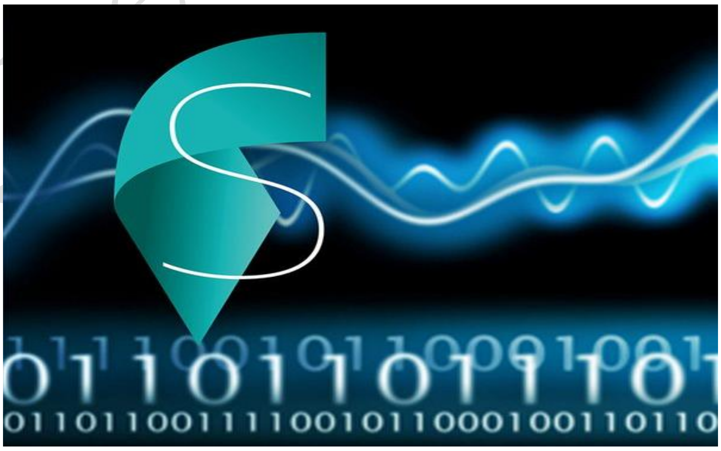
CYBER SECURITY TECHNOLOGY ENGINEERING DEPARTMENT

DIGITAL SIGNAL PROCESSING

THIRD STAGE

Lect.8 Digital Convolution and Deconvolution.



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Digital Convolution

Given a linear time-invariant system, we can determine its unit-impulse response *h(n)*, which relates the system input and output. To find the output sequence *y(n)* for any input sequence *x(n)*, we write the digital convolution as shown in Equation (3) as:

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$
$$= \ldots + h(-1)x(n+1) + h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + \ldots$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$
$$= \ldots + x(-1)h(n+1) + x(0)h(n) + x(1)h(n-1) + x(2)h(n-2) + \ldots \qquad \textbf{(4)}$$

Using a conventional notation, we express the digital convolution as

$$y(n) = h(n) * x(n). \qquad \textbf{(5)}$$

Note that for a causal system, which implies its impulse response

$$h(n) = 0 \text{ for } n < 0,$$

The lower limit of the convolution sum begins at 0 instead of 1, that is

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k) = \sum_{k=0}^{\infty} x(k)h(n-k).$$

We will focus on evaluating the convolution sum based on Equation (4). Let us examine first a few outputs from Equation (4):

$$y(0) = \sum_{k=-\infty}^{\infty} x(k)h(-k) = \ldots + x(-1)h(1) + x(0)h(0) + x(1)h(-1) + x(2)h(-2) + \ldots$$
$$y(1) = \sum_{k=-\infty}^{\infty} x(k)h(1-k) = \ldots + x(-1)h(2) + x(0)h(1) + x(1)h(0) + x(2)h(-1) + \ldots$$
$$y(2) = \sum_{k=-\infty}^{\infty} x(k)h(2-k) = \ldots + x(-1)h(3) + x(0)h(2) + x(1)h(1) + x(2)h(0) + \ldots$$
$$\ldots$$

Digital convolution using the graphical method.

Step 1. Obtain the reversed sequence $h(-k)$.

Step 2. Shift $h(-k)$ by $|n|$ samples to get $h(n-k)$. If $n \geq 0$, $h(-k)$ will be shifted to the right by n samples; but if $n < 0$, $h(-k)$ will be shifted to the left by $|n|$ samples.

Step 3. Perform the convolution sum that is the sum of the products of two sequences $x(k)$ and $h(n-k)$ to get $y(n)$.

Step 4. Repeat steps 1 to 3 for the next convolution value $y(n)$.

Example

Given a sequence,

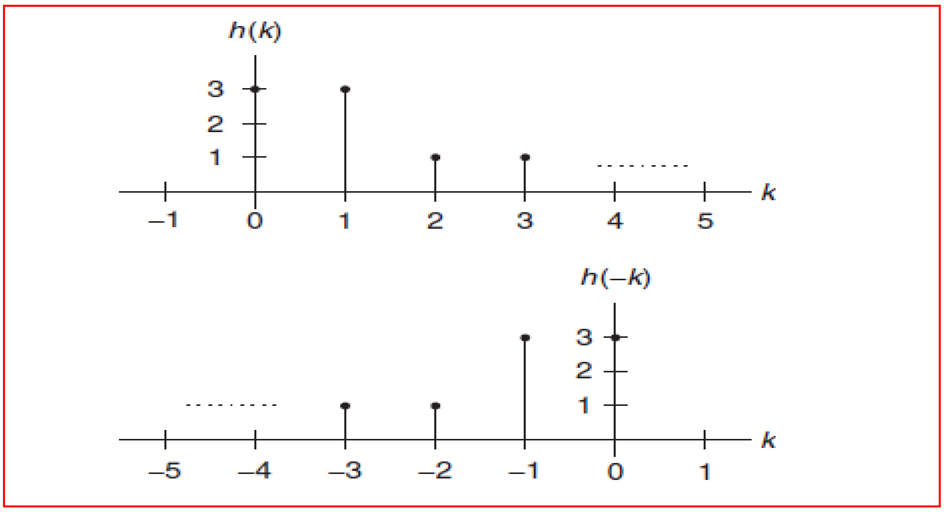
$$h(k) = \begin{cases} 3, & k = 0,1 \\ 1, & k = 2,3 \\ 0 & elsewhere \end{cases}$$

where k is the time index or sample number,

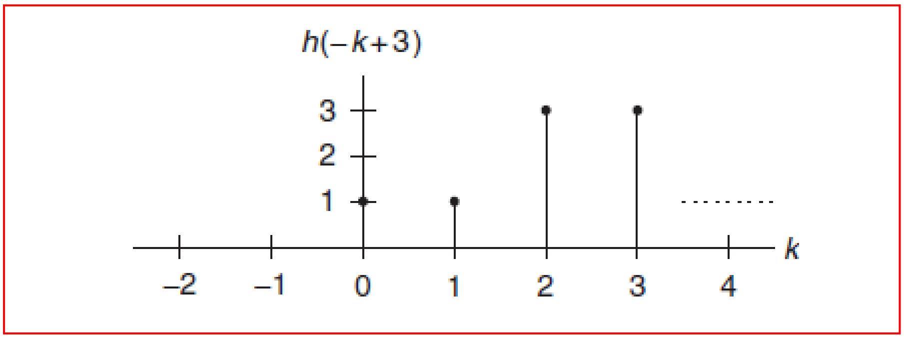
a) Sketch the sequence *h(k)* and reversed sequence h(- k).
b) Sketch the shifted sequences *h(-k +3)* and *h(- k - 2).*

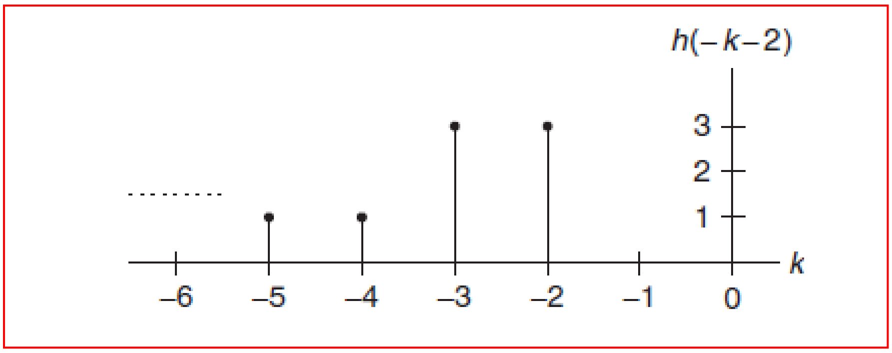
Solution:

a)



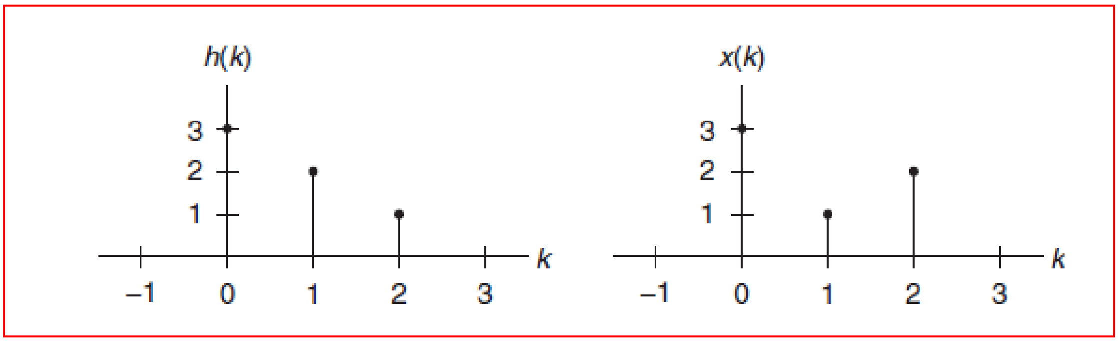
b)





Example

Using the following sequences defined in the Figure below, evaluate the digital convolution

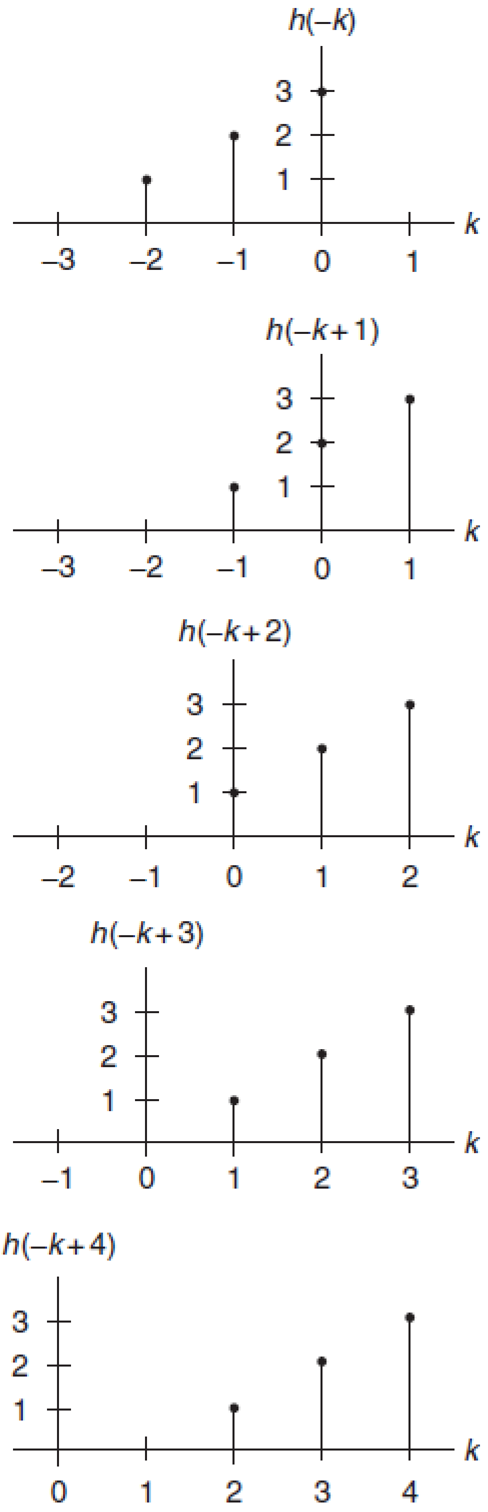


$$h(k) = [\underset{\uparrow \atop 0}{3} \quad 2 \quad 1] \qquad\qquad x(k) = [\underset{\uparrow \atop 0}{3} \quad 1 \quad 2]$$

a) By the graphical method.
b) By applying the formula directly.

Solution:

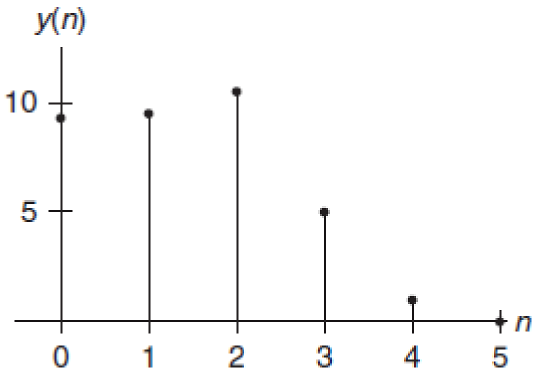
a)



h(−k)
3
2
1
k
−3
−2
−1
0
1
h(−k+1)
3
2
1
k
−3
−2
−1
0
1
h(−k+2)
3
2
1
k
−2
−1
0
1
2
h(−k+3)
3
2
1
k
−1
0
1
2
3
h(−k+4)
3
2
1
k
0
1
2
3
4

We can compute the convolution sum as:

sum of product of $x(k)$ and $h(-k)$: $y(0) = 3 \times 3 = 9$
sum of product of $x(k)$ and $h(1-k)$: $y(1) = 1 \times 3 + 3 \times 2 = 9$
sum of product of $x(k)$ and $h(2-k)$: $y(2) = 2 \times 3 + 1 \times 2 + 3 \times 1 = 11$
sum of product of $x(k)$ and $h(3-k)$: $y(3) = 2 \times 2 + 1 \times 1 = 5$
sum of product of $x(k)$ and $h(4-k)$: $y(4) = 2 \times 1 = 2$
sum of product of $x(k)$ and $h(5-k)$: $y(n) = 0$ for $n > 4$, since sequences $x(k)$ and $h(n-k)$ do not overlap.



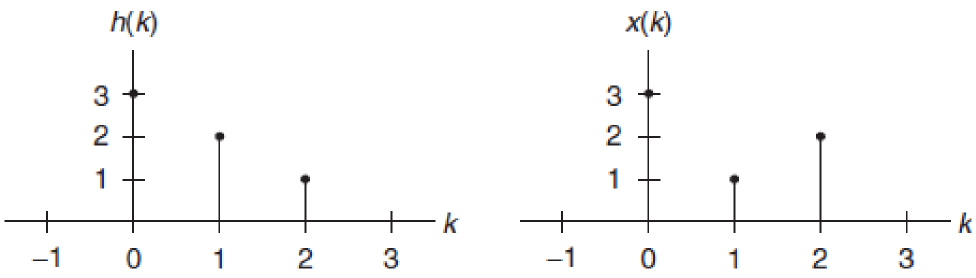
b) Applying Equation (4) with zero initial conditions leads to

$$y(n) = x(0)h(n) + x(1)h(n-1) + x(2)h(n-2)$$

$$\begin{aligned}
&n = 0,\ y(0) = x(0)h(0) + x(1)h(-1) + x(2)h(-2) = 3 \times 3 + 1 \times 0 + 2 \times 0 = 9,\\
&n = 1,\ y(1) = x(0)h(1) + x(1)h(0) + x(2)h(-1) = 3 \times 2 + 1 \times 3 + 2 \times 0 = 9,\\
&n = 2,\ y(2) = x(0)h(2) + x(1)h(1) + x(2)h(0) = 3 \times 1 + 1 \times 2 + 2 \times 3 = 11,\\
&n = 3,\ y(3) = x(0)h(3) + x(1)h(2) + x(2)h(1) = 3 \times 0 + 1 \times 1 + 2 \times 2 = 5.\\
&n = 4,\ y(4) = x(0)h(4) + x(1)h(3) + x(2)h(2) = 3 \times 0 + 1 \times 0 + 2 \times 1 = 2,\\
&n \geq 5,\ y(n) = x(0)h(n) + x(1)h(n-1) + x(2)h(n-2) = 3 \times 0 + 1 \times 0 + 2 \times 0 = 0.
\end{aligned}$$

Example:

Using the following sequences defined in the Figure below, evaluate the digital convolution



Convolve them using the table method.

Solution:

Digital convolution steps via the table.

Step 1. List the index k covering a sufficient range.

Step 2. List the input $x(k)$.

Step 3. Obtain the reversed sequence $h(-k)$, and align the rightmost element of $h(n-k)$ to the leftmost element of $x(k)$.

Step 4. Cross-multiply and sum the nonzero overlap terms to produce $y(n)$.

Step 5. Slide $h(n-k)$ to the right by one position.

Step 6. Repeat step 4; stop if all the output values are zero or if required.

Convolution sum using the table method.

k:	-2	-1	0	1	2	3	4	5	
$x(k)$:			**3**	**1**	**2**				
$h(-k)$:	1	2	3						$y(0) = 3 \times 3 = 9$
$h(1-k)$		1	2	3					$y(1) = 3 \times 2 + 1 \times 3 = 9$
$h(2-k)$			1	2	3				$y(2) = 3 \times 1 + 1 \times 2 + 2 \times 3 = 11$
$h(3-k)$				1	2	3			$y(3) = 1 \times 1 + 2 \times 2 = 5$
$h(4-k)$					1	2	3		$y(4) = 2 \times 1 = 2$
$h(5-k)$						1	2	3	$y(5) = 0$ (no overlap)

Deconvolution:

The digital Deconvolution can be performed by ***Iterative Approach***, ***Polynomial Approach***, and ***Graphical Method***. In the following subsection, the polynomial approach will be explained.

<u>Polynomial Approach:</u>

A long division process is applied between two polynomials. For causal system, the remainder is always *zero*.

Example: If $y(n)$ = [15 -8 -5 2] and h(n) = [-3 1] find x(n) .

Solution: $y = 15 - 8x - 5x^2 + 2x^3$, and h = $-3 + x$. Applying long division, we obtain

$$
\begin{array}{r|l}
 & 2x^2 + x - 5 \\
\hline
x-3 & 2x^3 - 5x^2 - 8x + 15 \\
 & \underline{2x^3 - 6x^2} \\
 & \quad\quad x^2 - 8x \\
 & \quad\quad \underline{x^2 - 3x} \\
 & \quad\quad\quad -5x + 15 \\
 & \quad\quad\quad \underline{-5x + 15} \\
 & \text{Remainder} \quad 0
\end{array}
$$

result = $-5 + x + 2x^2$. Then $x(n)$ = [-5 1 2]

Example: If $y(n)$ = [12 10 14 6] and h(n) = [4 2] find x(n) .

Solution: $y = 12 + 10x + 14x^2 + 6x^3$, and h = $4 + 2x$. Applying long division, we obtain

result = $3 + x + 3x^2$. Then $x(n)$ = [3 1 3]