UNIVERSITY OF CALIFORNIA, BERKELEY

EE-120: Lecture 8

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1 CTFT PROPERTIES

Convergence Theorem:

If $\int_{-\infty}^{\infty} |x(t)| dt < \infty$ then $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$ exists and is continuous. Furthermore, $X(\omega) \to 0$ as $\omega = \pm\infty$ Please note, that this is a sufficient condition for the existence of the Fourier Transform and not a necessary condition.

2 THEORY OF DISTRIBUTIONS

Definition: A *Distribution* T is a linear mapping $T : D(R) \to R$ where $D(R)$ is the set of *test functions* $\phi : \mathbb{R} \to \mathbb{R}$ having two properties:

- ϕ is infinitely differentiable
- ϕ has compact support.

However instead of denoting the relation of T acing on test function ϕ as $T(\phi)$ we by convention denote the operation as $\langle T, \phi \rangle$. Note that an example of this is our unit impulse δ which we can define as $\langle \delta, \phi \rangle = \phi(0)$ which we can interpret as delta evaluating a test function at 0. In general: Given a **test function** x and a **distribution** T operates on x to produce a number $\langle T, x \rangle$

2.1 Properties of Distributions

In general we define the distribution induced by a function g as the following:

$$\langle T_g, x\rangle = \int_{-\infty}^{\infty} g(t)^* x(t) dt \tag{2.1}$$

Note that because this function is linear, this implies the following. (Note how it behaves similarly to an inner product.

- $\langle T_{g_1+g_2}, x\rangle = \langle T_{g_1}, x\rangle + \langle T_{g_2}, x\rangle$
- $\langle T_g, \alpha x\rangle = \alpha\langle T_g, x\rangle$
- $\langle \alpha T_g, \alpha x\rangle = \alpha^*\langle T_g, x\rangle$

Note that we can know define the Fourier transform FT of a distribution T as:

$$\langle FT, X\rangle = 2\pi\langle T, x\rangle \tag{2.2}$$

As an example consider $FT_1 = 2\pi\delta$ This would imply the following:

$$\begin{aligned}
\langle FT_1, X\rangle &= 2\pi\langle T_1, x\rangle && (2.3)\\
&= \int_{-\infty}^{\infty} 1\cdot x(t)dt && (2.4)\\
&= \int_{-\infty}^{\infty} x(t)dt && (2.5)\\
&= 2\pi X(0) && (2.6)\\
&= \langle 2\pi\delta, X\rangle && (2.7)
\end{aligned}$$

3 Discrete Time Fourier Transforms

We define the discrete-time Fourier Transform as follows:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \tag{3.1}$$

If we note the similarity to the equation in the continuous case, we can continue to motivate both the following analysis equation, and the respective synthesis equation as the limit of periodic signal $\hat{x}$ with period T as $T\to\infty$ Remember though, that because in discrete time the finite quantity of Fourier coefficients a_k will translate to the synthesis equation only needing to be integrated over a length of 2π. This will give us the following analysis equation.

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n} d\omega \tag{3.2}$$

3.1 Properties of the DTFT

Again consider the following signal, fourier transform pairs x, X

- Time Shift: $x[n-n_0] \leftrightarrow e^{-j\omega n_0} X(e^{j\omega})$
- Frequency Shift: $e^{j\omega_0 n} x[n] \leftrightarrow X(e^{j(\omega-\omega_0)})$
- Time Reversal: $x[-n] \leftrightarrow X(e^{-j\omega})$