UNIVERSITY OF CALIFORNIA, BERKELEY

EE-120: Lecture 5

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1 Fourier Series in Discrete-Time

For discrete-time signals, x is periodic if there exists $N \in \mathbb{Z} \neq 0$ s.t. $x[n+N] = x[n] : \forall n$
Eg: $\cos(\omega_0 n)$ is periodic only when $\frac{\omega_0}{\pi}$ is rational. To find the fundamental period N of one of these signals, you need to find the smallest such $M, N \neq 0$ such that the following equation holds.

$$\omega_0 N = 2\pi M \tag{1.1}$$

1.1 Things to Note

- From the definition of period, the smallest period is technically 1 for constant signals.
- However for oscillatory signals, the smallest period is 2.
- This would imply that the fastest frequency for an oscillatory signal is $\omega_0 = \frac{2\pi}{T}\Big|_{T=2} = \pi$

If we define $\Phi_k[n]$ as the following.

$$\Phi_k[n] := e^{jk\omega_0 n}, k = 0 \pm 1, \ldots, \pm N - 1 \tag{1.2}$$

So, if we take note of the following:

$$e^{j(k+N)\omega_0 n} = e^{jk\omega_0 n}(e^{jN\omega_0 n}) = e^{jk\omega_0 n} \tag{1.3}$$

This means our functions Φ are modular in N, in other words:

$$\Phi_k[n] = \Phi_{k+iN}[n] : \forall i \in \mathbb{Z}_+ \neq 0 \tag{1.4}$$

This means we can express the discrete-time Fourier Series as the following finite sum:

$$x[n] = \sum_{k=\langle N\rangle} a_k \Phi_k[n] \tag{1.5}$$

We will call this the discrete-time **Synthesis Equation**

1.2 Properties of $\Phi_k[n]$

- Periodicity in n: $\Phi_k[n+N] = \Phi_k[n]$
- Periodicity in k: $\Phi_{k+N}[n] = \Phi_k[n]$
- $\sum_{\langle N\rangle} \Phi_k[n] = \begin{cases} N : k = 0 \mod N \\ 0 : k \neq 0 \mod N \end{cases}$
- $\Phi_k[n]\Phi_m[n] = \Phi_{k+m}[n]$

1.3 How do we find α_k

We can motivate our formula for a_k with the following:

$$\sum_{n=\langle N\rangle} x[n]\Phi_{-m}[n] = \sum_{n=\langle N\rangle}\sum_{k=\langle N\rangle} a_k \Phi_{k-m}[n] = N a_m \tag{1.6}$$

This means a_k is equal to the following:

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n] e^{-j\omega_0 kn} \tag{1.7}$$

As in continuous time, if x is real then $a_{-k} = a_k^*$ and if we take note of the modularity of the coefficients then we can conclude that $a_{N-k} = a_k^*$

1.4 Fourier Series as a Change of Basis

Let $x = \begin{bmatrix} x[0] & x[1] & \ldots & x[N-1] \end{bmatrix}^T$ and define $\Phi_k = \begin{bmatrix} 1 & e^{jk\frac{2\pi}{N}} & \ldots & e^{jk\frac{2\pi}{N}(N-1)} \end{bmatrix}^T$
Then we can view x as the following change of basis.

$$x = \sum_{i=0} \Phi_i a_i \tag{1.8}$$

Take note that from the zero sum property of the Φ function and the additive property of these Φ functions, we can also state that these vectors are orthogonal. Based on this we can arrive at the following simplified expression for a_k

$$x^T\Phi_k = \sum_i a_i \Phi_i^T \Phi_k = N a_k \tag{1.9}$$

$$a_k = \frac{1}{N} x^T \tag{1.10}$$