Machines C Answer Key
UChicago 2021

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Instructions:
Be sure to enter all answers into Scilympiad! For decimal answers, provide at least 3 sig figs if you're unsure. We will not penalize for extra significant figures. Feel free to leave fractions when convenient (but please reduce to lowest form for the sake of grading!). That said, most questions are multiple choice to speed up grading.
You can always assume $g = 9.8$ m/s^2.
We hope you enjoy and learn something new from this test! Best of luck!

1 Warm-up

Question 1 [6 pt]. Give the correct formula for the ideal mechanical advantage of each machine, only using d_{in} (`d_in`), d_{out} (`d_out`), h (`h`), L_{slope} (`L_slope`), L (`L`), $w_{separation}$ (`w_separation`), P (`P`) (pitch), π (`pi`), R_{wheel} (`R_wheel`), r_{axle} (`r_axle`), T_{in} (`T_in`) (teeth), T_{out} (`T_out`). Numbers may also be used if necessary.

1. Lever
2. Inclined Plane
3. Screw
4. Wedge
5. Wheel and Axle
6. Gears

Answer: 1. d_{in}/d_{out} 2. L_{slope}/h 3. $2\pi L/P$ 4. $L/w_{separation}$ 5. R_{wheel}/r_{axle} 6. T_{out}/T_{in}

Question 2 [6 pt]. What is the formula for efficiency of any machine (don't use the terms above)?

Answer: W_{out}/W_{in} or AMA/IMA

Question 3 [6 pt]. Write the letter of the term (left) that best corresponds with each numbered definition (right). Note that not all terms will be used.

a. Efficiency	1) Unit of work.
b. Friction	2) Unit of power.
c. Joule	3) Unit of force.
d. Mechanical Advantage	4) Product of force and velocity.
e. Newton	5) Reduces efficiency.
f. Output	6) Opposes effort.
g. Power	
h. Resistance	
i. Rotation	
j. Watt	
k. Work	

Answer: 1. c — 2. j — 3. e — 4. g — 5. b — 6. h

Question 4 [5 pt]. Each term on the left describes a phrase listed on the right. Write the letter of the correct matching for each phrase on the right.

a. Compound Machine	1) Two rigidly attached wheels of different diameters
b. Inclined Plane	2) Output force divided by input force
c. Lever	3) An inclined plane that moves
d. Mechanical Advantage	4) A rigid rod that rotates about a fulcrum
e. Pulley	5) A flat sloped surface.
f. Simple Machine	
g. Wedge	
h. Wheel and Axle	
i. Work	

Answer: 1. h — 2. d — 3. g — 4. c — 5. b

Question 5 [9 pt]. Determine what type of lever each of the following is. If the item described is not a lever, write the correct type of simple machine.

1. Tweezers
2. Boat Oars
3. Wheelbarrow
4. Nutcracker (the handheld one – ignore the figurine nutcracker)
5. Zipper
6. Hockey Stick (consider the top hand relatively fixed)
7. Bottle Opener
8. Broom
9. See Saw

Answer: 1. Third 2. First 3. Second 4. Second 5. Wedge 6. Third 7. Second 8. Third 9. First

Question 6 [2 pt]. What is the difference between a single start and a multiple start screw?

Answer: Something along the lines of longer lead distance for multiple start screw. Or number of threads (1 vs 2+).

Question 7 [2 pt]. In Figure 1 is the fine-tuner for a violin, which is used to make small adjustments to the force/tension on a string. What simple machine(s) is (are) used here? (if there's a lever, indicate which type; if there's a wheel-and-axle, indicate whether it's a gear)

Answer: Screw, Class 1 lever

Figure 1: Violin fine-tuner (Sources: a b)

2 Build-based theory questions

In lieu of a normal in-person device testing, I hope these theory questions will help give you a deeper understanding of the math behind the build-portion of the event, as well as strategies to maximize your efficiency in the future. (Some of these questions would not be fair to ask during an in-person competition, since everyone does the build portion at different times!)

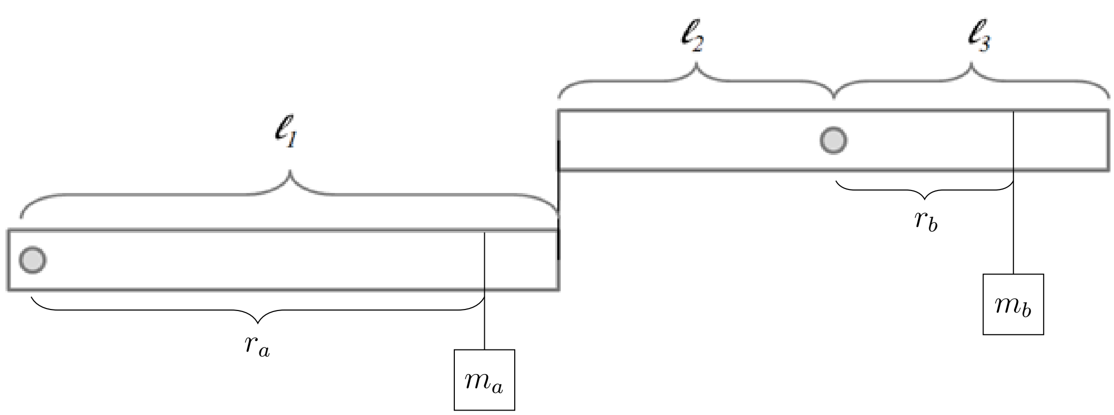

Figure 2: Simple Division C device with no counterweights. Each gray circle is a fulcrum, and a piece string connects the two levers. Note: this is not to scale.

2.1 Device Construction

Consider a device with two levers of equal length ($\ell_1 = \ell_2 + \ell_3$) and uniform density. For simplicity, assume that the fulcrum of the left lever is located perfectly at the left edge (ignore the offset).

Question 8 [10 pt]. What is ℓ_1/ℓ_2 for a balanced lever system when m_a and m_b are not present? Hint: it's a rational value (no square roots or anything nasty).

Moral: no additional counterweights are needed to stabilize levers if they are proportioned carefully (but note that we made several inaccurate approximations/assumptions, so if you use this ratio then you may need small adjustment weights like paperclips).

Answer: 3

2.2 Mass ratio calculation

Suppose we hang weights a (with mass m_a) and b (with mass m_b) at distances r_a and r_b from the left and right levers' fulcra, respectively, such that the system is balanced. Then the mass ratio m_a/m_b is given by:

$$\frac{m_a}{m_b} = \frac{\ell_1}{\ell_2}\frac{r_b}{r_a}$$

If you are unable to calculate ℓ_1/ℓ_2 or are unsure of your result, you can enter answers for later questions in terms of a constant $c = \ell_1/\ell_2$ (e.g. if your answer is $\frac{15}{4}c$, you can write `15/4 c` or `3.75c`).

Question 9 [2 pt]. What is the mass ratio m_a/m_b if $r_a = 15$ cm and $r_b = 35$ cm?
Answer: 7/3 c = 2.33 c = 7

2.3 Error analysis

Error analysis is the process of determining the uncertainty or error of a calculated quantity (e.g. mass ratio) based on uncertainties in the original measurements (e.g. r_b).

We want to find the mass ratio m_a/m_b accurately and quickly, but ensuring accurate measurements of r_a and r_b could take a long time. In this section, we will see how uncertainties on these distances affect the accuracy of the calculated mass ratio.

Again consider the left mass position $r_a = 15.0$, but this time we do not know r_b exactly. Suppose we know that it lies within 1.0 cm of 35 cm. I'll write this as $r_b = 35 \pm 1$ cm.

Question 10 [2 pt]. What is our lower bound on m_a/m_b?
Answer: $34/15c = 34/5 \approx 2.266c = 6.8$

Question 11 [2 pt]. What is our upper bound on m_a/m_b?
Answer: $36/15c = 12/5c = 36/5 \approx 2.400c = 7.2$

Question 12 [4 pt]. Suppose we estimate m_a/m_b using $r_a = 15$ cm and $r_b = 35$ cm. What is the largest possible fractional error of m_a/m_b? Please provide at least 3 significant figures (or a precise fraction). There should be no dependence on the value of c. Hint: this happens when the true mass ratio is at one of the bounds, and the error should be a fraction of the

true mass ratio.
Answer: $\frac{35/15-34/15}{34/15} = 1/34 \approx 0.02941 = 2.941\%$

This time, suppose $r_a = 15 \pm 1$ cm is uncertain and $r_b = 35.0$ cm is known exactly.

Question 13 [2 pt]. What is our lower bound on m_a/m_b?
Answer: $35/16c = 105/16 \approx 2.1875c = 6.5625$

Question 14 [2 pt]. What is our upper bound on m_a/m_b?
Answer: $35/14c = 105/14 \approx 2.5c = 7.5$

Question 15 [4 pt]. Suppose we estimate m_a/m_b using $r_a = 15$ cm and $r_b = 35$ cm. What is the largest possible fractional error of m_a/m_b? Please provide at least 3 significant figures (or a precise fraction). There should be no dependence on the value of c. Hint: this happens when the true mass ratio is at one of the bounds, and the error should be a fraction of the true mass ratio.

Answer: $\frac{35/15-35/16}{35/16} = 1/15 \approx 0.06666 = 6.666\%$

Moral of the story: try to get both weights as far from the fulcum as possible, then leave the closer weight alone (ideally on a whole centimeter/distance marking for better precision).

2.4 Binary Search

Binary search is a technique in computer science that can be used to efficiently find an element in a sorted list. It can also be used to estimate a number, assuming we have a way to compare it to other numbers. For example, we can estimate $\sqrt{2}$ only using squaring and comparison operations. We may not know the value of $\sqrt{2}$, but we can compare an arbitrary number x to $\sqrt{2}$ by comparing x^2 to 2 ($x^2 \leq 2 \iff x \leq \sqrt{2}$).

Maintain a range of possible values for $\sqrt{2}$ – call it $[\ell, u]$ – and compare the middle of this range against 2. Call the middle of the range $m = \frac{\ell+u}{2}$. If $m^2 \leq 2$, then the true value lies on the left side: $[\ell, m]$. Otherwise, the true value lies on the right side: $[m, u]$. In either case, the range of possible values shrinks by a factor of 2 with every square-and-compare step (=the uncertainty is halved).

Suppose we start with the conservative range of $1 \leq \sqrt{2} \leq 2$, or $\sqrt{2} \in [1, 2]$ (the $\in$ symbol means "is in"). After 6 iterations, we'd have an estimate of $\sqrt{2}$ that is correct to two decimal places (the range is length $1/2^6 = 1/64 = 0.01562$, and the estimate lies in the middle of the final range).

This probably sounds very abstract, but we can use the same reasoning to efficiently find the balance point of the build-portion machine (without the need to wait for the levers to balance!!).

2.5 Binary Search for Machines

The natural approach to the build portion of this event is:

1) Place masses on the lever
2) Move around the masses until the device is balanced
3) Read off the positions and plug into the calculator

Step 2 is generally the most time-consuming of the three, especially if your device is very sensitive. However, a useful insight is that you don't need to balance the device if you have a good estimate of what the balance point *would* be.

In this section, we'll analyze how binary search could be used to speed up that second step. For simplicity, leave the left mass in place (r_a is fixed) and only move around the right mass (search for the true r_b). Suppose we've placed the left mass such that we know $20 < r_b < 40$ cm.

Question 16 [2 pt]. If the left lever is tilting down and the right lever is tilting up, should we move mass b to the left or right?
Answer: Right

Question 17 [8 pt]. The worst-case fractional error occurs when the true value of r_b is small – in this case, let $r_b = 20$ cm. Call our estimated value of the position s, and call the uncertainty δ, such that $s - \delta < r_b < s + \delta$. In the terminology of the previous question, $r_b \in [s - \delta, s + \delta]$. What is the largest δ that ensures the estimated mass ratio is within 1% of the true mass ratio?
Ignore the possible restriction of $s > 20$ cm for simplicity (even though we said we know $r_b > 20$ cm). Hint: we want the error as a fraction of the *true* mass ratio (so the c will get canceled out).

Multiple choice:

a) 0.05 cm
b) 0.1 cm
c) 0.2 cm
d) 0.4 cm
e) 0.5 cm
f) 1 cm
g) 2 cm

Answer: 0.2 cm

Constraint: $s - \delta \leq r_b \leq s + \delta$

$$\begin{aligned}
error &= \frac{|(m_a/m_b)_{estimated} - (m_a/m_b)_{true}|}{(m_a/m_b)_{true}} \\
&= \frac{|s - r_b|}{r_b} \\
&\leq \delta / r_b \\
&\leq 0.01 \\
\delta &\leq 0.01\ r_b = 0.2 \text{ cm}
\end{aligned}$$

Question 18 [10 pt]. How many measurements (evaluations of values of r_b) would you need to try in the range [20 cm, 40 cm] to be guaranteed an error in mass ratio of at most 1%? Pretend you have perfect control over r_b.

Multiple choice:

a) 4
b) 5
c) 6
d) 7
e) 8

Answer: c) 6
Initial δ is 10 cm, and we halve this until dipping below 0.2 cm: $\log_2(10/0.2) \approx 5.643$ so 6 queries $(15 \cdot (1/2)^7 \approx 0.117$ is not quite enough).

If you drag the mass across the lever, the number of measurements involved is not very useful (you're constantly making measurements). However, this metric is relevant if you pick up the mass to move it (less friction, better for moving quickly).

2.6 Conclusion

This is a naive approach with grossly simplified analysis, since you will have a feeling of where the mass should lie (and it takes less time to move between close points!). I would actually discourage you from performing binary search in practice. Nevertheless, I find the framework helpful in formalizing intuitive strategies. Here are two main takeaways from this section:
1. Stop trying to get your device to balance perfectly! Every 16 seconds costs you 1% of the maximum event score (or equivalently, it acts like an extra 6.67% error in a mass ratio estimate).
2. By intentionally overshooting the balance point, you can efficiently find r_b while bounding the error of your final mass ratio estimate.

If you got through this section, congratulations on reading through two (probably) new concepts – error analysis and binary search! If you had trouble understanding my explanations, I invite you to look these topics up online after the test!

3 Efficiency and Friction

3.1 One more build-related question

Consider the simple lever for the Division B build portion.

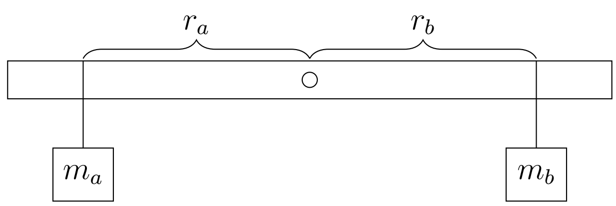

Figure 3: The single class-1 lever in the Division B device

Question 19 [3 pt]. Suppose the fulcrum is a little sticky, so the efficiency of the lever only works out to $19/20 = 0.95$. Consider the case where $m_a = m_b$. What is maximum ratio of distances r_a/r_b (and equivalently mass ratio) we might measure? Note: this is error built into the device and why it's important to have levers that turn smoothly.

Multiple choice:

a) $381/400$
b) $19/20$
c) 1
d) $20/19$
e) $400/381$

Answer: d) $20/19 \approx 1.0526$

3.2 Dangerous rock-climbing

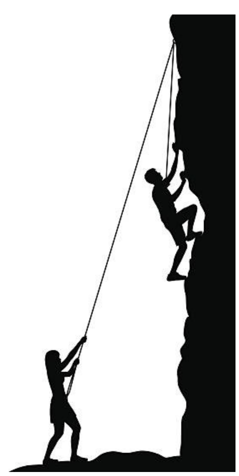

Figure 4: from iStock photos

Augustus is rock-climbing, as shown in Figure 4, belayed by with his wife Livia. From the ground, Livia supports Augustus with a rope that is fed through a single pulley at the top of the cliff (and is attached to Augustus's safety harness). The pulley has an efficiency of 0.5, and Augustus weighs 750 N.

Question 20 [3 pt]. If Augustus slips, what is the most force that Livia can use to pull on the rope that will not save Augustus? (Livia was rumored to have been responsible for Augustus's death by poisoning)

Multiple choice:

a) 112.5 N
b) 325 N
c) 562.5 N
d) 750 N
e) 1125 N
f) 1500 N

Answer: b) 325 N

Question 21 [3 pt]. Suppose Livia lets go completely, but Augustus manages to catch the rope on the opposite side of the pulley. If he wants to pull himself up, how much force will he need to supply? (Still assume he is no longer supported by the rocks) Multiple choice:

a) 112.5 N
b) 325 N
c) 562.5 N
d) 750 N
e) 1125 N
f) 1500 N

Answer: f) 1500 N

3.3 Well, well, well... pulling up a bucket

Question 22 [2 pt]. How much work is expended in raising a 5 kg bucket of water up a 5 m well?

Multiple Choice (rounded):

a) 25 J
b) 123 J
c) 221 J
d) 233 J

e) 245 J
f) 258 J
g) 272 J
h) 490 J

Answer: e) 245 J

Question 23 [3 pt]. How much work is expended if we use pulley system with an IMA of 2, but each of the two pulleys involved has an efficiency of 0.95?

Multiple Choice (rounded):

a) 25 J
b) 123 J
c) 221 J
d) 233 J
e) 245 J
f) 258 J
g) 272 J

Answer: g) $245/(0.95)^2 \approx 271.5$ J

Question 24 [3 pt]. Suppose the pulleys each have a heat capacity of 4.60 J/°C (10 grams of stainless steel). How much hotter do the pulleys get, assuming none of the energy lost to friction goes to the rope or surroundings?
Hint: a simple dimensional analysis guess would give you the right equation of $\Delta T = W/C$, where ΔT is the change in temperature, W is the work, and C is the heat capacity (not specific heat capacity).

Multiple Choice (rounded):

a) 1.33 °C
b) 1.40 °C
c) 2.60 °C
d) 2.88 °C

Answer: d) $\frac{245((1/0.95)^2-1)/2\ \text{J}}{4.60\text{J}/°\text{C}} \approx 2.877°\text{C}$ – need to divide by 2 to split the lost energy across the two pulleys; -1 point for double value

3.4 Slippery cardboard

Question 25 [3 pt]. It can be deceptively tiring to carry light objects with awkward shapes. Consider holding a flat piece of cardboard from the top, as in Figure 5. The piece of cardboard has a mass of 1 kg, and the surface has a coefficient of friction of $\mu_s = 0.3$. How

much force would someone need to supply to keep the cardboard from slipping? Add the (magnitudes of) forces from both sides and respond in Newtons.
Note: the difficulty also comes from the fact that this uses different muscles from carrying something with handles, but that's harder to reason about :).

Multiple Choice:

a) 2.94 N
b) 14.0 N
c) 16.3 N
d) 32.6 N

Answer: d) $\frac{1\text{ kg}\cdot 9.8\text{ m/s}^2}{0.3} = 32.6\text{ N}$

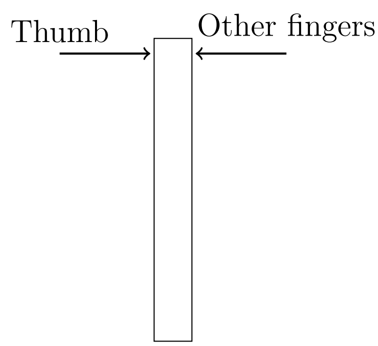

Figure 5: Trying to grip a slippery piece of cardboard.

4 Sisyphus and a boulder

Here is a quick version of the story, in case you have not heard it (for your personal knowledge, and also because we're at UChicago `:P`). Sisyphus was very clever but also mischievous and upset the gods on multiple occasions – he even chained Thanatos (the god of death) to a chair, at which point people on Earth could no longer die. His famous punishment in the afterworld was to push an enormous boulder up a hill. Right before reaching the top, the boulder always slips and rolls to the bottom of the hill so that Sisyphus will never be able to complete the task. Despite this, the absurdist writer Albert Camus says that Sisyphus continues this hopeless task in defiance of the gods...
And now for your tasks, which are very finite and (hopefully) manageable:

First, we will consider applying force to a spherical boulder with radius r on level ground. If we push very short distances, this system looks a lot like a lever (in general wheels are fancy levers).

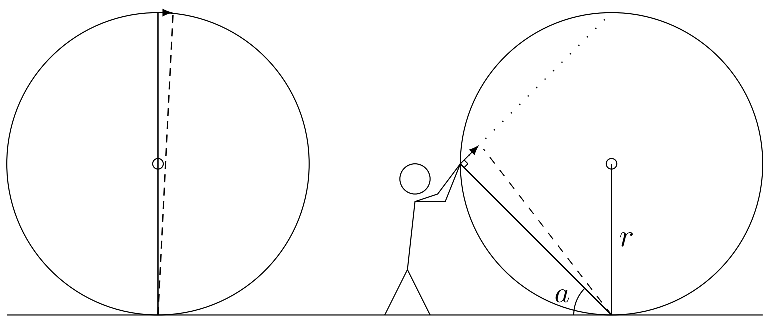

Figure 6: The left boulder is being pushed at the top, while the right boulder is being pushed at an angle a from the bottom.

Question 26 [2 pt]. If we push tangentially at the top, what is the IMA we achieve? Compare against the case of applying a horizontal force to the center of the sphere (this is impossible, but it lets us ignore angular mechanics). Pretend that the boulder rotates about its contact point with the ground for very short distances.
Hint: think about distances rather than forces.

Answer: 2

Question 27 [2 pt]. What class lever does the boulder behave like in this case?

Answer: Class 2

Since Sisyphus is likely shorter than the boulder, we want to see how the IMA changes depending where he pushes the boulder. Consider a contact point at angle a above the ground (as in the right side of Figure 6). For simplicity, suppose the force is directed toward the top of the boulder to ensure a right angle with the line connecting Sisyphus's contact point to the boulder's contact point with the ground.

Question 28 [4 pt]. What is the IMA of this system? Again compare against the case of pushing the center of the sphere.

Multiple choice:

a) $\sin a$
b) $\sin 2a$
c) $2 \sin a$
d) $\cos a$
e) $\cos 2a$
f) $2 \cos a$
g) $\sin a \cos a$

Answer: $2 \sin a$ – could use law of sines: $\sin(\pi/2)/2r = \sin(a)/\ell \implies d_{in}/d_{out} = \ell/r = 2 \sin a$ with ℓ as the unlabeled chord across the boulder connecting Sisyphus to the ground

Question 29 [9 pt]. Say the boulder is 1000 kg, and Sisyphus needs to push the boulder up a hill with angle of inclination 20°. How much force does Sisyphus need to push the boulder at a constant velocity if he pushes from angle $a = 15°$? (free response)

Note the angle a is relative to the hill, not an absolute level plane. Give your answer in Newtons.

Answer: 6475 N
On the slope, you need to provide $mg \sin \theta$ of force to the center of the boulder. Sisyphus just needs to provide

$$\frac{mg \sin \theta}{IMA} = \frac{mg \sin \theta}{2 \sin a} \approx \frac{1000 \text{ kg} \cdot 9.8 \text{m/s}^2 \cdot \sin(20°)}{2 \sin(15°)} \approx 6475.17 \text{ N}$$

of force to deliver the equivalent required force.
But I'm not totally sure because this direction would result in angular acceleration about the center of the boulder?

5 Planetary gear walkthrough

In this section, we'll look at the motion of gears within a planetary gear used almost like a ball bearing (center sun gear fixed, driven planetary carrier).

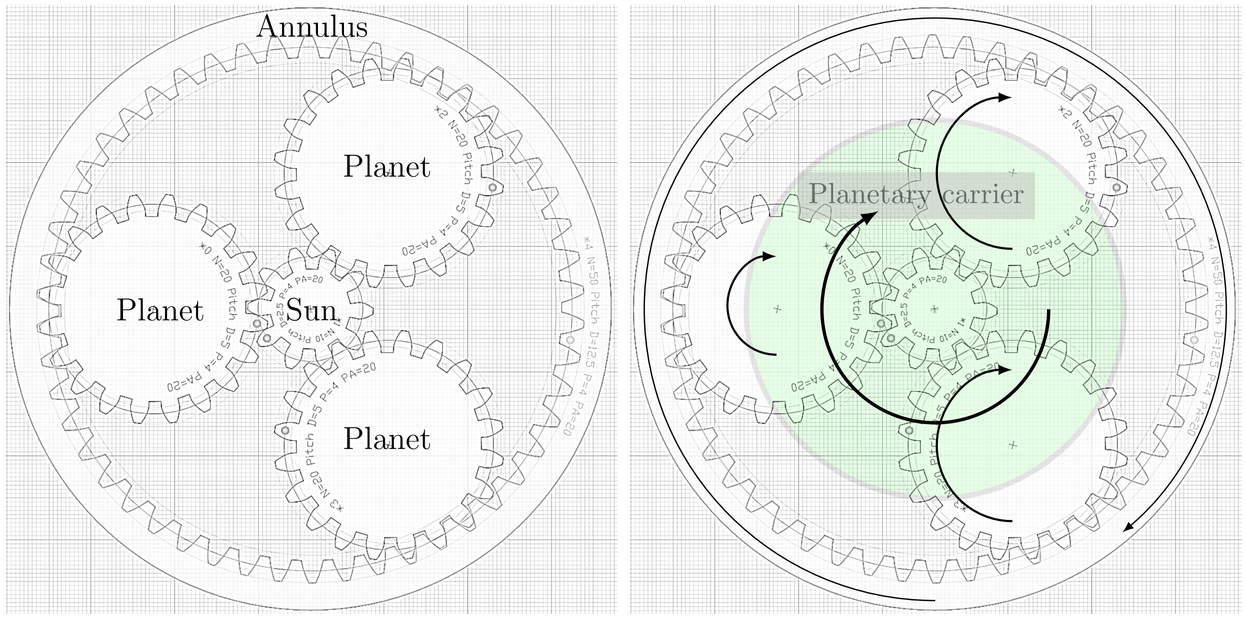

Figure 7: Planetary Gear, created with `geargenerator.com`. Each planet, the annulus, and the planetary carrier (in green) all rotate clockwise. The sun does not turn here.

Answer: Visualize here

Terminology:

- Sun: center gear - N_s teeth, radius r_s
- Planets: outer circular gears (connected with the sun) - N_p teeth, radius r_p
- Planetary carrier: structure holding the planets together - radius $(r_p + r_s)$
- Annulus: outer ring - N_a teeth, radius r_a

All the teeth are the same size, so $N_s/r_s = N_p/r_p = N_a/r_a$. For this problem we will consider the case where $r_s : r_p : r_a$ is 1:2:5.

I find it tricky to visualize the gear motion as the planetary carrier moves. After the questions, I suggest an optional approach intended to help break down the analysis.

5.1 Questions

Question 30 [5 pt]. For each revolution of the planetary carrier, how many revolutions does each planet gear turn relative to the page?

Multiple choice:

a) 2/3
b) 5/6
c) 1
d) 6/5
e) 3/2
f) 2
g) 5/2
h) 15/2

Answer: e) $\frac{15/2}{5} = 3/2 = 1.5$

Question 31 [5 pt]. For each revolution of the planetary carrier, how many revolutions does the annulus turn? (this is the gear ratio)

Multiple choice:

a) 2/3
b) 5/6
c) 1
d) 6/5
e) 3/2
f) 2
g) 5/2
h) 15/2

Answer: d) $6/5 = 1.2$

Question 32 [5 pt]. If the planetary carrier is driven with a torque of 120 $\mathrm{N} \cdot \mathrm{m}$, what is the torque that the annulus can deliver?

Multiple choice:

a) 80 N·m
b) 100 N·m
c) 120 N·m
d) 144 N·m
e) 180 N·m

Answer: b) 100 $\mathrm{N} \cdot \mathrm{m}$ – Even though the annulus has a different radius, the angular velocity is increased by 6/5, which corresponds to a drop in force of 5/6 for fixed radius and therefore 5/6 in torque.

5.2 Suggested walkthrough

I suggest that, for now, we pretend to drive the system by the annulus and keep the planetary carrier fixed. This means that the sun will end up rotating. After we've calculated the relative amounts that each gear rotates, we can rotate back into the frame where the sun is stationary by subtracting the sun's number of rotations from the other gears' numbers of rotations.

Below is a table to help keep track of the motion. The top row means: I rotate the annulus by one revolution, which results in $2\pi r_a$ of movement at a radius of r_a (different radius for each gear). You don't need to do the angle column if you don't want.

Keep in mind that the sun gear will rotate counter-clockwise, so we give it a negative path length, angle, and number of revolutions for convenience (allows us to add/subtract this quantity from everything else).

Gear	Path length turned	Angle	# of revolutions	# revs (sun-subtracted)
Annulus ($r_a = 5$)	$2\pi r_a = 10\pi$	2π	1	
Planets ($r_p = 2$)	$2\pi r_a = 10\pi$			
Sun ($r_s = 1$)	$-2\pi r_a = -10\pi$			0
Planetary Carrier	0	0	0	

Answer:

Gear	Path length	Angle	# of revolutions	# revs (sun-subtracted)
Annulus ($r_a = 5$)	$2\pi r_a$	2π	1	$1 + r_a/r_s = 6$
Planets ($r_p = 2$)	$2\pi r_a$	$2\pi r_a/r_p$	$r_a/r_p = 5/2$	$r_a/r_p + r_a/r_s = 15/2$
Sun ($r_s = 1$)	$-2\pi r_a$	$-2\pi r_a/r_s$	$-r_a/r_s = -5$	0
Planetary Carrier	0	0	0	$r_a/r_s = 5$

Now you have the ratios of the numbers of revolutions that the annulus, planets, and planetary carrier take, so it should be a matter of division to compute the requested values.

6 Architecture/engineering-inspired questions

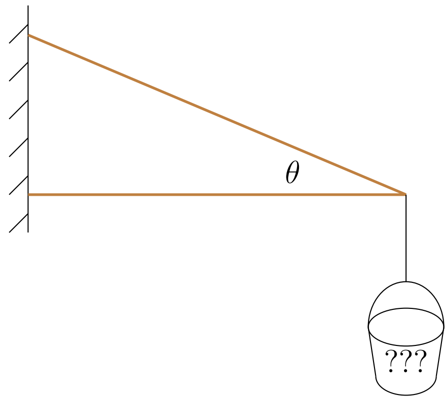

Figure 8: Simple cantilever

6.1 Cantilever

Question 33 [4 pt]. In Figure 8 is a cantilever with just two beams at an angle of angle $\theta = 25°$. If the top beam breaks when it carries 100 N of force, but the bottom beam does not break, what is the maximum load the cantilever can hold? (assume Earth gravity as usual)

Multiple choice (rounded):

a) 4.31 kg
b) 4.76 kg
c) 9.25 kg
d) 11.3 kg
e) 21.9 kg
f) 24.1 kg
g) 42.3 kg

Answer: a) $100 \sin\theta = 42.26$ N $\implies$ 4.31 kg

Question 34 [5 pt]. What is the maximum load if the bottom beam breaks when 100 N is placed on its tip, assuming the top beam does not break? (again in kilograms)

Multiple choice (rounded):

a) 4.31 kg
b) 4.76 kg
c) 9.25 kg
d) 11.3 kg
e) 21.9 kg

f) 24.1 kg
g) 42.3 kg

Answer: b) $100\text{ N} \cdot \tan 25° \approx 46.63\text{ N} \implies 4.76\text{ kg}$
Let the top bar carry force h and the bottom carry $h\cos\theta$ ($= 100$ N when it breaks). This means that we're interested in the horizontal force, $h\sin\theta = (h\cos\theta)\tan\theta = 100\text{ N} \cdot \tan\theta$

6.2 Arch

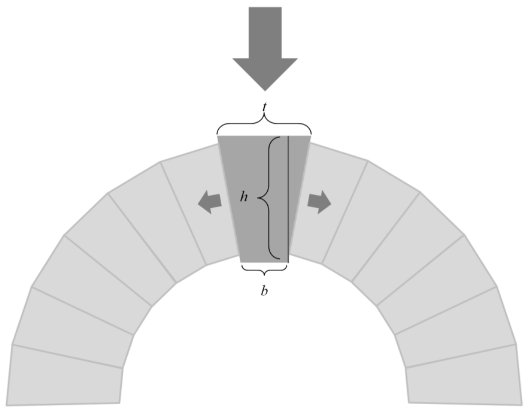

Figure 9: An arch's keystone redirects force to each side.

Question 35 [5 pt]. If an arch in Figure 9 needs to support a force F from above, how much horizontal force does it impart on each neighboring brick?

Multiple choice:

a) $\frac{2(t-b)}{hF}$
b) $\frac{2(t-b)F}{h}$
c) $\frac{(t-b)F}{2h}$
d) $\frac{hF}{2(t-b)}$
e) $\frac{hF}{t-b}$

Answer: e) $\frac{h}{t-b}F$ (Give $F/2$ to two wedges, each with IMA $\frac{t-b}{2h}$)

6.3 Flying Buttress

The buttress was a common architectural device to help walls to support heavy vaulted roofs, especially of churches and cathedrals. The flying buttress is a flashy variant that helps provide horizontal support while allowing the main building to get more light (by moving the support structures farther out).

For this question, we'll consider a simplified model of the flying buttress to get a rough idea of how this can be useful. In Figure 11, the roof's weight is responsible for a force

Figure 10: Several views of the Notre-Dame in Paris from the back, plus a profile view of a flying buttress. (Sources: a b c)

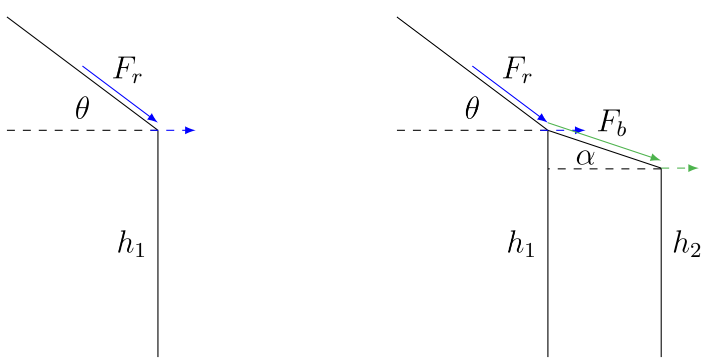

Figure 11: Simplified diagrams of a wall supporting a roof with and without a flying buttress.

F_r that acts on the walls at an angle $\theta = 30°$ from the horizontal. Suppose $F_r = 10^6$ N $= 1,000,000$ N.

Question 36 [3 pt]. What is the mass of the roof being supported by this buttress? (consider just the half of the roof seen in the diagram)

Multiple choice (rounded):

a) 1.18×10^5 kg
b) 1.77×10^5 kg
c) 2.04×10^5 kg
d) 5.10×10^5 kg
e) 8.84×10^5 kg

Answer: c) $F_r/\sin\theta = 1000/(1/2) = 2000$ kN$/(9.8$ N/m$) \approx 204,000$ kg

Question 37 [3 pt]. Given $F_r = 10^6$ N (as before), $\theta = 30°$, and $h_1 = 10$ m, what is the torque the inner buttress experiences if there is no flying buttress?

Multiple choice (rounded):

a) 5.77×10^5 m · N
b) 8.66×10^5 m · N
c) 1.73×10^6 m · N
d) 4.33×10^6 m · N
e) 5.00×10^6 m · N
f) 5.77×10^6 m · N
g) 8.66×10^6 m · N
h) 1.73×10^7 m · N

Answer: g) $h_1 \cdot F_r \cos\theta = 10 \cdot 10^6 \cdot \sqrt{3}/2 = 5\sqrt{3} \times 10^6 \approx 8.66 \times 10^6$ m · N $= 8,660,000$ m · N

Question 38 [4 pt]. Now suppose all the horizontal force is offloaded from the original buttress (the one with height h_1) through the flying buttress. Then if $h_2 = 5$ m, how much torque is applied to the outer pier (the pillar with height h_2)?

Hint: the value of α should not actually matter for this value but would influence the vertical force transferred; moreover, you do not need to calculate F_b. It is intended to help visualization.

Multiple choice (rounded):

a) 5.77×10^5 m · N
b) 8.66×10^5 m · N
c) 1.73×10^6 m · N
d) 4.33×10^6 m · N

e) 5.00×10^6 m · N

f) 5.77×10^6 m · N

g) 8.66×10^6 m · N

h) 1.73×10^7 m · N

Answer: (half the previous answer)
$h_2 \cdot F_r \cos\theta = 25\sqrt{3} \times 10^5 \approx 4.330 \times 10^6 \text{ m} \cdot \text{N} \approx 4,330,000 \text{ m} \cdot \text{N}$

In addition to having to bear a smaller torque, the outer pier can easily be made thicker and stronger without cluttering space near the windows.

Point total: 160