Conservation laws in a nutshell

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Let consider a time-varying control volume $\Omega = \Omega(t)$ with boundary $\partial\Omega$ and its normal vector $\boldsymbol{n}$, given by the mapping $\boldsymbol{y} = \boldsymbol{y}(X,t)$ from a reference configuration $\widehat{\Omega}$, namely

$$\Omega = \left(I_{\widehat{\Omega}} + \boldsymbol{y}\right)(\widehat{\Omega}, t) \ , \ \hat{\boldsymbol{w}}(X,t) = \partial_t \boldsymbol{y}(X,t) = \partial_t x(X,t) = \boldsymbol{w}(x,t)$$

and let us denote $\boldsymbol{u}$ the velocity of an infinitesimal particle at a point $x \in \Omega$, with $x(X,t) = X + \boldsymbol{y}(X,t)$.

Let $L = L(x(t),t) = \hat{L}(X,t)$ be a quantity to be conserved (for example the impluse $L = \rho\boldsymbol{u}$, mass $L = \rho$ or energy density $\rho\|\boldsymbol{u}\|^2$), with $\boldsymbol{u}(x(X,t),t)$. The general conservation law state that

$$\partial_t \int_\Omega L + \int_{\partial\Omega} L(\boldsymbol{u} - \boldsymbol{w}) \cdot \boldsymbol{n} + \int_{\partial\Omega} Q_b + \int_\Omega Q_v = 0 \tag{1}$$

with Q_b and Q_v boundary and volume source terms, respectively. We now will make use of the divergence theorem, namely

$$\int_{\partial\Omega} \boldsymbol{u} \cdot \boldsymbol{n} = \int_\Omega \boldsymbol{\nabla} \cdot \boldsymbol{u} \ , \ [\boldsymbol{\nabla}]_i = \frac{\partial}{\partial x_i} \ , \ i = 1, \ldots, d$$

and the so-called *geometric conservation law*

$$\partial_t \int_\Omega = \int_{\partial\Omega} \boldsymbol{w} \cdot \boldsymbol{n} \Rightarrow \partial_t J(X,t) = J \boldsymbol{\nabla} \cdot \boldsymbol{w}(x,t) \tag{2}$$

with

$$J(X,t) = \det(\boldsymbol{F}(X,t)) \ , \ \boldsymbol{F}(X,t) = \frac{\partial x}{\partial X} = \mathbb{1} + \widehat{\boldsymbol{\nabla}} \boldsymbol{y}(X,t) \ , \ [\widehat{\boldsymbol{\nabla}}]_i = \frac{\partial}{\partial X_i} \ , \ i = 1, \ldots, d$$

Note that the term $\boldsymbol{\nabla} \cdot \boldsymbol{w}$ is expressed in the coordinates x, but it has also a representation in X. However, for deriving the conservation laws, we do not need that representation.

We will assume now the structure $Q_b = \boldsymbol{b} \cdot \boldsymbol{n}$. Hence, we can conveniently rewrite (1) as

$$\begin{aligned}
\int_{\widehat{\Omega}} \partial_t(\hat{L}J) + \int_\Omega \boldsymbol{\nabla} \cdot (L(\boldsymbol{u} - \boldsymbol{w})) &= \int_{\partial\Omega} Q_b + \int_\Omega Q_v &(3)\\
\int_{\widehat{\Omega}} (\partial_t \hat{L} J + LJ \boldsymbol{\nabla} \cdot \boldsymbol{w} + \int_\Omega \boldsymbol{\nabla} \cdot (L(\boldsymbol{u} - \boldsymbol{w}))) &= \int_{\partial\Omega} Q_b + \int_\Omega Q_v &(4)\\
\int_\Omega (\partial_t L + L \boldsymbol{\nabla} \cdot \boldsymbol{w} + \boldsymbol{\nabla} \cdot (L(\boldsymbol{u} - \boldsymbol{w}))) &= \int_{\partial\Omega} Q_b + \int_\Omega Q_v &(5)\\
\int_\Omega (\partial_t L + L \boldsymbol{\nabla} \cdot \boldsymbol{w} + L \boldsymbol{\nabla} \cdot (\boldsymbol{u} - \boldsymbol{w}) + (\boldsymbol{u} - \boldsymbol{w}) \cdot \boldsymbol{\nabla} L) &= \int_{\partial\Omega} Q_b + \int_\Omega Q_v &(6)\\
\int_\Omega (\partial_t L + L \boldsymbol{\nabla} \cdot \boldsymbol{u} + (\boldsymbol{u} - \boldsymbol{w}) \cdot \boldsymbol{\nabla} L - \boldsymbol{\nabla} \cdot \boldsymbol{b}) &= \int_\Omega Q_v &(7)
\end{aligned}$$

and since this relation has to be true for all Ω it leads to the differential formulations

$$\partial_t L + L \boldsymbol{\nabla} \cdot \boldsymbol{u} + (\boldsymbol{u} - \boldsymbol{w}) \cdot \boldsymbol{\nabla} L - \boldsymbol{\nabla} \cdot \boldsymbol{b} = 0 \ \text{ in } \ \Omega \tag{8}$$

In the case of the mass conservation, (1) reads

$$\partial_t \rho + \rho \boldsymbol{\nabla} \cdot \boldsymbol{u} + (\boldsymbol{u} - \boldsymbol{w}) \cdot \boldsymbol{\nabla} \rho = 0 \quad \text{in} \quad \Omega \tag{9}$$

with $Q_b = Q_v = 0$.

The momentum conservation in the $i-th$ direction reads ($\boldsymbol{b} = \boldsymbol{\sigma}_{i,:}$, $Q_v = \boldsymbol{f}_i$)

$$\partial_t(\rho \boldsymbol{u}_i) + (\rho \boldsymbol{u}_i) \boldsymbol{\nabla} \cdot \boldsymbol{u} + (\boldsymbol{u} - \boldsymbol{w}) \cdot \boldsymbol{\nabla}(\rho \boldsymbol{u}_i) - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}_{i,:} = \boldsymbol{f}_i \quad \text{in} \quad \Omega. \tag{10}$$

Note that we can we can write up all terms obtaining

$$\begin{aligned}
\partial_t(\rho)\boldsymbol{u}_i + \rho \partial_t(\boldsymbol{u}_i) + \rho \boldsymbol{u}_i \boldsymbol{\nabla} \cdot \boldsymbol{u} + (\boldsymbol{u} - \boldsymbol{w}) \cdot \boldsymbol{\nabla}(\rho)\boldsymbol{u}_i + \rho(\boldsymbol{u} - \boldsymbol{w}) \cdot \boldsymbol{\nabla}(\boldsymbol{u}_i) - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}_{i,:} &= \boldsymbol{f}_i \quad \text{in} \quad \Omega \\
\boldsymbol{u}_i(\partial_t \rho + \rho \boldsymbol{\nabla} \cdot \boldsymbol{u} + (\boldsymbol{u} - \boldsymbol{w}) \cdot \boldsymbol{\nabla} \rho) + \rho \partial_t(\boldsymbol{u}_i) + \rho(\boldsymbol{u} - \boldsymbol{w}) \cdot \boldsymbol{\nabla}(\boldsymbol{u}_i) - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}_{i,:} &= \boldsymbol{f}_i \quad \text{in} \quad \Omega \\
\rho \partial_t(\boldsymbol{u}_i) + \rho(\boldsymbol{u} - \boldsymbol{w}) \cdot \boldsymbol{\nabla}(\boldsymbol{u}_i) - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}_{i,:} &= \boldsymbol{f}_i \quad \text{in} \quad \Omega
\end{aligned}$$

where from the second to the third line the general mass conservation equation (9) was used.

Finally, the energy density conservation reads ($Q_b = Q_v = 0$)

$$\partial_t(\rho \|\boldsymbol{u}\|^2/2) + \rho \|\boldsymbol{u}\|^2/2 \boldsymbol{\nabla} \cdot \boldsymbol{u} + (\boldsymbol{u} - \boldsymbol{w}) \cdot \boldsymbol{\nabla}(\rho \|\boldsymbol{u}\|^2/2) = 0 \quad \text{in} \quad \Omega \tag{11}$$

or in integral form by replacing $L = \rho \|\boldsymbol{u}\|^2/2$ directly in (1)

$$\partial_t \int_\Omega \rho \|\boldsymbol{u}\|^2/2 + \int_{\partial\Omega} \rho \|\boldsymbol{u}\|^2/2(\boldsymbol{u} - \boldsymbol{w}) \cdot \boldsymbol{n} = 0 \tag{12}$$

Integrating it in time

$$\frac{1}{2} \int_\Omega \rho \|\boldsymbol{u}\|^2 = \frac{1}{2} \int_{\widehat{\Omega}} \widehat{\rho} \widehat{\boldsymbol{u}}^2 - \frac{1}{2} \int_{\widehat{t}}^t \int_{\partial\Omega} \rho \|\boldsymbol{u}\|^2 (\boldsymbol{u} - \boldsymbol{w}) \cdot \boldsymbol{n} \tag{13}$$

where $\widehat{\boldsymbol{u}}$ denotes the velocity at $\widehat{t}$,is the energy storage of the internal forces.