Complex Analysis (LOCF)

Practical 17

A particular Contour Integral

Question : Perform the following Line integrals

(i) $\int_C \exp\left(\frac{2}{z}\right) dz$

(ii) $\int_C \frac{1}{z^4 + z^3 - 2z^2} dz$

where C is the unit circle with center at z = 0 taken in the positive sense.

The curve C can be parametrized as

$$C : z(t) = x(t) + i y(t), \ 0 \le t \le 2\pi$$

where x(t) = Cos[t] and y(t) = Sin[t].

In[24]:=
```
c[t_] = Cos[t] + ⅈ Sin[t];
```

(i) $\int_C \exp\left(\frac{2}{z}\right) dz$

In[25]:=
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f[z_] := ⅇ^(2/z)
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In[26]:= val = ∫_0^(2π) f[c[t]] * c'[t] ⅆt;
         Print[
           "The Value of the contour integration ",
           "∫_C ", f[z], "ⅆz is ", "∫_0^(2π)",
           "f[z[t]]z'[t]", "ⅆt ", "= ", val];
         Print["where C: z[t] = ",
           c[t], ",  for  0 ≤ t ≤ 2π"];
```

The Value of the contour integration

$$\int_C e^{2/z} dz \text{ is } \int_0^{2\pi} f[z[t]]z'[t]dt = 4 i \pi$$

where C: z[t] =

Cos[t] + ⅈ Sin[t], for 0 ≤ t ≤ 2π

(ii) $$\int_C \frac{1}{z^4 + z^3 - 2 z^2} dz$$

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In[30]:= g[z_] := 1/(z^4 + z^3 - 2 z^2)
         Solve[1/g[z] == 0, z]
```

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Out[31]= {{z → -2}, {z → 0}, {z → 0}, {z → 1}}
```

```
In[ ]:= Text[
          "The function g has poles at z = 0 of order
            2 and a pole at z = 1 of order 1."]
```

Out[ ]= "The function g has poles at z = 0 of order 2 and a pole at z = 1 of order 1."

In[32]:= **Apart[g[z]]**

Out[32]= $$\frac{1}{3\,(-1+z)}-\frac{1}{2\,z^2}-\frac{1}{4\,z}-\frac{1}{12\,(2+z)}$$

$$\frac{1}{z^4+z^3-2\,z^2}=$$
$$-\frac{1}{2\,z^2}-\frac{1}{4\,z}-\frac{1}{12\,(z+2)}+\frac{1}{3\,(z-1)}$$

So that

$$\int_C \frac{1}{z^4+z^3-2\,z^2}\,dz = -\int_C \frac{1}{2\,z^2}\,dz - \int_C \frac{1}{4\,z}\,dz - \int_C \frac{1}{12\,(z+2)}\,dz + \int_C \frac{1}{3\,(z-1)}\,dz$$

Using

(i) Cauchy's Integral formula: $\int_C \frac{h[z]}{z-a}\,dz = 2\pi i\,h[a]$

(ii) Derivative of an analytic function: $\int_C \frac{h[z]}{(z-a)^2}\,dz = 2\pi i\,h'[a]$

where a is any point inside or on C.

```
In[51]:= h1[z_] := 1/2
         f1[z_] := h1[z]/z^2
         h2[z_] := 1/4
         f2[z_] := h2[z]/z
         h3[z_] := 1/12
         f3[z_] := h3[z]/(z + 2)
         h4[z_] := 1/3
         f4[z_] := h4[z]/(z - 1)
In[66]:= Val1 = 2 π ⅈ (h1'[z] /. z → 0);
         Val2 = 2 π ⅈ (h2[z] /. z → 0);
         Val3 = 0;
         (* function is analytic inside and on C*)
         Val4 = 2 π ⅈ (h4[z] /. z → 1);
```

In rendered form:

$$h_1[z\_] := \frac{1}{2}$$
$$f_1[z\_] := \frac{h_1[z]}{z^2}$$
$$h_2[z\_] := \frac{1}{4}$$
$$f_2[z\_] := \frac{h_2[z]}{z}$$
$$h_3[z\_] := \frac{1}{12}$$
$$f_3[z\_] := \frac{h_3[z]}{z+2}$$
$$h_4[z\_] := \frac{1}{3}$$
$$f_4[z\_] := \frac{h_4[z]}{z-1}$$

In[70]:=
```
V = -Val1 - Val2 - Val3 + Val4;
Print[
  "The Value of the contour integration ",
  "∫_C ", g[z], "ⅆz is = ", V];
Print["where C: z[t] = ",
  c[t], ",  for  0 ≤ t ≤ 2π"];
```

The Value of the contour integration

$$\int_C \frac{1}{-2 z^2 + z^3 + z^4} dz \text{ is } = \frac{i \pi}{6}$$

where C: $z[t] = \frac{1}{3}[t]$, for $0 \le t \le 2\pi$

(ii) $\displaystyle \int_C \frac{1}{z^4 + z^3 - 2 z^2} dz$

Using Method of Residues

In[33]:= $g[z\_] := \dfrac{1}{z^4 + z^3 - 2 z^2}$

$\text{Solve}\left[\dfrac{1}{g[z]} == 0, z\right]$

Out[34]= {{z → -2}, {z → 0}, {z → 0}, {z → 1}}

In[79]:= a = Residue[g[z], {z, 0}]

Out[79]= $-\frac{1}{4}$

In[80]:= b = Residue[g[z], {z, 1}]

Out[80]= $\frac{1}{3}$

In[83]:= **Va1 = 2 ⅈ π (a + b)**

Out[83]= $\frac{i \pi}{6}$