'-L \geq 0 \end{cases} \\
&= \begin{cases} 0 : n' < 0 \\ \sum_{i=0}^{n'} x[i] : n' \geq 0 \end{cases} \\
&= \hat{y}[n']
\end{aligned}
$$

Therefore the system is time invariant.

4 Impulse and Frequency Responses

- The impulse response is defined as follows:

$$h[n] = 0.25\delta[n-1] + 0.5\delta[n] + 0.25\delta[n+1] \tag{4.1}$$

- The frequency response is the following:

$$
\begin{aligned}
H(e^{j\omega k}) &= \sum_{-\infty}^{\infty} e^{j\omega k} \\
&= \frac{e^{-j\omega}}{4} + \frac{1}{2} + \frac{e^{j\omega}}{4} \\
&= \frac{1}{2}(1 + cos(\omega))
\end{aligned}
$$

The sketch and response are on the written page.

5 Frequency Response of Circuit

The frequency of

$$
\begin{aligned}
RCd\frac{y(t)}{dt} + y(t) = x(t) \\
RCH(j\omega)e^{j\omega t} j\omega + H(j\omega)e^{j\omega t} = e^{j\omega t} \\
H(j\omega)(RCj\omega + 1) = 1 \\
H(j\omega) = \frac{1}{RCjw+1}
\end{aligned}
$$

$$
\begin{aligned}
|H(j\omega)| &= \sqrt{H(j\omega)H^*(j\omega)} \qquad (5.1) \\
&= \frac{1}{\sqrt{(RC\omega)^2 - 1}} : \omega \geq 0 \qquad (5.2)
\end{aligned}
$$