UNIVERSITY OF CALIFORNIA, BERKELEY

EE-120: Homework 5

Oscar Ortega

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PROBLEM 1

- a: $x(t) = e^{-t}\cos(2t)u(t)$ We know this is a pair derived from the lecture notes:

$$e^{-t}\cos(2t)u(t) \leftrightarrow \frac{s+1}{(s+1)^2+2^2} : \mathscr{R}(s) > -1 \tag{0.1}$$

- b: $x(t) = \sin(t)u(-t)$

  We know the signal $x'(t) = \sin(t)u(-t) \leftrightarrow \frac{1}{s^2+1^2} : \mathscr{R}(s) > 0$ from the lecture notes. Furthermore, our signal $x(t) = -x'(-t) = -\sin(-t)u(-t) = \sin(t)u(-t)$, so combining linearity and time shift properties of Laplace transforms, we arrive at the following Laplace transform pair.

$$x(t) \leftrightarrow -\frac{1}{s^2+1^2} : \mathscr{R}(s) < 0 \tag{0.2}$$

- c: $x(t) = t\cos(t)u(t)$

$$x(t) = t\cos(t)u(t) \tag{0.3}$$
$$= \frac{1}{2}t\left(e^{t}u(t) + e^{-t}u(t)\right) \tag{0.4}$$

This will have the following corresponding Laplace Transform pair

$$= -\frac{1}{2}\left(\frac{d}{ds}(\frac{1}{1-s}) + \frac{d}{ds}(\frac{1}{1+s})\right) \tag{0.5}$$

$$= \frac{1}{2(1-s)^2} + \frac{1}{2(1+s)^2} : \mathscr{R}(s) > 1 \bigcap \mathscr{R}(s) > -1 \tag{0.6}$$

$$= \frac{1}{2(1-s)^2} + \frac{1}{2(1+s)^2} : \mathscr{R}(s) > -1 \tag{0.7}$$

- d: We know the laplace transform of the $\delta(t)$ is $X(s) = 1$ combining this with the the time-shift properties.

$$\delta(t-1) \leftrightarrow e^{-s}1 : ROC = \mathbb{C} \tag{0.8}$$

PROBLEM 2

- a:

$$X(s) = \frac{s+2}{(s+2)^2+1} : \mathscr{R}(s) > -2 \tag{0.9}$$

As we can see from the lecture notes, this Laplace transform has the corresponding time-domain pair:

$$e^{-at}\cos(\omega_0 t)u(t) \leftrightarrow \frac{s+a}{(s+a)^2+\omega_0^2} : \mathscr{R}(s) > -a \tag{0.10}$$

$$e^{-2t}\cos(t)u(t) \leftrightarrow \frac{s+2}{(s+2)^2+1} : \mathscr{R}(s) > -2 \tag{0.11}$$

- b:

$$X(s) = \frac{se^{-s}}{s^2+1} : \mathscr{R}(s) > 0 \tag{0.12}$$

We know the following Laplace transform pair from the lecture notes:

$$\cos(\omega_0 t)u(t) \leftrightarrow \frac{s}{s^2+\omega_0^2} : \mathscr{R}(s) > 0 \tag{0.13}$$

From the time-shift properties of the Laplace transform we can then say the following:

$$\cos(\omega_0(t-t_0))u(t-t_0) \leftrightarrow \frac{se^{-st_0}}{s^2+\omega_0^2} : \mathscr{R}(s) > 0 \tag{0.14}$$

$$\cos(t-1)u(t-1) \leftrightarrow \frac{se^{-s}}{s^2+1} : \mathscr{R}(s) > 0 \tag{0.15}$$

- c:

$$X(s) = \frac{s}{(s^2+1)^2} : \mathscr{R}(s) > -3 \tag{0.16}$$

We know the following Laplace transform pair from the lecture notes.

$$\sin(\omega_0 t)u(t) \leftrightarrow \frac{\omega_0}{s^2+\omega_0^2} : \mathscr{R}(s) > 0 \tag{0.17}$$

$$\tag{0.18}$$

Combining linearity and frequency differentiation properties,
we can then say the following.

$$-t\sin(\omega_0 t)u(t) \leftrightarrow \frac{d}{ds}\frac{\omega_0}{s^2+\omega_0^2} = \frac{2s\omega_0}{(s^2+\omega_0^2)^2} : \mathscr{R}(s) > 0 \tag{0.19}$$

$$-\frac{t}{2}\sin(\omega_0 t)u(t) \leftrightarrow \frac{d}{ds}\frac{\omega_0}{2(s^2+\omega_0^2)} = \frac{s\omega_0}{(s^2+\omega_0^2)^2} : \mathscr{R}(s) > 0 \tag{0.20}$$

$$-\frac{t}{2}\sin(t)u(t) \leftrightarrow \frac{s}{(s^2+1)^2} : \mathscr{R}(s) > 0 \tag{0.21}$$

- d:

$$X(s) = \frac{s+2}{s^2+7s+12} : \mathscr{R}(s) > -3 \tag{0.22}$$

We can use partial fraction decomposition:

$$\begin{aligned}
X(s) &= \frac{s+2}{s^2+7s+12} && (0.23)\\
&= \frac{A}{s+4} + \frac{B}{s+3} && (0.24)\\
&= \frac{(A+B)s+(3A+4B)2}{(s+3)(s+4)} && (0.25)
\end{aligned}$$

We solve the system $A+B=1, 3A+4B=2 :\rightarrow A=2, B=-1$, which yields the following decomposition and inverse transform pair.

$$(2e^{-4t} - e^{-3t})u(t) \leftrightarrow \frac{2}{s+4} - \frac{1}{s+3} : \mathscr{R}(s) > -3 \tag{0.26}$$

Problem 3

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} : \zeta \in [0,1] \tag{0.27}$$

- a: Let $\zeta \in (0,1]$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \tag{0.28}$$
$$(s + \zeta\omega_n)^2 = -(1-\zeta^2)\omega_n^2 \tag{0.29}$$
$$s + \zeta\omega_n = \pm\sqrt{-(1-\zeta^2)\omega_n^2} \tag{0.30}$$
$$s = -\zeta\omega_n\left(\pm\omega_n\sqrt{1-\zeta^2}\right)j \tag{0.31}$$

Because the real part of the poles are strictly negative, this transfer function shows the impulse response is causal.

- b: Let $\zeta = 0$

$$s^2 + \omega_n^2 = 0 \tag{0.32}$$
$$s^2 = -\omega_n^2 \tag{0.33}$$
$$s = 0 \pm \omega_n j \tag{0.34}$$

As one can see, the poles are no longer strictly negative, therefore the impulse response can no longer be said to be causal.

- c:
Let $x(t) = \cos(t)u(t) : H(s) = \frac{1}{s^2+1}$

$$x(t) \leftrightarrow \frac{s}{s^2+1} : \mathscr{R}(s) > 0 \tag{0.35}$$
$$y(t) = (x * h)(t) \leftrightarrow X(s)H(s) \tag{0.36}$$
$$-\frac{t}{2}\sin(t)u(t) \leftrightarrow \frac{s}{(s^2+1)^2} : \mathscr{R}(s) > 0 \tag{0.37}$$
$$\tag{0.38}$$

Which is clearly unbounded as $t \to \infty$.

- d: Credit to the derivation in note 7

$$H(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n j\omega + \omega_n^2} \tag{0.39}$$
$$= \frac{1}{(\frac{j\omega}{\omega_n})^2 + 2\zeta\frac{j\omega}{\omega_n} + 1} \tag{0.40}$$
$$|H(j\omega)|^2 = \frac{1}{(\frac{\omega}{\omega_n})^4 + (4\zeta^2 - 2)(\frac{\omega}{\omega_n})^2 + 1} \tag{0.41}$$

This function will strictly increase in ω if $4\delta^2 - 2 \geq 0$ because the denominator will strictly increase in magnitude.

$$4\delta^2 - 2 \geq 0 \tag{0.42}$$

$$\delta \geq \frac{1}{\sqrt{2}} \tag{0.43}$$

Problem 4

This problem is in the back of the file.

Problem 5

- a:

$$\frac{dz_1(t)}{dt} = z_2(t) \tag{0.44}$$
$$\frac{dz_2(t)}{dt} = -a_0 z_1(t) - a_1 z_2(t) + x(t) \tag{0.45}$$
$$y(t) = b_0 z_1(t) + b_1 z_2(t) \tag{0.46}$$

This yields the following state-space equations: $z(t) = \begin{bmatrix} z_1(t) & z_2(t) \end{bmatrix}^T$

$$\frac{d}{dt} z(t) = \begin{bmatrix} 0 & 1 \\ -a_0 & a_1 \end{bmatrix} z(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x(t) \tag{0.47}$$
$$y(t) = \begin{bmatrix} b_0 & b_1 \end{bmatrix} z(t) \tag{0.48}$$

Taking laplace transforms:

$$sZ(s) = \begin{bmatrix} 0 & 1 \\ -a_0 & a_1 \end{bmatrix} Z(s) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} X(s) \tag{0.49}$$
$$Y(s) = \begin{bmatrix} b_0 & b_1 \end{bmatrix} Z(s) \tag{0.50}$$
$$(sI - \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}) Z(s) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} X(s) \tag{0.51}$$
$$Z(s) = (sI - \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix})^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} X(s) \tag{0.52}$$
$$Y(s) = \begin{bmatrix} b_0 & b_1 \end{bmatrix} (sI - \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix})^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} X(s) \tag{0.53}$$
$$H(s) = \begin{bmatrix} b_0 & b_1 \end{bmatrix} (sI - \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix})^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{0.54}$$
$$\tag{0.55}$$

- b:

Given the following equations:

$$\frac{dz_1(t)}{dt} = -a_0 z_2(t) + b_0 x(t) \tag{0.56}$$
$$\frac{dz_2(t)}{dt} = z_1(t) - a_1 z_2(t) + b_1 x(t) \tag{0.57}$$
$$y(t) = z_1(t) \tag{0.58}$$

This lends the following state space equations

$$\frac{d}{dt} z(t) = \begin{bmatrix} 0 & -a_0 \\ 1 & -a_1 \end{bmatrix} z(t) + \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} x(t) \tag{0.59}$$
$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} z(t) \tag{0.60}$$

Which after taking laplace transforms will yield the following transfer function by similar procedure to part a:

$$H(s) = \begin{bmatrix} 0 & 1 \end{bmatrix} \left(sI - \begin{bmatrix} 0 & -a_0 \\ 1 & -a_1 \end{bmatrix} \right)^{-1} \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} \tag{0.61}$$

PROBLEM 6

- a:

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t) \tag{0.62}$$

$$s^2Y(s) - sy(0-) - \frac{dy(0-)}{dt} + 3(sY(s) - y(0-)) + 2Y(s) = X(s) \tag{0.63}$$

$$(s^2 + 3s + 2)Y(s) = X(s) + (s+3)y(0-) + \frac{dy(0-)}{dt} \tag{0.64}$$

$$Y(s) = \frac{5s^2 + 21s + 20}{(s+1)(s+2)(s+3)} \tag{0.65}$$

From here we can solve a system of the form:

$$Y(s) = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3} \tag{0.66}$$

Solving for the system gives us $A = 7, B = -8, C = 6$, which implies our system is the following.

$$y(t) = (7e^{-t} - 8e^{-2t} + 6e^{-3t})u(t) \tag{0.67}$$

I have scratch work below.