University of California, Berkeley

EE-120: Lecture 6

Oscar Ortega

July 16, 2021

1 Continuous Time Fourier Transforms

Up until now with *fourier series*, we have only dealt with the periodic signals. However, the fourier transform is applicable to aperiodic signals and has the following form.

$$X(\omega) = \int_{\infty}^{\infty} x(t)e^{-j\omega t}dt \tag{1.1}$$

Consider $x, \hat{x} : \mathbb{F} \to \mathbb{F}$ and let $x = \lim_{T\to\infty} \hat{x}$ where $\hat{x}$ is a signal with period T. Note that as T increases $\omega_0 = \frac{2\pi}{T}$ decreases and the harmonic components $a_k = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}dt$ become closer in terms of the frequency ω_0 (take note of the difference in frequency between a_{k+1} and a_k).

2 Example

Let

$$x(t) = \begin{cases} 1 : t \leq |T_1| \\ 0 : t > |T_1| \end{cases} \tag{2.1}$$

And let $\hat{x}$ be the same width rectangle signal, with period T. As shown earlier in this class the fourier coefficients were as follows.

$$a_k = \begin{cases} \frac{2T_1}{T} : k = 0 \\ \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} : k \neq 0 \end{cases} \tag{2.2}$$

and take note of the fourier transform of x

$$X(\omega) = \begin{cases} 2T_1 : \omega = 0 \\ \frac{2\sin(\omega T_1)}{\omega} : \omega \neq 0 \end{cases} \tag{2.3}$$

This means $X(\omega)|_{\omega=k\omega_0} = Ta_k$ which shows that the fourier transform X is a continuum for the FS coefficients as $T \to \infty$.
Consider the following proof as motivation for the synthesis equation:

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(\omega)|_{w=k\omega_0} e^{jk\omega_0 t} \tag{2.4}$$

$$= \sum_{k=-\infty}^{\infty} \frac{1}{T} X(\omega) e^{j\omega t}|_{\omega=k\omega_0} \tag{2.5}$$

$$= \frac{1}{2\pi} \sum_{k=-\infty} \omega_0 (X(w) e^{j\omega t})|_{\omega=k\omega_0} \tag{2.6}$$

Which approaches the following as $T \to \infty$

$$\frac{1}{2\pi} \sum_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \tag{2.7}$$

This is our analysis equation for the Fourier Transform.

3 Properties of the Fourier Transform

Consider two signal fourier transform pairs $x(t) \leftrightarrow X(\omega)$, $y(t) \leftrightarrow Y(\omega)$

- Linearity: For any $a, b \in \mathbb{F}$
  $ax(t) + by(t) \leftrightarrow aX(w) + by(t)$
- Time Shift: $x(t - t_0) \leftrightarrow e^{-j\omega t_0} X(\omega)$
- Conjugation $x^*(t) \leftrightarrow X^*(-\omega)$
  Corollary: if x is real $X(\omega) = X^*(-\omega)$
- Time and Frequency Scaling: $x(at) \leftrightarrow \frac{1}{|a|} X(\frac{\omega}{\alpha})$
  Note applying this to sincs shows the following:
  $\frac{W}{\pi}\text{sinc}(\frac{W}{\pi} t) \leftrightarrow X(\frac{\pi}{W}\omega) = \begin{cases} 1 : |\omega| < W \\ 0 : |\omega| \geq W \end{cases}$
  Also note: this implies that if x is even symmetric and real valued, then $X(\omega)$ is real valued as well (think to the proof for fourier series).
- Convolution Property: $(x_1 * x_2(t)) \leftrightarrow X_1(\omega) X_2(\omega)$