Theorem: angular momentum is conserved iff linear momentum too and $\underline{\underline{\sigma}}' = \underline{\underline{\sigma}}$, i.e. the Cauchy stresses are symmetric.

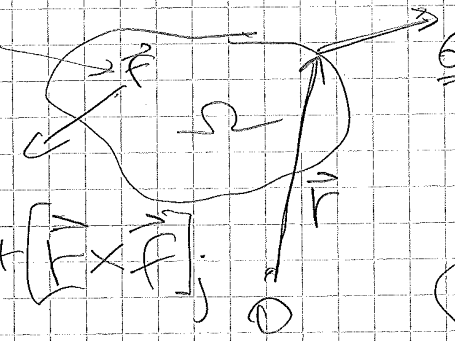
Proof: $\Leftarrow$ angular momentum vector

$$L_j = [\vec{r} \times \rho \vec{u}]_j , \quad \vec{r} = \sum_{i=1}^{3} x_i \vec{e}_i$$

Conservation equation:

$$(*) \qquad \partial_t L + L \nabla\cdot\vec{u} + (\vec{u}-\vec{w})\cdot\nabla L - \nabla\cdot\vec{b} = Q_V$$

"Source" of angular momentum: $Q_V, \vec{b}$?



$$\underline{Q_V} = M_j + [\vec{r} \times \vec{f}]_j$$

M_j: assume $\equiv 0$

$$\int_{\partial\Omega} [\vec{r} \times \underline{\underline{\sigma}}\vec{n}]_j \, ds$$

$$= \int_{\partial\Omega} \Big[\vec{r} \times \sum_{i=1}^{3} \vec{\sigma}_i n_i\Big]_j ds$$

$\vec{\sigma}_i$: i-th column of $\underline{\underline{\sigma}}$

$$= \int_{\partial\Omega} \sum_{i=1}^{3} [\vec{r} \times \vec{\sigma}_i]_j \, n_i \, ds$$

$$= \int_{\Omega} \underbrace{\sum_{i=1}^{3} \partial_i [\vec{r} \times \vec{\sigma}_i]_j}_{[\vec{b}_i]_j} \, dv$$

Inserting everything into (*)

$$\partial_t \left[\vec{r}\times\rho\vec{u}\right]_j + \left[\vec{r}\times\rho\vec{u}\right]_j \nabla\cdot\vec{u} + (\vec{u}-\vec{\omega})\cdot\nabla)\left[\vec{r}\times\rho\vec{u}\right]_j - \sum_{i=1}^{3}\partial_i\left[\vec{r}\times\vec{\sigma}_i\right]_j = \left[\vec{r}\times\vec{f}\right]_j$$

Next:

$$\left[\underbrace{\partial_t\vec{r}}_{=\vec{\omega}}\times\rho\vec{u}\right]_j + \left[\vec{r}\times\partial_t(\rho\vec{u})\right]_j$$

(from complementary reading)

$$+\left[\vec{r}\times(\rho\vec{u}(\nabla\cdot\vec{u}))\right]_j + \sum_{k=1}^{3}(u_k-w_k)\partial_k\left[\vec{r}\times\rho\vec{u}\right]_j$$

$$-\sum_{i=1}^{3}\left[\partial_i\vec{r}\times\vec{\sigma}_i\right]_j + \left[\vec{r}\times\partial_i\vec{\sigma}_i\right]_j = \left[\vec{r}\times\vec{f}\right]_j$$

Then, regrouping and expanding the rest:

$$\left[\vec{r}\times\left\{\partial_t(\rho\vec{u}) + \rho\vec{u}\nabla\cdot\vec{u} - \sum_{i=1}^{3}\partial_i\vec{\sigma}_i - \vec{f}\right\}\right]_j + \vec{\omega}\times\rho\vec{u}$$

$$+\left[\sum_{k=1}^{3}(u_k-w_k)\left\{\partial_k\vec{r}\times\rho\vec{u} + \vec{r}\times\partial_k(\rho\vec{u})\right\}\right]_j$$

$$-\left[\sum_{i=1}^{3}\partial_i\vec{r}\times\vec{\sigma}_i\right]_j = 0$$

"If linear momentum conservation (LCM) then also $\vec{r}\times LMC = 0$. Then

$$\vec{\omega}\times\rho\vec{u} + \sum_{k=1}^{3}(u_k-w_k)\partial_k\vec{r}\times\rho\vec{u} = \sum_{i=1}^{3}\partial_i\vec{r}\times\vec{\sigma}_i$$

Note that $\partial_i \vec{r} = \vec{e}_i$.

In fact, the left-hand-side is zero:

$$(u_1 - w_1)\vec{e}_1 \times (\rho u_1 \vec{e}_1 + \rho u_2 \vec{e}_2 + \rho u_3 \vec{e}_3)$$
$$+ (u_2 - w_2)\vec{e}_2 \times (\cdots) + (u_3 - w_3)\vec{e}_3 (\cdots) =$$

$$\cancel{u_1 u_2 \vec{e}_3} - \cancel{u_1 u_3 \vec{e}_2} - \cancel{u_2 u_1 \vec{e}_3} + \cancel{u_2 u_3 \vec{e}_1}$$
$$+ \cancel{u_3 u_1 \vec{e}_2} - \cancel{u_3 u_2 \vec{e}_1}$$

$$= -\vec{w} \times \rho \vec{u}!$$

Hence

$$\vec{e}_1 \times (\sigma_{11}\vec{e}_1 + \sigma_{21}\vec{e}_2 + \sigma_{31}\vec{e}_3)$$
$$+ \vec{e}_2 \times (\sigma_{12}\vec{e}_1 + \sigma_{22}\vec{e}_2 + \sigma_{32}\vec{e}_3)$$
$$+ \vec{e}_3 \times (\sigma_{13}\vec{e}_1 + \sigma_{23}\vec{e}_2 + \sigma_{33}\vec{e}_3) = 0$$

Computing the cross-products results:

$$\sigma_{21}\vec{e}_3 - \sigma_{31}\vec{e}_2 - \sigma_{12}\vec{e}_3 + \sigma_{32}\vec{e}_1$$
$$+ \sigma_{13}\vec{e}_2 - \sigma_{23}\vec{e}_1 = 0$$

$$\Longrightarrow \boxed{\sigma_{13} = \sigma_{31}, \ \sigma_{32} = \sigma_{23}, \ \sigma_{21} = \sigma_{12}}$$

Proof $\Leftarrow$: Start with symmetry and go back adding zeros.