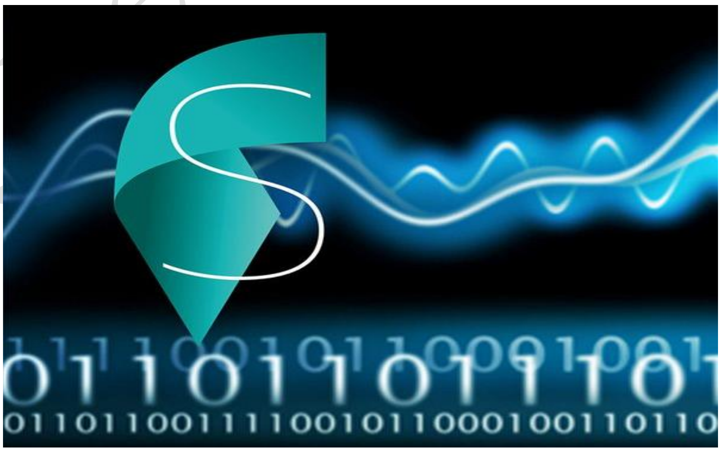
CYBER SECURITY TECHNOLOGY ENGINEERING DEPARTMENT

DIGITAL SIGNAL PROCESSING

THIRD STAGE

Lect.6,7 Analysis of Discrete Linear Time-Invariant Systems.



Asst. Lect. Haider Saad

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Difference Equations and Impulse Responses

Format of Difference Equation

A causal (The system is called causal because the current output depends only on the current input and past inputs and outputs, not on the future), linear time-invariant (The system is called Linear Time-Invariant (LTI) because it is linear and constant with time) system can be described by a difference equation having the following general form

$$\boxed{\begin{aligned} &y(n) + a_1 y(n-1) + \ldots + a_N y(n-N) \\ &= b_0 x(n) + b_1 x(n-1) + \ldots + b_M x(n-M), \end{aligned}} \qquad \textbf{(1)}$$

where $a_1, \ldots, a_N$ and $b_0, b_1, \ldots,$ b are the coefficients of the difference equation. Equation (1) can further be written as

$$\boxed{\begin{aligned} y(n) = &-a_1 y(n-1) - \ldots - a_N y(n-N) \\ &+ b_0 x(n) + b_1 x(n-1) + \ldots + b_M x(n-M) \end{aligned}} \qquad \textbf{(2)}$$

Or

$$\boxed{y(n) = -\sum_{i=1}^{N} a_i y(n-i) + \sum_{j=0}^{M} b_j x(n-j).} \qquad \textbf{(3)}$$

Notice that y(n) is the current output, which depends on the past output samples y(n _ 1), . . . , y(n _ N), the current input sample x(n), and the past input samples, x(n _ 1), . . . , x(n _ N).

We will examine the specific difference equations in the following examples.

Example

Given the following difference equation:

$$\boxed{y(n) = 0.25y(n-1) + x(n),}$$

Identify the nonzero system coefficients.

Solution:

Comparison with Equation (2) leads to

$$\boxed{\begin{aligned} b_0 &= 1 \\ -a_1 &= 0.25, \end{aligned}}$$

System Representation Using Its Impulse Response:

A linear time-invariant system can be completely described by its unit-impulse response, which is defined as the system response due to the impulse input δ(n) with zero initial conditions, depicted in Figure (1).

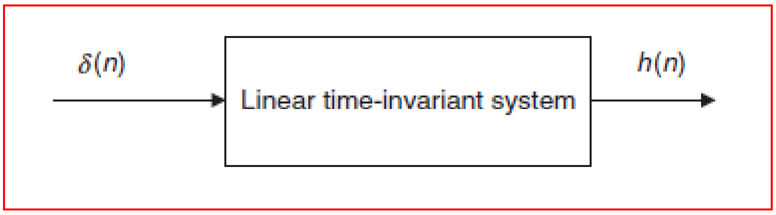


Figure (1): Unit-impulse response of the linear time-invariant system.

With the obtained unit-impulse response h(n), we can represent the linear time-invariant system in Figure (2).

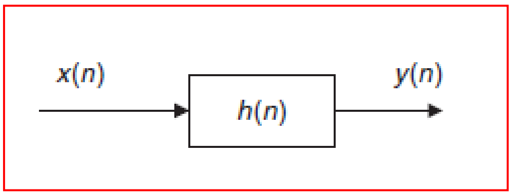
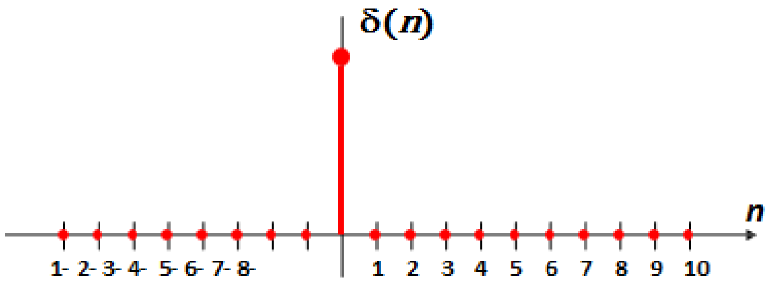


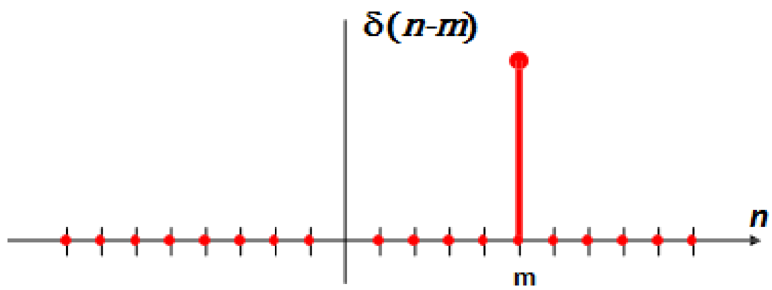
Figure (2): Representation of a linear time-invariant system using the impulse response.

- **delta function or unit-impulse (sample) sequence *δ(n)***

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

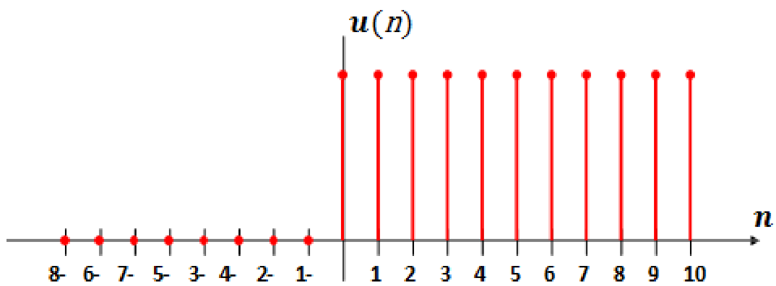


$$\delta(n\text{-}m) = \begin{cases} 1 & n = \mathrm{m} \\ 0 & n \neq \mathrm{m} \end{cases}$$

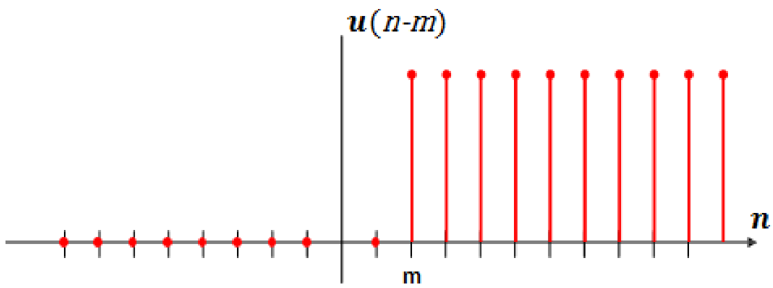


- **unit-step sequence *U(n)***

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



$$u(n\text{-}m) = \begin{cases} 1 & n \geq \mathrm{m} \\ 0 & n < \mathrm{m} \end{cases}$$



Example:

Given the linear time-invariant system

y(n) = 0.5x(n) + 0.25x(n - 1) with an initial condition *x(-1) = 0,*

a) Determine the unit-impulse response *h(n).*
b) Draw the system block diagram.
c) Write the output using the obtained impulse response.

Solution:

a) According to Figure 1, let $x(n) = \delta(n)$, then

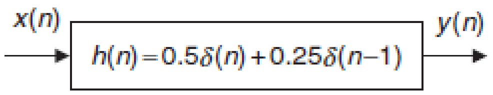
$$\boxed{h(n) = y(n) = 0.5x(n) + 0.25x(n-1) = 0.5\delta(n) + 0.25\delta(n-1).}$$

Thus, for this particular linear system, we have

$$\boxed{h(n) = \begin{cases} 0.5 & n = 0 \\ 0.25 & n = 1 \\ 0 & elsewhere \end{cases}}$$

note
unit response is h(n
unit sequence is d(n

b) The block diagram of the linear time-invariant system is shown as



c) The system output can be rewritten as

$$\boxed{y(n) = h(0)x(n) + h(1)x(n-1).}$$

In general, we can express the output sequence of a linear time-invariant system from its impulse response and inputs as

$$\boxed{y(n) = \ldots + h(-1)x(n+1) + h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + \ldots .} \quad \textbf{(3)}$$

Equation (3) is called the digital convolution sum, which will be explored in a later section. We can verify Equation (3) by substituting the impulse sequence $x(n) = \delta(n)$ to get the impulse response

$$h(n) = \ldots + h(-1)\delta(n+1) + h(0)\delta(n) + h(1)\delta(n-1) + h(2)\delta(n-2) + \ldots,$$

Example:

Given the difference equation

$$y(n) = 0.25y(n-1) + x(n) \text{ for } n \geq 0 \text{ and } y(-1) = 0,$$

a) Determine the unit-impulse response *h(n).*
b) Draw the system block diagram.
c) Write the output using the obtained impulse response.
d) For a step input *x(n) = u(n),* verify and compare the output responses for the first three output samples using the difference equation and digital convolution sum (Equation 3).

Solution:

a) Let $x(n) = \delta(n)$, then

$$h(n) = 0.25h(n-1) + \delta(n).$$

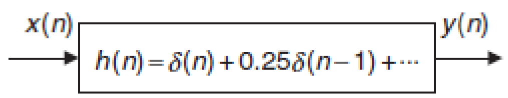
To solve for *h(n),* we evaluate

$$h(0) = 0.25h(-1) + \delta(0) = 0.25 \times 0 + 1 = 1$$
$$h(1) = 0.25h(0) + \delta(1) = 0.25 \times 1 + 0 = 0.25$$
$$h(2) = 0.25h(1) + \delta(2) = 0.25 \times 0.25 + 0 = 0.0625$$

With the calculated results, we can predict the impulse response as

$$h(n) = \delta(n) + 0.25\delta(n-1) + 0.0625\delta(n-2) + \ldots.$$

b) The system block diagram is given in Figure below.



c) The output sequence is a sum of infinite terms expressed as

$$\boxed{\begin{aligned} y(n) &= h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + \ldots \\ &= x(n) + 0.25x(n-1) + 0.0625x(n-2) + \ldots \end{aligned}}$$

d) From the difference equation and using the zero-initial condition, we have

$$\boxed{\begin{aligned} &y(n) = 0.25y(n-1) + x(n) \text{ for } n \geq 0 \text{ and } y(-1) = 0 \\ &n = 0, y(0) = 0.25y(-1) + x(0) = u(0) = 1 \\ &n = 1, y(1) = 0.25y(0) + x(1) = 0.25 \times u(0) + u(1) = 1.25 \\ &n = 2, y(2) = 0.25y(1) + x(2) = 0.25 \times 1.25 + u(2) = 1.3125 \\ &\ldots. \end{aligned}}$$

Applying the convolution sum in Equation (3) yields

$$\boxed{\begin{aligned} y(n) &= x(n) + 0.25x(n-1) + 0.0625x(n-2) + \ldots \\ n = 0,\ y(0) &= x(0) + 0.25x(-1) + 0.0625x(-2) + \ldots \\ &= u(0) + 0.25 \times u(-1) + 0.125 \times u(-2) + \ldots = 1 \\ n = 1,\ y(1) &= x(1) + 0.25x(0) + 0.0625x(-1) + \ldots \\ &= u(1) + 0.25 \times u(0) + 0.125 \times u(-1) + \ldots = 1.25 \\ n = 2,\ y(2) &= x(2) + 0.25x(1) + 0.0625x(0) + \ldots \\ &= u(2) + 0.25 \times u(1) + 0.0625 \times u(0) + \ldots = 1.3125 \end{aligned}}$$