UNIVERSITY OF CALIFORNIA, BERKELEY

EE-120: Lecture 2

Oscar Ortega

July 16, 2021

1 PROPERTIES OF CONVOLUTION

Recall the convolution operator:

$$(x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \tag{1.1}$$

Here are some properties of the convolution operator where x, δ, h, h_i are signals:

$$\textbf{Identity of unit impulse: } (x * \delta)[n] = x[n] \tag{1.2}$$

$$\textbf{Convolution with shifted Unit Impulse: } x[n] * \delta[n-N] = x[n-N] \tag{1.3}$$

$$\textbf{Commutative Property: } x * h[n] = h * x[n] \tag{1.4}$$

$$\textbf{Distributive Property: } x * (h_1 + h_2)[n] = (x * h_1) + (x * h_2) \tag{1.5}$$

$$\textbf{Associative Property: } x * (h_1 * h_2) = (x * h_1) * h_2) \tag{1.6}$$

2 IMPLICATIONS FOR LTI SYSTEMS

These properties of the convolution operator have important implications for LTI Systems.

- Consider two responses with impulse responses h_1 and h_2. Then the parallel combination of two LTI systems with impulse responses h_1 and h_2 can be represented as an equivalent LTI system with impulse response $h_1 + h_2$. This is equivalent to an LTI system $x * h_1 + x * h_2 \equiv x * (h_1 + h_2)$ where $h_1 + h_2$ is the parallel combination.

- Similarly, combining two LTI systems in series will also produce equivalent LTI systems through the associativity property $x * (h_1 * h_2) \equiv (x * h_1) * h_2$
- By the commutative property, we can also take note that swapping the order in which we convolve does no matter as well when we are dealing with LTI systems.

3 Determining Causality and Stability from the Impulse Response

Recall that a system is *causal* if the output signal is only a function of present or past inputs.
Preposition: A discrete-time LTI system is causal iff $h[n] = 0 \forall n < 0$
Proof:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

and let $h[k] \neq 0$ for some $k < 0$. Then $y[n]$ is dependent on $x[n-k]$ Which is a future value of the signal given $k < 0$.

Preposition: A discrete-time LTI system is stable if $\sum_{k=\infty}^{\infty} |h[k]| \leq B : B < \infty$
Proof $\rightarrow$: Let $\rightarrow$: $x[n] < \infty$

$$|y[n]| = |\sum_k x[n-k]h[k]| \leq \sum_k |x[n-k]||h[k]| \leq B \sum_k |h[k]| \leq \infty \tag{3.1}$$

Proof $\leftarrow$: Let $x[n] = \text{sgn}\{h[-n]\}$

$$y[0] = \sum_k h[k]x[-k] = \sum_k |h[k]| = \infty \tag{3.2}$$

4 Continuous LTI Systems

In continuous time, the unit impulse is defined as:

$$\delta(t) := \lim_{\Delta \to 0} \delta_\Delta(t) \tag{4.1}$$

Where $\delta_\Delta(t)$ is a pulse with width Δ, amplitude $\frac{1}{\Delta}$, and area equal to 1.

5 Properties of the unit impulse

- $$f(0)\delta(t) = f(t)\delta(t) \tag{5.1}$$

- $$f(t)\delta(t-T) = f(T)\delta(t-T) \tag{5.2}$$

- $$\delta(at) = \frac{1}{|a|}\delta(t) \tag{5.3}$$

6 Convolution Integral

Here is the convolution integral for $y = x * h$

$$y(t) = \int_{\infty}^{\infty} x(\tau)h(t-\tau)d\tau \tag{6.1}$$

$$= \lim_{\Delta \to 0} \sum_{k=\infty}^{\infty} x(k\Delta)h(t-k\Delta)\Delta \tag{6.2}$$

Without proof, take note that the convolution integral possesses similar properties to the convolution sum, and also maintains the same properties of determining causality and stability with LTI systems.