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Author(s): M. Lavine

Source: *Biometrika*, Vol. 82, No. 1 (Mar., 1995), pp. 220-222
Published by: Oxford University Press on behalf of Biometrika Trust
Stable URL: http://www.jstor.org/stable/2337641
Accessed: 11-10-2016 22:29 UTC

Biometrika (1995), **82**, 1, *pp.* 220–2
Printed in Great Britain

On an approximate likelihood for quantiles

By M. LAVINE

Institute of Statistics and Decision Sciences, Duke University, Box 90251, Durham, North Carolina 27708-0251, U.S.A.

Summary

Let $Z_1, \ldots, Z_n$ be a random sample from F, an uncertain one-dimensional distribution function, and suppose that a prior distribution is available only for θ, a vector of quantiles of F. Bayesian inference is difficult because the likelihood function is not fully specified. This paper considers a method of approximating the likelihood function and shows that it provides conservative inferences.

Some key words: Likelihood; Quantile.

1. Substitute likelihoods for quantiles

Let $Z=(Z_1, \ldots, Z_n)$ be a random sample from F, an uncertain one-dimensional distribution function, and suppose that a prior distribution is available only for θ, some of the quantiles of F. We use F^A and F^B to represent candidates for F, and f^A and f^B to denote their densities, assumed to exist. A posterior cannot be calculated because the likelihood function $f(Z|\theta)$ has not been fully specified. Approximations to the likelihood function, depending only on θ, would be useful, along with any information about the likely size or direction of the error. Two approximations have been proposed in the literature for special cases, one by Jeffreys (1961, § 4.4) and one by Boos & Monahan (1986). This is a note about Jeffreys's approximation. Throughout the paper we omit Z as an argument to functions and write, for example, $l(f)$ for the likelihood function $f(Z)$.

Let $p=(p_1, \ldots, p_m)$ satisfying $0<p_1<\ldots<p_m<1$ be given, and let

$$\theta=(-\infty=\theta_0, \theta_1, \ldots, \theta_{m+1}=\infty)$$

be the uncertain vector of quantiles such that $F(\theta_i)=p_i$. The approximation of Jeffreys (1961, § 4.4) is for inference about the median of F, the case where $m=1$ and $p_1=0{\cdot}5$. Define $k(\theta)=\sum 1_{(-\infty,\theta_1]}(Z_i)$, the number of observations to the left of θ. Jeffreys (1961, § 4.4) suggests using

$$s(\theta)=\binom{n}{k(\theta)}\left(\tfrac{1}{2}\right)^n$$

as a substitute likelihood function.

Jeffreys says that his suggestion yields a 'valid uncertainty'. Monahan & Boos (1992) point out that $s(\theta)$ is not the conditional distribution of the data given any statistic and say that inference based on $s(\theta)$ is 'invalid'. Nonetheless, Jeffreys's suggestion has obvious appeal and is easily generalised to inference for a vector of quantiles. For arbitrary m let $k(\theta)=(k_1(\theta), \ldots, k_{m+1}(\theta))$ be the numbers of observations falling into the $m+1$ bins into which the real line is divided by θ. More formally, $k_i(\theta)=\sum 1_{(\theta_{i-1},\theta_i]}(Z_j)$. Then base inference on the substitute likelihood function

$$s(\theta)=\binom{n}{k_1 k_2 \ldots k_{m+1}}\prod \Delta p_i^{k_i},$$

where $\Delta p=(p_1, p_2-p_1, \ldots, 1-p_m)$.

2. Conservative inference

When a prior distribution is specified for θ but not for other features of F then Bayes' theorem is not available and s might be useful as a substitute likelihood function. How does the substitution err?

Suppose that Z is generated from model F^A, the true value of F, and that F^A has quantiles θ^A. Let $\theta^B \neq \theta^A$ be another set of quantiles and F^B a distribution having those quantiles. If we restrict attention to F^A and F^B then the evidence in Z for distinguishing between θ^A and θ^B is the log-likelihood ratio $\log\{l(f^A)/l(f^B)\}$.

If $\log\{s(\theta^B)l(f^A)/s(\theta^A)l(f^B)\} \geqslant 0$ then s is distinguishing between θ^A and θ^B less well than the log-likelihood. The quantity $\log\{s(\theta^B)l(f^A)/s(\theta^A)l(f^B)\}$ depends on the sample and will not, in general, be nonnegative for all possible values of Z. If, for every $\theta^B \neq \theta^A$ and F^B with quantiles θ^B,

$$\liminf_{n\to\infty} \log\left\{\frac{s(\theta^B)l(f^A)}{s(\theta^A)l(f^B)}\right\} \geqslant 0$$

with probability one under F^A, we call s asymptotically conservative at the truth.

We shall show that s is asymptotically conservative at the truth. Let $\Delta q_i = F^A(\theta_i^B) - F^A(\theta_{i-1}^B)$. Let $X_i = X(Z_i)$ identify the bin into which Z_i falls; that is X_i satisfies $Z_i \in (\theta_{X_i-1}^B, \theta_{X_i}^B]$. And finally, let $\mathrm{KL}\,(F^A, F^B;X) = \sum \Delta q_i \log(\Delta q_i/\Delta p_i)$ be the Kullback–Leibler divergence between F^A and F^B in the experiment X. The limits in the following theorem and proof are almost surely under F^A; that notation is suppressed in the proof.

THEOREM 1. *We have*

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{s(\theta^A)}{s(\theta^B)} = \mathrm{KL}\,(F^A, F^B; X)$$

almost surely under F^A.

Proof. We have

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} \log \frac{s(\theta^A)}{s(\theta^B)} &= \lim_{n\to\infty} \frac{1}{n} \log\left\{\frac{\prod k_i(\theta^B)! \prod \Delta p_i^{k_i(\theta^A)}}{\prod k_i(\theta^A)! \prod \Delta p_i^{k_i(\theta^B)}}\right\} \\
&= \lim_{n\to\infty} \frac{1}{n}\left[\sum \log \frac{k_i(\theta^B)!}{k_i(\theta^A)!} + \sum\{k_i(\theta^A) - k_i(\theta^B)\}\log \Delta p_i\right] \\
&= \lim_{n\to\infty} \frac{1}{n}\left[\sum \log \frac{k_i(\theta^B)^{k_i(\theta^B)+\frac{1}{2}}}{k_i(\theta^A)^{k_i(\theta^A)+\frac{1}{2}}} + \sum\{k_i(\theta^A) - k_i(\theta^B)\}\log \Delta p_i\right] \\
&= \lim_{n\to\infty} [\textstyle\sum\{\Delta q_i \log k_i(\theta^B) - \Delta p_i \log k_i(\theta^A)\} + \sum(\Delta p_i - \Delta q_i)\log \Delta p_i] \\
&= \lim_{n\to\infty}\left[\sum\left\{\Delta q_i \log\frac{k_i(\theta^B)}{n} - \Delta p_i \log \frac{k_i(\theta^A)}{n}\right\} + \sum(\Delta p_i - \Delta q_i)\log \Delta p_i\right] \\
&= \textstyle\sum \Delta q_i(\log \Delta q_i - \log \Delta p_i) = \mathrm{KL}\,(F^A, F^B;X). \qquad \square
\end{aligned}$$

COROLLARY 1. *We have that s is asymptotically conservative at the truth.*

Proof. We have

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{l(f^A)s(\theta^B)}{l(f^B)s(\theta^A)} = \mathrm{KL}\,(F^A, F^B; X) + \mathrm{KL}\,(F^A, F^B; Z|X) + \lim \frac{1}{n} \log \frac{s(\theta^B)}{s(\theta^A)}.$$

The theorem shows that the first and last terms cancel each other. The middle term is a Kullback–Leibler divergence and hence nonnegative. $\square$

Acknowledgement

This research was supported by the National Science Foundation.

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[*Received April* 1993. *Revised July* 1994]