Sensitivity Analysis

Introduction

- Sensitivity analysis (or post-optimality analysis) is used to determine how the optimal solution is affected by changes, within specified ranges, in:

1. The objective function coefficients
2. The right-hand side (RHS) values

- Sensitivity analysis is important to the manager who must operate in a dynamic environment with imprecise estimates of the coefficients.

Example 1

$Max\ Z = 5X_1 + 7X_2$

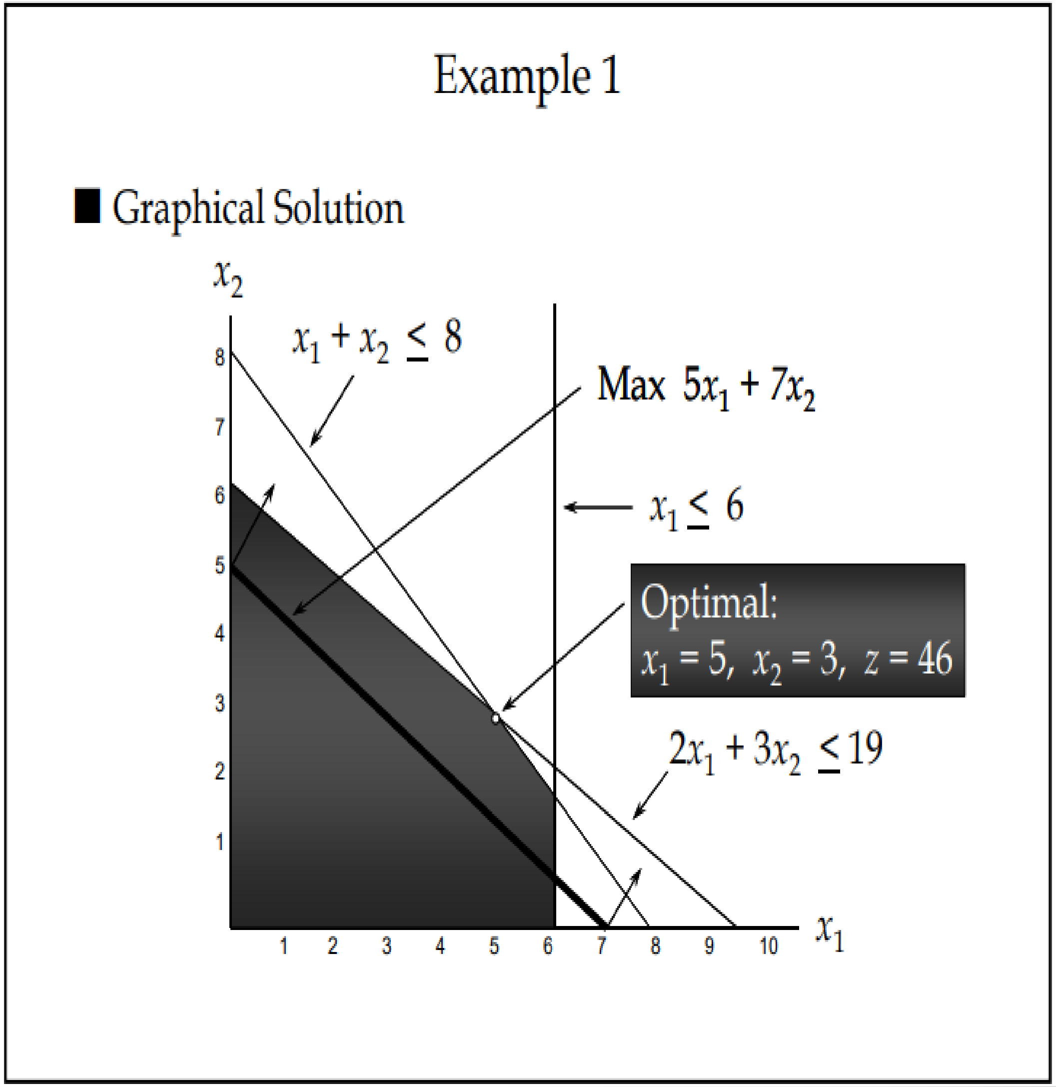
Subject to

$X_1 \leq 6 \ldots\ldots(1)$

$2X_1 + 3X_2 \leq 19 \ldots\ldots(2)$

$X_1 + X_2 \leq 8 \ldots\ldots(3)$

$X_1, X_2 \geq 0 \ldots\ldots(4)$



Objective Function Coefficients

- Let us consider how changes in the objective function coefficients might affect the optimal solution.
- The range of optimality for each coefficient provides the range of values over which the current solution will remain optimal.
- Managers should focus on those objective coefficients that have a narrow range of optimality and coefficients near the endpoints of the range.
- **Graphically**, the limits of a range of optimality are found by changing the slope of the objective function line within the limits of the slopes of the binding constraint lines.
- **The slope of an objective function line**,

  $Max\ Z = c_1 X_1 + c_2 X_2$, is $-c_1/c_2$
- **The slope of a constraint**, $a_1 X_1 + a_2 X_2$= b, is -a_1/a_2

- **Range of Optimality for c_1**

The slope of the objective function line is $-c_1/c_2$. The slope of the first binding constraint, $x_1 + x_2 = 8$, is -1 and the slope of the second binding constraint, $2x_1 + 3x_2 = 19$, is -2/3.

Find the range of values for c_1 (with c_2 staying 7) such that the objective function line slope lies between that of the two binding constraints:

$$-1 \leq -c_1/7 \leq -2/3$$

Multiplying through by -7 (and reversing the inequalities):

$$\boxed{14/3 \leq c_1 \leq 7}$$

■ Range of Optimality for c_2

Find the range of values for c_2 (with c_1 staying 5) such that the objective function line slope lies between that of the two binding constraints:

$$-1 \leq -5/c_2 \leq -2/3$$

Multiplying by -1: $\quad 1 \geq 5/c_2 \geq 2/3$

Inverting, $\quad 1 \leq c_2/5 \leq 3/2$

Multiplying by 5: $\quad \boxed{5 \leq c_2 \leq 15/2}$

Right-Hand Sides

- Let us consider how a change in the right-hand side for a constraint might affect the feasible region and perhaps cause a change in the optimal solution.
- The **improvement** in the value of the optimal solution per unit **increase** in the right-hand side is called the **dual price**.
- The range of feasibility is the range over which the dual price is applicable.
- As the RHS increases, other constraints will become binding and limit the change in the value of the objective function.
- **Graphically**, a dual price is determined by adding +1 to the right hand side value in question and then resolving for the optimal solution in terms of the same two binding constraints.

- **The dual price** is equal to the difference in the values of the objective functions between the new and original problems.

$$\textbf{Dual price} = \boldsymbol{Z_{new} - Z_{old}}$$

- The dual price for a **nonbinding** constraint is 0.
- A negative dual price indicates that the objective function will not improve if the RHS is increased

■ Dual Prices

Constraint 1: Since $x_1 \leq 6$ is not a binding constraint, its dual price is 0.

Constraint 2: Change the RHS value of the second constraint to 20 and resolve for the optimal point determined by the last two constraints:
$2x_1 + 3x_2 = 20$ and $x_1 + x_2 = 8$.

The solution is $x_1 = 4$, $x_2 = 4$, $z = 48$. Hence, the dual price = $z_{new} - z_{old} = 48 - 46 = 2$.

- **Dual Prices**

Constraint 3: Change the RHS value of the third constraint to 9 and resolve for the optimal point determined by the last two constraints: $2x_1 + 3x_2 = 19$ and $x_1 + x_2 = 9$.

The solution is: $x_1 = 8, x_2 = 1, z = 47$.

The dual price is $z_{new} - z_{old} = 47 - 46 = 1$.