UNIVERSITY OF CALIFORNIA, BERKELEY

EE-120: Homework 3

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PROBLEM 1

- a:
  Proof:
  For arbitrary signals $x(t)$ and $h(t)$ with corresponding fourier transform pairs $X(j\omega)$ and $H(j\omega)$ we know the following:

$$(x * h)(t) \leftrightarrow X(j\omega)H(j\omega) \tag{0.1}$$

$$\frac{d^k}{dt^k}x(t) \leftrightarrow (j\omega)^k X(j\omega) \tag{0.2}$$

  This implies the following if we combine the rules:

$$\frac{d^k}{dt^k}(h * x)(t) \leftrightarrow (j\omega)^k (H(j\omega)X(j\omega)) \tag{0.3}$$

$$\rightarrow ((j\omega)^k H(j\omega))X(j\omega) \tag{0.4}$$

$$((\frac{d^k}{dt^k}h) * x)(t) \leftarrow \tag{0.5}$$

$$(h_k * x)(t) = \tag{0.6}$$

  Where $h_k(t) = \frac{d^k}{dt^k}h(t)$

- b:

$$h_1(t) = \frac{d}{dt}h(t) \tag{0.7}$$
$$= \frac{d}{dt}\Big(\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}\Big) \tag{0.8}$$
$$= -\frac{1}{\sqrt{2\pi}}te^{\frac{-t^2}{2}} \tag{0.9}$$

Let $x(t) = e^{-t^2}$ the gaussian impulse we found the fourier transform of in HW2 and let $X(\omega)$ equal that fourier transform. Because we know $\alpha x(\beta t) \leftrightarrow \frac{\alpha}{|\beta|}X(\frac{\omega}{\beta})$ by properties of fourier transforms

$$h(t) = \frac{1}{\sqrt{2\pi}}x(\frac{1}{2}t) \leftrightarrow \sqrt{\frac{2}{\pi}}X(2\omega) \tag{0.10}$$

Because we know $h_1(t) = \frac{d}{dt}h(t)$ we know the corresponding FT pair is $j\omega H(\omega) = j\omega\sqrt{\frac{2}{\pi}}X(2\omega)$

$$j\omega\sqrt{\frac{2}{\pi}}X(2\omega) = j\omega\sqrt{\frac{2}{\pi}}\sqrt{\pi}\exp(-\frac{(2\omega)^2}{4}) \tag{0.11}$$
$$= j\omega\sqrt{2}\exp(-\omega^2) \tag{0.12}$$
$$= H_1(\omega) \tag{0.13}$$

$|H_1(\omega)| = \omega\sqrt{2}\exp(-\omega^2)$ rough sketch shown below.

- c:
  We know the differentiator is an lti system:

$$y(t) = (x * h)(t) \leftrightarrow H_{\text{diff}}(\omega)X(\omega) = Y(\omega) \tag{0.14}$$
$$\rightarrow H_{\text{diff}}(\omega) = \frac{Y(\omega)}{X(\omega)} \tag{0.15}$$

By the derivative property we also know the following:

$$H_{\text{diff}}(\omega) = \frac{j\omega X(\omega)}{X(\omega)} \tag{0.16}$$
$$= j\omega \tag{0.17}$$

$H_1(j\omega)$ is like the response of $H_{\text{diff}}(j\omega)$ if H_1 was evaluated at $\exp(-\omega^2)|_{\omega=0}$

PROBLEM 2

- a:

$$|H(e^{j\omega})| = \frac{|e^{-j\omega} - \alpha|}{|1 - \alpha e^{-j\omega}|} \tag{0.18}$$
$$= \frac{|(\cos(\omega) - \alpha) + i\sin(\omega)|}{|1 - \alpha\cos(\omega) - \alpha i \sin(\omega)|} \tag{0.19}$$
$$= \left(\frac{(\cos(\omega) - \alpha)^2 + \sin^2(\omega)}{(1 - \alpha\cos(\omega))^2 + \alpha^2\sin^2(\omega))}\right)^{\frac{1}{2}} \tag{0.20}$$
$$= \left(\frac{1 - 2\alpha\cos(\omega) + \alpha^2}{1 - 2\alpha\cos(\omega) + \alpha^2}\right)^{\frac{1}{2}} \tag{0.21}$$
$$= 1 \tag{0.22}$$

- b:

$$\angle H(e^{j\omega}) = \angle(e^{-j\omega} - \alpha) - \angle(1 - \alpha e^{-j\omega}) \tag{0.23}$$
$$= \angle(\cos(\omega) - \alpha) + i\sin(\omega)) - \angle(1 - \alpha\cos(\omega) - \alpha i \sin(\omega)) \tag{0.24}$$
$$= \tan^{-1}(\frac{\sin(\omega)}{\cos(\omega) - \alpha}) - \tan^{-1}(\frac{\alpha\sin(\omega)}{1 - \alpha\cos(\omega)})) \tag{0.25}$$

- c: We first find the Fourier Transform for x

$$x[n] = \cos(\frac{\pi}{6}n) + \cos(\pi n) \tag{0.26}$$
$$= \frac{1}{2}\left(\exp(-\frac{\pi}{6}n) + \exp(\frac{\pi}{6}n) + \exp(-\pi n) + \exp(\pi n)\right) \tag{0.27}$$
$$X(\omega) \to \pi\left(\delta[\omega - \frac{\pi}{6}] + \delta[\omega + \frac{\pi}{6}] + \delta[\omega + \pi] + \delta[\omega - \pi]\right) \tag{0.28}$$
$$\tag{0.29}$$

We know the system is LTI which tells us the fourier transform pair $y(t) = (x * h)(t) \leftrightarrow X(\omega)H(\omega)$

$$H(\omega)X(\omega) = \pi\left(\sum_{\omega \in \{\pi, -\pi, -\frac{\pi}{6}, \frac{\pi}{6}\}} H(e^{j\omega})\right) \tag{0.30}$$
$$\tag{0.31}$$

The $\pi, -\pi$ terms are both equal to -1 and the $-\frac{\pi}{6}, \frac{\pi}{6}$: terms are conjugates of one another.

$$\frac{a + bj}{c + dj} + \frac{a - bj}{c - dj} = \frac{2(ac + bd)}{c^2 + d^2} \tag{0.32}$$

So, plugging $H(e^{j\frac{\pi}{6}}), H(e^{-j\frac{\pi}{6}})$:
Solving for the top $2(ac+bd)$ with $\alpha = \frac{1}{\sqrt{3}}$

$$= 2((-\alpha + \cos(\frac{\pi}{6}))(1 - \alpha\cos(\frac{\pi}{6})) - \alpha\sin^2(\frac{\pi}{6}) \tag{0.33}$$
$$= -\alpha + (\alpha^2+1)\cos(\frac{\pi}{6}) - \alpha\cos^2(\frac{\pi}{6}) - \alpha\sin^2(\frac{\pi}{6}) \tag{0.34}$$
$$= -2\alpha + (\alpha^2+1)\cos(\frac{\pi}{6}) \tag{0.35}$$
$$= -\frac{2}{\sqrt{3}} + \frac{2}{\sqrt{3}} = 0 \tag{0.36}$$

Which tells us $Y(\omega) = -2\pi$ Which tells us $y[n] = -2\pi\delta[n]$

- d:

$$H(e^{j\omega}) = \frac{e^{-j\omega} - \alpha}{1 - \alpha e^{-j\omega}} \tag{0.37}$$
$$H(e^{j\omega})1 - \alpha e^{-j\omega} = e^{-j\omega} - \alpha \tag{0.38}$$
$$\rightarrow y[n] - \alpha y[n-1] = x[n] - \delta[n-\alpha] \tag{0.39}$$

PROBLEM 3

- a: The length of the output signal will be $\text{len}(x) + \text{len}(h) - 1 = 100 + 4 - 1 = 103$
- b: the codes and plots are in the ipython file

PROBLEM 4

- a:
  Recall that the impulse response $h[n_1, n_2]$ is the output of the system with $\delta[n, n]$ as the input.

$$h[n_1, n_2] = \frac{1}{5}(\delta[n_1, n_2] + \delta[n_1 - 1, n_2] + \delta[n_1 + 1, n_2] + \delta[n_1, n_2 + 1] + \delta[n_1, n_2 - 1]) \tag{0.40}$$

- b:

$$\begin{aligned}
H(e^{j\omega n_1}, e^{j\omega n_2}) &= \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} h[n_1, n_2] e^{-j\omega_1 n_1} e^{-j\omega_2 n_2} && (0.41)\\
&= \frac{1}{5} \sum_{i=-1}^{1} \sum_{k=-1}^{1} e^{-j\omega_1 i} e^{-j\omega_2 k} && (0.42)\\
&= \frac{1}{5}(\sum_{i=-1}^{1} e^{-j\omega_1 i} \sum_{k=-1}^{1} e^{-j\omega_2 k}) && (0.43)\\
&= \frac{1}{5}((1 + e^{-j\omega_1} + e^{j\omega_1})(1 + e^{j\omega_2} + e^{-j\omega_2})) && (0.44)\\
&= \frac{1}{5}((1 + 2\cos(\omega_1))(1 + 2\cos(\omega_2))) && (0.45)
\end{aligned}$$

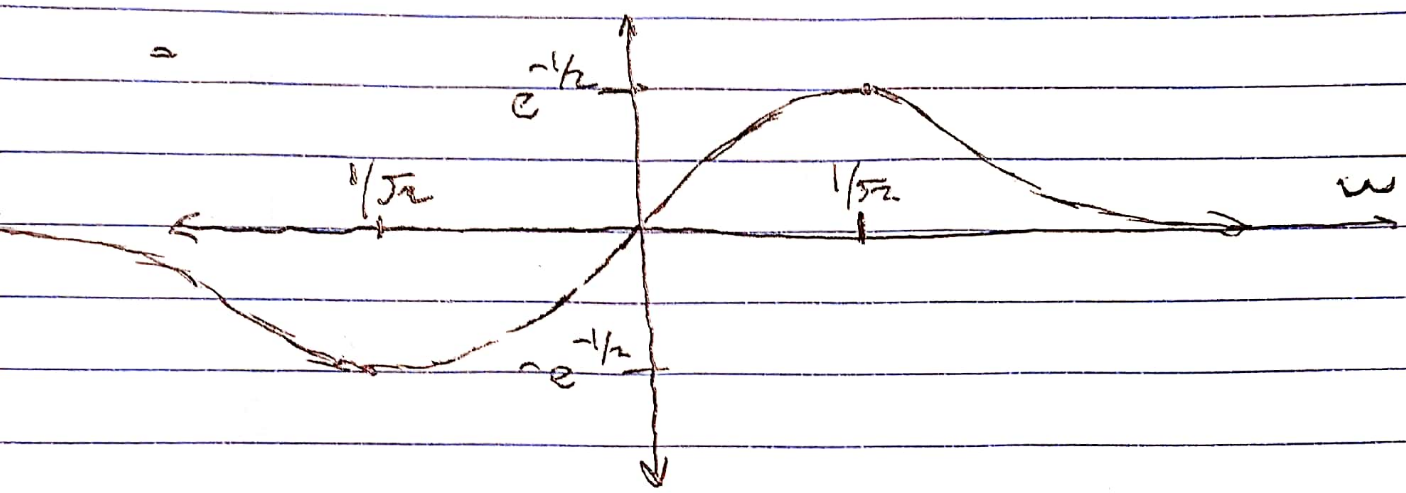
  The codes and plot are below:

PROBLEM 5

Consider the form of the 2d CTFT for 2d signal $x(n_1, n_2)$ and evaluate at $\omega_2 = 0$

$$
\begin{aligned}
X(j\omega_1, j\omega_2)\Big|_{\omega_2=0} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t_1,t_2)e^{j\omega_1 t_1}e^{j\omega_2 t_2}dt_1 dt_2\Big|_{\omega_2=0} && (0.46)\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t_1,t_2)e^{j\omega_1 t_1}dt_1 dt_2 && (0.47)\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t_1,t_2)dt_2 e^{j\omega_1 t_1}dt_1 && (0.48)\\
&= \int_{-\infty}^{\infty} x_0(t_1)e^{j\omega_1 t_1}dt_1 && (0.49)\\
&= X_0(j\omega_1) && (0.50)
\end{aligned}
$$

$|H_1(\omega)|$



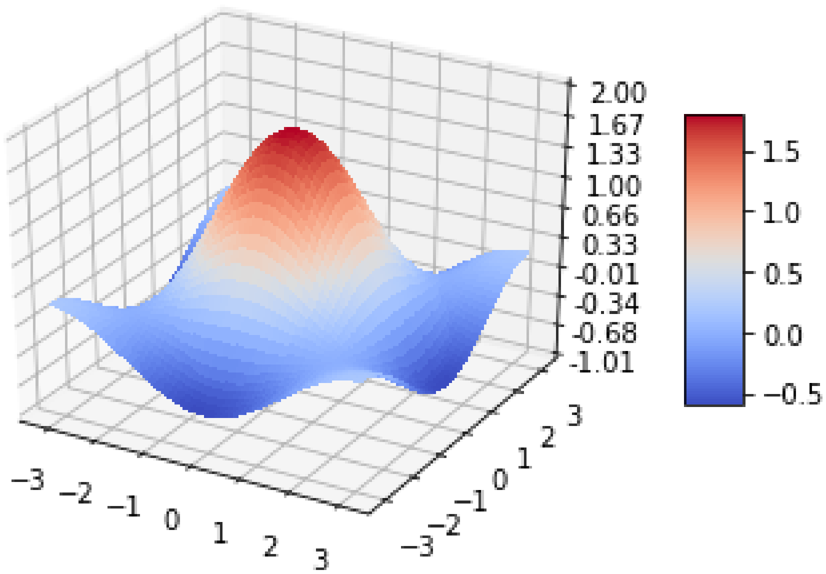
Homework 3

October 24, 2019

1 Homework 3:

1.1 Question 4 part B

[4]:
```
from mpl_toolkits.mplot3d import Axes3D  # noqa: F401 unused import

import matplotlib.pyplot as plt
from matplotlib import cm
from matplotlib.ticker import LinearLocator, FormatStrFormatter
import numpy as np


fig = plt.figure()
ax = fig.gca(projection='3d')

# Make data.
W_1 = np.arange(-np.pi,np.pi, 0.001)
W_2 = np.arange(-np.pi, np.pi, 0.001)
W_1, W_2 = np.meshgrid(W_1, W_2)
fResponse = (1 / 5) *  ((1 + 2 * np.cos(W_1)) * (1 + 2 * np.cos(W_2)))
# Plot the surface.
surf = ax.plot_surface(W_1, W_2, fResponse, cmap=cm.coolwarm,
                       linewidth=0, antialiased=False)

# Customize the z axis.
ax.set_zlim(-1.01, 2)
ax.zaxis.set_major_locator(LinearLocator(10))
ax.zaxis.set_major_formatter(FormatStrFormatter('%.02f'))

# Add a color bar which maps values to colors.
fig.colorbar(surf, shrink=0.5, aspect=5)

plt.show()
```



1.2 Question 2 Part B

[27]:
```
n = np.linspace(0, 99, 100, dtype = np.complex64)
x = np.cos(.1 * np.pi * n) + .5 * np.cos(np.pi * n)
x_pad =  np.concatenate((x, np.zeros(3, dtype = np.complex64)))
h = np.array([1/4, 1/4, 1/4, 1/4], dtype = np.complex64)
h_pad = np.concatenate((h, np.zeros(99, dtype = np.complex64)))
X = np.fft.fft(x_pad)
H = np.fft.fft(h_pad)
Y = H * X


y = np.fft.ifft(Y)
plt.plot(np.real(y))
plt.title("Sequence y computed via Convolution Property")
plt.show()
```

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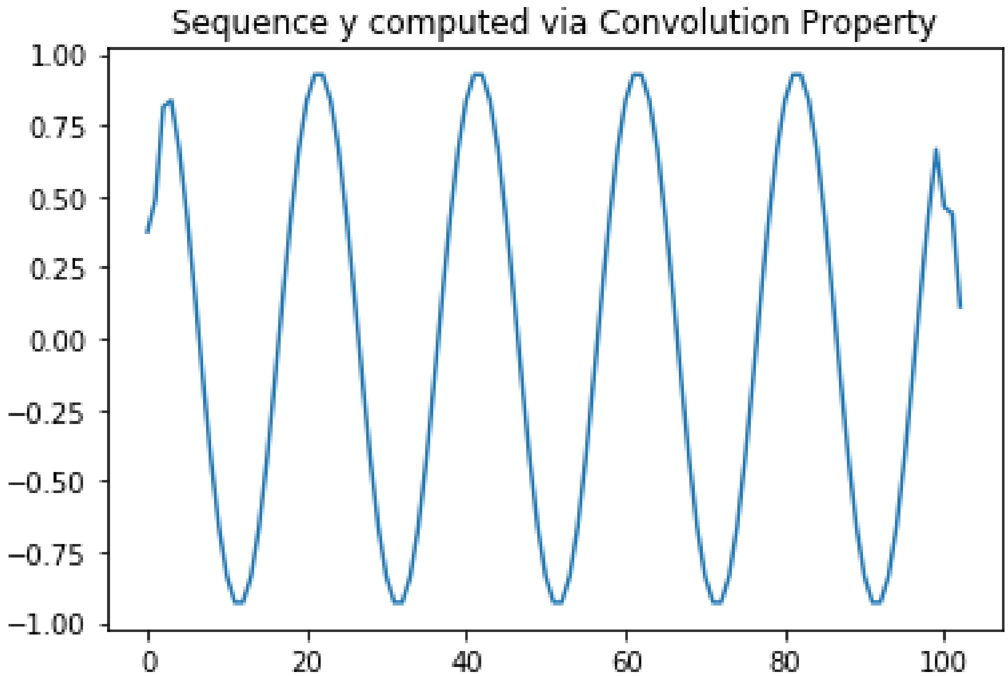
Sequence y computed via Convolution Property
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
0
20
40
60
80
100