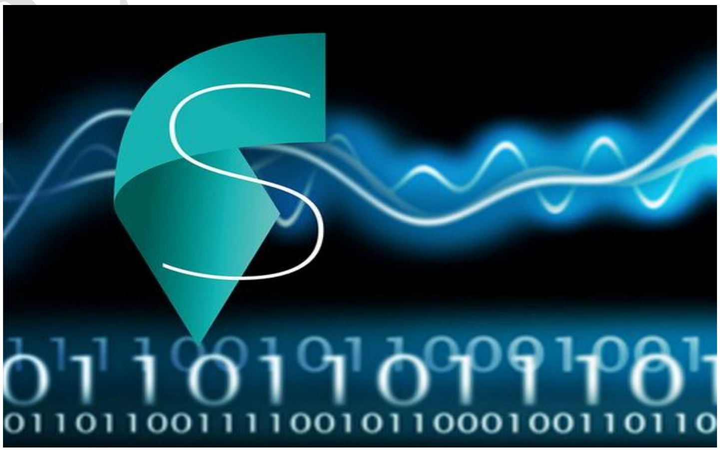
CYBER SECURITY TECHNOLOGY ENGINEERING DEPARTMENT

DIGITAL SIGNAL PROCESSING

THIRD STAGE

Lect.2 The Concept of Frequency in Continuous and Discrete Time Signal



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1. Introduction

Frequency represents the number of cycles a signal completes per unit of time. It plays a central role in understanding signal behavior in both continuous and discrete domains. In continuous-time, frequency is measured in Hz (cycles/second), while in discrete-time, it is measured in cycles per sample. Frequency is crucial in analyzing sinusoidal and periodic signals.

2. Continuous-Time Sinusoidal Signals

A continuous-time sinusoidal signal is represented by:

$x_a(t) = A\cos(\Omega t + \theta)$ (Eq. 1.3.1)

$$x_a(t) = A\cos(\Omega t + \theta), \quad -\infty < t < \infty \qquad (1.3.1)$$

where:

- A is the amplitude of the sinusoid,
- Ω is the angular frequency (radians/sec),
- θ is the phase (radians).
- $_a$ represent analog signal.

The angular frequency is related to the frequency in Hz (F) by:

$\Omega = 2\pi F$ (Eq. 1.3.2)

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Thus:

$x_a(t) = A\cos(2\pi F t + \theta)$ (Eq. 1.3.3)

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We will use both forms, (1.3.1) and (1.3.3), in representing sinusoidal signals.

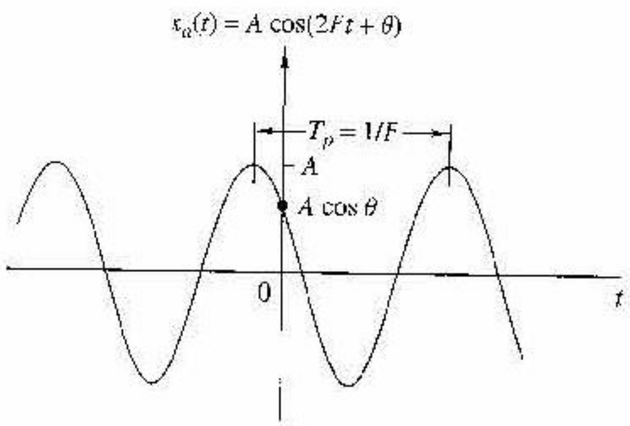

Figure 1.3.1
Example of an analog sinusoidal signal.

2.1 Properties of Continuous-Time Sinusoids

A1. For every fixed value of the frequency F, $x_a(t)$ is periodic. Indeed, it can easily be shown, using elementary trigonometry, that

$$x_a(t + T_p) = x_a(t)$$

where $T_p = 1/F$ is the fundamental period of the sinusoidal signal.

A2. Continuous-time sinusoidal signals with distinct (different) frequencies are themselves distinct

A3. Increasing the frequency F results in an increase in the rate of oscillation of the signal, in the sense that more periods are included in a given time interval.

We observe that for $F = 0$, the value $T_p = \infty$ is consistent with the fundamental relation $F = 1/T_p$. Due to continuity of the time variable t, we can increase the frequency F, without limit, with a corresponding increase in the rate of oscillation.

The relationships we have described for sinusoidal signals carry over to the class of complex exponential signals

$$x_a(t) = Ae^{j(\Omega t+\theta)} \tag{1.3.4}$$

This can easily be seen by expressing these signals in terms of sinusoids using the Euler identity

$$e^{\pm j\phi} = \cos\phi \pm j\sin\phi \tag{1.3.5}$$

By definition, frequency is an inherently positive physical quantity. This is obvious if we interpret frequency as the number of cycles per unit time in a periodic signal. However, in many cases, only for mathematical convenience, we need to introduce negative frequencies. To see this we recall that the sinusoidal signal (1.3.1) may be expressed as

$$x_a(t) = A\cos(\Omega t + \theta) = \frac{A}{2}\,e^{j(\Omega t+\theta)} + \frac{A}{2}\,e^{-j(\Omega t+\theta)} \tag{1.3.6}$$

- Reason for using this representation: Math becomes much simpler when dealing with exponents instead of sine/cosine (especially in transformations and spectral analysis).

3. Discrete-Time Sinusoidal Signals

A discrete-time sinusoid is given by:

$x(n) = A\cos(\omega n + \theta)$ (Eq. 1.3.7)

ω is the frequency in radians/sample. If we use **normalized frequency** f:

$\omega = 2\pi f$ (Eq. 1.3.8), so

$x(n) = A\cos(2\pi f n + \theta)$ (Eq. 1.3.9)

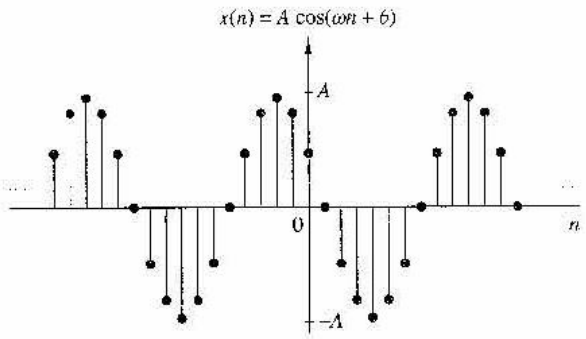

Figure 1.3.3
Example of a discrete-time sinusoidal signal ($\omega = \pi/6$ and $\theta = \pi/3$).

3.1 Properties of Discrete-Time Sinusoids

In contrast to continuous-time sinusoids, the discrete-time sinusoids are characterized by the following properties:

B1. *A discrete-time sinusoid is periodic only if its frequency f is a rational number.*

By definition, a discrete-time signal $x(n)$ is periodic with period $N (N > 0)$ if and only if

$$x(n+N) = x(n) \qquad \text{for all } n \tag{1.3.10}$$

The smallest value of N for which (1.3.10) is true is called the *fundamental period*.

The proof of the periodicity property is simple. For a sinusoid with frequency f_0 to be periodic, we should have

$$\cos[2\pi f_0(N+n)+\theta] = \cos(2\pi f_0 n + \theta)$$

This relation is true if and only if there exists an integer k such that

$$2\pi f_0 N = 2k\pi$$

or, equivalently,

$$f_0 = \frac{k}{N} \tag{1.3.11}$$

According to (1.3.11), a discrete-time sinusoidal signal is periodic only if its frequency f_0 can be expressed as the ratio of two integers (i.e., f_0 is rational).

To determine the fundamental period N of a periodic sinusoid, we express its frequency f_0 as in (1.3.11) and cancel common factors so that k and N are relatively prime. Then the fundamental period of the sinusoid is equal to N. Observe that a small change in frequency can result in a large change in the period. For example, note that $f_1 = 31/60$ implies that $N_1 = 60$, whereas $f_2 = 30/60$ results in $N_2 = 2$.

B2. *Discrete-time sinusoids whose frequencies are separated by an integer multiple of 2π are identical.*

To prove this assertion, let us consider the sinusoid $\cos(\omega_0 n + \theta)$. It easily follows that

$$\cos[(\omega_0 + 2\pi)n + \theta] = \cos(\omega_0 n + 2\pi n + \theta) = \cos(\omega_0 n + \theta) \tag{1.3.12}$$

As a result, all sinusoidal sequences

$$x_k(n) = A\cos(\omega_k n + \theta), \qquad k = 0, 1, 2, \ldots \tag{1.3.13}$$

where

$$\omega k = \omega_0 + 2k\pi, \qquad -\pi \le \omega_0 \le \pi$$

are *indistinguishable* (i.e., *identical*). Any sequence resulting from a sinusoid with a frequency $|\omega| > \pi$, or $|f| > \frac{1}{2}$, is identical to a sequence obtained from a sinusoidal signal with frequency $|\omega| < \pi$. Because of this similarity, we call the sinusoid having the frequency $|\omega| > \pi$ an *alias* of a corresponding sinusoid with frequency $|\omega| < \pi$. Thus we regard frequencies in the range $-\pi \le \omega \le \pi$, or $-\frac{1}{2} \le f \le \frac{1}{2}$, as unique and all frequencies $|\omega| > \pi$, or $|f| > \frac{1}{2}$, as aliases. The reader should notice the difference between discrete-time sinusoids and continuous-time sinusoids, where the latter result in distinct signals for Ω or F in the entire range $-\infty < \Omega < \infty$ or $-\infty < F < \infty$.

B3. *The highest rate of oscillation in a discrete-time sinusoid is attained when $\omega = \pi$ (or $\omega = -\pi$) or, equivalently, $f = \frac{1}{2}$ (or $f = -\frac{1}{2}$).*

To illustrate this property, let us investigate the characteristics of the sinusoidal signal sequence

$$x(n) = \cos \omega_0 n$$

when the frequency varies from 0 to π. To simplify the argument, we take values of $\omega_0 = 0, \pi/8, \pi/4, \pi/2, \pi$ corresponding to $f = 0, \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}$, which result in periodic sequences having periods $N = \infty, 16, 8, 4, 2$, as depicted in Fig. 1.3.4. We note that the period of the sinusoid decreases as the frequency increases. In fact, we can see that the rate of oscillation increases as the frequency increases.

To see what happens for $\pi \le \omega_0 \le 2\pi$, we consider the sinusoids with frequencies $\omega_1 = \omega_0$ and $\omega_2 = 2\pi - \omega_0$. Note that as ω_1 varies from π to 2π, ω_2 varies from π to 0. It can be easily seen that

$$\begin{aligned} x_1(n) &= A\cos\omega_1 n = A\cos\omega_0 n \\ x_2(n) &= A\cos\omega_2 n = A\cos(2\pi - \omega_0)n \\ &= A\cos(-\omega_0 n) = x_1(n) \end{aligned} \tag{1.3.14}$$

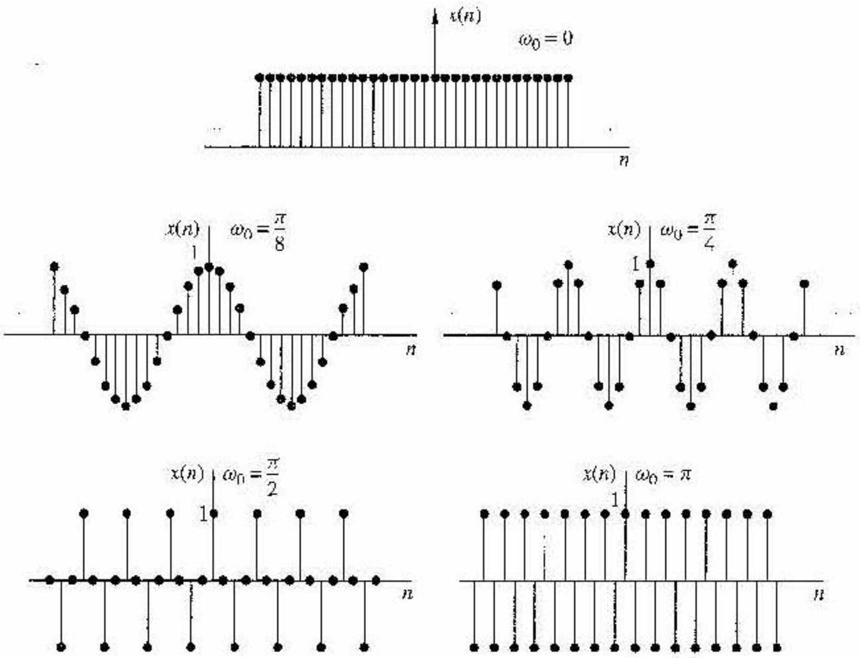

Figure 1.3.4 Signal $x(n) = \cos\omega_0 n$ for various values of the frequency ω_0.

Hence ω_2 is an alias of ω_1. If we had used a sine function instead of a cosine function, the result would basically be the same, except for a 180° phase difference between the sinusoids $x_1(n)$ and $x_2(n)$. In any case, as we increase the relative frequency ω_0 of a discrete-time sinusoid from π to 2π, its rate of oscillation decreases. For $\omega_0 = 2\pi$ the result is a constant signal, as in the case for $\omega_0 = 0$. Obviously, for $\omega_0 = \pi$ (or $f = \frac{1}{2}$) we have the highest rate of oscillation.

As for the case of continuous-time signals, negative frequencies can be introduced as well for discrete-time signals. For this purpose we use the identity

$$x(n) = A\cos(\omega n + \theta) = \frac{A}{2}e^{j(\omega n+\theta)} + \frac{A}{2}e^{-j(\omega n+\theta)} \tag{1.3.15}$$

4. Harmonically Related Complex Exponentials

These are complex exponentials with frequencies that are integer multiples of a fundamental frequency F_0.

Continuous-time: $S_k(t) = e^{(j2\pi kF_0t)}$ (Eq. 1.3.16)

Discrete-time: $S_k(n) = e^{(j2\pi kn/N)}$ (Eq. 1.3.18)

A periodic signal can be written as:

$x_a(t) = \Sigma\ ck * S_k(t)$ (Eq. 1.3.17)

$x(n) = \Sigma\ ck * S_k(n)$ (Eq. 1.3.20)

EXAMPLE 1.3.1

Stored in the memory of a digital signal processor is one cycle of the sinusoidal signal

$$x(n) = \sin\left(\frac{2\pi n}{N} + \theta\right)$$

where $\theta - 2\pi q/N$, where q and N are integers

(a) Determine how this table of values can be used to obtain values of harmonically related sinusoids having the same phase

(b) Determine how this table can be used to obtain sinusoids of the same frequency but different phase.

Solution.

(a) Let $x_k(n)$ denote the sinusoidal signal sequence

$$x_k(n) - \sin\left(\frac{2\pi nk}{N} + \theta\right)$$

This is a sinusoid with frequency $f_k = k/N$, which is harmonically related to $x(n)$. But $x_k(n)$ may be expressed as

$$x_k(n) = \sin\left[\frac{2\pi(kn)}{N} + \theta\right]$$
$$= x(kn)$$

Thus we observe that $x_k(0) = x(0)$, $x_k(1) = x(k)$, $x_k(2) = x(2k)$, and so on. Hence the sinusoidal sequence $x_k(n)$ can be obtained from the table of values of $x(n)$ by taking every kth value of $x(n)$, beginning with $x(0)$. In this manner we can generate the values of all harmonically related sinusoids with frequencies $f_k = k/N$ for $k = 0, 1, \ldots, N-1$

(b) We can control the phase θ of the sinusoid with frequency $f_k = k/N$ by taking the first value of the sequence from memory location $q = \theta N/2\pi$, where q is an integer Thus the initial phase θ controls the starting location in the table and we wrap around the table each time the index (kn) exceeds N.