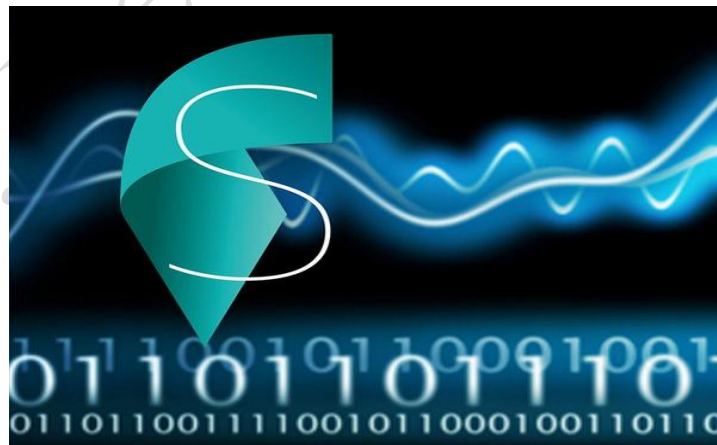# CYBER SECURITY TECHNOLOGY ENGINEERING DEPARTMENT

DIGITAL SIGNAL PROCESSING

Third Stage
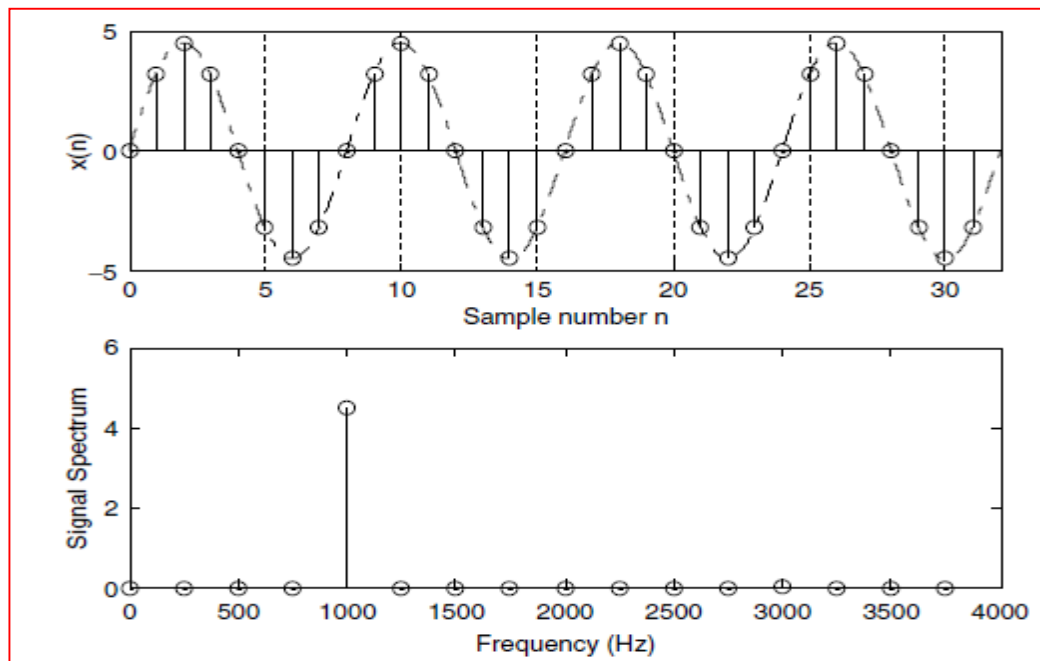## Lect.9 Discrete Fourier Transform

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Discrete Fourier Transform

Figure below illustrates the time domain representation of a 1,000-Hz sinusoid with 32 samples at a sampling rate of 8,000 Hz; the bottom plot shows the signal spectrum (frequency domain representation), where we can clearly observe that the amplitude peak is located at the frequency of 1,000 Hz in the calculated spectrum. Hence, the spectral plot better displays frequency information of a digital signal.


The algorithm transforming the time domain signal samples to the frequency domain components is known as the discrete Fourier transform, or DFT.

The DFT also establishes a relationship between the time domain representation and the frequency domain representation. Therefore, we can apply the DFT to perform frequency analysis of a time domain sequence.

Fourier Series Coefficients of Periodic Digital Signals

Let us look at a process in which we want to estimate the spectrum of a periodic digital signal $x(n)$ sampled at a rate of f_s Hz with the fundamental period $T_0 = NT$, as shown in Figure below.
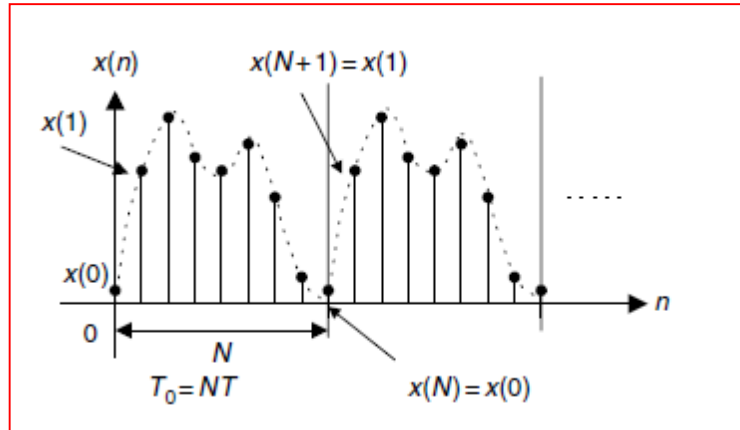
According to Fourier series analysis, the coefficients of the Fourier series expansion of a periodic signal $x(t)$ in a complex form is

$$c_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt \qquad -\infty < k < \infty,$$

where k is the number of harmonics corresponding to the harmonic frequency of kf_0 and $\omega_0 = 2\pi/T_0$ and $f_0 = 1/T_0$ are the fundamental frequency in radians per second and the fundamental frequency in Hz, respectively.

To apply above Equation, we substitute $T_0 = NT$, $\omega_0 = 2\pi/T_0$ and approximate the integration over one period using a summation by substituting $dt = T$ and $t = nT$. We obtain

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi kn}{N}}, \qquad -\infty < k < \infty.$$

Since the coefficients c_k are obtained from the Fourier series expansion in the complex form, the resultant spectrum c_k will have two sides. There is an important feature of above Equation in which the Fourier series coefficient c_k is periodic of N. We can verify this as follows

$$c_{k+N} = \frac{1}{N} \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi(k+N)n}{N}} = \frac{1}{N} \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi kn}{N}} e^{-j2\pi n}.$$

Since $e^{-j2\pi n} = \cos(2\pi n) - j\sin(2\pi n) = 1$, it follows that

$$c_{k+N} = c_k.$$

Therefore, the two-sided line amplitude spectrum $|c_k|$ is periodic, as shown in

Figure below.
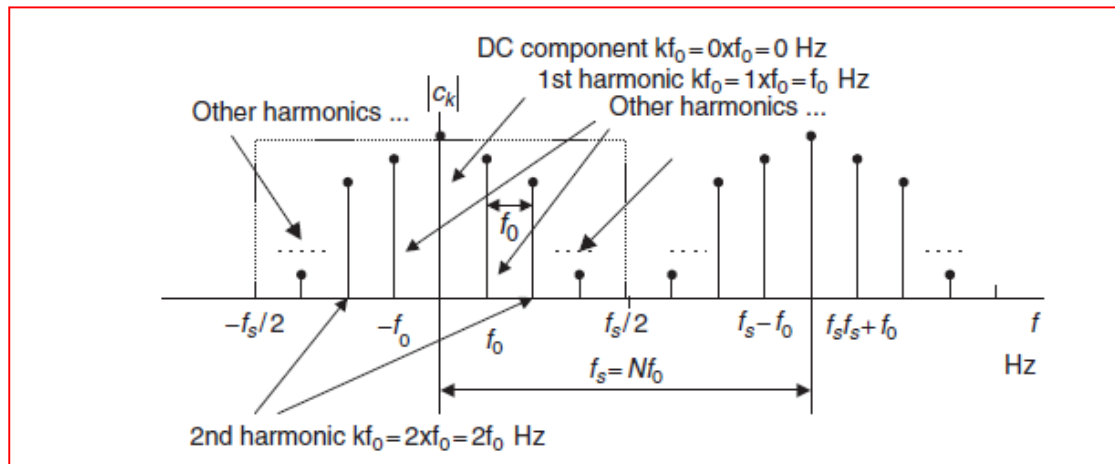
For convenience, we compute the spectrum over the range from 0 to f_s Hz with nonnegative indices, that is,

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi kn}{N}}, \quad k = 0, 1, \ldots, N-1.$$

Example

The periodic signal

$$x(t) = \sin(2\pi t)$$

is sampled using the rate $f_s = 4$ Hz.

    a. Compute the spectrum c_k using the samples in one period.

    b. Plot the two-sided amplitude spectrum $|c_k|$ over the range from -2 to 2 Hz.

Solution:

    a. From the analog signal, we can determine the fundamental frequency ω_0 = 2π radians per second and $f_0 = \frac{\omega_0}{2\pi} = \frac{2\pi}{2\pi} = 1 Hz$, and the fundamental period $T_0 = 1$ second. Since using the sampling interval $T = 1/f_s = 0.25$ second, we get the sampled signal as

$$x(n) = x(nT) = \sin(2\pi nT) = \sin(0.5\pi n)$$

and plot the first eight samples as shown in Figure
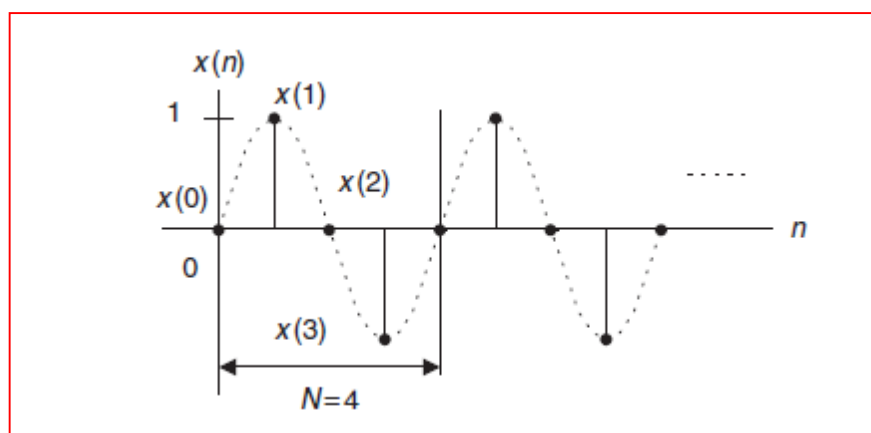
Choosing the duration of one period, $N = 4$, we have the sample values as follows

$$x(0) = 0; \; x(1) = 1; \; x(2) = 0; \; \text{and} \; x(3) = -1.$$

$$c_0 = \frac{1}{4}\sum_{n=0}^{3} x(n) = \frac{1}{4}(x(0) + x(1) + x(2) + x(3)) = \frac{1}{4}(0 + 1 + 0 - 1) = 0$$

$$c_1 = \frac{1}{4}\sum_{n=0}^{3} x(n)e^{-j2\pi \times 1n/4} = \frac{1}{4}\left(x(0) + x(1)e^{-j\pi/2} + x(2)e^{-j\pi} + x(3)e^{-j3\pi/2}\right)$$
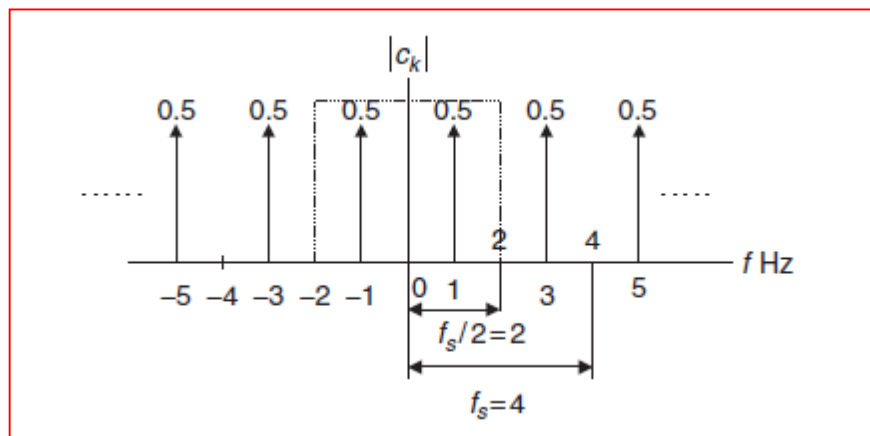
$$= \frac{1}{4}(x(0) - jx(1) - x(2) + jx(3) = 0 - j(1) - 0 + j(-1)) = -j0.5.$$

Similarly, we get

$$c_2 = \frac{1}{4}\sum_{k=0}^{3} x(n)e^{-j2\pi \times 2n/4} = 0, \text{ and } c_3 = \frac{1}{4}\sum_{n=0}^{3} x(k)e^{-j2\pi \times 3n/4} = j0.5.$$

Using periodicity, it follows that

b. The amplitude spectrum for the digital signal is sketched in Figure

As we know, the spectrum in the range of -2 to 2 Hz presents the information of the sinusoid with a frequency of 1 Hz and a peak value of $2|c_1| = 1$ which is converted from two sides to one side by doubling the spectral value. Note that we do not double the direct-current (DC) component, that is, c_0.

Discrete Fourier Transform Formulas

Now, let us concentrate on development of the DFT. Figure below shows one way to obtain the DFT formula.
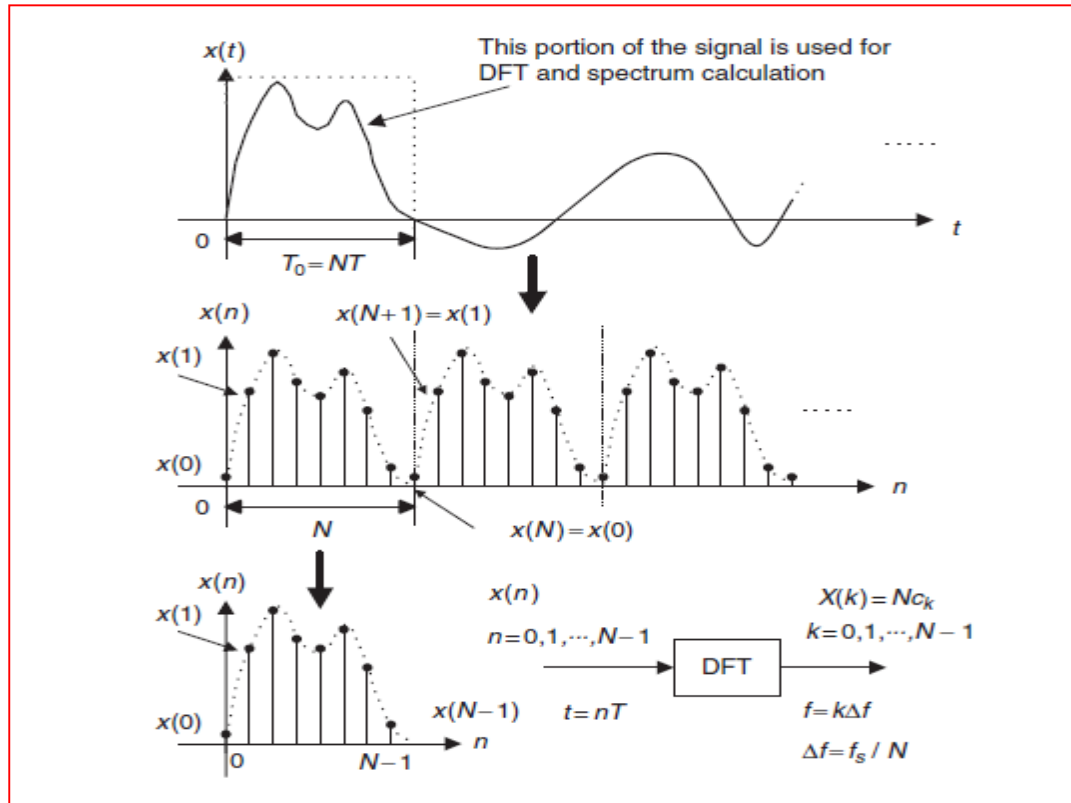
First, we assume that the process acquires data samples from digitizing the interested continuous signal for duration of T seconds. Next, we assume that a

periodic signal $x(n)$ is obtained by copying the acquired N data samples with the duration of T to itself repetitively.

We determine the Fourier series coefficients using one-period N data samples then we multiply the Fourier series coefficients by a factor of N to obtain

$$X(k) = Nc_k = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi kn}{N}}, \quad k = 0, 1, \ldots, N-1,$$

Now let us conclude the DFT definition. Given a sequence x(n), $0 \leq n \leq N - 1$, its DFT is defined as

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} = \sum_{n=0}^{N-1} x(n)W_N^{kn}, \text{ for } k = 0, 1, \ldots, N-1.$$

Above equation can be expanded as

$$X(k) = x(0)\, W_N^{k0} + x(1)\, W_N^{k1} + x(2)W_N^{k2} + \ldots + x(N-1)W_N^{k(N-1)},$$
$$\text{for } k = 0, 1, \ldots, N-1,$$

Where the factor W_N (called the twiddle factor in some textbooks) is defined as

$$W_N = e^{-j2\pi/N} = \cos\left(\frac{2\pi}{N}\right) - j\sin\left(\frac{2\pi}{N}\right).$$

The inverse DFT is given by

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi kn/N} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-kn}, \text{ for } n = 0,1, \ldots, N-1.$$

Similarly

$$x(n) = \frac{1}{N}\left(X(0)W_N^{-0n} + X(1)W_N^{-1n} + X(2)W_N^{-2n} + \ldots + X(N-1)W_N^{-(N-1)n}\right),$$
$$\text{for } n = 0, 1, \ldots, N-1.$$

Example

Given a sequence $x(n)$ for $0 \leq n \leq 3$, where $x(0) = 1$, $x(1) = 2$, $x(2) = 3$, and $x(3) = 4$,

Evaluate its DFT X(k).

Solution:

$$X(k) = \sum_{n=0}^{3} x(n) W_4^{kn} = \sum_{n=0}^{3} x(n) e^{-j\frac{\pi k n}{2}}.$$

Thus, for $k = 0$

$$
\begin{aligned}
X(0) &= \sum_{n=0}^{3} x(n) e^{-j0} = x(0)e^{-j0} + x(1)e^{-j0} + x(2)e^{-j0} + x(3)e^{-j0} \\
&= x(0) + x(1) + x(2) + x(3) \\
&= 1 + 2 + 3 + 4 = 10
\end{aligned}
$$

for $k = 1$

$$
\begin{aligned}
X(1) &= \sum_{n=0}^{3} x(n) e^{-j\frac{\pi n}{2}} = x(0)e^{-j0} + x(1)e^{-j\frac{\pi}{2}} + x(2)e^{-j\pi} + x(3)e^{-j\frac{3\pi}{2}} \\
&= x(0) - jx(1) - x(2) + jx(3) \\
&= 1 - j2 - 3 + j4 = -2 + j2
\end{aligned}
$$

for $k = 2$

$$
\begin{aligned}
X(2) &= \sum_{n=0}^{3} x(n) e^{-j\pi n} = x(0)e^{-j0} + x(1)e^{-j\pi} + x(2)e^{-j2\pi} + x(3)e^{-j3\pi} \\
&= x(0) - x(1) + x(2) - x(3) \\
&= 1 - 2 + 3 - 4 = -2
\end{aligned}
$$

and for $k = 3$

$$
\begin{aligned}
X(3) &= \sum_{n=0}^{3} x(n) e^{-j\frac{3\pi n}{2}} = x(0)e^{-j0} + x(1)e^{-j\frac{3\pi}{2}} + x(2)e^{-j3\pi} + x(3)e^{-j\frac{9\pi}{2}} \\
&= x(0) + jx(1) - x(2) - jx(3) \\
&= 1 + j2 - 3 - j4 = -2 - j2
\end{aligned}
$$

Example

Using the DFT coefficients X(k) for $0 \leq k \leq 3$ computed in previous example

Evaluate its inverse DFT to determine the time domain sequence x(n).

Solution:

$$x(n) = \frac{1}{4} \sum_{k=0}^{3} X(k) W_4^{-nk} = \frac{1}{4} \sum_{k=0}^{3} X(k) e^{j\frac{\pi kn}{2}}.$$

Then for $n = 0$

$$x(0) = \frac{1}{4} \sum_{k=0}^{3} X(k) e^{j0} = \frac{1}{4} \left(X(0) e^{j0} + X(1) e^{j0} + X(2) e^{j0} + X(3) e^{j0} \right)$$

$$= \frac{1}{4} (10 + (-2 + j2) - 2 + (-2 - j2)) = 1$$

for $n = 1$

$$x(1) = \frac{1}{4} \sum_{k=0}^{3} X(k) e^{j\frac{k\pi}{2}} = \frac{1}{4} \left(X(0) e^{j0} + X(1) e^{j\frac{\pi}{2}} + X(2) e^{j\pi} + X(3) e^{j\frac{3\pi}{2}} \right)$$

$$= \frac{1}{4} (X(0) + jX(1) - X(2) - jX(3))$$

$$= \frac{1}{4} (10 + j(-2 + j2) - (-2) - j(-2 - j2)) = 2$$

for $n = 2$

$$x(2) = \frac{1}{4} \sum_{k=0}^{3} X(k) e^{jk\pi} = \frac{1}{4} \left(X(0) e^{j0} + X(1) e^{j\pi} + X(2) e^{j2\pi} + X(3) e^{j3\pi} \right)$$

$$= \frac{1}{4} (X(0) - X(1) + X(2) - X(3))$$

$$= \frac{1}{4} (10 - (-2 + j2) + (-2) - (-2 - j2)) = 3$$

and for $n = 3$

$$x(3) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{\frac{jk\pi 3}{2}} = \frac{1}{4}\left(X(0)e^{j0} + X(1)e^{j\frac{3\pi}{2}} + X(2)e^{j3\pi} + X(3)e^{j\frac{9\pi}{2}} \right)$$

$$= \frac{1}{4}(X(0) - jX(1) - X(2) + jX(3))$$

$$= \frac{1}{4}(10 - j(-2 + j2) - (-2) + j(-2 - j2)) = 4$$

Now we explore the relationship between the frequency bin k and it's associated frequency.

The calculated N DFT coefficients $X(k)$ represent the frequency components ranging from 0 Hz (or radians/second) to f_s Hz (or ω_s radians/second), hence we can map the frequency bin k to its corresponding frequency as follows:

$$\omega = \frac{k\omega_s}{N} \text{ (radians per second),}$$

or in terms of Hz,

$$f = \frac{kf_s}{N} \text{ (Hz),}$$

where $\omega_s = 2\pi f_s$.

We can define the frequency resolution as the frequency step between two consecutive DFT coefficients to measure how fine the frequency domain presentation is and achieve

$$\Delta\omega = \frac{\omega_s}{N} \text{ (radians per second),}$$

or in terms of Hz, it follows that

$$\Delta f = \frac{f_s}{N} \text{ (Hz).}$$

Example

In previous examples, given a sequence *x(n)* for $0 \leq n \leq 3$, where *x(0)* = 1, *x(1)* = 2, *x(2)* = 3, and *x(3)* = 4, we have computed four DFT coefficients *X(k)* for $0 \leq k \leq 3$ as *X(0)* = 10, *X(1)* = -2 + j2, *X(2)* = -2, and *X(3)* = -2 - j2. If the sampling rate is 10 Hz,

    a. Determine the sampling period, time index, and sampling time instant for a digital sample x(3) in time domain.

    b. Determine the frequency resolution, frequency bin number, and mapped frequency for each of the DFT coefficients X(1) and X(3) in frequency domain.

Solution:

a. In time domain, we have the sampling period calculated as
$$T = 1/f_s = 1/10 = 0.1 \text{ second.}$$

For data *x(3)*, the time index is $n = 3$ and the sampling time instant is determined by
$$t = nT = 3 \cdot 0.1 = 0.3 \text{ second.}$$

b. In frequency domain, since the total number of DFT coefficients is four, the frequency resolution is determined by
$$\Delta f = \frac{f_s}{N} = \frac{10}{4} = 2.5 \text{ Hz.}$$

The frequency bin number for *X(1)* should be $k = 1$ and its corresponding frequency is determined by
$$f = \frac{kf_s}{N} = \frac{1 \times 10}{4} = 2.5 \text{ Hz.}$$

Similarly, for *X(3)* and $k = 3$,
$$f = \frac{kf_s}{N} = \frac{3 \times 10}{4} = 7.5 \text{ Hz.}$$