CHAPTER 1
Limits and Their Properties

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Section 1.1 A Preview of Calculus

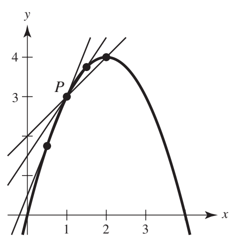
1. Precalculus: $(20 \text{ ft/sec})(15 \text{ seconds}) = 300$ feet

3. Calculus required: slope of tangent line at $x = 2$ is rate of change, and equals about 0.16.

5. Precalculus: Area $= \frac{1}{2}bh = \frac{1}{2}(5)(3) = \frac{15}{2}$ sq. units

7. $f(x) = 4x - x^2$

(a)



(b) $$\text{slope} = m = \frac{(4x - x^2) - 3}{x - 1}$$

$$= \frac{(x-1)(3-x)}{x-1} = 3 - x, \quad x \neq 1$$

$x = 2$: $m = 3 - 2 = 1$

$x = 1.5$: $m = 3 - 1.5 = 1.5$

$x = 0.5$: $m = 3 - 0.5 = 2.5$

(c) At $P(1, 3)$ the slope is 2.

You can improve your approximation of the slope at $x = 1$ by considering x-values very close to 1.

9. (a) $\text{Area} \approx 5 + \frac{5}{2} + \frac{5}{3} + \frac{5}{4} \approx 10.417$

$$\text{Area} \approx \frac{1}{2}\left(5 + \frac{5}{1.5} + \frac{5}{2} + \frac{5}{2.5} + \frac{5}{3} + \frac{5}{3.5} + \frac{5}{4} + \frac{5}{4.5}\right) \approx 9.145$$

(b) You could improve the approximation by using more rectangles.

11. (a) $D_1 = \sqrt{(5-1)^2 + (1-5)^2} = \sqrt{16 + 16} \approx 5.66$

(b) $D_2 = \sqrt{1 + \left(\frac{5}{2}\right)^2} + \sqrt{1 + \left(\frac{5}{2} - \frac{5}{3}\right)^2} + \sqrt{1 + \left(\frac{5}{3} - \frac{5}{4}\right)^2} + \sqrt{1 + \left(\frac{5}{4} - 1\right)^2}$

$\approx 2.693 + 1.302 + 1.083 + 1.031 \approx 6.11$

(c) Increase the number of line segments.

Section 1.2 Finding Limits Graphically and Numerically

1.

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	0.3448	0.3344	0.3334	0.3332	0.3322	0.3226

$$\lim_{x \to 2} \frac{x - 2}{x^2 - x - 2} \approx 0.3333 \qquad \left(\text{Actual limit is } \tfrac{1}{3}.\right)$$

3.

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	0.2911	0.2889	0.2887	0.2887	0.2884	0.2863

$\lim_{x\to 0} \frac{\sqrt{x+3}-\sqrt{3}}{x} \approx 0.2887$ (Actual limit is $1/(2\sqrt{3})$.)

5.

x	2.9	2.99	2.999	3.001	3.01	3.1
$f(x)$	-0.0641	-0.0627	-0.0625	-0.0625	-0.0623	-0.0610

$\lim_{x\to 3} \frac{[1/(x+1)]-(1/4)}{x-3} \approx -0.0625$ (Actual limit is $-\frac{1}{16}$.)

7.

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	0.9983	0.99998	1.0000	1.0000	0.99998	0.9983

$\lim_{x\to 0} \frac{\sin x}{x} \approx 1.0000$ (Actual limit is 1.) (Make sure you use radian mode.)

9. $\lim_{x\to 3} (4-x) = 1$

11. $\lim_{x\to 2} f(x) = \lim_{x\to 2} (4-x) = 2$

13. $\lim_{x\to 5} \frac{|x-5|}{x-5}$ does not exist. For values of x to the left of 5, $|x-5|/(x-5)$ equals -1, whereas for values of x to the right of 5, $|x-5|/(x-5)$ equals 1.

15. $\lim_{x\to 1} \sin \pi x = 0$

17. $\lim_{x\to 0} \cos(1/x)$ does not exist since the function oscillates between -1 and 1 as x approaches 0.

19. (a) $f(1)$ exists. The black dot at $(1, 2)$ indicates that $f(1) = 2$.

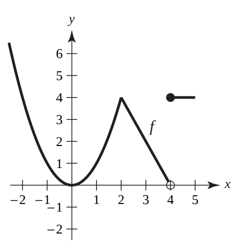
(b) $\lim_{x\to 1} f(x)$ does not exist. As x approaches 1 from the left, $f(x)$ approaches 2.5, whereas as x approaches 1 from the right, $f(x)$ approaches 1.

(c) $f(4)$ does not exist. The hollow circle at $(4, 2)$ indicates that f is not defined at 4.

(d) $\lim_{x\to 4} f(x)$ exists. As x approaches 4, $f(x)$ approaches 2: $\lim_{x\to 4} f(x) = 2$.

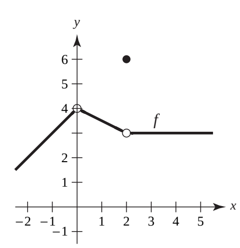
21. $\lim_{x\to c} f(x)$ exists for all $c \neq -3$. In particular, $\lim_{x\to 2} f(x) = 2$.

23.



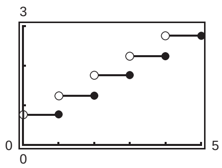
$\lim_{x\to c} f(x)$ exists for all values of $c \neq 4$.

25. One possible answer is



27. $C(t) = 0.75 - 0.50[\![-(t-1)]\!]$

(a)



(b)

t	3	3.3	3.4	3.5	3.6	3.7	4
C	1.75	2.25	2.25	2.25	2.25	2.25	2.25

$\lim_{t\to 3.5} C(t) = 2.25$

(c)

t	2	2.5	2.9	3	3.1	3.5	4
C	1.25	1.75	1.75	1.75	2.25	2.25	2.25

$\lim_{t\to 3} C(t)$ does not exist. The values of C jump from 1.75 to 2.25 at $t = 3$.

29. We need $|f(x) - 3| = |(x+1) - 3| = |x - 2| < 0.4$. Hence, take $\delta = 0.4$. If $0 < |x - 2| < 0.4$, then $|x - 2| = |(x+1) - 3| = |f(x) - 3| < 0.4$, as desired.

31. You need to find δ such that $0 < |x - 1| < \delta$ implies $|f(x) - 1| = \left|\frac{1}{x} - 1\right| < 0.1$. That is,

$$-0.1 < \frac{1}{x} - 1 < 0.1$$
$$1 - 0.1 < \frac{1}{x} < 1 + 0.1$$
$$\frac{9}{10} < \frac{1}{x} < \frac{11}{10}$$
$$\frac{10}{9} > x > \frac{10}{11}$$
$$\frac{10}{9} - 1 > x - 1 > \frac{10}{11} - 1$$
$$\frac{1}{9} > x - 1 > -\frac{1}{11}.$$

So take $\delta = \frac{1}{11}$. Then $0 < |x - 1| < \delta$ implies

$$-\frac{1}{11} < x - 1 < \frac{1}{11}$$
$$-\frac{1}{11} < x - 1 < \frac{1}{9}.$$

Using the first series of equivalent inequalities, you obtain

$$|f(x) - 1| = \left|\frac{1}{x} - 1\right| < 0.1.$$

33. $\lim_{x\to 2}(3x + 2) = 8 = L$

$$|(3x+2) - 8| < 0.01$$
$$|3x - 6| < 0.01$$
$$3|x - 2| < 0.01$$
$$0 < |x - 2| < \frac{0.01}{3} \approx 0.0033 = \delta$$

Hence, if $0 < |x - 2| < \delta = \frac{0.01}{3}$, you have

$$3|x - 2| < 0.01$$
$$|3x - 6| < 0.01$$
$$|(3x+2) - 8| < 0.01$$
$$|f(x) - L| < 0.01$$

35. $\lim_{x\to 2}(x^2 - 3) = 1 = L$

$$|(x^2 - 3) - 1| < 0.01$$
$$|x^2 - 4| < 0.01$$
$$|(x+2)(x-2)| < 0.01$$
$$|x + 2|\,|x - 2| < 0.01$$
$$|x - 2| < \frac{0.01}{|x+2|}$$

If we assume $1 < x < 3$, then $\delta = 0.01/5 = 0.002$.

Hence, if $0 < |x - 2| < \delta = 0.002$, you have

$$|x - 2| < 0.002 = \frac{1}{5}(0.01) < \frac{1}{|x+2|}(0.01)$$
$$|x + 2||x - 2| < 0.01$$
$$|x^2 - 4| < 0.01$$
$$|(x^2 - 3) - 1| < 0.01$$
$$|f(x) - L| < 0.01$$

37. $\lim_{x\to 2}(x+3) = 5$

Given $\varepsilon > 0$:

$$|(x+3)-5| < \varepsilon$$
$$|x-2| < \varepsilon = \delta$$

Hence, let $\delta = \varepsilon$.

Hence, if $0 < |x-2| < \delta = \varepsilon$, you have

$$|x-2| < \varepsilon$$
$$|(x+3)-5| < \varepsilon$$
$$|f(x)-L| < \varepsilon.$$

39. $\lim_{x\to -4}\left(\frac{1}{2}x - 1\right) = \frac{1}{2}(-4) - 1 = -3$

Given $\varepsilon > 0$:

$$\left|\left(\tfrac{1}{2}x - 1\right) - (-3)\right| < \varepsilon$$
$$\left|\tfrac{1}{2}x + 2\right| < \varepsilon$$
$$\tfrac{1}{2}|x-(-4)| < \varepsilon$$
$$|x-(-4)| < 2\varepsilon$$

Hence, let $\delta = 2\varepsilon$.

Hence, if $0 < |x-(-4)| < \delta = 2\varepsilon$, you have

$$|x-(-4)| < 2\varepsilon$$
$$\left|\tfrac{1}{2}x + 2\right| < \varepsilon$$
$$\left|\left(\tfrac{1}{2}x - 1\right) + 3\right| < \varepsilon$$
$$|f(x)-L| < \varepsilon.$$

41. $\lim_{x\to 6} 3 = 3$

Given $\varepsilon > 0$:

$$|3-3| < \varepsilon$$
$$0 < \epsilon$$

Hence, any $\delta > 0$ will work.

Hence, for any $\delta > 0$, you have

$$|3-3| < \varepsilon$$
$$|f(x)-L| < \varepsilon.$$

43. $\lim_{x\to 0}\sqrt[3]{x} = 0$

Given $\varepsilon > 0$: $\left|\sqrt[3]{x} - 0\right| < \varepsilon$

$$\left|\sqrt[3]{x}\right| < \varepsilon$$
$$|x| < \varepsilon^3 = \delta$$

Hence, let $\delta = \varepsilon^3$.

Hence for $0|x-0|\delta = \varepsilon^3$, you have

$$|x| < \varepsilon^3$$
$$\left|\sqrt[3]{x}\right| < \varepsilon$$
$$\left|\sqrt[3]{x} - 0\right| < \varepsilon$$
$$|f(x)-L| < \varepsilon.$$

45. $\lim_{x\to -2}|x-2| = |(-2)-2| = 4$

Given $\varepsilon > 0$:

$$\big||x-2|-4\big| < \varepsilon$$
$$|-(x-2)-4| < \varepsilon \quad (x-2<0)$$
$$|-x-2| = |x+2| = |x-(-2)| < \varepsilon$$

Hence, let $\delta = \varepsilon$.

Hence for $0 < |x-(-2)| < \delta = \varepsilon$, you have

$$|x+2| < \varepsilon$$
$$|-(x+2)| < \varepsilon$$
$$|-(x-2)-4| < \varepsilon$$
$$\big||x-2|-4\big| < \varepsilon \quad (\text{because } x-2<0)$$
$$|f(x)-L| < \varepsilon.$$

47. $\lim_{x\to 1}(x^2+1) = 2$

Given $\varepsilon > 0$:

$$|(x^2+1)-2| < \varepsilon$$
$$|x^2-1| < \varepsilon$$
$$|(x+1)(x-1)| < \varepsilon$$
$$|x-1| < \frac{\varepsilon}{|x+1|}$$

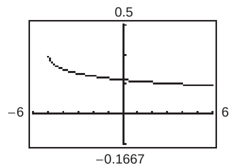
If we assume $0 < x < 2$, then $\delta = \varepsilon/3$.

Hence for $0 < |x-1| < \delta = \dfrac{\varepsilon}{3}$, you have

$$|x-1| < \frac{1}{3}\varepsilon < \frac{1}{|x+1|}\varepsilon$$
$$|x^2-1| < \varepsilon$$
$$|(x^2+1)-2| < \varepsilon$$
$$|f(x)-2| < \varepsilon.$$

49. $f(x) = \dfrac{\sqrt{x+5} - 3}{x - 4}$

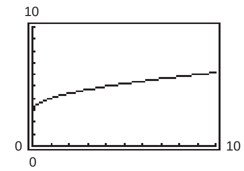
$\lim\limits_{x \to 4} f(x) = \dfrac{1}{6}$



The domain is $[-5, 4) \cup (4, \infty)$. The graphing utility does not show the hole at $\left(4, \frac{1}{6}\right)$.

51. $f(x) = \dfrac{x - 9}{\sqrt{x} - 3}$

$\lim\limits_{x \to 9} f(x) = 6$



The domain is all $x \geq 0$ except $x = 9$. The graphing utility does not show the hole at $(9, 6)$.

53. $\lim\limits_{x \to 8} f(x) = 25$ means that the values of f approach 25 as x gets closer and closer to 8.

55. No. The fact that $\lim\limits_{x \to 2} f(x) = 4$ has no bearing on the value of f at 2.

57. (a) $C = 2\pi r$

$$r = \frac{C}{2\pi} = \frac{6}{2\pi} = \frac{3}{\pi} \approx 0.9549 \text{ cm}$$

(b) If $C = 5.5$, $r = \dfrac{5.5}{2\pi} \approx 0.87535$ cm

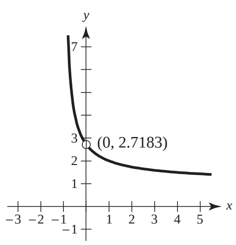
If $C = 6.5$, $r = \dfrac{6.5}{2\pi} \approx 1.03451$ cm

Thus $0.87535 < r < 1.03451$

(c) $\lim\limits_{r \to 3/\pi} (2\pi r) = 6$; $\varepsilon = 0.5$; $\delta \approx 0.0796$

59. $f(x) = (1 + x)^{1/x}$

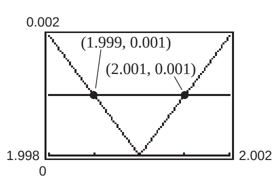
$\lim\limits_{x \to 0} (1 + x)^{1/x} = e \approx 2.71828$



x	$f(x)$
-0.1	2.867972
-0.01	2.731999
-0.001	2.719642
-0.0001	2.718418
-0.00001	2.718295
-0.000001	2.718283

x	$f(x)$
0.1	2.593742
0.01	2.704814
0.001	2.716942
0.0001	2.718146
0.00001	2.718268
0.000001	2.718280

61.



Using the zoom and trace feature, $\delta = 0.001$. That is, for

$$0 < |x - 2| < 0.001, \left|\frac{x^2 - 4}{x - 2} - 4\right| < 0.001.$$

63. False; $f(x) = (\sin x)/x$ is undefined when $x = 0$. From Exercise 7, we have

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

65. False; let

$$f(x) = \begin{cases} x^2 - 4x, & x \neq 4 \\ 10, & x = 4 \end{cases}.$$

$f(4) = 10$

$\lim\limits_{x \to 4} f(x) = \lim\limits_{x \to 4} (x^2 - 4x) = 0 \neq 10$

67. $f(x) = \sqrt{x}$

(a) $\lim\limits_{x \to 0.25} \sqrt{x} = 0.5$ is true.

As x approaches $0.25 = \frac{1}{4}$, $f(x) = \sqrt{x}$ approaches $\frac{1}{2} = 0.5$.

(b) $\lim\limits_{x \to 0} \sqrt{x} = 0$ is false.

$f(x) = \sqrt{x}$ is not defined on an open interval containing 0 because the domain of f is $x \geq 0$.

69. If $\lim_{x \to c} f(x) = L_1$ and $\lim_{x \to c} f(x) = L_2$, then for every $\varepsilon > 0$, there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that $|x - c| < \delta_1 \Rightarrow |f(x) - L_1| < \varepsilon$ and $|x - c| < \delta_2 \Rightarrow |f(x) - L_2| < \varepsilon$. Let δ equal the smaller of δ_1 and δ_2.

Then for $|x - c| < \delta$, we have $|L_1 - L_2| = |L_1 - f(x) + f(x) - L_2| \le |L_1 - f(x)| + |f(x) - L_2| < \varepsilon + \varepsilon$.

Therefore, $|L_1 - L_2| < 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows that $L_1 = L_2$.

71. $\lim_{x \to c} [f(x) - L] = 0$ means that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if

$$0 < |x - c| < \delta,$$

then

$$|(f(x) - L) - 0| < \varepsilon.$$

This means the same as $|f(x) - L| < \varepsilon$ when

$$0 < |x - c| < \delta.$$

Thus, $\lim_{x \to c} f(x) = L$.

73. Answers will vary.

75. The radius OP has a length equal to the altitude z of the triangle plus $\frac{h}{2}$. Thus, $z = 1 - \frac{h}{2}$.

$$\text{Area triangle} = \frac{1}{2}b\left(1 - \frac{h}{2}\right)$$

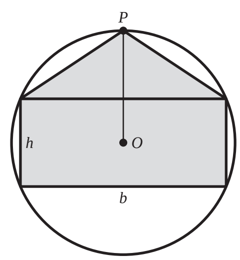
$$\text{Area rectangle} = bh$$

Since these are equal, $\frac{1}{2}b\left(1 - \frac{h}{2}\right) = bh$

$$1 - \frac{h}{2} = 2h$$

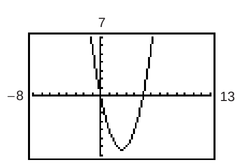
$$\frac{5}{2}h = 1$$

$$h = \frac{2}{5}$$



Section 1.3 Evaluating Limits Analytically

1.

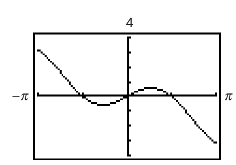


$h(x) = x^2 - 5x$

(a) $\lim_{x \to 5} h(x) = 0$

(b) $\lim_{x \to -1} h(x) = 6$

3.



$f(x) = x \cos x$

(a) $\lim_{x \to 0} f(x) = 0$

(b) $\lim_{x \to \pi/3} f(x) \approx 0.524$
$\left(= \frac{\pi}{6}\right)$

5. $\lim_{x \to 2} x^4 = 2^4 = 16$

7. $\lim_{x \to 0} (2x - 1) = 2(0) - 1 = -1$

9. $\lim_{x \to -3} (x^2 + 3x) = (-3)^2 + 3(-3) = 9 - 9 = 0$

11. $\lim_{x\to -3}(2x^2 + 4x + 1) = 2(-3)^2 + 4(-3) + 1 = 18 - 12 + 1 = 7$

13. $\lim_{x\to 2}\frac{1}{x} = \frac{1}{2}$

15. $\lim_{x\to 1}\frac{x-3}{x^2+4} = \frac{1-3}{1^2+4} = \frac{-2}{5} = -\frac{2}{5}$

17. $\lim_{x\to 7}\frac{5x}{\sqrt{x+2}} = \frac{5(7)}{\sqrt{7+2}} = \frac{35}{\sqrt{9}} = \frac{35}{3}$

19. $\lim_{x\to 3}\sqrt{x+1} = \sqrt{3+1} = 2$

21. $\lim_{x\to -4}(x+3)^2 = (-4+3)^2 = 1$

23. (a) $\lim_{x\to 1} f(x) = 5 - 1 = 4$

(b) $\lim_{x\to 4} g(x) = 4^3 = 64$

(c) $\lim_{x\to 1} g(f(x)) = g(f(1)) = g(4) = 64$

25. (a) $\lim_{x\to 1} f(x) = 4 - 1 = 3$

(b) $\lim_{x\to 3} g(x) = \sqrt{3+1} = 2$

(c) $\lim_{x\to 1} g(f(x)) = g(3) = 2$

27. $\lim_{x\to \pi/2}\sin x = \sin\frac{\pi}{2} = 1$

29. $\lim_{x\to 2}\cos\frac{\pi x}{3} = \cos\frac{2\pi}{3} = -\frac{1}{2}$

31. $\lim_{x\to 0}\sec 2x = \sec 0 = 1$

33. $\lim_{x\to 5\pi/6}\sin x = \sin\frac{5\pi}{6} = \frac{1}{2}$

35. $\lim_{x\to 3}\tan\left(\frac{\pi x}{4}\right) = \tan\frac{3\pi}{4} = -1$

37. (a) $\lim_{x\to c}[5g(x)] = 5\lim_{x\to c} g(x) = 5(3) = 15$

(b) $\lim_{x\to c}[f(x) + g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = 2 + 3 = 5$

(c) $\lim_{x\to c}[f(x)g(x)] = \left[\lim_{x\to c} f(x)\right]\left[\lim_{x\to c} g(x)\right] = (2)(3) = 6$

(d) $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{2}{3}$

39. (a) $\lim_{x\to c}[f(x)]^3 = \left[\lim_{x\to c} f(x)\right]^3 = (4)^3 = 64$

(b) $\lim_{x\to c}\sqrt{f(x)} = \sqrt{\lim_{x\to c} f(x)} = \sqrt{4} = 2$

(c) $\lim_{x\to c}[3f(x)] = 3\lim_{x\to c} f(x) = 3(4) = 12$

(d) $\lim_{x\to c}[f(x)]^{3/2} = \left[\lim_{x\to c} f(x)\right]^{3/2} = (4)^{3/2} = 8$

41. $f(x) = -2x + 1$ and $g(x) = \frac{-2x^2 + x}{x}$ agree except at $x = 0$.

(a) $\lim_{x\to 0} g(x) = \lim_{x\to 0} f(x) = 1$

(b) $\lim_{x\to -1} g(x) = \lim_{x\to -1} f(x) = 3$

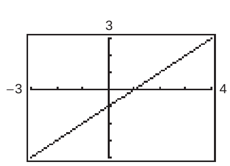
43. $f(x) = x(x+1)$ and $g(x) = \frac{x^3 - x}{x - 1}$ agree except at $x = 1$.

(a) $\lim_{x\to 1} g(x) = \lim_{x\to 1} f(x) = 2$

(b) $\lim_{x\to -1} g(x) = \lim_{x\to -1} f(x) = 0$

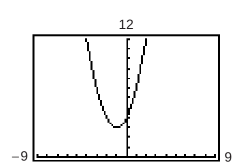
45. $f(x) = \frac{x^2 - 1}{x + 1}$ and $g(x) = x - 1$ agree except at $x = -1$.

$\lim_{x\to -1} f(x) = \lim_{x\to -1} g(x) = -2$



47. $f(x) = \frac{x^3 - 8}{x - 2}$ and $g(x) = x^2 + 2x + 4$ agree except at $x = 2$.

$\lim_{x\to 2} f(x) = \lim_{x\to 2} g(x) = 12$



49. $\lim\limits_{x\to5}\dfrac{x-5}{x^2-25}=\lim\limits_{x\to5}\dfrac{x-5}{(x+5)(x-5)}$

$=\lim\limits_{x\to5}\dfrac{1}{x+5}=\dfrac{1}{10}$

51. $\lim\limits_{x\to-3}\dfrac{x^2+x-6}{x^2-9}=\lim\limits_{x\to-3}\dfrac{(x+3)(x-2)}{(x+3)(x-3)}$

$=\lim\limits_{x\to-3}\dfrac{x-2}{x-3}=\dfrac{-5}{-6}=\dfrac{5}{6}$

53. $\lim\limits_{x\to0}\dfrac{\sqrt{x+5}-\sqrt{5}}{x}=\lim\limits_{x\to0}\dfrac{\sqrt{x+5}-\sqrt{5}}{x}\cdot\dfrac{\sqrt{x+5}+\sqrt{5}}{\sqrt{x+5}+\sqrt{5}}$

$=\lim\limits_{x\to0}\dfrac{(x+5)-5}{x\left(\sqrt{x+5}+\sqrt{5}\right)}=\lim\limits_{x\to0}\dfrac{1}{\sqrt{x+5}+\sqrt{5}}=\dfrac{1}{2\sqrt{5}}=\dfrac{\sqrt{5}}{10}$

55. $\lim\limits_{x\to4}\dfrac{\sqrt{x+5}-3}{x-4}=\lim\limits_{x\to4}\dfrac{\sqrt{x+5}-3}{x-4}\cdot\dfrac{\sqrt{x+5}+3}{\sqrt{x+5}+3}$

$=\lim\limits_{x\to4}\dfrac{(x+5)-9}{(x-4)\left(\sqrt{x+5}+3\right)}=\lim\limits_{x\to4}\dfrac{1}{\sqrt{x+5}+3}=\dfrac{1}{\sqrt{9}+3}=\dfrac{1}{6}$

57. $\lim\limits_{x\to0}\dfrac{\dfrac{1}{3+x}-\dfrac{1}{3}}{x}=\lim\limits_{x\to0}\dfrac{3-(3+x)}{(3+x)3(x)}=\lim\limits_{x\to0}\dfrac{-x}{(3+x)(3)(x)}=\lim\limits_{x\to0}\dfrac{-1}{(3+x)3}=-\dfrac{1}{9}$

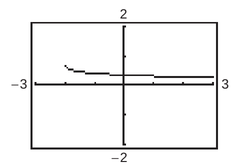
59. $\lim\limits_{\Delta x\to0}\dfrac{2(x+\Delta x)-2x}{\Delta x}=\lim\limits_{\Delta x\to0}\dfrac{2x+2\Delta x-2x}{\Delta x}$

$=\lim\limits_{\Delta x\to0}2=2$

61. $\lim\limits_{\Delta x\to0}\dfrac{(x+\Delta x)^2-2(x+\Delta x)+1-(x^2-2x+1)}{\Delta x}=\lim\limits_{\Delta x\to0}\dfrac{x^2+2x\Delta x+(\Delta x)^2-2x-2\Delta x+1-x^2+2x-1}{\Delta x}$

$=\lim\limits_{\Delta x\to0}(2x+\Delta x-2)=2x-2$

63. $\lim\limits_{x\to0}\dfrac{\sqrt{x+2}-\sqrt{2}}{x}\approx0.354$

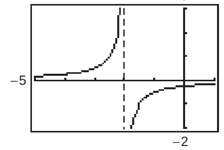


x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	0.358	0.354	0.354	?	0.354	0.353	0.349

Analytically, $\lim\limits_{x\to0}\dfrac{\sqrt{x+2}-\sqrt{2}}{x}=\lim\limits_{x\to0}\dfrac{\sqrt{x+2}-\sqrt{2}}{x}\cdot\dfrac{\sqrt{x+2}+\sqrt{2}}{\sqrt{x+2}+\sqrt{2}}$

$=\lim\limits_{x\to0}\dfrac{x+2-2}{x\left(\sqrt{x+2}+\sqrt{2}\right)}=\lim\limits_{x\to0}\dfrac{1}{\sqrt{x+2}+\sqrt{2}}=\dfrac{1}{2\sqrt{2}}=\dfrac{\sqrt{2}}{4}\approx0.354.$

65. $\lim\limits_{x\to0}\dfrac{\dfrac{1}{2+x}-\dfrac{1}{2}}{x}=-\dfrac{1}{4}$



x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	-0.263	-0.251	-0.250	?	-0.250	-0.249	-0.238

Analytically, $\lim\limits_{x\to0}\dfrac{\dfrac{1}{2+x}-\dfrac{1}{2}}{x}=\lim\limits_{x\to0}\dfrac{2-(2+x)}{2(2+x)}\cdot\dfrac{1}{x}=\lim\limits_{x\to0}\dfrac{-x}{2(2+x)}\cdot\dfrac{1}{x}=\lim\limits_{x\to0}\dfrac{-1}{2(2+x)}=-\dfrac{1}{4}.$

67. $\lim_{x\to 0}\frac{\sin x}{5x} = \lim_{x\to 0}\left[\left(\frac{\sin x}{x}\right)\left(\frac{1}{5}\right)\right] = (1)\left(\frac{1}{5}\right) = \frac{1}{5}$

69. $\lim_{x\to 0}\frac{\sin x(1-\cos x)}{2x^2} = \lim_{x\to 0}\left[\frac{1}{2}\cdot\frac{\sin x}{x}\cdot\frac{1-\cos x}{x}\right]$

$= \frac{1}{2}(1)(0) = 0$

71. $\lim_{x\to 0}\frac{\sin^2 x}{x} = \lim_{x\to 0}\left[\frac{\sin x}{x}\sin x\right] = (1)\sin 0 = 0$

73. $\lim_{h\to 0}\frac{(1-\cos h)^2}{h} = \lim_{h\to 0}\left[\frac{1-\cos h}{h}(1-\cos h)\right]$

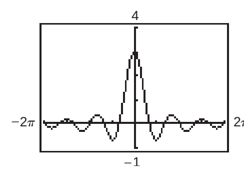
$= (0)(0) = 0$

75. $\lim_{x\to\pi/2}\frac{\cos x}{\cot x} = \lim_{x\to\pi/2}\sin x = 1$

77. $\lim_{t\to 0}\frac{\sin 3t}{2t} = \lim_{t\to 0}\left(\frac{\sin 3t}{3t}\right)\left(\frac{3}{2}\right) = (1)\left(\frac{3}{2}\right) = \frac{3}{2}$

79. $f(t) = \frac{\sin 3t}{t}$

t	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(t)$	2.96	2.9996	3	?	3	2.9996	2.96

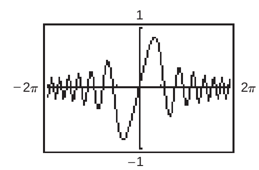


The limit appears to equal 3.

Analytically, $\lim_{t\to 0}\frac{\sin 3t}{t} = \lim_{t\to 0}3\left(\frac{\sin 3t}{3t}\right) = 3(1) = 3.$

81. $f(x) = \frac{\sin x^2}{x}$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	-0.099998	-0.01	-0.001	?	0.001	0.01	0.099998



Analytically, $\lim_{x\to 0}\frac{\sin x^2}{x} = \lim_{x\to 0}x\left(\frac{\sin x^2}{x^2}\right) = 0(1) = 0.$

83. $\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x\to 0}\frac{2(x+\Delta x)+3-(2x+3)}{\Delta x} = \lim_{\Delta x\to 0}\frac{2x+2\Delta x+3-2x-3}{\Delta x} = \lim_{\Delta x\to 0}\frac{2\Delta x}{\Delta x} = 2$

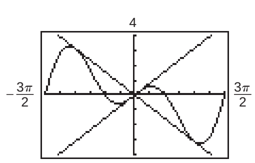
85. $\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x\to 0}\frac{\frac{4}{x+\Delta x}-\frac{4}{x}}{\Delta x} = \lim_{\Delta x\to 0}\frac{4x-4(x+\Delta x)}{(x+\Delta x)x\,\Delta x} = \lim_{\Delta x\to 0}\frac{-4}{(x+\Delta x)x} = \frac{-4}{x^2}$

87. $\lim_{x\to 0}(4-x^2) \le \lim_{x\to 0}f(x) \le \lim_{x\to 0}(4+x^2)$

$4 \le \lim_{x\to 0}f(x) \le 4$

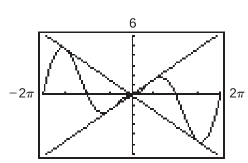
Therefore, $\lim_{x\to 0}f(x) = 4.$

89. $f(x) = x\cos x$



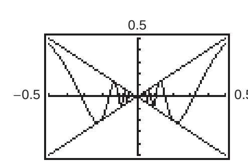
$\lim_{x\to 0}(x\cos x) = 0$

91. $f(x) = |x|\sin x$



$\lim_{x\to 0}|x|\sin x = 0$

93. $f(x) = x\sin\frac{1}{x}$



$\lim_{x\to 0}\left(x\sin\frac{1}{x}\right) = 0$

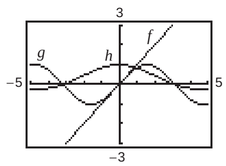
95. We say that two functions f and g agree at all but one point (on an open interval) if $f(x) = g(x)$ for all x in the interval except for $x = c$, where c is in the interval.

97. An indeterminant form is obtained when evaluating a limit using direct substitution produces a meaningless fractional expression such as $0/0$. That is,

$$\lim_{x \to c} \frac{f(x)}{g(x)}$$

for which $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$

99. $f(x) = x,\ g(x) = \sin x,\ h(x) = \dfrac{\sin x}{x}$



When you are "close to" 0 the magnitude of f is approximately equal to the magnitude of g. Thus, $|g|/|f| \approx 1$ when x is "close to" 0.

101. $s(t) = -16t^2 + 1000$

$$\lim_{t \to 5} \frac{s(5) - s(t)}{5 - t} = \lim_{t \to 5} \frac{600 - (-16t^2 + 1000)}{5 - t} = \lim_{t \to 5} \frac{16(t + 5)(t - 5)}{-(t - 5)} = \lim_{t \to 5} -16(t + 5) = -160 \text{ ft/sec.}$$

Speed = 160 ft/sec

103. $s(t) = -4.9t^2 + 150$

$$\lim_{t \to 3} \frac{s(3) - s(t)}{3 - t} = \lim_{t \to 3} \frac{-4.9(3^2) + 150 - (-4.9t^2 + 150)}{3 - t} = \lim_{t \to 3} \frac{-4.9(9 - t^2)}{3 - t}$$

$$= \lim_{t \to 3} \frac{-4.9(3 - t)(3 + t)}{3 - t} = \lim_{t \to 3} -4.9(3 + t) = -29.4 \text{ m/sec}$$

105. Let $f(x) = 1/x$ and $g(x) = -1/x$. $\lim_{x \to 0} f(x)$ and $\lim_{x \to 0} g(x)$ do not exist.

$$\lim_{x \to 0} [f(x) + g(x)] = \lim_{x \to 0} \left[\frac{1}{x} + \left(-\frac{1}{x}\right)\right] = \lim_{x \to 0} [0] = 0$$

107. Given $f(x) = b$, show that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - b| < \varepsilon$ whenever $|x - c| < \delta$. Since $|f(x) - b| = |b - b| = 0 < \varepsilon$ for any $\varepsilon > 0$, then any value of $\delta > 0$ will work.

109. If $b = 0$, then the property is true because both sides are equal to 0. If $b \neq 0$, let $\varepsilon > 0$ be given. Since $\lim_{x \to c} f(x) = L$, there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon/|b|$ whenever $0 < |x - c| < \delta$. Hence, wherever $0 < |x - c| < \delta$, we have

$$|b||f(x) - L| < \varepsilon \quad \text{or} \quad |bf(x) - bL| < \varepsilon$$

which implies that $\lim_{x \to c} [bf(x)] = bL$.

111. $$-M|f(x)| \le f(x)g(x) \le M|f(x)|$$

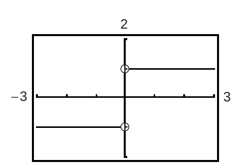
$$\lim_{x \to c} (-M|f(x)|) \le \lim_{x \to c} f(x)g(x) \le \lim_{x \to c} (M|f(x)|)$$

$$-M(0) \le \lim_{x \to c} f(x)g(x) \le M(0)$$

$$0 \le \lim_{x \to c} f(x)g(x) \le 0$$

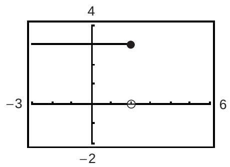
Therefore, $\lim_{x \to c} f(x)g(x) = 0$.

113. False. As x approaches 0 from the left, $\dfrac{|x|}{x} = -1$.



115. True

117. False. The limit does not exist.



119. Let

$$f(x) = \begin{cases} 4, & \text{if } x \geq 0 \\ -4, & \text{if } x < 0 \end{cases}$$

$$\lim_{x \to 0} |f(x)| = \lim_{x \to 0} 4 = 4.$$

$\lim_{x \to 0} f(x)$ does not exist since for $x < 0$, $f(x) = -4$ and for $x \geq 0$, $f(x) = 4$.

121. $f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$

$g(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ x, & \text{if } x \text{ is irrational} \end{cases}$

$\lim_{x \to 0} f(x)$ does not exist.

No matter how "close to" 0 x is, there are still an infinite number of rational and irrational numbers so that $\lim_{x \to 0} f(x)$ does not exist.

$\lim_{x \to 0} g(x) = 0$

When x is "close to" 0, both parts of the function are "close to" 0.

123. (a) $$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{1 - \cos x}{x^2} \cdot \frac{1 + \cos x}{1 + \cos x}$$

$$= \lim_{x \to 0} \frac{1 - \cos^2 x}{x^2(1 + \cos x)}$$

$$= \lim_{x \to 0} \frac{\sin^2 x}{x^2} \cdot \frac{1}{1 + \cos x}$$

$$= (1)\left(\frac{1}{2}\right) = \frac{1}{2}$$

(b) Thus, $\frac{1 - \cos x}{x^2} \approx \frac{1}{2} \implies 1 - \cos x \approx \frac{1}{2}x^2$

$\implies \cos x \approx 1 - \frac{1}{2}x^2$ for $x \approx 0$.

(c) $\cos(0.1) \approx 1 - \frac{1}{2}(0.1)^2 = 0.995$

(d) $\cos(0.1) \approx 0.9950$, which agrees with part (c).

Section 1.4 Continuity and One-Sided Limits

1. (a) $\lim_{x \to 3^+} f(x) = 1$

(b) $\lim_{x \to 3^-} f(x) = 1$

(c) $\lim_{x \to 3} f(x) = 1$

The function is continuous at $x = 3$.

3. (a) $\lim_{x \to 3^+} f(x) = 0$

(b) $\lim_{x \to 3^-} f(x) = 0$

(c) $\lim_{x \to 3} f(x) = 0$

The function is NOT continuous at $x = 3$.

5. (a) $\lim_{x \to 4^+} f(x) = 2$

(b) $\lim_{x \to 4^-} f(x) = -2$

(c) $\lim_{x \to 4} f(x)$ does not exist

The function is NOT continuous at $x = 4$.

7. $\lim_{x \to 5^+} \frac{x - 5}{x^2 - 25} = \lim_{x \to 5^+} \frac{1}{x + 5} = \frac{1}{10}$

9. $\lim_{x \to -3^-} \frac{x}{\sqrt{x^2 - 9}}$ does not exist because $\frac{x}{\sqrt{x^2 - 9}}$ decreases without bound as $x \to -3^-$.

11. $\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = -1$

13. $$\lim_{\Delta x \to 0^-} \frac{\frac{1}{x + \Delta x} - \frac{1}{x}}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{x - (x + \Delta x)}{x(x + \Delta x)} \cdot \frac{1}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{-\Delta x}{x(x + \Delta x)} \cdot \frac{1}{\Delta x}$$
$$= \lim_{\Delta x \to 0^-} \frac{-1}{x(x + \Delta x)}$$
$$= \frac{-1}{x(x + 0)} = -\frac{1}{x^2}$$

15. $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \frac{x + 2}{2} = \frac{5}{2}$

17. $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x + 1) = 2$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^3 + 1) = 2$

$\lim_{x \to 1} f(x) = 2$

19. $\lim_{x \to \pi} \cot x$ does not exist since $\lim_{x \to \pi^+} \cot x$ and $\lim_{x \to \pi^-} \cot x$ do not exist.

21. $\lim_{x \to 4^-} (3[\![x]\!] - 5) = 3(3) - 5 = 4$
$([\![x]\!] = 3$ for $3 \le x < 4)$

23. $\lim_{x \to 3} (2 - [\![-x]\!])$ does not exist because

$\lim_{x \to 3^-} (2 - [\![-x]\!]) = 2 - (-3) = 5$

and

$\lim_{x \to 3^+} (2 - [\![-x]\!]) = 2 - (-4) = 6.$

25. $f(x) = \frac{1}{x^2 - 4}$

has discontinuities at $x = -2$ and $x = 2$ since $f(-2)$ and $f(2)$ are not defined.

27. $f(x) = \frac{[\![x]\!]}{2} + x$

has discontinuities at each integer k since $\lim_{x \to k^-} f(x) \neq \lim_{x \to k^+} f(x)$.

29. $g(x) = \sqrt{25 - x^2}$ is continuous on $[-5, 5]$.

31. $\lim_{x \to 0^-} f(x) = 3 = \lim_{x \to 0^+} f(x)$.
f is continuous on $[-1, 4]$.

33. $f(x) = x^2 - 2x + 1$ is continuous for all real x.

35. $f(x) = 3x - \cos x$ is continuous for all real x.

37. $f(x) = \frac{x}{x^2 - x}$ is not continuous at $x = 0, 1$. Since $\frac{x}{x^2 - x} = \frac{1}{x - 1}$ for $x \neq 0$, $x = 0$ is a removable discontinuity, whereas $x = 1$ is a nonremovable discontinuity.

39. $f(x) = \frac{x}{x^2 + 1}$ is continuous for all real x.

41. $f(x) = \frac{x + 2}{(x + 2)(x - 5)}$

has a nonremovable discontinuity at $x = 5$ since $\lim_{x \to 5} f(x)$ does not exist, and has a removable discontinuity at $x = -2$ since

$$\lim_{x \to -2} f(x) = \lim_{x \to -2} \frac{1}{x - 5} = -\frac{1}{7}.$$

43. $f(x) = \frac{|x + 2|}{x + 2}$

has a nonremovable discontinuity at $x = -2$ since $\lim_{x \to -2} f(x)$ does not exist.

45. $f(x) = \begin{cases} x, & x \le 1 \\ x^2, & x > 1 \end{cases}$

has a **possible** discontinuity at $x = 1$.

1. $f(1) = 1$

2. $\left.\begin{array}{l} \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x = 1 \\ \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x^2 = 1 \end{array}\right\} \lim_{x\to 1} f(x) = 1$

3. $f(1) = \lim_{x\to 1} f(x)$

f is continuous at $x = 1$, therefore, f is continuous for all real x.

47. $f(x) = \begin{cases} \frac{x}{2} + 1, & x \le 2 \\ 3 - x, & x > 2 \end{cases}$

has a **possible** discontinuity at $x = 2$.

1. $f(2) = \frac{2}{2} + 1 = 2$

2. $\begin{array}{l} \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \left(\frac{x}{2} + 1\right) = 2 \\ \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (3 - x) = 1 \end{array}$

Therefore, f has a nonremovable discontinuity at $x = 2$.

49. $f(x) = \begin{cases} \tan \frac{\pi x}{4}, & |x| < 1 \\ x, & |x| \ge 1 \end{cases}$

$= \begin{cases} \tan \frac{\pi x}{4}, & -1 < x < 1 \\ x, & x \le -1 \text{ or } x \ge 1 \end{cases}$

has **possible** discontinuities at $x = -1$, $x = 1$.

1. $f(-1) = -1$ $\quad f(1) = 1$

2. $\lim_{x\to -1} f(x) = -1$ $\quad \lim_{x\to 1} f(x) = 1$

3. $f(-1) = \lim_{x\to -1} f(x)$ $\quad f(1) = \lim_{x\to 1} f(x)$

f is continuous at $x = \pm 1$, therefore, f is continuous for all real x.

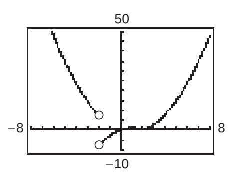
51. $f(x) = \csc 2x$ has nonremovable discontinuities at integer multiples of $\pi/2$.

53. $f(x) = [\![x - 1]\!]$ has nonremovable discontinuities at each integer k.

55. $\lim_{x\to 0^+} f(x) = 0$

$\lim_{x\to 0^-} f(x) = 0$

f is not continuous at $x = -2$.



57. $f(2) = 8$

Find a so that $\lim_{x\to 2^+} ax^2 = 8 \implies a = \frac{8}{2^2} = 2$.

59. Find a and b such that $\lim_{x\to -1^+} (ax + b) = -a + b = 2$ and $\lim_{x\to 3^-} (ax + b) = 3a + b = -2$.

$$\begin{aligned} a - b &= -2 \\ (+)\ 3a + b &= -2 \\ \hline 4a \quad &= -4 \\ a &= -1 \\ b &= 2 + (-1) = 1 \end{aligned}$$

$$f(x) = \begin{cases} 2, & x \le -1 \\ -x + 1, & -1 < x < 3 \\ -2, & x \ge 3 \end{cases}$$

61. $f(g(x)) = (x - 1)^2$

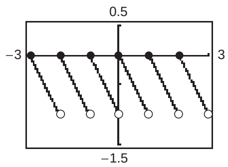
Continuous for all real x.

63. $f(g(x)) = \dfrac{1}{(x^2 + 5) - 6} = \dfrac{1}{x^2 - 1}$

Nonremovable discontinuities at $x = \pm 1$

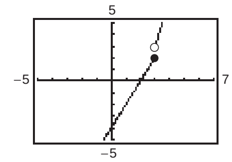
65. $y = [\![x]\!] - x$

Nonremovable discontinuity at each integer



67. $f(x) = \begin{cases} 2x - 4, & x \le 3 \\ x^2 - 2x, & x > 3 \end{cases}$

Nonremovable discontinuity at $x = 3$



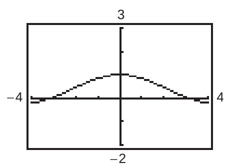
69. $f(x) = \dfrac{x}{x^2 + 1}$

Continuous on $(-\infty, \infty)$

71. $f(x) = \sec \dfrac{\pi x}{4}$

Continuous on:
$\ldots, (-6, -2), (-2, 2), (2, 6), (6, 10), \ldots$

73. $f(x) = \dfrac{\sin x}{x}$



The graph **appears** to be continuous on the interval $[-4, 4]$.
Since $f(0)$ is not defined, we know that f has a discontinuity at $x = 0$. This discontinuity is removable so it does not show up on the graph.

75. $f(x) = \frac{1}{16}x^4 - x^3 + 3$ is continuous on $[1, 2]$.

$f(1) = \frac{33}{16}$ and $f(2) = -4$. By the Intermediate Value Theorem, $f(c) = 0$ for at least one value of c between 1 and 2.

77. $f(x) = x^2 - 2 - \cos x$ is continuous on $[0, \pi]$.

$f(0) = -3$ and $f(\pi) = \pi^2 - 1 > 0$. By the Intermediate Value Theorem, $f(c) = 0$ for the least one value of c between 0 and π.

79. $f(x) = x^3 + x - 1$

$f(x)$ is continuous on $[0, 1]$.

$f(0) = -1$ and $f(1) = 1$

By the Intermediate Value Theorem, $f(x) = 0$ for at least one value of c between 0 and 1. Using a graphing utility, we find that $x \approx 0.6823$.

81. $g(t) = 2 \cos t - 3t$

g is continuous on $[0, 1]$.

$g(0) = 2 > 0$ and $g(1) \approx -1.9 < 0$.

By the Intermediate Value Theorem, $g(t) = 0$ for at least one value c between 0 and 1. Using a graphing utility, we find that $t \approx 0.5636$.

83. $f(x) = x^2 + x - 1$

f is continuous on $[0, 5]$.

$f(0) = -1$ and $f(5) = 29$

$$-1 < 11 < 29$$

The Intermediate Value Theorem applies.

$$x^2 + x - 1 = 11$$
$$x^2 + x - 12 = 0$$
$$(x + 4)(x - 3) = 0$$
$$x = -4 \text{ or } x = 3$$
$$c = 3 \text{ (}x = -4 \text{ is not in the interval.)}$$

Thus, $f(3) = 11$.

85. $f(x) = x^3 - x^2 + x - 2$

f is continuous on $[0, 3]$.

$f(0) = -2$ and $f(3) = 19$

$$-2 < 4 < 19$$

The Intermediate Value Theorem applies.

$$x^3 - x^2 + x - 2 = 4$$
$$x^3 - x^2 + x - 6 = 0$$
$$(x - 2)(x^2 + x + 3) = 0$$
$$x = 2$$

$(x^2 + x + 3$ has no real solution.)

$$c = 2$$

Thus, $f(2) = 4$.

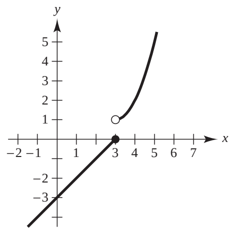
87. (a) The limit does not exist at $x = c$.

(b) The function is not defined at $x = c$.

(c) The limit exists at $x = c$, but it is not equal to the value of the function at $x = c$.

(d) The limit does not exist at $x = c$.

89.



The function is not continuous at $x = 3$ because $\lim_{x\to 3^+} f(x) = 1 \neq 0 = \lim_{x\to 3^-} f(x)$.

91. True

1. $f(c) = L$ is defined.

2. $\lim_{x\to c} f(x) = L$ exists.

3. $f(c) = \lim_{x\to c} f(x)$

All of the conditions for continuity are met.

93. False; a rational function can be written as $P(x)/Q(x)$ where P and Q are polynomials of degree m and n, respectively. It can have, at most, n discontinuities.

95. $\lim_{t\to 4^-} f(t) \approx 28$

$\lim_{t\to 4^+} f(t) \approx 56$

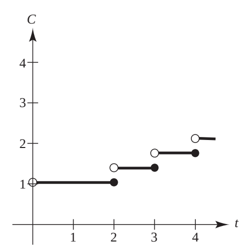
At the end of day 3, the amount of chlorine in the pool has decreased to about 28 oz. At the beginning of day 4, more chlorine was added, and the amount was about 56 oz.

97. $C = \begin{cases} 1.04, & 0 < t \le 2 \\ 1.04 + 0.36[\![t - 1]\!], & t > 2,\ t \text{ is not an integer} \\ 1.04 + 0.36(t - 2), & t > 2,\ t \text{ is an integer} \end{cases}$

Nonremovable discontinuity at each integer greater than or equal to 2.

You can also write C as

$$C = \begin{cases} 1.04, & 0 < t \le 2 \\ 1.04 - 0.36[\![2 - t]\!], & t > 2 \end{cases}.$$



99. Let $s(t)$ be the position function for the run up to the campsite. $s(0) = 0$ ($t = 0$ corresponds to 8:00 A.M., $s(20) = k$ (distance to campsite)). Let $r(t)$ be the position function for the run back down the mountain: $r(0) = k$, $r(10) = 0$. Let $f(t) = s(t) - r(t)$.

When $t = 0$ (8:00 A.M.), $f(0) = s(0) - r(0) = 0 - k < 0$.

When $t = 10$ (8:10 A.M.), $f(10) = s(10) - r(10) > 0$.

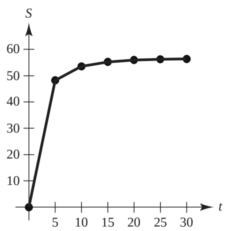
Since $f(0) < 0$ and $f(10) > 0$, then there must be a value t in the interval $[0, 10]$ such that $f(t) = 0$. If $f(t) = 0$, then $s(t) - r(t) = 0$, which gives us $s(t) = r(t)$. Therefore, at some time t, where $0 \le t \le 10$, the position functions for the run up and the run down are equal.

101. Suppose there exists x_1 in $[a, b]$ such that $f(x_1) > 0$ and there exists x_2 in $[a, b]$ such that $f(x_2) < 0$. Then by the Intermediate Value Theorem, $f(x)$ must equal zero for some value of x in $[x_1, x_2]$ (or $[x_2, x_1]$ if $x_2 < x_1$). Thus, f would have a zero in $[a, b]$, which is a contradiction. Therefore, $f(x) > 0$ for all x in $[a, b]$ or $f(x) < 0$ for all x in $[a, b]$.

103. If $x = 0$, then $f(0) = 0$ and $\lim_{x\to 0} f(x) = 0$. Hence, f is continuous at $x = 0$.

If $x \neq 0$, then $\lim_{t\to x} f(t) = 0$ for x rational, whereas $\lim_{t\to x} f(t) = \lim_{t\to x} kt = kx \neq 0$ for x irrational. Hence, f is not continuous for all $x \neq 0$.

105. (a)



(b) There appears to be a limiting speed and a possible cause is air resistance.

107. $f(x) = \begin{cases} 1 - x^2, & x \le c \\ x, & x > c \end{cases}$

f is continuous for $x < c$ and for $x > c$. At $x = c$, you need $1 - c^2 = c$. Solving $c^2 + c - 1$, you obtain

$$c = \frac{-1 \pm \sqrt{1 + 4}}{2} = \frac{-1 \pm \sqrt{5}}{2}.$$

109. $f(x) = \dfrac{\sqrt{x + c^2} - c}{x}, \; c > 0$

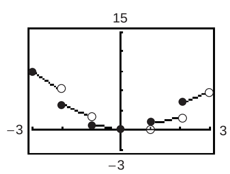
Domain: $x + c^2 \ge 0 \implies x \ge -c^2$ and $x \ne 0$, $[-c^2, 0) \cup (0, \infty)$

$$\lim_{x\to 0} \frac{\sqrt{x + c^2} - c}{x} = \lim_{x\to 0} \frac{\sqrt{x + c^2} - c}{x} \cdot \frac{\sqrt{x + c^2} + c}{\sqrt{x + c^2} + c}$$

$$= \lim_{x\to 0} \frac{(x + c^2) - c^2}{x\left[\sqrt{x + c^2} + c\right]} = \lim_{x\to 0} \frac{1}{\sqrt{x + c^2} + c} = \frac{1}{2c}$$

Define $f(0) = 1/(2c)$ to make f continuous at $x = 0$.

111. $h(x) = x[\![x]\!]$



h has nonremovable discontinuities at

$x = \pm 1, \pm 2, \pm 3, \ldots .$

113. The statement is true.

If $y \ge 0$ and $y \le 1$, then $y(y - 1) \le 0 \le x^2$, as desired. So assume $y > 1$. There are now two cases.

Case 1: If $x \le y - \frac{1}{2}$, then $2x + 1 \le 2y$ and

$$\begin{aligned} y(y - 1) &= y(y + 1) - 2y \\ &\le (x + 1)^2 - 2y \\ &= x^2 + 2x + 1 - 2y \\ &\le x^2 + 2y - 2y \\ &= x^2 \end{aligned}$$

Case 2: If $x \ge y - \frac{1}{2}$

$$\begin{aligned} x^2 &\ge \left(y - \tfrac{1}{2}\right)^2 \\ &= y^2 - y + \tfrac{1}{4} \\ &> y^2 - y \\ &= y(y - 1) \end{aligned}$$

In both cases, $y(y - 1) \le x^2$.

Section 1.5 Infinite Limits

1. $\displaystyle\lim_{x\to -2^+} 2\left|\frac{x}{x^2 - 4}\right| = \infty$

$\displaystyle\lim_{x\to -2^-} 2\left|\frac{x}{x^2 - 4}\right| = \infty$

3. $\displaystyle\lim_{x\to -2^+} \tan\frac{\pi x}{4} = -\infty$

$\displaystyle\lim_{x\to -2^-} \tan\frac{\pi x}{4} = \infty$

5. $f(x) = \dfrac{1}{x^2 - 9}$

x	-3.5	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9	-2.5
$f(x)$	0.308	1.639	16.64	166.6	-166.7	-16.69	-1.695	-0.364

$$\lim_{x \to -3^-} f(x) = \infty$$

$$\lim_{x \to -3^+} f(x) = -\infty$$

7. $f(x) = \dfrac{x^2}{x^2 - 9}$

x	-3.5	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9	-2.5
$f(x)$	3.769	15.75	150.8	1501	-1499	-149.3	-14.25	-2.273

$$\lim_{x \to -3^-} f(x) = \infty$$

$$\lim_{x \to -3^+} f(x) = -\infty$$

9. $\displaystyle\lim_{x \to 0^+} \frac{1}{x^2} = \infty = \lim_{x \to 0^-} \frac{1}{x^2}$

Therefore, $x = 0$ is a vertical asymptote.

11. $\displaystyle\lim_{x \to 2^+} \frac{x^2 - 2}{(x - 2)(x + 1)} = \infty$

$$\lim_{x \to 2^-} \frac{x^2 - 2}{(x - 2)(x + 1)} = -\infty$$

Therefore, $x = 2$ is a vertical asymptote.

$$\lim_{x \to -1^+} \frac{x^2 - 2}{(x - 2)(x + 1)} = \infty$$

$$\lim_{x \to -1^-} \frac{x^2 - 2}{(x - 2)(x + 1)} = -\infty$$

Therefore, $x = -1$ is a vertical asymptote.

13. $\displaystyle\lim_{x \to -2^-} \frac{x^2}{x^2 - 4} = \infty$ and $\displaystyle\lim_{x \to -2^+} \frac{x^2}{x^2 - 4} = -\infty$

Therefore, $x = -2$ is a vertical asymptote.

$\displaystyle\lim_{x \to 2^-} \frac{x^2}{x^2 - 4} = -\infty$ and $\displaystyle\lim_{x \to 2^+} \frac{x^2}{x^2 - 4} = \infty$

Therefore, $x = 2$ is a vertical asymptote.

15. No vertical asymptote since the denominator is never zero.

17. $f(x) = \tan 2x = \dfrac{\sin 2x}{\cos 2x}$ has vertical asymptotes at

$$x = \frac{(2n + 1)\pi}{4} = \frac{\pi}{4} + \frac{n\pi}{2}, \ n \text{ any integer.}$$

19. $\displaystyle\lim_{t \to 0^+} \left(1 - \frac{4}{t^2}\right) = -\infty = \lim_{t \to 0^-} \left(1 - \frac{4}{t^2}\right)$

Therefore, $t = 0$ is a vertical asymptote.

21. $\displaystyle\lim_{x\to -2^+} \frac{x}{(x+2)(x-1)} = \infty$

$\displaystyle\lim_{x\to -2^-} \frac{x}{(x+2)(x-1)} = -\infty$

Therefore, $x = -2$ is a vertical asymptote.

$\displaystyle\lim_{x\to 1^+} \frac{x}{(x+2)(x-1)} = \infty$

$\displaystyle\lim_{x\to 1^-} \frac{x}{(x+2)(x-1)} = -\infty$

Therefore, $x = 1$ is a vertical asymptote.

23. $f(x) = \dfrac{x^3+1}{x+1} = \dfrac{(x+1)(x^2-x+1)}{x+1}$

has no vertical asymptote since

$\displaystyle\lim_{x\to -1} f(x) = \lim_{x\to -1} (x^2 - x + 1) = 3.$

The graph has a hole at $x = -1$.

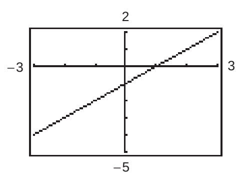
25. $f(x) = \dfrac{(x-5)(x+3)}{(x-5)(x^2+1)} = \dfrac{x+3}{x^2+1}, x \neq 5$

No vertical asymptotes. The graph has a hole at $x = 5$.

27. $s(t) = \dfrac{t}{\sin t}$ has vertical asymptotes at $t = n\pi$, n a nonzero integer. There is no vertical asymptote at $t = 0$ since

$\displaystyle\lim_{t\to 0} \frac{t}{\sin t} = 1.$

29. $\displaystyle\lim_{x\to -1} \frac{x^2-1}{x+1} = \lim_{x\to -1} (x-1) = -2$

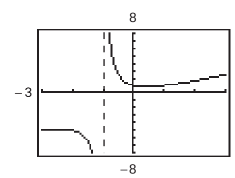


Removable discontinuity at $x = -1$

31. $\displaystyle\lim_{x\to -1^+} \frac{x^2+1}{x+1} = \infty$

$\displaystyle\lim_{x\to -1^-} \frac{x^2+1}{x+1} = -\infty$

Vertical asymptote at $x = -1$



33. $\displaystyle\lim_{x\to 2^+} \frac{x-3}{x-2} = -\infty$

35. $\displaystyle\lim_{x\to 3^+} \frac{x^2}{(x-3)(x+3)} = \infty$

37. $\displaystyle\lim_{x\to -3^-} \frac{x^2+2x-3}{x^2+x-6} = \lim_{x\to -3^-} \frac{(x-1)(x+3)}{(x-2)(x+3)} = \lim_{x\to -3^-} \frac{x-1}{x-2} = \frac{4}{5}$

39. $\displaystyle\lim_{x\to 1} \frac{x^2-x}{(x^2+1)(x-1)} = \lim_{x\to 1} \frac{x}{x^2+1} = \frac{1}{2}$

41. $\displaystyle\lim_{x\to 0^-} \left(1 + \frac{1}{x}\right) = -\infty$

43. $\displaystyle\lim_{x\to 0^+} \frac{2}{\sin x} = \infty$

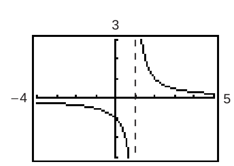
45. $\displaystyle\lim_{x\to \pi} \frac{\sqrt{x}}{\csc x} = \lim_{x\to \pi} \left(\sqrt{x}\sin x\right) = 0$

47. $\displaystyle\lim_{x\to (1/2)^-} x\sec(\pi x) = \infty$ and $\displaystyle\lim_{x\to (1/2)^+} x\sec(\pi x) = -\infty$.

Therefore, $\displaystyle\lim_{x\to (1/2)} x\sec(\pi x)$ does not exist.

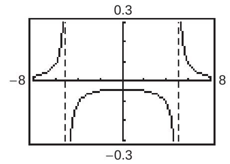
49. $f(x) = \dfrac{x^2+x+1}{x^3-1} = \dfrac{x^2+x+1}{(x-1)(x^2+x+1)}$

$\displaystyle\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{1}{x-1} = \infty$



51. $f(x) = \dfrac{1}{x^2 - 25}$

$\lim\limits_{x \to 5^-} f(x) = -\infty$



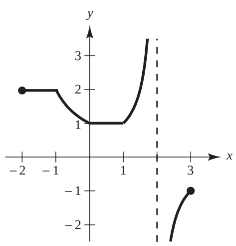
53. A limit in which $f(x)$ increases or decreases without bound as x approaches c is called an infinite limit. ∞ is not a number. Rather, the symbol

$$\lim_{x \to c} f(x) = \infty$$

says how the limit fails to exist.

55. One answer is $f(x) = \dfrac{x - 3}{(x - 6)(x + 2)} = \dfrac{x - 3}{x^2 - 4x - 12}$.

57.



59. (a) $r = 50\pi \sec^2 \dfrac{\pi}{6} = \dfrac{200\pi}{3}$ ft/sec

(b) $r = 50\pi \sec^2 \dfrac{\pi}{3} = 200\pi$ ft/sec

(c) $\lim\limits_{\theta \to (\pi/2)^-} [50\pi \sec^2 \theta] = \infty$

61. $m = \dfrac{m_0}{\sqrt{1 - (v^2/c^2)}}$

$$\lim_{v \to c^-} m = \lim_{v \to c^-} \frac{m_0}{\sqrt{1 - (v^2/c^2)}} = \infty$$

63. (a) Average speed $= \dfrac{\text{Total distance}}{\text{Total time}}$

$$50 = \frac{2d}{(d/x) + (d/y)}$$

$$50 = \frac{2xy}{y + x}$$

$$50y + 50x = 2xy$$

$$50x = 2xy - 50y$$

$$50x = 2y(x - 25)$$

$$\frac{25x}{x - 25} = y$$

Domain: $x > 25$

(b)

x	30	40	50	60
y	150	6.667	50	42.857

(c) $\lim\limits_{x \to 25^+} \dfrac{25x}{x - 25} = \infty$

As x gets close to 25 mph, y becomes larger and larger.

65. (a) $A = \dfrac{1}{2}bh - \dfrac{1}{2}r^2\,\theta = \dfrac{1}{2}(10)(10 \tan \theta) - \dfrac{1}{2}(10)^2\theta$

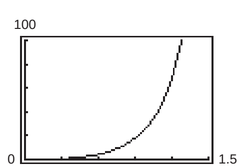
$= 50 \tan \theta - 50\,\theta$

Domain: $\left(0, \dfrac{\pi}{2}\right)$

(b)

θ	0.3	0.6	0.9	1.2	1.5
$f(\theta)$	0.47	4.21	18.0	68.6	630.1

(c)



(d) $\lim\limits_{\theta \to \pi/2^-} A = \infty$

67. False; for instance, let

$$f(x) = \frac{x^2 - 1}{x - 1}.$$

The graph of f has a hole at $(1, 2)$, not a vertical asymptote.

69. True

71. Let $f(x) = \dfrac{1}{x^2}$ and $g(x) = \dfrac{1}{x^4}$, and $c = 0$.

$\lim\limits_{x\to 0} \dfrac{1}{x^2} = \infty$ and $\lim\limits_{x\to 0} \dfrac{1}{x^4} = \infty$, but

$$\lim_{x\to 0} \left(\frac{1}{x^2} - \frac{1}{x^4}\right) = \lim_{x\to 0} \left(\frac{x^2 - 1}{x^4}\right) = -\infty \neq 0.$$

73. Given $\lim\limits_{x\to c} f(x) = \infty$, let $g(x) = 1$. then $\lim\limits_{x\to c} \dfrac{g(x)}{f(x)} = 0$ by Theorem 1.15.

75. $f(x) = \dfrac{1}{x - 3}$ is defined for all $x > 3$. Let $M > 0$ be given.

We need $\delta > 0$ such that $f(x) = \dfrac{1}{x - 3} > M$ whenever $3 < x < 3 + \delta$

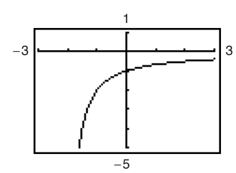
Equivalently, $x - 3 < \dfrac{1}{M}$ whenever $|x - 3| < \delta, x > 3$.

So take $\delta = \dfrac{1}{M}$. Then for $x > 3$ and $|x - 3| < \delta, \dfrac{1}{x - 3} > \dfrac{1}{8} = M$ and hence $f(x) > M$.

Review Exercises for Chapter 1

1. Calculus required. Using a graphing utility, you can estimate the length to be 8.3. Or, the length is slightly longer than the distance between the two points, approximately 8.25.

3. $f(x) = \dfrac{\dfrac{4}{x + 2} - 2}{x}$



x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	-1.0526	-1.0050	-1.0005	-0.9995	-0.9950	-0.9524

$\lim\limits_{x\to 0} f(x) \approx -1.0$

5. $h(x) = \dfrac{x^2 - 2x}{x}$

(a) $\lim\limits_{x\to 0} h(x) = -2$

(b) $\lim\limits_{x\to -1} h(x) = -3$

7. $\lim\limits_{x\to 1} (3 - x) = 3 - 1 = 2$

Let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon$. Then for $0 < |x - 1| < \delta = \varepsilon$, you have

$$|x - 1| < \varepsilon$$
$$|1 - x| < \varepsilon$$
$$|(3 - x) - 2| < \varepsilon$$
$$|f(x) - L| < \varepsilon.$$

9. $\lim\limits_{x\to 2}(x^2-3)=1$

Let $\varepsilon > 0$ be given. We need $|x^2-3-1| < \varepsilon \Rightarrow |x^2-4| = |(x-2)(x+2)| < \varepsilon \Rightarrow |x-2| < \dfrac{1}{|x+2|}\varepsilon.$

Assuming, $1 < x < 3$, you can choose $\delta = \varepsilon/5$. Hence, for $0 < |x-2| < \delta = \varepsilon/5$ you have

$$\begin{aligned}|x-2| &< \frac{\varepsilon}{5} < \frac{1}{|x+2|}\varepsilon\\ |x-2||x+2| &< \varepsilon\\ |x^2-4| &< \varepsilon\\ |(x^2-3)-1| &< \varepsilon\\ |f(x)-L| &< \varepsilon.\end{aligned}$$

11. $\lim\limits_{t\to 4}\sqrt{t+2} = \sqrt{4+2} = \sqrt{6} \approx 2.45$

13. $\lim\limits_{t\to -2}\dfrac{t+2}{t^2-4} = \lim\limits_{t\to -2}\dfrac{1}{t-2} = -\dfrac{1}{4}$

15. $$\begin{aligned}\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4} &= \lim_{x\to 4}\frac{\sqrt{x}-2}{\left(\sqrt{x}-2\right)\left(\sqrt{x}+2\right)}\\ &= \lim_{x\to 4}\frac{1}{\sqrt{x}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{4}\end{aligned}$$

17. $\lim\limits_{x\to 0}\dfrac{[1/(x+1)]-1}{x} = \lim\limits_{x\to 0}\dfrac{1-(x+1)}{x(x+1)} = \lim\limits_{x\to 0}\dfrac{-1}{x+1} = -1$

19. $$\begin{aligned}\lim_{x\to -5}\frac{x^3+125}{x+5} &= \lim_{x\to -5}\frac{(x+5)(x^2-5x+25)}{x+5}\\ &= \lim_{x\to -5}(x^2-5x+25) = 75\end{aligned}$$

21. $\lim\limits_{x\to 0}\dfrac{1-\cos x}{\sin x} = \lim\limits_{x\to 0}\left(\dfrac{x}{\sin x}\right)\left(\dfrac{1-\cos x}{x}\right) = (1)(0) = 0$

23. $$\begin{aligned}\lim_{\Delta x\to 0}\frac{\sin[(\pi/6)+\Delta x]-(1/2)}{\Delta x} &= \lim_{\Delta x\to 0}\frac{\sin(\pi/6)\cos\Delta x+\cos(\pi/6)\sin\Delta x-(1/2)}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{1}{2}\cdot\frac{(\cos\Delta x-1)}{\Delta x} + \lim_{\Delta x\to 0}\frac{\sqrt{3}}{2}\cdot\frac{\sin\Delta x}{\Delta x} = 0 + \frac{\sqrt{3}}{2}(1) = \frac{\sqrt{3}}{2}\end{aligned}$$

25. $\lim\limits_{x\to c}[f(x)\cdot g(x)] = \left(-\frac{3}{4}\right)\left(\frac{2}{3}\right) = -\frac{1}{2}$

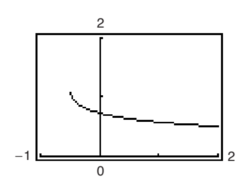
27. $f(x) = \dfrac{\sqrt{2x+1}-\sqrt{3}}{x-1}$

(a)

x	1.1	1.01	1.001	1.0001
$f(x)$	0.5680	0.5764	0.5773	0.5773

$\lim\limits_{x\to 1^+}\dfrac{\sqrt{2x+1}-\sqrt{3}}{x-1} \approx 0.577$ $\left(\text{Actual limit is } \sqrt{3}/3.\right)$

(b)



—CONTINUED—

27. —CONTINUED—

(c) $\lim_{x\to1^+} \frac{\sqrt{2x+1}-\sqrt{3}}{x-1} = \lim_{x\to1^+} \frac{\sqrt{2x+1}-\sqrt{3}}{x-1} \cdot \frac{\sqrt{2x+1}+\sqrt{3}}{\sqrt{2x+1}+\sqrt{3}}$

$= \lim_{x\to1^+} \frac{(2x+1)-3}{(x-1)\left(\sqrt{2x+1}+\sqrt{3}\right)}$

$= \lim_{x\to1^+} \frac{2}{\sqrt{2x+1}+\sqrt{3}}$

$= \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

29. $\lim_{t\to a} \frac{s(a)-s(t)}{a-t} = \lim_{t\to4} \frac{(-4.9(4)^2+200)-(-4.9t^2+200)}{4-t}$

$= \lim_{t\to4} \frac{4.9(t-4)(t+4)}{4-t}$

$= \lim_{t\to4} -4.9(t+4) = -39.2$ m/sec

31. $\lim_{x\to3^-} \frac{|x-3|}{x-3} = \lim_{x\to3^-} \frac{-(x-3)}{x-3}$

$= -1$

33. $\lim_{x\to2^-} f(x) = 0$

35. $\lim_{t\to1} h(t)$ does not exist because

$\lim_{t\to1^-} h(t) = 1 + 1 = 2$ and

$\lim_{t\to1^+} h(t) = \frac{1}{2}(1+1) = 1.$

37. $f(x) = [\![x+3]\!]$

$\lim_{x\to k^+} [\![x+3]\!] = k+3$ where k is an integer.

$\lim_{x\to k^-} [\![x+3]\!] = k+2$ where k is an integer.

Nonremovable discontinuity at each integer k

Continuous on $(k, k+1)$ for all integers k

39. $f(x) = \frac{3x^2-x-2}{x-1} = \frac{(3x+2)(x-1)}{x-1}$

$\lim_{x\to1} f(x) = \lim_{x\to1} (3x+2) = 5$

Removable discontinuity at $x = 1$

Continuous on $(-\infty, 1) \cup (1, \infty)$

41. $f(x) = \frac{1}{(x-2)^2}$

$\lim_{x\to2} \frac{1}{(x-2)^2} = \infty$

Nonremovable discontinuity at $x = 2$

Continuous on $(-\infty, 2) \cup (2, \infty)$

43. $f(x) = \frac{3}{x+1}$

$\lim_{x\to1^-} f(x) = -\infty$

$\lim_{x\to1^+} f(x) = \infty$

Nonremovable discontinuity at $x = -1$

Continuous on $(-\infty, -1) \cup (-1, \infty)$

45. $f(x) = \csc \frac{\pi x}{2}$

Nonremovable discontinuities at each even integer.

Continuous on

$(2k, 2k+2)$

for all integers k.

47. $f(2) = 5$

Find c so that $\lim_{x\to2^+} (cx+6) = 5$.

$c(2) + 6 = 5$

$2c = -1$

$c = -\frac{1}{2}$

49. f is continuous on $[1, 2]$.
$f(1) = -1 < 0$ and
$f(2) = 13 > 0$. Therefore by the Intermediate Value Theorem, there is at least one value c in $(1, 2)$ such that $2c^3 - 3 = 0$.

51. $f(x) = \frac{x^2-4}{|x-2|} = (x+2)\left[\frac{x-2}{|x-2|}\right]$

(a) $\lim_{x\to2^-} f(x) = -4$ (b) $\lim_{x\to2^+} f(x) = 4$ (c) $\lim_{x\to2} f(x)$ does not exist.

53. $g(x) = 1 + \dfrac{2}{x}$

Vertical asymptote at $x = 0$

55. $f(x) = \dfrac{8}{(x-10)^2}$

Vertical asymptote at $x = 10$

57. $\displaystyle\lim_{x\to -2^-} \frac{2x^2 + x + 1}{x + 2} = -\infty$

59. $\displaystyle\lim_{x\to -1^+} \frac{x+1}{x^3+1} = \lim_{x\to -1^+} \frac{1}{x^2 - x + 1} = \frac{1}{3}$

61. $\displaystyle\lim_{x\to 1^-} \frac{x^2 + 2x + 1}{x - 1} = -\infty$

63. $\displaystyle\lim_{x\to 0^+} \left(x - \frac{1}{x^3}\right) = -\infty$

65. $\displaystyle\lim_{x\to 0^+} \frac{\sin 4x}{5x} = \lim_{x\to 0^+} \left[\frac{4}{5}\left(\frac{\sin 4x}{4x}\right)\right] = \frac{4}{5}$

67. $\displaystyle\lim_{x\to 0^+} \frac{\csc 2x}{x} = \lim_{x\to 0^+} \frac{1}{x \sin 2x} = \infty$

69. $C = \dfrac{80{,}000p}{100 - p},\ 0 \le p < 100$

(a) $C(15) \approx \$14{,}117.65$ (b) $C(50) = \$80.000$

(c) $C(90) = \$720{,}000$ (d) $\displaystyle\lim_{p\to 100^-} \frac{80{,}000p}{100 - p} = \infty$

Problem Solving for Chapter 1

1. (a) $\text{Perimeter } \triangle PAO = \sqrt{x^2 + (y-1)^2} + \sqrt{x^2 + y^2} + 1$

$$= \sqrt{x^2 + (x^2-1)^2} + \sqrt{x^2 + x^4} + 1$$

$\text{Perimeter } \triangle PBO = \sqrt{(x-1)^2 + y^2} + \sqrt{x^2 + y^2} + 1$

$$= \sqrt{(x-1)^2 + x^4} + \sqrt{x^2 + x^4} + 1$$

(b) $r(x) = \dfrac{\sqrt{x^2 + (x^2-1)^2} + \sqrt{x^2 + x^4} + 1}{\sqrt{(x-1)^2 + x^4} + \sqrt{x^2 + x^4} + 1}$

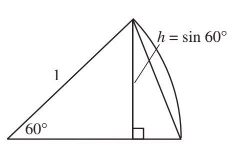
x	4	2	1	0.1	0.01
Perimeter $\triangle PAO$	33.02	9.08	3.41	2.10	2.01
Perimeter $\triangle PBO$	33.77	9.60	3.41	2.00	2.00
$r(x)$	0.98	0.95	1	1.05	1.005

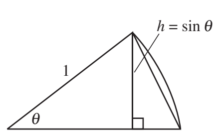
(c) $\displaystyle\lim_{x\to 0^+} r(x) = \frac{1+0+1}{1+0+1} = \frac{2}{2} = 1$

3. (a) There are 6 triangles, each with a central angle of $60° = \pi/3$. Hence,

$$\text{Area hexagon} = 6\left[\frac{1}{2}bh\right] = 6\left[\frac{1}{2}(1)\sin\frac{\pi}{3}\right]$$

$$= \frac{3\sqrt{3}}{2} \approx 2.598.$$





Error: $\pi - \dfrac{3\sqrt{3}}{2} \approx 0.5435$

(b) There are n triangles, each with central angle of $\theta = 2\pi/n$. Hence,

$$A_n = n\left[\frac{1}{2}bh\right] = n\left[\frac{1}{2}(1)\sin\frac{2\pi}{n}\right] = \frac{n\sin(2\pi/n)}{2}.$$

(c)

n	6	12	24	48	96
A_n	2.598	3	3.106	3.133	3.139

(d) As n gets larger and larger, $2\pi/n$ approaches 0.

Letting $x = 2\pi/n$,

$$A_n = \frac{\sin(2\pi/n)}{2/n} = \frac{\sin(2\pi/n)}{(2\pi/n)}\pi = \frac{\sin x}{x}\pi$$

which approaches $(1)\pi = \pi$.

5. (a) Slope $= -\dfrac{12}{5}$

(b) Slope of tangent line is $\dfrac{5}{12}$.

$$y + 12 = \frac{5}{12}(x - 5)$$

$$y = \frac{5}{12}x - \frac{169}{12} \quad \text{Tangent line}$$

(c) $Q = (x, y) = \left(x, -\sqrt{169 - x^2}\right)$

$$m_x = \frac{-\sqrt{169 - x^2} + 12}{x - 5}$$

(d) $$\lim_{x\to 5} m_x = \lim_{x\to 5} \frac{12 - \sqrt{169 - x^2}}{x - 5} \cdot \frac{12 + \sqrt{169 - x^2}}{12 + \sqrt{169 - x^2}}$$

$$= \lim_{x\to 5} \frac{144 - (169 - x^2)}{(x - 5)\left(12 + \sqrt{169 - x^2}\right)}$$

$$= \lim_{x\to 5} \frac{x^2 - 25}{(x - 5)\left(12 + \sqrt{169 - x^2}\right)}$$

$$= \lim_{x\to 5} \frac{(x + 5)}{12 + \sqrt{169 - x^2}} = \frac{10}{12 + 12} = \frac{5}{12}$$

This is the same slope as part (b).

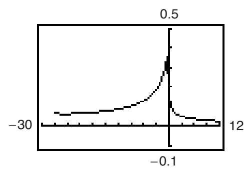
7. (a) $3 + x^{1/3} \geq 0$

$$x^{1/3} \geq -3$$

$$x \geq -27$$

Domain: $x \geq -27, x \neq 1$

(b)



(c) $$\lim_{x\to -27^+} f(x) = \frac{\sqrt{3 + (-27)^{1/3}} - 2}{-27 - 1}$$

$$= \frac{-2}{-28} = \frac{1}{14} \approx 0.0714$$

(d) $$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{\sqrt{3 + x^{1/3}} - 2}{x - 1} \cdot \frac{\sqrt{3 + x^{1/3}} + 2}{\sqrt{3 + x^{1/3}} + 2}$$

$$= \lim_{x\to 1} \frac{3 + x^{1/3} - 4}{(x - 1)\left(\sqrt{3 + x^{1/3}} + 2\right)}$$

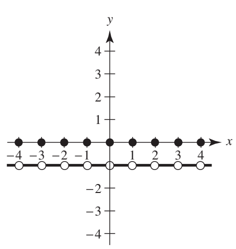
$$= \lim_{x\to 1} \frac{x^{1/3} - 1}{(x^{1/3} - 1)(x^{2/3} + x^{1/3} + 1)\left(\sqrt{3 + x^{1/3}} + 2\right)}$$

$$= \lim_{x\to 1} \frac{1}{(x^{2/3} + x^{1/3} + 1)\left(\sqrt{3 + x^{1/3}} + 2\right)}$$

$$= \frac{1}{(1 + 1 + 1)(2 + 2)} = \frac{1}{12}$$

9. (a) $\lim_{x\to 2} f(x) = 3$: g_1, g_4 (b) f continuous at 2: g_1 (c) $\lim_{x\to 2^-} f(x) = 3$: g_1, g_3, g_4

11.



(a) $f(1) = [\![1]\!] + [\![-1]\!] = 1 + (-1) = 0$

$f(0) = 0$

$f\left(\frac{1}{2}\right) = 0 + (-1) = -1$

$f(-2.7) = -3 + 2 = -1$

(b) $\lim_{x\to 1^-} f(x) = -1$

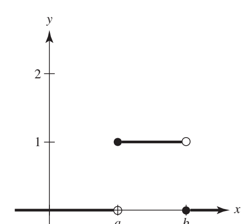
$\lim_{x\to 1^+} f(x) = -1$

$\lim_{x\to 1/2} f(x) = -1$

(c) f is continuous for all real numbers except

$x = 0, \pm 1, \pm 2, \pm 3, \ldots$

13. (a)



(b) (i) $\lim_{x\to a^+} P_{a,b}(x) = 1$

(ii) $\lim_{x\to a^-} P_{a,b}(x) = 0$

(iii) $\lim_{x\to b^+} P_{a,b}(x) = 0$

(iv) $\lim_{x\to b^-} P_{a,b}(x) = 1$

(c) $P_{a,b}$ is continuous for all positive real numbers except $x = a, b$.

(d) The area under the graph of U, and above the x-axis, is 1.