Machines B Answer Key
UChicago 2021

Oliver Tsang and Derek Li

Saturday, January 23, 2021

Instructions:

1. Be sure to enter all answers into Scilympiad! For decimal answers, provide at least 3 sig figs if you're unsure. We will not penalize for extra significant figures. Feel free to leave fractions when convenient (but please reduce to lowest form for the sake of grading!).
2. Many problems are multiple-choice to make grading easier. Not all the options make physical sense, though!! Whenever possible, check that your answers make sense before moving on!
3. Make sure to ask on scilympiad if you have any questions about the test.
4. We hope you enjoy and learn something new from this test! Best of luck!

1 Warm-up

Question 1 [5 pt]. Give the correct formula for the ideal mechanical advantage of each machine, only using d_{in} (`d_in`), d_{out} (`d_out`), h (`h`), L_{slope} (`L_slope`), L (`L`), $w_{separation}$ (`w_separation`), π (`pi`), R_{wheel} (`R_wheel`), r_{axle} (`r_axle`), T_{in} (`T_in`) (teeth), T_{out} (`T_out`). Other constants may also be used if necessary.

1. Lever
2. Inclined Plane
3. Wedge
4. Wheel and Axle
5. Gears

Answer: 1. d_{in}/d_{out} 2. L_{slope}/h 3. $L/w_{separation}$ 4. R_{wheel}/r_{axle} 5. T_{out}/T_{in}

Question 2 [6 pt]. What is the formula for efficiency of any machine (don't use the terms above)?

Answer: W_{out}/W_{in} or AMA/IMA

Question 3 [6 pt]. Write the letter of the term that best corresponds with each numbered definition. Note that not all terms will be used.

a. Efficiency
b. Friction
c. Joule
d. Mechanical Advantage
e. Newton
f. Output
g. Power
h. Resistance
i. Rotation
j. Watt
k. Work

1) Unit of work.
2) Unit of power.
3) Unit of force.
4) Product of force and velocity.
5) Reduces efficiency.
6) Opposes effort.

Answer: 1. c — 2. j — 3. e — 4. g — 5. b — 6. h (c-j-e-g-b-h)

Question 4 [5 pt]. Each term on the left describes a phrase listed on the right. Write the letter of the correct matching.

a. Compound Machine	1) Two rigidly attached wheels of different diameters
b. Inclined Plane	2) Output force divided by input force
c. Lever	3) An inclined plane that moves
d. Mechanical Advantage	4) A rigid rod that rotates about a fulcrum
e. Pulley	5) A flat sloped surface.
f. Simple Machine	
g. Wedge	
h. Wheel and Axle	
i. Work	

Answer: 1. h — 2. d — 3. g — 4. c — 5. b (h-d-g-c-b)

Question 5 [9 pt]. Determine what type of lever each of the following is. If the item described is not a lever, write the correct type of simple machine.

1. Tweezers
2. Boat Oars
3. Wheelbarrow
4. Nutcracker (ignore the figurine nutcracker)
5. Zipper
6. Hockey Stick (consider the top hand relatively fixed)
7. Bottle Opener
8. Broom
9. See Saw

Answer: 1. Third 2. First 3. Second 4. Second 5. Wedge 6. Third 7. Second 8. Third 9. First (3-1-2-1-wedge-3-2-3-1)

2 Build-based theory questions

In place of a normal in-person device testing, I hope these theory questions will help give you a deeper understanding of the math behind the build-portion of the event, as well as strategies to maximize your efficiency in the future. (Some of these questions would not be fair to ask during an in-person competition, since everyone does the build portion at different times!)

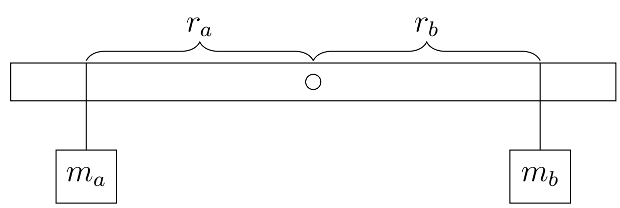

Figure 1: The single class-1 lever in the Division B device

2.1 Mass ratio calculation

Suppose we hang weights a (with mass m_a) and b (with mass m_b) at distances r_a and r_b from the left and right levers' fulcra, respectively, such that the system is balanced. Then the mass ratio m_a/m_b is given by:

$$\frac{m_a}{m_b} = \frac{r_b}{r_a}$$

Question 6 [2 pt]. What is the mass ratio m_a/m_b if $r_a = 15$ cm and $r_b = 35$ cm? (Please don't get a and b confused!!)

Answer: $7/3 = 2.333$

2.2 Note on device construction

Question 7 [3 pt]. Suppose the fulcrum is a little sticky, so the efficiency of the lever only works out to 0.95. Consider the case where $m_a = m_b$. What is **maximum** possible ratio of distances r_b/r_a (and equivalently mass ratio) we might could measure for a balanced lever? Moral: if your lever doesn't turn smoothly, there's some built-in uncertainty in your measurements!

Multiple choice:

a) $361/400$
b) $19/20$
c) 1
d) $20/19$
e) $400/361$

Answer: d) $20/19 \approx 1.0526$

2.3 Error analysis

Error analysis is the process of determining the uncertainty or error of a calculated quantity (e.g. mass ratio) based on uncertainties in the original measurements (e.g. r_b).

We want to find the mass ratio m_a/m_b accurately and quickly, but ensuring accurate measurements of r_a and r_b could take a long time. In this section, we will see how uncertainty on these distances affect the accuracy of the calculated mass ratio.

Again consider the mass positions $r_a = 15.0$ cm and $r_b = 35.0$ cm, but this time we do not know r_b exactly. Suppose we know that it lies within 1.0 cm of 35 cm. I'll write this as $r_b = 35 \pm 1$ cm.

Question 8 [2 pt]. When $r_a = 15.0$ cm and $r_b = 35.0 \pm 1.0$ cm, what is our **lower** bound on m_a/m_b?
Multiple choice:

a) $34/16 = 2.125$
b) $35/16 = 2.1875$
c) $34/15 \approx 2.266$
d) $36/15 = 2.4$
e) $35/14 = 2.5$
f) $36/14 \approx 2.571$

Answer: c) $34/15 \approx 2.266$

Question 9 [2 pt]. When $r_a = 15.0$ cm and $r_b = 35.0 \pm 1.0$ cm, what is our **upper** bound on m_a/m_b?
Hint: it can't be lower than your answer for the previous question!!
Multiple choice:

a) $34/16 = 2.125$
b) $35/16 = 2.1875$
c) $34/15 \approx 2.266$
d) $36/15 = 2.4$
e) $35/14 = 2.5$
f) $36/14 \approx 2.571$

Answer: d) $36/15 = 12/5 = 2.4$

Question 10 [4 pt]. Suppose we estimate m_a/m_b with $r_a = 15$ cm and $r_b = 35$ cm. What is the largest possible fractional error of m_a/m_b? Hint: this happens when the true mass ratio is at one of the bounds, and the error should be a fraction of the true mass ratio. Please provide at least 3 significant figures (or a precise fraction).
Answer: $\frac{35/15-34/15}{34/15} = 1/34 \approx 0.2941 = 2.941\%$

This time, suppose $r_a = 15 \pm 1$ cm is uncertain and $r_b = 35$ cm is known exactly.

Question 11 [2 pt]. When $r_a = 15.0 \pm 1.0$ cm and $r_b = 35.0$ cm, what is our **lower** bound on m_a/m_b?
Multiple choice:

a) $34/16 = 2.125$
b) $35/16 = 2.1875$
c) $34/15 \approx 2.266$
d) $36/15 = 2.4$
e) $35/14 = 2.5$
f) $36/14 \approx 2.571$

Answer: b) $35/16 \approx 2.1875$

Question 12 [2 pt]. When $r_a = 15.0 \pm 1.0$ cm and $r_b = 35.0$ cm, what is our **upper** bound on m_a/m_b?
Multiple choice:

a) $34/16 = 2.125$
b) $35/16 = 2.1875$
c) $34/15 \approx 2.266$
d) $36/15 = 2.4$
e) $35/14 = 2.5$
f) $36/14 \approx 2.571$

Answer: e) $35/14 \approx 2.5$

Question 13 [4 pt]. Suppose we estimate m_a/m_b with $r_a = 15$ cm and $r_b = 35$ cm. What is the largest possible fractional error of m_a/m_b? Hint: this happens when the true mass ratio is at one of the bounds, and the error should be a fraction of the true mass ratio.

Answer: $\frac{35/15 - 35/16}{35/16} = 1/15 \approx 0.06666 = 6.666\%$

Moral of the story: try to get both weights as far from the fulcum as possible, then leave the closer weight alone.

2.4 Strategic tips

No question is asked – feel free to revisit after the test, but they're here because I find them important.

1. Speed is important! Every 16 seconds costs you 1% of the maximum event score (or equivalently, it acts like an extra 6.67% error in a mass ratio estimate – this works out to 0.42 % error per second!). It could be worth writing a program in your calculator to save a couple precious extra seconds.

2. It's unnecessary to get your device to balance perfectly! This is typically time-consuming, and we can still estimate the balance point without actually balancing the masses.

   Instead, try to find tight bounds on each value: for example, suppose I hang the left mass at $r_a = 20.0$ cm exactly and, after moving around the right mass a bit, I know that placing it at 31.6 cm causes the right side of the lever to tilt up, while 32.6 cm causes the right side to tilt down. Then, I could use 32.1 cm as my value of r_b (it's somewhere in the middle of the range). I would estimate a mass ratio $m_a/m_b = 32.1/20 \approx 1.605$. The worst case for me is if the true value of r_b should have been just over 31.6 cm, which yields a mass ratio of 1.580. Taking this as the truth would give me 1.6% as the worst-case fractional error.

   That's actually a nice result. Depending how far the masses are from the fulcrum, it's possible to guarantee at most 2% error only knowing the location of the balance point to within a centimeter!

3. To find the mass ratio, think of your task as trying to restrict the range of values that r_b can take on (say r_a was fixed at the beginning). A quick way to do this is to find an upper and lower bound, then to chop the range in half repeatedly by observing the lever's behavior in the middle of the range.

   Going back to the example above: if I wanted a more accurate answer, I could test whether my current estimate, 32.1 cm, is too high or too low. If it's too high, the new estimate would become $r_b \approx 31.85$ cm (halfway between our current estimate and the current lower bound of 31.6). If it were too low, the new estimate would be $r_b \approx 32.35$ cm. In either case, the error would drop to at most $\sim 0.79\%$. One measurement halves the error! Note that this process would need to take less than two seconds to improve the overall build-portion score.

4. Practice a lot! You'll get a feel for how to use your device more efficiently and accurately with a wide range of masses. Also, try to make practicing as efficient/easy as possible (e.g. automate score calculation with Excel or python or something).

3 Efficiency and Friction

3.1 Belayed and betrayed

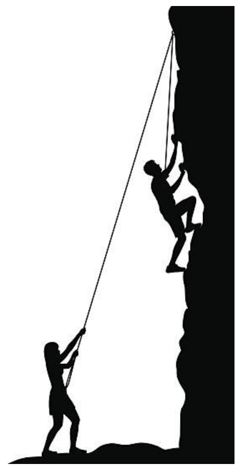

Figure 2: from iStock photos

Augustus is rock-climbing, belayed by with his wife Livia. From the ground, Livia supports Augustus with a rope that is fed through a single pulley at the top of the cliff (and is attached to Augustus's safety harness). The pulley has an efficiency of 0.5, and Augustus weighs 750 N.

Question 14 [3 pt]. If Augustus slips, what is the most force that Livia can use to pull on the rope that will not save Augustus? (Livia was rumored to have been responsible for Augustus's death by poisoning – maybe she would call this an "accident")
Multiple choice:

a) 112.5 N
b) 325 N
c) 562.5 N
d) 750 N
e) 1125 N
f) 1500 N

Answer: b) 325 N

Question 15 [3 pt]. Suppose Livia lets go completely, but Augustus manages to catch the rope on the opposite side of the pulley. If he wants to pull himself up, how much force will he need to supply? (Still assume he is no longer supported by the rocks)

Multiple choice:

a) 112.5 N
b) 325 N

c) 562.5 N
d) 750 N
e) 1125 N
f) 1500 N

Answer: f) 1500 N

3.2 Well, well, well... a bucket

Question 16 [2 pt]. How much work is expended in raising a 5 kg bucket of water up a 5 m well? (Give your answer in Joules)

Multiple Choice (rounded):

a) 123 J
b) 221 J
c) 233 J
d) 245 J
e) 258 J
f) 272 J

Answer: d) 245 J

Question 17 [3 pt]. How much work is expended if we use pulley system with an IMA of 2, but **each** of the two pulleys involved has an efficiency of 0.95? (Again, in Joules)

Multiple Choice (rounded):

a) 123 J
b) 221 J
c) 233 J
d) 245 J
e) 258 J
f) 272 J

Answer: e) $245/(0.95)^2 \approx 271.5$ J; half credit for $245/0.95 \approx 258$ J

Question 18 [3 pt]. Suppose the pulleys each have a heat capacity of 4.60 J/°C (10 grams of stainless steel). How much hotter do the pulleys get, assuming none of the energy lost to friction goes to the rope or surroundings? (Answer in °C)
Hint: a simple guess of dimensional analysis would tell you the right equation of $\Delta T = W/C$, where ΔT is the change in temperature, W is the work, and C is the heat capacity (not specific heat capacity).

Multiple Choice (rounded):

a) 1.33 °C
b) 1.40 °C
c) 2.60 °C
d) 2.88 °C

Answer: d) $\frac{245((1/0.95)^2-1)/2 \text{ J}}{4.60\text{J}/°\text{C}} \approx 2.877°\text{C}$ – need to divide by 2 to split the lost energy across the two pulleys

3.3 Slippery cardboard

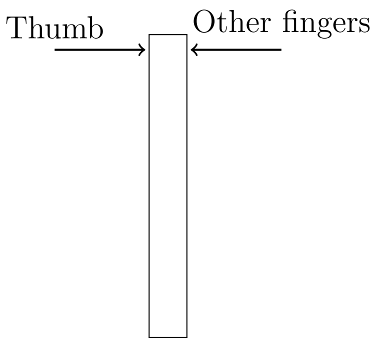

Figure 3: Trying to grip a slippery piece of cardboard.

Question 19 [3 pt]. It can be deceptively tiring to carry light objects with awkward shapes. Consider holding a flat piece of cardboard from the top, as in Figure 3. The piece of cardboard has a mass of 1 kg, and the surface has a coefficient of friction of $\mu_s = 0.3$. How much force would someone need to supply to keep the cardboard from slipping? Add the (magnitudes of) forces from both sides and respond in Newtons.
Note: the difficulty also comes from the fact that this uses different muscles from carrying something with handles, but that's harder to reason about :).

Multiple Choice:

a) 2.94 N
b) 14.0 N
c) 16.3 N
d) 32.6 N

Answer: $\frac{1 \text{ kg}\cdot 9.8 \text{ m/s}^2}{0.3} = 32.6$ N

4 Sisyphus and a boulder

Here is a quick version of the story, in case you have not heard it (for your personal knowledge, and also because we're at UChicago `:P`). Sisyphus was very clever but also mischievous and upset the gods on multiple occasions – he even chained Thanatos (the god of death) to a chair, at which point people on Earth could no longer die. His famous punishment in the afterworld was to push an enormous boulder up a hill. Right before reaching the top, the boulder always slips and rolls to the bottom of the hill so that Sisyphus will never be able to complete the task. Despite this, the absurdist writer Albert Camus says that Sisyphus continues this hopeless task in defiance of the gods...
And now for your tasks, which are very finite and (hopefully) manageable:

First, we will consider applying force to a spherical boulder with radius r on level ground. If we push very short distances, this system looks a lot like a lever (in general wheels are fancy levers).

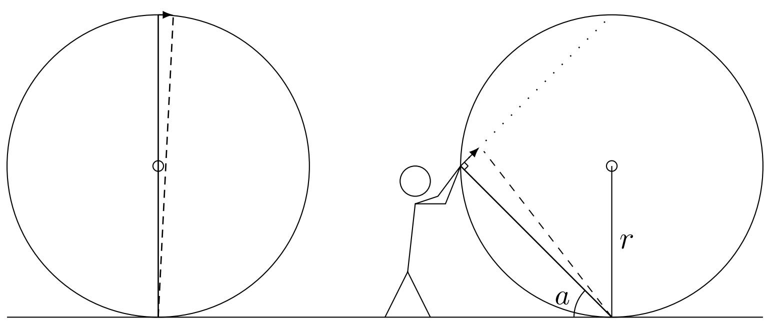

Figure 4: The left boulder is being pushed at the top, while the right boulder is being pushed at an angle a from the bottom.

Question 20 [2 pt]. If we push tangentially at the top, what is the IMA we achieve? Compare against the case of applying a horizontal force to the center of the sphere (this is impossible, but it lets us ignore angular mechanics for now). Pretend that the boulder rotates about its contact point with the ground for very short distances. (free response)
Hint: think about distances rather than forces.

Answer: 2

Question 21 [2 pt]. What class lever does the boulder behave like in this case? (Class 1, 2, or 3)

Answer: Class 2

Since Sisyphus is likely shorter than the boulder, we want to see how the IMA changes depending where he pushes the boulder. Consider a contact point at angle a above the ground (as in the right side of Figure 4). For simplicity, suppose the force is directed toward the top of the boulder to ensure a right angle with the line connecting Sisyphus's contact point to the boulder's contact point with the ground.

Question 22 [3 pt]. What is the IMA of this system? Again compare against the case of pushing the center of the sphere. You could use the law of sines or just brute-force check each option.
Hint: check your answer with several simple cases like $a = \pi/2 = 90°$ (should agree with your answer from above), $a = \pi/4 = 45°$ (as pictured), and $a = 0$.

Multiple choice:

a) $\sin a$
b) $\sin 2a$
c) $2 \sin a$
d) $\cos a$
e) $\cos 2a$
f) $2 \cos a$

Answer: c) $2 \sin a$ – could use law of sines: $\sin(\pi/2)/2r = \sin(a)/\ell \implies d_{in}/d_{out} = \ell/r = 2 \sin a$ with ℓ as the unlabeled chord across the boulder connecting Sisyphus to the ground

Question 23 [5 pt]. Suppose a boulder is 1000 kg and lying on a hill with an angle of inclination of 20°. What is the magnitude of the normal force provided by the hill (this force keeps it from falling through the ground)? (free response)
Even though this is the underworld, let's just assume $g = 9.8$ m/s^2.
Hint: confused whether to use sin or cos? Calculate the normal force if the slope were actually flat (note: $\sin(0) = 0, \cos(0) = 1$), to see if you equation makes sense.

Answer: $mg \cos\theta = 1000 \cdot 9.8 \cos(20°)$ N $= 3351.8$ N.

Question 24 [7 pt]. Say the boulder is 1000 kg, and Sisyphus needs to push the boulder up a hill with angle of inclination 20°. How much force does Sisyphus need to push the boulder at a constant velocity if he pushes from angle $a = 15°$? (free response – show work for possible partial credit)

Note: the angle a is relative to the hill, not an absolute level plane. Give your answer in Newtons.

Note 2: Sisyphus needs to provide force **not** supplied by the hill (from the last question). This means that if you used cos last time, you should use sin this time (and vice versa).

Answer: 6475 N
On the slope, you need to provide $mg \sin\theta$ of force to the center of the boulder. Sisyphus just needs to provide

$$\frac{mg \sin\theta}{IMA} = \frac{mg \sin\theta}{2 \sin a} \approx \frac{1000 \text{ kg} \cdot 9.8\text{m/s}^2 \cdot \sin(20°)}{2 \sin(15°)} \approx 6475.17 \text{ N}$$

of force to deliver the equivalent required force.
But I'm not totally sure because this direction would result in angular acceleration about the center of the boulder?

5 Architecture/engineering-inspired questions

5.1 Cantilever

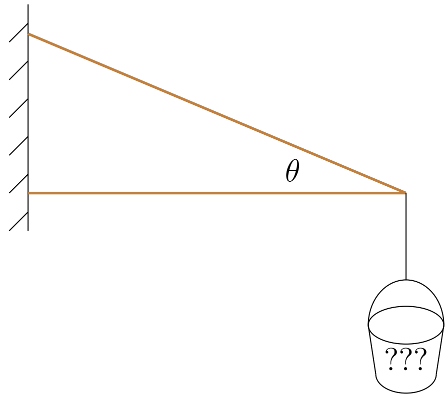

Figure 5: Simple cantilever

Question 25 [4 pt]. In the figure above (5) is a cantilever with just two beams at an angle of angle $\theta = 25°$. If the top beam breaks when it carries 100 N of force, but the bottom beam does not break, what is the maximum load the cantilever can hold?
Multiple choice (rounded):

a) 4.31 kg
b) 4.76 kg
c) 9.25 kg
d) 11.3 kg
e) 21.9 kg
f) 24.1 kg

Answer: a) $100 \sin\theta = 42.26$ N $\implies$ 4.31 kg

Question 26 [5 pt]. What is the maximum load if the bottom beam breaks when it supplies 100 N of vertical force at its tip, assuming the top beam does not break?
Multiple choice (rounded):

a) 4.31 kg
b) 4.76 kg
c) 9.25 kg
d) 11.3 kg
e) 21.9 kg
f) 24.1 kg

Answer: b) $100 \text{ N} \cdot \tan 25° \approx 46.63$ N $\implies$ 4.76 kg
Let the top bar carry force h and the bottom carry $h \cos\theta$ ($= 100$ N when it breaks). This means that we're interested in the horizontal force, $h \sin\theta = (h\cos\theta)\tan\theta = 100 \text{ N} \cdot \tan\theta$

5.2 Arch

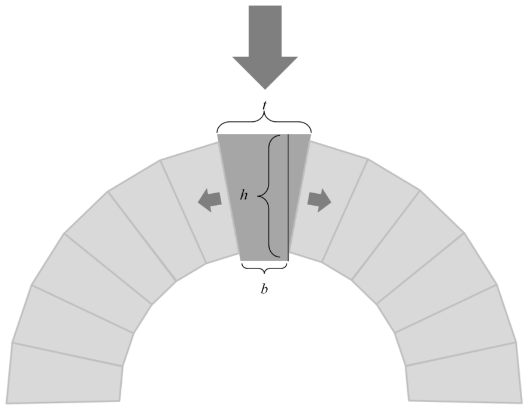

Figure 6: An arch's keystone redirects force to each side.

Question 27 [5 pt]. If an arch in Figure 6 needs to support a force F from above, how much horizontal force does it impart on each neighboring brick?

Multiple choice:

a) $\frac{2(t-b)}{hF}$
b) $\frac{2(t-b)F}{h}$
c) $\frac{(t-b)F}{2h}$
d) $\frac{hF}{2(t-b)}$
e) $\frac{hF}{t-b}$

Answer: e) $\frac{h}{t-b}F$ (Give $F/2$ to two wedges, each with IMA $\frac{t-b}{2h}$)

5.3 Flying Buttress

The buttress was a common architectural device to help walls to support heavy vaulted roofs, especially of churches and cathedrals. The flying buttress is a flashy variant that helps provide horizontal support while allowing the main building to get more light (by moving the support structures farther out).

For this question, we'll consider a simplified model of the flying buttress to get a rough idea of how this can be useful. In Figure 8, the roof's weight is responsible for a force F_r that acts on the walls at an angle θ from the horizontal. Suppose $F_r = 10^6$ N $= 1,000,000$ N.

Figure 7: Several views of the Notre-Dame in Paris from the back, plus a profile view of a flying buttress. (Sources: a b c)

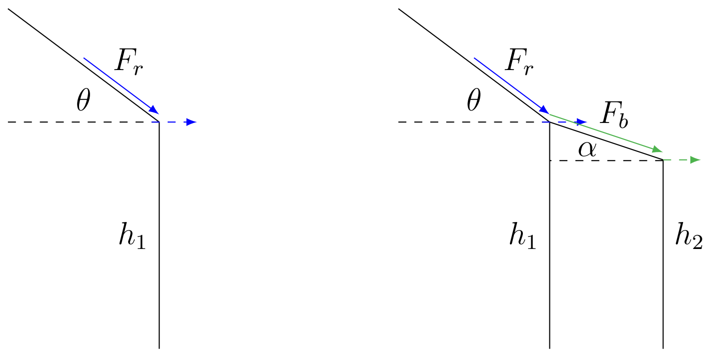

Figure 8: Simplified diagrams of a wall supporting a roof with and without a flying buttress.

Question 28 [3 pt]. What is the mass of the roof being supported by this buttress? (consider just the half of the roof seen in the diagram)

Multiple choice (rounded):

a) 1.18×10^5 kg
b) 1.77×10^5 kg
c) 2.04×10^5 kg
d) 5.10×10^5 kg
e) 8.84×10^5 kg

Answer: c) $F_r / \sin\theta = 1000/(1/2) = 2000$ kN$/(9.8$ N/m$) \approx 204,000$ kg

Question 29 [3 pt]. Given $F_r = 10^6$ N (as before), $\theta = 30°$, and $h_1 = 10$ m, what is the (magnitude of) torque the inner buttress experiences if there is no flying buttress? Call this (magnitude of) torque τ_i for the next question. (answer in m $\cdot$ N)

Hint: $F_r \cos\theta$ is the horizontal force imparted by the roof.

Multiple choice (rounded):

a) 8.66×10^5 N · m
b) 1.73×10^6 N · m
c) 4.33×10^6 N · m
d) 5.00×10^6 N · m
e) 5.77×10^6 N · m
f) 8.66×10^6 N · m

Answer: f) $h_1 \cdot F_r \cos\theta = 10 \cdot 10^6 \cdot \sqrt{3}/2 = 5\sqrt{3} \times 10^6 \approx 8.66 \times 10^6 \text{ m} \cdot \text{N} = 8,660,000 \text{ m} \cdot \text{N}$

Question 30 [4 pt]. Now suppose all the horizontal force is offloaded from the original buttress (the one with height h_1) through the flying buttress. Call the resulting torque on the outer pier τ_o. If $h_2 = 5$ m, then what is τ_o/τ_i? In other words, how much were we able to reduce the torque?
Hint: the value of α should not actually matter for this value but would influence the vertical force transferred; moreover, you do not need to calculate F_b. It is intended to help visualization.

Another hint: consider how the horizontal force on the outer pier is related to the force on the inner pier (with the knowledge that α never figures into the picture).

Multiple choice:

a) 1/4
b) 1/3
c) 1/2
d) 4/5
e) 1
f) 3

Answer: c. 1/2 (half the previous answer because of same horizontal force with a the radius halved)
$h_2 \cdot F_r \cos\theta = 25\sqrt{3} \times 10^5 \approx 4.330 \times 10^6 \text{ m} \cdot \text{N} \approx 4,330,000 \text{ m} \cdot \text{N}$

In addition to having to bear a smaller torque, the outer pier can easily be made thicker and stronger without cluttering space near the windows.

Point total: 112