3.2. Conservation of angular momentum (5 points)

We saw in the lectures and in the complementary reading that the general conservation law reads:

$$\partial_t L + L \sum_{i=1}^{3} \partial_{x_i} \mathbf{u}_i + \sum_{k=1}^{3} (\mathbf{u}_k - \mathbf{w}_k)\, \partial_{x_k} L - \sum_{\ell=1}^{3} \partial_{x_\ell} \mathbf{b}_\ell = 0 \tag{7}$$

The goal is to proof that the Cauchy stresses need to be symmetric, i.e. $\sigma = \sigma^\top \in \mathbb{R}^{3\times 3}$ as stated in the lecture, by assuming that the fluid flow satisfies the conservation of angular momentum. For this we will proceed in some steps:

1. **(1 point)** As you have seen, on a control volume Ω, the vector forces acting on the boundary are $\sigma\mathbf{n}$, and hence the angular momentum vector produced by those forces is $\mathbf{r} \times \sigma\mathbf{n}$, with $\mathbf{r}_i = x_i(X, t)$ and $\mathbf{w} = \partial_t \mathbf{r}$. In order to use the differential relation (7), we need to convert the external boundary source into a volume source. Therefore, for the j-th component of the angular momentum vector, show that:

$$\int_{\partial\Omega} [\mathbf{r} \times \sigma\mathbf{n}]_j = \int_{\Omega} \sum_{\ell=1}^{3} \partial_{x_\ell} \left([\mathbf{r} \times \sigma_{:,\ell}]_j \right) \tag{8}$$

with $\sigma_{:,i}$ the ith-column of σ (recall the notations from the lecture and complementary material on conservation laws in moving domains).

2. **(2 points)** Asssume that Equation (7) holds for linear momentum (i,.e for $L = \rho\mathbf{u}_1, \rho\mathbf{u}_2$ and $\rho\mathbf{u}_3$ with $\mathbf{b}_\ell = \sigma_{\ell,1}, \sigma_{\ell,2}, \sigma_{\ell,3}$, respectively). Then, prove that if $L = [\mathbf{r} \times \rho\mathbf{u}]_j, \mathbf{b}_i = [\mathbf{r} \times \sigma_{:,i}]_j$, Equation (7) for $j = 1, 2, 3$ leads to:

$$(\partial_t \mathbf{r}) \times \rho\mathbf{u} + \left\{ \sum_{k=1}^{3} (\mathbf{u}_k - \mathbf{w}_k) \left(\partial_{x_k} \mathbf{r}\right) \right\} \times \rho\mathbf{u} = \sum_{i=1}^{3} \left(\partial_{x_i} \mathbf{r}\right) \times \sigma_{:,i} \tag{9}$$

3. **(1 point)** Show now that the left-hand-side of Equation (9) is zero.

4. **(1 point)** Show now that the Cauchy stress matrix is symmetric. You have to deliver all the steps written on paper for grading.

Theorem: angular momentum is conserved if linear momentum too and $\sigma^T = \sigma$, i.e. the cauchy stresses are symmetric

Proof: <= angular momentum vector $L = [\vec{r} \times \rho\vec{u}]_j, \quad \vec{r} = \sum_{i=1}^{3} x_i \vec{e}_i$

Conservation equation: (*) $\partial_t L + L\nabla \cdot \vec{u} + (\vec{u} - \vec{\omega}) \cdot \nabla L - \nabla \cdot \vec{b} = Q_v$

"Source" of angular momentum: $Q_v / \vec{b}$?

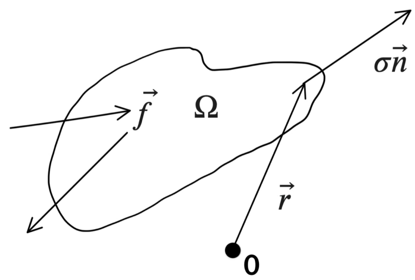

Figure 2:

$Q_v = M_j + [\vec{r} \times \vec{f}\,]_j$

assume $\equiv 0$

$\int_{\partial\Omega} [\vec{r} \times \sigma\vec{n}]_j ds =$

$= \int_{\partial\Omega} [\vec{r} \times \sum_{i=1}^{3} \sigma_i \vec{n}_i]_j ds =$

$= \int_{\partial\Omega} \sum_{i=1}^{3} [\vec{r} \times \sigma_i]_j \vec{n}_i ds =$

$= \int_{\Omega} \sum_{i=1}^{3} \partial_i [\vec{r} \times \sigma_i]_j \vec{n}_i dV$

$[\vec{r} \times \sigma_i]_j = b_i$

Inserting everything into (*)

$\partial_t \left[\vec{r} \times \rho\vec{u}\right]_j + [\vec{r} \times \rho\vec{u}]_j \nabla \cdot \vec{u} + (\vec{u} - \vec{\omega}) \cdot \nabla [\vec{r} \times \rho\vec{u}]_j - \sum_{i=1}^{3} \partial_i \left[\vec{r} \times \vec{\sigma}_i\right]_j = [\vec{r} \times \vec{f}\,]_j$

Next:

$\left[\partial_t \vec{r} \times \rho\vec{u}\right]_j + \left[\vec{r} \times d_t(\rho\vec{u})\right]_j + \left[\vec{r} \times \left(\rho\vec{u}\,(\nabla \cdot \vec{u})\right)\right]_j + \sum_{k=1}^{3} (u_k - \omega_k)\, \partial_k [\vec{r} \times \rho\vec{u}]_j - \sum_{i=1}^{3} \left[\left[\partial_i \vec{r} \times \vec{\sigma}_i\right]_j + \left[\vec{r} \times \partial_i \vec{\sigma}_i\right]_j\right] = [\vec{r} \times \vec{f}\,]_j$

Then, regrouping and expanding the rest:

$\left[\vec{r} \times \left\{ \partial_t(\rho\vec{u}) + \rho\vec{u}\nabla \cdot \vec{u} - \sum_{i=1}^{3} \partial_i \vec{\sigma}_i - \vec{f} \right\}\right]_j + \vec{\omega} \times \rho\vec{u} + \left[\sum_{k=1}^{3} (u_k - \omega_k) \left\{ \partial_k \vec{r} \times \rho\vec{u} + \vec{r} \times \partial_k \left(\rho\vec{u}\right)\right\}\right]_j - \left[\sum_{i=1}^{3} \partial_i \vec{r} \times \sigma_i\right]_j = 0$

If linear monentum conservatron (LCM) then also $\vec{r} \times LMC = 0$. Then

$\vec{\omega} \times \rho\vec{u} + \sum_{k=1}^{3} (u_k - \omega_k)\, \partial_k \vec{r} \times \rho\vec{u} = \sum_{i=1}^{3} \partial_i \vec{r} \times \vec{\sigma}_i$

Note that $\partial_i \vec{r} = \vec{e}_i$

In fact, the left-hand-side is zero:

$(u_1 - \omega_1)\, \vec{e}_1 \times \left(\rho u_1 \vec{e}_1 + \rho u_2 \vec{e}_2 + pu_3 \vec{e}_3\right) + (u_2 - w_2)\, \vec{e}_2 \times (\cdots) + (u_3 - w_3)\, \vec{e}_3(\cdots) = u_1 u_2 \vec{e_3} - u_1 u_3 \vec{e}_2 - u_2 u_1 \vec{e}_3 + u_2 u_3 \vec{e}_1 + u_3 u_1 \vec{e}_2 - u_3 x_2 e_1$

$\equiv -\vec{\omega} \times \rho\vec{u}.$

Hence $\vec{e}_1 \times \left(\sigma_{11}\vec{e}_1 + \sigma_{21}\vec{e}_2 + \sigma_{31}\vec{e}_3\right) + \vec{e}_2 \times \left(\sigma_{12}\vec{e_1} + \sigma_{22}\vec{e}_2 + \sigma_{32}\vec{e_3}\right) + \vec{e}_3 \times \left(\sigma_{13}\vec{e_1} + \sigma_{23}\vec{e}_2 + \sigma_{33}\vec{e_3}\right) = 0$

Computing the gross-products results: $\sigma_{21}\vec{e}_3 - \sigma_{31}\vec{e}_2 - b_{12}\vec{e}_3 + \sigma_{32}\vec{e}_1 + \sigma_{23}\vec{e}_2 - \sigma_{23}\vec{e}_1 = 0$

$\Leftrightarrow \sigma_{13} = \sigma_{31}, \sigma_{21} = \sigma_{12}, \sigma_{32} = \sigma_{23}$

Proof $\Rightarrow$: Start with symmetry and go back adding zero