UNIVERSITY OF CALIFORNIA, BERKELEY

EE-120: Homework 2

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We can say the signal $\hat{x}$ is equal to the following.

$$\hat{x}(t) = x(t - \frac{1}{2}) - x(t + \frac{1}{2}) \tag{0.1}$$

Where x is the square wave.
By linearity: We know that given the two signals $x(t-\frac{1}{2})$ and $x(t+\frac{1}{2})$ with FS coefficients a'_k, b'_k then $\hat{x}$ will have FS coefficients equal to $a'_k - b'_k$
By time shift on the signal $x(t-\frac{1}{2})$ we know that $a'_k = a_k e^{-jk\frac{\pi}{2}}$ Where a_k are the FS coefficients of the square wave.

Similarly, by time shift on the signal $x(t+\frac{1}{2})$ we know that $b'_k = a_k e^{jk\frac{\pi}{2}}$
This implies that the fourier series coefficients for the signal $\hat{x}$ are $a'_k - b'_k$:

$$= a_k e^{-jk\frac{\pi}{2}} - a_k e^{jk\frac{\pi}{2}} \tag{0.2}$$

$$= -a_k\Big(-e^{-jk\frac{\pi}{2}} + e^{jk\frac{\pi}{2}}\Big) \tag{0.3}$$

$$= -a_k\Big(e^{jk\frac{\pi}{2}} + -e^{-jk\frac{\pi}{2}}\Big) \tag{0.4}$$

$$= -2ja_k \sin(k\frac{\pi}{2}) \tag{0.5}$$

$$= \frac{-2j}{k\pi}\sin^2(\frac{\pi k}{2}) \tag{0.6}$$

$$= \begin{cases} 0 : k \text{ is even} \\ \frac{-2j}{k\pi} : k \text{ is odd} \end{cases} \tag{0.7}$$

- a: Let $x : \mathbb{R} \to \mathbb{R}$ have FS coefficients a_k and let $x(t) = x(-t) \forall t$
  Because x is real valued then $x^* = x$ which tells us $a_k = a^*_{-k}$ and Because x is even symmetric, $a_k = a_{-k}$. Because $a_{-k} = a^*_{-k} \forall k$
  This implies the fourier series coefficients are purely real.

- b: Lemma: $x = a + bi \in \mathbb{C} - x = x^* \to a = 0$
  By definition we define the $x^* = a - bi$

$$x^* = a - bi = -a - bi = -x \tag{0.8}$$

  Which only holds true for $a = 0$

  Let x be defined as above with FS coefficients and let $x(t) = -x(-t)$.
  Combining the time shift and linearity properties of FS coefficients, $a_k = -a_{-k}$ and because x is real valued, then $a_k = a^*_{-k}$
  This implies $a_{-k} = a^*_{-k} : \forall k$. This implies the Fourier series coefficients are purely imaginary.

- c: Let x be defined as above with FS coefficients and let $x(t) = -x(t + \frac{T}{2})$
  By the linearity and time-shift properties of FS coefficients, this implies that $a_k = -a_k e^{-jk\omega_0 t_0}$, with $\omega_0 = \frac{2\pi}{T}$ and $t_0 = -\frac{T}{2}$
  This implies $a_k = -a_k e^{-jk\pi}$
  Let $k = 2a : a \in \mathbb{Z}$
  $a_k = a_{2a} = -a_{2a} e^{-j2a\pi} = -a_{2a} * 1$ Which is only true for $a_{2a} = 0$

- d: Consider $t \in [0, \frac{T}{2}]$ and let $x(t) = 1$
  $|t| < T_1 \to |-t| < T_1 \to x(-t) = 1$
  Now, if we let x(t) = 0
  $T_1 < |t| \leq \frac{T}{2} \to T_1 < |-t| \leq \frac{T}{2} \to x(-t) = 0$. Because this is true over all t in a given period, Then the square wave is even symmetric, and as we can see, the Fourier series coefficients a_k for the square wave are purely real.

  Now, consider the signal $\hat{x}(t) = x(t - \frac{1}{2}) - x(t + \frac{1}{2})$, where x is the square wave defined earlier.

$$\begin{aligned}
-\hat{x}(-t) &= -x(-t - \frac{1}{2}) + x(-t + \frac{1}{2}) && (0.9)\\
&= -x(-(t + \frac{1}{2})) + x(-(t - \frac{1}{2})) && (0.10)\\
&= x(t + \frac{1}{2}) + x(t - \frac{1}{2}) && (0.11)\\
&= \hat{x}(t) && (0.12)
\end{aligned}$$

  Which means $\hat{x}$ is odd symmetric which we can see from the FS coefficients which are all imaginary. Setting $T = 2$, we can see that property 3 holds.

$$\begin{aligned}
-x(t+1) &= -x((t+1)-\frac{1}{2}) + x((t+1)+\frac{1}{2}) && (0.13)\\
&= -x(t+\frac{1}{2}) + x(t+\frac{3}{2}) && (0.14)\\
&= -x(t+\frac{1}{2}) + x(t-\frac{1}{2}) && (0.15)\\
&= \hat{x}(t) && (0.16)
\end{aligned}$$

And as we can see, the following signal has $a_k = 0$ for even k.

- a: Let $W := \begin{bmatrix} \Phi_0 & \Phi_1 & \cdots & \Phi_{N-1} \end{bmatrix}$
  $\Phi_k = \begin{bmatrix} 1 \\ e^{\frac{-jk2\pi}{N}} \\ \vdots \\ e^{\frac{-jk2\pi}{N}(N-1)} \end{bmatrix}$
- b:
  Done in iPython Notebook. The Fourier series coefficients are $a_0 = a_1 = a_2 = \frac{1}{3}$
- c:
  Given the analysis equation in class we can define $W^{-1} = \frac{1}{N} \begin{bmatrix} \Phi_0 & \Phi_1^{-1} & \cdots & \Phi_{N-1}^{-1} \end{bmatrix}$
  $\Phi_k^{-1} = \begin{bmatrix} 1 \\ e^{\frac{jk2\pi}{N}} \\ \vdots \\ e^{\frac{jk2\pi}{N}(N-1)} \end{bmatrix}$
- d: Performed in iPython Notebook.

- a: Given $X(\omega)$ equal to the following:

$$X(\omega) = \int_{-\infty}^{\infty} e^{-t^2} e^{-j\omega t} dt \tag{0.17}$$

Then $\frac{\partial X(\omega)}{\partial \omega}$ is equal to the following:

$$\frac{\partial X(\omega)}{\partial \omega} = \int_{-\infty}^{\infty} -jte^{-t^2} e^{-j\omega t} dt \tag{0.18}$$
$$= \frac{j}{2}\left(e^{-jwt} e^{-t^2} \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-t^2} j\omega e^{-j\omega t} dt \right) \tag{0.19}$$
$$= \frac{j^2}{2}\left(\int_{-\infty}^{\infty} e^{-t^2} \omega e^{-j\omega t} dt \right) \tag{0.20}$$
$$= \frac{\omega}{2}\left(\int_{-\infty}^{\infty} e^{-t^2} e^{-j\omega t} dt \right) \tag{0.21}$$
$$= -\frac{\omega}{2} X(\omega) \tag{0.22}$$

- b: To find α, β given $X(\omega) = \alpha e^{\frac{-\omega^2}{\beta}}$ we need to find the unique solutions to the following system of equations.

$$X(0) = \sqrt{\pi} \tag{0.23}$$
$$\frac{dX(\omega)}{d\omega} = -\frac{\omega}{2} X(\omega) \tag{0.24}$$

Starting from eq. 0.23

$$X(0) = \alpha e^{\frac{0}{\beta}} = \alpha = \sqrt{\pi} \tag{0.25}$$

Now with eq. 0.24

$$\frac{dX(\omega)}{d\omega} = -\frac{\omega}{2} X(\omega) \tag{0.26}$$
$$-\frac{2\alpha\omega}{\beta} e^{-\frac{\omega^2}{\beta}} - = -\frac{\omega}{2} \alpha e^{-\frac{\omega^2}{\beta}} \tag{0.27}$$
$$\frac{2}{\beta} = \frac{1}{2} \tag{0.28}$$
$$\beta = 4 \tag{0.29}$$

Which means $\alpha = \sqrt{\pi}, \beta = 4$

Given the following modulation scheme:

$$r(t) = s(t)\cos(\omega_0 t + \phi) \tag{0.30}$$
$$= s(t)\left(\frac{e^{-j\omega_0 t + \phi} + e^{j(\omega_0 t + \phi)}}{2}\right) \tag{0.31}$$
$$\rightarrow R(\omega) = \frac{1}{2}e^{-j\phi}S(w + w_0) + \frac{1}{2}e^{j\phi}S(w - \omega_0) \tag{0.32}$$
$$g(t) = r(t)\cos(\omega_0 t) \tag{0.33}$$
$$\rightarrow G(\omega) = \frac{1}{2}R(\omega - \omega_0) + \frac{1}{2}R(w + \omega_0) \tag{0.34}$$
$$= \frac{1}{2}\left(\frac{1}{2}(e^{-j\phi}S(\omega) + e^{j\phi}S(\omega) + e^{-j\phi}S(\omega + 2\omega_0) + e^{j\phi}S(\omega - 2\omega_0)\right) \tag{0.35}$$
$$= \frac{1}{2}\cos(\phi)S(\omega) + \frac{1}{4}(e^{-j\phi}S(w + 2\omega_0) + e^{j\phi}S(w - 2\omega_0)) \tag{0.36}$$

After g(t) is run through an ideal lowpass filter, with passband gain of 2 $Y(\omega)$ is equal to the following:

$$= cos(\phi)S(\omega) \tag{0.37}$$
$$\rightarrow y(t) = cos(\phi)s(t) \tag{0.38}$$

In the case $\phi = \frac{\pi}{2}$, then the receiver does not receive $s(t)$ because the signal was attenuated completely. The signal is completely recovered when ϕ is an integer multiple of 2π

HW3 question 3

September 26, 2019

[1]:
```
import numpy as np
```

[2]:
```
phi_0 = np.exp(np.array([0,
                         0,
                         0]))
phi_1 = np.exp(np.array([0,
                         0+2 * np.pi / 3j,
                         0+4 * np.pi / 3j]))
phi_2 = np.exp(np.array([0,
                         0+4 * np.pi / 3j,
                         0+8 * np.pi / 3j]))
```

[3]:
```
W = np.array([phi_0, phi_1, phi_2]).T
impulse_train = np.array([1,0,0])
```

[4]:
```
fs_coefficients = np.linalg.solve(W, impulse_train)
```

[5]:
```
fs_coefficients
```

[5]:
```
array([0.33333333+1.94289029e-16j, 0.33333333-8.32667268e-17j,
       0.33333333-1.11022302e-16j])
```

[6]:
```
W @ fs_coefficients
```

[6]:
```
array([ 1.00000000e+00+0.00000000e+00j, -1.66533454e-16+2.22044605e-16j,
        0.00000000e+00+5.55111512e-17j])
```

[7]:
```
phi_0_inv = np.exp(np.array([0,
                         0,
                         0]))
phi_1_inv = np.exp(np.array([0,
                         0-2 * np.pi / 3j,
                         0-4 * np.pi / 3j]))
phi_2_inv = np.exp(np.array([0,
                         0-4 * np.pi / 3j,
                         0-8 * np.pi / 3j]))
```

[8]:
```
W_inv = np.array([phi_0_inv, phi_1_inv, phi_2_inv]).T / 3
```

As we can see, we can say with a more than reasonable degree of certainty that this is the identity, confirming that W_inv is an inverse to the W matrix defined.

[9]:
```
np.allclose(W @ W_inv, np.eye(3), 1e-15)
```

[9]:
```
True
```

[ ]:
```
```