UNIVERSITY OF CALIFORNIA, BERKELEY

EE-120: Lecture 12

Oscar Ortega

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1 SAMPLING

Consider a continuous time signal $x(t)$ and let us define $x_d[t]$ as follows:

$$x_d[n] = x(nT) : T : \text{sampling period} \tag{1.1}$$

A natural question to ask is whether or not we can recover x from x_d.

2 SHANNON-NYQUIST SAMPLING THEOROM

If $x(t)$ is bandlimited (finite range in frequency domain) with $X(j\omega) = 0$ for $|\omega| > \omega_m$ and $\omega_s > 2\omega_M$ where $\omega_s = \frac{2\pi}{T}$, then x(t) is uniquely determined by its sample signal x_d!

3 EXPLANATION - PROOF OF THE THEOROM

$$x_d[n] = x(t)p(t) \tag{3.1}$$

Where we define p as the impulse train with period T: By the multiplicity property of Fourier transforms this implies the following:

$$X_d(j\omega) = \frac{1}{2\pi} X(j\omega) * P(\omega) \tag{3.2}$$

But what is the Fourier Transform of $p(t)$? Recall that because the impulse train is a periodic signal $p(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ by the fourier series expansion.

$$
\begin{aligned}
a_k &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} p(t)e^{-jk\omega_0 t}dt && (3.3)\\
&= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \Big(\sum_{k=-\infty}^{\infty}\delta(t-kT)\Big)e^{-jk\omega_0 t}dt && (3.4)\\
&= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t)e^{-jk\omega_0 t}dt && (3.5)\\
&= \frac{1}{T}e^{-jk\omega_0 t}|_{t=0} && (3.6)\\
&= \frac{1}{T} && (3.7)
\end{aligned}
$$

This means our signal $p(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} e^{-jk\omega_0 t}$ By linearity and time shift this gives us the following for $P(j\omega)$:

$$P(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(w-k\omega_0) \tag{3.8}$$

Also an impulse train!

$$X_d(j\omega) = \frac{1}{T}X(j\omega) * \hat{P}(j\omega) \tag{3.9}$$

Where $\hat{P}(j\omega) = \frac{T}{2\pi}P(j\omega)$, the normalized impulse train with period ω_0. Note that the convolution of the two signals will simply produce numerous copies of the signal $X(j\omega)$ that are height scaled by a factor of $\frac{1}{T}$ and the copy pasting interval is determined by ω_0, how often the impulses go through your signal $X(j\omega)$.

This is why if ω_0 if are not spread far enough apart, the copy pastes superimpose on one another leading us to the condition that $\omega_s > 2\omega_M$: the width of the non-zero frequency portion of $X(j\omega)$. This superposition effect is known as *aliasing*.

4 How to recover the signal

Assuming the conditions for sampling have been met, to recover our signal x all we need to do is use a lowpass filter with a gain of T and a cutoff frequency that ensures we only retrieve one of the copypasted signals. In other words, establishing a cutoff frequency of $\frac{|\omega_s|}{2}$. We call this filter our *reconstruction filter*.

5 Reconstruction Filter

Take note that our definition of our reconstruction filter is simply a scaled sinc! More precisely, in our time-domain our reconstruction filter $h_r(t) = \text{sinc}(\frac{t}{T})$.