UNIVERSITY OF CALIFORNIA, BERKELEY

EE-120: Lecture 3

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1 REFRESHER ON COMPLEX NUMBERS

For this class, we will have the following notation for complex numbers:

$$z = \sigma + j\omega \tag{1.1}$$

$$j = \sqrt{-1} \tag{1.2}$$

We define the **conjugate** of $z \in \mathbb{C}$ as follows:

$$z^* = \sigma - j\omega \tag{1.3}$$

$$zz^* = \sigma^2 + \omega^2 = |z|^2 \tag{1.4}$$

Where $|z|$ is the **magnitude** of z

2 EULER'S FORMULA

We can define the angle θ of a complex number z as follows:

$$\theta = \tan^{-1}\left(\frac{\omega}{\sigma}\right) \tag{2.1}$$

With this definition, we can now motivate Euler's formula

$$e^{j\theta} = \cos(\theta) + j\sin(\theta) \tag{2.2}$$

$$e^{-j\theta} = \cos(\theta) - j\sin(\theta) \tag{2.3}$$

Remember these identities (they are used a lot in proofs for this class):

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \tag{2.4}$$

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j} \tag{2.5}$$

3 Roots of Unity

An **nth root of unity** $n \in \mathbf{Z}$ is a number z satisfying

$$z^n = 1 \tag{3.1}$$

The nth roots of unity are as follows:

$$x_k = e^{j2\pi\frac{k}{n}} : k \in \{0, 1, ..., n\} \tag{3.2}$$

Take note of the following:

$$x^n_{k=k'} = e^{j2\pi k'} \tag{3.3}$$

$$= \cos(2k'\pi) + i\sin(2k'\pi) = 1 \tag{3.4}$$

4 Response of LTI Systems to Complex Exponentials

We define the **Complex Exponential Signal** as follows:

$$x(t) = e^{st}, s \in \mathbf{C} := e^{\sigma t} e^{j\omega t} \tag{4.1}$$

We denote the real-valued component as the **envelope** of the signal.

$$x[n] = z^n, z = re^{j\omega} \in \mathbb{C} := r^n e^{j\omega n} \tag{4.2}$$

Preposition: The response to a complex exponential is the same complex exponential, scaled by a factor. Which by definition makes the complex exponential an **eigenfunction**.

$$y(t) = (h * e^{st})(t) = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau = \left(\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau\right)e^{st}$$

$$H(s) := \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$$

$$y[n] = (h * z)[n] = \sum_{-\infty}^{\infty} h[k]z^{[n-k]} = \left(\sum_{-\infty}^{\infty} h[k]z^{-k}\right)z^n$$

$$H[z] := \sum_{-\infty}^{\infty} h[k]z^{-k}$$

Where we call the functions H, as the **transfer functions** or also **system functions**. Allowing $s = j\omega, z = e^{j\omega}$, these are then known as the frequency responses of a signal.

5 Filtering

Definition: **Frequency Response**: how system responds to sinusoids of different frequencies
Amplitude of e^{jwn} changed by $|H(e^{jw})|$ - magnitude response
Phase of e^{jwn} changed by angle$H(e^{jw})$ - phase response.

6 FIR vs. IIR Systems

A causal **Finite Impulse Response** system has the form

$$y[n] = \sum_{i=0}^{M} b_i x[n-i] \tag{6.1}$$

Its easy to determine the impulse responses of these signals ('replace x with h'). Note these systems are also always stable.

7 Constant-coefficient Linear Difference Equations

Systems like the accumulator:

$$y[n] - y[n-1] = x[n], y[-1] = 0, x[n] = 0 \forall n_{<0} \tag{7.1}$$

are part of a more general class of systems that are defined by their **constant-coefficient linear difference equations** defined by the following form.

$$\sum_{i=1}^{N} a_i y[n-i] = \sum_{j=1}^{M} b_i x[n-j] \tag{7.2}$$

In general, these systems are causal if $a_0 \neq 0$ (We usually make $a_0 = 1$ We can usually implement these systems with following recurrence relation:

$$y[n] = -\sum_{i=1}^{N} a_i y[n-i] + \sum_{j=1}^{M} b_i x[n-j] \tag{7.3}$$