Weak Hamiltonian, CP Violation and Rare Decays *

Andrzej J. Buras

Technische Universität München
Physik Department
D-85748 Garching, Germany

Abstract

These lectures describe in detail the effective Hamiltonians for weak decays of mesons constructed by means of the operator product expansion and the renormalization group method. We calculate Wilson coeffcients of local operators, discuss mixing of operators under renormalization, the anomalous dimensions of operators and anomalous dimension matrices. We elaborate on the renormalization scheme and renormalization scale dependences and their cancellations in physical amplitudes. In particular we discuss the issue of γ_5 in D-dimensions and the role of evanescent operators in the calculation of two-loop anomalous dimensions. We present an explicit calculation of the 6×6 one-loop anomalous dimension matrix involving current-current and QCD-penguin operators and we give some hints how to properly calculate two-loop anomalous dimensions of these operators. In the phenonomenological part of these lectures we discuss in detail: CKM matrix, the unitarity triangle and its determination, two-body non-leptonic B-decays and the generalized factorization, the ratio ε'/ε , $B \to X_s \gamma$, $K^+ \to \pi^+ \nu \bar{\nu}$, $K_L \to \pi^0 \nu \bar{\nu}$, $B \to X_s \nu \bar{\nu}$, $B_s \to \mu \bar{\mu}$ and some aspects of CP violation in B-decays.

^{*}Dedicated to my father Bronislaw Buras (1915-1994).

To appear in "Probing the Standard Model of Particle Interactions", F. David and R. Gupta, eds, 1998 Elsevier Science B.V.

Contents

1	Intr	roduction	1
	1.1	General View	1
	1.2	Strategy	6
	1.3	Whats New in these Lectures	9
2	Firs	st Steps	10
	2.1	The Basic Lagrangian	10
	2.2	Elementary Vertices	12
	2.3	CKM Matrix	15
		2.3.1 General Remarks	15
		2.3.2 Standard Parametrization	16
		2.3.3 Wolfenstein Parameterization	17
		2.3.4 Wolfenstein Parametrization beyond LO	17
		2.3.5 Unitarity Triangle	19
		2.3.6 Final Comments	21
3	FC	NC Processes	23
	3.1	General Remarks	23
	3.2	Effective Vertices	24
		3.2.1 Penguin vertices	24
		3.2.2 Box vertices	24
		3.2.3 Effective Feynman Rules	24
		3.2.4 Basic Functions	27
		3.2.5 Explicit Calculation of the Box Diagrams	30
	3.3	Effective Hamiltonians for FCNC Processes	31
		3.3.1 An Example	31
		3.3.2 Penguin-Box Expansion	32
	3.4	More about GIM	33
	3.5	Final Comments	35
4	Ren	normalization and Renormalization Group	35
	4.1	General Remarks	35
	4.2	QCD Lagrangian	35
	4.3	Dimensional Regularization	36
	4.4	The Issue of γ_5 in D Dimensions	37
		4.4.1 Preliminaries	37

		4.4.2	Naive Dimensional Regularization	37
		4.4.3	Dimensional Reduction	38
		4.4.4	The 't Hooft–Veltman Rules	39
	4.5	Renor	malization	41
		4.5.1	General Remarks	41
		4.5.2	The Counter–term Method	41
		4.5.3	MS and $\overline{\rm MS}$ Renormalization Schemes	42
		4.5.4	Renormalization of Green Functions	43
	4.6	Renor	malization Group Equations	44
		4.6.1	The Basic Equations	44
		4.6.2	Compendium of Useful Results	46
	4.7	Runni	ng Coupling Constant	46
	4.8	Runni	ng Quark Mass	48
	4.9	RG In	aproved Perturbation Theory	49
	4.10	Final (Comments	50
_	0	, .		
5	_		Product Expansion in Weak Decays	50
	5.1		ninaries	50
	5.2		Idea	50
	5.3		Approach	51
	5.4		and Short Distance QCD Effects	54
		5.4.1	Preliminaries	54
		5.4.2	Calculation of Wilson Coefficients	55
		5.4.3	A Different Look	59
		5.4.4	Operator Mixing and Diagonalization	59
		5.4.5	Factorization of SD and LD	60
		5.4.6	Independence of C_i from External States	61
	5.5		and the Renormalization Group	61
		5.5.1	Preliminaries	61
		5.5.2	Renormalization Group Equations for C_{\pm}	62
		5.5.3	Choice of the Matching Scale	64
		5.5.4	Threshold Effects in LO	64
		5.5.5	RGE for C_i : Case of Operator Mixing	64
	5.6		ary of Basic Formalism	66
	5.7	Future	e Generalizations	68
	5.8	Motiva	ations for NLO	68

6	\mathbf{Wil}	son Coefficients Beyond Leading Order	69
	6.1	Preliminaries	69
	6.2	The Case without Operator Mixing	69
	6.3	The Case of Operator Mixing	71
		6.3.1 Preliminaries	71
		6.3.2 Determination of $\vec{C}(\mu_W)$	71
		6.3.3 Renormalization Group Evolution	72
		6.3.4 Final Result for $\vec{C}(\mu)$	74
	6.4	The Calculation of the Anomalous Dimensions	74
		6.4.1 Master Formulae	74
		6.4.2 How to Use One-Loop Master Formulae	75
		6.4.3 How to Use Two-Loop Master Formulae	76
		6.4.4 A Warning on the HV Scheme	78
	6.5	Explicit Calculation of 2×2 Anomalous Dimension Matrix	79
		6.5.1 Current-Current Insertions: Generalities	79
		6.5.2 Anomalous Dimensions of Q_1 and Q_2	80
	6.6	Mixing of Operators with different Dimensions	82
	6.7	Renormalization Scheme Dependence	82
	6.8	Renormalization Scale Dependence	85
	6.9	Evanescent Operators	86
		6.9.1 Origin of Evanescent Operators	86
		6.9.2 Including Evanescent Operators in the Master Formulae	88
		6.9.3 How to Use the Improved Master Formulae $\dots \dots \dots \dots$	91
		6.9.4 Evanescent Scheme Dependences	93
7	The	Effective $\Delta F = 1$ Hamiltonian: Current-Current	
	Ope	rators	95
	7.1	Basic Formalism	95
	7.2	Numerical Results for B-Decays	97
	7.3	Numerical Results for D- and K-Decays	97
	7.4	Discussion	98
8	Ger	eralizations	98
	8.1	Preliminaries	98
	8.2	$K^+ \to \pi^+ \nu \bar{\nu} \dots \dots$	98
	8.3	$B_d^0 - \bar{B}_d^0$ Mixing	100
		8.3.1 Proliminaries	100

	8.3.2	LO Analysis	101
	8.3.3	NLO Analysis	102
8.4	QCD I	Penguins	104
	8.4.1	Operators	104
	8.4.2	Effective Hamiltonian	105
	8.4.3	Wilson Coefficients	105
	8.4.4	Matching Conditions for QCD Penguins	107
	8.4.5	Numerical Values for $C_i(\mu_b)$	110
	8.4.6	Threshold Effects in the Presence of Penguins	110
8.5	Explic	it Calculation of 6×6 Anomalous Dimension Matrix	111
	8.5.1	Preliminaries	111
	8.5.2	Penguin Insertions: Generalities	111
	8.5.3	Explicit Calculation of Penguin Insertions	113
	8.5.4	Explicit Calculation of Current-Current Insertions	115
	8.5.5	Putting Things together	117
	8.5.6	An Advice	117
8.6	Electro	oweak Penguins	117
	8.6.1	Operators	117
	8.6.2	Wilson Coefficients	118
	8.6.3	Numerical Values	120
8.7	Magne	etic Penguins	121
8.8	Semi-I	Leptonic Operators	122
8.9	Charm	Quarks in Electroweak Loops	125
8.10	Pengu	in–Box Expansion from OPE	126
8.11	Status	of NLO Calculations	127
8.12	Final 1	Remarks	128
Non	-Lepto	onic Two-body Decays and Factorization	129
9.1	-	inaries	129
9.2		ization	129
9.3		alized Factorization	131
9.4		erent Formulation	134
9.5		rical Analysis	137
9.6		alized Factorization and N^{eff}	140
9.7		ary	142

9

10 ε_K ,	B^0 - \bar{B}^0 Mixing and the Unitarity Triangle	143
10.1	Preliminaries	143
10.2	Express Review of $K^0 - \bar{K}^0$ Mixing	144
10.3	The First Look at ε and ε'	147
10.4	Basic Formula for ε	149
10.5	Basic Formula for B^0 - \bar{B}^0 Mixing	151
10.6	Standard Analysis of the Unitarity Triangle	153
10.7	Numerical Results	156
	10.7.1 Input Parameters	156
	10.7.2 $ V_{ub}/V_{cb} $, $ V_{cb} $ and ε_K	157
	10.7.3 Output of the Standard Analysis	157
	10.7.4 Correlation between ε_K and ΔM_d	160
10.8	Final Remarks	161
11 ε'/ε	in the Standard Model	162
11.1	Preliminaries	162
11.2	History of ε'/ε	162
11.3	Basic Formulae	163
11.4	Hadronic Matrix Elements	166
	11.4.1 Preliminaries	166
	11.4.2 $(V - A) \otimes (V - A)$ Operators	167
	11.4.3 $(V - A) \otimes (V + A)$ Operators	
	11.4.4 $B_6^{(1/2)}$ and $B_8^{(3/2)}$ from Lattice	168
	11.4.5 $B_6^{(1/2)}$ and $B_8^{(3/2)}$ from the 1/N Approach	169
	11.4.6 $B_6^{(1/2)}$ and $B_8^{(3/2)}$ from the Chiral Quark Model	172
	11.4.7 Strategy for $(V - A) \otimes (V + A)$ Operators	172
11.5	An Analytic Formula for ε'/ε	174
11.6	The Status of the Strange Quark Mass	176
11.7	Numerical Results for ε'/ε	177
11.8	Summary	179
12 B -	$ ightarrow X_s \gamma$	181
12.1	General Remarks	181
12.2	The Decay $B \to X_s \gamma$ in the Leading Log Approximation	183
	12.2.1 Anomalous Dimension Matrix	183
	12.2.2 Renormalization Group Evolution	185
	12.2.3 Scale Uncertainties at LO	187

	12.3	$B \to X_s \gamma$ Beyond Leading Logarithms
		12.3.1 Master Formulae
		12.3.2 Going Beyond the Spectator Model
		12.3.3 Numerical Analysis at NLO
	12.4	Recent Developments
	12.5	Experimental Status
	12.6	A Look Beyond the Standard Model
	12.7	Summary and Outlook
13	Rai	$\mathbf{re}\ K$ - and B -Decays
		General Remarks
		The Decay $K^+ \to \pi^+ \nu \bar{\nu}$
		13.2.1 The effective Hamiltonian
		13.2.2 Deriving the Branching Ratio
		13.2.3 Basic Phenomenology
		13.2.4 Numerical Analysis of $K^+ \to \pi^+ \nu \bar{\nu}$
		13.2.5 $ V_{td} $ from $K^+ \to \pi^+ \nu \bar{\nu} \dots \dots$
		13.2.6 Summary and Outlook
	13.3	The Decay $K_{\rm L} \to \pi^0 \nu \bar{\nu}$
		13.3.1 The effective Hamiltonian
		13.3.2 Deriving the Branching Ratio
		13.3.3 Master Formulae for $Br(K_L \to \pi^0 \nu \bar{\nu})$
		13.3.4 $ V_{cb} $ and $\mathrm{Im}\lambda_t$ from $K_L \to \pi^0 \nu \bar{\nu}$
		13.3.5 Numerical Analysis of $K_{\rm L} \to \pi^0 \nu \bar{\nu}$
		13.3.6 Summary and Outlook
	13.4	Unitarity Triangle and $\sin 2\beta$ from $K \to \pi \nu \bar{\nu}$
	13.5	$K \to \pi \nu \bar{\nu}$ Beyond the Standard Model
	13.6	The Decays $B \to X_{s,d} \nu \bar{\nu}$
		13.6.1 Effective Hamiltonian
		13.6.2 The Branching Ratios
	13.7	The Decays $B_{s,d} \to l^+ l^-$
		13.7.1 The Effective Hamiltonian
		13.7.2 The Branching Ratios
		13.7.3 Outlook
	13.8	Higher Order Electroweak Effects in Rare Decays

14 Future Visions			
	14.1	Preliminaries	223
	14.2	CP-Asymmetries in B-Decays	223
	14.3	CP-Asymmetries in B-Decays versus $K \to \pi \nu \bar{\nu}$	227
	14.4	Unitarity Triangle from $K_{\rm L} \to \pi^0 \nu \bar{\nu}$ and $\sin 2\alpha$	229
	14.5	Unitarity Triangle and $ V_{cb} $ from $\sin 2\alpha$, $\sin 2\beta$ and $K_L \to \pi^0 \nu \bar{\nu}$	230
	14.6	Unitarity Triangle from R_t and $\sin 2\beta$	230
15	Sum	amary and Outlook	23 0
16	Fina	d Messages	233

1 Introduction

1.1 General View

The basic starting point for any serious phenomenology of weak decays of hadrons is the effective weak Hamiltonian which has the following generic structure

$$\mathcal{H}_{eff} = \frac{G_F}{\sqrt{2}} \sum_{i} V_{\text{CKM}}^i C_i(\mu) Q_i . \qquad (1.1)$$

Here G_F is the Fermi constant and Q_i are the relevant local operators which govern the decays in question. The Cabibbo-Kobayashi-Maskawa factors V_{CKM}^i [1, 2] and the Wilson Coefficients C_i [3, 4] describe the strength with which a given operator enters the Hamiltonian.

In the simplest case of the β -decay, \mathcal{H}_{eff} takes the familiar form

$$\mathcal{H}_{eff}^{(\beta)} = \frac{G_F}{\sqrt{2}} \cos \theta_c [\bar{u}\gamma_\mu (1 - \gamma_5)d \otimes \bar{e}\gamma^\mu (1 - \gamma_5)\nu_e] , \qquad (1.2)$$

where V_{ud} has been expressed in terms of the Cabibbo angle. In this particular case the Wilson Coefficient is equal unity and the local operator, the object between the square brackets, is given by a product of two V - A currents. This local operator is represented by the diagram (b) in fig. 1. Equation (1.2) represents the Fermi theory for β -decays as formulated by

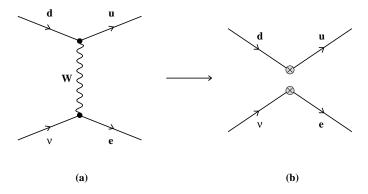


Figure 1: β -decay at the quark level in the full (a) and effective (b) theory.

Sudarshan and Marshak [5] and Feynman and Gell-Mann [6] forty years ago, except that in (1.2) the quark language has been used and following Cabibbo a small departure of V_{ud} from unity has been incorporated. In this context the basic formula (1.1) can be regarded as a generalization of the Fermi Theory to include all known quarks and leptons as well as their strong and electroweak interactions as summarized by the Standard Model. It should be stressed that the formulation of weak decays in terms of effective Hamiltonians is very suitable for the inclusion of new physics effects. We will discuss this issue briefly in these lectures.

Now, I am aware of the fact that the formal operator language used here is hated by experimentalists and frequently disliked by more phenomenological minded theorists. Consequently the literature on weak decays, in particular on B-meson decays, is governed by Feynman diagram drawings with W-, Z- and top quark exchanges, rather than by the operators in (1.1). In the case of the β -decay we have the diagram (a) in fig. 1. Yet such Feynman diagrams with full W-propagators, Z-propagators and top-quark propagators really represent the situation at very short distance scales $\mathcal{O}(M_{W,Z}, m_t)$, whereas the true picture of a decaying hadron with masses $\mathcal{O}(m_b, m_c, m_K)$ is more properly described by effective point-like vertices which are represented by the local operators Q_i . The Wilson coefficients C_i can then be regarded as coupling constants associated with these effective vertices.

Thus \mathcal{H}_{eff} in (1.1) is simply a series of effective vertices multiplied by effective coupling constants C_i . This series is known under the name of the operator product expansion (OPE) [3, 4, 7]. Due to the interplay of electroweak and strong interactions the structure of the local operators (vertices) is much richer than in the case of the β -decay. They can be classified with respect to the Dirac structure, colour structure and the type of quarks and leptons relevant for a given decay. Of particular interest are the operators involving quarks only. They govern the non-leptonic decays.

Now what about the couplings $C_i(\mu)$ and the scale μ ? The important point is that $C_i(\mu)$ summarize the physics contributions from scales higher than μ and due to asymptotic freedom of QCD they can be calculated in perturbation theory as long as μ is not too small. C_i include the top quark contributions and contributions from other heavy particles such as W, Z-bosons and charged Higgs particles or supersymmetric particles in the supersymmetric extensions of the Standard Model. At higher orders in the electroweak coupling the neutral Higgs may also contribute. Consequently $C_i(\mu)$ depend generally on m_t and also on the masses of new particles if extensions of the Standard Model are considered. This dependence can be found by evaluating so-called box and penguin diagrams with full W-, Z-, top- and new particles exchanges and properly including short distance QCD effects. The latter govern the μ -dependence of the couplings $C_i(\mu)$.

The value of μ can be chosen arbitrarily. As we will see below it serves to separate the physics contributions to a given decay amplitude into short-distance contributions at scales higher than μ and long-distance contributions corresponding to scales lower than μ . It is customary to choose μ to be of the order of the mass of the decaying hadron. This is $\mathcal{O}(m_b)$ and $\mathcal{O}(m_c)$ for B-decays and D-decays respectively. In the case of K-decays the typical choice is $\mu = \mathcal{O}(1-2 \ GeV)$ instead of $\mathcal{O}(m_K)$, which is much too low for any perturbative calculation of the couplings C_i .

Now due to the fact that $\mu \ll M_{W,Z}$, m_t , large logarithms $\ln M_W/\mu$ compensate in the

evaluation of $C_i(\mu)$ the smallness of the QCD coupling constant α_s and terms $\alpha_s^n(\ln M_W/\mu)^n$, $\alpha_s^n(\ln M_W/\mu)^{n-1}$ etc. have to be resummed to all orders in α_s before a reliable result for C_i can be obtained. This can be done very efficiently by means of the renormalization group methods [8, 9, 10]. The resulting renormalization group improved perturbative expansion for $C_i(\mu)$ in terms of the effective coupling constant $\alpha_s(\mu)$ does not involve large logarithms and is more reliable.

It should be stressed at this point that the construction of the effective Hamiltonian \mathcal{H}_{eff} by means of the operator product expansion and the renormalization group methods can be done fully in the perturbative framework. The fact that the decaying hadrons are bound states of quarks is irrelevant for this construction. Consequently the coefficients $C_i(\mu)$ are independent of the particular decay considered in the same manner in which the usual gauge couplings are universal and process independent.

So far so good. Having constructed the effective Hamiltonian we can proceed to evaluate the decay amplitudes. An amplitude for a decay of a given meson M=K,B,... into a final state $F=\pi\nu\bar{\nu},\ \pi\pi,\ DK$ is simply given by

$$A(M \to F) = \langle F | \mathcal{H}_{eff} | M \rangle = \frac{G_F}{\sqrt{2}} \sum_{i} V_{CKM}^i C_i(\mu) \langle F | Q_i(\mu) | M \rangle, \tag{1.3}$$

where $\langle F|Q_i(\mu)|M\rangle$ are the hadronic matrix elements of Q_i between M and F. As indicated in (1.3) these matrix elements depend similarly to $C_i(\mu)$ on μ . They summarize the physics contributions to the amplitude $A(M \to F)$ from scales lower than μ .

We realize now the essential virtue of OPE: it allows to separate the problem of calculating the amplitude $A(M \to F)$ into two distinct parts: the *short distance* (perturbative) calculation of the couplings $C_i(\mu)$ and the *long-distance* (generally non-perturbative) calculation of the matrix elements $\langle Q_i(\mu) \rangle$. The scale μ , as advertised above, separates then the physics contributions into short distance contributions contained in $C_i(\mu)$ and the long distance contributions contained in $\langle Q_i(\mu) \rangle$. By evolving this scale from $\mu = \mathcal{O}(M_W)$ down to lower values one simply transforms the physics contributions at scales higher than μ from the hadronic matrix elements into $C_i(\mu)$. Since no information is lost this way the full amplitude cannot depend on μ . Therefore the μ -dependence of the couplings $C_i(\mu)$ has to cancel the μ -dependence of $\langle Q_i(\mu) \rangle$. In other words it is a matter of choice what exactly belongs to $C_i(\mu)$ and what to $\langle Q_i(\mu) \rangle$. This cancellation of μ -dependence involves generally several terms in the expansion in (1.3).

Clearly, in order to calculate the amplitude $A(M \to F)$, the matrix elements $\langle Q_i(\mu) \rangle$ have to be evaluated. Since they involve long distance contributions one is forced in this case to use non-perturbative methods such as lattice calculations, the 1/N expansion (N is the number of colours), QCD sum rules, hadronic sum rules, chiral perturbation theory and so

on. In the case of certain B-meson decays, the *Heavy Quark Effective Theory* (HQET) also turns out to be a useful tool. Needless to say, all these non-perturbative methods have some limitations. Consequently the dominant theoretical uncertainties in the decay amplitudes reside in the matrix elements $\langle Q_i(\mu) \rangle$.

The fact that in most cases the matrix elements $\langle Q_i(\mu) \rangle$ cannot be reliably calculated at present, is very unfortunate. One of the main goals of the experimental studies of weak decays is the determination of the CKM factors V_{CKM} and the search for the physics beyond the Standard Model. Without a reliable estimate of $\langle Q_i(\mu) \rangle$ this goal cannot be achieved unless these matrix elements can be determined experimentally or removed from the final measurable quantities by taking the ratios or suitable combinations of amplitudes or branching ratios. However, this can be achieved only in a handful of decays and generally one has to face directly the calculation of $\langle Q_i(\mu) \rangle$.

Now in the case of semi-leptonic decays, in which there is at most one hadron in the final state, the chiral perturbation theory in the case of K-decays and HQET in the case of B-decays have already provided useful estimates of the relevant matrix elements. This way it was possible to achieve satisfactory determinations of the CKM elements V_{us} and V_{cb} in $K \to \pi e \nu$ and $B \to D^* e \nu$ respectively. We will also see that some rare decays like $K \to \pi \nu \bar{\nu}$ and $B \to \mu \bar{\mu}$ can be calculated very reliably.

The case of non-leptonic decays in which the final state consists exclusively out of hadrons is a completely different story. Here even the matrix elements entering the simplest decays, the two-body decays like $K \to \pi\pi$, $D \to K\pi$ or $B \to DK$ cannot be calculated in QCD reliably at present. For this reason approximative schemes for these decays can be found in the literature. One of such schemes, the factorization scheme for matrix elements has been popular for some time among experimentalists and phenomenologists. The other approach is the diagrammatic approach [11], in which the decay amplitudes are decomposed into various contributions corresponding to certain flavour-flow topologies which in the literature appear unter the names of "trees", "colour-suppressed trees", "penguins", "annihilation" etc. Supplemented by isospin symmetry, the approximate SU(3) flavour symmetry and various "plausible" dynamical assumptions the diagrammatic approach has been used extensively for non-leptonic B-decays during the nineties.

As we will see in these lectures the factorization approach has several limitations and an improved treatment of non-leptonic B-decays, beyond this approach, is called for. We will have no time to discuss the diagrammatic approach, which goes beyond the factorization approach, but also here improvements are necessary. For K-decays some progress in this direction has been done by means of the 1/N approach, hadronic sum rules, chiral perturbation theory and lattice calculations. However, these techniques will not be discussed here as they

are the subjects of other lectures at this summer school. We will only collect necessary results obtained in these approaches.

Returning to the Wilson coefficients $C_i(\mu)$ it should be stressed that similar to the effective coupling constants they do not depend only on the scale μ but also on the renormalization scheme used: this time on the scheme for the renormalization of local operators. That the local operators undergo renormalization is not surprising. After all they represent effective vertices and as the usual vertices in a field theory they have to be renormalized when quantum corrections like QCD or QED corrections are taken into account. As a consequence of this, the hadronic matrix elements $\langle Q_i(\mu) \rangle$ are renormalization scheme dependent and this scheme dependence must be cancelled by the one of $C_i(\mu)$ so that the physical amplitudes are renormalization scheme independent. Again, as in the case of the μ -dependence, the cancellation of the renormalization scheme dependence involves generally several terms in the expansion (1.3).

Now the μ and the renormalization scheme dependences of the couplings $C_i(\mu)$ can be evaluated efficiently in the renormalization group improved perturbation theory. Unfortunately the incorporation of these dependences in the non-perturbative evaluation of the matrix elements $\langle Q_i(\mu) \rangle$ remains as an important challenge and most of the non-perturbative methods on the market are insensitive to these dependences. The consequence of this unfortunate situation is obvious: the resulting decay amplitude are μ and renormalization scheme dependent which introduces potential theoretical uncertainty in the predictions. On the other hand we will see in the course of these lectures that in certain decays these dependences are rather mild.

So far I have discussed only *exclusive* decays. It turns out that in the case of *inclusive* decays of heavy mesons, like B-mesons, things turn out to be easier. In an inclusive decay one sums over all (or over a special class) of accessible final states so that the amplitude for an inclusive decay takes the form:

$$A(B \to X) = \frac{G_F}{\sqrt{2}} \sum_{f \in X} V_{\text{CKM}}^i C_i(\mu) \langle f | Q_i(\mu) | B \rangle . \tag{1.4}$$

At first sight things look as complicated as in the case of exclusive decays. It turns out, however, that the resulting branching ratio can be calculated in the expansion in inverse powers of m_b with the leading term described by the spectator model in which the B-meson decay is modelled by the decay of the b-quark:

$$Br(B \to X) = Br(b \to q) + \mathcal{O}(\frac{1}{m_b^2}). \tag{1.5}$$

This formula is known under the name of the Heavy Quark Expansion (HQE) [12]-[14]. Since the leading term in this expansion represents the decay of the quark, it can be calculated in perturbation theory or more correctly in the renormalization group improved perturbation theory. It should be realized that also here the basic starting point is the effective Hamiltonian (1.1) and that the knowledge of the couplings $C_i(\mu)$ is essential for the evaluation of the leading term in (1.5). But there is an important difference relative to the exclusive case: the matrix elements of the operators Q_i can be "effectively" evaluated in perturbation theory. This means, in particular, that their μ and renormalization scheme dependences can be evaluated and the cancellation of these dependences by those present in $C_i(\mu)$ can be investigated.

Clearly in order to complete the evaluation of $Br(B \to X)$ also the remaining terms in (1.5) have to be considered. These terms are of a non-perturbative origin, but fortunately they are suppressed by at least two powers of m_b . They have been studied by several authors in the literature with the result that they affect various branching ratios by less then 10% and often by only a few percent. Consequently the inclusive decays give generally more precise theoretical predictions at present than the exclusive decays. On the other hand their measurements are harder. There are of course some important theoretical issues related to the validity of HQE in (1.5) which appear in the literature under the name of quark-hadron duality. Since these matters are discussed in detail by Mark Wise in his lectures, I will not discuss them here and will use HQE as God given.

We have learned now that the matrix elements of Q_i are easier to handle in inclusive decays than in the exclusive ones. On the other hand the evaluation of the couplings $C_i(\mu)$ is equally difficult in both cases although as stated above it can be done in a perturbative framework. Still in order to achieve sufficient precision for the theoretical predictions it is desirable to have accurate values of these couplings. Indeed it has been realized at the end of the eighties that the leading term (LO) in the renormalization group improved perturbation theory, in which the terms $\alpha_s^n(\ln M_W/\mu)^n$ are summed, is generally insufficient and the inclusion of next-to-leading corrections (NLO) which correspond to summing the terms $\alpha_s^n(\ln M_W/\mu)^{n-1}$ is necessary. In particular, unphysical left-over μ -dependences in the decay amplitudes and branching ratios resulting from the truncation of the perturbative series are considerably reduced by including NLO corrections. These corrections are known by now for the most important and interesting decays and will constitute a considerable part of these lectures.

1.2 Strategy

Like in any serious climb we need some strategy for these lectures. We will see that some parts of our tour will be rather easy, some other parts rather technical and difficult. The tour consists of seven sections (2–8) devoted to the basic formalism of weak decays and seven sections (9–15), which present in detail some selected applications of this formalism. The

map of our route is given the contents. Here we go.

As always we begin with *First Steps*. They will allow us to collect most elementary ingredients of the Standard Model including elementary vertices, propagators and the corresponding Feynman rules. This is also the place to discuss the CKM matrix, its most convenient parametrizations and the related unitarity triangle.

Having this general information at hand we will move next to discuss FCNC processes in rather general terms. The idea here is to collect most important effective vertices resulting from penguin and box diagrams and give for them Feynman rules. We will then see that there are seven basic m_t -dependent functions which enter the effective vertices in question and thereby determine the strength of FCNC transitions.

We will illustrate the derivation of effective rules by calculating explicitly two simplest vertices. In this part we will also have a first look at effective weak Hamiltonians which are the main objectives of these lectures. This will allow us to discuss briefly GIM mechanism [15] and give a description of the so-called penguin-box expansion [16]: version of OPE particularly suitable for the study of the $m_{\rm t}$ -dependence of FCNC processes. Here, obviously, the seven basic functions mentioned above will play the crucial role.

The two first parts just described actually have an introductory character. They are really like a gentle hike to our base camp. From now on the matters begin to be more difficult. Particularly difficult are sections 4,5,6 and 8 which are really at the heart of these lectures. Any serious student who wants to learn the field of weak Hamiltonians at the level needed for professional applications should study these sections in great detail. The reason being that it is not sufficient for a good phenomenology to simply copy from some papers the values of Wilson coefficients and insert them in some formulae also copied from still other papers. There are so many subtleties in this field that without a sufficient understanding of section 4,5,6 and 8 it will be difficult to avoid errors in phenomenological applications of the formalism presented there. So what can be found in the basic sections 4,5,6 and 8?

Section 4 is devoted to the renormalization and the renormalization group in QCD. In particular we will discuss the dimensional regularization paying some tribute to the issue of γ_5 in $D \neq 4$ dimensions. We will discuss the MS and $\overline{\rm MS}$ renormalization schemes giving the list of the most important renormalization constants. Some of these constants will be calculated. Subsequently we will move to discuss renormalization group equations offering several explicit derivations. This section culminates in the analysis of the running QCD coupling, the analysis of the running quark mass and the introduction of the concept of the renormalization group improved perturbation theory.

The formal discussion of weak hamiltonians is given first in sections 5 and 6. Section 5 introduces the concept of the operator product expansion and discusses in great detail the

implementation of the renormalization group techniques into OPE in the leading logarithmic approximation (LO). Section 6 can be considered as the generalization of section 5 to include next-to-leading logarithmic corrections (NLO). The basic actors of sections 5 and 6 are so-called current-current operators Q_1 and Q_2 . Only these operators are discussed in these two sections. My strategy here is to present in the simplest setting most important concepts of this field. Thus we will discuss Wilson coefficients of local operators, mixing of operators under renormalization, anomalous dimensions and anomalous dimension matrices, matching of the full theory to the effective theory, renormalization scheme and renormalization scale dependences and their cancellations, the issue of evanescent operators in the calculation of two-loop anomalous dimensions and several other things. In particular we will give master formulae for anomalous dimensions and a procedure for a correct calculation of Wilson coefficient functions including NLO corrections. We will also present an explicit calculation of the anomalous dimension matrix for the operators (Q_1, Q_2) at the one-loop level and we will give some hints how to properly calculate two-loop anomalous dimensions of these operators.

After a short break (section 7), in which a numerical study of the results of sections 5 and 6 will be presented, we will move to section 8 which can be considered as a generalization of sections 5 and 6 to include other operators: the penguin operators of various sorts and the operators originating in the box diagrams. In particular we will derive the proper matching for penguin operators and we will provide explicit calculations of the 6×6 one-loop anomalous dimension matrix for the current-current and QCD penguin operators. The material of this section should allow the reader to follow without difficulties the applications of OPE and renormalization group to any decay at the NLO level present in the literature.

The remaining sections of these lectures amount simply to the applications of the formalism developed in sections 4,5,6 and 8. It will also turn out that our brief numerical analysis of section 7 was not accidental.

I would like to stress that sections 9–15 should not be considered as a comprehensive review of the phenomenology of the full field of weak decays. Certainly not! There are several issues which I have omitted completely: one of them is the $\Delta I=1/2$ issue in non-leptonic decays, the other two are the popular rare decays $K_L \to \pi^0 e^- e^+$ and $B \to X_s \mu \bar{\mu}$. Moreover the presentation of CP-asymmetries in B-decays is very superficial and the D-meson decays are completely omitted. Yet there are many excellent reviews of these topics and I will from time to time give references where this material can be found. My choice of topics for sections 9-15 was motivated by the wish to present the techniques and methods developed in the previous sections in some representative, phenomenologically important settings. Here is the choice I have made.

Section 9 deals with two-body non-leptonic B-decays. The purpose of this section is to

make a critical look at the existing analyses of these decays in the framework of factorization and the so-called generalized factorization. I have decided to discuss this topic here as it offers an excellent arena for various issues analyzed already in sections 5,6 and 7.

Section 10 deals with the issues of $K^0 - \bar{K}^0$ and $B^0_{d,s} - \bar{B}^0_{d,s}$ mixings, indirect CP violation in $K_L \to \pi\pi$ and with the standard construction of the unitarity triangle.

Section 11 deals with a sad story: the efforts to calculate the CP violating ratio ε'/ε . Here I will first summarize the work done in my group at the Technical University in Munich. Subsequently I will make a brief review of other efforts including most recent developments.

Section 12 deals with much more successful efforts: the calculations of the branching ratio for the inclusive $B \to X_s \gamma$ decay. Here the matrix elements of the relevant operators can be effectively calculated and various issues related to scale dependences which we have discussed formally in sections 6 and 8 can be analyzed in explicit terms.

Section 13 is my love story: the rare decays $K_L \to \pi^0 \nu \bar{\nu}$ and $K^+ \to \pi^+ \nu \bar{\nu}$. Since these decays are theoretically very clean, also here various formal issues discussed in previous sections can be analyzed with concrete examples. We will demonstrate the great potential of $K \to \pi \nu \bar{\nu}$ in the determination of the CKM parameters: in particular of V_{td} , $\text{Im}V_{ts}^*V_{td}$ and $\sin 2\beta$. We will also discuss the rare B-decays $B \to X_{s,d}\nu\bar{\nu}$ and $B_{s,d} \to l^+l^-$. This section ends with a description of two-loop electroweak contributions to rare K- and B-decays in the large m_t limit.

Section 14 offers some future visions. First we will discuss briefly CP-asymmetries in B-decays and their potential in the determination of the angles of the unitarity triangle. This determination of the CKM parameters will be confronted with the determination by means of $K \to \pi \nu \bar{\nu}$ decays. Subsequently a number of other strategies for a clean determination of the CKM matrix will be briefly discussed.

Section 15 offers a brief outlook of the field of weak decays for the next ten years. Finally in section 16 a few general messages on the Les Houches summer school 1997 will be made.

1.3 Whats New in these Lectures

In writing these lectures I benefited enormously from a review on NLO QCD corrections to weak decays written in collaboration with Gerhard Buchalla and Markus Lautenbacher in 1995 [17], from a review on CP violation and rare decays written in collaboration with Robert Fleischer in the spring of 1997 [18] and from several courses on the renormalization of QCD and the renormalization group methods in weak decays I have given at the Technical University in Munich during the nineties. It is unavoidable that there is some overlap between the present lectures and the reviews in [17, 18]. On the other hand there are several differences and many things which are covered here but cannot be found there. In particular:

- We discuss the renormalization of QCD and the renormalization group in more detail offering several derivations.
- We cover the issue of γ_5 in D dimensions.
- We analyze the role of evanescent operators in the calculation of two-loop anomalous dimensions of local operators.
- We present an explicit calculation of the 6 × 6 one-loop anomalous dimension matrix involving current-current and QCD penguin operators.
- We calculate explicitly a counter-diagram in the evaluation of two-loop anomalous dimensions in order to exhibit the role of evanescent operators.
- We discuss critically the hypothesis of the generalized factorization in two-body non-leptonic B-decays.
- We review the present status of the calculation of the non-perturbative factors $B_6^{(1/2)}$ and $B_8^{(3/2)}$ relevant for the calculation of ε'/ε and present an updated analysis of this ratio.
- We review in detail the present status of the radiative $B \to X_s \gamma$ decay including most recent developments.
- While our discussion of $K^+ \to \pi^+ \nu \bar{\nu}$ and $K_L \to \pi^0 \nu \bar{\nu}$ borrows a lot from [17, 18], it contains a few derivations absent there as well as new numerical estimates. Moreover we discuss briefly two–loop electroweak contributions to these decays.
- Our discussion of CP violation in B-decays is very superficial. On the other hand we make some comments on the recent hot issue: the role of the final state interactions in the determination of the CP-phases from CP-asymmetries in B-decays.

2 First Steps

2.1 The Basic Lagrangian

Throughout these lectures we will dominantly work in the context of the Standard Model with three generations of quarks and leptons and the interactions described by the gauge group $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ which undergoes the spontaneous breakdown:

$$SU(3)_C \otimes SU(2)_L \otimes U(1)_Y \to SU(3)_C \otimes U(1)_Q$$
 (2.1)

Here Y and Q denote the weak hypercharge and the electric charge generators respectively. $SU(3)_C$ stands for QCD.

There are excellent text books on the dynamics of the Standard Model and on the Field Theory. My favourites are listed in [20]–[29]. Let us therfore collect here only those ingredients of this model which are fundamental for the subject of weak decays.

- The strong interactions are mediated by eight gluons G_a , the electroweak interactions by W^{\pm} , Z^0 , γ and the neutral Higgs boson H^0 . In the non-physical gauges also other exchanges have to be included. In particular, the contributions from fictictious Higgs particles ϕ^{\pm} have to be taken into account to obtain gauge independent results.
- The dynamics of the theory is described by the fundamental Lagrangian:

$$\mathcal{L} = \mathcal{L}(QCD) + \mathcal{L}(SU(2)_L \otimes U(1)_Y) + \mathcal{L}(Higgs)$$
(2.2)

from which - after quantization and spontaneous symmetry breaking - the Feynman rules can be derived.

• Concerning *Electroweak Interactions*, the left-handed leptons and quarks are put into $SU(2)_L$ doublets:

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L \qquad \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L \qquad \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L \tag{2.3}$$

$$\begin{pmatrix} u \\ d' \end{pmatrix}_{L} \qquad \begin{pmatrix} c \\ s' \end{pmatrix}_{L} \qquad \begin{pmatrix} t \\ b' \end{pmatrix}_{L} \tag{2.4}$$

with the corresponding right-handed fields transforming as singlets under $SU(2)_L$. The primes in (2.4) are discussed a few pages below. The electroweak interactions are summarized by the Lagrangian

$$\mathcal{L}_{\text{int}}^{\text{EW}} = \mathcal{L}_{\text{CC}} + \mathcal{L}_{\text{NC}},$$
 (2.5)

with \mathcal{L}_{CC} and \mathcal{L}_{NC} describing *charged* and *neutral* current interactions respectively. Concentrating on the fermion-gauge-boson electroweak interactions we have:

• Charged Current Interactions:

$$\mathcal{L}_{CC} = \frac{g_2}{2\sqrt{2}} (J_{\mu}^+ W^{+\mu} + J_{\mu}^- W^{-\mu}), \tag{2.6}$$

where

$$J_{\mu}^{+} = (\bar{u}d')_{V-A} + (\bar{c}s')_{V-A} + (\bar{t}b')_{V-A} + (\bar{\nu}_{e}e)_{V-A} + (\bar{\nu}_{\mu}\mu)_{V-A} + (\bar{\nu}_{\tau}\tau)_{V-A}$$
 (2.7)

denotes the charged current and g_2 is the $SU(2)_L$ coupling constant.

• Neutral Current Interactions:

$$\mathcal{L}_{NC} = -eJ_{\mu}^{em}A^{\mu} + \frac{g_2}{2\cos\Theta_W}J_{\mu}^0Z^{\mu},$$
 (2.8)

where e is the QED coupling constant and Θ_{W} the Weinberg angle. The neutral electromagnetic and weak currents are given by

$$J_{\mu}^{\rm em} = \sum_{f} Q_f \bar{f} \gamma_{\mu} f \tag{2.9}$$

$$J_{\mu}^{0} = \sum_{f} \bar{f} \gamma_{\mu} (v_f - a_f \gamma_5) f, \qquad (2.10)$$

where

$$v_f = T_3^f - 2Q_f \sin^2 \Theta_W, \qquad a_f = T_3^f.$$
 (2.11)

Here Q_f and T_3^f denote the charge and the third component of the weak isospin of the left-handed fermion f_L , respectively. These electroweak charges are collected in table 1.

Table 1: Electroweak Quantum Numbers.

	$ u_L^e$	e_L^-	e_R^-	u_L	d_L	u_R	d_R
Q	0	-1	-1	2/3	-1/3	2/3	-1/3
T_3	1/2	-1/2	0	1/2	-1/2	0	0
Y	-1	-1	-2	1/3	1/3	4/3	-2/3

2.2 Elementary Vertices

Let us next recall those elementary interaction vertices which govern the physics of weak decays. They are shown in fig. 2. The following comments should be made:

- The indices i, j denote flavour: i, j = u, d, c, t, ...
- In non-physical gauges also vertices involving fictitious Higgs particles in place of W^{\pm} , Z^0 have to be included in this list.
- The quartic electroweak couplings will not enter these lectures.
- The flavour is conserved in vertices involving neutral gauge bosons Z^0 , γ and G. This fact implies the absence of flavour changing neutral current (FCNC) transitions at the tree level. This striking property of neutral interactions in the Standard Model is quarantieed by the GIM mechanism [15]. We will return to this mechanism later on.

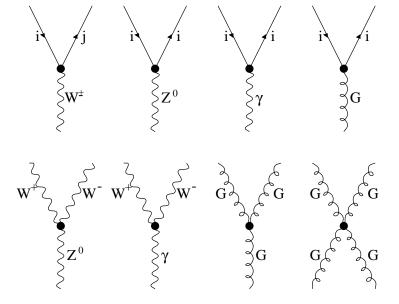


Figure 2: Elementary Vertices

• The charged current processes mediated by W^{\pm} are obviously flavour violating with the strength of violation given by the gauge coupling g_2 and effectively at low energies by the Fermi constant

$$\frac{G_{\rm F}}{\sqrt{2}} = \frac{g_2^2}{8M_W^2} \tag{2.12}$$

and a unitary 3×3 CKM matrix.

• The CKM matrix [1, 2, 15] connects the weak eigenstates (d', s', b') and the corresponding mass eigenstates d, s, b through

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix} = \hat{V}_{CKM} \begin{pmatrix} d \\ s \\ b \end{pmatrix}. \tag{2.13}$$

In the leptonic sector the analogous mixing matrix is a unit matrix due to the masslessness of neutrinos in the Standard Model. The unitarity of the CKM matrix assures the absence of elementary FCNC vertices. It is consequently at the basis of the GIM mechanism. On the other hand, the fact that the V_{ij} 's can a priori be complex numbers allows CP violation in the Standard Model. The structure of the CKM matrix is discussed in detail in the next subsection.

• The most important Feynman rules are given in figs. 3 and 4. It should be noted that the photonic and gluonic vertices are vectorlike (V), the W^{\pm} vertices are purely V - A and the Z^0 vertices involve both V - A and V + A structures.

• The vertices with fictitious Higgs ϕ^{\pm} have not been shown. Of particular interest are the vertices involving the top quark. Setting $m_d = 0$ we have for instance the following Feynman rules:

$$\bar{t}\phi^+d: \qquad -\frac{g_2}{2\sqrt{2}}V_{td}(1-\gamma_5)\frac{m_{\rm t}}{M_{\rm W}}$$
 (2.14)

$$\bar{d}\phi^- t: \frac{g_2}{2\sqrt{2}} V_{td}^* (1+\gamma_5) \frac{m_t}{M_W}$$
 (2.15)

Due to the proportionality to m_t these vertices play important role in rare and CP violating decays and transitions.

• Finally the vertex involving the neutral Higgs H^0 and the fermions is diagonal in flavour, again due to the GIM mechanism. It is given by

$$\bar{f}H^0f: -i\frac{g_2}{2}\frac{m_f}{M_W}$$
 (2.16)

The effects of H^0 on these lectures are only at two-loops in electroweak interactions.

Figure 3: Feynman Rules (Propagators)

With the help of the elementary vertices of fig. 2, the propagators and Feynman rules at hand, one can build physically interesting processes and subsequently evaluate them. The simplest of such processes, which forms the basis for subsequent considerations, is the W^{\pm} exchange between two fermion lines like the one shown in fig. 1a. Neglecting the momentum of the

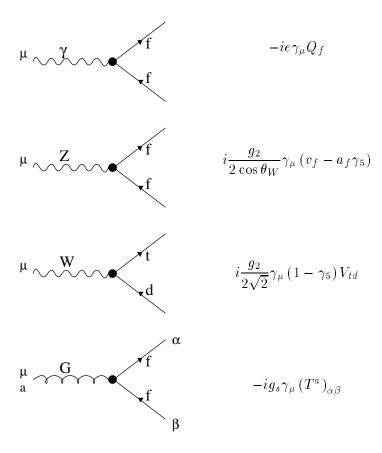


Figure 4: Feynman Rules (Vertices)

W-propagator relative to $M_{\rm W}$ and multiplying the result by "i", this process, generalized to arbitrary fermions in the Standard Model, gives the following tree level effective Hamiltonian describing the charged weak interactions of quarks and leptons:

$$\mathcal{H}_{\text{eff}}^{\text{tree}} = \frac{G_{\text{F}}}{\sqrt{2}} \, \mathcal{J}_{\mu}^{+} \mathcal{J}^{-\mu} \tag{2.17}$$

with \mathcal{J}_{μ}^{+} given in (2.7).

2.3 CKM Matrix

2.3.1 General Remarks

Let us next discuss the stucture of the quark-mixing-matrix \hat{V}_{CKM} defined by (2.13) in more detail. We know from the text books that this matrix can be parametrized by three angles and a single complex phase. This phase leading to an imaginary part of the CKM matrix is a necessary ingredient to describe CP violation within the framework of the Standard Model.

Many parametrizations of the CKM matrix have been proposed in the literature. We will

use two parametrizations in these lectures: the standard parametrization [30] recommended by the particle data group [31] and the Wolfenstein parametrization [32].

2.3.2 Standard Parametrization

With $c_{ij} = \cos \theta_{ij}$ and $s_{ij} = \sin \theta_{ij}$ (i, j = 1, 2, 3), the standard parametrization is given by:

$$\hat{V}_{CKM} = \begin{pmatrix}
c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\
-s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\
s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -s_{23}c_{12} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13}
\end{pmatrix},$$
(2.18)

where δ is the phase necessary for CP violation. c_{ij} and s_{ij} can all be chosen to be positive and δ may vary in the range $0 \le \delta \le 2\pi$. However, the measurements of CP violation in K decays force δ to be in the range $0 < \delta < \pi$.

From phenomenological applications we know that s_{13} and s_{23} are small numbers: $\mathcal{O}(10^{-3})$ and $\mathcal{O}(10^{-2})$, respectively. Consequently to an excellent accuracy $c_{13} = c_{23} = 1$ and the four independent parameters are given as

$$s_{12} = |V_{us}|, \quad s_{13} = |V_{ub}|, \quad s_{23} = |V_{cb}|, \quad \delta.$$
 (2.19)

The first three can be extracted from tree level decays mediated by the transitions $s \to u$, $b \to u$ and $b \to c$ respectively. The phase δ can be extracted from CP violating transitions or loop processes sensitive to $|V_{td}|$. The latter fact is based on the observation that for $0 \le \delta \le \pi$, as required by the analysis of CP violation in the K system, there is a one-to-one correspondence between δ and $|V_{td}|$ given by

$$|V_{td}| = \sqrt{a^2 + b^2 - 2ab\cos\delta}, \qquad a = |V_{cd}V_{cb}|, \qquad b = |V_{ud}V_{ub}|.$$
 (2.20)

The main phenomenological advantages of (2.18) over other parametrizations proposed in the literature are basically these two [33]:

- s_{12} , s_{13} and s_{23} being related in a very simple way to $|V_{us}|$, $|V_{ub}|$ and $|V_{cb}|$ respectively, can be measured independently in three decays.
- The CP violating phase is always multiplied by the very small s_{13} . This shows clearly the suppression of CP violation independently of the actual size of δ .

For numerical evaluations the use of the standard parametrization is strongly recommended. However once the four parameters in (2.19) have been determined it is often useful to make a change of basic parameters in order to see the structure of the result more transparently. This brings us to the Wolfenstein parametrization [32] and its generalization given in [34].

2.3.3 Wolfenstein Parameterization

The Wolfenstein parametrization is an approximate parametrization of the CKM matrix in which each element is expanded as a power series in the small parameter $\lambda = |V_{us}| = 0.22$,

$$\hat{V} = \begin{pmatrix}
1 - \frac{\lambda^2}{2} & \lambda & A\lambda^3(\varrho - i\eta) \\
-\lambda & 1 - \frac{\lambda^2}{2} & A\lambda^2 \\
A\lambda^3(1 - \varrho - i\eta) & -A\lambda^2 & 1
\end{pmatrix} + \mathcal{O}(\lambda^4), \tag{2.21}$$

and the set (2.19) is replaced by

$$\lambda, \qquad A, \qquad \varrho, \qquad \eta. \tag{2.22}$$

Because of the smallness of λ and the fact that for each element the expansion parameter is actually λ^2 , it is sufficient to keep only the first few terms in this expansion.

The Wolfenstein parametrization is certainly more transparent than the standard parametrization. However, if one requires sufficient level of accuracy, the higher order terms in λ have to be included in phenomenological applications. This can be done in many ways. The point is that since (2.21) is only an approximation the exact definition of the parameters in (2.22) is not unique by terms of the neglected order $\mathcal{O}(\lambda^4)$. This situation is familiar from any perturbative expansion, where different definitions of expansion parameters (coupling constants) are possible. This is also the reason why in different papers in the literature different $\mathcal{O}(\lambda^4)$ terms in (2.21) can be found. They simply correspond to different definitions of the parameters in (2.22). Since the physics does not depend on a particular definition, it is useful to make a choice for which the transparency of the original Wolfenstein parametrization is not lost. Here we present one way of achieving this.

2.3.4 Wolfenstein Parametrization beyond LO

An efficient and systematic way of finding higher order terms in λ is to go back to the standard parametrization (2.18) and to define the parameters (λ, A, ρ, η) through [34, 35]

$$s_{12} = \lambda, \qquad s_{23} = A\lambda^2, \qquad s_{13}e^{-i\delta} = A\lambda^3(\varrho - i\eta)$$
 (2.23)

to all orders in λ . It follows that

$$\varrho = \frac{s_{13}}{s_{12}s_{23}}\cos\delta, \qquad \eta = \frac{s_{13}}{s_{12}s_{23}}\sin\delta. \tag{2.24}$$

(2.23) and (2.24) represent simply the change of variables from (2.19) to (2.22). Making this change of variables in the standard parametrization (2.18) we find the CKM matrix as a function of $(\lambda, A, \varrho, \eta)$ which satisfies unitarity exactly! Expanding next each element in

powers of λ we recover the matrix in (2.21) and in addition find explicit corrections of $\mathcal{O}(\lambda^4)$ and higher order terms:.

$$V_{ud} = 1 - \frac{1}{2}\lambda^2 - \frac{1}{8}\lambda^4 + \mathcal{O}(\lambda^6)$$
 (2.25)

$$V_{us} = \lambda + \mathcal{O}(\lambda^7) \tag{2.26}$$

$$V_{ub} = A\lambda^3(\varrho - i\eta) \tag{2.27}$$

$$V_{cd} = -\lambda + \frac{1}{2} A^2 \lambda^5 [1 - 2(\varrho + i\eta)] + \mathcal{O}(\lambda^7)$$
 (2.28)

$$V_{cs} = 1 - \frac{1}{2}\lambda^2 - \frac{1}{8}\lambda^4(1 + 4A^2) + \mathcal{O}(\lambda^6)$$
 (2.29)

$$V_{cb} = A\lambda^2 + \mathcal{O}(\lambda^8) \tag{2.30}$$

$$V_{td} = A\lambda^3 \left[1 - (\varrho + i\eta)(1 - \frac{1}{2}\lambda^2) \right] + \mathcal{O}(\lambda^7)$$
(2.31)

$$V_{ts} = -A\lambda^2 + \frac{1}{2}A(1 - 2\varrho)\lambda^4 - i\eta A\lambda^4 + \mathcal{O}(\lambda^6)$$
(2.32)

$$V_{tb} = 1 - \frac{1}{2}A^2\lambda^4 + \mathcal{O}(\lambda^6)$$
 (2.33)

We note that by definition V_{ub} remains unchanged and the corrections to V_{us} and V_{cb} appear only at $\mathcal{O}(\lambda^7)$ and $\mathcal{O}(\lambda^8)$, respectively. Consequently to an an excellent accuracy we have:

$$V_{us} = \lambda, \qquad V_{cb} = A\lambda^2, \tag{2.34}$$

$$V_{ub} = A\lambda^3(\varrho - i\eta), \qquad V_{td} = A\lambda^3(1 - \bar{\varrho} - i\bar{\eta})$$
(2.35)

with

$$\bar{\varrho} = \varrho(1 - \frac{\lambda^2}{2}), \qquad \bar{\eta} = \eta(1 - \frac{\lambda^2}{2}).$$
(2.36)

The advantage of this generalization of the Wolfenstein parametrization over other generalizations found in the literature is the absence of relevant corrections to V_{us} , V_{cb} and V_{ub} and an elegant change in V_{td} which allows a simple generalization of the so-called unitarity triangle beyond LO.

Finally let us collect useful analytic expressions for $\lambda_i = V_{id}V_{is}^*$ with i = c, t:

$$Im\lambda_t = -Im\lambda_c = \eta A^2 \lambda^5 = |V_{ub}| |V_{cb}| \sin \delta$$
(2.37)

$$Re\lambda_c = -\lambda(1 - \frac{\lambda^2}{2}) \tag{2.38}$$

$$\operatorname{Re}\lambda_t = -(1 - \frac{\lambda^2}{2})A^2\lambda^5(1 - \bar{\varrho}). \tag{2.39}$$

Expressions (2.37) and (2.38) represent to an accuracy of 0.2% the exact formulae obtained using (2.18). The expression (2.39) deviates by at most 2% from the exact formula in the

full range of parameters considered. For ϱ close to zero this deviation is below 1%. A careful reader may note that a small $\mathcal{O}(\lambda^7)$ has been dropped in deriving (2.39). This has been done both for artistic reasons and in order to increase the accuracy of this formula. After inserting the expressions (2.37)–(2.39) in the exact formulae for quantities of interest, a further expansion in λ should not be made.

2.3.5 Unitarity Triangle

The unitarity of the CKM-matrix implies various relations between its elements. In particular, we have

$$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0. (2.40)$$

Phenomenologically this relation is very interesting as it involves simultaneously the elements V_{ub} , V_{cb} and V_{td} which are under extensive discussion at present.

The relation (2.40) can be represented as a "unitarity" triangle in the complex $(\bar{\varrho}, \bar{\eta})$ plane. The invariance of (2.40) under any phase-transformations implies that the corresponding triangle is rotated in the $(\bar{\varrho}, \bar{\eta})$ plane under such transformations. Since the angles and the sides (given by the moduli of the elements of the mixing matrix) in these triangles remain unchanged, they are phase convention independent and are physical observables. Consequently they can be measured directly in suitable experiments. The area of the unitarity triangle is related to the measure of CP violation J_{CP} [36]:

$$|J_{\rm CP}| = 2 \cdot A_{\Delta}, \tag{2.41}$$

where A_{Δ} denotes the area of the unitarity triangle.

The construction of the unitarity triangle proceeds as follows:

• We note first that

$$V_{cd}V_{cb}^* = -A\lambda^3 + \mathcal{O}(\lambda^7). \tag{2.42}$$

Thus to an excellent accuracy $V_{cd}V_{cb}^*$ is real with $|V_{cd}V_{cb}^*| = A\lambda^3$.

• Keeping $\mathcal{O}(\lambda^5)$ corrections and rescaling all terms in (2.40) by $A\lambda^3$ we find

$$\frac{1}{A\lambda^3}V_{ud}V_{ub}^* = \bar{\varrho} + i\bar{\eta}, \qquad \frac{1}{A\lambda^3}V_{td}V_{tb}^* = 1 - (\bar{\varrho} + i\bar{\eta}) \qquad (2.43)$$

with $\bar{\rho}$ and $\bar{\eta}$ defined in (2.36).

• Thus we can represent (2.40) as the unitarity triangle in the complex $(\bar{\varrho}, \bar{\eta})$ plane as shown in fig. 5.

Let us collect useful formulae related to this triangle:

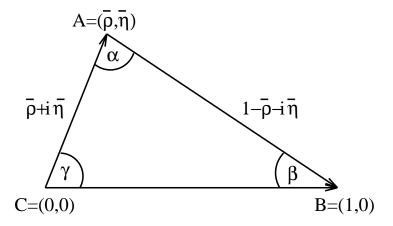


Figure 5: Unitarity Triangle.

• Using simple trigonometry one can express $\sin(2\phi_i)$, $\phi_i = \alpha, \beta, \gamma$, in terms of $(\bar{\varrho}, \bar{\eta})$ as follows:

$$\sin(2\alpha) = \frac{2\bar{\eta}(\bar{\eta}^2 + \bar{\varrho}^2 - \bar{\varrho})}{(\bar{\varrho}^2 + \bar{\eta}^2)((1 - \bar{\varrho})^2 + \bar{\eta}^2)}$$
(2.44)

$$\sin(2\beta) = \frac{2\bar{\eta}(1-\bar{\varrho})}{(1-\bar{\varrho})^2 + \bar{\eta}^2} \tag{2.45}$$

$$\sin(2\gamma) = \frac{2\bar{\varrho}\bar{\eta}}{\bar{\varrho}^2 + \bar{\eta}^2} = \frac{2\varrho\eta}{\varrho^2 + \eta^2}.$$
 (2.46)

• The lengths CA and BA in the rescaled triangle to be denoted by R_b and R_t , respectively, are given by

$$R_b \equiv \frac{|V_{ud}V_{ub}^*|}{|V_{cd}V_{cb}^*|} = \sqrt{\bar{\varrho}^2 + \bar{\eta}^2} = (1 - \frac{\lambda^2}{2}) \frac{1}{\lambda} \left| \frac{V_{ub}}{V_{cb}} \right|$$
(2.47)

$$R_t \equiv \frac{|V_{td}V_{tb}^*|}{|V_{cd}V_{cb}^*|} = \sqrt{(1-\bar{\varrho})^2 + \bar{\eta}^2} = \frac{1}{\lambda} \left| \frac{V_{td}}{V_{cb}} \right|. \tag{2.48}$$

• The angles β and γ of the unitarity triangle are related directly to the complex phases of the CKM-elements V_{td} and V_{ub} , respectively, through

$$V_{td} = |V_{td}|e^{-i\beta}, \quad V_{ub} = |V_{ub}|e^{-i\gamma}.$$
 (2.49)

• The angle α can be obtained through the relation

$$\alpha + \beta + \gamma = 180^{\circ} \tag{2.50}$$

expressing the unitarity of the CKM-matrix.

The triangle depicted in fig. 5 together with $|V_{us}|$ and $|V_{cb}|$ gives a full description of the CKM matrix. Looking at the expressions for R_b and R_t , we observe that within the Standard Model the measurements of four CP conserving decays sensitive to $|V_{us}|$, $|V_{ub}|$, $|V_{cb}|$ and $|V_{td}|$ can tell us whether CP violation ($\eta \neq 0$) is predicted in the Standard Model. This is a very remarkable property of the Kobayashi-Maskawa picture of CP violation: quark mixing and CP violation are closely related to each other.

2.3.6 Final Comments

What do we know about the CKM matrix and the unitarity triangle on the basis of *tree level* decays? A detailed answer to this question can be found in the reports of the Particle Data Group [31] as well as other reviews. Here I would like to quote only a few numbers without going into details how they have been obtained. They are

$$|V_{us}| = \lambda = 0.2205 \pm 0.0018$$
 $|V_{cb}| = 0.040 \pm 0.003,$ (2.51)

$$\frac{|V_{ub}|}{|V_{cb}|} = 0.08 \pm 0.02, \qquad |V_{ub}| = (3.2 \pm 0.8) \cdot 10^{-3}. \tag{2.52}$$

The value for $|V_{us}|$ follows from $K^+ \to \pi^0 e^+ \nu_e$, $K_{\rm L}^0 \to \pi^- e^+ \nu_e$ and semileptonic hyperon decays. The chiral perturbation theory [37, 38] plays an important role in these determinations. $|V_{cb}|$ follows from exclusive and inclusive semileptonic B decays governed by the transition $b \to c$. Here the recent improved data [39] combined with HQET in the case of exclusive decays and HQE in the case of inclusive decays played important role [40, 41, 42, 43]. $|V_{ub}|$ follows from from exclusive and inclusive semileptonic B decays governed by the transition $b \to u$ [39, 44].

Setting $\lambda = 0.22$, scanning $|V_{cb}|$ and $|V_{ub}|$ in the ranges (2.51) and $\cos \delta$ in the range $-1 \le \cos \delta \le 1$, we find [18]:

$$4.5 \cdot 10^{-3} \le |V_{td}| \le 13.7 \cdot 10^{-3}, \qquad 0.0353 \le |V_{ts}| \le 0.0429$$
 (2.53)

and

$$0.9991 \le |V_{tb}| \le 0.9993, \qquad 0.9736 \le |V_{cs}| \le 0.9750.$$
 (2.54)

From (2.53) we observe that the unitarity of the CKM matrix requires approximate equality of $|V_{ts}|$ and $|V_{cb}|$: $0.954 \le |V_{ts}|/|V_{cb}| \le 0.997$ which is evident if one compares (2.34) with (2.32). Moreover $|V_{tb}|$ is predicted to be very close to unity. The experimental value from top-quark decays obtained by CDF is $|V_{tb}| = 0.99 \pm 0.15$.

Let us then see what these results imply for the unitarity triangle of fig. 5. To this end it is sufficient to insert the first result in (2.52) into (2.47) to find

$$R_b = 0.36 \pm 0.09 \tag{2.55}$$

This tells us only that the apex A of the unitarity triangle lies in the band shown in fig. 6. In order to answer the question where the apex A lies on this "unitarity clock" we have to look at different decays. Most promising in this respect are the so-called "loop induced" decays and transitions which are the subject of several sections in these lectures and CP asymmetries in B-decays which will be briefly discussed in Section 14. These two different

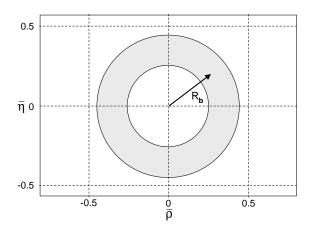


Figure 6: "Unitarity Clock".

routes for explorations of the CKM matrix and of the related unitarity triangle may answer the important question, whether the Kobayashi-Maskawa picture of CP violation is correct and more generally whether the Standard Model offers a correct description of weak decays of hadrons. Indeed, in order to answer these important questions it is essential to calculate as many branching ratios as possible, measure them experimentally and check if they all can be described by the same set of the parameters $(\lambda, A, \varrho, \eta)$. In the language of the unitarity triangle this means that the various curves in the $(\bar{\varrho}, \bar{\eta})$ plane extracted from different decays should cross each other at a single point as shown in fig. 7. Moreover the angles (α, β, γ) in the resulting triangle should agree with those extracted one day from CP-asymmetries in B-decays. For artistic reasons the value of $\bar{\eta}$ in fig. 7 has been chosen to be higher than the fitted central value $\bar{\eta} \approx 0.35$.

Since the CKM matrix is only a parametrization of quark mixing and of CP violation and does not offer the explanation of these two very important phenomena, many physicists hope that a new physics while providing a dynamical origin of quark mixing and CP violation will also change the picture given in fig. 7. That is, the different curves based on the Standard Model expressions, will not cross each other at a single point and the angles (α, β, γ) extracted

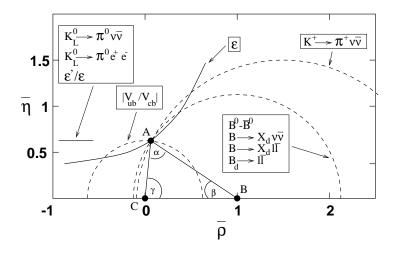


Figure 7: The ideal Unitarity Triangle.

one day from CP-asymmetries in B-decays will disagree with the ones determined from rare K and B decays.

Clearly the plot in fig. 7 is highly idealized because in order to extract such nice curves from various decays one needs perfect experiments and perfect theory. One of the goals of these lectures is to describe the present status of the theory of weak decays and to identify those decays for which at least the theory is under control. For such decays, if they can be measured with a sufficient precision, the curves in fig. 7 are not unrealistic.

The formal discussion of the theory of weak decays is however a real steep climb and it is advisable to do first a long but gentle hike by discussing loop induced decays in general terms. They are known under the name of *Flavour Chanqing Neutral Current* (FCNC) processes.

3 FCNC Processes

3.1 General Remarks

The flavour diagonal structure of the basic vertices involving γ , Z and G in fig. 4 forbidds the appearance of FCNC processes at the tree level. With the help of the flavour-changing W^{\pm} -vertex one can, however, construct one-loop and higher order diagrams which mediate FCNC processes. The fact that these processes take place only as loop effects makes them particularly useful for testing the quantum structure of the theory and in the search of the physics beyond the Standard Model. At the one-loop level they can be described by a set of basic triple and quartic effective vertices. In the literature they appear under the names of penguin and box diagrams, respectively.

3.2 Effective Vertices

3.2.1 Penguin vertices

These vertices involve only quarks and can be depicted as in fig. 8, where i and j have the same charge but different flavour and k denotes the internal quark whose charge is different from that of i and j. These effective vertices can be calculated by using the elementary vertices and propagators of figs. 3 and 4. Important examples are given in fig. 9. The diagrams with fictitious Higgs exchanges in place of W^{\pm} have not been shown. Strictly speaking, also self-energy corrections on external lines have to be included to make the effective vertices finite.

3.2.2 Box vertices

These vertices involve in general both quarks and leptons and can be depicted as in fig. 10, where again i, j, m, n stand for external quarks or leptons and k and l denote the internal quarks and leptons. In the vertex (a) the flavour violation takes place on both sides (left and right) of the box, whereas in (b) the right-hand side is flavour conserving. These effective quartic vertices can also be calculated using the elementary vertices and propagators of figs. 3 and 4. We have for instance the vertices in fig. 11 which contribute to $B_d^0 - \bar{B}_d^0$ mixing and $K^+ \to \pi^+ \nu \bar{\nu}$, respectively. The fictitious Higgs exchanges have not been shown. Other interesting examples will be discussed in the course of these lectures.

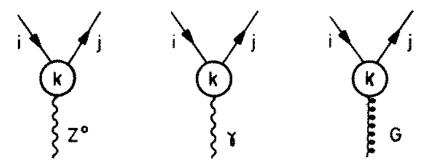


Figure 8: Penguin vertices

3.2.3 Effective Feynman Rules

With the help of the elementary vertices and propagators shown in figs. 3 and 4, one can derive "Feynman rules" for the effective vertices discussed above by calculating simply the diagrams on the r.h.s. of the equations in figs. 9 and 11. In fig. 9 the Z^0 , γ and gluon are off-shell. In the case of inclusive decays $B \to X\gamma$ and $B \to XG$ we need also corresponding

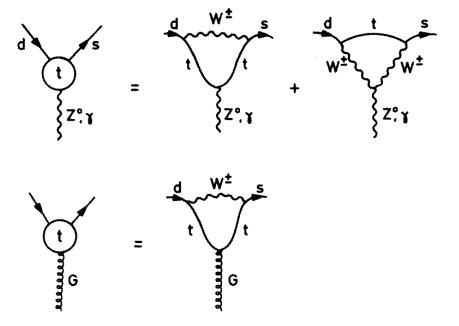


Figure 9: Penguin vertices resolved in terms of basic vertices

vertices with on-shell photons and gluons. For these two cases it is essential to keep the mass of the external b-quark as otherwise the corresponding vertices would vanish.

The rules for effective vertices are given in the 't Hooft–Feynman gauge for the W^{\pm} propagator as follows:

$$Box(\Delta S = 2) = \lambda_i^2 \frac{G_F^2}{16\pi^2} M_W^2 S_0(x_i) (\bar{s}d)_{V-A} (\bar{s}d)_{V-A}$$
 (3.1)

$$Box(T_3 = 1/2) = \lambda_i \frac{G_F}{\sqrt{2}} \frac{\alpha}{2\pi \sin^2 \Theta_W} [-4B_0(x_i)] (\bar{s}d)_{V-A} (\bar{\nu}\nu)_{V-A}$$
(3.2)

$$Box(T_3 = -1/2) = \lambda_i \frac{G_F}{\sqrt{2}} \frac{\alpha}{2\pi \sin^2 \Theta_W} B_0(x_i) (\bar{s}d)_{V-A} (\bar{\mu}\mu)_{V-A}$$
(3.3)

$$\bar{s}Zd = i\lambda_i \frac{G_F}{\sqrt{2}} \frac{e}{2\pi^2} M_Z^2 \frac{\cos\Theta_W}{\sin\Theta_W} C_0(x_i) \bar{s}\gamma_\mu (1 - \gamma_5) d$$
(3.4)

$$\bar{s}\gamma d = -i\lambda_i \frac{G_F}{\sqrt{2}} \frac{e}{8\pi^2} D_0(x_i) \bar{s}(q^2 \gamma_\mu - q_\mu \not q) (1 - \gamma_5) d$$
 (3.5)

$$\bar{s}G^a d = -i\lambda_i \frac{G_F}{\sqrt{2}} \frac{g_s}{8\pi^2} E_0(x_i) \bar{s}_\alpha (q^2 \gamma_\mu - q_\mu \not q) (1 - \gamma_5) T^a_{\alpha\beta} d_\beta$$
 (3.6)

$$\bar{s}\gamma'b = i\bar{\lambda}_i \frac{G_F}{\sqrt{2}} \frac{e}{8\pi^2} D_0'(x_i) \bar{s} [i\sigma_{\mu\lambda} q^{\lambda} [m_b(1+\gamma_5)]]b$$
(3.7)

$$\bar{s}G^{\prime a}b = i\bar{\lambda}_i \frac{G_F}{\sqrt{2}} \frac{g_s}{8\pi^2} E_0^{\prime}(x_i) \bar{s}_{\alpha} [i\sigma_{\mu\lambda}q^{\lambda}[m_b(1+\gamma_5)]] T_{\alpha\beta}^a b_{\beta}, \qquad (3.8)$$

where

$$\lambda_i = V_{is}^* V_{id} \qquad \bar{\lambda}_i = V_{is}^* V_{ib} \tag{3.9}$$

In these rules q_{μ} is the *outgoing* gluon or photon momentum and T_3 indicates whether $\nu\bar{\nu}$ or l^+l^- leaves the box diagram. The first rule involves quarks only. The last two rules involve on-shell photon and gluon. We have set $m_s=0$ in these rules.

These rules for effective vertices together with the rules of figs. 3 and 4 allow the calculation of the effective Hamiltonians for FCNC processes, albeit without the inclusion of QCD corrections. The way these rules should be used requires some care:

- The penguin vertices should be used in the same manner as the elementary vertices of fig. 4 which follow from $i\mathcal{L}$. Once a mathematical expression corresponding to a given diagram has been found, the contribution of this diagram to the relevant effective Hamiltonian is obtained by multiplying this mathematical expression by "i".
- Our conventions for the box vertices are such that they directly give the contributions to the effective Hamiltonians.

We will give an example below by calculating the internal top contributions to $K^+ \to \pi^+ \nu \bar{\nu}$. First, however, let us make a few general remarks emphasizing the new features of these effective vertices as compared to the ones of fig. 2.

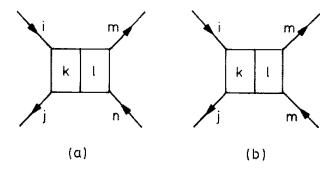


Figure 10: Box vertices

- They are higher order in the gauge couplings and consequently suppressed relative to elementary transitions. This is consistent with experimental findings which show very strong suppression of FCNC transitions relative to tree level processes.
- Because of the internal W^{\pm} exchanges all penguin vertices in fig. 9 are purely V-A, i.e. the effective vertices involving γ and G are parity violating as opposed to their elementary interactions in fig. 1! Also the structure of the Z^0 coupling changes since now only V-A couplings are involved. The box vertices are of the $(V-A)\otimes (V-A)$ type.

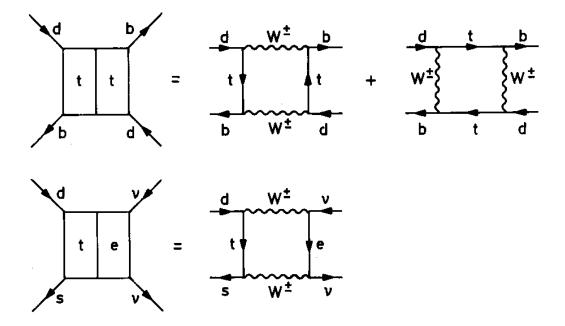


Figure 11: Box vertices resolved in terms of elementary vertices

• The effective vertices depend on the masses of internal quarks or leptons and consequently are calculable functions of

$$x_i = \frac{m_i^2}{M_W^2}, \qquad i = u, c, t.$$
 (3.10)

A set of basic universal functions can be found. These functions govern the physics of all FCNC processes. They are given below.

- The effective vertices depend on elements of the CKM matrix and this dependence can be found directly from the diagrams of figs. 9 and 11.
- The dependences of a given vertex on the CKM factors and the masses of internal fermions govern the strength of the vertex in question.
- Another new feature of the effective vertices as compared with the elementary vertices is their dependence on the gauge used for the W^{\pm} propagator. We will return to this point below.

3.2.4 Basic Functions

The basic functions present in (3.1)-(3.8) were calculated by various authors, in particular by Inami and Lim [45]. They are given explicitly as follows:

$$B_0(x_t) = \frac{1}{4} \left[\frac{x_t}{1 - x_t} + \frac{x_t \ln x_t}{(x_t - 1)^2} \right]$$
 (3.11)

$$C_0(x_t) = \frac{x_t}{8} \left[\frac{x_t - 6}{x_t - 1} + \frac{3x_t + 2}{(x_t - 1)^2} \ln x_t \right]$$
 (3.12)

$$D_0(x_t) = -\frac{4}{9} \ln x_t + \frac{-19x_t^3 + 25x_t^2}{36(x_t - 1)^3} + \frac{x_t^2(5x_t^2 - 2x_t - 6)}{18(x_t - 1)^4} \ln x_t$$
 (3.13)

$$E_0(x_t) = -\frac{2}{3} \ln x_t + \frac{x_t^2 (15 - 16x_t + 4x_t^2)}{6(1 - x_t)^4} \ln x_t + \frac{x_t (18 - 11x_t - x_t^2)}{12(1 - x_t)^3}$$
(3.14)

$$D_0'(x_t) = -\frac{(8x_t^3 + 5x_t^2 - 7x_t)}{12(1 - x_t)^3} + \frac{x_t^2(2 - 3x_t)}{2(1 - x_t)^4} \ln x_t$$
 (3.15)

$$E_0'(x_t) = -\frac{x_t(x_t^2 - 5x_t - 2)}{4(1 - x_t)^3} + \frac{3}{2} \frac{x_t^2}{(1 - x_t)^4} \ln x_t$$
 (3.16)

$$S_0(x_t) = \frac{4x_t - 11x_t^2 + x_t^3}{4(1 - x_t)^2} - \frac{3x_t^3 \ln x_t}{2(1 - x_t)^3}$$
(3.17)

$$S_0(x_c) = x_c \tag{3.18}$$

$$S_0(x_c, x_t) = x_c \left[\ln \frac{x_t}{x_c} - \frac{3x_t}{4(1 - x_t)} - \frac{3x_t^2 \ln x_t}{4(1 - x_t)^2} \right].$$
 (3.19)

We would like to make a few comments:

- In the last two expressions we have kept only linear terms in $x_c \ll 1$, but of course all orders in x_t . The last function generalizes $S_0(x_t)$ in (3.17) to include box diagrams with simultaneous top-quark and charm-quark exchanges.
- The subscript "0" indicates that these functions do not include QCD corrections to the relevant penguin and box diagrams. These corrections will be discussed in detail in subsequent sections.
- In writing the expressions in (3.11)-(3.19) we have omitted x_t -independent terms which do not contribute to decays due to the GIM mechanism. We will discuss this issue in more detail below. Moreover

$$S_0(x_t) \equiv F(x_t, x_t) + F(x_u, x_u) - 2F(x_t, x_u)$$
(3.20)

and

$$S_0(x_i, x_j) = F(x_i, x_j) + F(x_u, x_u) - F(x_i, x_u) - F(x_j, x_u), \tag{3.21}$$

where $F(x_i, x_j)$ is the true function corresponding to a given box diagram with i and j quark exchanges. These particular combinations can be found by drawing all possible box diagrams (also those with u-quark exchanges), setting $m_u = 0$ and using unitarity of the CKM-matrix which implies in particular the relation:

$$\lambda_u + \lambda_c + \lambda_t = 0. (3.22)$$

In this way the effective Hamiltonians for FCNC transitions can be directly obtained by summing only over t and c quarks.

• The expressions given for $B_0(x_t)$, $C_0(x_t)$ and $D_0(x_t)$ correspond to the 't Hooft– Feynman gauge ($\xi = 1$). In an arbitrary R_{ξ} gauge they look differently. In phenomenological applications it is useful therefore to work instead with the following gauge independent combinations [16]:

$$C_0(x_t,\xi) - 4B_0(x_t,\xi,1/2) = C_0(x_t) - 4B_0(x_t) = X_0(x_t)$$
(3.23)

$$C_0(x_t, \xi) - B_0(x_t, \xi, -1/2) = C_0(x_t) - B_0(x_t) = Y_0(x_t)$$
(3.24)

$$C_0(x_t,\xi) + \frac{1}{4}D_0(x_t,\xi) = C_0(x_t) + \frac{1}{4}D_0(x_t) = Z_0(x_t).$$
 (3.25)

- $X_0(x_t)$ and $Y_0(x_t)$ are linear combinations of the V-A components of Z^0 -penguin and box-diagrams with final quarks or leptons having weak isospin T_3 equal to 1/2 and -1/2, respectively.
- $Z_0(x_t)$ is a linear combination of the vector component of the Z^0 -penguin and the γ -penguin.
- These new functions are given explicitly as follows:

$$X_0(x_t) = \frac{x_t}{8} \left[\frac{x_t + 2}{x_t - 1} + \frac{3x_t - 6}{(x_t - 1)^2} \ln x_t \right]$$
 (3.26)

$$Y_0(x_t) = \frac{x_t}{8} \left[\frac{x_t - 4}{x_t - 1} + \frac{3x_t}{(x_t - 1)^2} \ln x_t \right]$$
 (3.27)

$$Z_0(x_t) = -\frac{1}{9} \ln x_t + \frac{18x_t^4 - 163x_t^3 + 259x_t^2 - 108x_t}{144(x_t - 1)^3} + \frac{32x_t^4 - 38x_t^3 - 15x_t^2 + 18x_t}{72(x_t - 1)^4} \ln x_t.$$
(3.28)

Thus the set of gauge independet basic functions which govern the FCNC processes is given by:

$$S_0(x_t), \quad X_0(x_t), \quad Y_0(x_t), \quad Z_0(x_t), \quad E_0(x_t), \quad D_0'(x_t), \quad E_0'(x_t).$$
 (3.29)

Finally, we give approximate formulae for the basic functions:

$$S_0(x_t) = 0.784 \ x_t^{0.76}, \quad X_0(x_t) = 0.660 \ x_t^{0.575},$$
 (3.30)

$$Y_0(x_t) = 0.315 \ x_t^{0.78}, \quad Z_0(x_t) = 0.175 \ x_t^{0.93}, \quad E_0(x_t) = 0.564 \ x_t^{-0.51},$$
 (3.31)

$$D_0'(x_t) = 0.244 \ x_t^{0.30}, \qquad E_0'(x_t) = 0.145 \ x_t^{0.19}.$$
 (3.32)

In the range $150 \,\text{GeV} \le m_t \le 200 \,\text{GeV}$ these approximations reproduce the exact expressions to an accuracy better than 1%. We have then

$$S_0(x_t) = 2.46 \left(\frac{m_t}{170 \,\text{GeV}}\right)^{1.52},$$
 (3.33)

$$X_0(x_t) = 1.57 \left(\frac{m_t}{170 \,\text{GeV}}\right)^{1.15}, \qquad Y_0(x_t) = 1.02 \left(\frac{m_t}{170 \,\text{GeV}}\right)^{1.56},$$
 (3.34)

$$Z_0(x_t) = 0.71 \left(\frac{m_t}{170 \,\text{GeV}}\right)^{1.86}, \qquad E_0(x_t) = 0.26 \left(\frac{m_t}{170 \,\text{GeV}}\right)^{-1.02},$$
 (3.35)

$$D_0'(x_t) = 0.38 \left(\frac{m_t}{170 \,\text{GeV}}\right)^{0.60}, \qquad E_0'(x_t) = 0.19 \left(\frac{m_t}{170 \,\text{GeV}}\right)^{0.38}.$$
 (3.36)

These formulae will allow us to exhibit elegantly the $m_{\rm t}$ dependence of various branching ratios in the phenomenological sections of these lectures.

3.2.5 Explicit Calculation of the Box Diagrams

Let us derive the rules (3.2) and (3.3). Beginning with (3.2) we consider the relevant box diagram in fig. 11. Setting $m_{\nu} = m_e = 0$ the contributions of the fictitious Higgs exchanges ϕ^{\pm} are also set to zero and we are left only with the W^{\pm} exchanges. Concentrating first on the internal top-quark contribution and using the Feynman rules of fig. 3 and 4 we have, after simple manipulations of Dirac matrices, the following expression for the diagram in fig. 11

$$\mathcal{D}(\nu\bar{\nu}) = \left(\frac{g_2}{2\sqrt{2}}\right)^4 \lambda_t T_{\sigma\tau} R^{\sigma\tau} \tag{3.37}$$

where

$$T_{\sigma\tau} = -4\bar{s}\gamma_{\mu}\gamma_{\sigma}\gamma_{\nu}(1-\gamma_{5})d\otimes\bar{\nu}\gamma^{\mu}\gamma_{\tau}\gamma^{\nu}(1-\gamma_{5})\nu \tag{3.38}$$

and

$$R^{\sigma\tau} = \int \frac{d^4k}{(2\pi)^4} \frac{k^{\sigma}k^{\tau}}{[k^2 - m_{\rm t}^2]k^2[k^2 - M_{\rm W}^2]^2}.$$
 (3.39)

Note that in view of very massive internal propagators we can set all external momenta to zero.

The integral $R^{\sigma\tau}$ can be easily evaluated by means of the standard methods. As the box diagram is finite we do not have to introduce any regulators and we find

$$R^{\sigma\tau} = \frac{g^{\sigma\tau}}{4} \int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2 - m_t^2][k^2 - M_W^2]^2} = -\frac{g^{\sigma\tau}}{64\pi^2} \frac{i}{M_W^2} [4B_0(x_t) + 1] , \qquad (3.40)$$

where $B_0(x_t)$ is given in (3.11) with $x_t = m_t^2/M_W^2$.

Next using the standard rules for Dirac matrices we find

$$T_{\sigma\tau}g^{\sigma\tau} = -64(\bar{s}\gamma_{\mu}(1-\gamma_{5})d) \otimes (\bar{\nu}\gamma^{\mu}(1-\gamma_{5})\nu) \equiv -64(\bar{s}d)_{V-A}(\bar{\nu}\nu)_{V-A}$$
(3.41)

We will develop in section 6 a simple method for evaluating in no time the expressions like (3.41).

Collecting all these results we find

$$\mathcal{D}(\nu\bar{\nu}) = \lambda_t \frac{g_2^4}{64\pi^2} \frac{i}{M_W^2} [4B_0(x_t) + 1]$$
(3.42)

We can drop "1" in the square brackets as the inclusion of u-quark and c-quark exchanges will cancel it after the unitarity of the CKM matrix has been used $(\lambda_u + \lambda_c + \lambda_t = 0)$.

In order to obtain the final result for the effective Hamiltonian corresponding to the last diagram in fig. 11 we multiply (3.42) by "i" and use

$$\frac{g_2^4}{64\pi^2} \frac{1}{M_W^2} = \frac{G_F}{\sqrt{2}} \frac{\alpha}{2\pi \sin^2 \Theta_W}$$
 (3.43)

to obtain

$$H_{eff}(T_3 = 1/2) = \lambda_t \frac{G_F}{\sqrt{2}} \frac{\alpha}{2\pi \sin^2 \Theta_W} [-4B_0(x_t)] (\bar{s}d)_{V-A} (\bar{\nu}\nu)_{V-A}$$
(3.44)

which is simply the rule in (3.2).

Having this result it is straightforward to derive the rule (3.3). In this case the charge flow on the lepton line is opposite to the one in fig. 11 and the Dirac structure in (3.38) is replaced by

$$\tilde{T}_{\sigma\tau} = 4\bar{s}\gamma_{\mu}\gamma_{\sigma}\gamma_{\nu}(1-\gamma_{5})d \otimes \bar{\mu}\gamma^{\nu}\gamma_{\tau}\gamma^{\mu}(1-\gamma_{5})\mu \tag{3.45}$$

with all other expressions unchanged. Consequently

$$\tilde{T}_{\sigma\tau}g^{\sigma\tau} = 16(\bar{s}d)_{V-A}(\bar{\mu}\mu)_{V-A} = -\frac{1}{4}T_{\sigma\tau}g^{\sigma\tau}$$
 (3.46)

Thus replacing in (3.44) $(\bar{\nu}\nu)_{V-A}$ by $(\bar{\mu}\mu)_{V-A}$ and multiplying it by -1/4 we recover the rule (3.3).

3.3 Effective Hamiltonians for FCNC Processes

3.3.1 An Example

With the help of the Feynman rules given in figs. 3 and 4 and the effective rules in (3.2) and (3.4) it is an easy matter to construct the effective Hamiltonian for the decay $K^+ \to \pi^+ \bar{\nu}_e \nu_e$ to which the diagrams in fig. 12 contribute.

Replacing the Z^0 propagator by $ig_{\mu\nu}/M_Z^2$ and multiplying the first diagram by "i", we find the well-known result for the top contribution to this decay:

$$\mathcal{H}_{\text{eff}}(K^+ \to \pi^+ \nu_e \bar{\nu}_e) = \frac{G_F}{\sqrt{2}} \frac{\alpha}{2\pi \sin^2 \Theta_W} V_{ts}^* V_{td} X_0(x_t) (\bar{s}d)_{V-A} (\bar{\nu}_e \nu_e)_{V-A}. \tag{3.47}$$

Here we have expressed the combination $C_0(x_t) - 4B_0(x_t)$ through the function $X_0(x_t)$.

Figure 12: Calculation of $\mathcal{H}_{\text{eff}}(K^+ \to \pi^+ \nu_e \bar{\nu}_e)$

3.3.2 Penguin-Box Expansion

One can generalize such calculations to other processes in which other basic effective vertices are present. For decays involving photonic and/or gluonic penguin vertices, the $1/q^2$ in the propagator cancels the q^2 in the vertex and the resulting effective Hamiltonian can again be written in terms of local four–fermion operators. Thus generally an effective Hamiltonian for any decay considered can be written in the absence of QCD corrections as

$$\mathcal{H}_{\text{eff}}^{\text{FCNC}} = \sum_{k} C_k O_k, \tag{3.48}$$

where O_k denote local operators such as $(\bar{s}d)_{V-A}(\bar{s}d)_{V-A}$, $(\bar{s}d)_{V-A}(\bar{u}u)_{V-A}$ etc. The coefficients C_k of these operators are simply linear combinations of the functions of (3.29) times the corresponding CKM factors which can be read off from our rules. Consequently it is possible to write the amplitudes for all FCNC decays and transitions as linear combinations of the basic, process independent m_t -dependent functions $F_r(x_t)$ of (3.29) with corresponding coefficients P_r characteristic for the decay under consideration. This "Penguin Box Expansion" [16] takes the following general form:

$$A(\text{decay}) = P_0(\text{decay}) + \sum_r P_r(\text{decay}) F_r(x_t), \qquad (3.49)$$

where the sum runs over all possible functions contributing to a given amplitude. P_0 summarizes contributions stemming from internal quarks other than the top, in particular the charm quark. As we will demonstrate in the course of these lectures the general expansion in (3.49) can be derived from the Operator Product Expansion and is valid also in the presence of QCD corrections. We will encounter many examples of the expansion (3.49) in the course of these lectures. We will see that similarly to $K^+ \to \pi^+ \nu \bar{\nu}$, there are other decays which depend only on a single function. However, generally, several basic functions contribute to a given decay. In particular, we have the following correspondence between the most interesting FCNC processes and the basic functions:

$$K^{0} - \bar{K}^{0} - \text{mixing} \qquad S_{0}(x_{t}), \, S_{0}(x_{c}, x_{t})$$

$$B^{0} - \bar{B}^{0} - \text{mixing} \qquad S_{0}(x_{t})$$

$$K \to \pi \nu \bar{\nu}, \, B \to X_{d,s} \nu \bar{\nu} \qquad X_{0}(x_{t})$$

$$K_{L} \to \mu \bar{\mu}, \, B \to l \bar{l} \qquad Y_{0}(x_{t})$$

$$K_{L} \to \pi^{0} e^{+} e^{-} \qquad Y_{0}(x_{t}), \, Z_{0}(x_{t}), \, E_{0}(x_{t})$$

$$\varepsilon' \qquad X_{0}(x_{t}), \, Y_{0}(x_{t}), \, Z_{0}(x_{t}), \, E_{0}(x_{t})$$

$$B \to X_{s} \gamma \qquad D'_{0}(x_{t}), \, E'_{0}(x_{t})$$

$$B \to X_{s} \mu^{+} \mu^{-} \qquad Y_{0}(x_{t}), \, Z_{0}(x_{t}), \, E_{0}(x_{t}), \, E'_{0}(x_{t})$$

3.4 More about GIM

At this stage it is useful to return to the GIM mechanism [15] which did not allow tree level FCNC transitions. This mechanism is also felt in the Hamiltonian of (3.48) and in fact it is fully effective when the masses of internal quarks of a given charge in loop diagrams are set to be equal, e.g. $m_u = m_c = m_t$. Indeed the CKM factors in any FCNC process enter in the combinations

$$C_k \propto \sum_{i=u,c,t} \lambda_i F(x_i)$$
 or $\sum_{i,j=u,c,t} \lambda_i \lambda_j \tilde{F}(x_i,x_j),$ (3.50)

where F, \tilde{F} denote any of the functions of (3.29), and the λ_i are given in the case of K and B meson decays and particle—antiparticle mixing as follows:

$$\lambda_{i} = \begin{cases} V_{is}^{*}V_{id} & K-\text{decays}, \quad K^{0} - \bar{K}^{0} \\ V_{ib}^{*}V_{id} & B-\text{decays}, \quad B_{d}^{0} - \bar{B}_{d}^{0} \\ V_{ib}^{*}V_{is} & B-\text{decays}, \quad B_{s}^{0} - \bar{B}_{s}^{0} \end{cases}$$
(3.51)

They satisfy the unitarity relation

$$\lambda_u + \lambda_c + \lambda_t = 0, (3.52)$$

which implies vanishing coefficients C_k in (3.50) if $x_u = x_c = x_t$. For this reason the mass–independent terms in the calculation of the basic functions in (3.29) can always be omitted. In this limit, FCNC decays and transitions are absent. Thus beyond tree level the conditions for a complete GIM cancellation of FCNC processes are:

- Unitarity of the CKM matrix
- Exact horizontal flavour symmetry which assures the equality of quark masses of a given charge.

Now in nature such a horizontal symmetry, even if it exists at very short distance scales, is certainly broken at low energies by the disparity of masses of quarks of a given charge.

This in fact is the origin of the breakdown of the GIM mechanism at the one-loop level and the appearance of FCNC transitions. The size of this breakdown, and consequently the size of FCNC transitions, depends on the disparity of masses, on the behaviour of the basic functions of (3.29), and can be affected by QCD corrections as we will see in the course of these lectures. Let us make two observations:

• For small $x_i \ll 1$, relevant for $i \neq t$, the functions (3.11)-(3.19) behave as follows:

$$S_0(x_i) \propto x_i, \quad B_0(x_i) \propto x_i \ln x_i, \quad C_0(x_i) \propto x_i \ln x_i$$
 (3.53)

$$D_0(x_i) \propto \ln x_i$$
, $E_0(x_i) \propto \ln x_i$, $D'_0(x_i) \propto x_i$, $E'_0(x_i) \propto x_i$. (3.54)

This implies "hard" (quadratic) GIM suppression of FCNC processes governed by the functions $S_0, B_0, C_0, D'_0, E'_0$ provided the top quark contributions due to small CKM factors can be neglected. In the case of $D_0(x_i)$ and $E_0(x_i)$ only "soft" (logarithmic) GIM suppression is present.

• For large x_t we have

$$S_0(x_t) \propto x_t, \quad B_0(x_t) \propto \text{const}, \quad C_0(x_t) \propto x_t$$
 (3.55)

$$D_0(x_t) \propto \ln x_t$$
, $E_0(x_t) \propto \text{const}$, $D_0'(x_t) \propto \text{const}$, $E_0'(x_t) \propto \text{const}$. (3.56)

Thus for FCNC processes governed by top quark contributions, the GIM suppression is not effective at the one loop level and in fact in the case of decays and transitions receiving contributions from $S_0(x_t)$ and $C_0(x_t)$ important enhancements are possible.

The latter property emphasizes the special role of K and B decays with regard to FCNC transitions. In these decays the appearance of the top quark in the internal loop with $m_t > M_W \gg m_c, m_u$ removes the GIM suppression, making K and B decays a particularly useful place to test FCNC transitions and to study the physics of the top quark. Of course the hierarchy of various FCNC transitions is also determined by the hierarchy of the elements of the CKM matrix allowing this way to perform sensitive tests of this sector of the Standard Model.

The FCNC decays of D-mesons are much stronger suppressed because only d, s, and b quarks with $m_d, m_s, m_b \ll M_W$ enter internal loops and the GIM mechanism is much more effective. Also the known structure of the CKM matrix is less favorable than in K and B decays. For these reasons we will restrict our presentation to the latter. In the extensions of the Standard Model, FCNC transitions are possible at the tree level and the hierarchies discussed here may not apply. Reviews of FCNC transitions in D-decays can be found in [46].

3.5 Final Comments

The discussion presented in this and the previous section left out completely the QCD effects in weak decays. In particular, we have not shown how to translate the calculations done in terms of quarks into predictions for the decays of their bound states, the hadrons. Similary we did not include short distance QCD corrections. Still, I hope that this discussion has shown the richness of the field of weak decays and of FCNC processes and motivated the reader to learn more about the more technical part of this field. Such motivation is clearly necessary as from now on our gentle hike is turning quickly into a real climb. This climb will be rather steep and in certain parts technically difficult. It will last with small breaks until we reach section 9. From section 9 on it will be easy again.

4 Renormalization and Renormalization Group

4.1 General Remarks

This section collects those basic facts about QCD, its renormalization and the renormalization group, which are indispensable for our climb. In particular we discuss the dimensional regularization, the MS and $\overline{\rm MS}$ renormalization schemes and renormalization group equations for the running QCD coupling and the running quark masses. We recall solutions of these equations and present numerical examples. At the end of this section we explain what is meant by the renormalization group improved perturbation theory.

4.2 QCD Lagrangian

The Lagrangian density of QCD, omitting the ghosts and setting the gauge parameter to $\xi = 1$, reads

$$\mathcal{L}_{QCD} = -\frac{1}{4} (\partial_{\mu} A^{a}_{\nu} - \partial_{\nu} A^{a}_{\mu}) (\partial^{\mu} A^{a\nu} - \partial^{\nu} A^{a\mu}) - \frac{1}{2} (\partial^{\mu} A^{a}_{\mu})^{2}$$

$$+ \bar{q}_{\alpha} (i \partial - m_{q}) q_{\alpha} - g_{s} \bar{q}_{\alpha} T^{a}_{\alpha\beta} \gamma^{\mu} q_{\beta} A^{a}_{\mu}$$

$$+ \frac{g_{s}}{2} f^{abc} (\partial_{\mu} A^{a}_{\nu} - \partial_{\nu} A^{a}_{\mu}) A^{b\mu} A^{c\nu} - \frac{g_{s}^{2}}{4} f^{abe} f^{cde} A^{a}_{\mu} A^{b}_{\nu} A^{c\mu} A^{d\nu}$$

$$(4.1)$$

Here A^a_μ are the gluon fields with (a,b,c=1,..8) and $q=(q_1,q_2,q_3)$ is the color triplet of quark flavor $q,\,q=u,\,d,\,s,\,c,\,b,\,t.\,g_s$ is the QCD coupling so that

$$\alpha_s = \frac{g_s^2}{4\pi}.\tag{4.2}$$

Finally T^a and f^{abc} are the generators and structure constants of $SU(3)_C$, respectively. From this Lagrangian one can derive the Feynman rules for QCD. Some of these rules have been given in figs. 3 and 4.

4.3 Dimensional Regularization

In order to deal with divergences that appear in loop corrections to Green functions we have to regularize the theory to have an explicit parametrization of the singularities. In these lectures we will employ dimensional regularization (DR). In this regularization Feynman diagrams are evaluated in $D=4-2\,\varepsilon$ space-time dimensions and singularities are extracted as poles for $\varepsilon\to 0$. Thus the results of one-loop or two-loop calculations have the following general structure:

One Loop Result =
$$\frac{a_1}{\varepsilon} + b_1$$
, (4.3)

Two Loop Result =
$$\frac{a_2}{\varepsilon^2} + \frac{b_2}{\varepsilon} + c_2$$
, (4.4)

where a_i , b_i and c_2 are finite.

Several useful formulae for the evaluation of Feynman diagrams in $D=4-2\varepsilon$ dimensions are collected in the appendix A of my review in [47] and in Muta's book [20]. Here we only stress the following important point. Let us consider the second term in the second line in (4.1). The mass dimensions of q_i , A^a_μ and \mathcal{L} are (D-1)/2, (D-2)/2 and D respectively. Consequently, the dimension of g_s in $D=4-2\varepsilon$ dimensions is simply equal to ε . It is more useful to work with a dimensionless coupling constant in arbitrary D dimensions. To this end we make the replacement in (4.1):

$$g_s \rightarrow g_s \mu^{\varepsilon}$$
 (4.5)

where μ is an arbitrary parameter with the dimensions of mass and g_s on the r.h.s is dimensionless. The appearence of the scale μ has profound impact on these lectures.

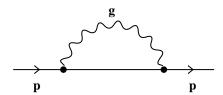


Figure 13: Quark-Self-Energy Diagram

As an example, consider the calculation of the one-loop self-energy diagram of fig. 13. Setting $m_q = 0$ and denoting the external quark momentum by p (with $p^2 < 0$), we arrive by means of standard techniques at

$$i\Sigma_{\alpha\beta} = i \not\!\! b C_F \delta_{\alpha\beta} g_s^2 [2(1-\varepsilon)] P_{div} B(2-\varepsilon, 1-\varepsilon)$$

$$(4.6)$$

where $C_F = 4/3$ is the relevant colour factor. Next

$$P_{div} \equiv \frac{\Gamma(\varepsilon)}{(4\pi)^{2-\varepsilon}} \left(\frac{\mu^2}{-p^2}\right)^{\varepsilon} = \frac{1}{16\pi^2} \left[\frac{1}{\varepsilon} + \ln 4\pi - \gamma_E + \ln \frac{\mu^2}{-p^2} + \mathcal{O}(\varepsilon)\right]$$
(4.7)

and Γ and B are the known Euler functions. In expanding P_{div} around $\varepsilon = 0$ we have used

$$\Gamma(\varepsilon) = \frac{1}{\varepsilon} - \gamma_E + O(\varepsilon) \qquad \gamma_E = 0.5772...$$
 (4.8)

where γ_E is the Euler constant. Since

$$B(2 - \varepsilon, 1 - \varepsilon) = \frac{1}{2}(1 + 2\varepsilon) + \mathcal{O}(\varepsilon^2)$$
(4.9)

we arrive at

$$i\Sigma_{\alpha\beta} = i \not\!p C_F \delta_{\alpha\beta} \frac{\alpha_s}{4\pi} \left[\frac{1}{\varepsilon} + \ln 4\pi - \gamma_E + \ln \frac{\mu^2}{-p^2} + 1 \right]$$
(4.10)

where $\mathcal{O}(\varepsilon)$ terms have been set to zero. We have thus extracted the singularity as a $1/\varepsilon$ pole and have obtained a well-defined finite part. The appearance of the first four terms in the square bracket in (4.10) originating from P_{div} in (4.7) is characteristic for all divergent one-loop calculations.

The dimensional regularization is the favourite regularization in gauge theories as it preserves all symmetries of the theory. Possible problems are connected with the treatment of γ_5 in $D \neq 4$ dimensions, which clearly is of relevance for the study of weak interactions. Let us discuss this issue now. We follow here [48]

4.4 The Issue of γ_5 in D Dimensions

4.4.1 Preliminaries

Let us describe the three distinct sets of computational rules, for the manipulation of covariants and Dirac matrices, most commonly used in perturbative calculations in the Standard Model. These schemes all employ the method of dimensional regularization of the Feynman integrals [49, 50], and in each case $D = 4 - 2\varepsilon$ denotes the number of dimensions. We will not discuss other regularization schemes such as BPHZ and lattice. These work directly in 4-dimensions and hence don't have algebraic consistency problems with respect to γ_5 , but their use introduces other subtleties and two-loop calculations therewith are extremely tedious.

4.4.2 Naive Dimensional Regularization

The most commonly used set of rules is one we shall call 'naive dimensional regularization' (NDR). Only the D- dimensional metric tensor g is introduced satisfying,

$$g_{\mu\nu} = g_{\nu\mu}, \qquad g_{\mu\rho}g^{\rho}_{\nu} = g_{\mu\nu}, \qquad g^{\mu}_{\mu} = D,$$
 (4.11)

and the Dirac matrices γ_{μ} obey

$$\{\gamma_{\mu}, \gamma_{\nu}\} = 2g_{\mu\nu}.\tag{4.12}$$

It is standard (but inessential) to set the trace of the unit matrix to equal 4; we shall adopt this convention in this and the schemes below. When γ_5 appears in the Feynman vertices the manipulation rule adopted in this scheme is that it anticommutes with the Dirac matrices,

$$\{\gamma_{\mu}, \gamma_5\} = 0. \tag{4.13}$$

It has repeatedly been emphasized in the literature [50, 51] that this rule leads to obvious algebraic inconsistencies. Nevertheless this scheme has been most widely employed for most calculations because of its ease to incorporate standard software in computer programs. It is known to lead to incorrect results in certain cases, e.g. the axial anomaly is not reproduced. On the other hand in many cases it does reproduce the correct results. A necessary condition for this seems to be that the calculated amplitude does not involve the evaluation of a closed odd parity fermion loop. Indeed, with the NDR rules one does not know how to unambiguously handle the expression $\text{Tr}(\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\lambda)$. Beginning with the work of Peter Weisz and myself [48] it has been demonstrated in many explicit calculations that the NDR scheme gives correct results, consistent with the schemes without the γ_5 problems, provided one can avoid the calculations of traces like the one given above. In fact all the higher order QCD calculations for weak decays performed in the NDR scheme in the last nine years and listed in table 10 could avoid the direct calculation of such traces. To this end it is often necessary at the intermediate stages of the calculation to work with a special basis of local operators which differs from the standard basis discussed in sections 5-8 of these lectures. Examples of such strategies valid only at two-loop level can be found in [68, 69, 70]. An approach valid apparently to all orders is presented in [71].

4.4.3 Dimensional Reduction

A second set of manipulation rules initially introduced by Siegel [52] for the renormalization of supersymmetric theories goes under the name of dimensional reduction. Here the Dirac matrices $\tilde{\gamma}$ are taken to be in 4-dimensions, thus

$$\{\tilde{\gamma}_{\mu}, \tilde{\gamma}_{\nu}\} = 2\tilde{g}_{\mu\nu},\tag{4.14}$$

where \tilde{g} is the 4-dimensional metric tensor,

$$\tilde{g}_{\mu\nu} = \tilde{g}_{\nu\mu}, \qquad \tilde{g}_{\mu\rho}\tilde{g}^{\rho}_{\nu} = \tilde{g}_{\mu\nu}, \qquad \tilde{g}^{\mu}_{\mu} = 4.$$
 (4.15)

When evaluating the Feynman integrals the D- dimensional $g_{\mu\nu}$ inevitably makes its appearence and it is necessary to supplement the rules with one which stipulates the result of

contraction of the 4- and D- dimensional metric tensors. In order to preserve gauge invariance and in apparent concord with the reduction to D < 4 dimensions the rule employed is

$$\tilde{g}_{\mu\rho}g^{\rho}_{\nu} = g_{\mu\nu}.\tag{4.16}$$

The advantage of the scheme is that the 4-dimensional Dirac algebra can be used to reduce the algebraic complexity of the amplitudes. However there is a price to be paid which involves a number of field theoretical subtleties some of which are already present in the pure QCD part of the dynamics. These are discussed in [53]. Again, this scheme has been criticized [51, 54] since it leads to similar difficulties as the naive dimensional regularization described above. In particular it implies that identities homogeneous in the metric tensor in 4-dimensions are also satisfied in generic D-dimensions, which is manifestly algebraically inconsistent. Although the axial anomaly can be reproduced [55], and although there is to our knowledge as yet no known explicit calculation using DRED which gives the wrong result, it has not yet been established as a consistent scheme and thus maintains at present merely the status of a prescription.

In the field of weak decays the DRED scheme has been used in [53] for the calculation of higher order QCD corrections to $\Delta S = 1$ decays. This result has been confirmed in [48] and shown to be compatible with the NDR scheme and the 't Hooft-Veltman scheme discussed below. Similarly the initial problem of calculating higher order QCD corrections to the $B \to X_s \gamma$ in the DRED scheme [56] has been resolved by Misiak [57]. These days the DRED scheme is less popular and the most calculations of QCD corrections are in the NDR scheme and the 't Hooft-Veltman scheme to which we turn now our attention.

4.4.4 The 't Hooft-Veltman Rules

The third set of rules is the one originally proposed by 't Hooft and Veltman [49] and by Akyeampong and Delbourgo [58] and systematized by Breitenlohner and Maison [50]. The latter authors showed that this is a consistent formulation of dimensional regularization even when γ_5 couplings are present.

To write down the rules it is convenient to introduce in addition to the D- and 4- dimensional metric tensors g and \tilde{g} satisfying (4.11) and (4.15) respectively, the -2ε - dimensional tensor \hat{g} satisfying,

$$\hat{g}_{\mu\nu} = \hat{g}_{\nu\mu}, \qquad \hat{g}_{\mu\rho}\hat{g}^{\rho}_{\nu} = \hat{g}_{\mu\nu}, \qquad \hat{g}^{\mu}_{\mu} = -2\varepsilon.$$
 (4.17)

The important difference with respect to dimensional reduction is that instead of the rule (4.16) for contracting the different metric tensors one imposes

$$\tilde{g}_{\mu\rho}g^{\rho}_{\nu} = \tilde{g}_{\mu\nu},\tag{4.18}$$

which does not lead to manifest algebraic inconsistencies. In addition to (4.18) one has,

$$\hat{g}_{\mu\rho}g^{\rho}_{\nu} = \hat{g}_{\mu\nu}, \qquad \hat{g}_{\mu\rho}\tilde{g}^{\rho}_{\nu} = 0.$$
 (4.19)

The D-dimensional Dirac matrix is now split into a 4- and -2ε -dimensional parts,

$$\gamma_{\mu} = \tilde{\gamma}_{\mu} + \hat{\gamma}_{\mu},\tag{4.20}$$

with γ and $\tilde{\gamma}$ obeying the anticommutation relations (4.12) and (4.14) respectively. $\hat{\gamma}$ on the other hand satisfies

$$\{\hat{\gamma}_{\mu}, \hat{\gamma}_{\nu}\} = 2\hat{g}_{\mu\nu},\tag{4.21}$$

and it anticommutes with $\tilde{\gamma}$

$$\{\hat{\gamma}_{\mu}, \tilde{\gamma}_{\nu}\} = 0. \tag{4.22}$$

Note also by virtue of (4.19) it follows

$$\hat{\gamma}_{\mu}\tilde{\gamma}^{\mu} = 0, \qquad \hat{g}^{\nu}_{\mu}\tilde{\gamma}_{\nu} = 0, \qquad \tilde{g}^{\nu}_{\mu}\hat{\gamma}_{\nu} = 0.$$
 (4.23)

In [50] it is shown that a γ_5 can be introduced which anticommutes with $\tilde{\gamma}$ but commutes with $\hat{\gamma}$,

$$\gamma_5^2 = 1, \qquad \{\gamma_5, \tilde{\gamma}_\nu\} = 0, \qquad [\gamma_5, \hat{\gamma}_\nu] = 0.$$
 (4.24)

Since γ_5 does not have simple commutation properties with γ_{μ} it is important to consistently define the coupling to chiral fields in a model such as the Standard Model; e.g. for coupling to left-handed fields the symmetrically defined vertex

$$\frac{1}{2}(1+\gamma_5)\gamma_{\mu}(1-\gamma_5) = \tilde{\gamma}_{\mu}(1-\gamma_5), \tag{4.25}$$

should be used [59].

This scheme has admittedly some rather unattractive features. In particular it is more inconvenient to implement in algebraic computer programs than the NDR scheme. Nevertheless it must be stressed again that it is to date the only known scheme (within the framework of dimensional regularization) which has been demonstrated to be consistent [50, 51], and thus its inconvenience must be tolerated. For this reason a computer package for Dirac algebra manipulation in the HV and NDR schemes called TRACER has been developed in my group at the Technical University in Munich by Jamin and Lautenbacher [60]. Using this program one can appreciate the simplicity of the NDR scheme compared with the HV scheme for which the computer calculations can be sometimes really time consuming.

4.5 Renormalization

4.5.1 General Remarks

In order to eliminate the divergences in Green functions one has to renormalize the fields and parameters in the Lagrangian through

$$A_{0\mu}^{a} = Z_{3}^{1/2} A_{\mu}^{a}$$
 $q_{0} = Z_{q}^{1/2} q$ $q_{0,s} = Z_{g} g_{s} \mu^{\varepsilon}$ $m_{0} = Z_{m} m$ (4.26)

The index "0" indicates unrenormalized quantities. A^a_μ and q are renormalized fields, g_s is the renormalized QCD coupling and m the renormalized quark mass. The factors Z are the renormalization constants. They are divergent quantities, chosen in such a manner that the divergences disappear once the Greens functions have been expressed in terms of renormalized quantities only.

It should be stressed that the unrenormalized parameters $g_{0,s}$ and m_0 are independent of the scale μ . This implies, in particular, that g_s must be μ -dependent. Since Z_i have a perturbative expansion in g_s they must also depend on μ . Consequently also the renormalized mass m is μ -dependent.

4.5.2 The Counter-term Method

A straightforward way to implement renormalization is provided by the counter-term method. Thereby parameters and fields in the original Lagrangian, considered as unrenormalized (bare) quantities, are reexpressed through renormalized ones by means of (4.26). Thus

$$\mathcal{L}_{QCD}^0 = \mathcal{L}_{QCD} + \mathcal{L}_C \tag{4.27}$$

where \mathcal{L}_{QCD} is given in (4.1). \mathcal{L}_{QCD}^0 is also given by (4.1) but with q replaced by q_0 and similarly for A_{μ}^a , g_s and m. \mathcal{L}_C is the counter-term Lagrangian. It is simply defined by (4.27). For instance:

$$\mathcal{L}_{q} = \bar{q}_{0}i \ \partial q_{0} - m_{0}\bar{q}_{0}q_{0} \equiv \bar{q}i \ \partial q - m\bar{q}q + (Z_{q} - 1)\bar{q}i \ \partial q - (Z_{q}Z_{m} - 1)m\bar{q}q \ . \tag{4.28}$$

 \mathcal{L}_{QCD} given entirely in terms of renormalized quantities leads to the usual Feynman rules of figs. 3 and 4. The counter-terms ($\sim (Z-1)$) can be formally treated as new interaction terms that contribute to Green functions calculated in perturbation theory. For these new interactions also Feynman rules can be derived. For instance, the Feynman rule for the counter-terms in (4.28) reads (p is the quark momentum)

$$i\delta_{\alpha\beta}[(Z_q-1) \not\!p - (Z_qZ_m-1)m]$$
 (4.29)

The constants Z_i are determined such that the the contributions from these new interactions cancel the divergences in the Green functions resulting from the calculations based on \mathcal{L}_{QCD} in (4.27) only. There is some arbitrariness how this can be done because a given renormalization prescription can in general subtract not only the divergences but also finite parts. The subtractions of finite parts is, however, not uniquely defined which results in the renormalization scheme dependence of Z_i and of the renormalized fields and parameters. We will elaborate on this scheme dependence and its cancellation in physical quantities at later stages of these lectures.

4.5.3 MS and $\overline{\rm MS}$ Renormalization Schemes

The simplest renormalization scheme is the *Minimal Subtraction Scheme MS* [9] in which only divergences are subtracted. In this scheme, the renormalization constants are given by

$$Z_{i} = \frac{\alpha_{s}}{4\pi} \frac{a_{1i}}{\varepsilon} + \left(\frac{\alpha_{s}}{4\pi}\right)^{2} \left(\frac{a_{2i}}{\varepsilon^{2}} + \frac{b_{2i}}{\varepsilon}\right) + \mathcal{O}(\alpha_{s}^{3})$$

$$(4.30)$$

where a_{ji} and b_{ji} are μ -independent constants. The fact that in this scheme the renormalization constants do not have any explicit μ -dependence and depend on μ only through g_s is an important virtue of this scheme. This, in particular, in the context of renormalization group equations discussed below. Similarly the renormalization constants Z_i do not depend on masses. Therefore the MS-scheme and the schemes discussed below belong to the class of mass independent renormalization schemes [10].

Now, starting with the MS scheme, one can construct a whole class of subtraction schemes which differ from MS by a different continuation of the renormalized coupling constant to D dimensions. For these MS-like schemes we have

$$g_{0,s} = Z_g^k g_s^k \mu_k^{\varepsilon} \qquad \mu_k = \mu f_k \tag{4.31}$$

where f_k is an arbitrary number which defines the particular scheme "k". Since different schemes in this class differ from the MS scheme only by a schift in μ , the renormalization constants for these schemes can be obtained from (4.30) by replacing α_s by α_s^k chareacteristic for a given scheme. The constants a_{ji} and b_{ji} , being μ -independent, remain unchanged.

Of particular interest is the so-called MS scheme [61] in which

$$\mu_{\overline{MS}} = \mu e^{\gamma_E/2} (4\pi)^{-1/2} \tag{4.32}$$

and P_{div} in (4.7) is replaced by

$$\bar{P}_{div} \equiv \frac{\Gamma(\varepsilon)}{(4\pi)^{2-\varepsilon}} \left(\frac{\mu_{\overline{MS}}^2}{-p^2}\right)^{\varepsilon} = \frac{1}{16\pi^2} \left[\frac{1}{\varepsilon} + \ln\frac{\mu^2}{-p^2} + \mathcal{O}(\varepsilon)\right]$$
(4.33)

We observe that in this scheme the terms $\ln 4\pi - \gamma_E$, the artifacts of the dimensional regularization, are absent!

In summary then:

$$\{MS \to \overline{MS}\} \equiv \{\mu \to \mu_{\overline{MS}}\}$$
 (4.34)

$$\{Z_i^{MS} \to Z_i^{\overline{MS}}\} \equiv \{\alpha_s^{MS} \to \alpha_s^{\overline{MS}}\}.$$
 (4.35)

In these lectures, we will exclusively work with the $\overline{\rm MS}$ scheme. In order to simplify the notation we will denote $\mu_{\overline{MS}}$ simply by μ and simultaneously drop the $\ln 4\pi - \gamma_E$ terms in any finite contribution. Similarly α_s in these lectures will always stand for $\alpha_s^{\overline{\rm MS}}$.

As an example let us find Z_q and Z_m . To this end we repeat the calculation of the self-energy diagram of fig. 13, this time keeping the quark mass m. Dropping the finite terms, which are of no concern for finding Z_i in the $\overline{\text{MS}}$ scheme, we find

$$(i\Sigma_{\alpha\beta})_{div} = iC_F \delta_{\alpha\beta} \frac{\alpha_s}{4\pi} (\not p - 4m) \frac{1}{\varepsilon} + \mathcal{O}(\alpha_s^2)$$
(4.36)

Adding to this result the counter-term (4.29) and requiring the final result to be zero we readly find

$$Z_q = 1 - \frac{\alpha_s}{4\pi} C_F \frac{1}{\varepsilon} + \mathcal{O}(\alpha_s^2) \tag{4.37}$$

$$Z_m = 1 - \frac{\alpha_s}{4\pi} 3C_F \frac{1}{\varepsilon} + \mathcal{O}(\alpha_s^2)$$
 (4.38)

Similarly Z_3 and Z_g can be found by calculating one-loop corrections to the gluon propagator and the gluon- $\bar{q}q$ vertex, respectively. One finds:

$$Z_3 = 1 - \frac{\alpha_s}{4\pi} \left[\frac{2}{3} f - \frac{5}{3} N \right] \frac{1}{\varepsilon} + \mathcal{O}(\alpha_s^2)$$
 (4.39)

$$Z_g = 1 - \frac{\alpha_s}{4\pi} \left[\frac{11}{6} N - \frac{2}{6} f \right] \frac{1}{\varepsilon} + \mathcal{O}(\alpha_s^2)$$
 (4.40)

where N denotes the number of colours (N=3 in QCD) and f stands for the number of quark flavours.

4.5.4 Renormalization of Green Functions

Let us denote by

$$G^{(n_F, n_G)}(p_i, g_s, m, \mu, \varepsilon) \equiv \langle 0 | T(q_1, ... q_{n_F}, A_1^{\mu}, ... A_{n_G}^{\mu}) | 0 \rangle$$
(4.41)

a connected renormalized Green function with n_F quark and n_G gluon external legs carrying momenta p_j . Here m indicates general dependence on masses. The corresponding amputated renormalized one-particle irreducible Green function is given by

$$\Gamma^{(n_F, n_G)} = \frac{G^{(n_F, n_G)}}{\prod^{n_F} G^{(2,0)} \prod^{n_G} G^{(0,2)}} . \tag{4.42}$$

Similar expressions exist for the unrenormalized Green functions $G_0^{(n_F,n_G)}$ and $\Gamma_0^{(n_F,n_G)}$ with all renormalized parameters and fields replaced by the corresponding bare quantities. With (4.26), $\Gamma^{(n_F,n_G)}$ and $\Gamma_0^{(n_F,n_G)}$ are related to each other by

$$\Gamma^{(n_F,n_G)}(p_j, g_s, m, \mu, \varepsilon) = Z_q^{n_F/2} Z_3^{n_G/2} \Gamma_0^{(n_F,n_G)}(p_j, g_{0,s}, m_0, \varepsilon) . \tag{4.43}$$

The renormalization then means that when $g_{0,s}$ and m_0 on the r.h.s of (4.43) are expressed through g and m according to (4.26), $\Gamma^{(n_F,n_G)}$ are finite and the limit

$$\lim_{\varepsilon \to 0} \Gamma^{(n_F, n_G)}(p_j, g_s, m, \mu, \varepsilon) = \Gamma^{(n_F, n_G)}(p_j, g_s, m, \mu)$$
(4.44)

exists.

As an example consider the result for the quark self-energy in (4.10). In the notation of (4.43) its divergent part added to the "tree level" propagator is given by

$$\Gamma_0^{(2,0)} = iC_F \delta_{\alpha\beta} \not p \left(1 + \frac{\alpha_s}{4\pi} \frac{1}{\varepsilon}\right). \tag{4.45}$$

The corresponding renormalized two-point function is given by

$$\Gamma^{(2,0)} = Z_q \Gamma_0^{(2,0)} \tag{4.46}$$

which with (4.37) is indeed finite. In this case at $\mathcal{O}(\alpha_s)$ only quark field renormalization is needed to obtain finite result. Coupling renormalization is necessary first at $\mathcal{O}(\alpha_s^2)$.

4.6 Renormalization Group Equations

4.6.1 The Basic Equations

In the process of renormalization we have introduced an arbitrary mass parameter μ . The μ -dependence of the renormalized coupling constant g_s and of the renormalized quark mass m is governed by the renormalization group equations. These equations are derived from the definitions (4.26) using the fact that bare quantities are μ -independent. One finds ($g \equiv g_s$):

$$\frac{dg(\mu)}{d\ln\mu} = \beta(g(\mu), \varepsilon) \tag{4.47}$$

$$\frac{dm(\mu)}{d\ln\mu} = -\gamma_m(g(\mu))m(\mu) \tag{4.48}$$

where

$$\beta(q,\varepsilon) = -\varepsilon q + \beta(q),\tag{4.49}$$

$$\beta(g) = -g \frac{1}{Z_g} \frac{dZ_g}{d \ln \mu}, \qquad \gamma_m(g) = \frac{1}{Z_m} \frac{dZ_m}{d \ln \mu}. \tag{4.50}$$

(4.49) is valid in arbitrary dimensions. In four dimensions $\beta(g,\varepsilon)$ reduces to $\beta(g)$. Let us prove (4.49) [62]. Using (4.26) we have

$$\beta(g,\varepsilon) = g_0 \mu \frac{d}{d\mu} [\mu^{-\varepsilon} Z_g^{-1}] = g_0 \left[-\varepsilon \mu^{-\varepsilon} Z_g^{-1} + \mu^{-\varepsilon+1} \frac{dZ_g^{-1}}{d\mu} \right]$$

$$= -\varepsilon g - g_0 \mu^{-\varepsilon+1} \frac{1}{Z_g^2} \frac{dZ_g}{d\mu} = -\varepsilon g - g \mu \frac{dZ_g}{d\mu} \frac{1}{Z_g}. \tag{4.51}$$

Similarly one can derive the expression for γ_m in (4.50) by inserting $m = m_0/Z_m$ into (4.48).

 $\beta(g)$ and $\gamma(g)$ are called renormalization group functions. $\beta(g)$ governs the μ -dependence of $g(\mu)$. γ_m , the anomalous dimension of the mass operator, governs the μ -dependence of $m(\mu)$. In the MS ($\overline{\text{MS}}$)-scheme they depend only on g. In particular they carry no explicit μ -dependence and are independent of masses. Writing

$$Z_i = 1 + \sum_{k=1}^{\infty} \frac{1}{\varepsilon^k} Z_{i,k}(g)$$

$$(4.52)$$

and using (4.49) and (4.50) one finds

$$\beta(g) = 2g^3 \frac{dZ_{g,1}(g)}{dq^2},\tag{4.53}$$

$$\gamma_m(g) = -2g^2 \frac{dZ_{m,1}(g)}{dq^2}. (4.54)$$

Thus $\beta(g)$ and $\gamma_m(g)$ can be directly obtained from the $1/\varepsilon$ -pole parts of the renormalization constants Z_g and Z_m , respectively. This is a very useful property of the MS-like schemes. Let us demonstrate that (4.53) is indeed true. We follow Muta [20] and write

$$\beta(g,\varepsilon) = -\varepsilon g - gf(g), \qquad f(g) = \frac{\mu}{Z_g} \frac{dZ_g}{d\mu}.$$
 (4.55)

Specializing the expansion (4.52) to Z_g and inserting it into formula for f(g) gives

$$f(g)\left(1 + \frac{Z_{g,1}}{\varepsilon} + \frac{Z_{g,2}}{\varepsilon^2} + \dots\right) = \frac{1}{\varepsilon}\beta(g,\varepsilon)\left(\frac{dZ_{g,1}}{dg} + \frac{1}{\varepsilon}\frac{dZ_{g,2}}{dg} + \dots\right) . \tag{4.56}$$

Now finitness of $\beta(g)$ implies finitness of f(g). Consequently the equality (4.56) should hold for each coefficient of the power $1/\varepsilon$. In particular the non-singular terms give

$$f(g) = -g\frac{dZ_{g,1}}{dg} , \qquad (4.57)$$

which with $\beta(g) = -gf(g)$ gives (4.53). The proof of (4.54) can be done in an analogous manner using the finitness of γ_m . It is left as a homework problem.

With Z_g and Z_m in (4.40) and (4.38) respectively, the formulae (4.53) and (4.54) give immediately the leading terms for $\beta(g)$ and $\gamma_m(g)$:

$$\beta(g) = -\frac{g^3}{16\pi^2} \left[\frac{11}{3} N - \frac{2}{3} f \right], \tag{4.58}$$

$$\gamma_m(g) = \frac{g^2}{16\pi^2} 6C_F. \tag{4.59}$$

With this technique it is also easy to show that the anomalous dimensions of the quark field (γ_q) and the quant field (γ_G) defined by

$$\gamma_q(g) = \frac{1}{2} \frac{1}{Z_q} \frac{dZ_q}{d \ln \mu}, \qquad \gamma_G(g) = \frac{1}{2} \frac{1}{Z_3} \frac{dZ_3}{d \ln \mu},$$
(4.60)

are given by

$$\gamma_i(g) = -g^2 \frac{dZ_{i,1}(g)}{dg^2} \qquad (i = q, G).$$
(4.61)

4.6.2 Compendium of Useful Results

It will be useful to have a collection of results for $\beta(g)$, $\gamma(\alpha_s)$ and $Z_{q,1}(\alpha_s)$ including also two-loop contributions. They are:

$$\beta(g) = -\beta_0 \frac{g^3}{16\pi^2} - \beta_1 \frac{g^5}{(16\pi^2)^2} \tag{4.62}$$

$$\gamma_m(\alpha_s) = \gamma_m^{(0)} \frac{\alpha_s}{4\pi} + \gamma_m^{(1)} \left(\frac{\alpha_s}{4\pi}\right)^2 \tag{4.63}$$

$$Z_{q,1}(\alpha_s) = a_1 \frac{\alpha_s}{4\pi} + a_2 \left(\frac{\alpha_s}{4\pi}\right)^2 \tag{4.64}$$

where

$$\beta_0 = \frac{11N - 2f}{3} \qquad \beta_1 = \frac{34}{3}N^2 - \frac{10}{3}Nf - 2C_F f \tag{4.65}$$

$$\gamma_m^{(0)} = 6C_F \qquad \gamma_m^{(1)} = C_F \left(3C_F + \frac{97}{3}N - \frac{10}{3}f \right)$$
 (4.66)

$$a_1 = -C_F$$
 $a_2 = C_F \left(\frac{3}{4}C_F - \frac{17}{4}N + \frac{1}{2}f\right)$ (4.67)

$$C_F = \frac{N^2 - 1}{2N}. (4.68)$$

These results are valid in the MS ($\overline{\text{MS}}$) scheme. N is the number of colours and f the number of quark flavors. Whereas β_0 , β_1 , $\gamma_m^{(0)}$, $\gamma_m^{(1)}$ are gauge independent, a_1 and a_2 given here have been obtained in the $\xi = 1$ gauge.

4.7 Running Coupling Constant

With the expansion (4.62), the renormalization group equation (4.47) for $g(\mu)$ can be written as follows:

$$\frac{d\alpha_s}{d\ln\mu} = -2\beta_0 \frac{\alpha_s^2}{4\pi} - 2\beta_1 \frac{\alpha_s^3}{(4\pi)^2}$$
 (4.69)

Solving it, one finds [61]:

$$\frac{\alpha_s(\mu)}{4\pi} = \frac{1}{\beta_0 \ln(\mu^2/\Lambda_{\overline{MS}}^2)} - \frac{\beta_1}{\beta_0^3} \frac{\ln \ln(\mu^2/\Lambda_{\overline{MS}}^2)}{\ln^2(\mu^2/\Lambda_{\overline{MS}}^2)}.$$
 (4.70)

Let us make a few comments:

• $\Lambda_{\overline{MS}}$ is a QCD scale characteristic for the $\overline{\rm MS}$ scheme. It can be determined by measuring $\alpha_s(\mu)$ at a single value of μ . To this end the quantity used to determine $\alpha_s(\mu)$ has to be calculated in the $\overline{\rm MS}$ scheme. Strictly speaking $\alpha_s(\mu)$ should really read $\alpha_{s,\overline{\rm MS}}$ but we will work exclusively in the $\overline{\rm MS}$ scheme and this complication of the notation is unnecessary. Yet it is useful to quote the relation to the MS scheme. Using (see (4.32))

$$\mu \equiv \mu_{\overline{MS}} = \mu_{MS} e^{\gamma_E/2} (4\pi)^{-1/2} \tag{4.71}$$

in (4.70) one finds the relation between α_s in the MS and $\overline{\text{MS}}$ schemes:

$$\alpha_{s,MS} = \alpha_{s,\overline{MS}} \left(1 + \beta_0 (\gamma_E - \ln 4\pi) \frac{\alpha_{s,\overline{MS}}}{4\pi} \right)$$
 (4.72)

or

$$\Lambda_{\overline{MS}}^2 = 4\pi e^{-\gamma_E} \Lambda_{MS}^2 \tag{4.73}$$

• $\Lambda_{\overline{MS}}$ and $\alpha_s(\mu)$ depend on f, the number of "effective" flavours present in β_0 and β_1 . What "effective" f really means will be explained in the next section. For the time being we adopt the following working procedure:

$$f = \begin{cases} 6 & \mu \ge m_{\rm t} \\ 5 & m_{\rm b} \le \mu \le m_{\rm t} \\ 4 & m_{\rm c} \le \mu \le m_{\rm b} \\ 3 & \mu \le m_{\rm c}. \end{cases}$$
(4.74)

Denoting by $\alpha_s^{(f)}$ the effective coupling constant for a theory with f effective flavours and by $\Lambda_{\overline{MS}}^{(f)}$ the corresponding QCD scale parameter, we have the following boundary conditions which follow from the continuity of α_s :

$$\alpha_s^{(6)}(m_t) = \alpha_s^{(5)}(m_t), \qquad \alpha_s^{(5)}(m_b) = \alpha_s^{(4)}(m_b), \qquad \alpha_s^{(4)}(m_c) = \alpha_s^{(3)}(m_c).$$
 (4.75)

The above continuity conditions allow to find values of $\Lambda_{\overline{MS}}^{(f)}$ for different f once one particular $\Lambda_{\overline{MS}}^{(f)}$ is known. In table 2 we show different $\alpha_s^{(f)}(\mu)$ and $\Lambda_{\overline{MS}}^{(f)}$ corresponding to

$$\alpha_s^{(5)}(M_{\rm Z}) = 0.118 \pm 0.005,$$
(4.76)

which is in the ball park of the present world average extracted from different processes [63]. To this end we have set $m_c = 1.3$ GeV, $m_b = 4.4$ GeV and $m_t = 170$ GeV. We

observe that for $\mu \geq m_c$ the values of $\alpha_s(\mu)$ are sufficiently small that the effects of strong interactions can be treated in perturbation theory. When one moves to low energy scales, α_s increases and at $\mu \approx \mathcal{O}(1 \text{ GeV})$ and high values of $\Lambda_{\overline{MS}}^{(3)}$ one finds $\alpha_s^{(3)}(\mu) > 0.5$. This signals breakdown of perturbation theory for scales lower than 1 GeV. Yet it is gratifying that strong interaction contributions to weak decays coming from scales higher than 1 GeV can be treated by perturbative methods.

$\alpha_s^{(6)}(m_{ m t})$	0.1037	0.1054	0.1079	0.1104	0.1120
$\Lambda_{\overline{\mathrm{MS}}}^{(6)}[\mathrm{MeV}]$	66	76	92	110	123
$\alpha_s^{(5)}(M_{ m Z})$	0.113	0.115	0.118	0.121	0.123
$\Lambda_{\overline{\mathrm{MS}}}^{(5)}[\mathrm{MeV}]$	169	190	226	267	296
$\alpha_s^{(5)}(m_{ m b})$	0.204	0.211	0.222	0.233	0.241
$\Lambda_{\overline{\mathrm{MS}}}^{(4)}[\mathrm{MeV}]$	251	278	325	376	413
$\alpha_s^{(4)}(m_{ m c})$	0.336	0.357	0.396	0.443	0.482
$\Lambda_{\overline{\mathrm{MS}}}^{(3)}[\mathrm{MeV}]$	297	325	372	421	457
(9)		1	1	1	

Table 2: Values of $\alpha_s^{(f)}(\mu)$ and $\Lambda_{\overline{MS}}^{(f)}$ corresponding to given values of $\alpha_s^{(5)}(M_Z)$.

Finally we would like to give an equivalent expression for α_s , which allows to calculate $\alpha_s(\mu)$ directly from the experimental value given in (4.76):

0.444

 $\alpha_s^{(3)}(1 \, \text{GeV}) = 0.409$

$$\alpha_s(\mu) = \frac{\alpha_s(M_Z)}{v(\mu)} \left[1 - \frac{\beta_1}{\beta_0} \frac{\alpha_s(M_Z)}{4\pi} \frac{\ln v(\mu)}{v(\mu)} \right],\tag{4.77}$$

0.514

where

$$v(\mu) = 1 - \beta_0 \frac{\alpha_s(M_Z)}{2\pi} \ln\left(\frac{M_Z}{\mu}\right), \tag{4.78}$$

0.605

0.690

Strictly speaking (4.77) is valid for the f = 5 theory. In order to find $\alpha_s(\mu)$ for $f \neq 5$ one has to proceed as in (4.74) and (4.75).

4.8 Running Quark Mass

Let us next find the μ -dependence of $m(\mu)$. With $dg/d \ln \mu = \beta(g)$ the solution of

$$\frac{dm(\mu)}{d\ln\mu} = -\gamma_m(g)m(\mu) \tag{4.79}$$

is obviously

$$m(\mu) = m(\mu_0) \exp\left[-\int_{g(\mu_0)}^{g(\mu)} dg' \frac{\gamma_m(g')}{\beta(g')}\right].$$
 (4.80)

Here $m(\mu_0)$ is the value of the running mass at the scale μ_0 . For instance: $m_s(2 \text{ GeV})$. Inserting the expansions for $\gamma_m(g)$ and $\beta(g)$ into (4.80) and expanding in α_s gives:

$$m(\mu) = m(\mu_0) \left[\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)} \right]^{\frac{\gamma_m^{(0)}}{2\beta_0}} \left[1 + \left(\frac{\gamma_m^{(1)}}{2\beta_0} - \frac{\beta_1 \gamma_m^{(0)}}{2\beta_0^2} \right) \frac{\alpha_s(\mu) - \alpha_s(\mu_0)}{4\pi} \right]. \tag{4.81}$$

In the literature the running quark mass is often denoted by $\overline{m}(\mu)$. In these lectures we will use both notations: $\overline{m}(\mu) \equiv m(\mu)$.

Since formulae similar to (4.79)–(4.81) will often appear in these lectures, it is useful to derive at least the leading term in (4.81). Keeping the leading terms in $\gamma_m(g)$ and $\beta(g)$ we have

$$-\int_{g(\mu_0)}^{g(\mu)} dg' \frac{\gamma_m(g')}{\beta(g')} = \int_{g(\mu_0)}^{g(\mu)} dg' \frac{\gamma_m^{(0)}}{\beta_0} \frac{1}{g'} = \frac{1}{2} \frac{\gamma_m^{(0)}}{\beta_0} \ln \frac{g^2(\mu)}{g^2(\mu_0)}$$
(4.82)

which inserted in (4.80) gives the leading term in (4.81). Keeping also the NLO terms in $\gamma_m(g)$ and $\beta(g)$ and proceeding in a similar manner one readily finds the NLO term in (4.81).

Let us practice a bit the formula (4.81). Since the power $\gamma_m^{(0)}/2\beta_0$ is positive, $m(\mu)$ similarly to α_s decreases with increasing μ . Using $\Lambda_{\overline{\rm MS}}^{(4)}=325\,$ MeV and $m_{\rm c}=1.3\,$ GeV we find for instance a dictionary between the values of the strange quark mass m_s evaluated at different scales using as the input the values $m_s(m_{\rm c})$ with $m_{\rm c}=1.3\,$ GeV. We note a rather sizable dependence of m_s on μ .

Table 3: The dictionary between the values of $m_s(\mu)$ in units of MeV.

$m_{ m s}(m_{ m c})$	75	100	125	150	175
$m_{\rm s}(2~{ m GeV})$	64	86	107	129	150
$m_{\rm s}(1~{ m GeV})$	86	115	144	173	202

On the other hand the μ dependence of the top quark mass $m_{\rm t}(\mu_t)$ is much weaker. Taking $m_{\rm t}(170\,{\rm GeV})=170\,{\rm GeV},~\alpha_s^{(5)}(M_{\rm Z})=0.118$ and scanning μ_t in the range $100\,{\rm GeV}\leq\mu_t\leq 300\,{\rm GeV}$ we find

$$163.0 \,\text{GeV} \le m_{\text{t}}(\mu_t) \le 177.4 \,\text{GeV} \ .$$
 (4.83)

4.9 RG Improved Perturbation Theory

The structure of (4.77) and (4.81) makes it clear that RG approach goes beyond the usual perturbation theory. In order to see what is going on, let us consider the leading term in (4.77):

$$\alpha_s(\mu) = \frac{\alpha_s(M_Z)}{1 - \beta_0 \frac{\alpha_s(M_Z)}{2\pi} \ln\left(\frac{M_Z}{\mu}\right)} . \tag{4.84}$$

Expanding it in $\alpha_s(M_Z)$ we find:

$$\alpha_s(\mu) = \alpha_s(M_Z) \left[1 + \sum_{n=1}^{\infty} \left(\beta_0 \frac{\alpha_s(M_Z)}{2\pi} \ln \frac{M_Z}{\mu} \right)^n \right]$$
(4.85)

We conclude that the solution of the renormalization group equations sums automatically large logarithms $\log(M_{\rm Z}/\mu)$ which appear for $\mu << M_{\rm Z}$. More generally

LO: Summation of
$$\left(\alpha_s(M_Z) \ln \frac{M_Z}{\mu}\right)^n$$
, (4.86)

NLO: Summation of
$$\alpha_s(M_Z)^n \left(\ln \frac{M_Z}{\mu}\right)^{n-1}$$
. (4.87)

In particular we note that the expansion (4.81) in terms of $\alpha_s(\mu)$ does not involve large logarithms and a few terms suffice to obtain reliable result. (4.81) is an example of a *Renormalization Group Improved Perturbative Expansion*. We will encounter similar expansions for other quantities in the course of these lectures

4.10 Final Comments

We have collected certain information about QCD and tools like renormalization group methods which allow to sum large logarithms. We have also discussed the μ dependences of the running QCD coupling and the running quark masses. Yet all these nice and powerful tools are still insufficient to attack the question of Weak Decays. Yes, what we still need is the operator product expansion.

5 Operator Product Expansion in Weak Decays

5.1 Preliminaries

Weak Decays of Hadrons are mediated through weak interactions of quarks, whose strong interactions, binding the quarks into hadrons, are characterized by typical hadronic energy scale of $\mathcal{O}(1 \text{ GeV})$, much lower than the scale of weak interactions: $\mathcal{O}(M_{W,Z})$. Our goal is therfore to derive an effective low energy theory describing the weak interactions of quarks. The formal framework to achieve this is precisely the Operator Product Expansion (OPE) [3, 4, 7].

5.2 Basic Idea

Consider the quark level transition $c \to su\bar{d}$. Disregarding QCD effects for the moment, the corresponding tree-level W-exchange amplitude (fig. 14a multiplied by "i") is given by

$$A = -\frac{G_F}{\sqrt{2}} V_{cs}^* V_{ud} \frac{M_W^2}{k^2 - M_W^2} (\bar{s}c)_{V-A} (\bar{u}d)_{V-A}$$

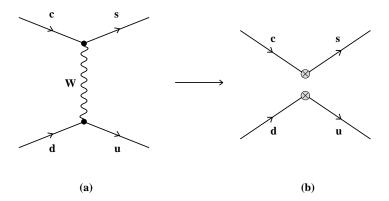


Figure 14: $c \to su\bar{d}$ at the Tree-Level.

$$= \frac{G_F}{\sqrt{2}} V_{cs}^* V_{ud}(\bar{s}c)_{V-A}(\bar{u}d)_{V-A} + \mathcal{O}(\frac{k^2}{M_W^2})$$
 (5.1)

where

$$(\bar{s}c)_{V-A} \equiv \bar{s}\gamma_{\mu}(1-\gamma_5)c . \tag{5.2}$$

Since k, the momentum transfer through the W propagator, is very small as compared to $M_{\rm W}$, terms of the order $\mathcal{O}(k^2/M_W^2)$ can safely be neglected and the full amplitude A can be approximated by the first term on the r.h.s of (5.1). Now the result in (5.1) may also be obtained from

$$\mathcal{H}_{eff} = \frac{G_F}{\sqrt{2}} V_{cs}^* V_{ud}(\bar{s}c)_{V-A}(\bar{u}d)_{V-A} + \text{High D Operators}, \tag{5.3}$$

where the higher dimension operators, typically involving derivative terms, correspond to the terms $\mathcal{O}(k^2/M_W^2)$ in (5.1). Neglecting the latter terms corresponds to the neglect of higher dimensional operators. In what follows we will always neglect the higher dimensional operators keeping only the operators with dimensions five and six. This simple example illustrates the basic idea of OPE: the product of two charged current operators is expanded into a series of local operators, whose contributions are weighted by effective coupling constants, the Wilson coefficients. In this particular example the leading operator has the dimension 6 and its Wilson coefficient in the normalization of the pilot formula (1.1) is simply equal unity. This value will be changed by QCD corrections as we will see few pages below. Moreover QCD corrections to the diagrams in fig. 14 will generate another operator.

5.3 Formal Approach

Let us be a bit more formal for a moment and investigate whether the same result can be obtained using the path integral formalism. We will see that this is indeed the case. This

discussion will on the one hand provide a formal basis for the simple procedure given above and on the other hand will give us more insight in the virtues of OPE. Simultaneously we will discover that there is no need to be very formal for the rest of the lectures and we can proceed by simply generalizing our simple procedure of section 5.2 to more complicated situations in which also QCD effects and more complicated diagrams are present.

Our formal discussion follows [17] and consists of four steps.

Step 1

Consider the generating functional for Green functions in the path integral formalism. The relevant part for our discussion is

$$Z_W \sim \int [dW^+][dW^-] \exp(i \int d^4x \mathcal{L}_W)$$
 (5.4)

where

$$\mathcal{L}_{W} = -\frac{1}{2} (\partial_{\mu} W_{\nu}^{+} - \partial_{\nu} W_{\mu}^{+}) (\partial^{\mu} W^{-\nu} - \partial^{\nu} W^{-\mu}) + M_{W}^{2} W_{\mu}^{+} W^{-\mu} + \frac{g_{2}}{2\sqrt{2}} (J_{\mu}^{+} W^{+\mu} + J_{\mu}^{-} W^{-\mu}), \tag{5.5}$$

$$J_{\mu}^{+} = V_{pn}\bar{p}\gamma_{\mu}(1-\gamma_{5})n \qquad p = (u,c,t) \quad n = (d,s,b) \qquad J_{\mu}^{-} = (J_{\mu}^{+})^{\dagger}.$$
 (5.6)

Step 2:

We use the unitary gauge for the W field. Introducing the operator:

$$K_{\mu\nu}(x,y) = \delta^{(4)}(x-y) \left[g_{\mu\nu}(\partial^2 + M_W^2) - \partial_\mu \partial_\nu \right]$$
 (5.7)

we have, after discarding a total derivative in the W kinetic term,

$$Z_W \sim \int [dW^+][dW^-] \exp\left[i \int d^4x d^4y W_\mu^+(x) K^{\mu\nu}(x,y) W_\nu^-(y) + i \frac{g_2}{2\sqrt{2}} \int d^4x J_\mu^+ W^{+\mu} + J_\mu^- W^{-\mu}\right].$$
 (5.8)

The inverse of $K_{\mu\nu}$, denoted by $\Delta_{\mu\nu}$, and defined through

$$\int d^4 y K_{\mu\nu}(x,y) \Delta^{\nu\lambda}(y,z) = g_\mu^{\lambda} \delta^{(4)}(x-z)$$
(5.9)

is the W propagator in the unitary gauge

$$\Delta_{\mu\nu}(x,y) = \int \frac{d^4k}{(2\pi)^4} \Delta_{\mu\nu}(k) e^{-ik(x-y)}$$
 (5.10)

$$\Delta_{\mu\nu}(k) = \frac{-1}{k^2 - M_W^2} \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{M_W^2} \right) . \tag{5.11}$$

Step 3:

Performing the gaussian functional integration over $W^{\pm}(x)$ in (5.8) explicitly, we arrive at

$$Z_W \sim \exp\left[-i\int \frac{g_2^2}{8} J_{\mu}^{-}(x) \Delta^{\mu\nu}(x,y) J_{\nu}^{+}(y) d^4x d^4y\right]$$
 (5.12)

This result implies a nonlocal action functional for the quarks:

$$S_{nl} = \int d^4x \mathcal{L}_{kin} - \frac{g_2^2}{8} \int d^4x d^4y J_{\mu}^-(x) \Delta^{\mu\nu}(x,y) J_{\nu}^+(y)$$
 (5.13)

where the second term represents charged current interactions of quarks.

Step 4:

Finally, we expand this second, nonlocal term in powers of $1/M_W^2$ to obtain a series of local interaction operators of dimensions that increase with the order in $1/M_W^2$. To lowest order

$$\Delta^{\mu\nu}(x,y) \approx \frac{g^{\mu\nu}}{M_W^2} \delta^{(4)}(x-y) \tag{5.14}$$

and the second term in (5.13) becomes

$$-\frac{g_2^2}{8M_W^2} \int d^4x J_\mu^-(x) J^{+\mu}(x)$$
 (5.15)

corresponding to the usual effective charged current interaction Lagrangian

$$\mathcal{L}_{int,eff} = -\frac{G_F}{\sqrt{2}} J_{\mu}^{-}(x) J^{+\mu}(x) = -\frac{G_F}{\sqrt{2}} V_{pn}^* V_{p'n'}(\bar{n}p)_{V-A}(\bar{p}'n')_{V-A}$$
 (5.16)

which contains, among other terms, the leading contribution to (5.3).

Let us note several basic aspects of this approach:

- Formally, the procedure to approximate the interaction term in (5.13) by (5.15) is an example of short distance OPE. The product of the local operators $J_{\mu}^{-}(x)$ and $J_{\nu}^{+}(y)$, to be taken at short-distances due to the convolution with the massive, short-range W propagator $\Delta^{\mu\nu}(x,y)$, is expanded into a series of composite local operators. The leading term is shown in (5.15).
- The dominant contributions in the short-distance expansion come from the operators of lowest dimension (six in the present example). The operators of higher dimensions can usually be neglected in weak decays.
- OPE series is equivalent to the original theory, when considered to all orders in $1/M_W^2$. The truncation of the operator series yields a systematic approximation scheme for low energy processes, neglecting contributions suppressed by powers of k^2/M_W^2 .

- In going from the full to the effective theory the W boson is removed as an explicit, dynamical degree of freedom: it is "integrated out" in step 3 of our procedure. Alternatively in the canonical operator formalism the W field gets "contracted out" through the application of Wick's theorem. From the point of view of low energy dynamics, the effects of a short-range exchange force mediated by a heavy boson approximately corresponds to a point interaction familiar from the Fermi Theory.
- Similarly one can "integrate out" or "contract out" heavy quarks. This gives Effective f-quark theories where f denotes the "light" quarks which have not been integrated out. We now understand what the effective number of flavours introduced in connection with the formula (4.74) really means. By going from higher to lower μ scales one integrates out systematically flavours with masses higher than the actual value of μ . However, as we will stress below, in connection with renormalization group ideas, there is some freedom at which μ a given flavour is integrated out. For instance one can extend the five flavour theory down to $\mu = m_b/2$.

All this was a bit formal but fortunately we make still another observation. The approach of evaluating the relevant Green functions (or amplitudes) directly in order to construct the OPE, as in (5.1), gives the same result as the more formal technique employing path integrals. Consequently we can return, putting aside path integrals, to our Feynman diagram calculations. Our first task is to investigate how (5.1) or (5.3) changes when QCD effects are included.

5.4 OPE and Short Distance QCD Effects

5.4.1 Preliminaries

Due to the asymptotic freedom of QCD, the short distance QCD corrections to weak decays, that is the contribution of hard gluons at energies of the order $\mathcal{O}(M_W)$ down to hadronic scales $\mathcal{O}(1\,\mathrm{GeV})$, can be treated in the renormalization group (RG) improved perturbation theory. We will illustrate this on a simple example of the $c \to su\bar{d}$ transition beginning with the ordinary perturbation theory, subsequently summing leading logarithms by the RG method and finally generalizing the result to include next-to-leading logarithms. We will do this in some detail emphasizing certain characteristic features of this approach. In particular we will discuss at length the scale and renormalization scheme dependences advertised in the pilot section of these lectures. Once all these features are well understood it will be straightforward to proceed to other transitions and to generalize the approach to more exciting situations involving penguins and boxes.

For the $c \to su\bar{d}$ transition we had without QCD effects

$$\mathcal{H}_{eff}^{(0)} = \frac{G_F}{\sqrt{2}} V_{cs}^* V_{ud}(\bar{s}_{\alpha} c_{\alpha})_{V-A} (\bar{u}_{\beta} d_{\beta})_{V-A}$$
 (5.17)

where the summation over repeated color indices is understood.

With QCD effects $\mathcal{H}_{eff}^{(0)}$ is generalized to

$$\mathcal{H}_{eff} = \frac{G_F}{\sqrt{2}} V_{cs}^* V_{ud} (C_1(\mu) Q_1 + C_2(\mu) Q_2)$$
 (5.18)

where

$$Q_1 = (\bar{s}_{\alpha}c_{\beta})_{V-A}(\bar{u}_{\beta}d_{\alpha})_{V-A} \tag{5.19}$$

$$Q_2 = (\bar{s}_{\alpha} c_{\alpha})_{V-A} (\bar{u}_{\beta} d_{\beta})_{V-A} \tag{5.20}$$

The essential features of this Hamiltonian are:

In addition to the original operator Q₂ (with index 2 for historical reasons) a new operator Q₁ with the same flavour form but different colour structure is generated. That a new operator has to be introduced is evident if we inspect the colour structure of the diagrams (b) and (c) in fig. 15. They contain the product of the color charges T^a_{αβ} and T^a_{γδ} which using the colour algebra can be rewritten as follows

$$T^{a}_{\alpha\beta}T^{a}_{\gamma\rho} = -\frac{1}{2N}\delta_{\alpha\beta}\delta_{\gamma\delta} + \frac{1}{2}\delta_{\alpha\delta}\delta_{\gamma\beta}$$
 (5.21)

The first term on the r.h.s gives a correction to the coefficient of the operator Q_2 and the second term gives life to the new operator Q_1 .

- The Wilson coefficients C_1 and C_2 , the coupling constants for the interaction terms Q_1 and Q_2 , become calculable nontrivial functions of α_s , M_W and the renormalization scale μ .
- If QCD is neglected, $C_1 = 0$, $C_2 = 1$ and (5.18) reduces to (5.17).

5.4.2 Calculation of Wilson Coefficients

Our first task is the calculation of the coefficients $C_{1,2}$ in the ordinary perturbation theory. $C_{1,2}$ can be determined by the requirement that the amplitude A_{full} in the full theory be reproduced by the corresponding amplitude in the effective theory (5.18):

$$A_{full} = A_{eff} = \frac{G_F}{\sqrt{2}} V_{cs}^* V_{ud} (C_1 \langle Q_1 \rangle + C_2 \langle Q_2 \rangle)$$
 (5.22)

This procedure is called "the *matching* of the full theory onto the effective theory". We recall that the full theory is the one in which all particles appear as dynamical degrees of

freedom. In the case at hand the effective theory is constructed by integrating out the W field only. The matching procedure which gives the values of C_1 and C_2 proceeds in three steps [61]. The explicit three steps presented below are sufficient for the subsequent summation of the leading logarithms or equivalently for the leading term of the RG improved perturbation theory. We will generalize these steps in the next section in order to be able to include also the NLO term in this expansion.

Here we go:

Step 1: Calculation of A_{full}

The current-current diagrams of fig. 15 (a)–(c) and their symmetric counterparts, give for the full amplitude A_{full} to $\mathcal{O}(\alpha_s)$ ($m_i = 0, p^2 < 0$):

$$A_{full} = \frac{G_F}{\sqrt{2}} V_{cs}^* V_{ud} \left[\left(1 + 2C_F \frac{\alpha_s}{4\pi} \left(\frac{1}{\varepsilon} + \ln \frac{\mu^2}{-p^2} \right) \right) S_2 + \frac{3}{N} \frac{\alpha_s}{4\pi} \ln \frac{M_W^2}{-p^2} S_2 \right]$$

$$-3 \frac{\alpha_s}{4\pi} \ln \frac{M_W^2}{-p^2} S_1$$
(5.23)

Here:

$$S_1 \equiv \langle Q_1 \rangle_{tree} = (\bar{s}_{\alpha} c_{\beta})_{V-A} (\bar{u}_{\beta} d_{\alpha})_{V-A} \tag{5.24}$$

$$S_2 \equiv \langle Q_2 \rangle_{tree} = (\bar{s}_{\alpha} c_{\alpha})_{V-A} (\bar{u}_{\beta} d_{\beta})_{V-A} \tag{5.25}$$

are just the tree level matrix elements of Q_1 and Q_2 . A few comments should be made.

• We use the term "amplitude" in the meaning of an "amputated Green function" (multiplied by "i"). Correspondingly operator matrix elements are amputated Green functions with operator insertion. Thus gluonic self energy corrections on external legs are not included.

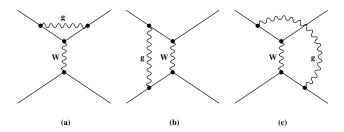


Figure 15: One-loop current-current diagrams in the full theory.

• For simplicity we have chosen all external momenta p to be equal and set all quark masses to zero. As we will see below this choice has no impact on the coefficients C_i .

- We have kept only logarithmic corrections $\sim \alpha_s \cdot \log$ and discarded constant contributions of order $\mathcal{O}(\alpha_s)$, which corresponds to the leading log approximation (LO).
- The singularity $1/\varepsilon$ can be removed by the quark field renormalization. This is, however, not necessary for finding C_i as we will see soon.

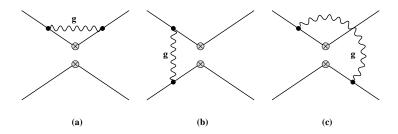


Figure 16: One loop current-current diagrams in the effective theory. The 4-vertex " $\otimes \otimes$ " denotes the insertion of a 4-fermion operator Q_i .

Step 2: Calculation of Matrix Elements $\langle Q_i \rangle$

The unrenormalized current-current matrix elements of Q_1 and Q_2 are found at $\mathcal{O}(\alpha_s)$ by calculating the diagrams in fig. 16 (a)-(c) and their symmetric counter-parts. Adding the contributions without QCD corrections (S_1 and S_2 respectively) and using the same assumptions about the external legs as in step 1, we have

$$\langle Q_1 \rangle^{(0)} = \left(1 + 2C_F \frac{\alpha_s}{4\pi} \left(\frac{1}{\varepsilon} + \ln \frac{\mu^2}{-p^2} \right) \right) S_1 + \frac{3}{N} \frac{\alpha_s}{4\pi} \left(\frac{1}{\varepsilon} + \ln \frac{\mu^2}{-p^2} \right) S_1$$

$$-3 \frac{\alpha_s}{4\pi} \left(\frac{1}{\varepsilon} + \ln \frac{\mu^2}{-p^2} \right) S_2$$

$$(5.26)$$

$$\langle Q_2 \rangle^{(0)} = \left(1 + 2C_F \frac{\alpha_s}{4\pi} \left(\frac{1}{\varepsilon} + \ln \frac{\mu^2}{-p^2} \right) \right) S_2 + \frac{3}{N} \frac{\alpha_s}{4\pi} \left(\frac{1}{\varepsilon} + \ln \frac{\mu^2}{-p^2} \right) S_2$$

$$-3 \frac{\alpha_s}{4\pi} \left(\frac{1}{\varepsilon} + \ln \frac{\mu^2}{-p^2} \right) S_1$$
(5.27)

The divergences in the first terms can again be eliminated through the quark field renormalization. However, in contrast to the full amplitude in (5.23), the resulting expressions are still divergent after this renormalization. To remove these additional divergences multiplicative renormalization, referred to as *operator renormalization*, is necessary:

$$Q_i^{(0)} = Z_{ij}Q_i {5.28}$$

We observe that the renormalization constant is in this case a 2×2 matrix \hat{Z} . Using (4.43) with $(n_F, n_G) = (4, 0)$, we find the relation between the unrenormalized $(\langle Q_i \rangle^{(0)})$ and the

renormalized amputated Green functions ($\langle Q_i \rangle$):

$$\langle Q_i \rangle^{(0)} = Z_q^{-2} Z_{ij} \langle Q_j \rangle . \tag{5.29}$$

 Z_q^{-2} removes the $1/\varepsilon$ divergences in the first terms in (5.26) and (5.27). Z_{ij} remove the remaining divergences. From (5.26), (5.27) we read off ($\overline{\text{MS}}$ -scheme)

$$\hat{Z} = 1 + \frac{\alpha_s}{4\pi} \frac{1}{\varepsilon} \begin{pmatrix} 3/N & -3\\ -3 & 3/N \end{pmatrix}$$
 (5.30)

Thus the renormalized matrix elements $\langle Q_i \rangle$ are given by

$$\langle Q_1 \rangle = \left(1 + 2C_F \frac{\alpha_s}{4\pi} \ln \frac{\mu^2}{-p^2} \right) S_1 + \frac{3}{N} \frac{\alpha_s}{4\pi} \ln \frac{\mu^2}{-p^2} S_1 - 3\frac{\alpha_s}{4\pi} \ln \frac{\mu^2}{-p^2} S_2 , \qquad (5.31)$$

$$\langle Q_2 \rangle = \left(1 + 2C_F \frac{\alpha_s}{4\pi} \ln \frac{\mu^2}{-p^2} \right) S_2 + \frac{3}{N} \frac{\alpha_s}{4\pi} \ln \frac{\mu^2}{-p^2} S_2 - 3\frac{\alpha_s}{4\pi} \ln \frac{\mu^2}{-p^2} S_1 . \tag{5.32}$$

Step 3: Extraction of C_i

Inserting $\langle Q_i \rangle$ into (5.22) and comparing with (5.23) we can now extract the coefficients C_1 and C_2 . Yet, we have to be a bit careful. In the full theory we did not perform any quark field renormalization whereas we did this renormalization in the effective theory as seen in (5.29). This is clearly inconsistent and this inconsistency is signalled by the divergent Wilson coefficient which is clearly wrong. To proceed correctly we have to either remove the divergence in (5.23) by performing quark field renormalization as in (5.29) or to leave (5.23) as it is and remove the quark field renormalization from (5.29). In both cases the matching (5.22) gives the same result

$$C_1(\mu) = -3\frac{\alpha_s}{4\pi} \ln \frac{M_W^2}{\mu^2} , \qquad C_2(\mu) = 1 + \frac{3}{N} \frac{\alpha_s}{4\pi} \ln \frac{M_W^2}{\mu^2} .$$
 (5.33)

This simple example shows that it is essential in the process of matching to treat the external states in the full and the effective theory in the same manner in order to obtain the correct result for the Wilson coefficients. In this example we were lucky. The inconsistency, which I made for pedagogical reasons, was signalled by a leftover divergence. In the case of NLO calculations were also finite non-logarithmic corrections have to be kept, a possible inconsistency in matching is much harder to see and it is crucial that at all stages of the matching the treatment of the external legs on both sides of (5.22) is the same. For this reason we are free to decide whether we perform external field renormalization or not. In the latter case the left-over divergences in the full and the effective theory will simply cancel each other in the process of matching. I discussed here the issue of the cancellation of the *ultraviolet* divergences related to external fields. The same comments apply to the *infrared* divergences. For strategic reasons I will now discuss something else and will return to the issue of infrared divergences in the context of matching a few pages below.

5.4.3 A Different Look

The renormalization of the interaction terms C_iQ_i in the effective theory can also be achieved in a different, but equivalent, way by using the standard counter-term method. Here C_i are treated as coupling constants, which have to be renormalized. We follow here [17].

To this end let us consider \mathcal{H}_{eff} as the starting point with fields and "coupling constants" C_i regardes as bare quantities. They are renormalized according to (q=s, c, u, d)

$$q^{(0)} = Z_q^{1/2} q C_i^{(0)} = Z_{ij}^c C_j (5.34)$$

where \hat{Z}^c denotes the renormalization matrix for the couplings C_j . It is evident that \hat{Z}^c must be somehow related to the renormalization matrix \hat{Z} in (5.30). Let us find this relation.

Omitting the factor $\frac{G_F}{\sqrt{2}}V_{cs}^*V_{ud}$ we have

$$\mathcal{H}_{eff} = C_i^{(0)} Q_i(q^{(0)}) \equiv Z_q^2 Z_{ij}^c C_j Q_i \equiv C_i Q_i + (Z_q^2 Z_{ij}^c - \delta_{ij}) C_j Q_i$$
 (5.35)

where the first term on the r.h.s is written in terms of renormalized couplings and fields (C_iQ_i) and the second term is a counter-term. The argument $q^{(0)}$ on the l.h.s of (5.35) indicates that the interaction vertices Q_i are composed of bare fields. Using (5.35) we get the finite renormalized result

$$A_{eff} = Z_q^2 Z_{ij}^c C_j \langle Q_i \rangle^{(0)} \tag{5.36}$$

On the other hand using (5.29), we have

$$A_{eff} = C_i \langle Q_i \rangle = C_i Z_{ii}^{-1} Z_a^2 \langle Q_i \rangle^{(0)}$$

$$(5.37)$$

Hence comparing the last two equations we finally find the relation

$$Z_{ij}^c = Z_{ji}^{-1}. (5.38)$$

This relation will turn out to be very useful in deriving the renormalization group equations for the couplings C_i .

5.4.4 Operator Mixing and Diagonalization

We have just seen, that gluonic corrections to the matrix element of the original operator Q_2 are not just proportional to Q_2 itself, but involve the additional structure Q_1 . Therefore, besides a Q_2 -counter-term, a counter-term $\sim Q_1$ is needed to renormalize this matrix element. Similarly the renormalization of Q_1 requires both Q_1 and Q_2 counter-terms. We say that the operators Q_1 and Q_2 mix under renormalization.

For the study of the renormalization group properties of the system (Q_1, Q_2) it is useful to diagonalize it by going to a different operator basis defined by

$$Q_{\pm} = \frac{Q_2 \pm Q_1}{2} \qquad C_{\pm} = C_2 \pm C_1 \ .$$
 (5.39)

The new operators Q_+ and Q_- are renormalized independently of each other:

$$Q_{\pm}^{(0)} = Z_{\pm}Q_{\pm} \tag{5.40}$$

where

$$Z_{\pm} = 1 + \frac{\alpha_s}{4\pi} \frac{1}{\varepsilon} \left(\mp 3 \frac{N \mp 1}{N} \right) . \tag{5.41}$$

In this new basis the OPE reads

$$A \equiv A_{+} + A_{-} = \frac{G_{F}}{\sqrt{2}} V_{cs}^{*} V_{ud}(C_{+}(\mu)\langle Q_{+}(\mu)\rangle + C_{-}(\mu)\langle Q_{-}(\mu)\rangle) , \qquad (5.42)$$

where $(S_{\pm} = (S_2 \pm S_1)/2)$

$$A_{\pm} = \frac{G_F}{\sqrt{2}} V_{cs}^* V_{ud} \left[\left(1 + 2C_F \frac{\alpha_s}{4\pi} \ln \frac{\mu^2}{-p^2} \right) S_{\pm} + \left(\frac{3}{N} \mp 3 \right) \frac{\alpha_s}{4\pi} \ln \frac{M_W^2}{-p^2} S_{\pm} \right]$$
 (5.43)

and

$$\langle Q_{\pm}(\mu) \rangle = \left(1 + 2C_F \frac{\alpha_s}{4\pi} \ln \frac{\mu^2}{-p^2} \right) S_{\pm} + \left(\frac{3}{N} \mp 3 \right) \frac{\alpha_s}{4\pi} \ln \frac{\mu^2}{-p^2} S_{\pm} ,$$
 (5.44)

$$C_{\pm}(\mu) = 1 + (\frac{3}{N} \mp 3) \frac{\alpha_s}{4\pi} \ln \frac{M_W^2}{\mu^2}$$
 (5.45)

5.4.5 Factorization of SD and LD

We have just witnessed in explicit terms the most important feature of the OPE, advertised already at the beginning of these lectures: factorization of short-distance (coefficients) and long-distance (operator matrix elements) contributions. Schematically, this factorization has the following structure:

$$(1 + \alpha_s G \ln \frac{M_W^2}{-p^2}) \doteq (1 + \alpha_s G \ln \frac{M_W^2}{\mu^2}) \cdot (1 + \alpha_s G \ln \frac{\mu^2}{-p^2})$$
 (5.46)

which is achieved by the following splitting of the logarithm

$$\ln \frac{M_W^2}{-p^2} = \ln \frac{M_W^2}{\mu^2} + \ln \frac{\mu^2}{-p^2}$$
 (5.47)

or from the point of view of the integration over some virtual momenta through the splitting

$$\int_{-p^2}^{M_W^2} \frac{dk^2}{k^2} = \int_{\mu^2}^{M_W^2} \frac{dk^2}{k^2} + \int_{-p^2}^{\mu^2} \frac{dk^2}{k^2}.$$
 (5.48)

In particular the last formula makes it clear that the Wilson coefficients contain the contributions from large virtual momenta of the loop correction from scales $\mu = \mathcal{O}(1\,\text{GeV})$ to M_{W} , whereas the low energy contributions are separated into the matrix elements. The renormalization scale μ acts as the scale at which the full contribution to the amplitude is separated into a low energy and a high energy part.

5.4.6 Independence of C_i from External States

Let us next return to the issue of the infrared divergences in the process of matching. In the matching discussed explicitly above they are regulated by taking $p^2 \neq 0$. They appear both in A_{full} and A_{eff} . Yet as we have shown above the dependence of A_{full} on p^2 , representing the long-distance structure of A is, from the point of view of the effective theory, fully contained in $\langle Q_i \rangle$ and the Wilson coefficients C_i are free from this dependence.

Since the coefficient functions do not depend on the external states, any external state can be used for their extraction, the only requirement being that the infrared (and mass) singularities are properly regularized. In our example an off-shell momentum p for massless external quarks has been used, but such a choice is clearly one of several possibilities. In general one could work with any other arbitrary momentum configuration, on-shell or off-shell, with or without external quark mass, with infrared divergences regulated by off-shell momenta, quark masses, a fictitious gluon mass or by dimensional regularization. All these methods would give the same results for C_i .

In particular the dimensional regularization of infrared divergences is very convenient as many integrals simplify considerably. Older discussions of dimensional infrared regularization can be found in Muta's book [20] and also in a paper by Marciano [64]. Recently this method has been used in calculating NLO corrections to $K \to \pi \nu \bar{\nu}$ [65] and also for the matching conditions in $B \to X_s \gamma$ [66, 67]. In particular as we stressed in [67], the distinction of $1/\varepsilon$ ultraviolet divergences from the infrared ones is not necessary because after proper renormalization of ultraviolet singularities, the left-over divergences are of infrared origin only. These singularities cancel then automatically in the process of matching. To this end, however, it is essential to perform the matching at all stages in $D = 4 - 2\varepsilon$ dimensions. This implies that already at the NLO level, $\mathcal{O}(\varepsilon)$ terms in Wilson coefficients have to be kept at the intermediate stages of the calculation. More details on this efficient technique can be found in the papers quoted above.

5.5 OPE and the Renormalization Group

5.5.1 Preliminaries

So far we have computed

$$C_{\pm}(\mu) = 1 + (\frac{3}{N} \mp 3) \frac{\alpha_s}{4\pi} \ln \frac{M_W^2}{\mu^2}$$
 (5.49)

in ordinary perturbation theory. Unfortunately for $\mu = 1 \,\mathrm{GeV}$ the first order correction term amounts to 65 – 130% although $\alpha_s/4\pi \approx 4\%$. This finding illustrates explicitly the breakdown of the naive perturbative expansion caused by the appearance of large logarithms originating in the presence of largely disparate scales M_W and μ .

Clearly, the result in (5.49) can only be used for $\mu = \mathcal{O}(M_{\rm W})$. For $\mu \ll M_{\rm W}$ we have to sum the large logarithms to all orders of perturbation theory before we can trust our result for C_{\pm} . Fortunately we have developed in section 4 a very powerful technique to sum such logarithms and we know exactly what we have to do. Yes, in order to sum these large logs we have to find renormalization group equations for C_{\pm} and solve them.

5.5.2 Renormalization Group Equations for C_{\pm}

The renormalization group equations for C_{\pm} follow from the fact, that the unrenormalized Wilson coefficients $C_{\pm}^{(0)}$ do not depend on μ . Using the relation (5.38), properly adapted to the diagonal basis, we have first

$$C_{\pm} = Z_{\pm} C_{\pm}^{(0)} \qquad Q_{\pm}^{(0)} = Z_{\pm} Q_{\pm}$$
 (5.50)

and subsequently

$$\frac{dC_{\pm}(\mu)}{d\ln\mu} = \gamma_{\pm}(g)C_{\pm}(\mu). \tag{5.51}$$

Here γ_{\pm} is the anomalous dimension of the operator Q_{\pm} and given by

$$\gamma_{\pm}(g) = \frac{1}{Z_{+}} \frac{dZ_{\pm}}{d \ln \mu}.$$
 (5.52)

Comparing (5.51) and (5.52) with (4.48) and (4.50), respectively we see great similarities with the case of the running quark mass. The only modification is the opposite sign in (5.51). Consequently many relevant formulae of section 4 can be immediately employed. Here we go:

• In the MS $(\overline{\text{MS}})$ -scheme

$$Z_{\pm} = 1 + \sum_{k=1}^{\infty} \frac{1}{\varepsilon^k} Z_{\pm,k}(g)$$
 (5.53)

and consequently

$$\gamma_{\pm}(g) = -2g^2 \frac{\partial Z_{\pm,1}(g)}{\partial g^2} . \tag{5.54}$$

• Using then

$$Z_{\pm} = 1 + \frac{\alpha_s}{4\pi} \frac{1}{\varepsilon} \left(\mp 3 \frac{N \mp 1}{N} \right) \tag{5.55}$$

as obtained in (5.41) gives the one-loop anomalous dimensions of Q_{\pm} :

$$\gamma_{\pm}(\alpha_s) = \frac{\alpha_s}{4\pi} \gamma_{\pm}^{(0)} \qquad \gamma_{\pm}^{(0)} = \pm 6 \frac{N \mp 1}{N}.$$
(5.56)

• The solution of (5.51) is given as for $m(\mu)$ in (4.80):

$$C_{\pm}(\mu) = U_{\pm}(\mu, \mu_W)C_{\pm}(\mu_W)$$
 (5.57)

where $\mu_W = \mathcal{O}(M_W)$ and $U_{\pm}(\mu, \mu_W)$ is the evolution function:

$$U_{\pm}(\mu, \mu_W) = \exp\left[\int_{g(\mu_W)}^{g(\mu)} dg' \frac{\gamma_{\pm}(g')}{\beta(g')}\right]$$
 (5.58)

• Using (5.56) and $\beta(g) = -\beta_0 g^3/16\pi^2$ we can now find $C_{\pm}(\mu)$ by using the leading term in the formula (4.81) for $m(\mu)$. Setting $\mu_0 = M_{\rm W}$ and taking into account the relative sign between (4.79) and (5.51) we have

$$C_{\pm}(\mu) = \left[\frac{\alpha_s(M_W)}{\alpha_s(\mu)}\right]^{\frac{\gamma_{\pm}^{(0)}}{2\beta_0}} C_{\pm}(M_W)$$

$$(5.59)$$

• In order to complete the calculation we use the fact that at $\mu = M_{\rm W}$ no large logarithms are present and $C_{\pm}(M_{\rm W})$ can be calculated in ordinary perturbation theory. From (5.45) we have in LO

$$C_{\pm}(M_W) = 1 \tag{5.60}$$

and consequently for $\mu = \mu_b = \mathcal{O}(m_b)$

$$C_{\pm}(\mu_b) = \left[\frac{\alpha_s(M_W)}{\alpha_s(\mu_b)}\right]^{\frac{\gamma_{\pm}^{(0)}}{2\beta_0}} . \tag{5.61}$$

We have now summed all leading logarithms and the important formula (5.61) gives the coefficients C_{\pm} in the leading log approximation or in other words the leading term of the RG improved perturbation theory. For instance, specializing to the case of f = 5 and $\mu_b = \mathcal{O}(m_b)$ we obtain

$$C_{+}(\mu_{b}) = \left[\frac{\alpha_{s}(M_{W})}{\alpha_{s}(\mu_{b})}\right]^{\frac{6}{23}} \qquad C_{-}(\mu_{b}) = \left[\frac{\alpha_{s}(M_{W})}{\alpha_{s}(\mu_{b})}\right]^{\frac{-12}{23}}$$
(5.62)

with α_s given by the leading expression (4.84). For $\mu_b = 5.0$ GeV and $\Lambda_{\overline{\rm MS}}^{(5)} = 225$ GeV one finds $C_+(\mu_b) = 0.847$ and $C_-(\mu_b) = 1.395$, i.e. suppression of C_+ and an enhancement of C_- relative to $C_- = C_+ = 1$ without QCD corrections. The corresponding enhancements and suppressions for scales $\mathcal{O}(1 \text{ GeV})$ reflect to some extent the dominance of the $\Delta I = 1/2$ transitions over $\Delta 3/2$ transitions in $K \to \pi\pi$ decays (the $\Delta I = 1/2$ rule) first analyzed in QCD in [72]. These short distance effects are insufficient, however, to explain the dominance of $\Delta I = 1/2$ transitions observed experimentally. We will return briefly to this issue in section 11.

5.5.3 Choice of the Matching Scale

In calculating (5.59) we have set the high energy matching scale to $M_{\rm W}$. The choice of the high energy matching scale, to be denoted by μ_W , is of course not unique. The only requirement is that $\mu_W = \mathcal{O}(M_{\rm W})$ in order to avoid large logarithms $\ln(M_W/\mu_W)$. However, we know from (5.49) that in the LO approximation, in which $\mathcal{O}(\alpha_s)$ terms are dropped in $C_{\pm}(\mu_W)$, we have using (5.60)

$$C_{\pm}(\mu_W) = C_{\pm}(M_W) + \mathcal{O}(\alpha_s) = 1.$$
 (5.63)

Consequently in this approximation we also have

$$C_{\pm}(\mu_b) = \left[\frac{\alpha_s(\mu_W)}{\alpha_s(\mu_b)}\right]^{\frac{\gamma_{\pm}^{(0)}}{2\beta_0}} = \left[\frac{\alpha_s(M_W)}{\alpha_s(\mu_b)}\right]^{\frac{\gamma_{\pm}^{(0)}}{2\beta_0}} (1 + \mathcal{O}(\alpha_s))$$

$$(5.64)$$

which differs from (5.61) by $\mathcal{O}(\alpha_s)$ corrections.

We observe that a change of μ_W around the value of M_W causes an ambiguity of $\mathcal{O}(\alpha_s)$ in the coefficient. This ambiguity represents a theoretical uncertainty in the determination of $C_{\pm}(\mu_b)$. In order to reduce it, it is necessary to go beyond the leading order. We will do this in the following section. Similar ambiguity exists in the choice of the low energy scale μ_b as we will discuss at various places in these lectures, in particular in connection with $B \to X_s \gamma$ decay.

5.5.4 Threshold Effects in LO

The evolution function U depends on f through $\alpha_s^{(f)}$ and β_0 in the exponent. One can generalize the renormalization group evolution from $M_{\rm W}$ down to say $\mu_c = \mathcal{O}(m_{\rm c})$ to include the threshold effect of the b-quark as follows

$$C_{\pm}(\mu_c) = U_{+}^{(f=4)}(\mu_c, \mu_b) U_{+}^{(f=5)}(\mu_b, M_W) C_{\pm}(M_W)$$
(5.65)

which is valid in LO. Here $\mu_b = \mathcal{O}(m_b)$. Thus (5.62) generalizes to

$$C_{+}(\mu_{c}) = \left[\frac{\alpha_{s}^{(4)}(\mu_{b})}{\alpha_{s}^{(4)}(\mu_{c})}\right]^{\frac{6}{25}} \left[\frac{\alpha_{s}^{(5)}(M_{W})}{\alpha_{s}^{(5)}(\mu_{b})}\right]^{\frac{6}{23}}, \qquad C_{-}(\mu_{c}) = \left[\frac{\alpha_{s}^{(4)}(\mu_{b})}{\alpha_{s}^{(4)}(\mu_{c})}\right]^{\frac{-12}{25}} \left[\frac{\alpha_{s}^{(5)}(M_{W})}{\alpha_{s}^{(5)}(\mu_{b})}\right]^{\frac{-12}{23}}.$$

$$(5.66)$$

Again also here there is an ambiguity in μ_c which can only be reduced by going to NLO.

5.5.5 RGE for C_i : Case of Operator Mixing

The coefficients $C_i(\mu)$ can be now calculated by inverting (5.39) with the result

$$C_1(\mu) = \frac{C_+(\mu) - C_-(\mu)}{2}$$
, $C_2(\mu) = \frac{C_+(\mu) + C_-(\mu)}{2}$, (5.67)

where $C_{\pm}(\mu)$ is given in (5.61) or (5.66).

Yet, it is instructive to derive (5.67) by using a procedure which one can also apply to more complicated situations in which several operators mix under renormalization. To this end we write

$$\vec{C}^T = (C_1, C_2) , \qquad \vec{Q}^T = (Q_1, Q_2).$$
 (5.68)

Then

$$\vec{C}^{(0)} = \hat{Z}_c \vec{C} \qquad \vec{Q}^{(0)} = \hat{Z} \vec{Q}$$
 (5.69)

with $\hat{Z}_c^T = \hat{Z}^{-1}$. Defining next the anomalous dimension matrix $\hat{\gamma}$ by

$$\hat{\gamma} = \hat{Z}^{-1} \frac{d\hat{Z}}{d\ln \mu},\tag{5.70}$$

the μ -independence of $\vec{C}^{(0)}$ implies

$$\frac{d\vec{C}(\mu)}{d\ln\mu} = \hat{\gamma}^T(\alpha_s)\vec{C}(\mu). \tag{5.71}$$

The solution of this equation is

$$\vec{C}(\mu) = \hat{U}(\mu, M_W)\vec{C}(M_W)$$
 (5.72)

where

$$\hat{U}(\mu, M_{\mathrm{W}}) = \exp\left[\int_{g(M_{\mathrm{W}})}^{g(\mu)} dg' \frac{\hat{\gamma}^{T}(g')}{\beta(g')}\right]$$
(5.73)

is the μ -evolution matrix.

In the MS $(\overline{\text{MS}})$ -scheme we have

$$\hat{Z} = \hat{1} + \sum_{k=1}^{\infty} \frac{1}{\varepsilon^k} \hat{Z}_k(g)$$

$$(5.74)$$

and

$$\hat{\gamma}(g) = -2g^2 \frac{\partial \hat{Z}_1(g)}{\partial g^2} \tag{5.75}$$

Consequently using

$$\hat{Z} = 1 + \frac{\alpha_s}{4\pi} \frac{1}{\varepsilon} \begin{pmatrix} 3/N & -3\\ -3 & 3/N \end{pmatrix}$$
 (5.76)

we have to first order in α_s [72]

$$\hat{\gamma}(\alpha_s) = \frac{\alpha_s}{4\pi} \hat{\gamma}^{(0)} = \frac{\alpha_s}{4\pi} \begin{pmatrix} -6/N & 6\\ 6 & -6/N \end{pmatrix}. \tag{5.77}$$

In order to find $C_i(\mu)$ let us write the LO evolution matrix as

$$\hat{U}^{(0)}(\mu, M_{W}) = \hat{V}\left(\left[\frac{\alpha_{s}(M_{W})}{\alpha_{s}(\mu)}\right]^{\frac{\vec{\gamma}^{(0)}}{2\beta_{0}}}\right) \hat{V}^{-1}$$
(5.78)

where \hat{V} diagonalizes $\hat{\gamma}^{(0)T}$

$$\hat{\gamma}_D^{(0)} = \hat{V}^{-1} \hat{\gamma}^{(0)T} \hat{V} \tag{5.79}$$

and $\vec{\gamma}^{(0)}$ is the vector containing the diagonal elements of the diagonal matrix :

$$\hat{\gamma}_D^{(0)} = \begin{pmatrix} \gamma_+^{(0)} & 0\\ 0 & \gamma_-^{(0)} \end{pmatrix} \tag{5.80}$$

with $\gamma_{\pm}^{(0)}$ given in (5.56). Using

$$\hat{V} = \hat{V}^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1\\ 1 & -1 \end{pmatrix}$$
 (5.81)

and

$$\vec{C}^T(M_W) = (C_1(M_W), C_2(M_W)) = (0, 1)$$
 (5.82)

we reproduce (5.67) with $C_{\pm}(\mu)$ given by (5.62).

The threshold effects can be incorporated as in (5.65)

$$\vec{C}(\mu_c) = \hat{U}^{(f=4)}(\mu_c, \mu_b)\hat{U}^{(f=5)}(\mu_b, \mu_W)\vec{C}(\mu_W). \tag{5.83}$$

It is evident that this procedure is valid for arbitrary number of operators mixing under renormalization. However for more complicated situations one has to use computer programs like *Mathematica* to obtain analytic formulae like (5.67). We will give some examples later on.

5.6 Summary of Basic Formalism

It is a good moment to make a break and to summarize what we have achieved in our climb so far. This will also allow us to make a strategy for the next steps, which as we will see are technically more advanced.

Ultimately our goal is the evaluation of weak decay amplitudes involving hadrons in the framework of a low energy effective theory, of the form

$$\langle \mathcal{H}_{eff} \rangle = \frac{G_F}{\sqrt{2}} V_{CKM} \langle \vec{Q}^T(\mu) \rangle \vec{C}(\mu),$$

where μ denotes a scale of the order of the mass of the decaying hadron. The procedure for this calculation can be divided into the following three steps.

Step 1: Matching in Perturbation Theory

Calculation of Wilson coefficients $\vec{C}(\mu_W)$ at $\mu_W = \mathcal{O}(M_W)$ to the desired order in α_s . Since logarithms of the form $\ln(\mu_W/M_W)$ are not large, this can be performed in ordinary perturbation theory. In the case of the operators $Q_{1,2}$ and in the LO approximation we simply have $C_1(\mu_W) = 0$, $C_2(\mu_W) = 1$ or $C_{\pm}(\mu_W) = 1$.

This step amounts to matching the full theory onto a five quark effective theory. In this process W^{\pm} , Z^0 , the top-quark and generally all heavy particles with masses higher than $M_{\rm W}$ are integrated out. In the case of $Q_{1,2}$ analyzed so far, the effect of integrating out the top-quark has only been seen in that for $\mu \leq M_{\rm W}$ we have used $\alpha^{(5)}(\mu)$ instead of $\alpha^{(6)}(\mu)$. Later when we move to other decays, the effect of integrating out the top-quark will be more profound.

The matching in question is achieved using the following procedure:

- Calculation of the amplitude in the full theory,
- Calculation of the operator matrix elements,
- Extraction of $C_i(\mu_W)$ from $A_{full} = A_{eff}$.

The resulting $C_i(\mu_W)$ depend generally on the masses of the heavy particles which have been integrated out. Again in the special case of $Q_{1,2}$ this dependence is absent.

Step 2: RG Improved Perturbation Theory

- Calculation of the anomalous dimensions of the operators,
- Solution of the renormalization group equation for $\vec{C}(\mu)$,
- Evolution of the coefficients from μ_W down to the appropriate low energy scale μ

$$\vec{C}(\mu) = \hat{U}(\mu, \mu_W)\vec{C}(\mu_W) .$$

Step 3: Non-Perturbative Regime

Calculation of hadronic matrix elements $\langle \vec{Q}(\mu) \rangle$, normalized at the appropriate low energy scale μ , by means of some non-perturbative method.

Important issues in this procedure are:

- Factorization of short- and long distance contributions:
 - $-\vec{C}(\mu)$: contributions from scales higher than μ
 - $-\langle \vec{Q}(\mu)\rangle$: contributions from scales lower than μ
 - Cancellation of the μ -dependence between $C_i(\mu)$ and $\langle Q_i(\mu) \rangle$.
- Summation of large logs by means of the RG method. More specifically, in the n-the order of RG improved perturbation theory the terms

$$\alpha_s^n(\mu) \left(\alpha_s(\mu) \ln \frac{M_W}{\mu}\right)^k$$

are summed to all orders in k (k=0, 1, 2,...). This approach is justified as long as $\alpha_s(\mu)$ is small enough. The leading order corresponds in most cases to n = 0, the NLO to n = 1. In certain processes these canonical values of n may change.

5.7 Future Generalizations

Until now, our application of the basic formalism summarized above, concentrated on the current-current operators Q_1 and Q_2 in the LO approximation. In the following sections we will generalize this discussion in several aspects:

- We will generalize the calculation of the couplings $C_i(\mu)$ beyond the LO approximation,
- We will include new operators originating in penguin diagrams of various sort (Gluon-penguins, Photon-penguins, Z^0 -penguins). These operators are generally called *Penguin Operators*. This generalization will bring the m_t dependence into $C_i(\mu_W)$ of these new operators.
- We will also include new operators originating in *Box Diagrams*. This generalization will also bring the m_t dependence into $C_i(\mu_W)$ of these new operators.
- In the process of including operators it will turn out to be necessary to consider also renormalization group equations involving simultaneously C_i of order $\mathcal{O}(1)$, $\mathcal{O}(\alpha_s)$ and $\mathcal{O}(\alpha)$ with $\alpha = \alpha_{QED}$.
- Finally we will develop efficient methods for the calculation of the anomalous dimensions of the operators Q_i .

We begin these generalizations by including NLO QCD corrections to $C_{\pm}(\mu)$. We will do this in such a manner that the generalization of the formulae listed below to more complicated processes will be straightforward.

5.8 Motivations for NLO

Going beyond the LO approximation is certainly an important but a non-trivial step. For this reason we need some motivations to perform this step. Here are the main reasons for going beyond LO:

- The NLO is first of all necessary to test the validity of the renormalization group improved perturbation theory.
- Without going to NLO the QCD scale $\Lambda_{\overline{MS}}$ extracted from various high energy processes cannot be used meaningfully in weak decays.

- Due to renormalization group invariance the physical amplitudes do not depend on the scales μ present in α_s or in the running quark masses, in particular $m_t(\mu)$, $m_b(\mu)$ and $m_c(\mu)$. However, in perturbation theory this property is broken through the truncation of the perturbative series. Consequently one finds sizable scale ambiguities in the leading order, which can be reduced considerably by going to NLO. An example of such an ambiguity is the choice of the high energy matching scale μ_W discussed above.
- The Wilson Coefficients are renormalization scheme dependent quantities. This scheme dependence appears first at NLO. For a proper matching of the short distance contributions to the long distance matrix elements obtained from lattice calculations it is essential to calculate NLO. The same is true for inclusive heavy quark decays in which the hadron decay can be modeled by a decay of a heavy quark and the matrix elements of Q_i can be effectively calculated in an expansion in $1/m_b$.
- In several cases the central issue of the top quark mass dependence is strictly a NLO effect.

6 Wilson Coefficients Beyond Leading Order

6.1 Preliminaries

We will now generalize the formulae of the previous section beyond the LO approximation concentrating on the Wilson coefficients C_{\pm} and $C_{1,2}$. We will begin with the case without operator mixing. Subsequently we will generalize our discussion to the case of the operator mixing. Next we will develop methods for the calculation of anomalous dimensions generalizing our previous discussion to the mixing of operators with different canonical dimensions. In particular the mixing between six and five dimensional operators. While we do not have space to present an explicit two-loop calculation of anomalous dimensions, we will derive explicitly the one-loop anomalous dimension matrix (5.77). In section 8.5 we will generalize this calculation to include the penguin operators. A detailed discussion of renormalization scheme and renormalization scale dependences and of their cancellations in physical amplitudes is an important part of this section. Finally, we will discuss the issue of the so-called evanescent operators which have to be taken into account in a proper calculation of the anomalous dimensions at the two-loop level.

6.2 The Case without Operator Mixing

Let us consider the coefficients $C_{\pm}(\mu)$ for which we have the general expression:

$$C_{\pm}(\mu) = U_{\pm}(\mu, M_{\rm W})C_{\pm}(M_{\rm W})$$
 (6.1)

where

$$U_{\pm}(\mu, M_{\rm W}) = \exp\left[\int_{g(M_{\rm W})}^{g(\mu)} dg' \frac{\gamma_{\pm}(g')}{\beta(g')}\right]$$
 (6.2)

and we have set $\mu_W = M_W$ in order to simplify the formulae below. This restriction will be relaxed whenever it will turn out to be appropriate.

At NLO we use:

$$C_{\pm}(M_W) = 1 + \frac{\alpha_s(M_W)}{4\pi}B_{\pm}$$
 (6.3)

$$\gamma_{\pm}(\alpha_s) = \gamma_{\pm}^{(0)} \frac{\alpha_s}{4\pi} + \gamma_{\pm}^{(1)} \left(\frac{\alpha_s}{4\pi}\right)^2 \tag{6.4}$$

$$\beta(g) = -\beta_0 \frac{g^3}{16\pi^2} - \beta_1 \frac{g^5}{(16\pi^2)^2} \tag{6.5}$$

Inserting the last two formulae into (6.2) and expanding in α_s we find

$$U_{\pm}(\mu, M_{\mathrm{W}}) = \left[1 + \frac{\alpha_s(\mu)}{4\pi} J_{\pm}\right] \left[\frac{\alpha_s(M_W)}{\alpha_s(\mu)}\right]^{d_{\pm}} \left[1 - \frac{\alpha_s(M_W)}{4\pi} J_{\pm}\right]$$
(6.6)

with

$$J_{\pm} = \frac{d_{\pm}}{\beta_0} \beta_1 - \frac{\gamma_{\pm}^{(1)}}{2\beta_0} \qquad d_{\pm} = \frac{\gamma_{\pm}^{(0)}}{2\beta_0}. \tag{6.7}$$

This is similar to the μ -dependence of the quark mass discussed in section 4.8 and given in (4.81) except that we have written the evolution function in a particular way: the couplings $\alpha_s(\mu)$ increase by going from right to left. This is clearly not necessary but is useful for the future generalization to the case of operator mixing.

Inserting (6.6) and (6.3) into (6.1) we find an important formula for $C_{\pm}(\mu)$ in the NLO approximation:

$$C_{\pm}(\mu) = \left[1 + \frac{\alpha_s(\mu)}{4\pi}J_{\pm}\right] \left[\frac{\alpha_s(M_W)}{\alpha_s(\mu)}\right]^{d_{\pm}} \left[1 + \frac{\alpha_s(M_W)}{4\pi}(B_{\pm} - J_{\pm})\right] . \tag{6.8}$$

Let us next outline the procedure for finding B_{\pm} . Since the operators Q_{+} and Q_{-} do not mix under renormalization, B_{+} and B_{-} can be found separately. The procedure for finding B_{\pm} amounts to the generalization of the matching procedure in LO to include in addition to logarithms also constant $\mathcal{O}(\alpha_s)$ terms.

Step 1:

$$A_{full}^{\pm} = \frac{G_F}{\sqrt{2}} \left(1 + \frac{\alpha_s(\mu_W)}{4\pi} \left[-\frac{\gamma_{\pm}^{(0)}}{2} \ln \frac{M_W^2}{-p^2} + \tilde{A}_{\pm}^{(1)} \right] \right) S_{\pm} , \qquad (6.9)$$

where S_{\pm} are the tree matrix elements.

Step 2:

$$A_{eff}^{\pm} = \frac{G_F}{\sqrt{2}} C_{\pm}(\mu_W) \langle Q_{\pm}(\mu_W) \rangle$$

$$= \frac{G_F}{\sqrt{2}} C_{\pm}(\mu_W) (1 + \frac{\alpha_s(\mu_W)}{4\pi} \left[\frac{\gamma_{\pm}^{(0)}}{2} \ln \frac{-p^2}{\mu_W^2} + \tilde{r}_{\pm} \right]) S_{\pm} .$$
(6.10)

Step 3:

Comparison of (6.9) and (6.10) yields

$$C_{\pm}(\mu_W) = 1 + \frac{\alpha_s(\mu_W)}{4\pi} \left[-\frac{\gamma_{\pm}^{(0)}}{2} \ln \frac{M_W^2}{\mu_W^2} + B_{\pm} \right] , \qquad (6.11)$$

where

$$B_{\pm} = \tilde{A}_{+}^{(1)} - \tilde{r}_{\pm} \ . \tag{6.12}$$

Setting $\mu_W = M_W$ we reproduce (6.3). Any infrared dependence like $\ln -p^2$ or any special properties of the external quark states present in $\tilde{A}_{\pm}^{(1)}$ and \tilde{r}_{\pm} cancel in the difference (6.12) so that B_{\pm} are just numerical constants independent of external states. We will give the numerical values of B_{\pm} in the next section.

6.3 The Case of Operator Mixing

6.3.1 Preliminaries

Let us generalize the preceding discussion to the case of operator mixing. Now

$$\mathcal{H}_{eff} = \frac{G_F}{\sqrt{2}} \sum_{i} C_i(\mu) Q_i(\mu) \equiv \frac{G_F}{\sqrt{2}} \vec{Q}^T(\mu) \vec{C}(\mu) , \qquad (6.13)$$

where the index i runs over all contributing operators, in our example Q_1 and Q_2 .

The Wilson coefficient functions are given then by

$$\vec{C}(\mu) = \hat{U}(\mu, \mu_W)\vec{C}(\mu_W)$$
 (6.14)

Our goal is to find $\vec{C}(\mu_W)$ and the evolution matrix $\hat{U}(\mu, \mu_W)$ keeping NLO corrections.

6.3.2 Determination of $\vec{C}(\mu_W)$

The procedure for finding $\vec{C}(\mu_W)$ proceeds again in three steps:

Step 1:

The amplitude in the full theory after field renormalization is given by:

$$A_{full} = \frac{G_F}{\sqrt{2}} \vec{S}^T (\vec{A}^{(0)} + \frac{\alpha_s(\mu_W)}{4\pi} \vec{A}^{(1)}). \tag{6.15}$$

Here \vec{S} denotes the tree level matrix elements of the operators \vec{Q} . In order to simplify the presentation we have absorbed the logarithms in the $\mathcal{O}(\alpha_s)$ term $\vec{A}^{(1)}$ which also contains non-logarithmic terms as in (6.9). Later on we will discuss a specific example which will exhibit the detail structure of (6.15) more transparently.

Step 2:

In the effective theory, after quark field renormalization and the renormalization of the operators through

$$\vec{Q}^{(0)} = \hat{Z}\vec{Q},\tag{6.16}$$

the renormalized matrix elements of the operators are

$$\langle \vec{Q}(\mu_W) \rangle = (\hat{1} + \frac{\alpha_s(\mu_W)}{4\pi} \hat{r}) \vec{S}$$
(6.17)

and consequently

$$A_{eff} = \frac{G_F}{\sqrt{2}} \vec{S}^T (1 + \frac{\alpha_s(\mu_W)}{4\pi} \hat{r}^T) \vec{C}(\mu_W).$$
 (6.18)

Again \hat{r} contains the relevant logarithms together with the non-logarithmic terms as in (6.10).

Step 3:

Equating (6.15) and (6.18) we obtain

$$\vec{C}(\mu_W) = \vec{A}^{(0)} + \frac{\alpha_s(\mu_W)}{4\pi} (\vec{A}^{(1)} - r^T \vec{A}^{(0)}). \tag{6.19}$$

6.3.3 Renormalization Group Evolution

The renormalization group equation for \hat{C}

$$\frac{d\vec{C}(\mu)}{d\ln\mu} = \hat{\gamma}^T(g)\vec{C}(\mu) \tag{6.20}$$

has to be solved now with the boundary condition (6.19).

The general solution can be written down iteratively

$$\hat{U}(\mu, \mu_W) = 1 + \int_{g(\mu_W)}^{g(\mu)} dg_1 \frac{\hat{\gamma}^T(g_1)}{\beta(g_1)} + \int_{g(\mu_W)}^{g(\mu)} dg_1 \int_{g(\mu_W)}^{g_1} dg_2 \frac{\hat{\gamma}^T(g_1)}{\beta(g_1)} \frac{\hat{\gamma}^T(g_2)}{\beta(g_2)} + \dots$$
 (6.21)

which using $dg/d \ln \mu = \beta(g)$ solves

$$\frac{d}{d\ln\mu}\hat{U}(\mu,\mu_W) = \hat{\gamma}^T(g)\hat{U}(\mu,\mu_W) \ . \tag{6.22}$$

The series in (6.21) can be written more compactly:

$$\hat{U}(\mu, \mu_W) = T_g \exp \int_{g(\mu_W)}^{g(\mu)} dg' \frac{\hat{\gamma}^T(g')}{\beta(g')} , \qquad (6.23)$$

where in the case $g(\mu) > g(\mu_W)$ the g-ordering operator T_g is defined through

$$T_g f(g_1) \dots f(g_n) = \sum_{perm} \Theta(g_{i_1} - g_{i_2}) \dots \Theta(g_{i_{n-1}} - g_{i_n}) f(g_{i_1}) \dots f(g_{i_n}).$$
 (6.24)

It brings ordering of the functions $f(g_i)$ such that the coupling constants increase from right to left. The sum in (6.24) runs over all permutations $\{i_1, \ldots, i_n\}$ of $\{1, 2, \ldots, n\}$. The T_g ordering is necessary because at NLO $[\hat{\gamma}(g_1), \hat{\gamma}(g_2)] \neq 0$. Indeed the matrices $\hat{\gamma}^{(0)}$ and $\hat{\gamma}^{(1)}$ in the perturbative expansion of the anomalous dimension matrix

$$\hat{\gamma}(\alpha_s) = \hat{\gamma}^{(0)} \frac{\alpha_s}{4\pi} + \hat{\gamma}^{(1)} \left(\frac{\alpha_s}{4\pi}\right)^2 . \tag{6.25}$$

do not commute with each other.

Inserting (6.25) and the expansion (6.5) for $\beta(g)$ into (6.23) we can write the evolution matrix in analogy to (6.6) as

$$\hat{U}(\mu, \mu_W) = \left[1 + \frac{\alpha_s(\mu)}{4\pi} \hat{J} \right] \hat{U}^{(0)}(\mu, \mu_W) \left[1 - \frac{\alpha_s(\mu_W)}{4\pi} \hat{J} \right]$$
(6.26)

Now it is clear why we have written (6.6) in a special manner. It can be nicely generalized to the mixing case where the ordering of matrices matters.

 $\hat{U}^{(0)}$ in (6.26) is the leading evolution matrix which we already discussed in the LO section:

$$\hat{U}^{(0)}(\mu, \mu_W) = \hat{V} \left(\left[\frac{\alpha_s(\mu_W)}{\alpha_s(\mu)} \right]^{\frac{\vec{\gamma}^{(0)}}{2\beta_0}} \right)_D \hat{V}^{-1}$$
(6.27)

where \hat{V} diagonalizes $\hat{\gamma}^{(0)T}$

$$\hat{\gamma}_D^{(0)} = \hat{V}^{-1} \gamma^{(0)T} \hat{V} \tag{6.28}$$

and $\vec{\gamma}^{(0)}$ is the vector containing the diagonal elements of the diagonal matrix $\hat{\gamma}_D^{(0)}$.

The derivation of the analytic expression for the matrix \hat{J} follows [47] and is also given in the appendix of the first paper in [68]. Here we give only the final result. In order to write down the expression for the matrix \hat{J} , we define the matrix

$$\hat{G} = \hat{V}^{-1} \hat{\gamma}^{(1)T} \hat{V} \tag{6.29}$$

and a matrix \hat{H} whose elements are

$$H_{ij} = \delta_{ij} \gamma_i^{(0)} \frac{\beta_1}{2\beta_0^2} - \frac{G_{ij}}{2\beta_0 + \gamma_i^{(0)} - \gamma_i^{(0)}}.$$
 (6.30)

Then

$$\hat{J} = \hat{V}\hat{H}\hat{V}^{-1}. (6.31)$$

6.3.4 Final Result for $\vec{C}(\mu)$

Putting all things together we obtain the final result

$$\vec{C}(\mu) = \left(1 + \frac{\alpha_s(\mu)}{4\pi}\hat{J}\right)\hat{U}^{(0)}(\mu, \mu_W)(\vec{A}^{(0)} + \frac{\alpha_s(\mu_W)}{4\pi}[\vec{A}^{(1)} - (\hat{r}^T + \hat{J})\vec{A}^{(0)}])$$
(6.32)

which we will discuss in more detail below.

In the case of (Q_1, Q_2) the inclusion of the flavour thresholds in (6.14) is very similar to the LO case:

$$\vec{C}(\mu) = \hat{U}_3(\mu, \mu_c)\hat{U}_4(\mu_c, \mu_b)\hat{U}_5(\mu_b, \mu_W)\vec{C}(\mu_W)$$
(6.33)

where \hat{U}_f is the evolution matrix for f effective flavors given in (6.26). This formula has to be slightly modified if the penguin operators are present. We will return to this point at a suitable moment of these lectures.

6.4 The Calculation of the Anomalous Dimensions

6.4.1 Master Formulae

In the previous section we have calculated the anomalous dimension matrix in the process of the matching of the full and effective theories. In fact looking back one can see that the anomalous dimensions can be read off from the coefficients of the logarithms in the matrix elements $\langle Q_{1,2} \rangle$ in (5.31) and (5.32). In more complicated situations, where many operators are present, such a method is not very useful and it is important to develop an efficient method for the calculation of anomalous dimensions. Here it comes:

ullet The evaluation of the amputated Green functions with insertion of the operators \vec{Q} as in fig. 16 gives the relation

$$\langle \vec{Q} \rangle^{(0)} = Z_q^{-2} \hat{Z} \langle \vec{Q} \rangle \equiv \hat{Z}_{GF} \langle \vec{Q} \rangle$$
 (6.34)

where \hat{Z} is the renormalization constant matrix of the operators \vec{Q} and \hat{Z}_{GF} is just defined above.

• Next, the anomalous dimension matrix is given by

$$\hat{\gamma}(g) = \hat{Z}^{-1} \frac{d\hat{Z}}{d \ln \mu} \tag{6.35}$$

• In the MS (or $\overline{\text{MS}}$) scheme we have

$$\hat{Z} = 1 + \sum_{k=1}^{\infty} \frac{1}{\varepsilon^k} \hat{Z}_k(g)$$
(6.36)

and consequently as derived in the previous section

$$\hat{\gamma}(g) = -2g^2 \frac{\partial \hat{Z}_1(g)}{\partial g^2} = -2\alpha_s \frac{\partial \hat{Z}_1(\alpha_s)}{\partial \alpha_s}.$$
 (6.37)

• For Z_q and \hat{Z}_{GF} we have

$$Z_q = 1 + \sum_{k=1}^{\infty} \frac{1}{\varepsilon^k} Z_{q,k}(g)$$
(6.38)

$$\hat{Z}_{GF} = 1 + \sum_{k=1}^{\infty} \frac{1}{\varepsilon^k} \hat{Z}_{GF,k}(g)$$

$$(6.39)$$

As the matrix elements $\langle \vec{Q} \rangle$ are finite, the singularities in \hat{Z}_{GF} are found directly from the calculation of the unrenormalized Green functions (6.34).

• From (6.34), (6.36), (6.38), (6.39) we find

$$\hat{Z}_1 = 2Z_{q,1}\hat{1} + \hat{Z}_{GF,1}. (6.40)$$

• With

$$Z_{q,1} = a_1 \frac{\alpha_s}{4\pi} + a_2 \left(\frac{\alpha_s}{4\pi}\right)^2 \tag{6.41}$$

and

$$\hat{Z}_{GF,1} = \hat{b}_1 \frac{\alpha_s}{4\pi} + \hat{b}_2 \left(\frac{\alpha_s}{4\pi}\right)^2 \tag{6.42}$$

we obtain by means of (6.37) *Master Formulae* for the one- and two-loop anomalous dimension matrices:

$$\gamma_{ij}^{(0)} = -2[2a_1\delta_{ij} + (b_1)_{ij}], \qquad \gamma_{ij}^{(1)} = -4[2a_2\delta_{ij} + (b_2)_{ij}]. \tag{6.43}$$

• In the case without mixing between operators these expressions reduce to:

$$\gamma^{(0)} = -2[2a_1 + b_1], \qquad \gamma^{(1)} = -4[2a_2 + b_2].$$
 (6.44)

6.4.2 How to Use One-Loop Master Formulae

Let us illustrate how the first formula in (6.43) can be used to obtain the one-loop anomalous dimension matrix (5.77). From (5.26) we extract the coefficients of the $1/\varepsilon$ singularities to be

$$(b_1)_{11} = 2C_F + \frac{3}{N}, \qquad (b_1)_{12} = -3.$$
 (6.45)

Similarly from (5.27) we find

$$(b_1)_{21} = -3, (b_1)_{22} = 2C_F + \frac{3}{N}.$$
 (6.46)

Now $a_1 = -C_F$. Consequently the term $2a_1$ in (6.43) cancels precisely the $2C_F$ term present in $(b_1)_{11}$ and $(b_1)_{22}$. The leftover entries give the one-loop anomalous dimension matrix (5.77).

The fact that the renormalization of the external quark fields cancels the terms $2C_F$ in $(b_1)_{11}$ and $(b_1)_{22}$ is by no means accidental. It is a consequence of the vanishing of the anomalous dimension of the conserved weak current. Indeed the master formula for the anomalous dimension of a current can be obtained by considering a two-point Green function instead of four-point functions considered in deriving the master formulae (6.44). One finds this time

$$\gamma_c^{(0)} = -2[a_1 + b_1^c], \qquad \gamma_c^{(1)} = -4[a_2 + b_2^c]$$
(6.47)

where b_1^c and b_2^c are obtained by calculating the relevant one-loop and two-loop diagrams, respectively. b_1^c is simply obtained by calculating the one-loop upper vertex of the diagram (a) in fig. 16. b_2^c is found by calculating the corresponding two-loop generalization of this vertex. One finds $b_1^c = C_F$. The factor of two in $(b_1)_{11}$ in front of C_F in (6.45) comes from a symmetric diagram to the diagram (a) in fig. 16 with the gluon exchanged between the lower quark legs. With $a_1 = -C_F$ we find $\gamma_c^{(0)} = 0$ as it should be.

We get the following useful message from this discussion. The only diagrams responsible for the non-vanishing anomalous dimensions of current-current operators Q_1 and Q_2 are the diagrams in which the gluons connect the quark legs belonging to different weak currents. At the one-loop level these are the diagrams (b) and (c) in fig. 16 and the corresponding symmetric diagrams. One should stress that this simple rule is not valid for the insertion of penguin operators into current-current diagrams. We will see this explicitly in section 8.

6.4.3 How to Use Two-Loop Master Formulae

The calculation of $(b_2)_{ij}$ in (6.43) is a bit tricker and technically more difficult. To this end one has to calculate first two-loop diagrams with Q_i insertions. Examples are given in fig. 17. Next the corresponding two-loop counter-diagrams have to be *subtracted*. In the MS-like schemes the latter are obtained by retaining only the $1/\varepsilon$ parts in the subdiagrams. The counter-diagrams corresponding to the two-loop diagrams in fig. 17 are shown in fig. 18 where the small boxes stand for the singular parts of the corresponding subdiagrams in fig. 17. For instance the Feynman rule for the box in fig. 18b is, in accordance with (4.6) given by

$$iC_F \delta_{\alpha\beta} \frac{\alpha_s}{4\pi} \not p \frac{1}{\varepsilon}$$
 (6.48)

where p is the momentum of the quark. Since MS-like schemes are the so-called mass-independent renormalization schemes the quark masses can be set to zero in evaluating anomalous dimensions.

Dropping colour factors and Dirac tensors the result for each diagram including the counter-diagram has the structure

$$I - I_C = \left(\frac{\alpha}{4\pi}\right)^2 \left[\frac{\mu^2}{-p^2}\right]^{2\varepsilon} \left[\frac{F}{\varepsilon^2} + \frac{G}{\varepsilon} + \dots\right] - \left(\frac{\alpha}{4\pi}\right)^2 \left[\frac{\mu^2}{-p^2}\right]^{\varepsilon} \left[\frac{F_C}{\varepsilon^2} + \frac{G_C}{\varepsilon} + \dots\right]$$
(6.49)

where the second term represents the counter-diagram. Note that the power of μ^2 in the counter-term is ε and not 2ε as in I. It turns out that for diagrams with non-vanishing F one has diagram by diagram the relation $F_C = 2F$ and consequently the pole part does not depend on μ as it should be:

$$I - I_C = \left(\frac{\alpha}{4\pi}\right)^2 \left[-\frac{F}{\varepsilon^2} + \frac{G - G_C}{\varepsilon} + \dots \right]$$
 (6.50)

The coefficient $G - G_C$ can then be identified with the contribution of a given diagram (after the inclusion of colour factors) to the coefficient $(b_2)_{ij}$ entering the master formula (6.43).

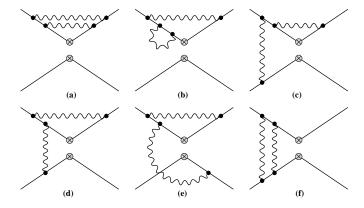


Figure 17: Examples of two loop current-current diagrams contributing to the NLO anomalous dimensions of the operators Q_1 and Q_2 .

Again, as in the one-loop case, the diagrams with gluons exchanged only between quark legs belonging to the same weak current can be omitted in the evaluation of two-loop anomalous dimensions of the operators Q_1 and Q_2 , provided the anomalous dimension of the weak current vanishes at two-loop level. In this case their contributions to $(b_2)_{ij}$ are canceled by the a_2 term in the master formula (6.43). It should be stressed that this feature might not be preserved by some regularization schemes. In particular, it depends on the treatment of γ_5 in $D \neq 4$ dimensions. In the NDR scheme, in which γ_5 anticommutes with γ_μ in $D \neq 4$ dimensions, one has indeed $\gamma_c^{(1)} = 0$. But this is not true in the HV scheme, where γ_5 has more complicated properties. Indeed one finds [48]

$$[\gamma_c^{(1)}]_{HV} = 4C_F \beta_0. \tag{6.51}$$

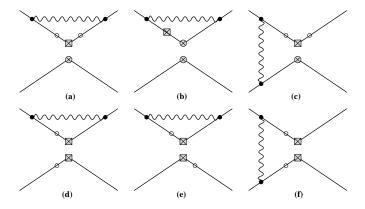


Figure 18: Counter-diagrams to the diagrams of fig. 17.

We will return to this issue in a moment. For the time being we give the two-loop generalizations of the one-loop anomalous dimension matrix (5.77) in the NDR scheme [48]:

$$\hat{\gamma}_{NDR}^{(1)} = \begin{pmatrix} -\frac{22}{3} - \frac{57}{2N^2} - \frac{2f}{3N} & \frac{39}{N} - \frac{19N}{6} + \frac{2f}{3} \\ \frac{39}{N} - \frac{19N}{6} + \frac{2f}{3} & -\frac{22}{3} - \frac{57}{2N^2} - \frac{2f}{3N} \end{pmatrix}$$
(6.52)

and in the HV scheme [48]:

$$\hat{\gamma}_{HV}^{(1)} = \begin{pmatrix} -\frac{110}{3} - \frac{57}{2N^2} + \frac{44N^2}{3} + \left(\frac{14}{3N} - \frac{8N}{3}\right)f & \frac{39}{N} + \frac{23N}{2} - 2f \\ \frac{39}{N} + \frac{23N}{2} - 2f & -\frac{110}{3} - \frac{57}{2N^2} + \frac{44N^2}{3} + \left(\frac{14}{3N} - \frac{8N}{3}\right)f \end{pmatrix}$$
(6.53)

The corresponding result in the DRED scheme has been first calculated in [53] and confirmed in [48]. We observe substantial renormalization scheme dependence. In particular the diagonal elements in (6.53) contain terms $\mathcal{O}(N^2)$ whereas such terms are absent in (6.52). The origin of these terms can be traced back to the non-vanishing of $[\gamma_c^{(1)}]_{HV}$. Indeed these terms cancel in the difference $\hat{\gamma}_{HV}^{(1)} - 2[\gamma_c^{(1)}]_{HV}\hat{1} \equiv [\hat{\gamma}_{HV}^{(1)}]_{\text{eff}}$.

As we will discuss in subsection 6.7, such a shift of two-loop anomalous dimensions is always possible provided also the matching conditions for the Wilson coefficients at μ_W are appropriately changed. Yet eventually this shift modifies the Wilson coefficients and this modification will be compensated by the corresponding change in the matrix elements of the operators so that physical quantities are independent of these manipulations. All this should be clearer after subsection 6.7 where the cancellation of this scheme dependences in physical quantities are discussed in explicit terms.

6.4.4 A Warning on the HV Scheme

In this context we should warn the reader that the numerical values of the Wilson coefficients in the HV scheme presented here and in [17, 73] correspond to the choice $[\hat{\gamma}_{HV}^{(1)}]_{\text{eff}}$. This

differs from the treatment of my Italian friends [74, 75] who use $\hat{\gamma}_{HV}^{(1)}$ of (6.53) instead. For this reason the NLO corrections to Wilson coefficients in the HV scheme presented here are generally smaller than the ones found by the Rome group. The final physical results are, however, the same.

6.5 Explicit Calculation of 2×2 Anomalous Dimension Matrix

6.5.1 Current-Current Insertions: Generalities

The set of six diagrams contributing to one—loop anomalous dimension matrix through operator insertions into current-current topologies is given by the diagrams in fig. 16 and their symmetric counterparts. We begin by developing the technology for the calculation of insertions of any operator with arbitrary colour and Dirac structure into the diagrams of fig. 16. This will allow us to calculate later also insertions of penguin operators into the current-current topologies of fig. 16.

Let us then denote the colour and Dirac structure of any operator by

$$\hat{V}_1 \otimes \hat{V}_2, \qquad \Gamma_1 \otimes \Gamma_2 , \qquad (6.54)$$

respectively, so that an operator can be generally written as follows:

$$O = (\bar{s}_{\alpha} \Gamma_1 \hat{V}_1^{\alpha \beta} c_{\beta}) \otimes (\bar{u}_{\gamma} \Gamma_2 \hat{V}_2^{\gamma \delta} d_{\delta})$$

$$(6.55)$$

Here we have made specific choice of quark flavours adapted to the operators discussed in section 5, but it is trivial to generalize the following discussion to any other choice of flavours.

Let us consider a few examples:

$$\hat{V}_1^{\alpha\beta} \otimes \hat{V}_2^{\gamma\delta} = \delta_{\alpha\beta} \otimes \delta_{\gamma\delta} \equiv \mathbf{1}_{\alpha\beta,\gamma\delta} , \qquad (6.56)$$

$$\hat{V}_1^{\alpha\beta} \otimes \hat{V}_2^{\gamma\delta} = \delta_{\alpha\delta} \otimes \delta_{\gamma\beta} \equiv \tilde{\mathbf{1}}_{\alpha\beta,\gamma\delta} , \qquad (6.57)$$

$$\hat{V}_1^{\alpha\beta} \otimes \hat{V}_2^{\gamma\delta} = (T^a)_{\alpha\beta} \otimes (T^a)_{\gamma\delta} \equiv \mathbf{\Pi}_{\alpha\beta,\gamma\delta} \ . \tag{6.58}$$

Then the colour identity (5.21) is simply given by

$$\mathbf{\Pi} = \frac{1}{2} \left(\tilde{\mathbf{1}} - \frac{1}{N} \mathbf{1} \right) . \tag{6.59}$$

In this notation the operators Q_1 and Q_2 have both the structure $\Gamma_1 = \Gamma_2 = \gamma_{\mu}(1 - \gamma_5)$. With the ordering of flavours as in (6.55), the colour structure of Q_1 is $\tilde{\mathbf{1}}$. The one of Q_2 is $\mathbf{1}$. In order to gain some insight into the calculation of the diagrams in fig. 16, let us consider the diagram (a) with the insertion of the operator Q_2 . The flavour labels are as in the diagram of fig. 14. For this diagram we have then

$$\mathcal{D}_{a}^{(1)} = -ig^{2}\mu^{2\varepsilon}C_{F} \int \frac{d^{D}k}{(2\pi)^{D}} \frac{[\bar{s}T_{\nu}c] \otimes [\bar{u}\Gamma^{\nu}d]}{k^{2}[(k+p)^{2}]^{2}}$$
(6.60)

where

$$T_{\nu} = \gamma_{\mu}(\not k + \not p)\Gamma_{\nu}(\not k + \not p)\gamma^{\mu} = \gamma_{\mu}\gamma_{\rho}\Gamma_{\nu}\gamma_{\sigma}\gamma^{\mu}(k+p)^{\rho}(k+p)^{\sigma}$$

$$(6.61)$$

and $\Gamma_{\nu} = \gamma_{\nu}(1 - \gamma_5)$. We have used $T^a T^a = C_F$.

Keeping only the divergent part in the relevant D-dimensional integral we find

$$\int \frac{d^D k}{(2\pi)^D} \frac{(k+p)^{\rho} (k+p)^{\sigma}}{k^2 [(k+p)^2]^2} = ig^{\rho\sigma} \frac{1}{16\pi^2} \frac{1}{4} \frac{1}{\varepsilon} + \text{finite}$$
 (6.62)

Thus the divergent part of $\mathcal{D}_a^{(1)}$ is given by

$$\mathcal{D}_{a}^{(1)} = C_{F} \frac{\alpha_{s}}{4\pi} \left[\frac{1}{4} \frac{1}{\varepsilon} \right] \bar{s} \gamma_{\mu} \gamma_{\rho} \Gamma_{\nu} \gamma^{\rho} \gamma^{\mu} c \otimes \bar{u} \Gamma^{\nu} d \tag{6.63}$$

It is straightforward to extend this calculation to other diagrams in fig. 16 and to the arbitrary operator given in (6.55). To this end we fix the ordering of the four flavours as in (6.55) and drop the external spinors. We find then

$$\mathcal{D}_{a} = \frac{\alpha_{s}}{4\pi} \left[\frac{1}{4} \frac{1}{\varepsilon} \right] \left(\mathcal{C}_{a}^{(1)} \gamma_{\mu} \gamma_{\rho} \Gamma_{1} \gamma^{\rho} \gamma^{\mu} \otimes \Gamma_{2} + \mathcal{C}_{a}^{(2)} \Gamma_{1} \otimes \gamma_{\mu} \gamma_{\rho} \Gamma_{2} \gamma^{\rho} \gamma^{\mu} \right)$$
(6.64)

$$\mathcal{D}_{b} = -\frac{\alpha_{s}}{4\pi} \left[\frac{1}{4} \frac{1}{\varepsilon} \right] \left(\mathcal{C}_{b}^{(1)} \Gamma_{1} \gamma_{\rho} \gamma_{\mu} \otimes \Gamma_{2} \gamma^{\rho} \gamma^{\mu} + \mathcal{C}_{b}^{(2)} \gamma_{\mu} \gamma_{\rho} \Gamma_{1} \otimes \gamma^{\mu} \gamma^{\rho} \Gamma_{2} \right)$$
(6.65)

$$\mathcal{D}_{c} = \frac{\alpha_{s}}{4\pi} \left[\frac{1}{4} \frac{1}{\varepsilon} \right] \left(\mathcal{C}_{c}^{(1)} \Gamma_{1} \gamma_{\rho} \gamma_{\mu} \otimes \gamma^{\mu} \gamma^{\rho} \Gamma_{2} + \mathcal{C}_{c}^{(2)} \gamma_{\mu} \gamma_{\rho} \Gamma_{1} \otimes \Gamma_{2} \gamma^{\rho} \gamma^{\mu} \right)$$
(6.66)

where the index (1) stands for the diagrams shown in fig. 16 and the index (2) for their symmetric counter-parts. The colour factors are given by

$$C_a^{(1)} = T^a \hat{V}_1 T^a \otimes \hat{V}_2 \qquad C_a^{(2)} = \hat{V}_1 \otimes T^a \hat{V}_2 T^a$$
(6.67)

$$C_b^{(1)} = \hat{V}_1 T^a \otimes \hat{V}_2 T^a \qquad C_b^{(2)} = T^a \hat{V}_1 \otimes T^a \hat{V}_2$$
 (6.68)

$$C_c^{(1)} = \hat{V}_1 T^a \otimes \hat{T}^a V_2 \qquad C_c^{(2)} = T^a \hat{V}_1 \otimes \hat{V}_2 T^a$$
(6.69)

6.5.2 Anomalous Dimensions of Q_1 and Q_2

Let us apply these general formulae to the case of the operator Q_2 for which we have

$$\hat{V}_1 \otimes \hat{V}_2 = \mathbf{1}, \qquad \Gamma_1 = \Gamma_2 = \gamma_\nu (1 - \gamma_5) \equiv \Gamma.$$
 (6.70)

Since we are interested only in the $1/\varepsilon$ singularity we can use in one-loop formulae the rules for γ -algebra valid in four dimensions. Then

$$\gamma_{\mu}\gamma_{\rho}\Gamma\gamma^{\rho}\gamma^{\mu}\otimes\Gamma=\Gamma\otimes\gamma_{\mu}\gamma_{\rho}\Gamma\gamma^{\rho}\gamma^{\mu}=4\Gamma\otimes\Gamma\tag{6.71}$$

$$\Gamma \gamma_{\rho} \gamma_{\mu} \otimes \Gamma \gamma^{\rho} \gamma^{\mu} = \gamma_{\mu} \gamma_{\rho} \Gamma \otimes \gamma^{\mu} \gamma^{\rho} \Gamma = 16 \Gamma \otimes \Gamma$$

$$(6.72)$$

$$\Gamma \gamma_{\rho} \gamma_{\mu} \otimes \gamma^{\mu} \gamma^{\rho} \Gamma = \gamma_{\mu} \gamma_{\rho} \Gamma \otimes \Gamma \gamma^{\rho} \gamma^{\mu} = 4\Gamma \otimes \Gamma \tag{6.73}$$

These results can be most efficiently found by using a trick which I will call the *Greek Method* [76] from now on. Let me illustrate this method by deriving (6.72). Following [76] let us write

$$\gamma_{\mu}\gamma_{\rho}\Gamma\otimes\gamma^{\mu}\gamma^{\rho}\Gamma=\Lambda\ \Gamma\otimes\Gamma\ ,\tag{6.74}$$

where A is the coefficient we are looking for. In order to find it we replace \otimes in (6.74) by a matrix γ_{τ} to obtain

$$\gamma_{\mu}\gamma_{\rho}\Gamma\gamma_{\tau}\gamma^{\mu}\gamma^{\rho}\Gamma = A \Gamma\gamma_{\tau}\Gamma. \tag{6.75}$$

Inserting Γ of (6.70) into this equality and contracting indices we determine A to be 16.

Using (6.71)–(6.73) in our master formulae (6.64)–(6.66) and summing all diagrams we find

$$\sum_{i} \mathcal{D}_{i} = \frac{\alpha_{s}}{4\pi} \frac{1}{\varepsilon} \Gamma \otimes \Gamma \left[\mathcal{C}_{a}^{(1)} + \mathcal{C}_{a}^{(2)} - 4(\mathcal{C}_{b}^{(1)} + \mathcal{C}_{b}^{(2)}) + \mathcal{C}_{c}^{(1)} + \mathcal{C}_{c}^{(2)} \right]. \tag{6.76}$$

For the colour structure in (6.70), the colour structures in this formula can be easily found to be

$$C_a^{(1)} = C_a^{(2)} = C_F \mathbf{1}, \tag{6.77}$$

$$C_b^{(1)} = C_b^{(2)} = C_c^{(1)} = C_c^{(2)} = \frac{1}{2} \left(\tilde{\mathbf{1}} - \frac{1}{N} \mathbf{1} \right). \tag{6.78}$$

Now $\tilde{\mathbf{1}}$ stands for the operator Q_1 . Consequently inserting (6.77) and (6.78) into (6.76) and comparing the coefficient of $1/\varepsilon$ with (6.42) we extract

$$(b_1)_{21} = -3, (b_1)_{22} = 2C_F + \frac{3}{N}.$$
 (6.79)

The insertion of the operator Q_1 represented by $\tilde{\mathbf{1}}$ into diagrams of fig. 16 can be evaluated in an analogous manner by using the master formulae (6.64)–(6.66). Because the colour structure is more complicated the calculation is now a bit more involved. In order to avoid this complication it is useful to make a Fierz reordering in Q_1 and Q_2 so that

$$Q_1 = (\bar{s}_{\alpha} d_{\alpha})_{V-A} (\bar{u}_{\beta} c_{\beta})_{V-A} \qquad Q_2 = (\bar{s}_{\alpha} d_{\beta})_{V-A} (\bar{u}_{\beta} c_{\alpha})_{V-A} . \tag{6.80}$$

Now the roles of Q_1 and Q_2 are interchanged: Q_1 is $\mathbf{1}$ and Q_2 is $\mathbf{1}$. Since gluons are flavour blind we find immidiately

$$(b_1)_{11} = 2C_F + \frac{3}{N}, \qquad (b_1)_{12} = -3.$$
 (6.81)

(6.79) and (6.81) are precisely the values given in (6.46) and (6.45) respectively. Upon inserting them into the one-loop master formula (6.43) and using $a_1 = -C_F$ we finally reproduce the anomalous dimension matrix (5.77). We will extend this calculation to penguin operators in subsection 8.5.

6.6 Mixing of Operators with different Dimensions

It is useful to know the following properties of mixing of operators with different canonical dimensions

- The operators of a given dimension mix only into operators of the same or lower dimension. In a more formal terminology: to renormalize an operator of a given dimension one needs only operators as counter-terms of the same or lower dimension.
- This means in particular that the operators of dimension six, as Q_1 and Q_2 , can mix into other six dimensional operators and five dimensional magnetic penguin operators of sections 8.7 and 12. On the other hand the magnetic penguin operators cannot mix into dimension six operators.
- Consequently whereas the Q_1 and Q_2 operators influence the Wilson coefficients of the magnetic penguin operators, the latter operators have no impact on C_1 and C_2 .

The proof of these properties is based essentially on dimensional analysis. It can be found on page 149 of the book by Collins [29].

Here comes another useful remark. As we will discuss in section 12 the mixing between the operators (Q_1, Q_2) and the magnetic penguin operators appears first at the two-loop level. That is the leading anomalous dimension is obtained by calculating two-loop diagrams and not one-loop diagrams as discussed sofar. The next-to-leading anomalous dimensions are then obtained from three-loop calculations. In this particular case our master formulae in (6.43) change to $(i \neq j)$

$$\gamma_{ij}^{(0)} = -4[(b_2)_{ij}], \qquad \gamma_{ij}^{(1)} = -6[(b_3)_{ij}].$$
(6.82)

with $(b_2)_{ij}$ and $(b_3)_{ij}$ obtained from $1/\varepsilon$ singularities in two-loop and three-loop diagrams respectively.

6.7 Renormalization Scheme Dependence

At NLO various quantities like the Wilson coefficients and the anomalous dimensions depend on the renormalization scheme for operators. This dependence arises because the renormalization prescription involves an arbitrariness in the finite parts to be subtracted along with the ultraviolet singularities. Two different schemes are then related by a finite renormalization. A particular example of the RS dependence is the dependence on the treatment of γ_5 in D dimensions. We have seen in (6.52) and (6.53) that the two-loop anomalous dimension matrix for the operators (Q_1, Q_2) in the NDR scheme differs from the one in the HV scheme.

Returning back to our discussion of the NLO corrections to the Wilson coefficients of subsection 6.3, one finds that

$$\beta_0, \quad \beta_1, \quad \gamma^{(0)}, \quad \vec{A}^{(0)}, \quad \vec{A}^{(1)}, \quad \hat{r}^T + \hat{J}, \quad \langle \vec{Q} \rangle^T \vec{C}$$
 (6.83)

are scheme independent, whereas

$$\hat{r}, \quad \gamma^{(1)}, \quad \hat{J}, \quad \vec{C}, \quad \langle \vec{Q} \rangle$$
 (6.84)

are scheme dependent. Let us demonstrate this.

First of all, it is clear that the product

$$\langle \vec{Q}(\mu) \rangle^T \vec{C}(\mu) \tag{6.85}$$

representing the full amplitude is independent of RS. The factorization of the amplitude into \vec{C} and $\langle \vec{Q} \rangle$ makes them, however, scheme dependent. Explicitly, for two different schemes (primed and unprimed) we have

$$\langle \vec{Q} \rangle' = (1 + \frac{\alpha_s}{4\pi} \hat{s}) \langle \vec{Q} \rangle , \qquad \vec{C}' = (1 - \frac{\alpha_s}{4\pi} \hat{s}^T) \vec{C} , \qquad (6.86)$$

where \hat{s} is a constant matrix representing a finite renormalization of \vec{C} and $\langle \vec{Q} \rangle$.

Having the relations (6.86) at hand it is straightforward to find relations between various quantities in the primed and unprimed schemes. From

$$\langle \vec{Q}(\mu_W) \rangle = (\hat{1} + \frac{\alpha_s(\mu_W)}{4\pi} \hat{r}) \vec{S}, \qquad (6.87)$$

where \vec{S} is a vector of tree level matrix elements, we immediately obtain

$$\hat{r}' = \hat{r} + \hat{s} \tag{6.88}$$

Next from

$$\langle \vec{Q}(\mu) \rangle^T \vec{C}(\mu) \equiv \langle \vec{Q}(\mu) \rangle^T \hat{U}(\mu, M_W) \vec{C}(M_W)$$
(6.89)

we have

$$\hat{U}'(\mu, M_W) = (1 - \frac{\alpha_s(\mu)}{4\pi} \hat{s}^T) \hat{U}(\mu, M_W) (1 + \frac{\alpha_s(M_W)}{4\pi} \hat{s}^T). \tag{6.90}$$

A comparison with

$$\hat{U}(\mu, \mu_W) = (1 + \frac{\alpha_s(\mu)}{4\pi} \hat{J}) \hat{U}^{(0)}(\mu, \mu_W) (1 - \frac{\alpha_s(\mu_W)}{4\pi} \hat{J})$$
(6.91)

yields then

$$\hat{J}' = \hat{J} - \hat{s}^T \tag{6.92}$$

Next from (5.29) and (6.86) we clearly have

$$\hat{Z}' = \hat{Z} \left(\hat{1} - \frac{\alpha_s}{4\pi} \hat{s} \right) \tag{6.93}$$

Using next the defintion of the anomalous dimension matrix (6.35) and the expansion (6.25) we find

$$\hat{\gamma}^{(0)\prime} = \hat{\gamma}^{(0)} \qquad \hat{\gamma}^{(1)\prime} = \hat{\gamma}^{(1)} + [\hat{s}, \hat{\gamma}^{(0)}] + 2\beta_0 \hat{s} \tag{6.94}$$

Let us make a few observations:

- From (6.88) and (6.92) follows the scheme independence of $\hat{r}^T + \hat{J}$. Next, $\vec{A}^{(0)}$ and $\vec{A}^{(1)}$, obtained from the calculation in the full theory, are clearly independent of the renormalization of operators. Consequently, the factor on the right hand side of $\hat{U}^{(0)}$ in $\vec{C}(\mu)$ in (6.32), related to the "upper end" of the evolution, is independent of RS.
- The same is true for $\hat{U}^{(0)}$ as $\hat{\gamma}^{(0)}$ and β_0 are scheme independent.
- \vec{C} depends on RS through \hat{J} to the left of $\hat{U}^{(0)}$. This dependence is compensated for by the corresponding scheme dependence of $\langle \vec{Q} \rangle$ in (6.86).

In the absence of operator mixing the relations between various quantities in two different schemes simplify. Going back to $C_{\pm}(\mu)$ in (6.8) we have

$$\gamma_{\pm}^{(1)'} = \gamma_{\pm}^{(1)} + 2\beta_0 s_{\pm} , \qquad B_{\pm}' = B_{\pm} - s_{\pm} , \qquad J_{\pm}' = J_{\pm} - s_{\pm} , \qquad (6.95)$$

where s_{\pm} are constant numbers analogous to \hat{s} in (6.86).

Recalling

$$J_{\pm} = \frac{1}{2\beta_0} \left(\frac{\beta_1}{\beta_0} \gamma_{\pm}^{(0)} - \gamma_{\pm}^{(1)} \right) \tag{6.96}$$

we verify the scheme independence of $B_{\pm} - J_{\pm}$ in (6.8). Again the scheme dependence of $C_{\pm}(\mu)$ originates in the scheme dependence of J_{\pm} present in the first factor in (6.8).

We should emphasize that the renormalization scheme dependence discussed here refers to the renormalization of operators and should be distinguished from the renormalization scheme dependence of α_s discussed in section 4.7. The issue of the latter scheme dependence in the context of OPE is discussed in detail at the end of section III in [17] and will not be repeated here.

6.8 Renormalization Scale Dependence

A physical amplitude cannot depend on the arbitrary renormalization scale μ . The μ -dependence of the Wilson coefficients has to be canceled by the μ -dependence of the matrix elements $\langle Q_i(\mu) \rangle$. Due to the mixing under renormalization this cancelation may involve simultaneously several operators. Now, whereas $C_i(\mu)$ can be calculated in perturbation theory, this is not the case for the matrix elements $\langle Q_i(\mu) \rangle$. Unfortunately, the existing non-perturbative methods are still insufficient to study the μ -dependence of $\langle Q_i(\mu) \rangle$ and to verify the μ -independence of physical amplitudes in explicit terms. Notable exceptions are inclusive decays of heavy mesons like $B \to X_s \gamma$ and $B \to X_s e^+e^-$, where one can analyze the cancellation of the μ -dependence using perturbative calculations of the relevant matrix elements $\langle Q_i(\mu) \rangle$. We refer to [78, 77] for the full exposition of this issue in $B \to X_s \gamma$ and $B \to X_s e^+e^-$. Here it sufficies to illustrate the cancellation of the μ dependence by considering a toy model in which only a single operator Q is present and its matrix element $\langle Q(\mu) \rangle$ is calculated in perturbation theory.

Let us consider then the amplitude

$$A = \langle \mathcal{H}_{eff} \rangle = \frac{G_F}{\sqrt{2}} \langle Q(\mu_b) \rangle C(\mu_b)$$
 (6.97)

with

$$C(\mu_b) = U(\mu_b, \mu_W)C(\mu_W)$$
 (6.98)

where $\mu_b = \mathcal{O}(m_b)$ and $\mu_W = \mathcal{O}(M_W)$. We want to discuss the cancelation of the μ_b and μ_W dependences in (6.97) and (6.98) in explicit terms.

Beginning with the leading logarithmic approximation we have

$$U(\mu_b, \mu_W) = U^{(0)}(\mu_b, \mu_W) = \left[\frac{\alpha_s(\mu_W)}{\alpha_s(\mu_b)}\right]^{\frac{\gamma^{(0)}}{2\beta_0}}, \qquad C(\mu_W) = 1.$$
 (6.99)

Moreover $\langle Q(\mu_b) \rangle = \langle Q \rangle_{tree}$ carries no μ_b dependence. Consequently in LO the amplitude depends on μ_b and μ_W . Since $\alpha_s(\mu_b) \gg \alpha_s(\mu_W)$, the μ_b -dependence is stronger than the μ_W -dependence. If $\gamma^{(0)}/2\beta_0$ is $\mathcal{O}(1)$ the μ_b -dependence of the resulting amplitudes and branching ratios may be rather disturbing. Known example of such a situation is the strong μ_b -dependence of the branching ratio $Br(B \to X_s \gamma)$ at LO. We will discuss this in detail in section 12.

Let us next include NLO corrections. Now the various entries in (6.99) are generalized as follows:

$$U(\mu_b, \mu_W) = \left(1 + \frac{\alpha_s(\mu_b)}{4\pi}J\right)U^{(0)}(\mu_b, \mu_W)\left(1 - \frac{\alpha_s(\mu_W)}{4\pi}J\right)$$
(6.100)

$$C(\mu_W) = 1 + \frac{\alpha_s(\mu_W)}{4\pi} \left(\frac{\gamma^{(0)}}{2} \ln \frac{\mu_W^2}{M_W^2} + B \right)$$
 (6.101)

$$\langle Q(\mu_b) \rangle = \langle Q \rangle_{tree} \left[1 + \frac{\alpha_s(\mu_b)}{4\pi} \left(\frac{\gamma^{(0)}}{2} \ln \frac{m_b^2}{\mu_b^2} + \tilde{r} \right) \right]$$
 (6.102)

We can now show that the amplitude A in (6.97) is independent of μ_b and μ_W in $\mathcal{O}(\alpha_s)$. Using the following useful formula

$$\frac{\alpha_s(m_1)}{\alpha_s(m_2)} = 1 + \frac{\alpha_s}{4\pi} \beta_0 \ln \frac{m_2^2}{m_1^2}$$
(6.103)

and keeping only logarithmic terms we can rewrite (6.100) as

$$U(\mu_b, \mu_W) = \left(1 + \frac{\alpha_s(\mu_b)}{4\pi} \frac{\gamma^{(0)}}{2} \ln \frac{\mu_b^2}{m_b^2}\right) \left[\frac{\alpha_s(M_W)}{\alpha_s(m_b)}\right]^{\frac{\gamma^{(0)}}{2\beta_0}} \left(1 + \frac{\alpha_s(\mu_W)}{4\pi} \frac{\gamma^{(0)}}{2} \ln \frac{M_W^2}{\mu_W^2}\right)$$
(6.104)

Inserting (6.101), (6.102) and (6.104) into (6.97) we find that μ_b and μ_W dependences cancel at $\mathcal{O}(\alpha_s)$.

This simple example illustrates very clearly the virtue of NLO corrections. They reduce considerably various μ -dependences present in the LO approximation. On the other hand we recover the well known fact that at fixed order of perturbation theory there remain unphysical μ -dependences which are of the order of the neglected higher order contributions. In our simple example the leftover μ_b and μ_W dependences can be investigated numerically by inserting expressions (6.100)–(6.102) into (6.97) and varying μ_b and μ_W say in the ranges $m_b/2 \le \mu_b \le 2m_b$ and $M_W/2 \le \mu_W \le 2M_W$, respectively. By comparing the result of this exercise with an analogous exercise in LO one can on the one hand appreciate the importance of NLO calculations. On the other hand the leftover μ_W and μ_b dependences at NLO give a rough estimate of the theoretical uncertainty due to the truncation of the perturbative series. We will illustrate all this with numerical examples at later stages in these lectures.

The μ -dependences discussed here are related to the renormalization group evolution of the Wilson coefficients from high to low energy scales. This evolution originates in the non-vanishing of the anomalous dimensions of the corresponding operators Q_i . On the other hand, as we have seen in section 4, the nonvanishing of the anomalous dimension γ_m of the mass operator implies the μ -dependence of the quark masses, in particular $m_t(\mu_t)$, $m_b(\mu_b)$ and $m_c(\mu_c)$. These μ -dependences and their cancellation in decay amplitudes will be briefly discussed in section 8 and in the phenomenological sections of these lectures.

6.9 Evanescent Operators

6.9.1 Origin of Evanescent Operators

In evaluating the anomalous dimensions of Q_1 and Q_2 we have used the Greek Method to reduce the complicated Dirac structures given in (6.71)–(6.73) to $\Gamma \otimes \Gamma$. Since we were only

interested in the $1/\varepsilon$ singularity in a one-loop diagram, this reduction has been performed in D=4 dimensions. In the case of two-loop calculations, in which the diagrams of fig. 16 are subdiagrams of the diagrams in fig. 17, this reduction has to be performed in arbitrary D-dimensions. Indeed, in a two-loop diagram, the leading singularity is $1/\varepsilon^2$. The $\mathcal{O}(\varepsilon)$ terms arising from reductions like (6.71)–(6.73) in arbitrary D-dimensions, when multiplied by $1/\varepsilon^2$, will contribute to the $1/\varepsilon$ singularity relevant for the calculation of the two-loop anomalous dimensions.

The question then is how to find $\mathcal{O}(\varepsilon)$ corrections to (6.71)–(6.73). I will follow here the work done almost ten years ago in collaboration with Peter Weisz [48]. In this paper we have proposed a simple method for finding these terms. Although more general methods have been developed subsequently, I still think that our method is most useful for practical purposes. Yet other methods [79, 80, 82, 81], in particular the one of Herrlich and Nierste [80, 82, 81], give a deeper inside into these matters and I will briefly discuss them at the end of this subsection.

The simplest method to find the $\mathcal{O}(\varepsilon)$ terms in question would be to apply the Greek Method in D-dimensions. That is, evaluate (6.75) in D-dimensions in order to determine the coefficient A. For instance in the case of (6.72) we would find $16 - 4\varepsilon$ instead of 16 when using the NDR scheme for γ_5 . This is what has been done in [76]. Yet as pointed out in [48], the mere replacement of 16 in (6.72) by $16 - 4\varepsilon$ with analogous replacements in (6.71) and (6.73) would eventually give incorrect two-loop anomalous dimensions. As demonstrated in [48] the correct procedure is to supplement the Greek Method in D dimensions by the addition of other operators to the r.h.s of (6.71)–(6.73) which vanish in D = 4 dimensions. Such operators are called evanescent operators.

We will explain the role of evanescent operators in the calculation of two-loop anomalous dimensions below. First, however, we want to give the generalizations of (6.71)–(6.73) to $D \neq 4$ dimensions. In the case of the NDR scheme for γ_5 they are given as follows [48]

$$\gamma_{\mu}\gamma_{\rho}\Gamma\gamma^{\rho}\gamma^{\mu}\otimes\Gamma=4(1-2\varepsilon)\Gamma\otimes\Gamma\tag{6.105}$$

$$\Gamma \gamma_{\rho} \gamma_{\mu} \otimes \Gamma \gamma^{\rho} \gamma^{\mu} = 4(4 - \varepsilon) \Gamma \otimes \Gamma + E^{\text{NDR}}$$
(6.106)

$$\Gamma \gamma_{\rho} \gamma_{\mu} \otimes \gamma^{\mu} \gamma^{\rho} \Gamma = 4(1 - 2\varepsilon) \Gamma \otimes \Gamma - E^{\text{NDR}} , \qquad (6.107)$$

where $\mathcal{O}(\varepsilon^2)$ terms have been dropped. Identical results are found for the structures in the second column of the set (6.71)–(6.73). E^{NDR} stands for the evanescent operator given explicitly by

$$E^{\text{NDR}} = \frac{1}{2} [\gamma_{\mu} \gamma_{\rho} \Gamma \gamma^{\rho} \gamma^{\mu} \otimes \Gamma + \Gamma \otimes \gamma_{\mu} \gamma_{\rho} \Gamma \gamma^{\rho} \gamma^{\mu} - \Gamma \gamma_{\rho} \gamma_{\mu} \otimes \gamma^{\mu} \gamma^{\rho} \Gamma - \gamma_{\mu} \gamma_{\rho} \Gamma \otimes \Gamma \gamma^{\rho} \gamma^{\mu}] \quad (6.108)$$

As one can verify using the Greek Method, $E^{\rm NDR}$ vanishes in D=4. In the case of HV and DRED schemes the formulae (6.105)–(6.107) are modified and the evanescent operators have more complicated structures. They can be found in [48]. It should be noted that there is no contribution from evanescent operators to (6.105) in the NDR scheme.

From calculational point of view the insertions of evanescent operators into the relevant diagrams are most efficiently evaluated by defining $E^{\rm NDR}$ simply as the difference between the structures on the l.h.s of (6.106) and (6.107) and the respective terms on the r.h.s involving $\Gamma \otimes \Gamma$. We will demonstrate this explicitly below.

6.9.2 Including Evanescent Operators in the Master Formulae

In subsection 6.4 we have derived the master formulae (6.43) and (6.44) for the computation of two loop anomalous dimensions. This derivation did not take into account the presence of evanescent operators. Therefore in cases in which the contributions of these operators matter, our formulae are strictly speaking incomplete. It is the purpose of the next few pages to correct for it and to derive a procedure for the calculation of two-loop anomalous dimensions which takes into account the evanescent operators. I follow here again my work in collaboration with Peter Weisz [48].

Let us go back to the equation (6.49) which invoves the coefficients (F,G) and (F_C,G_C) in the singularities of the two-loop diagrams and the corresponding counter-diagrams respectively. The evaluation of F and G is still straightforward. Having the final result for a two-loop diagram with complicated Dirac structure one can simply project on the space of physical operators, denoted generically by $\Gamma \otimes \Gamma$, by using the Greek method. In this way one can easily deduce the coefficients of the terms proportional to $\Gamma \otimes \Gamma$. As an example let us consider the Dirac structure resulting from the diagram (f) in fig. 17. Then the projection by means of the Greek Method gives:

$$\Gamma \gamma_{\mu} \gamma_{\nu} \gamma_{\rho} \gamma_{\tau} \otimes \Gamma \gamma^{\mu} \gamma^{\nu} \gamma^{\rho} \gamma^{\tau} = 16 (16 - 14\varepsilon) \Gamma \otimes \Gamma . \tag{6.109}$$

The treatment of counter-diagrams needs more care. After the evaluation of the subdiagrams the $1/\varepsilon$ contributions are multiplied by the structures in (6.105)–(6.107) i.e. they include E operators. Making projection onto $\Gamma \otimes \Gamma$ already at this stage would be incorrect. Indeed inserting E into counter-diagrams of fig. 18, generates back the original operator $\Gamma \otimes \Gamma$ and introduces a correction to G_C and consequently a correction to two-loop anomalous dimension of the original operator. It is precisely this correction which we have neglected in our master formulae. We will now find how our method has to be modified in order to include the effects of E-operators.

For two-loop computation it is sufficient to consider the effects of mixing with the evanescent operators specified in the previous subsubsection. However, higher loop computations would require, in the NDR and HV schemes, consideration of an ever increasing number of independent operators. Thus generally the renormalized operators in generic *D*- dimensions are given by,

$$O_i = (\hat{Z}^{-1})_{ij} O_j^{(0)} . (6.110)$$

Let us consider the case where the set $O_i^{(0)}$ includes the initial bare operators Q_+ and Q_- introduced in section 5 which are expected not to mix under renormalization. In what follows we will denote them by $O_j^{(0)}$ with j=1,2, respectively. All operators O_j with j>2 correspond to evanescent operators. It is convenient to choose the basis such that the operators $O_j^{(0)}$ with j=3,4 respectively are the evanescent operators E^+,E^- . As mentioned above it is, for our purposes, not necessary to specify the basis further nor to give explicit formulae of E^{\pm} .

The renormalization matrix \hat{Z} has a perturbative expansion of the form,

$$\hat{Z} = \hat{1} + \frac{\alpha_s}{4\pi} \hat{Z}^{(1)} + \frac{\alpha_s^2}{4\pi} \hat{Z}^{(2)} + \dots$$
 (6.111)

Only the first four columns and first four rows of these (a priori infinite dimensional) matrices are of interest here. The understanding of the form of the matrices $\hat{Z}^{(1)}$, $\hat{Z}^{(2)}$ is crucial. First we have

$$\hat{Z}^{(1)} = \begin{pmatrix} * & 0 & * & 0 \\ 0 & * & 0 & * \\ * & * & - & - \\ * & * & - & - \end{pmatrix}$$

$$(6.112)$$

where a * denotes non-zero entries and the elements "-" are of no interest to us. In particular we have, $Z_{12}^{(1)} = Z_{21}^{(1)} = 0$. This situation need not, however, continue at higher loops, since in generic *D*-dimensions the bare operators $Q_{\pm}^{(0)}$ do not have definite Fierz transformation properties in the NDR and HV schemes. Hence it can, and in fact does in the NDR and HV schemes, happen that,

$$Z_{12}^{(2)} \neq 0, \qquad Z_{21}^{(2)} \neq 0 \ . \tag{6.113}$$

At the same time, at the one-loop level not only do we have $Z_{31}^{(1)} \neq 0$, $Z_{42}^{(1)} \neq 0$ but to define renormalized evanescent operators which can really be neglected in precisely 4-dimensions one must, in general, take into account the mixing with operators of differing 'naive' Fierz symmetry i.e. it can happen that

$$\hat{Z}_{32}^{(1)} \neq 0, \qquad \hat{Z}_{41}^{(1)} \neq 0 \ . \tag{6.114}$$

We will soon see that this is necessary so that the renormalized operators when restricted to precisely 4-dimensions have the correct Fierz symmetry.

Consider now the renormalization group equations for Green functions containing one renormalized operator O_j insertion, in the regularized D-dimensional theory. They take the standard form but due to the mixing with the evanescent bare operators, an anomalous dimension matrix occurs

$$\hat{\gamma} = \hat{Z}^{-1} \mu \frac{\partial}{\partial \mu} \hat{Z} = \hat{Z}^{-1} (-\varepsilon g + \beta(g)) \frac{\partial}{\partial g} \hat{Z}. \tag{6.115}$$

Expanding $\beta(g)$, $\hat{\gamma}$ and \hat{Z} in powers of the renormalized coupling g as in the previous sections we obtain from (6.115)

$$\hat{\gamma}^{(0)} = -2\varepsilon \hat{Z}^{(1)},\tag{6.116}$$

and at two loops,

$$\hat{\gamma}^{(1)} = -4\varepsilon \hat{Z}^{(2)} - 2\beta_0 \hat{Z}^{(1)} + 2\varepsilon \hat{Z}^{(1)} \hat{Z}^{(1)}. \tag{6.117}$$

Expanding the $\hat{Z}^{(r)}$ in inverse powers of ε ,

$$\hat{Z}^{(1)} = \hat{Z}_0^{(1)} + \frac{1}{\varepsilon} \hat{Z}_1^{(1)} \qquad \hat{Z}^{(2)} = \hat{Z}_0^{(2)} + \frac{1}{\varepsilon} \hat{Z}_1^{(2)} + \frac{1}{\varepsilon^2} \hat{Z}_2^{(2)}$$
(6.118)

and using the fact that the anomalous dimension matrix has a finite limit for $\varepsilon \to 0$ we must have the relation,

$$4\hat{Z}_{2}^{(2)} + 2\beta_0 \hat{Z}_{1}^{(1)} - 2\hat{Z}_{1}^{(1)} \hat{Z}_{1}^{(1)} = 0, (6.119)$$

which has been explicitly checked for the physical \pm submatrix in [48]. We also get

$$\gamma^{(1)} = -4\hat{Z}_1^{(2)} - 2\beta_0 \hat{Z}_0^{(1)} + 2(\hat{Z}_1^{(1)} \hat{Z}_0^{(1)} + \hat{Z}_0^{(1)} \hat{Z}_1^{(1)}). \tag{6.120}$$

Note that we have introduced in (6.118) the nonsingular terms $\hat{Z}_0^{(1)}$ and $\hat{Z}_0^{(2)}$ in order to be able to incorporate the effects of evanescent operators. In particular the presence of the finite renormalization $\hat{Z}_0^{(1)}$ allows in the approach of [48] to remove the finite contributions from evenescent operators to the matrix elements of physical operators. On the other hand, as we will see in a moment, this finite renormalization has an impact on the two-loop anomalous dimensions of the physical operators and consequently on their Wilson coefficients. In this context we note that

$$(\hat{Z}_0^{(1)})_{ij} = 0 \quad (i, j = 1, 2) , \qquad (\hat{Z}_1^{(1)})_{31} = (\hat{Z}_1^{(1)})_{42} = 0.$$
 (6.121)

The latter property assures that $1/\varepsilon^2$ terms are not affected by the evanescent operators at the two-loop level. Finally using the properties (6.112) and (6.121) in (6.120) we find

$$\gamma_{+}^{(1)} = \gamma_{11}^{(1)} = -4(\hat{Z}_{1}^{(2)})_{11} + 2(\hat{Z}_{1}^{(1)})_{13}(\hat{Z}_{0}^{(1)})_{31}$$

$$(6.122)$$

$$\gamma_{-}^{(1)} = \gamma_{22}^{(1)} = -4(\hat{Z}_{1}^{(2)})_{22} + 2(\hat{Z}_{1}^{(1)})_{24}(\hat{Z}_{0}^{(1)})_{42}$$
(6.123)

$$\gamma_{+-}^{(1)} = \gamma_{12}^{(1)} = -4(\hat{Z}_1^{(2)})_{12} + 2(\hat{Z}_1^{(1)})_{13}(\hat{Z}_0^{(1)})_{32} \tag{6.124}$$

$$\gamma_{-+}^{(1)} = \gamma_{21}^{(1)} = -4(\hat{Z}_{1}^{(2)})_{21} + 2(\hat{Z}_{1}^{(1)})_{24}(\hat{Z}_{0}^{(1)})_{41} \tag{6.125}$$

The first term in (6.122) and (6.123) represents (after addition of wave function renormalization) our master formula (6.43) which was obtained neglecting the mixing with the E-operators. The remaining terms reflecting the mixing in question are the corrections we were looking for.

Without these corrections and corresponding corrections in eqs. (6.124) and (6.125) the renormalized operators Q_{\pm} would not have the correct Fierz symmetry and they would mix under renormalization at the two-loop level i.e. $\gamma_{+-}^{(1)}$ and $\gamma_{-+}^{(1)}$ would be non-zero. The inclusion of E-operators restores the Fierz symmetry and removes this mixing i.e. $\gamma_{+-}^{(1)} = \gamma_{-+}^{(1)} = 0$. This is explicitly demonstrated in [48].

Looking at the 'extra contribution' from the evanescent operators, one realizes that it is proportional to the contribution that the counter-terms involving an evanescent operator insertion yield to the computation of $\hat{Z}^{(2)}$. This is precisely what we stated at the beginning of this subsubsection. Note however, that the correction terms in eq. (6.122) and (6.123) are by factor 2 smaller than the corresponding counter-terms (involving Q_{\pm} operators) present in the main terms. In the language of diagrams the result just means that in calculating $\gamma^{(1)}$ the contributions to counter-term diagrams involving an evanescent operator should be multiplied by a factor 1/2.

6.9.3 How to Use the Improved Master Formulae

We can now summarize the improved procedure for the calculation of the two-loop anomalous dimensions.

Step 1:

Calculate the full two-loop diagrams and project the Dirac structures onto the physical operators by means of the Greek Method. This gives in particular the coefficient G in (6.49).

Step 2:

Calculate the usual contribution to the counter-term by taking the relevant subdiagram of a given two-loop diagram, projecting it on to the physical operators $\Gamma \otimes \Gamma$ by means of the Greek Method, inserting the result of this projection into the remaining subdiagram of a given two-loop diagram and projecting the resulting expression again on to the physical operators by means of the Greek Method. This step gives the first part of G_C in (6.49). We will denote it by G_C^a .

Step 3:

Calculate the contribution of the evanescent operator to the counter diagram by simply inserting the difference of the two structures in (6.106) or (6.107), which define E, into the remaining one-loop subdiagram of a given two-loop diagram and project the result onto the physical operators by means of the Greek Method. This step gives the correction to the counter-term. We will denote the coefficient of $1/\varepsilon$ from this part by G_C^b .

Then the two loop anomalous dimension matrix is found by calculating

$$\gamma_{ij}^{(1)} = -4[2a_2\delta_{ij} + (G - G_C^a - \frac{1}{2}G_C^b)_{ij}]. \tag{6.126}$$

Note the factor 1/2 in the evanescent contribution. Formula (6.126) generalizes the master formula (6.43) to include the contributions of evanescent operators.

Let us illustrate this procedure by calculating the contribution of the diagram (f) in fig. 17 and of its counter-diagram (f) in fig. 18 to the two-loop anomalous dimension of the operator with the Dirac structure $\gamma_{\mu}(1-\gamma_5) \otimes \gamma_{\mu}(1-\gamma_5)$. We drop the colour factors in what follows.

Step 1:

Calculating the diagram 17f, using the projection (6.109) and multiplying by two (inclusion of the symmetric counterpart) one finds [48]

$$F = 16$$
, $G = 66$ (6.127)

with (F,G) defined in (6.49)

Step 2:

We first calculate the diagram 16b, as in (6.65). We find

$$\mathcal{D}_b^{(1)} = -\frac{\alpha_s}{4\pi} \left[\frac{1}{4} \frac{1}{\varepsilon} \right] (1 + 2\varepsilon) \left[\Gamma \gamma_\rho \gamma_\mu \otimes \Gamma \gamma^\rho \gamma^\mu \right], \tag{6.128}$$

where $(1+2\varepsilon)$ is an additional correction to the integral (6.62), which has to be kept now.

In order to find G_C^a we first project $\mathcal{D}_b^{(1)}$ on $\Gamma \otimes \Gamma$ by using the Greek Method and keeping only the divergent part:

$$[\mathcal{D}_b^{(1)}]_{div} = -\frac{\alpha_s}{4\pi} \left[\frac{1}{4} \frac{1}{\varepsilon} \right] 16 \ \Gamma \otimes \Gamma \ . \tag{6.129}$$

Inserting this into diagram 18f gives by means of (6.128)

$$I_C^{(a)} = \left(\frac{\alpha_s}{4\pi}\right)^2 \left[\frac{1}{4}\frac{1}{\varepsilon}\right]^2 (1+2\varepsilon) \left[\Gamma\gamma_\rho\gamma_\mu \otimes \Gamma\gamma^\rho\gamma^\mu\right]. \tag{6.130}$$

Using next the projection

$$\Gamma \gamma_{\rho} \gamma_{\mu} \otimes \Gamma \gamma^{\rho} \gamma^{\mu} = 4(4 - \varepsilon) \Gamma \otimes \Gamma \tag{6.131}$$

and including the symmetric counterpart of 18f gives

$$I_C^{(a)} = \left(\frac{\alpha_s}{4\pi}\right)^2 \left[\frac{32}{\varepsilon^2} + \frac{56}{\varepsilon}\right] \Gamma \otimes \Gamma \tag{6.132}$$

where finite terms have been dropped. Consequently

$$F_C^a = 32 \; , \qquad G_C^a = 56$$
 (6.133)

Step 3:

In order to calculate G_C^b we take first the evanescent part of $\mathcal{D}_b^{(1)}$. Dropping the $\mathcal{O}(\varepsilon)$ from the integral (it contributes only to finite parts of $I_C^{(b)}$) we have

$$[\mathcal{D}_b^{(1)}]_{ev} = -\frac{\alpha_s}{4\pi} \left[\frac{1}{4} \frac{1}{\varepsilon} \right] \left[\Gamma \gamma_\rho \gamma_\mu \otimes \Gamma \gamma^\rho \gamma^\mu - 4(4 - \varepsilon) \Gamma \otimes \Gamma \right] , \qquad (6.134)$$

where we have used (6.106). Inserting $[\mathcal{D}_b^{(1)}]_{ev}$ into the diagram 18f and multiplying by two for the symmetric counterpart we get

$$I_C^{(b)} = 2\left(\frac{\alpha_s}{4\pi}\right)^2 \left[\frac{1}{4}\frac{1}{\varepsilon}\right]^2 (1+2\varepsilon) \left[\Gamma\gamma_\rho\gamma_\mu\gamma_\nu\gamma_\tau \otimes \Gamma\gamma^\rho\gamma^\mu\gamma^\nu\gamma^\tau - 4(4-\varepsilon)\Gamma\gamma_\nu\gamma_\tau \otimes \Gamma\gamma^\nu\gamma^\tau\right] . \quad (6.135)$$

Projecting on $\Gamma \otimes \Gamma$ by means of (6.109) and (6.106) we obtain

$$I_C^{(b)} = \left(\frac{\alpha_s}{4\pi}\right)^2 \left[-\frac{12}{\varepsilon}\right] \Gamma \otimes \Gamma \tag{6.136}$$

and

$$F_C^b = 0$$
, $G_C^b = -12$. (6.137)

We observe that the $1/\varepsilon^2$ singularity is unaffected by the evanescent contribution. We can now calculate the relevant combination in (6.126) to be

$$G - G_C^a - \frac{1}{2}G_C^b = 16 , (6.138)$$

which is precisely the $1/\varepsilon$ singularity given in table 3 (diagram 5) of [48]. Also the $1/\varepsilon^2$ singularity $F - F_C^a = -16$ agrees with [48] and $F_C = 2F$ as promised after (6.49). Great! Everything works! I hope you are now motivated to calculate the remaining 26 two-loop diagrams and corresponding counter-diagrams necessary to reproduce the matrix (6.52). Actually the calculation of counter-diagrams is rather straightforward. The difficult part is the calculation of the two-loop diagrams, like the ones in fig. 17.

6.9.4 Evanescent Scheme Dependences

The definition of evanescent operators is not unique as stressed by Dugan and Grinstein [79] and in particular by Herrlich and Nierste [80]. As an example consider the Dirac structure on the l.h.s of (6.106). Following [80] we can generalize this formula to

$$\Gamma \gamma_{\rho} \gamma_{\mu} \otimes \Gamma \gamma^{\rho} \gamma^{\mu} = (16 + a\varepsilon)\Gamma \otimes \Gamma + E^{\text{NDR}}(a)$$
(6.139)

where "a" is an arbitrary parameter, which defines the evanescent operator $E^{\text{NDR}}(a)$. For a = -4 the definition in (6.106) is chosen.

Now as the preceding discussion has shown, the presence of evanescent operators influences the two-loop anomalous dimensions of physical operators $\Gamma \otimes \Gamma$. Consequently as emphasized in [80], the arbitrariness in the definition of the evanescent operators translates into an additional scheme dependence of two-loop anomalous dimensions which can be effectively parametrized by "a" in (6.139). Therefore, when giving the results for two-loop anomalous dimensions, it is not sufficient to state simply that they correspond to NDR, HV or any other renormalization scheme. One has to specify in addition the definition of evanescent operators. This is essential as this scheme dependence of two-loop anomalous dimensions can only be cancelled in physical amplitudes by the corresponding scheme dependences present in the matching conditions (for instance B_{\pm}) at scales $\mathcal{O}(M_{\rm W})$ and by the one present in the finite matrix elements of operators at scales $\mathcal{O}(\mu)$.

This means that the treatment of evanescent operators in the process of matching and in the calculation of matrix elements of operators at scales $\mathcal{O}(\mu)$ must be consistent with the one used in the calculation of two-loop anomalous dimensions. This issue is elaborated at length in [80, 81, 82] and in the appendix B of [100]. See also the appendix of [102].

There are two virtues of the definition of evanescent operators proposed by Weisz and myself and discussed in detail above:

- The evanescent operators defined in [48] influence only two-loop anomalous dimensions. By definition they do not contribute to the matching and to the finite corrections to matrix elements at scales $\mathcal{O}(\mu)$. They are simply subtracted away in the process of renormalization.
- As a consequence of this, the Fierz symmetry is preserved separately in two-loop anomalous dimensions, matching conditions and matrix elements at scales $\mathcal{O}(\mu)$.

The second property assures that the operators Q_+ and Q_- do not mix under renormalization separately in two-loop anomalous dimensions, in the matching conditions and in the matrix elements so that objects like B_{+-} , B_{-+} , $\gamma_{+-}^{(1)}$, $\gamma_{-+}^{(1)}$ are assured to vanish in this scheme. In other schemes (see [79]) this is not the case and the Fierz symmetry is only recovered after the two-loop anomalous dimensions are combined with the matching conditions which makes the calculations unnecessarily rather involved.

Now comes the most important message of this subsection: most of the existing NLO calculations adopt the definition of evanescent operators in [48] and all the two-loop anomalous dimensions and matching conditions given in these lectures and in the review [17] correspond to this definition.

7 The Effective $\Delta F = 1$ Hamiltonian: Current-Current Operators

7.1 Basic Formalism

Let us summarize the results for the coefficients $C_{1,2}(\mu)$ of the current-current operators $Q_{1,2}$ discussed extensively in the previous two sections and let us evaluate them for the cases of $\Delta B = 1$, $\Delta C = 1$ and $\Delta S = 1$ decays.

To be specific let us consider

$$Q_1 = (\bar{b}_{\alpha}c_{\beta})_{V-A}(\bar{u}_{\beta}d_{\alpha})_{V-A} \qquad Q_2 = (\bar{b}_{\alpha}c_{\alpha})_{V-A}(\bar{u}_{\beta}d_{\beta})_{V-A} \tag{7.1}$$

$$Q_1 = (\bar{s}_{\alpha}c_{\beta})_{V-A}(\bar{u}_{\beta}d_{\alpha})_{V-A} \qquad Q_2 = (\bar{s}_{\alpha}c_{\alpha})_{V-A}(\bar{u}_{\beta}d_{\beta})_{V-A}$$

$$(7.2)$$

$$Q_1 = (\bar{s}_{\alpha} u_{\beta})_{V-A} (\bar{u}_{\beta} d_{\alpha})_{V-A} \qquad Q_2 = (\bar{s}_{\alpha} u_{\alpha})_{V-A} (\bar{u}_{\beta} d_{\beta})_{V-A}$$

$$(7.3)$$

for $\Delta B = 1$, $\Delta C = 1$ and $\Delta S = 1$ decays respectively.

The corresponding effective Hamiltonians are given by

$$H_{eff}(\Delta B = 1) = \frac{G_F}{\sqrt{2}} V_{cb}^* V_{ud} [C_1(\mu)Q_1 + C_2(\mu)Q_2] \qquad (\mu = O(m_b))$$
 (7.4)

$$H_{eff}(\Delta C = 1) = \frac{G_F}{\sqrt{2}} V_{cs}^* V_{ud} [C_1(\mu)Q_1 + C_2(\mu)Q_2] \qquad (\mu = O(m_c))$$
 (7.5)

$$H_{eff}(\Delta S = 1) = \frac{G_F}{\sqrt{2}} V_{us}^* V_{ud} [C_1(\mu) Q_1 + C_2(\mu) Q_2] \qquad (\mu = O(1 \,\text{GeV}))$$
 (7.6)

In subsequent sections the Hamiltonian (7.6) will be generalized to include also penguin operators. The Hamiltonians (7.4) and (7.5) having operators built out of four different flavours are unaffected by penguin contributions. On the other hand there are other $\Delta B = 1$ and $\Delta C = 1$ Hamiltonians which contain important penguin contributions. A well known example is the Hamiltonian for $B \to X_s \gamma$ decay which will be analyzed in detail in section 12. However, the inclusion of penguin operators does not change the Wilson coefficients $C_{1,2}(\mu)$.

In a numerical analysis it is convenient to work with the operators Q_{\pm} and their coefficients z_{\pm} defined by

$$Q_{\pm} = \frac{1}{2}(Q_2 \pm Q_1)$$
 $z_{\pm} = C_{\pm} = C_2 \pm C_1$. (7.7)

 Q_{+} and Q_{-} do not mix under renormalization and

$$z_{\pm}(\mu) = \left[1 + \frac{\alpha_s(\mu)}{4\pi}J_{\pm}\right] \left[\frac{\alpha_s(M_W)}{\alpha_s(\mu)}\right]^{d_{\pm}} \left[1 + \frac{\alpha_s(M_W)}{4\pi}(B_{\pm} - J_{\pm})\right]$$
(7.8)

with

$$J_{\pm} = \frac{d_{\pm}}{\beta_0} \beta_1 - \frac{\gamma_{\pm}^{(1)}}{2\beta_0}, \qquad d_{\pm} = \frac{\gamma_{\pm}^{(0)}}{2\beta_0}, \qquad (7.9)$$

$$\gamma_{\pm}^{(0)} = \pm 12 \frac{N \mp 1}{2N} , \qquad (7.10)$$

$$\gamma_{\pm}^{(1)} = \frac{N \mp 1}{2N} \left[-21 \pm \frac{57}{N} \mp \frac{19}{3} N \pm \frac{4}{3} f - 2\beta_0 \kappa_{\pm} \right] , \qquad (7.11)$$

$$B_{\pm} = \frac{N \mp 1}{2N} \left[\pm 11 + \kappa_{\pm} \right] . \tag{7.12}$$

The parameters κ_{\pm} , introduced in [89], distinguish between various renormalization schemes. In particular

$$\kappa_{\pm} = \begin{cases}
0 & \text{NDR} \\
\mp 4 & \text{HV} \\
\mp 6 - 3 & \text{DRED}
\end{cases}$$
(7.13)

We recall that $B_{\pm} - J_{\pm}$ is scheme independent and the scheme dependence of $z_{\pm}(\mu)$ originates entirely from the scheme dependence of J_{\pm} at the lower end of the evolution in (7.8). Using N=3 one has

$$J_{\pm} = (J_{\pm})_{\text{NDR}} + \frac{3 \mp 1}{6} \kappa_{\pm} = (J_{\pm})_{\text{NDR}} \pm \frac{\gamma_{\pm}^{(0)}}{12} \kappa_{\pm} . \tag{7.14}$$

In order to exhibit the μ dependence at the same footing as the scheme dependence, it is useful to rewrite (7.8) in the case of B-decays as follows:

$$z_{\pm}(\mu) = \left[1 + \frac{\alpha_s(m_b)}{4\pi}\tilde{J}_{\pm}(\mu)\right] \left[\frac{\alpha_s(M_W)}{\alpha_s(m_b)}\right]^{d_{\pm}} \left[1 + \frac{\alpha_s(M_W)}{4\pi}(B_{\pm} - J_{\pm})\right]$$
(7.15)

with

$$\tilde{J}_{\pm}(\mu) = (J_{\pm})_{NDR} \pm \frac{\gamma_{\pm}^{(0)}}{12} \kappa_{\pm} + \frac{\gamma_{\pm}^{(0)}}{2} \ln(\frac{\mu^2}{m_b^2})$$
 (7.16)

summarizing both the renormalization scheme dependence and the μ -dependence. Note that in the first parenthesis in (7.15) we have set $\alpha_s(\mu) = \alpha_s(m_b)$ as the difference in the scales in this correction is still of a higher order. We also note that the scheme and the μ -dependent terms are both proportional to $\gamma_{\pm}^{(0)}$. This implies that the change of renormalization scheme can be compensated by the change in μ . From (7.16) we find generally

$$\mu_i^{\pm} = \mu_{NDR} \exp\left(\mp \frac{\kappa_{\pm}^{(i)}}{12}\right) \tag{7.17}$$

where i denotes a given scheme. From (7.13) we have then

$$\mu_{HV} = \mu_{NDR} \exp\left(\frac{1}{3}\right) \qquad \mu_{DRED}^{\pm} = \mu_{NDR} \exp\left(\frac{2\pm 1}{4}\right) \tag{7.18}$$

Evidently whereas the change in μ relating HV and NDR is the same for z_+ and z_- and consequently for $C_i(\mu)$, the relation between NDR and DRED is more involved. In any case μ_{HV} and μ_{DRED}^{\pm} are larger than μ_{NDR} . This discussion shows that a meaningful analysis of the μ dependence of $C_i(\mu)$ can only be made simultaneously with the analysis of the scheme dependence. This observation will be important for the analysis of two-body B-decays in section 9.

7.2 Numerical Results for B-Decays

For B-decays we have simply:

$$C_1(\mu) = \frac{z_+(\mu) - z_-(\mu)}{2}$$
 $C_2(\mu) = \frac{z_+(\mu) + z_-(\mu)}{2}$ (7.19)

We set f = 5 in the formulae above and use the two-loop $\alpha_s(\mu)$ of (4.70) with $\Lambda_{\overline{\text{MS}}}^{(5)}$. The results for LO and NLO in NDR and HV schemes are shown in the table 4.

	$\Lambda_{\overline{ ext{MS}}}^{(5)} = 160 ext{MeV}$			$\Lambda_{\overline{ ext{MS}}}^{(5)} = 225 ext{MeV}$			$\Lambda_{\overline{ ext{MS}}}^{(5)} = 290 ext{MeV}$		
Scheme	LO	NDR	HV	LO	NDR	HV	LO	NDR	HV
C_1	-0.270	-0.169	-0.206	-0.295	-0.184	-0.226	-0.317	-0.198	-0.243
C_2	1.119	1.071	1.089	1.132	1.078	1.100	1.144	1.085	1.109

Table 4: The coefficients $C_{1,2}(\mu)$ for B-decays at $\mu = \overline{m}_b(m_b) = 4.40 \,\text{GeV}$

7.3 Numerical Results for D- and K-Decays

In the case of D-decays and K-decays the relevant scales are $\mu = \mathcal{O}(m_c)$ and $\mu = \mathcal{O}(1\,\mathrm{GeV})$, respectively and consequently a more complicated formula with thresholds should in principle be used. Yet it is possible to use the following trick which avoids these complications. We can simply use the master formulae given above with $\Lambda_{\overline{\mathrm{MS}}}^{(5)}$ replaced by $\Lambda_{\overline{\mathrm{MS}}}^{(4)}$ and an "effective" number of active flavours f=4.15. The latter effective value for f allows to obtain an agreement with the exact results to better than 1.5%. The results are shown in the table 5 for $\Lambda_{\overline{\mathrm{MS}}}^{(4)}=325\,\mathrm{MeV}$. The calculation for different values of $\Lambda_{\overline{\mathrm{MS}}}^{(4)}$ is left as a homework problem.

Table 5: $C_{1,2}(\mu)$ for K-decays and D-decays and $\Lambda_{\overline{\rm MS}}^{(4)}=325\,{\rm MeV}.$

		$C_1(\mu)$		$C_2(\mu)$			
$\mu [{\rm GeV}]$	LO	NDR	HV	LO	NDR	HV	
1.00	-0.742	-0.510	-0.631	1.422	1.275	1.358	
1.25	-0.636	-0.430	-0.523	1.346	1.221	1.282	
1.50	-0.565	-0.378	-0.457	1.298	1.188	1.237	
1.75	-0.514	-0.340	-0.410	1.264	1.165	1.207	
2.00	-0.475	-0.311	-0.375	1.239	1.148	1.185	

7.4 Discussion

Let us make a few remarks:

- The scheme dependence of the Wilson coefficients is sizable. In particular for C_1 which vanishes in the absence of QCD corrections.
- The differences between LO and NLO results in the case of C_1 are large showing the importance of NLO corrections. Roughly half of this difference is due to the fact that for the chosen values of $\Lambda_{\overline{\rm MS}}$ one has $\alpha_s^{(LO)}(M_Z)=0.135$ to be compared with $\alpha_s(M_Z)=0.118\pm0.005$.
- It is important to keep in mind that these features are specific to the schemes chosen.

We will use the results of this section in the analysis of two-body non-leptonic B-decays in section 9.

8 Generalizations

8.1 Preliminaries

In section 5.7 we have made a strategy for the generalizations of the simple LO effective Hamiltonian involving only the current-current operators Q_1 and Q_2 . In the previous two sections we have extended the analysis of section 5 by calculating NLO corrections to the Wilson coefficients $C_{1,2}(\mu)$. The goal of the present section is a description of further generalizations of weak effective Hamiltonians which will include other operators mentioned in Section 3. Except for an explicit calculation of one-loop anomalous dimensions of penguin operators, we will mainly discuss new features skeeping often derivations. Indeed, this section should be considered as a guide to the weak effective Hamiltonians. Further details can be found in the cited literature and in the phenomenological sections of these lectures.

8.2
$$K^+ \rightarrow \pi^+ \nu \bar{\nu}$$

Let us begin with the rare decay $K^+ \to \pi^+ \nu \bar{\nu}$. It proceeds through penguin and box diagrams with internal charm and top exchanges. The internal u-quark contribution is needed only for the GIM mechanism but can otherwise be neglected. Let us concentrate here on the internal top contribution. We will briefly discuss the charm contribution in sections 8.9 and 13.

The relevant effective Hamiltonian without QCD corrections has been constructed already in Section 3. It is given in (3.47). Since the relevant operator is a product of a quark and a leptonic current, its anomalous dimension vanishes. This simplifies the QCD analysis

considerably. There is no renormalization group evolution from high-energy scales to low energy scales and the calculation of the relevant Wilson coefficient amounts to the matching of the full to the effective theory at scales $\mathcal{O}(M_{\rm W},m_{\rm t})$. The most difficult task here is the calculation of gluon corrections to the Z^0 -penguin and the relevant box diagram. It is a two-loop calculation with massive W^{\pm} and top quark propagators. In order to keep gauge invariance, fictitious ϕ^{\pm} Goldstone bosons, in place of W^{\pm} propagators have to be also included. Moreover, looking back at the Feynman rules (2.14) and (2.15), it is evident that for a heavy top quark these are precisely the dominant contributions. Examples of contributing two-loop diagrams are given in fig. 19. This calculation, involving roughly 40 two-loop diagrams, is rather tedious, but can be done analytically due to the fact that the presence of heavy internal propagators allows to set the external quark momenta and masses to zero. The infrared divergences can then be regulated dimensionally.

The effect of the inclusion of QCD corrections to the effective Hamiltonian (3.47) amounts to the replacement of the function $X_0(x_t)$ by a corrected function $X(x_t)$ given below. We have then

$$\mathcal{H}_{\text{eff}} = \frac{G_{\text{F}}}{\sqrt{2}} \frac{\alpha}{2\pi \sin^2 \Theta_{\text{W}}} \sum_{l=e,\mu,\tau} V_{ts}^* V_{td} X(x_t) \ Q(\nu \bar{\nu})$$
(8.1)

where

$$Q(\nu\bar{\nu}) = (\bar{s}d)_{V-A}(\bar{\nu}_l\nu_l)_{V-A} \tag{8.2}$$

and

$$X(x_t) = X_0(x_t) + \frac{\alpha_s}{4\pi} X_1(x_t)$$
 (8.3)

with

$$x_t = \frac{\overline{m}_t^2(\mu_t)}{M_W^2}, \qquad \mu_t = \mathcal{O}(m_t). \tag{8.4}$$

Here $\overline{m}_{t}^{2}(\mu_{t})$ is the running top quark mass defined at the scale μ_{t} . Next [45]

$$X_0(x_t) = \frac{x_t}{8} \left[\frac{x_t + 2}{x_t - 1} + \frac{3x_t - 6}{(x_t - 1)^2} \ln x_t \right]$$
 (8.5)

is the leading contribution considered before and [65, 98]

$$X_{1}(x_{t}) = -\frac{23x_{t} + 5x_{t}^{2} - 4x_{t}^{3}}{3(1 - x_{t})^{2}} + \frac{x_{t} - 11x_{t}^{2} + x_{t}^{3} + x_{t}^{4}}{(1 - x_{t})^{3}} \ln x_{t}$$

$$+ \frac{8x_{t} + 4x_{t}^{2} + x_{t}^{3} - x_{t}^{4}}{2(1 - x_{t})^{3}} \ln^{2} x_{t} - \frac{4x_{t} - x_{t}^{3}}{(1 - x_{t})^{2}} L_{2}(1 - x_{t})$$

$$+ \gamma_{m}^{(0)} x_{t} \frac{\partial X_{0}(x_{t})}{\partial x_{t}} \ln \frac{\mu_{t}^{2}}{M_{W}^{2}}$$

$$(8.6)$$

with

$$L_2(1-x) = \int_1^x dt \frac{\ln t}{1-t}.$$
 (8.7)

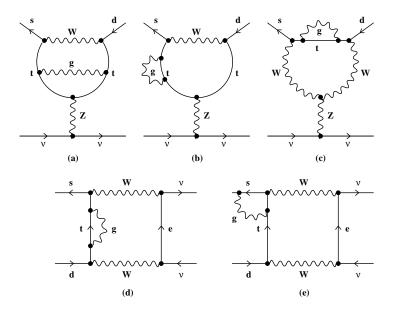


Figure 19: Examples of two-loop diagrams contributing to $K^+ \to \pi^+ \nu \bar{\nu}$.

results from the two-loop calculation discussed briefly above.

The μ_t -dependence of the last term in (8.6) cancels at $\mathcal{O}(\alpha_s)$ the μ_t -dependence of the leading term $X_0(x_t(\mu_t))$. Varying μ_t in the range 100 GeV $\leq \mu_t \leq 300$ GeV gives a theoretical uncertainty of $\pm 10\%$ in Br($K^+ \to \pi^+ \nu \bar{\nu}$) at LO which is reduced to $\pm 1\%$ when the QCD correction in (8.3) is included. For $\mu_t = m_t$, the complete function $X(x_t)$ can be written as

$$X(x_t) = \eta_X \cdot X_0(x_t), \qquad \eta_X = 0.985,$$
 (8.8)

with the QCD factor η_X practically independent of m_t and $\Lambda_{\overline{MS}}$. Thus for this choice of μ_t the QCD corrections turn out to be small. They are larger for $\mu_t = \mathcal{O}(M_W)$ without changing the final result. That is, the contributions of X_0 and X_1 to the full function X depend on the particular value of μ_t but the full result is practically independent of μ_t after the NLO corrections have been included. We will return to the phenomenological aspects of this decay in section 13, where we will derive the expression for $Br(K^+ \to \pi^+ \nu \bar{\nu})$.

8.3 $B_d^0 - \bar{B}_d^0$ Mixing

8.3.1 Preliminaries

The next generalization on our list is the $B_d^0 - \bar{B}_d^0$ mixing, which proceeds to an excellent approximation only through box diagrams with internal top quark exchanges. The contributions of the internal u and c quarks are only needed for GIM mechanism. Otherwise they can be set to zero due to the smallness of m_u and m_c relative to m_t . The effective Hamiltonian

 $\mathcal{H}_{\text{eff}}(\Delta B=2)$ for $B_d^0 - \bar{B}_d^0$ mixing, relevant for scales $\mu_b = \mathcal{O}(m_b)$ is given by

$$\mathcal{H}_{\text{eff}}^{\Delta B=2} = \frac{G_{\text{F}}^2}{16\pi^2} M_W^2 \left(V_{tb}^* V_{td}\right)^2 C_Q(\mu_b) Q(\Delta B = 2) + h.c. \tag{8.9}$$

where

$$Q(\Delta B = 2) = (\bar{b}_{\alpha} d_{\alpha})_{V-A} (\bar{b}_{\beta} d_{\beta})_{V-A} . \tag{8.10}$$

(8.9) can be easily derived by using the rules of Section 3. In the absence of QCD corrections one finds

$$C_Q = S_0(x_t) \tag{8.11}$$

where

$$S_0(x_t) = \frac{4x_t - 11x_t^2 + x_t^3}{4(1 - x_t)^2} - \frac{3x_t^3 \ln x_t}{2(1 - x_t)^3}$$
(8.12)

is a function analogous to $X_0(x_t)$ in (8.5).

8.3.2 LO Analysis

The study of QCD corrections to $B_d^0 - \bar{B}_d^0$ mixing is more involved than in the case of the top contribution to $K^+ \to \pi^+ \nu \bar{\nu}$ as the operator $Q(\Delta B=2)$, in contrast to $Q(\nu \bar{\nu})$, carries a non-vanishing anomalous dimension γ_Q . This anomalous dimension can be shown to be equal to the anomalous dimension of the operator Q_+ considered in the previous sections. Indeed, in the case at hand the Q_1 and Q_2 operators are given by

$$Q_1(\Delta B = 2) = (\bar{b}_{\alpha}d_{\beta})_{V-A}(\bar{b}_{\beta}d_{\alpha})_{V-A}, \qquad Q_2(\Delta B = 2) = Q(\Delta B = 2).$$
 (8.13)

and using Fierz symmetry we have

$$Q_1(\Delta B = 2) = Q(\Delta B = 2).$$
 (8.14)

Consequently

$$Q_{+} = \frac{Q_{2} + Q_{1}}{2} = Q(\Delta B = 2) , \qquad Q_{-} = \frac{Q_{2} - Q_{1}}{2} = 0$$
 (8.15)

and $\gamma_Q = \gamma_+$. In particular, in LO

$$\gamma_Q = \gamma_Q^{(0)} \frac{\alpha_s}{4\pi} , \qquad \gamma_Q^{(0)} = \gamma_+^{(0)} = 4 .$$
 (8.16)

This in turn implies

$$C_Q(\mu_b) = U^{(0)}(\mu_b, \mu_W)C_Q(\mu_W),$$
 (8.17)

$$U^{(0)}(\mu_b, \mu_W) = \left[\frac{\alpha_s(\mu_W)}{\alpha_s(\mu_b)}\right]^{\frac{\gamma_Q^{(0)}}{2\beta_0}}, \qquad C_Q(\mu_W) = S_0(x_t)$$
(8.18)

where $\beta_0 = 23/3$. Thus in LO the Wilson coefficient $C_Q(\mu_b)$ is given by

$$C_Q(\mu_b) = \left[\frac{\alpha_s(\mu_W)}{\alpha_s(\mu_b)}\right]^{6/23} S_0(x_t). \tag{8.19}$$

Before going to the NLO case, let us calculate the matrix element

$$\langle \bar{B}^0 | \mathcal{H}_{\text{eff}}^{\Delta B=2} | B^0 \rangle = \frac{G_F^2}{16\pi^2} M_W^2 \left(V_{tb}^* V_{td} \right)^2 C_Q(\mu_b) \langle \bar{B}^0 | Q(\Delta B = 2)(\mu_b) | B^0 \rangle \tag{8.20}$$

where

$$\langle \bar{B}^0 | Q(\Delta B = 2)(\mu_b) | B^0 \rangle \equiv \frac{8}{3} B_B(\mu_b) F_B^2 m_B^2$$
 (8.21)

and F_B is the B-meson decay constant. (8.20) will be an important quantity in Section 10.

The μ_b -dependent parameter $B_B(\mu_b)$ parametrizes the non-perturbative effects in the hadronic matrix element of the operator $Q(\Delta B)$. Its value is $\mathcal{O}(1)$. In phenomanological applications it is useful to define two μ_b -independent quantities:

$$\eta_B^{(0)} = \left[\alpha_s(\mu_W)\right]^{6/23}, \qquad \hat{B}_B^{(0)} = B_B(\mu_b) \left[\alpha_s(\mu_b)\right]^{-6/23}.$$
(8.22)

Then:

$$\langle \bar{B}^0 | \mathcal{H}_{\text{eff}}^{\Delta B=2} | B^0 \rangle = \frac{G_F^2}{6\pi^2} M_W^2 \left(V_{tb}^* V_{td} \right)^2 \hat{B}_B^{(0)} F_B^2 m_B^2 \eta_B^{(0)} S_0(x_t)$$
 (8.23)

We note that there is a left over μ_W -dependence in η_B and μ_t dependence in $S_0(x_t(\mu_t))$. In order to reduce these dependences we have to include NLO corrections.

8.3.3 NLO Analysis

Applying the standard procedure of matching one finds [90]

$$C_Q(\mu_W) = S_0(x_t) + \frac{\alpha_s(\mu_W)}{4\pi} \left[S_1(x_t) + F(\mu_W, \mu_t) S_0(x_t) + B_t S_0(x_t) \right]$$
(8.24)

$$F(\mu_W, \mu_t) = \frac{\gamma_Q^{(0)}}{2} \ln \frac{\mu_W^2}{M_W^2} + \gamma_m^{(0)} x_t \frac{\partial S_0(x_t)}{\partial x_t} \ln \frac{\mu_t^2}{M_W^2}$$
(8.25)

$$B_t = 5\frac{N-1}{2N} + 3\frac{N^2 - 1}{2N} \qquad (NDR)$$
 (8.26)

$$S_1(x_t) = \text{Complicated Function}$$
 (8.27)

The function $S_1(x_t)$ given in (XII.12) of ref. [17] is a result of a two-loop calculation [90] involving gluon corrections to the box diagrams. Typical diagrams are shown in fig. 20. The interested reader should consult the detailed analysis in [90] where a spectacular cancellation of infrared divergences and gauge dependences present in the diagrams of the full theory is achieved by the corresponding diagrams in the effective theory.

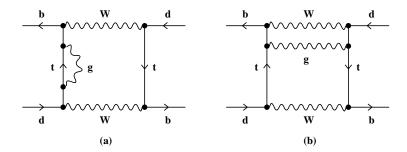


Figure 20: Examples of two-loop diagrams contributing to $B_d^0 - \bar{B}_d^0$ mixing.

The second log in $F(\mu_W, \mu_t)$ cancels the μ_t dependence in $S_0(x_t)$ in analogy to a similar logarithm in (8.6). The first logarithm in (8.25) cancels the μ_W dependence present in $U^{(0)}(\mu_b, \mu_W)$. For $\mu_W = \mu_t$ the formulae given above reduce to the ones given in [90] and [17]. But as discussed already there, μ_W and μ_t can differ from each other and for pedagogical reasons we do not put them equal here.

The NLO evolution function is given simply by

$$U(\mu_b, \mu_W) = \left[1 + \frac{\alpha_s(\mu_b)}{4\pi} J_5 \right] U^{(0)}(\mu_b, \mu_W) \left[1 - \frac{\alpha_s(\mu_W)}{4\pi} J_5 \right]$$
(8.28)

with

$$J_5 = J_+ = 1.627$$
 (NDR, f = 5). (8.29)

We can now define μ_b and μ_W independent quantities at the NLO level, which moreover are renormalization scheme independent:

$$\eta_B = \left[\alpha_s(\mu_W)\right]^{6/23} \left[1 + \frac{\alpha_s(\mu_W)}{4\pi} \left(\frac{S_1(x_t)}{S_0(x_t)} + F(\mu_W, \mu_t) + B_t - J_5\right)\right],\tag{8.30}$$

$$\hat{B}_B = B_B(\mu_b) \left[\alpha_s^{(5)}(\mu_b) \right]^{-6/23} \left[1 + \frac{\alpha_s^{(5)}(\mu_b)}{4\pi} J_5 \right] . \tag{8.31}$$

Then:

$$\langle \bar{B}^0 | \mathcal{H}_{\text{eff}}^{\Delta B=2} | B^0 \rangle = \frac{G_F^2}{6\pi^2} M_W^2 \left(V_{tb}^* V_{td} \right)^2 \hat{B}_B F_B^2 m_B^2 \eta_B S_0(x_t) . \tag{8.32}$$

It should be noted that both η_B and $S_0(x_t)$ depend on μ_t but the product $\eta_B \cdot S_0(x_t)$ is μ_t -independent in $\mathcal{O}(\alpha_s)$ as the second logarithm in (8.25) cancels the μ_t dependence in $S_0(x_t(\mu_t))$. If one varies μ_t in the range 100 GeV $\leq \mu_t \leq$ 300 GeV, the μ_t dependence of $\langle \bar{B}^0 | \mathcal{H}_{\text{eff}}^{\Delta B=2} | B^0 \rangle$ amounts in LO to $\pm 9\%$ and is reduced to $\pm 1\%$ in NLO.

It is customary to evaluate η_B at $\mu_t = \mu_W = m_t$, then practically η_B is independent of $m_{\rm t}$ and the full $m_{\rm t}$ dependence of $B_d^0 - \bar{B}_d^0$ mixing resides in $S_0(x_t)$ with $\overline{m}_{\rm t}(m_{\rm t})$. Then for $\alpha_s(M_{\rm Z}) = 0.118 \pm 0.03$ one has

$$\eta_B = 0.55 \pm 0.01 \tag{8.33}$$

where the error includes also the leftover scale uncertainties, which can only be reduced by calculating $\mathcal{O}(\alpha_s^2)$ corrections.

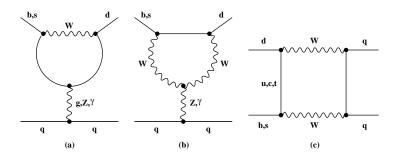


Figure 21: One loop penguin and box diagrams in the full theory.

8.4 QCD Penguins

8.4.1 Operators

The next generalization involves the inclusion of penguin operators in $\Delta F = 1$ transitions. They originate in the QCD penguin diagram (a) of fig. 21. Evaluating this diagram one can clearly see that there are two colour structures as in the case of Q_1 and Q_2 . They follow simply from the decomposition

$$T^{a}_{\alpha\beta}T^{a}_{\gamma\rho} = -\frac{1}{2N}\delta_{\alpha\beta}\delta_{\gamma\delta} + \frac{1}{2}\delta_{\alpha\delta}\delta_{\gamma\beta}$$
 (8.34)

The upper effective FCNC vertex in the penguin diagram has V-A structure as seen in the rules of Section 3. The lower vertex is vectorial (V). It turns out however that the renormalization of the operators with the Dirac structure $(V-A)\otimes V$ requires the introduction of two new operators with the Dirac structure $(V-A)\otimes A$ and the colour structures as the $(V-A)\otimes V$ operators. Indeed, when the $(V-A)\otimes V$ operators are inserted into the Green functions of fig. 22 two new operators in question are generated. The full set of operators which closes under renormalization consists then of two current-current operators (Q_1,Q_2) and the four penguin operators. It is customery to work with the $(V-A)\otimes (V-A)$ and $(V-A)\otimes (V+A)$ penguin operators rather than with the $(V-A)\otimes V$ and $(V-A)\otimes A$ structures. Then the basis of the operators necessary for the description of $\Delta B=1$ decays with $\Delta S=1$ is given (in the limit $\alpha\equiv\alpha_{QED}=0$) as follows:

Current-Current:

$$Q_1 = (\bar{c}_{\alpha}b_{\beta})_{V-A} (\bar{s}_{\beta}c_{\alpha})_{V-A} \qquad Q_2 = (\bar{c}b)_{V-A} (\bar{s}c)_{V-A}$$
(8.35)

QCD-Penguins:

$$Q_3 = (\bar{s}b)_{V-A} \sum_{q=u,d,s,c,b} (\bar{q}q)_{V-A} \qquad Q_4 = (\bar{s}_{\alpha}b_{\beta})_{V-A} \sum_{q=u,d,s,c,b} (\bar{q}_{\beta}q_{\alpha})_{V-A}$$
(8.36)

$$Q_5 = (\bar{s}b)_{V-A} \sum_{q=u,d,s,c,b} (\bar{q}q)_{V+A} \qquad Q_6 = (\bar{s}_{\alpha}b_{\beta})_{V-A} \sum_{q=u,d,s,c,b} (\bar{q}_{\beta}q_{\alpha})_{V+A}$$
(8.37)

The corresponding operators for other B–decays and the D- and K-decays can be obtained from this basis by an appropriate change of flavours.

8.4.2 Effective Hamiltonian

The effective Hamiltonian for $\Delta B = 1$ decays with $\Delta S = 1$ is given then by

$$H_{eff}(\Delta B = 1) = \frac{G_F}{\sqrt{2}} \left[\lambda_u (C_1(\mu_b) Q_1^u + C_2(\mu_b) Q_2^u) + \lambda_c (C_1(\mu_b) Q_1^c + C_2(\mu_b) Q_2^c) - \lambda_t \sum_{i=3}^{6} C_i(\mu) Q_i \right], \tag{8.38}$$

where

$$\lambda_q = V_{qs}^* V_{qb} \tag{8.39}$$

and

$$Q_1^q = (\bar{q}_{\alpha}b_{\beta})_{V-A}(\bar{s}_{\beta}q_{\alpha})_{V-A} , \qquad Q_2^q = (\bar{q}_{\alpha}b_{\alpha})_{V-A}(\bar{s}_{\beta}q_{\beta})_{V-A} . \tag{8.40}$$

In particular $Q_i^c = Q_i$ in (8.35).

8.4.3 Wilson Coefficients

The calculation of the Wilson coefficients $C_i(\mu_b)$ of the QCD penguin operators proceeds as outlined in Section 6.3.

The matching at $\mu_W = M_W$ gives, in the presence of the penguin diagrams, the values of $\vec{C}(M_W)$. In the NDR scheme they are given by:

$$C_1(M_{\rm W}) = \frac{11}{2} \frac{\alpha_s(M_{\rm W})}{4\pi},$$
 (8.41)

$$C_2(M_{\rm W}) = 1 - \frac{11}{6} \frac{\alpha_s(M_{\rm W})}{4\pi},$$
 (8.42)

$$C_3(M_W) = -\frac{\alpha_s(M_W)}{24\pi} \tilde{E}_0(x_t),$$
 (8.43)

$$C_4(M_{\rm W}) = \frac{\alpha_s(M_{\rm W})}{8\pi} \widetilde{E}_0(x_t), \qquad (8.44)$$

$$C_5(M_{\rm W}) = -\frac{\alpha_s(M_{\rm W})}{24\pi} \tilde{E}_0(x_t),$$
 (8.45)

$$C_6(M_{\rm W}) = \frac{\alpha_s(M_{\rm W})}{8\pi} \tilde{E}_0(x_t),$$
 (8.46)

where

$$E_0(x_t) = -\frac{2}{3} \ln x_t + \frac{x_t (18 - 11x_t - x_t^2)}{12(1 - x_t)^3} + \frac{x_t^2 (15 - 16x_t + 4x_t^2)}{6(1 - x_t)^4} \ln x_t, \qquad (8.47)$$

$$\widetilde{E}_0(x_t) = E_0(x_t) - \frac{2}{3}$$
 (8.48)

with

$$x_t = \frac{m_t^2}{M_W^2} \,. {8.49}$$

 $C_{1,2}(M_{\rm W})$ are simply obtained using (6.3) and (7.12). We will derive the QCD penguin coefficients $C_i(M_{\rm W})$ (i=4-6) two pages below. The constant -2/3 in (8.48) is characteristic for the NDR scheme. It is absent in the HV scheme. In LO $C_2(M_{\rm W})=1$ with all remaining coefficients set to zero. We observe that the $m_{\rm t}$ -dependence in the case at hand enters first at the NLO level.

The anomalous dimension matrix is 6×6 :

$$\hat{\gamma}_s(\alpha_s) = \hat{\gamma}_s^{(0)} \frac{\alpha_s}{4\pi} + \hat{\gamma}_s^{(1)} \left(\frac{\alpha_s}{4\pi}\right)^2. \tag{8.50}$$

The one loop coefficient $\hat{\gamma}_s^{(0)}$ is given for N=3 by [84]

$$\hat{\gamma}_s^{(0)} = \begin{pmatrix} -2 & 6 & 0 & 0 & 0 & 0 \\ 6 & -2 & \frac{-2}{9} & \frac{2}{3} & \frac{-2}{9} & \frac{2}{3} \\ 0 & 0 & \frac{-22}{9} & \frac{22}{3} & \frac{-4}{9} & \frac{4}{3} \\ 0 & 0 & 6 - \frac{2f}{9} & -2 + \frac{2f}{3} & \frac{-2f}{9} & \frac{2f}{3} \\ 0 & 0 & 0 & 0 & 2 & -6 \\ 0 & 0 & \frac{-2f}{9} & \frac{2f}{3} & \frac{-2f}{9} & -16 + \frac{2f}{3} \end{pmatrix}$$
(8.51)

The explicit calculation of this matrix is given in subsection 8.5.

The two-loop anomalous dimension matrix $\hat{\gamma}_s^{(1)}$ in the NDR scheme looks truly horrible:

$$\begin{pmatrix} -\frac{21}{2} - \frac{2f}{9} & \frac{7}{2} + \frac{2f}{3} & \frac{79}{9} & -\frac{7}{3} & -\frac{65}{9} & -\frac{7}{3} \\ \frac{7}{2} + \frac{2f}{3} & -\frac{21}{2} - \frac{2f}{9} & -\frac{202}{243} & \frac{1354}{81} & -\frac{1192}{243} & \frac{904}{81} \\ 0 & 0 & -\frac{5911}{486} + \frac{71f}{9} & \frac{5983}{162} + \frac{f}{3} & -\frac{2384}{243} - \frac{71f}{9} & \frac{1808}{81} - \frac{f}{3} \\ 0 & 0 & \frac{379}{18} + \frac{56f}{243} & -\frac{91}{6} + \frac{808f}{81} & -\frac{130}{9} - \frac{502f}{243} & -\frac{14}{3} + \frac{646f}{81} \\ 0 & 0 & \frac{-61f}{9} & -\frac{11f}{3} & \frac{71}{3} + \frac{61f}{9} & -99 + \frac{11f}{3} \\ 0 & 0 & \frac{-682f}{243} & \frac{106f}{81} & -\frac{225}{2} + \frac{1676f}{243} & -\frac{1343}{6} + \frac{1348f}{81} \end{pmatrix}$$

$$(8.52)$$

The corresponding matrix in the HV scheme can be found in [68]. These two loop matrices have been first calculated in [68, 74, 75]. The result in the NDR scheme has been confirmed subsequently in [110].

With all these results at hand one can now evaluate $C_i(\mu_b)$ by using

$$\vec{C}(\mu_b) = \hat{U}_5(\mu_b, M_W)\vec{C}(M_W)$$
 (8.53)

with $\hat{U}_5(\mu_b, M_{\rm W})$ given in (6.26). With the help of Mathematica we find

$$C_j(\mu_b) = C_j^{(0)}(\mu_b) + \frac{\alpha_s(\mu_b)}{4\pi} C_j^{(1)}(\mu_b)$$
(8.54)

where

$$C_j^{(0)}(\mu_b) = \sum_{i=3}^8 k_{ji} \eta^{a_i}$$
(8.55)

$$C_j^{(1)}(\mu_b) = \sum_{i=3}^8 [e_{ji}\eta E_0(x_t) + f_{ji} + g_{ji}\eta]\eta^{a_i}$$
(8.56)

with

$$\eta = \left[\frac{\alpha_s(M_{\rm W})}{\alpha_s(\mu_b)}\right] \ . \tag{8.57}$$

The magic numbers a_i , k_{ij} , e_{ij} , f_{ij} and g_{ij} are collected in tables 6 and 7. The indices i = 1, 2 in these tables are reserved for magnetic penguin operators discussed in sections 8.7 and 12. These tables have been calculated by means of the methods developed in sections 6 and 7.

Table 6: Magic Numbers.

i	3	4	5	6	7	8
a_i	$\frac{6}{23}$	$-\frac{12}{23}$	0.4086	-0.4230	-0.8994	0.1456
k_{1i}	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	0
k_{2i}	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0
k_{3i}	$-\frac{1}{14}$	$\frac{1}{6}$	0.0510	-0.1403	-0.0113	0.0054
k_{4i}	$-\frac{1}{14}$	$-\frac{1}{6}$	0.0984	0.1214	0.0156	0.0026
k_{5i}	0	0	-0.0397	0.0117	-0.0025	0.0304
k_{6i}	0	0	0.0335	0.0239	-0.0462	-0.0112

8.4.4 Matching Conditions for QCD Penguins

It is instructive to derive the matching conditions for QCD penguin operators in (8.43)–(8.46). In particular it is useful to see how the scheme dependent constant -2/3 is generated.

Table 7: More Magic Numbers.

i	3	4	5	6	7	8
a_i	$\frac{6}{23}$	$-\frac{12}{23}$	0.4086	-0.4230	-0.8994	0.1456
e_{1i}	0	0	0	0	0	0
f_{1i}	0.8136	0.7142	0	0	0	0
g_{1i}	1.0197	2.9524	0	0	0	0
e_{2i}	0	0	0	0	0	0
f_{2i}	0.8136	-0.7142	0	0	0	0
g_{2i}	1.0197	-2.9524	0	0	0	0
e_{3i}	0	0	0.1494	-0.3726	0.0738	-0.0173
f_{3i}	-0.0766	-0.1455	-0.8848	0.4137	-0.0114	0.1722
g_{3i}	-0.1457	-0.9841	0.2303	1.4672	0.0971	-0.0213
e_{4i}	0	0	0.2885	0.3224	-0.1025	-0.0084
f_{4i}	-0.2353	-0.0397	0.4920	-0.2758	0.0019	-0.1449
g_{4i}	-0.1457	0.9841	0.4447	-1.2696	-0.1349	-0.0104
e_{5i}	0	0	-0.1163	0.0310	0.0162	-0.0975
f_{5i}	0.0397	0.0926	0.7342	-0.1262	-0.1209	-0.1085
g_{5i}	0	0	-0.1792	-0.1221	0.0213	-0.1197
e_{6i}	0	0	0.0982	0.0634	0.3026	0.0358
f_{6i}	-0.1191	-0.2778	-0.5544	0.1915	-0.2744	0.3568
g_{6i}	0	0	0.1513	-0.2497	0.3983	0.0440

Afterall we stated in section 3 that all mass independent constants in the evaluation of penguin vertices involving m_t -dependent functions like $E_0(x_t)$ can be dropped because of GIM mechanism. Yet as we will see in a moment such statements are valid only in the full theory. In the effective theory the top quark is absent as a dynamical degree of freedom, GIM is no longer true and constants like -2/3 remain.

In order to demonstrate this explicitly let us consider first the tree level Hamiltonian for $\Delta B=1$ decays:

$$H_{eff}^{(0)}(\Delta B = 1) = \frac{G_F}{\sqrt{2}} \left[\lambda_u Q_2^u + \lambda_c Q_2^c + \lambda_t Q_2^t \right]$$
 (8.58)

where λ_q and Q_2^q are defined in (8.39) and (8.40) respectively. Note the appearance of the operator Q_2^t .

Next let us include QCD corrections and perform matching of the full theory to an effective

five quark theory in which the top quark is no longer a dynamical degree of freedom. Since we are only interested in the penguin coefficients we can leave out the QCD corrections to Q_2^q operators and also drop the Q_1^q operators. Calculating then the usual QCD penguin diagram 21a with full W^{\pm} and internal u, c, t quarks and adding the result to the tree level matrix element of the Hamiltonian (8.58) we find the amplitude in the full theory:

$$\mathcal{A}_{full} = \frac{G_F}{\sqrt{2}} \left(\lambda_u \left[\langle Q_2^u \rangle^0 - \frac{\alpha_s(M_W)}{8\pi} G_u(m_u) \langle Q_P \rangle^0 \right] + \lambda_c \left[\langle Q_2^c \rangle^0 - \frac{\alpha_s(M_W)}{8\pi} G_c(m_c) \langle Q_P \rangle^0 \right] + \lambda_t \left[-\frac{\alpha_s(M_W)}{8\pi} E_0(x_t) \langle Q_P \rangle^0 \right] \right). \tag{8.59}$$

Note that as a preparation for the matching we have already removed the tree level matrix element of Q_2^t in which the top quark field is a dynamical degree of freedom. Next

$$Q_P = Q_4 + Q_6 - \frac{1}{3}(Q_3 + Q_5) \tag{8.60}$$

where Q_i with i = 3 - 6 are the penguin operators defined in (8.36) and (8.37).

The functions $G_i(m_i)$ result from calculating penguin diagrams with internal u and c quarks. They are given explicitly in the appendix of the first paper in [68]. As we will see in a moment they will cancel out in the process of matching and their analytic expression is not needed here.

Now the effective theory involves only Q_2^u , Q_2^c and Q_2^c . Calculating the insertions of Q_2^u and Q_2^c into QCD penguin diagrams of fig. 22 and adding the tree level contributions of Q_2^q operators as in the full theory, we find

$$\mathcal{A}_{eff} = \frac{G_F}{\sqrt{2}} \left(\lambda_u \left[\langle Q_2^u \rangle^0 - \frac{\alpha_s(M_W)}{8\pi} (G_u(m_u) - r) \langle Q_P \rangle^0 \right] + \lambda_c \left[\langle Q_2^c \rangle^0 - \frac{\alpha_s(M_W)}{8\pi} (G_c(m_c) - r) \langle Q_P \rangle^0 \right] - \lambda_t C_P \langle Q_P \rangle^0 \right], \tag{8.61}$$

where C_P is the coefficient we are looking for. The minus sign in front of λ_t is a convention which has no impact on physics. Next r is a scheme dependent constant equal to 2/3 and 0 for NDR and HV schemes respectively. Finally it should be remarked that the insertions of penguin operators into penguin diagrams contribute only at $\mathcal{O}(\alpha_s^2)$ to (8.61) and do not contribute at this order.

Comparing (8.59) and (8.61) and using the unitarity relation $\lambda_u + \lambda_c = -\lambda_t$, we determine C_P to be

$$C_P = \frac{\alpha_s(M_W)}{8\pi} [E_0(x_t) - r]$$
 (8.62)

Inserting this result into (8.61), using the expression for Q_P in (8.60) and comparing the coefficient of λ_t with the one of (8.38) we derive the matching conditions (8.43)–(8.46).

8.4.5 Numerical Values for $C_i(\mu_b)$

In table 8 we give numerical values of the coefficients $C_i(\mu_b)$ in LO and the two NLO schemes in question.

	$\Lambda_{\overline{\rm MS}}^{(5)} = 160 {\rm MeV}$			$\Lambda_{\overline{\rm MS}}^{(5)} = 225{\rm MeV}$			$\Lambda_{\overline{\rm MS}}^{(5)} = 290{\rm MeV}$		
Scheme	LO	NDR	HV	LO	NDR	HV	LO	NDR	HV
C_1	-0.270	-0.169	-0.206	-0.295	-0.184	-0.226	-0.317	-0.198	-0.243
C_2	1.119	1.071	1.089	1.132	1.078	1.100	1.144	1.085	1.109
C_3	0.012	0.012	0.011	0.013	0.013	0.012	0.015	0.014	0.013
C_4	-0.028	-0.032	-0.026	-0.030	-0.035	-0.029	-0.032	-0.038	-0.031
C_5	0.008	0.008	0.008	0.008	0.009	0.009	0.009	0.009	0.010
C_6	-0.034	-0.037	-0.029	-0.038	-0.041	-0.033	-0.041	-0.044	-0.035

Table 8: $\Delta B = 1$ Wilson coefficients at $\mu = \overline{m}_b(m_b) = 4.40 \,\text{GeV}$ for $m_t = 170 \,\text{GeV}$.

Let us make just a few observations:

- Penguin coefficients are much smaller than C_1 and C_2 .
- The largest penguin coefficients are C_4 and C_6 . In the NDR and HV schemes their values are by a factor of 6-7 smaller than the coefficient C_1 .
- A numerical analysis shows that in the range $m_{\rm t} = (170 \pm 15) \,\text{GeV}$ the $m_{\rm t}$ dependence of the QCD penguin coefficients can be neglected.

8.4.6 Threshold Effects in the Presence of Penguins

In (6.33) we have given a formula for $\vec{C}(\mu)$ in the presence of flavour thresholds. This formula implies in particular that $\vec{C}_{f-1}(\mu_f) = \vec{C}_f(\mu_f)$ where μ_f is the threshold between an effective f-flavour theory and an effective theory with f-1 flavours. In the presence of penguin operators the matching is more involved. One finds now

$$\vec{C}_{f-1}(\mu_f) = \hat{M}(\mu_f)\vec{C}_f(\mu_f)$$
(8.63)

where $\hat{M}(\mu_f)$ is a matching matrix given by

$$\hat{M}(\mu_f) = \hat{1} + \frac{\alpha_s(\mu_f)}{4\pi} \delta \hat{r}^T \tag{8.64}$$

The matrix $\delta \hat{r}^T$ can be found in section VID of [17]. With (8.63) the formula (6.33) generalizes to

$$\vec{C}(\mu) = \hat{U}_3(\mu, \mu_c) \hat{M}(\mu_c) \hat{U}_4(\mu_c, \mu_b) \hat{M}(\mu_b) \hat{U}_5(\mu_b, \mu_W) \vec{C}(\mu_W). \tag{8.65}$$

8.5 Explicit Calculation of 6×6 Anomalous Dimension Matrix

8.5.1 Preliminaries

It is time to do a real climb by calculating the matrix (8.51). This involves the operator insertions into the penguin diagrams and into the current-current diagrams. Since master formulae for the latter insertions have already been derived and applied for the case of (Q_1, Q_2) in section 6.5, we begin our climb by discussing the penguin insertions.

8.5.2 Penguin Insertions: Generalities

The two diagrams contributing to the anomalous dimension matrix of $(Q_1,...,Q_6)$ through the penguin insertions are given in fig. 22. We observe that two types of insersions of a given operator into a penguin diagram are possible. Type A insertions represented by the diagram (a) are constructed by joining two quarks belonging to two different disconnected parts of an operator and attaching the gluon to the resulting internal quark line. For instance in the case of $(\bar{c}b)_{V-A}(\bar{s}c)_{V-A}$ one can join \bar{c} and c into one line. Type B insertions represented by the diagram (b) are constructed by joining two quarks belonging to the same part of a given operator and attaching the gluon to the resulting quark loop. For instance in the case of $(\bar{c}c)_{V-A}(\bar{s}b)_{V-A}$ we have a c-quark loop. Since qluons conserve flavour, penguin insertions are only possible if a given operator contains at least two quarks with the same flavour. Note that in the case of $(\bar{s}b)_{V-A}(\bar{s}s)_{V-A}$ both types of insertions are possible and have to be taken into account. Finally the bottom quark line attached to the lower end of the gluon represents any quark present in the effective theory. In calculating the contribution of a given diagram one has to sum over all quark flavours. In this manner the penguin operators are generated from insertions into penguin diagrams. It is clear from the last statement that the insertion of any operator $(Q_1,...,Q_6)$ into the penguin diagrams of fig. 22 always results in a linear combination of penguin operators. That is penguin operators mix under renormalization among themselves and the current-current operators mix into penguin operators but the mixing of penguin operators into Q_1 and Q_2 does not take place. This last feature is not affected by the insertions of penguin operators into the current-current diagrams of fig. 16

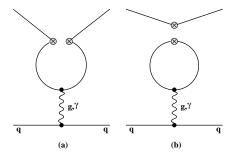


Figure 22: One loop penguin diagrams in the effective theory.

as we will see explicitly few pages below. Consequently without any calculation we can state that

$$(\hat{\gamma})_{i1} = (\hat{\gamma})_{i2} = 0 \qquad i = 3, ..6$$
 (8.66)

and this is also true for the electroweak penguins discussed in subsection 8.6.

After these general remarks let us derive two master formulae for penguin insertions which are analogous to the three master formulae for current-current insertions given in (6.64)–(6.66).

As in the case of current-current insertions, we consider an arbitrary operator with the colour structure $\hat{V}_1 \otimes \hat{V}_2$ and the Dirac structure $\Gamma_1 \otimes \Gamma_2$. Dropping the external spinors, the insertion of this operator into the penguin vertex of fig. 22a gives

$$W_{\lambda} = -ig\mu^{\varepsilon} \hat{V}_{1} T^{a} \hat{V}_{2} I^{\mu\nu} T^{\lambda}_{\mu\nu} \tag{8.67}$$

where

$$T^{\lambda}_{\mu\nu} = \Gamma_1 \gamma_{\nu} \gamma_{\lambda} \gamma_{\mu} \Gamma_2 \tag{8.68}$$

and

$$I^{\mu\nu} = \int \frac{d^D k}{(2\pi)^D} \frac{k^{\nu} (k-q)^{\mu}}{k^2 (k-q)^2} = -\frac{i}{16\pi^2} \left[\frac{1}{\varepsilon} \right] \left[\frac{1}{6} q^{\mu} q^{\nu} + q^2 \frac{g^{\mu\nu}}{12} \right]$$
(8.69)

with q being the gluon momentum. In evaluating $I^{\mu\nu}$ we have kept only the divergent part.

The master formula for type A insertions is then obtained by including the gluon propagator and the lower vertex in the penguin diagram (a) of fig. 22. We find

$$\mathcal{P}_A = -\mathcal{C}_A \frac{\alpha_s}{4\pi} \left[\frac{1}{\varepsilon} \right] \left[\frac{1}{6} \frac{q^\mu q^\nu}{q^2} + \frac{g^{\mu\nu}}{12} \right] \Gamma_1 \gamma_\nu \gamma_\lambda \gamma_\mu \Gamma_2 \otimes \gamma^\lambda , \qquad (8.70)$$

where the colour factor is given by

$$C_A = \hat{V}_1 T^a \hat{V}_2 \otimes T^a \ . \tag{8.71}$$

It is understood that the Dirac and colour structures on the l.h.s of \otimes are sandwiched between free spinors belonging to the inserted operator and $\gamma^{\lambda}T^{a}$ standing on the r.h.s of \otimes between

the spinors representing the bottom line of the penguin diagram. Formula (8.70) serves to calculate the coefficients $(b_1)_{ij}$ in the master formula (6.43). We will demonstrate this explicitly below.

The master formula for type B insertions can be derived in an analogous manner. We find

$$\mathcal{P}_{B} = \mathcal{C}_{B} \frac{\alpha_{s}}{4\pi} \left[\frac{1}{\varepsilon} \right] \left[\frac{1}{6} \frac{q^{\mu} q^{\nu}}{q^{2}} + \frac{g^{\mu\nu}}{12} \right] \operatorname{Tr}(\Gamma_{1} \gamma_{\mu} \gamma_{\lambda} \gamma_{\nu}) \Gamma_{2} \otimes \gamma^{\lambda}$$
(8.72)

where

$$C_B = \text{Tr}(\hat{V}_1 T^a) \hat{V}_2 \otimes T^a \tag{8.73}$$

with "Tr" in (8.73) standing for the trace in the colour space. Note that in this formula we have closed the part $\hat{V}_1\Gamma_1$ of the inserted operator in the loop. If the part $\hat{V}_2\Gamma_2$ is closed instead, the indices "1" and "2" in (8.72) and (8.73) should be interchanged. The rules for the incorporation of the external spinors into (8.72) should be evident in view of the comments made after (8.71). The difference in the overall sign compared to (8.70) is a consequence of "-1" for the fermion loop.

8.5.3 Explicit Calculation of Penguin Insertions

Let us apply our master formulae to the case of the operator

$$Q_2 = (\bar{c}_{\alpha}b_{\alpha})_{V-A}(\bar{s}_{\beta}c_{\beta})_{V-A} \tag{8.74}$$

for which we have

$$\hat{V}_1 \otimes \hat{V}_2 = \mathbf{1} \qquad \Gamma_1 = \Gamma_2 = \gamma_\tau (1 - \gamma_5) \tag{8.75}$$

The flavour structure in (8.74) tells us that only type A insertions are possible. We use therefore the master formula (8.70). As we are only interested in the coefficient of $1/\varepsilon$, we calculate all Dirac structures in D=4 dimensions. This gives

$$\left[\frac{1}{6} \frac{q^{\mu} q^{\nu}}{q^2} + \frac{g^{\mu\nu}}{12} \right] \Gamma_1 \gamma_{\nu} \gamma_{\lambda} \gamma_{\mu} \Gamma_2 = \frac{4}{3} \left[\gamma_{\lambda} - \frac{q_{\lambda} \not q}{q^2} \right]$$
(8.76)

where we have used the identity

$$\not q \gamma_{\lambda} \not q = 2q_{\lambda} \not q - q^2 \gamma_{\lambda}. \tag{8.77}$$

The term $q_{\lambda} \not q/q^2$ does not contribute here as using the Dirac equation one has $\bar{s} \not q b = 0$ for massless quarks. More care is needed when magnetic penguins of sections 8.7 and 12 are considered and m_b has to be kept.

Dropping then the second term on the r.h.s of (8.76), using

$$C_A = T^a \otimes T^a = \frac{1}{2} \left[\tilde{\mathbf{1}} - \frac{1}{N} \mathbf{1} \right] \tag{8.78}$$

with $\tilde{\mathbf{1}}$ defined in (6.57) and inserting the relevant spinors we arrive at

$$\mathcal{P}_A(Q_2) = -\frac{\alpha_s}{4\pi} \left[\frac{1}{\varepsilon} \right] \left[\frac{2}{3} \right] \left[\tilde{\mathbf{1}} - \frac{1}{N} \mathbf{1} \right] \left[\bar{s}_{\alpha} \gamma_{\lambda} (1 - \gamma_5) b_{\beta} \right] \otimes \sum_q \bar{q}_{\gamma} \gamma^{\lambda} q_{\delta}$$
(8.79)

Next we decompose γ^{λ} on the r.h.s of \otimes into V-A and V+A parts

$$\gamma^{\lambda} = \frac{1}{2}\gamma^{\lambda}(1 - \gamma_5) + \frac{1}{2}\gamma^{\lambda}(1 + \gamma_5) \tag{8.80}$$

which allows to express (8.79) in terms of the penguin operators

$$\mathcal{P}_A(Q_2) = -\frac{\alpha_s}{4\pi} \left[\frac{1}{\varepsilon} \right] \left[\frac{1}{3} \right] \left[Q_4 + Q_6 - \frac{1}{N} (Q_3 + Q_5) \right]$$
(8.81)

The contribution of penguin insertions to the coefficients $(b_1)_{2j}$ relevant for the master formula (6.43) are consequently given by

$$(b_1)_{23}^P = (b_1)_{25}^P = \frac{1}{3N}, \qquad (b_1)_{24}^P = (b_1)_{26}^P = -\frac{1}{3}.$$
 (8.82)

We next consider Q_1 and rewrite it using Fierz reordering as

$$Q_1 = (\bar{c}_{\alpha}c_{\alpha})_{V-A}(\bar{s}_{\beta}b_{\beta})_{V-A} \tag{8.83}$$

so that the colour and Dirac structures are again given by (8.75). This time only type B insertions are possible. However, $Tr(T^a) = 0$ and consequently the colour factor in (8.73) vanishes. Thus

$$\mathcal{P}_B(Q_1) = 0 \tag{8.84}$$

implying

$$(b_1)_{13}^P = (b_1)_{14}^P = (b_1)_{15}^P = (b_1)_{16}^P = 0.$$
 (8.85)

In the case of Q_3 , the type B insertion vanishes as in the case of Q_1 but now two type-A insertions are possible. One involves the internal b-quark, the other the s-quark. Since gluons are flavour-blind and Q_3 has the same colour and Dirac structures as Q_2 we can find immediately

$$\mathcal{P}_A(Q_3) = 2\mathcal{P}_A(Q_2) \tag{8.86}$$

Consequently using (8.82) we find

$$(b_1)_{33}^P = (b_1)_{35}^P = \frac{2}{3N}, \qquad (b_1)_{34}^P = (b_1)_{36}^P = -\frac{2}{3}.$$
 (8.87)

Next comes Q_4 . Performing Fierz reordering we have

$$Q_4 = \sum_{q} (\bar{s}_{\alpha} q_{\alpha})_{V-A} (\bar{q}_{\beta} b_{\beta})_{V-A}. \tag{8.88}$$

The type B insertions involving s and b quarks vanish as in the case of Q_1 . On the other hand we have f type A insertions involving all quark flavours. Thus

$$\mathcal{P}_A(Q_4) = f\mathcal{P}_A(Q_2) \tag{8.89}$$

and

$$(b_1)_{43}^P = (b_1)_{45}^P = \frac{f}{3N}, \qquad (b_1)_{44}^P = (b_1)_{46}^P = -\frac{f}{3}.$$
 (8.90)

The penguin insertions of Q_5 vanish. The type B insertions vanish because of $Tr(T^a) = 0$. The type A insertions vanish because now

$$\Gamma_1 = \gamma_\tau (1 - \gamma_5) , \qquad \Gamma_2 = \gamma_\tau (1 + \gamma_5)$$

$$(8.91)$$

and the Dirac structure in the master formula (8.70) vanishes. Thus

$$(b_1)_{53}^P = (b_1)_{54}^P = (b_1)_{55}^P = (b_1)_{56}^P = 0$$
 (8.92)

Finally the insertions of Q_6 have to be considered. Performing Fierz reordering we have

$$Q_6 = -2\sum_{q} (\bar{s}_{\alpha}(1+\gamma_5)q_{\alpha})(\bar{q}_{\beta}(1-\gamma_5)b_{\beta}), \tag{8.93}$$

implying

$$\hat{V}_1 \otimes \hat{V}_2 = \mathbf{1}$$
 $\Gamma_1 = (1 + \gamma_5)$ $\Gamma_2 = (1 - \gamma_5)$ (8.94)

Again as in the case of Q_4 the type B insertions vanish. The type A insertion of Q_6 is expected at first sight to give different result than the one of Q_4 because of the different Dirac structure. However, application of the master formula (8.70) gives

$$\mathcal{P}_A(Q_6) = \mathcal{P}_A(Q_4) \tag{8.95}$$

and consequently

$$(b_1)_{63}^P = (b_1)_{65}^P = \frac{f}{3N}$$
 $(b_1)_{64}^P = (b_1)_{66}^P = -\frac{f}{3}.$ (8.96)

8.5.4 Explicit Calculation of Current-Current Insertions

In section 6.5 we have calculated the 2×2 anomalous dimension matrix for the pair (Q_1, Q_2) by inserting these operators into the current-current diagrams of fig. 16. Using the master formulae (6.64)–(6.66) for these diagrams together with the basic formula (6.43) we have found the matrix (5.77) which as seen in the left upper corner of (8.51) constitutes a part of the 6×6 matrix we are trying to reproduce.

What remains to be done are the insertions of the penguin operators into the currentcurrent diagrams. The case of the pair (Q_3, Q_4) is simple. From the point of view of currentcurrent insertions the pair (Q_3, Q_4) behaves as (Q_1, Q_2) and we can write immediately

$$(b_1)_{33}^{cc} = 2C_F + \frac{3}{N}, \qquad (b_1)_{34}^{cc} = -3,$$
 (8.97)

$$(b_1)_{43}^{cc} = -3, (b_1)_{44}^{cc} = 2C_F + \frac{3}{N}.$$
 (8.98)

The case of Q_5 and Q_6 operators is different as they have the $(V-A)\otimes(V+A)$ structure. Let us consider Q_5 first. Using master formulae (6.64)–(6.66) for

$$\hat{V}_1 \otimes \hat{V}_2 = 1, \qquad \Gamma_1 = \gamma_\tau (1 - \gamma_5), \qquad \Gamma_2 = \gamma_\tau (1 + \gamma_5) , \qquad (8.99)$$

we arrive at

$$\sum_{i} \mathcal{D}_{i}(Q_{5}) = \frac{\alpha_{s}}{4\pi} \frac{1}{\varepsilon} \Gamma_{1} \otimes \Gamma_{2} \left[\mathcal{C}_{a}^{(1)} + \mathcal{C}_{a}^{(2)} - (\mathcal{C}_{b}^{(1)} + \mathcal{C}_{b}^{(2)}) + 4(\mathcal{C}_{c}^{(1)} + \mathcal{C}_{c}^{(2)}) \right]$$
(8.100)

with colour factors $C_i^{(j)}$ given in (6.77) and (6.78).

In order to perform the reduction of Dirac structures in the master formulae (6.64)–(6.66) we had to generalize the Greek Method to the $(V-A)\otimes(V+A)$ operators. In this case \otimes should be replaced by 1 as otherwise the Dirac structures would identically vanish. Now the coefficients (4,16,4) in (6.71)–(6.72) are replaced by (4,4,16) respectively, which implies a different weighting of the colour factors in (8.100) relative to (6.76). Noting that Q_5 is represented by 1 and Q_6 by $\tilde{\bf 1}$ we obtain from (8.100)

$$(b_1)_{55}^{cc} = 2C_F - \frac{3}{N}, \qquad (b_1)_{56}^{cc} = 3.$$
 (8.101)

Finally we consider Q_6 . Here it is useful to use the form (8.93). As Γ_i are now given by (8.94) one easily finds that the usual Greek Method with $\otimes = \gamma_{\tau}$ applies. The master formulae (6.64)–(6.66) then give

$$\sum_{i} \mathcal{D}_{i}(Q_{6}) = \frac{\alpha_{s}}{4\pi} \frac{1}{\varepsilon} \left[-2\Gamma_{1} \otimes \Gamma_{2} \right] \left[4(\mathcal{C}_{a}^{(1)} + \mathcal{C}_{a}^{(2)}) - (\mathcal{C}_{b}^{(1)} + \mathcal{C}_{b}^{(2)}) + (\mathcal{C}_{c}^{(1)} + \mathcal{C}_{c}^{(2)}) \right] . \tag{8.102}$$

The weighting of colour factors differs from the cases Q_2 and Q_5 . In particular using (6.78) we find that the two last terms in the square bracket cancel each other. Effectively then only the insertions of Q_6 into the diagrams (a) of fig. 16 and its symmetric counterpart contribute. However, contrary to the case of the operators Q_{1-5} this contribution will not be canceled by the δ_{ij} term in (6.43) as now $\mathcal{C}_a^{(j)}$ are multiplied by 4 instead of 1. Consequently noting that in this case Q_6 is represented by 1 and Q_5 by $\tilde{\mathbf{1}}$ we find

$$(b_1)_{65}^{cc} = 0, (b_1)_{66}^{cc} = 8C_F.$$
 (8.103)

8.5.5 Putting Things together

Let us add the results for penguin and current-current insertions obtained above. Setting N=3 we find the matrix \hat{b}_1 in (6.43):

$$\hat{b}_{1} = \begin{pmatrix} 2C_{F} + 1 & -3 & 0 & 0 & 0 & 0 \\ -3 & 2C_{F} + 1 & \frac{1}{9} & -\frac{1}{3} & \frac{1}{9} & -\frac{1}{3} \\ 0 & 0 & 2C_{F} + \frac{11}{9} & -\frac{11}{3} & \frac{2}{9} & -\frac{2}{3} \\ 0 & 0 & -3 + \frac{f}{9} & 2C_{F} + 1 - \frac{f}{3} & \frac{f}{9} & -\frac{f}{3} \\ 0 & 0 & 0 & 0 & 2C_{F} - 1 & 3 \\ 0 & 0 & \frac{f}{9} & -\frac{f}{3} & \frac{f}{9} & 8C_{F} - \frac{f}{3} \end{pmatrix}$$

$$(8.104)$$

Inserting this matrix into the one-loop master formula (6.43) we reproduce the full 6×6 matrix in (8.51). Fantastic! We have reproduced all the magic numbers in this matrix. This is almost like reaching the top of Mont Blanc.

8.5.6 An Advice

I hope that this long exercise and the exercise in subsection 6.5 were useful for those students who have never calculated anomalous dimension matrices. But there is another lesson from these exercises. The corresponding two-loop calculations of current-current and penguin insertions involving many more diagrams, more complicated colour factors and evanescent operators are truly horrible. They are not like climbing Mont Blanc but rather Mount Everest. They take several months rather than a day or two. Consequently it is advisable for beginners to take an experienced guide in order to climb these Himalayas. Fortunately the guides in physics, as opposed to those in the real Himalayas, are doing it for free.

Yet one useful advice is mandatory here. In our field there are unfortunately sponsors (thesis supervisors) who send their students to climb "Mount Everest" without having the slightest idea how difficult this climb is. Moreover they are of little help once the student gets stuck in the middle of the climb. Here is my advice. If you are not experienced in such Mount Everest calculations and your sponsor is as described above, there are only two solutions: either your sponsor has to provide you with a strong sherpa who has climbed Everest at least once, or you have to find another sponsor before it is too late [103]!

8.6 Electroweak Penguins

8.6.1 Operators

The inclusion of the electroweak penguins and box diagrams of fig. 21 generates two additional operators Q_7 and Q_9 . With respect to the colour structure they are analogous to Q_5 and

 Q_3 operators, respectively. When QCD effects are also taken into account, two additional operators Q_8 and Q_{10} are needed to close the system under renormalization. They are analogous to Q_6 and Q_4 , respectively. The full set of operators necessary for the description of $\Delta F = 1$ decays including electroweak effects consists then of 10 operators. The 4 electroweak penguin operators relevant for $\Delta B = 1$ decays with $\Delta S = 1$ are given by

$$Q_7 = \frac{3}{2} (\bar{s}b)_{V-A} \sum_{q=u,d,s,c,b} e_q (\bar{q}q)_{V+A}$$
(8.105)

$$Q_8 = \frac{3}{2} (\bar{s}_{\alpha} b_{\beta})_{V-A} \sum_{q=u,d,s,c,b} e_q(\bar{q}_{\beta} q_{\alpha})_{V+A}$$
(8.106)

$$Q_9 = \frac{3}{2} (\bar{s}b)_{V-A} \sum_{q=u,d,s,c,b} e_q(\bar{q}q)_{V-A}$$
(8.107)

$$Q_{10} = \frac{3}{2} (\bar{s}_{\alpha} b_{\beta})_{V-A} \sum_{q=u,d,s,c,b} e_q (\bar{q}_{\beta} q_{\alpha})_{V-A}$$
(8.108)

The overall factor 3/2 is introduced for convenience. The charge e_q is the charge of the quark coupled to the lower vertex of the photon or Z^0 -propagator.

8.6.2 Wilson Coefficients

In order to generate the electroweak penguin operators $Q_7 - Q_{10}$ it is sufficient to include the photon penguin together with the relevant QCD renormalization. However, in order to keep the gauge invariance also Z^0 -penguins and box-diagrams have to be included at the NLO level. The latter two sets of diagrams involving only heavy fields (W^{\pm}, Z^0, t) contribute only to the Wilson coefficients at $\mu = \mathcal{O}(M_{\rm W})$ and have no impact on the renormalization group evolution down to low energy scales. On the other hand the inclusion of Z^0 -penguins introduces a strong m_t -dependence into Wilson coefficients of the electroweak penguin operators, which in several cases has important phenomenological implications. We will discuss several of them in the phenomenological sections of these lectures. Here we give some information on the Wilson coefficients of electroweak penguin operators.

The matching at $\mu_W = M_W$ gives in the presence of the electroweak penguin and box diagrams the values of $C_i(M_W)$ with i = 1, ... 10. In the NDR scheme they are given by:

$$C_1(M_{\rm W}) = \frac{11}{2} \frac{\alpha_s(M_{\rm W})}{4\pi},$$
 (8.109)

$$C_2(M_{\rm W}) = 1 - \frac{11}{6} \frac{\alpha_s(M_{\rm W})}{4\pi} - \frac{35}{18} \frac{\alpha}{4\pi},$$
 (8.110)

$$C_3(M_W) = -\frac{\alpha_s(M_W)}{24\pi} \widetilde{E}_0(x_t) + \frac{\alpha}{6\pi} \frac{1}{\sin^2 \theta_W} \left[2B_0(x_t) + C_0(x_t) \right], \qquad (8.111)$$

$$C_4(M_W) = \frac{\alpha_s(M_W)}{8\pi} \tilde{E}_0(x_t),$$
 (8.112)

$$C_5(M_{\rm W}) = -\frac{\alpha_s(M_{\rm W})}{24\pi} \tilde{E}_0(x_t),$$
 (8.113)

$$C_6(M_{\rm W}) = \frac{\alpha_s(M_{\rm W})}{8\pi} \widetilde{E}_0(x_t), \qquad (8.114)$$

$$C_7(M_W) = \frac{\alpha}{6\pi} \left[4C_0(x_t) + \tilde{D}_0(x_t) \right],$$
 (8.115)

$$C_8(M_{\rm W}) = 0,$$
 (8.116)

$$C_9(M_{\rm W}) = \frac{\alpha}{6\pi} \left[4C_0(x_t) + \widetilde{D}_0(x_t) + \frac{1}{\sin^2 \theta_W} (10B_0(x_t) - 4C_0(x_t)) \right], \quad (8.117)$$

$$C_{10}(M_{\rm W}) = 0,$$
 (8.118)

We recall (see Section 3) that

$$B_0(x_t) = \frac{1}{4} \left[\frac{x_t}{1 - x_t} + \frac{x_t \ln x_t}{(x_t - 1)^2} \right], \tag{8.119}$$

$$C_0(x_t) = \frac{x_t}{8} \left[\frac{x_t - 6}{x_t - 1} + \frac{3x_t + 2}{(x_t - 1)^2} \ln x_t \right], \tag{8.120}$$

$$D_0(x_t) = -\frac{4}{9} \ln x_t + \frac{-19x_t^3 + 25x_t^2}{36(x_t - 1)^3} + \frac{x_t^2(5x_t^2 - 2x_t - 6)}{18(x_t - 1)^4} \ln x_t, \qquad (8.121)$$

$$\widetilde{D}_0(x_t) = D_0(x_t) - \frac{4}{9}.$$
 (8.122)

 $B_0(x_t)$ results from the evaluation of the box diagrams, $C_0(x_t)$ from the Z^0 -penguin, $D_0(x_t)$ from the photon penguin and $E_0(x_t)$ discussed already in the previous subsection from the gluon penguin diagram. The constant -4/9 in (8.122) is characteristic for the NDR scheme. It is absent in the HV scheme. We note that the presence of electroweak effects modifies the values of $C_2(M_{\rm W})$ and $C_3(M_{\rm W})$ by small $\mathcal{O}(\alpha)$ corrections. We also note that $C_8(M_{\rm W}) = C_{10}(M_{\rm W}) = 0$. For $\mu \neq M_{\rm W}$ non-vanishing C_8 and C_{10} are generated through QCD effects.

The anomalous dimension matrices are 10×10 :

$$\hat{\gamma}(\alpha_s, \alpha) = \hat{\gamma}_s^{(0)} \frac{\alpha_s}{4\pi} + \hat{\gamma}_e^{(0)} \frac{\alpha}{4\pi} + \hat{\gamma}_s^{(1)} \left(\frac{\alpha_s}{4\pi}\right)^2 + \hat{\gamma}_{se}^{(1)} \frac{\alpha_s}{4\pi} \frac{\alpha}{4\pi}$$
(8.123)

with $\gamma_s^{(0)}$ and $\gamma_s^{(1)}$ being 10×10 generalizations of the corresponding 6×6 matrices considered previously. Since now $\mathcal{O}(\alpha)$ effects are included in the coefficients at scales $\mathcal{O}(M_{\rm W})$, the anomalous dimension matrix must also include $\mathcal{O}(\alpha)$ contributions which are represented by $\hat{\gamma}_e^{(0)}$ and $\hat{\gamma}_{se}^{(1)}$ at LO and NLO respectively. The four matrices in (8.123) can be found in [17], where the references to the original literature is given. See also table 10.

The calculation of the 6×6 submatrix of $\hat{\gamma}_s^{(0)}$ has been presented in detail in section 8.5. The evaluation of $\hat{\gamma}_e^{(0)}$ proceeds in an analogous manner except that the colour factors have to be properly replaced by electric charges and the closed fermion loops coupled to the photon have to be multiplied by N=3. Any reader, who succeeded in calculating $\hat{\gamma}_s^{(0)}$ should have no difficulties in calculating within two hours the matrix $\hat{\gamma}_e^{(0)}$. This is a very nice exercise

indeed. The calculations of $\hat{\gamma}_s^{(1)}$ and $\hat{\gamma}_{se}^{(1)}$ are even nicer but take more time. Typically six months for $\hat{\gamma}_s^{(1)}$ and then a month for $\hat{\gamma}_{se}^{(1)}$.

Due to the simultaneous apperance of α and α_s , the RG analysis is more involved than the one discussed until now. In particular the evolution matrix takes now the general form

$$\hat{U}(m_1, m_2, \alpha) = \hat{U}(m_1, m_2) + \frac{\alpha}{4\pi} \hat{R}(m_1, m_2). \tag{8.124}$$

Here $\hat{U}(m_1, m_2)$ represents the pure QCD evolution matrix given in (6.26). $\hat{R}(m_1, m_2)$ describes the additional evolution in the presence of electromagnetic interactions. It includes both LO and NLO corrections. Let us recall that $\hat{U}(m_1, m_2)$ sums the logarithms $(\alpha_s t)^n$ and $\alpha_s(\alpha_s t)^n$ with $t = \ln(m_2^2/m_1^2)$. On the other hand $\hat{R}(m_1, m_2)$ sums the logarithms $t(\alpha_s t)^n$ and $(\alpha_s t)^n$. The expresion for $\hat{R}(m_1, m_2)$ is rather complicated. It can be found in [17, 73]. The Wilson coefficients are then found by using

$$\vec{C}(\mu_b) = \hat{U}_5(\mu_b, M_W, \alpha) \vec{C}(M_W)$$
 (8.125)

with $\vec{C}(M_{
m W})$ given in (8.109)–(8.109).

8.6.3 Numerical Values

In table 9 we give numerical values of the coefficients $C_i(\mu_b)$ in LO and the two NLO schemes in question. Let us make just a few observations:

Table 9: $\Delta B = 1$ Wilson coefficients at $\mu = \overline{m}_{\rm b}(m_{\rm b}) = 4.40\,{\rm GeV}$ for $m_{\rm t} = 170\,{\rm GeV}$.

	$\Lambda_{\overline{\mathrm{MS}}}^{(5)} = 160 \mathrm{MeV}$			$\Lambda_{\overline{ m MS}}^{(5)} = 225{ m MeV}$			$\Lambda_{\overline{\rm MS}}^{(5)} = 290 \mathrm{MeV}$		
Scheme	LO	NDR	HV	LO	NDR	HV	LO	NDR	HV
C_1	-0.283	-0.171	-0.209	-0.308	-0.185	-0.228	-0.331	-0.198	-0.245
C_2	1.131	1.075	1.095	1.144	1.082	1.105	1.156	1.089	1.114
C_3	0.013	0.013	0.012	0.014	0.014	0.013	0.016	0.016	0.014
C_4	-0.028	-0.033	-0.027	-0.030	-0.035	-0.029	-0.032	-0.038	-0.032
C_5	0.008	0.008	0.008	0.009	0.009	0.009	0.009	0.009	0.010
C_6	-0.035	-0.037	-0.030	-0.038	-0.041	-0.033	-0.041	-0.045	-0.036
C_7/α	0.043	-0.003	0.006	0.045	-0.002	0.005	0.047	-0.002	0.005
C_8/α	0.043	0.049	0.055	0.048	0.054	0.060	0.053	0.059	0.065
C_9/α	-1.268	-1.283	-1.273	-1.280	-1.292	-1.283	-1.290	-1.300	-1.293
C_{10}/α	0.302	0.243	0.245	0.328	0.263	0.266	0.352	0.281	0.284

- Electroweak penguin coefficients being O(α) are smaller than C₁₋₆ coefficients. Notable exception is the coefficient C₉ which is in the ball park of the smallest QCD penguin coefficients C₃ and C₅. It is the operator Q₉ which is the dominant electroweak penguin in B-decays [83]. With decreasing μ the coefficient C₈ increases considerably. While its role in B-decays can be fully neglected, it plays considerable role in the CP violation in K → ππ decays where also its hadronix matrix element is large.
- A numerical analysis shows that in contrast to C_1, \ldots, C_6 , the additional coefficients C_7, \ldots, C_{10} increase strongly with m_t . This strong m_t dependence originates in the Z^0 -penguin represented by the function $C_0(x_t)$ in (8.120). Even in the range $m_t = (170 \pm 15) \,\text{GeV}$ with in/decreasing m_t there is a relative variation of $\mathcal{O}(\pm 19\%)$ and $\mathcal{O}(\pm 10\%)$ for the absolute values of C_8 and $C_{9,10}$, respectively.

8.7 Magnetic Penguins

The inclusive decay $B \to X_s \gamma$ with an on-shell γ is governed by the operator $Q_{7\gamma}$ which originates in the photon-penguin vertex with $q^2=0$, where q_μ is the momentum of the emitted photon. In order to obtain a non-vanishing result one has to keep external b-quark mass as well as external momenta. This mass insertion together with the expansion to second order in external momenta generates $Q_{7\gamma}$, which due to the appearance of $\sigma^{\mu\nu}$ is known under the name of a magnetic photon penguin. The corresponding gluon-penguin vertex with $q^2=0$ results in a magnetic gluon penguin operator Q_{8G} which plays the dominant role in the inclusive $B\to X_s$ gluon decay. The magnetic penguins are given by

$$Q_{7\gamma} = \frac{e}{8\pi^2} m_b \bar{s}_{\alpha} \sigma^{\mu\nu} (1 + \gamma_5) b_{\alpha} F_{\mu\nu} \qquad Q_{8G} = \frac{g}{8\pi^2} m_b \bar{s}_{\alpha} \sigma^{\mu\nu} (1 + \gamma_5) T^a_{\alpha\beta} b_{\beta} G^a_{\mu\nu}. \tag{8.126}$$

The renormalization group analysis of $B \to X_s \gamma$ involves in addition to $Q_{7\gamma}$ and Q_{8G} also the operators $Q_1...Q_6$ discussed previously. The peculiar feature of this analysis is the vanishing of the mixing under renormalization between the sets $(Q_{7\gamma}, Q_{8G})$ and $(Q_1...Q_6)$ at the one loop level. That is in order to calculate the leading entry (LO), representing this mixing, in the relevant anomalous dimension matrix one is forced to perform two-loop calculations. At NLO the corresponding three loop calculations are necessary. Because this mixing has a very important impact on the resulting decay rate, these calculations are mandatory before a meaningful theoretical prediction for $B \to X_s \gamma$ can be obtained.

The decay $B \to X_s \gamma$ is one of the central decays in the rare decays phenomenology. Therefore, we will devote to it section 12 where both technical and phenomenological aspects of $B \to X_s \gamma$ will be reviewed.

8.8 Semi-Leptonic Operators

In the case of $K^+ \to \pi^+ \nu \bar{\nu}$ we have encountered the operator $Q(\nu \bar{\nu}) = (\bar{s}d)_{V-A}(\bar{\nu}\nu)_{V-A}$. This operator governs also the decay $K_L \to \pi^0 \nu \bar{\nu}$. An analogous operator

$$Q^{B}(\nu\bar{\nu}) = (\bar{s}b)_{V-A}(\bar{\nu}\nu)_{V-A} \tag{8.127}$$

governs the inclusive decay $B \to X_s \nu \bar{\nu}$. We will briefly discuss this decay in section 13. $Q(\nu \bar{\nu})$ and $Q^B(\nu \bar{\nu})$ have no anomalous dimensions and the RG analysis of their Wilson coefficients in the case of internal top contributions is very simple. We have discussed this in the case of $K^+ \to \pi^+ \nu \bar{\nu}$ at the beginning of this section. This simplification is caused by the fact that neutrinos do not couple neither to gluons nor photons.

Now, in the case of $B \to X_s \mu^+ \mu^-$ and $K_L \to \pi^0 e^+ e^-$ the following operators play the dominant role:

$$Q_{9V} = (\bar{s}b)_{V-A}(\bar{\mu}\mu)_V \qquad Q_{10A} = (\bar{s}b)_{V-A}(\bar{\mu}\mu)_A \tag{8.128}$$

and

$$Q_{7V} = (\bar{s}d)_{V-A}(e^+e^-)_V \qquad Q_{7A} = (\bar{s}d)_{V-A}(e^+e^-)_A \tag{8.129}$$

respectively. We will discuss here briefly Q_{9V} and Q_{10A} . The analysis of Q_{7V} and Q_{7A} is analogous, but more involved because of lower renormalization scales involved and related threshold effects. Detailed expositions of NLO analyses of $B \to X_s \mu^+ \mu^-$ and $K_L \to \pi^0 e^+ e^-$ can be found in [101, 78] and [100] respectively.

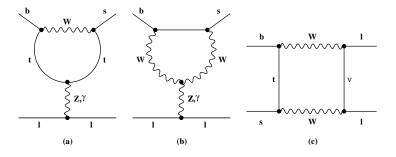


Figure 23: One loop diagrams in the full theory contributing to rare decays with charged leptons in the final state.

As in the case of $Q(\nu\bar{\nu})$ and $Q^B(\nu\bar{\nu})$, the semi-leptonic operators Q_{9V} and Q_{10A} have vanishing anomalous dimensions. However, the fact that charged leptons couple to photons makes the RG analysis of their coefficients more involved. Indeed these operators originate in the diagrams of fig. 23. Moreover, in the effective theory the diagrams in fig. 24 have to be

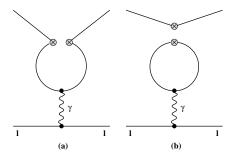


Figure 24: One loop diagrams in the effective theory contributing to rare decays with charged leptons in the final state.

considered, where the inserted operators are our good friends $(Q_1, ..., Q_6)$. The operator Q_{10A} involving the $\gamma_{\mu}\gamma_5$ current is uneffected by the diagrams involving photons and its Wilson coefficient can be calculated in the same manner as for the case of $Q(\nu\bar{\nu})$. Including gluon corrections to the one-loop diagrams in fig. 23 one finds:

$$C_{10A}(M_{\rm W}) = -\frac{\alpha}{2\pi} \frac{Y(x_t)}{\sin^2 \Theta_W}$$
 (8.130)

where

$$Y(x_t) = Y_0(x_t) + \frac{\alpha_s}{4\pi} Y_1(x_t)$$
(8.131)

with the one-loop function given by [45]

$$Y_0(x_t) = \frac{x_t}{8} \left[\frac{4 - x_t}{1 - x_t} + \frac{3x_t}{(1 - x_t)^2} \ln x_t \right]$$
 (8.132)

and

$$Y_{1}(x_{t}) = \frac{4x_{t} + 16x_{t}^{2} + 4x_{t}^{3}}{3(1 - x_{t})^{2}} - \frac{4x_{t} - 10x_{t}^{2} - x_{t}^{3} - x_{t}^{4}}{(1 - x_{t})^{3}} \ln x_{t}$$

$$+ \frac{2x_{t} - 14x_{t}^{2} + x_{t}^{3} - x_{t}^{4}}{2(1 - x_{t})^{3}} \ln^{2} x_{t} + \frac{2x_{t} + x_{t}^{3}}{(1 - x_{t})^{2}} L_{2}(1 - x_{t})$$

$$+ 8x \frac{\partial Y_{0}(x)}{\partial x} \ln \frac{\mu_{t}^{2}}{M_{W}^{2}}$$

$$(8.133)$$

resulting from two-loop calculations [98].

The μ_t -dependence of the last term in (8.133) cancels to the considered order the μ_t -dependence of the leading term $Y_0(x_t(\mu_t))$. The leftover μ_t dependence in $Y(x_t)$ is below 1%. For $\mu_t = m_t$, the complete function $Y(x_t)$ can be written as

$$Y(x_t) = \eta_Y \cdot Y_0(x_t), \qquad \eta_Y = 1.026 \pm 0.006,$$
 (8.134)

with the QCD factor η_Y practically independent of $m_{\rm t}$. The range in (8.134) corresponds to 150 GeV $\leq m_{\rm t} \leq 190$ GeV. The dependence on $\Lambda_{\overline{MS}}$ can be neglected.

The fate of the operator Q_{9V} is different. Now the diagrams with photon exchanges contribute in an important way. The presence of the photon penguin diagrams in the full theory brings in the function $D_0(x_t)$ of (8.120). On the other hand, the presence of $Q_1 - Q_6$ insertions into the photon penguin diagrams of fig. 24 introduces the mixing between $Q_1, ...Q_6$ operators and Q_{9V} under renormalization. That is the insertion of any of the four-quark operators into the diagrams in fig. 24 results in Q_{9V} multiplied by a certain coefficient. From these coefficients the entries $\gamma_{i9}^{(0)}$ with i = 1, ...6 in a 7 × 7 anomalous dimension matrix involving $(Q_1, ..., Q_6, Q_{9V})$ can be found. Including gluon corrections to fig. 24 gives $\gamma_{i9}^{(1)}$. Now, $\gamma_{i9}^{(0)}$ and $\gamma_{i9}^{(1)}$ are coefficients of α and $\alpha\alpha_s$ respectively. On the other hand the entries $\gamma_{ij}^{(0)}$ and $\gamma_{ij}^{(1)}$ with i, j = 1, ...6 are the coefficients of α_s and α_s^2 respectively. In order to work with an anomalous dimension matrix which has a usual expansion in α_s , it is convenient to introduce a new operator

$$Q'_{9V} = \frac{\alpha}{\alpha_s(\mu)} Q_{9V}, \qquad C'_{9V}(\mu) = \frac{\alpha_s(\mu)}{\alpha} C_{9V}(\mu)$$
 (8.135)

and perform the RG evolution for the set $(C_1, ... C_6, C'_{9V})$. Once $C'_{9V}(\mu)$ has been calculated by means of the standard method developed in sections 5 and 6, the coefficient $C_{9V}(\mu)$ can be obtained by using (8.135).

That the rescaling trick in (8.135) works at all, is related to the fact that Q_{9V} cannot mix back into the set $(Q_1, ..., Q_6)$. That is $\gamma_{9i}^{(0)} = \gamma_{9i}^{(1)} = 0$ for i = 1, ...6. This trick cannot be used in the case of electroweak penguin contributions to $\Delta F = 1$ decays discussed in subsection 8.6. There the mixing between the QCD penguin and electroweak penguin operators takes place in both directions and one does not gain anything by making a rescaling of electroweak four-quark operators. Fortunately the electroweak penguin operators in (8.105) and (8.107) contribute to $B \to X_s \mu^+ \mu^-$ and $K_L \to \pi^0 e^+ e^-$ first at $\mathcal{O}(\alpha^2)$ and consequently they can be fully neglected in these decays.

The anomalous dimensions $\gamma_{i9}^{(0)}$ and $\gamma_{i9}^{(1)}$ can be found in the formulae (VIII.11) and (VIII.12) of ref. [17]. One should note that, in contrast to Q_{9V} , the rescaled operator Q'_{9V} has effectively a non-vanishing anomalous dimension resulting from the presence of $\alpha_s(\mu)$ in (8.135):

$$\gamma_{99}^{(0)} = -2\beta_0, \qquad \gamma_{99}^{(1)} = -2\beta_1.$$
(8.136)

For completeness we give the result for $C_{9V}(\mu_b)$ including NLO corrections in the NDR scheme:

$$C_9^{NDR}(\mu_b) = \frac{\alpha}{2\pi} \left[P_0^{NDR} + \frac{Y_0(x_t)}{\sin^2 \Theta_W} - 4Z_0(x_t) \right], \tag{8.137}$$

where a negligible contribution proportional to $E_0(x_t)$ has been omitted. P_0^{NDR} is a m_t independent constant which for $\mu_b = 5.0 \,\text{GeV}$ and $\alpha_s(M_Z) = 0.118$ equals 2.59. The renormalization group improved perturbative expansion for P_0^{NDR} and $C_9^{NDR}(\mu_b)$ has the structure

 $1/\alpha_s + \mathcal{O}(1) + \mathcal{O}(\alpha_s)$..., as seen explicitly in the analytic formula (X.6) in [17]. For this reason in an NLO analysis of $B \to X_s \mu \bar{\mu}$ and also $K_L \to \pi^0 e^- e^+$ only the leading terms in $Y(x_t)$ and $Z(x_t)$, i.e. $Y_0(x_t)$ and $Z_0(x_t)$, contribute.

8.9 Charm Quarks in Electroweak Loops

Our discussion of QCD effects in penguin and box diagram contributions to rare decays concentrated on diagrams with internal top quark propagators. In the process of matching of the full theory onto effective five quark theory the top quark is integrated out together with W^{\pm} and Z^0 bosons and the resulting operators are local. The evolution of their coefficients down to low energy scales proceeds in the standard manner as discussed in the preceding sections.

In the case of penguin and box diagrams with internal charm quarks the situation is more complicated. After the matching at scales $\mathcal{O}(M_{\mathrm{W}})$ charm quarks remain as dynamical degrees of freedom and after W^{\pm} and Z^0 bosons have been integrated out one has to deal with bi-local structures rather than with local operators. An example is shown for the case of $K^+ \to \pi^+ \nu \bar{\nu}$ in fig. 25a. It results from Z^0 -penguin contributing to $K^+ \to \pi^+ \nu \bar{\nu}$. These structures have, in contrast to $Q(\nu \bar{\nu})$, anomalous dimensions which makes the renormalization group analysis non-trivial. Another important example are box diagram contributions to $K^0 - \bar{K}^0$ mixing where two internal charm propagators (see fig. 25b) or one charm propagator and one top propagator may appear simultaneously.

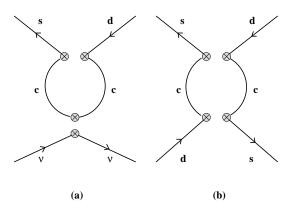


Figure 25: Bilocal Structures.

In all these cases the RG evolution from scales $\mu_W = \mathcal{O}(M_W)$ down to $\mu_c = \mathcal{O}(m_c)$ is more involved than the one presented sofar. In the process of matching of four-quark theory onto three-quark theory charm is integrated out and the effective theory below μ_c involves only local operators which can be analyzed in the standard manner. The renormalization

group analysis of bi-local structures is beyond the scope of these lectures. On the other hand the tools collected in this and preceding sections are sufficient for following detailed expositions of this subject without great difficulties. The internal charm contributions to $K^+ \to \pi^+ \nu \bar{\nu}$ are calculated in detail in [91] where the full two-loop renormalization group evolution is performed. A brief account of this analysis can be found in chapter XIB of the review [17]. We will use the numerical results obtained there in section 13. Similarly the contributions of charm to $K^0 - \bar{K}^0$ mixing are analyzed in detail in [92, 93]. A brief account of these papers can be found in chapters XIIC and XIID of the review [17]. The issue of internal charm contributions to non-leptonic two-body B-decays in the form of the so-called "charming penguins" is discussed in [104, 105].

8.10 Penguin–Box Expansion from OPE

In section 3 prior to the discussion of QCD effects we have formulated the FCNC decays in terms of effective vertices corresponding to various penguin and box diagrams. These effective vertices depend on a set of basic universal (process independent) m_t -dependent functions $F_r(x_t)$ listed in (3.29). We have also stated that any decay amplitude can be written as

$$A(\text{decay}) = P_0(\text{decay}) + \sum_r P_r(\text{decay}) F_r(x_t), \qquad (8.138)$$

where the coefficients P_r are process dependent. It is straightforward to derive these *Penguin-Box Expansion* [16] from OPE. To this end we use OPE and and rewrite a given decay amplitude $A(M \to F)$ as follows

$$A(M \to F) = \frac{G_{\rm F}}{\sqrt{2}} V_{\rm CKM} \sum_{i,k} \langle F \mid O_k(\mu) \mid M \rangle \, \hat{U}_{ki} \left(\mu, M_{\rm W}\right) \, C_i(M_{\rm W}), \tag{8.139}$$

where $\hat{U}_{kj}(\mu, M_W)$ is the renormalization group transformation from M_W down to μ given already at several places in these lectures.

Now, as we have seen in several examples in this section, $C_i(M_W)$ are linear combinations of the basic functions $F_r(x_t)$ so that we can write

$$C_i(M_W) = c_i + \sum_r h_{ir} F_r(x_t)$$
 (8.140)

where c_i and h_{ir} are m_t -independent constants. Inserting (8.140) into (8.139) and summing over i and k we recover (8.138) with

$$P_0(\text{decay}) = \sum_{i,k} \langle F \mid O_k(\mu) \mid M \rangle \, \hat{U}_{ki} (\mu, M_W) c_i , \qquad (8.141)$$

$$P_r(\text{decay}) = \sum_{i,k} \langle F \mid O_k(\mu) \mid M \rangle \, \hat{U}_{ki} (\mu, M_W) h_{ir} , \qquad (8.142)$$

where we have suppressed the overall factor $(G_F/\sqrt{2})V_{CKM}$.

The process dependence of P_0 and P_r enters through $\langle F \mid O_k(\mu) \mid M \rangle$. In certain cases like $K \to \pi \nu \bar{\nu}$ these matrix elements are very simple implying simple formulae for the coefficients P_0 and P_r . In other situations, like ε'/ε discussed in section 11, this is not the case.

From the perspective of the formula (8.139) the relation between the usual OPE and its PBE-version is clear. OPE puts the last two factors together by summing over "i" to obtain $C_k(\mu)$. The PBE is realized on the other hand by putting the first two factors together through the summation over "k" and subsequent rewriting of $C_i(M_W)$ in terms of $F_r(x_t)$ as explicitly shown in (6.59). Equivalently PBE is obtained by setting $\mu = M_W$ in (8.139) as \hat{U}_{ki} (μ, M_W) = δ_{ki} .

PBE is very well suited for the study of the extentions of the Standard Model in which new particles are exchanged in the loops. We know already that these particles are heavier than W-bosons and consequently they can be integrated out together with the weak bosons and the top quark. If there are no new local operators the mere change is to modify the functions $F_r(x_t)$ which now acquire the dependence on the masses of new particles such as charged Higgs particles and supersymmetric particles. The process dependent coefficients P_0 and P_r remain unchanged unless new effective operators with different Dirac and colour structures have to be introduced. Examples of the applications of PBE to physics beyond the Standard Model can be found in [86, 87, 88].

The universality of the functions $F_r(x_t)$ listed in (3.29) can be violated partly when QCD corrections to one loop penguin and box diagrams are included. For instance in the case of semi-leptonic FCNC transitions there is no gluon exchange in a Z^0 -penguin diagram parallel to the Z^0 -propagator but such an exchange takes place in non-leptonic decays in which the bottom line is a quark-line. Thus the general universality of $F_r(x_t)$ present at one loop level is reduced to two universality classes relevant for semi-leptonic and non-leptonic transitions. However, as we have seen in the case of $K \to \pi \nu \bar{\nu}$, the $\mathcal{O}(\alpha_s)$ corrections to the function $X_0(x_t)$ could be absorbed into an overall QCD factor η_X which moreover with a proper definition of m_t turned out to be essentially m_t -independent. Similar situations take place in other decays so that to a very good approximation the top mass dependence is governed even at the two-loop level through the one-loop functions and the inclusion of QCD effects plays mainly the role in reducing the μ_t -dependences.

8.11 Status of NLO Calculations

We end this section by listing all existing NLO calculations for weak decays in table 10. Further details on these calculations can be found in the original papers and in the review [17]. Some of the implications of these calculations will be analyzed in detail in subsequent

sections.

Table 10: References to NLO Calculations

Decay	Reference					
$\Delta F = 1$ Decays						
current-current operators	[53, 48]					
QCD penguin operators	[68, 73, 74, 75]					
electroweak penguin operators	[69, 73, 74, 75]					
magnetic penguin operators	[94, 110]					
$Br(B)_{SL}$	[53, 95, 96]					
inclusive $\Delta S = 1$ decays	[97]					
Particle-Antiparticle M	Mixing					
η_1	[92]					
$\eta_2,~\eta_B$	[90]					
η_3	[93]					
Rare K - and B -Meson	Decays					
$K_L^0 \to \pi^0 \nu \bar{\nu}, B \to l^+ l^-, B \to X_{\rm s} \nu \bar{\nu}$	[65, 98]					
$K^+ \to \pi^+ \nu \bar{\nu}, K_{\mathrm{L}} \to \mu^+ \mu^-$	[91]					
$K^+ o \pi^+ \mu \bar{\mu}$	[99]					
$K_{ m L} ightarrow \pi^0 e^+ e^-$	[100]					
$B o X_s \mu^+ \mu^-$	[101, 78]					
$B o X_s \gamma$	[106]-[112], [66, 67]					

8.12 Final Remarks

We are roughly half way through these lectures. The last seven sections dealt with the basic formalism of weak decays. The next seven sections will present some phenomenological applications of this formalism.

9 Non-Leptonic Two-body Decays and Factorization

9.1 Preliminaries

We will begin the applications of the formalism developed in the previous seven sections by discussing two-body non-leptonic decays. Although our discussion will concentrate on two-body B-decays, it can be generalized in a straightforward manner to D-decays.

I should state from the beginning that it is not my intention to give here a review of two-body decays and present detailed comparision with the available data. My intention is rather to reanalyze critically the concepts of the *factorization* hypothesis and in particular of the *generalized factorization* hypothesis discussed in the literature. As we will see soon, it is an excellent battle field for the formalism developed in previous sections.

Now comes a rather unfortunate move. In this and only this section we have to modify slightly our notation by interchanging the indices 1 and 2 in current-current operators. This we have to do in order to conform to the notation used in the literature on two-body non-leptonic decays. Thus we introduce the operators

$$O_1 = Q_2, \qquad O_2 = Q_1, \tag{9.1}$$

and their respective coefficients

$$\bar{C}_1(\mu) = C_2(\mu), \qquad \bar{C}_2(\mu) = C_1(\mu).$$
 (9.2)

Correspondingly

$$O_{\pm} = \frac{O_1 \pm O_2}{2}, \qquad z_{\pm} = \bar{C}_1(\mu) \pm \bar{C}_2(\mu),$$
 (9.3)

and

$$\bar{C}_1(\mu) = \frac{z_+(\mu) + z_-(\mu)}{2}, \qquad \bar{C}_2(\mu) = \frac{z_+(\mu) - z_-(\mu)}{2}, \qquad (9.4)$$

with all formulae (7.8)–(7.14) unchanged. We have introduced a "bar", omitted in the literature, in order to avoid possible confusion.

This section is based on [89] and the recent collaboration with Luca Silvestrini [113]. We do not cover here more dynamical approaches to non-leptonic decays like QCD sum reles. A very nice review of the applications of QCD sum rules to non-leptonic decays has been presented this year by Khodjamirian and Rückl [114] and is strongly recommended.

9.2 Factorization

In the factorization approach to non-leptonic meson decays [115, 116] one can distinguish three classes of decays for which the amplitudes have the following general structure [117, 118]:

$$A_{\rm I} = \frac{G_F}{\sqrt{2}} V_{CKM} a_1(\mu) \langle O_1 \rangle_F \qquad \text{(Class I)}, \tag{9.5}$$

$$A_{\rm II} = \frac{G_F}{\sqrt{2}} V_{CKM} a_2(\mu) \langle O_2 \rangle_F \qquad \text{(Class II)}, \tag{9.6}$$

$$A_{\text{III}} = \frac{G_F}{\sqrt{2}} V_{CKM}[a_1(\mu) + x a_2(\mu)] \langle O_1 \rangle_F \qquad \text{(Class III)}. \tag{9.7}$$

Here V_{CKM} denotes symbolically the CKM factor characteristic for a given decay. $\langle O_i \rangle_F$ are factorized hadronic matrix elements of the operators O_i given as products of matrix elements of quark currents and x is a non-perturbative factor equal to unity in the flavour symmetry limit. Finally $a_i(\mu)$ are QCD factors which are given as follows

$$a_1(\mu) = \bar{C}_1(\mu) + \frac{1}{N}\bar{C}_2(\mu), \qquad a_2(\mu) = \bar{C}_2(\mu) + \frac{1}{N}\bar{C}_1(\mu).$$
 (9.8)

We will soon give explicit examples and we will rederive these formulae as limiting cases of the generalized factorization hypothesis. First, however, we would like to make a few general comments on the weak points of this approach.

At first sight the simplicity of this approach is very appealing. Once the matrix elements $\langle O_i \rangle_F$ have been expressed in terms of various meson decay constants and generally model dependent formfactors, predictions for non-leptonic heavy meson decays can be made. Moreover relations between non-leptonic and semi-leptonic decays can be found which allow to test factorization in a model independent manner. An incomplete list of analyses of this type is given in [124, 118] and will be extended below.

On the other hand, it is well known that non-factorizable contributions must be present in the hadronic matrix elements of the current–current operators O_1 and O_2 in order to cancel the μ dependence of $\bar{C}_i(\mu)$ or $a_i(\mu)$ so that the physical amplitudes do not depend on the arbitrary renormalization scale μ . $\langle O_i \rangle_F$ being products of matrix elements of conserved currents are μ -independent and the cancellation of the μ dependence in (9.5)–(9.7) does not take place. Consequently from the point of view of QCD the factorization approach can be at best correct at a single value of μ , the so-called factorization scale μ_f . Although the approach itself does not provide the value of μ_f , the proponents of factorization expect $\mu_f = O(m_b)$ and $\mu_f = O(m_c)$ for B-decays and D-decays respectively.

Here we would like to point out that beyond the leading logarithmic approximation for $\bar{C}_i(\mu)$ a new complication arises. As we have discussed in previous sections, at next to leading level in the renormalization group improved perturbation theory the coefficients $\bar{C}_i(\mu)$ depend on the renormalization scheme for operators. Again only the presence of non-factorizable contributions in $\langle O_i \rangle$ can remove this scheme dependence in the physical amplitudes. However $\langle O_i \rangle_F$ are renormalization scheme independent and the factorization approach is of course unable to tell us whether it works better with an anti-commuting γ_5 in $D \neq 4$ dimensions (NDR scheme) or with another definition of γ_5 such as used in HV or DRED schemes. Moreover there are other renormalization schemes parametrized by κ_{\pm} in (7.11)–(7.14). The

renormalization scheme dependence emphasized here is rather annoying from the factorization point of view as it precludes a unique phenomenological determination of μ_f as we will show explicitly below.

On the other hand, arguments have been given [119, 120, 118], that factorization approach could be approximately true in the case of two-body decays with high energy release [119], or in certain kinematic regions [120, 122, 123]. We will not repeat here these arguments, which can be found in the original papers. Needless to say the issue of factorization does not only involve the short distance gluon corrections discussed here but also final state interactions as stressed in particular in [123].

It is difficult to imagine that factorization can hold even approximately in all circumstances. In spite of this, it became fashonable these days to test this idea, to some extent, by using certain set of formfactors to calculate $\langle O_i \rangle_F$ and by making global fits of the formulae (9.5)–(9.7) to the data treating a_1 and a_2 as free independent parameters. As an example we give the result of a recent analysis of this type for non-leptonic two-body B-decays [121]

$$a_1 \approx 1.08 \pm 0.04$$
 $a_2 \approx 0.21 \pm 0.05$ (9.9)

which is compatible with other analyses [125, 126, 124, 127, 128]. At the level of accuracy of the existing experimental data and because of strong model dependence in the relevant formfactors it is not yet possible to conclude on the basis of these analyses whether the factorization approach is a useful approximation in general or not. It is certainly conceivable that factorization may apply better to some non-leptonic decays than to others [118]-[122] and using all decays in a global fit may misrepresent the true situation.

The fact that $\langle O_i \rangle_F$ are μ -independent but $a_i(\mu)$ are μ -dependent, which is clearly inconsistent, inspired a number of authors [125, 126, 121, 127, 128] to generalize the concept of factorization. The presentation given in the next subsection, done in collaboration with Silvestrini [113], follows closely the generalization due to Neubert and Stech [121] which deals exclusively with the operators O_1 and O_2 . The generalization presented in [125, 127, 128] are similar in spirit but includes also the penguin contributions. I will discuss it briefly at the end of this section. In particular the very recent analysis of Ali, Kramer and Lü [128] is very informative.

9.3 Generalized Factorization

In the generalized factorization framework the formulae (9.5)–(9.7) are simply replaced by

$$A_{\rm I} = \frac{G_F}{\sqrt{2}} V_{CKM} a_1^{\rm eff} \langle O_1 \rangle_F \qquad \text{(Class I)}, \tag{9.10}$$

$$A_{\rm II} = \frac{G_F}{\sqrt{2}} V_{CKM} a_2^{\rm eff} \langle O_2 \rangle_F \qquad \text{(Class II)}, \tag{9.11}$$

$$A_{\text{III}} = \frac{G_F}{\sqrt{2}} V_{CKM} [a_1^{\text{eff}} + x a_2^{\text{eff}}] \langle O_1 \rangle_F \qquad \text{(Class III)}, \tag{9.12}$$

where a_i^{eff} are μ -independent and renormalization scheme independent parameters to be extracted from experimental data. From phenomenological point of view there is no change here relative to the standard factorization as only $a_i(\mu)$ have been replaced by a_i^{eff} . On the other hand, as stressed in particular in [121], the new formulation should allow in principle some insight into the importance of non-factorizable contributions.

In this context I should remark that in the recent literature mainly the μ -dependence of the non-factorizable contributions has been emphasized. Their scheme dependence has been only discussed in [89]. It is the latter issue which will be important in the discussion below. Let us then derive the formulae for a_i^{eff} including NLO corrections.

In order to describe generalized factorization in explicit terms let us consider the decay $\bar{B}^0 \to D^+\pi^-$. Then the relevant effective Hamiltonian is given by

$$H_{eff} = \frac{G_F}{\sqrt{2}} V_{cb} V_{ud}^* [\bar{C}_1(\mu) O_1 + \bar{C}_2(\mu) O_2] , \qquad (9.13)$$

where

$$O_1 = (\bar{d}_{\alpha}u_{\alpha})_{V-A}(\bar{c}_{\beta}b_{\beta})_{V-A} \qquad O_2 = (\bar{d}_{\alpha}u_{\beta})_{V-A}(\bar{c}_{\beta}b_{\alpha})_{V-A} . \tag{9.14}$$

 $\bar{C}_1(\mu)$ and $\bar{C}_2(\mu)$ are computed at the renormalization scale $\mu = O(m_b)$. Since all four quark flavours entering the operators in (9.14) are different from each other, no penguin operators contribute to this decay.

Using Fierz reordering and colour identities one can rewrite the amplitude for $\bar{B}^0 \to D^+\pi^-$ as

$$A(\bar{B}^0 \to D^+ \pi^-) = \frac{G_F}{\sqrt{2}} V_{cb} V_{ud}^* a_1^{\text{eff}} \langle O_1 \rangle_F$$
 (9.15)

where

$$\langle O_1 \rangle_F = \langle \pi^- \mid (\bar{d}u)_{V-A} \mid 0 \rangle \langle D^+ \mid (\bar{c}b)_{V-A} \mid \bar{B}^0 \rangle$$

$$(9.16)$$

is the factorized matrix element of the operator O_1 and summation over colour indices in each current is understood.

The effective parameter a_1^{eff} is then given by [121]

$$a_1^{\text{eff}} = \left(\bar{C}_1(\mu) + \frac{1}{N}\bar{C}_2(\mu)\right) \left[1 + \varepsilon_1^{(BD,\pi)}(\mu)\right] + \bar{C}_2(\mu)\varepsilon_8^{(BD,\pi)}(\mu). \tag{9.17}$$

 $\varepsilon_1^{(BD,\pi)}(\mu)$ and $\varepsilon_8^{(BD,\pi)}(\mu)$ are two hadronic parameters defined by

$$\varepsilon_1^{(BD,\pi)}(\mu) \equiv \frac{\langle \pi^- D^+ | (\bar{d}u)_{V-A}(\bar{c}b)_{V-A} | \bar{B}^0 \rangle}{\langle O_1 \rangle_F} - 1 \tag{9.18}$$

and

$$\varepsilon_8^{(BD,\pi)}(\mu) \equiv 2 \frac{\langle \pi^- D^+ | (\bar{d}T^a u)_{V-A} (\bar{c}T^a b)_{V-A} | \bar{B}^0 \rangle}{\langle O_1 \rangle_F}$$
(9.19)

with T^a denoting the colour matrices in the standard Feynman rules. $\varepsilon_i(\mu)$ parametrize the non-factorizable contributions to the hadronic matrix elements of operators. In the case of strict factorization ε_i vanish and a_1^{eff} reduces to $a_1(\mu)$.

It should be emphasized that no approximation has been made in (9.15). Since the matrix element $\langle O_1 \rangle_F$ is scale and renormalization scheme independent this must also be the case for the effective coefficient a_1^{eff} . Indeed the scale and scheme dependences of the coefficients $\bar{C}_1(\mu)$ and $\bar{C}_2(\mu)$ are cancelled by those present in the hadronic parameters $\varepsilon_i(\mu)$. We will give explicit formulae for the latter dependences below.

A similar exercise with the amplitude for $\bar{B}^0 \to D^0 \pi^0$ gives

$$A(\bar{B}^0 \to D^0 \pi^0) = \frac{G_F}{\sqrt{2}} V_{cb} V_{ud}^* a_2^{\text{eff}} \langle O_2 \rangle_F,$$
 (9.20)

where

$$\langle O_2 \rangle_F = \langle D^0 \mid (\bar{c}u)_{V-A} \mid 0 \rangle \langle \pi^0 \mid (\bar{d}b)_{V-A} \mid \bar{B}^0 \rangle \tag{9.21}$$

is the factorized matrix element of the operator O_2 .

The effective parameter a_2^{eff} is given by [121]

$$a_2^{\text{eff}} = \left(\bar{C}_2(\mu) + \frac{1}{N}\bar{C}_1(\mu)\right) \left[1 + \varepsilon_1^{(B\pi,D)}(\mu)\right] + \bar{C}_1(\mu)\varepsilon_8^{(B\pi,D)}(\mu). \tag{9.22}$$

 $\varepsilon_1^{(B\pi,D)}(\mu)$ and $\varepsilon_8^{(B\pi,D)}(\mu)$ are two hadronic parameters defined by

$$\varepsilon_1^{(B\pi,D)}(\mu) \equiv \frac{\langle \pi^0 D^0 | (\bar{c}u)_{V-A} (\bar{d}b)_{V-A} | \bar{B}^0 \rangle}{\langle O_2 \rangle_F} - 1 \tag{9.23}$$

and

$$\varepsilon_8^{(B\pi,D)}(\mu) \equiv 2 \frac{\langle \pi^0 D^0 | (\bar{c} T^a u)_{V-A} (\bar{d} T^a b)_{V-A} | \bar{B}^0 \rangle}{\langle O_2 \rangle_F} . \tag{9.24}$$

Again the μ and scheme dependences of ε_i in (9.23) and (9.24) cancel the corresponding dependences in $\bar{C}_i(\mu)$ so that the effective coefficient a_2^{eff} is μ and scheme independent. Similarly one can derive the formula (9.12) by using $B^- \to D^0 K^-$ or other decay belonging to class III.

Following section 5.1 of [73] and using the experience accumulated in previous sections it is straightforward to find the explicit μ and scheme dependences of the hadronic parameters $\varepsilon_i(\mu)$. To this end we note that the μ dependence of the matrix elements of the operators O_{\pm} is given by

$$\langle O_{\pm}(\mu) \rangle = U_{\pm}(m_{\rm b}, \mu) \langle O_{\pm}(m_{\rm b}) \rangle , \qquad (9.25)$$

where the evolution function $U_{\pm}(m_{\rm b}, \mu)$ including NLO QCD corrections is given as in (6.6) by

$$U_{\pm}(m_{\rm b}, \mu) = \left[1 + \frac{\alpha_s(m_{\rm b})}{4\pi}J_{\pm}\right] \left[\frac{\alpha_s(\mu)}{\alpha_s(m_{\rm b})}\right]^{d_{\pm}} \left[1 - \frac{\alpha_s(\mu)}{4\pi}J_{\pm}\right]$$
(9.26)

with J_{\pm} and d_{\pm} in (7.9). Note the different ordering of scales in (9.25) from the one in the evolution of Wilson coefficients in (8.53).

Having these formulae at hand it is straightforward to show that the μ -dependence of $\varepsilon_1(\mu)$ and $\varepsilon_8(\mu)$ is governed by the following equations:

$$1 + \varepsilon_{1}(\mu) = \frac{1}{2} \left[\left(1 + \frac{1}{N} \right) \left[1 + \varepsilon_{1}(m_{b}) \right] + \varepsilon_{8}(m_{b}) \right] U_{+}(m_{b}, \mu)$$

$$+ \frac{1}{2} \left[\left(1 - \frac{1}{N} \right) \left[1 + \varepsilon_{1}(m_{b}) \right] - \varepsilon_{8}(m_{b}) \right] U_{-}(m_{b}, \mu),$$
(9.27)

$$\varepsilon_{8}(\mu) = \frac{1}{2} \left[\left(1 - \frac{1}{N} \right) \varepsilon_{8}(m_{b}) + \left(1 - \frac{1}{N^{2}} \right) \left[1 + \varepsilon_{1}(m_{b}) \right] \right] U_{+}(m_{b}, \mu)
+ \frac{1}{2} \left[\left(1 + \frac{1}{N} \right) \varepsilon_{8}(m_{b}) - \left(1 - \frac{1}{N^{2}} \right) \left[1 + \varepsilon_{1}(m_{b}) \right] \right] U_{-}(m_{b}, \mu).$$
(9.28)

It is a very good exercise to derive these formulae and any student who wants to test her (his) skills in this field should try it.

These formulae reduce to the ones given in [121] when J_{\pm} in (9.26) are set to zero. They give both the μ -dependence and renormalization scheme dependence of ε_i . The latter dependence has not been considered in [121]. We will return to these expressions in a moment. First, however, we would like to formulate the generalized factorization in a more transparent manner.

9.4 A Different Formulation

In order to be able to discuss the relation of our presentation [113] to the one of [121] we have used until now, as in [121], the hadronic parameters $\varepsilon_1(\mu)$ and $\varepsilon_8(\mu)$ to describe non-factorizable contributions. It appears to us that it is more convenient to work instead with two other parameters defined simply by [113]

$$a_1^{\text{eff}} = a_1(\mu) + \xi_1^{\text{NF}}(\mu), \qquad a_2^{\text{eff}} = a_2(\mu) + \xi_2^{\text{NF}}(\mu),$$
 (9.29)

where $a_i(\mu)$ are defined in (9.8). Comparison with (9.17) and (9.22) gives

$$\xi_1^{\text{NF}}(\mu) = \varepsilon_1(\mu)a_1(\mu) + \varepsilon_8(\mu)\bar{C}_2(\mu),$$
(9.30)

$$\xi_2^{\text{NF}}(\mu) = \bar{\varepsilon}_1(\mu)a_2(\mu) + \bar{\varepsilon}_8(\mu)\bar{C}_1(\mu),$$
(9.31)

where

$$\varepsilon_1(\mu) = \varepsilon_1^{(BD,\pi)}, \qquad \varepsilon_8(\mu) = \varepsilon_8^{(BD,\pi)},$$
 (9.32)

$$\bar{\varepsilon}_1(\mu) = \varepsilon_1^{(B\pi,D)}, \qquad \bar{\varepsilon}_8(\mu) = \varepsilon_8^{(B\pi,D)}.$$
 (9.33)

In the framework of the strict factorization hypothesis $\xi_i^{\rm NF}(\mu)$ are set to zero. Their μ and scheme dependences can in principle be found by using the dependences of $\bar{C}_i(\mu)$ given in section 7 and of $\varepsilon_i(\mu)$ in (9.27) and (9.28). To this end, however, one needs the determination of the non-perturbative parameters $\varepsilon_i(\mu)$ and $\bar{\varepsilon}_i(\mu)$ at a single value of μ . If, as done in [121], $a_i^{\rm eff}$ are universal parameters, the determination of $\varepsilon_i(\mu)$ and $\bar{\varepsilon}_i(\mu)$ is only possible if one also makes the following universality assumptions:

$$\varepsilon_1(\mu) = \bar{\varepsilon}_1(\mu), \qquad \varepsilon_8(\mu) = \bar{\varepsilon}_8(\mu).$$
 (9.34)

In [121] such an assumption was unnecessary as $\varepsilon_1(\mu)$ has been set to zero and only $\varepsilon_8(\mu)$ has been extracted from the data.

With the assumptions in (9.34), $\varepsilon_1(\mu)$ and $\varepsilon_8(\mu)$ can indeed be found once the effective parameters a_i^{eff} have been determined experimentally. Using (9.17) and (9.22) together with (9.34) we find

$$\varepsilon_1(\mu) = \frac{\bar{C}_1(\mu)a_1^{\text{eff}} - \bar{C}_2(\mu)a_2^{\text{eff}}}{\bar{C}_1^2(\mu) - \bar{C}_2^2(\mu)} - 1 , \qquad (9.35)$$

$$\varepsilon_8(\mu) = \frac{a_2^{\text{eff}}}{\bar{C}_1(\mu)} - \left(\frac{\bar{C}_2(\mu)}{\bar{C}_1(\mu)} + \frac{1}{N}\right) [1 + \varepsilon_1(\mu)] . \tag{9.36}$$

On the other hand $\xi_i^{\rm NF}(\mu)$ can be determined without the universality assumption (9.34) from two decays simply as follows

$$\xi_1^{\text{NF}}(\mu) = a_1^{\text{eff}} - a_1(\mu) , \qquad \xi_2^{\text{NF}}(\mu) = a_2^{\text{eff}} - a_2(\mu) .$$
 (9.37)

Formulae in (9.37) make it clear that the strict factorization in which $\xi_i^{\text{NF}}(\mu)$ vanish can be at best correct at a single value of μ , the so-called factorization scale μ_f . In the first studies of factorization $\mu_f = m_{\text{b}}$ has been assumed. It has been concluded that such a choice is not in accord with the data. The idea of the generalized factorization as formulated in [125, 126, 121] is to allow μ_f to be different from m_{b} and to extract first non-factorizable parameters $\varepsilon_i(m_{\text{b}})$ from the data. Subsequently factorization scale μ_f can be found by requiring these parameters to vanish.

In the numerical analysis of this procedure done in [121] one additional assumption has been made. Using large N arguments it has been argued that $\varepsilon_1(\mu)$ can be set to zero while $\varepsilon_8(\mu)$ can be sizable. The resulting expressions for a_i^{eff} are then

$$a_1^{\text{eff}} = \bar{C}_1(m_b), \qquad a_2^{\text{eff}} = a_2(m_b) + \bar{C}_1(m_b)\varepsilon_8(m_b) , \qquad (9.38)$$

where additional small terms have been dropped in order to obtain the formula for $a_1^{\rm eff}$. Using subsequently the extracted value $a_2^{\rm eff}=0.21\pm0.05$ together with the coefficients $\bar{C}_i(m_{\rm b})$

from [48] one finds $\varepsilon_8(m_b) = 0.12 \pm 0.05$ [121]. Next assuming $\varepsilon_8(\mu_f) = 0$ one can find the factorization scale μ_f by inverting the formula [121]

$$\varepsilon_8(m_b) = -\frac{4\alpha_s(m_b)}{3\pi} \ln \frac{m_b}{\mu_f} , \qquad (9.39)$$

which follows from (9.28) with $\varepsilon_8(\mu_f) = 0$ and $\varepsilon_1(m_b) = 0$. Thus

$$\mu_f = m_{\rm b} \exp\left[\frac{3\pi\varepsilon_8(m_{\rm b})}{4\alpha_s(m_{\rm b})}\right] . \tag{9.40}$$

Taking $m_{\rm b}=4.8$ GeV and $\alpha_s(m_b)=0.21$ (corresponding to $\alpha_s(M_{\rm Z})=0.118$) we find using $\varepsilon_8(m_{\rm b})=0.12\pm0.05$ a rather large factorization scale $\mu_f=(15.9+11.3-6.6)$ GeV, by roughly a factor of 3-4 higher than $m_{\rm b}$. This implies that non-factorizable contributions in hadronic matrix elements at scales close to $m_{\rm b}$ are sizable. This is also signalled by the value of $\varepsilon_8(m_{\rm b})\approx0.12$ which is larger than the factorizable contribution $a_2(m_{\rm b})=0.09$ to the effective parameter $a_2^{\rm eff}=0.21\pm0.05$.

We would like to emphasize that such an interpretation of the analysis of Neubert and Stech [121] would be misleading. As stressed in [89] the coefficient $a_2(\mu)$ is very strongly dependent on the renormalization scheme. Consequently for a given value of a_2^{eff} also $\xi_2^{NF}(m_b)$ and $\varepsilon_8(m_b)$ are strongly scheme dependent. This shows, that a meaningful analysis of the μ -dependences in non-leptonic decays, such as the search for the factorization scale μ_f , cannot be made without simultaneously considering the scheme dependence. This is evident if one recalls that any variation of μ_f in the leading logarithm is equivalent to a shift in constant non-logarithmic terms. The latter represent NLO contributions in the renormalization group improved perturbation theory and must be included for a meaningful extraction of μ_f or any other scale like $\Lambda_{\overline{\text{MS}}}$. However, once the NLO contributions are taken into account, the renormalization scheme dependence enters the analysis and consequently the factorization scale μ_f at which the non-factorizable hadronic parameters $\xi_i^{NF}(\mu_f)$ or $\varepsilon_i(\mu_f)$ vanish is renormalization scheme dependent. Formula (7.16) exhibits all these statements very clearly.

From this discussion it becomes clear that for any chosen scale $\mu_f = \mathcal{O}(m_b)$, it is always possible to find a renormalization scheme for which

$$\xi_1^{NF}(\mu_f) = \xi_2^{NF}(\mu_f) = 0. (9.41)$$

Indeed as seen in (9.37) $\xi_i^{NF}(\mu)$ depend through $a_i(\mu)$ on κ_{\pm} (see section 7) which characterize a given renormalization scheme. The choice of κ_{\pm} corresponds to a particular finite renormalization of the operators O_{\pm} in addition to the renormalization in the NDR scheme. It is then straightforward to find the values of κ_{\pm} which assure that for a chosen scale μ_f the conditions in (9.41) are satisfied. We find [113]

$$\kappa_{+} = 3 \left[\frac{3}{4} \frac{a_{1}^{\text{eff}} + a_{2}^{\text{eff}}}{W_{+}(\mu_{f})} - 1 \right] \frac{4\pi}{\alpha_{s}(\mu_{f})} - 3(J_{+})_{\text{NDR}} , \qquad (9.42)$$

$$\kappa_{-} = \frac{3}{2} \left[\frac{3}{2} \frac{a_{1}^{\text{eff}} - a_{2}^{\text{eff}}}{W_{-}(\mu_{f})} - 1 \right] \frac{4\pi}{\alpha_{s}(\mu_{f})} - \frac{3}{2} (J_{-})_{\text{NDR}} , \qquad (9.43)$$

where

$$W_{\pm}(\mu_f) = \left[\frac{\alpha_s(M_W)}{\alpha_s(\mu_f)} \right]^{d_{\pm}} \left[1 + \frac{\alpha_s(M_W)}{4\pi} (B_{\pm} - J_{\pm}) \right]$$
(9.44)

with $(J_{\pm})_{\text{NDR}}$ being the values of J_{\pm} in the NDR scheme. $W_{\pm}(\mu_f)$ are clearly renormalization scheme independent as $B_{\pm} - J_{\pm}$ and d_{\pm} are scheme independent.

9.5 Numerical Analysis

Before presenting the numerical analysis of the formulae derived in the preceding subsections, it is important to clarify the difference between the Wilson coefficients in (9.4) used by us and the ones employed in [121]. In [121] scheme independent coefficients $\tilde{z}_{\pm}(\mu)$ of [48] instead of $z_{\pm}(\mu)$ have been used. These are obtained by multiplying $z_{\pm}(\mu)$ by $(1 - B_{\pm}\alpha_s(\mu)/4\pi)$ so that

$$\tilde{z}_{\pm}(\mu) = \left[\frac{\alpha_s(M_W)}{\alpha_s(\mu)}\right]^{d_{\pm}} \left[1 + \frac{\alpha_s(M_W) - \alpha_s(\mu)}{4\pi} (B_{\pm} - J_{\pm})\right]. \tag{9.45}$$

These coefficients are clearly not the coefficients of the operators O_{\pm} . In order to be consistent, the matrix elements $\langle O_{\pm} \rangle$ should then be replaced by

$$\langle \tilde{O}_{\pm} \rangle = (1 + B_{\pm} \alpha_s(\mu)/4\pi) \langle O_{\pm} \rangle.$$
 (9.46)

This, however, has not been done in [121]. This explains, to a large extent, why our results for $\varepsilon_8(m_b)$ differ considerably from the ones quoted in [121]. We strongly advice the practitioners of non-leptonic decays not to use the scheme independent coefficients of [48] in phenomenological applications. These coefficients have been introduced to test the compatibility of different renormalization schemes and can only be used for phenomenology together with $\langle \tilde{O}_{\pm} \rangle$. This would however unnecessarily complicate the analysis and it is therefore advisable to work with the true coefficients $\bar{C}_i(\mu)$ of the operators O_i as given in (9.4).

In [121] the values of a_i^{eff} given in (9.9) have been extracted from existing data on two-body B-decays. In order to illustrate various points made until now, we take the central values of a_i^{eff} in (9.9) and calculate $\varepsilon_i(\mu)$ and $\xi_i^{NF}(\mu)$ as functions of μ in the range 2.5 GeV $\leq \mu \leq$ 10 GeV for the NDR and HV schemes. The results are shown in fig. 26 and fig. 27. We observe that $\varepsilon_1(\mu)$ and $\xi_1^{\text{NF}}(\mu)$ are only weakly μ and scheme dependent in accordance with the findings in [89], where these dependences have been studied for $a_i(\mu)$ defined in (9.8). The strong μ and scheme dependences of $a_2(\mu)$ found there translate into similar strong dependences of $\varepsilon_8(\mu)$ and $\xi_2^{\text{NF}}(\mu)$.

We make the following observations:

• $\varepsilon_1(\mu)$ and $\xi_1^{\rm NF}(\mu)$ are non-zero in the full range of μ considered.

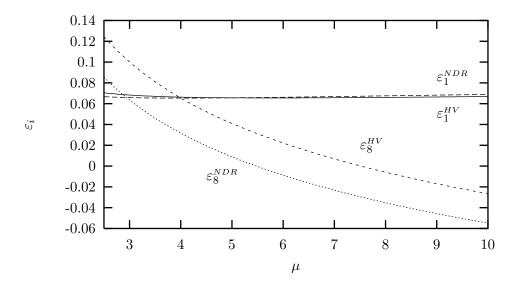


Figure 26: $\varepsilon_{1,8}(\mu)$ in the NDR and HV schemes.

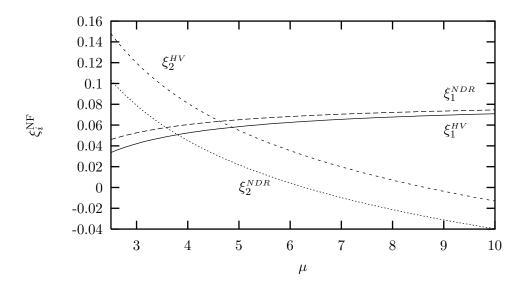


Figure 27: $\xi_{1,2}^{\rm NF}(\mu)$ in the NDR and HV schemes.

- $\varepsilon_8(\mu)$ and $\xi_2^{\text{NF}}(\mu)$ vary strongly with μ and vanish in the NDR scheme for $\mu = 5.5 \text{ GeV}$ and $\mu = 6.3 \text{ GeV}$ respectively. The corresponding values in the HV scheme are $\mu = 7.5 \text{ GeV}$ and $\mu = 8.6 \text{ GeV}$.
- There is no value of $\mu = \mu_f$ in the full range considered for which $\varepsilon_1(\mu)$ and $\varepsilon_8(\mu)$ or equivalently $\xi_1^{\text{NF}}(\mu)$ and $\xi_2^{\text{NF}}(\mu)$ simultaneously vanish. We also observe contrary to expectations in [121] that $\varepsilon_1(\mu)$ is not necessarily smaller than $\varepsilon_8(\mu)$. In fact the large N arguments presented in [121] that $\varepsilon_1(\mu) = \mathcal{O}(1/N^2)$ and $\varepsilon_8(\mu) = \mathcal{O}(1/N)$, imply strictly speaking only that the μ -dependence of $\varepsilon_8(\mu)$ is much stronger than that of $\varepsilon_1(\mu)$, which we indeed see in figs. 26 and 27. The hierarchy of their actual values is a dynamical question. Even if the large N-counting-rules $\varepsilon_1(\mu) = \mathcal{O}(1/N^2)$ and $\varepsilon_8(\mu) = \mathcal{O}(1/N)$ are true independently of the factorization hypothesis [129, 130], it follows from our analysis that once the generalized factorization hypothesis is made, the extracted values of ε_i violate for some range of μ the large-N rule $\varepsilon_1 \ll \varepsilon_8$.

We can next investigate for which renormalization scheme characterized by κ_{\pm} the factorization is exact at $\mu_f = m_b = 4.8$ GeV. We call this choice the "factorization scheme" (FS). Using the central values in (9.9) and $\Lambda_{\overline{\rm MS}}^{(5)} = 225\,{\rm MeV}$ we find by means of (9.42) and (9.43)

$$\kappa_{+} = 13.5, \qquad \kappa_{-} = 3.9 \qquad \text{(FS)}.$$
(9.47)

These values deviate considerably from the NDR values $\kappa_{\pm} = 0$ and the HV values $\kappa_{\pm} = \mp 4$. Yet one can verify that for these values $J_{+} = 6.13$ and $J_{-} = 1.17$ and consequently in this scheme the NLO corrections at $\mu = m_{\rm b}$ remain perturbative. In table 11 we give the values of $\xi_{i}^{\rm NF}(\mu)$ for the NDR,HV and FS schemes.

The discussion of this subsection casts some doubts on the usefulness of the formulation in [121] with respect to the study of non-factorizable contributions to non-leptonic decays.

Table 11: $\xi_{1,2}^{\rm NF}(\mu)$ as functions of μ for different schemes and $\Lambda_{\overline{\rm MS}}^{(5)}=225\,{\rm MeV}$.

	$\xi_1^{ m NF}(\mu)$			$\xi_2^{ m NF}(\mu)$		
$\mu [{\rm GeV}]$	NDR	HV	FS	NDR	HV	FS
2.5	0.046	0.035	-0.033	0.102	0.144	0.075
5.0	0.065	0.059	0.001	0.022	0.055	-0.004
7.5	0.071	0.067	0.014	-0.016	0.013	-0.041
10.0	0.074	0.071	0.021	-0.039	-0.013	-0.064

9.6 Generalized Factorization and N^{eff}

The generalized factorization presented in [125, 127, 128] is similar in spirit but includes more dynamics than the formulation in [121]. Unfortunately, as we will demonstrate below, also this approach has its weak points. Let us then briefly describe the basic idea.

As pointed sometime ago in [68, 131] and recently discussed in [125, 127, 128], it is always possible to calculate the scale and scheme dependence of the hadronic matrix elements in perturbation theory by simply calculating the matrix elements of the relevant operators between the quark states. Combining these scheme and scale dependent contributions with the Wilson coefficients $C_i(\mu)$ one obtaines the effective coefficients C_i^{eff} which are free from these dependences. If one neglects in addition final state interactions and other possible non-factorizable contributions the decay amplitudes can be generally written as follows

$$A = \langle H_{eff} \rangle = \frac{G_F}{\sqrt{2}} V_{CKM} [C_1^{\text{eff}} \langle O_1 \rangle^{\text{tree}} + C_2^{\text{eff}} \langle O_2 \rangle^{\text{tree}}] , \qquad (9.48)$$

where $\langle O_i \rangle^{\text{tree}}$ denote tree level matrix elements. The proposal in [125, 127, 128] is to use (9.48) and to apply the idea of the factorization to the tree level matrix elements. In this approach then the effective parameters $a_{1,2}^{\text{eff}}$ are given by

$$a_1^{\text{eff}} = C_1^{\text{eff}} + \frac{1}{N^{\text{eff}}} C_2^{\text{eff}} \qquad a_2^{\text{eff}} = C_2^{\text{eff}} + \frac{1}{N^{\text{eff}}} C_1^{\text{eff}}$$
 (9.49)

with analogous expressions for a_i^{eff} (i=3-10) parametrizing penguin contributions. Here N^{eff} is treated as a phenomenological parameter which models those non-factorizable contributions to the hadronic matrix elements which have not been included in C_i^{eff} . In particular it has been suggested in [125, 127, 128] that the values for N^{eff} extracted from the data on two-body non-leptonic decays should teach us about the pattern of non-factorizable contributions.

In particular when calculating the effective coefficients C_i^{eff} , the authors of [127, 128] have included a subset of contributions to the perturbative matrix elements, which is sufficient to cancel the scale and scheme dependence of the Wilson coefficients. Unfortunately the results of such calculations are generally gauge dependent and suffer from the dependence on the infrared regulator and generally on the assumptions about the external momenta. We have discussed this already in detail in section 6 but it is instructive to discuss this briefly once more in the context of the analyses in [125, 127, 128].

The Green function of the renormalized operator O, for a given choice of the ultraviolet regularization (NDR or HV for example), a choice of the external momenta p and of the gauge parameter λ , is given by

$$\Gamma_O^{\lambda}(p) = 1 + \frac{\alpha_s}{4\pi} \left(-\frac{\gamma^{(0)}}{2} \ln(\frac{-p^2}{\mu^2}) + \hat{r} \right),$$
(9.50)

with

$$\hat{r} = \hat{r}^{NDR,HV} + \lambda \hat{r}^{\lambda}. \tag{9.51}$$

The matrices $\hat{r}^{NDR,HV}$ depend on the choice of the external momenta and on the ultraviolet regularization, while \hat{r}^{λ} is regularization- and gauge-independent, but depends on the external momenta. It is clearly possible to define a renormalization scheme in which, for given external momenta and gauge parameter, $\Gamma_O^{\lambda}(p) = 1$, or in other words $\langle O \rangle_{p,\lambda} = \langle O \rangle^{\text{tree}}$ (this corresponds to the RI scheme discussed in [131]). However, the definition of the renormalized operators will now depend on the choice of the gauge and of the external momenta. If one were able, for example by means of lattice QCD, to compute the matrix element of the operator using the same renormalization prescription, the dependences on the gauge and on the external momenta would cancel between the Wilson coefficient and the matrix element. If, on the contrary, the matrix elements are estimated using factorization, no trace is kept of the renormalization prescription and the final result is gauge and infrared dependent.

In [127, 128] scale- and scheme-independent effective Wilson coefficients C_i^{eff} have been obtained by adding to $C_i(\mu)$ the contributions coming from vertex-type quark matrix elements, denoted by \hat{r}_V and $\hat{\gamma}_V$. In particular

$$C_{1}^{\text{eff}} = C_{1}(\mu) + \frac{\alpha_{s}}{4\pi} \left(r_{V}^{T} + \gamma_{V}^{T} \log \frac{m_{b}}{\mu} \right)_{1j} C_{j}(\mu),$$

$$C_{2}^{\text{eff}} = C_{2}(\mu) + \frac{\alpha_{s}}{4\pi} \left(r_{V}^{T} + \gamma_{V}^{T} \log \frac{m_{b}}{\mu} \right)_{2j} C_{j}(\mu)$$
(9.52)

where the index j runs through all contributing operators, also penguin operators considered in [125, 127, 128].

It is evident from the above discussion that \hat{r}_V depends not only on the external momenta, but also on the gauge chosen. For example, in [127, 128] the following result for \hat{r}_V is quoted:

$$\hat{r}_{V} = \begin{pmatrix} \frac{7}{3} & -7 & 0 & 0 & 0 & 0 \\ -7 & \frac{7}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{7}{3} & -7 & 0 & 0 \\ 0 & 0 & -7 & \frac{7}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{3} & 1 \\ 0 & 0 & 0 & 0 & -3 & \frac{35}{3} \end{pmatrix}.$$
(9.53)

This result is valid in the Landau gauge ($\lambda = 0$); in an arbitrary gauge, with the same choice of external momenta used to obtain (9.53) one would get

$$\hat{r}_V = \hat{r}_V(\lambda = 0) + \lambda r_V^{\lambda},\tag{9.54}$$

with $\hat{r}_V(\lambda=0)$ given in (9.53) and

$$r_V^{\lambda} = \begin{pmatrix} -\frac{5}{6} & -\frac{3}{2} & 0 & 0 & 0 & 0\\ -\frac{3}{2} & -\frac{5}{6} & 0 & 0 & 0 & 0\\ 0 & 0 & -\frac{5}{6} & -\frac{3}{2} & 0 & 0\\ 0 & 0 & -\frac{3}{2} & -\frac{5}{6} & 0 & 0\\ 0 & 0 & 0 & 0 & -\frac{11}{6} & \frac{3}{2}\\ 0 & 0 & 0 & 0 & 0 & \frac{8}{3} \end{pmatrix}. \tag{9.55}$$

The expressions for the full 10×10 \hat{r} matrices in the NDR and HV schemes and in the Feynman and Landau gauges are given in [131], for a different choice of the external momenta. The results for the Landau gauge are given in [68], where also penguin diagrams have been included.

Equation (9.54) shows that the definition of the effective coefficients advocated in [125, 127, 128] is gauge-dependent. In addition, it also depends on the choice of the external momenta. This implies that the effective number of colors extracted in [125, 127, 128] is also gauge-dependent, and therefore it cannot have any physical meaning. This finding casts some doubts on the usefulness of the formulation in [125, 127, 128] with respect to the study of non-factorizable contributions to non-leptonic decays.

The gauge dependences and infrared dependences discussed here appear in any calculation of matrix elements of operators between quark states necessary in the process of matching of the full theory onto an effective theory as we have seen in section 6. Another example can be found in [90] where the full gauge dependence of the quark matrix element of the operator $(\bar{s}d)_{V-A}(\bar{s}d)_{V-A}$ has been calculated. However, in the process of matching such unphysical dependences in the effective theory are cancelled by the corresponding contributions in the full theory so that the Wilson coefficients are free of such dependences. Similarly in the case of inclusive decays of heavy quarks, where the spectator model can be used, they are cancelled by gluon bremsstrahlung. In exclusive hadron decays there is no meaningful way to include such effects in a perturbative framework and one is left with the gauge and infrared dependences in question.

9.7 Summary

In this section we have critically analyzed the hypothesis of the generalized factorization. While the parametrization of the data in terms of a set of effective parameters discussed in [121, 125, 127, 128], may appear to be useful, we do not think that this approach offers convincing means to analyze the physics of non-factorizable contributions to non-leptonic decays. In particular:

- The renormalization scheme dependence of the non-factorizable contributions to hadronic matrix elements precludes the determination of the factorization scale μ_f .
- Consequently for any chosen value of $\mu = \mathcal{O}(m_b)$ it is possible to find a renormalization scheme for which the non-perturbative parameters $\varepsilon_{1,8}$ used in [121] to characterize the size of non-factorizable contributions vanish. The same applies to $\xi_{1,2}^{\text{NF}}(\mu)$ introduced in (9.37).
- We point out that the recent extractions of the effective number of colours N^{eff} from two-body non-leptonic B-decays, presented in [125, 127, 128], while μ and renormalization scheme independent suffer from gauge dependences and infrared regulator dependences.

Our analysis [113] demonstrates clearly the need for an approach to non-leptonic decays which goes beyond the generalized factorization discussed recently in the literature. Some possibilities are offered by dynamical approaches like QCD sum rules as recently reviewed in [114]. However, even a phenomenological approach which does not suffer from the weak points of factorization discussed here, would be a step forward. Some ideas in this direction will be presented in [132].

10 ε_K , B^0 - \bar{B}^0 Mixing and the Unitarity Triangle

10.1 Preliminaries

Let us next discuss particle–antiparticle mixing which in the past has been of fundamental importance in testing the Standard Model and often has proven to be an undefeatable challenge for suggested extensions of this model. Particle–antiparticle mixing is responsible for the small mass differences between the mass eigenstates of neutral mesons. Being an FCNC process it involves heavy quarks in loops and consequently it is a perfect testing ground for heavy flavour physics. Let us just recall that from the calculation of the $K_{\rm L}-K_{\rm S}$ mass difference, Gaillard and Lee [133] were able to estimate the value of the charm quark mass before charm discovery. On the other hand $B_d^0 - \bar{B}_d^0$ mixing [134] gave the first indication of a large top quark mass. Finally, particle–antiparticle mixing in the $K^0 - \bar{K}^0$ system offers within the Standard Model a plausible description of CP violation in $K_L \to \pi\pi$ discovered in 1964 [135].

In this section we will predominantly discuss the parameter ε describing the *indirect* CP violation in the K system and the mass differences $\Delta M_{d,s}$ which describe the size of $B_{d,s}^0 - \bar{B}_{d,s}^0$ mixings. In the Standard Model all these phenomena appear first at the one–loop level and as such they are sensitive measures of the top quark couplings $V_{ti}(i=d,s,b)$ and of the top quark mass.

We have seen in section 2 that tree level decays and the unitarity of the CKM matrix give us already a good information about V_{tb} and V_{ts} : $V_{tb} \approx 1$ and $|V_{ts}| \approx |V_{cb}|$. Similarly the value of the top quark mass measured by CDF and D0 (see below) is known within $\pm 4\%$. Consequently the main new information to be gained from the quantities discussed here are the values of $|V_{td}|$ and of the phase $\delta = \gamma$ in the CKM matrix. This will allow us to construct the unitarity triangle which has been introduced in subsection 2.3.

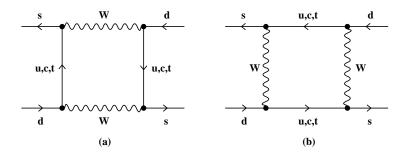


Figure 28: Box diagrams contributing to $K^0 - \bar{K}^0$ mixing in the Standard Model.

First, however, let us briefly recall the formalism of particle—antiparticle mixing. We will begin with the K-system. Subsequently we will give some formulae for $B_{d,s}^0 - \bar{B}_{d,s}^0$ mixings, necessary for the analysis of the unitarity triangle. A very detailed discussion of $B_{d,s}^0 - \bar{B}_{d,s}^0$ mixings can be found in section 8 of a review by Robert Fleischer and myself [18] and in his review [136]. The following subsection borrows a lot from [137] and [138].

10.2 Express Review of $K^0 - \bar{K}^0$ Mixing

 $K^0 = (\bar{s}d)$ and $\bar{K}^0 = (s\bar{d})$ are flavour eigenstates which in the Standard Model may mix via weak interactions through the box diagrams in fig. 28. We will choose the phase conventions so that

$$CP|K^0\rangle = -|\bar{K}^0\rangle, \qquad CP|\bar{K}^0\rangle = -|K^0\rangle.$$
 (10.1)

In the absence of mixing the time evolution of $|K^0(t)\rangle$ is given by

$$|K^{0}(t)\rangle = |K^{0}(0)\rangle \exp(-iHt) , \qquad H = M - i\frac{\Gamma}{2} ,$$
 (10.2)

where M is the mass and Γ the width of K^0 . Similar formula for \bar{K}^0 exists.

On the other hand, in the presence of flavour mixing the time evolution of the $K^0 - \bar{K}^0$ system is described by

$$i\frac{d\psi(t)}{dt} = \hat{H}\psi(t) \qquad \psi(t) = \begin{pmatrix} |K^0(t)\rangle \\ |\bar{K}^0(t)\rangle \end{pmatrix}$$
 (10.3)

where

$$\hat{H} = \hat{M} - i\frac{\hat{\Gamma}}{2} = \begin{pmatrix} M_{11} - i\frac{\Gamma_{11}}{2} & M_{12} - i\frac{\Gamma_{12}}{2} \\ M_{21} - i\frac{\Gamma_{21}}{2} & M_{22} - i\frac{\Gamma_{22}}{2} \end{pmatrix}$$
(10.4)

with \hat{M} and $\hat{\Gamma}$ being hermitian matrices having positive (real) eigenvalues in analogy with M and Γ . M_{ij} and Γ_{ij} are the transition matrix elements from virtual and physical intermediate states respectively. Using

$$M_{21} = M_{12}^*$$
, $\Gamma_{21} = \Gamma_{12}^*$, (hermiticity) (10.5)

$$M_{11} = M_{22} \equiv M , \qquad \Gamma_{11} = \Gamma_{22} \equiv \Gamma , \quad (CPT)$$
 (10.6)

we have

$$\hat{H} = \begin{pmatrix} M - i\frac{\Gamma}{2} & M_{12} - i\frac{\Gamma_{12}}{2} \\ M_{12}^* - i\frac{\Gamma_{12}^*}{2} & M - i\frac{\Gamma}{2} \end{pmatrix} . \tag{10.7}$$

We can next diagonalize the system to find:

Eigenstates:

$$K_{L,S} = \frac{(1+\bar{\varepsilon})K^0 \pm (1-\bar{\varepsilon})\bar{K}^0}{\sqrt{2(1+|\bar{\varepsilon}|^2)}}$$
(10.8)

where $\bar{\varepsilon}$ is a small complex parameter given by

$$\frac{1-\bar{\varepsilon}}{1+\bar{\varepsilon}} = \sqrt{\frac{M_{12}^* - i\frac{1}{2}\Gamma_{12}^*}{M_{12} - i\frac{1}{2}\Gamma_{12}}} \ . \tag{10.9}$$

Eigenvalues:

$$M_{LS} = M \pm \text{Re}Q$$
 $\Gamma_{LS} = \Gamma \mp 2\text{Im}Q$ (10.10)

where

$$Q = \sqrt{(M_{12} - i\frac{1}{2}\Gamma_{12})(M_{12}^* - i\frac{1}{2}\Gamma_{12}^*)}.$$
(10.11)

Consequently we have

$$\Delta M = M_L - M_S = 2 \text{Re} Q \qquad \Delta \Gamma = \Gamma_L - \Gamma_S = -4 \text{Im} Q.$$
 (10.12)

It should be noted that the mass eigenstates K_S and K_L differ from CP eigenstates

$$K_1 = \frac{1}{\sqrt{2}}(K^0 - \bar{K}^0), \qquad CP|K_1\rangle = |K_1\rangle, \qquad (10.13)$$

$$K_2 = \frac{1}{\sqrt{2}}(K^0 + \bar{K}^0), \qquad CP|K_2\rangle = -|K_2\rangle, \qquad (10.14)$$

by a small admixture of the other CP eigenstate:

$$K_{\rm S} = \frac{K_1 + \bar{\varepsilon}K_2}{\sqrt{1 + |\bar{\varepsilon}|^2}}, \qquad K_{\rm L} = \frac{K_2 + \bar{\varepsilon}K_1}{\sqrt{1 + |\bar{\varepsilon}|^2}}$$
 (10.15)

with $\bar{\varepsilon}$ defined in (10.9). $\bar{\varepsilon}$ can also be written as

$$\frac{1-\bar{\varepsilon}}{1+\bar{\varepsilon}} = \frac{\Delta M - i\frac{1}{2}\Delta\Gamma}{2M_{12} - i\Gamma_{12}} \equiv r\exp(i\kappa) \ . \tag{10.16}$$

It should be stressed that the small parameter $\bar{\varepsilon}$ depends on the phase convention chosen for K^0 and \bar{K}^0 . Therefore it may not be taken as a physical measure of CP violation. On the other hand Re $\bar{\varepsilon}$ and r are independent of phase conventions. In particular the departure of r from 1 measures CP violation in the $K^0 - \bar{K}^0$ mixing:

$$r = 1 + \frac{2|\Gamma_{12}|^2}{4|M_{12}|^2 + |\Gamma_{12}|^2} \operatorname{Im}\left(\frac{M_{12}}{\Gamma_{12}}\right) . \tag{10.17}$$

Since $\bar{\varepsilon}$ is $\mathcal{O}(10^{-3})$, we find, using (10.9), that

$$Im M_{12} \ll Re M_{12}, \qquad Im \Gamma_{12} \ll Re \Gamma_{12} .$$
 (10.18)

Consequently to a very good approximation:

$$\Delta M_K = 2 \text{Re} M_{12}, \qquad \Delta \Gamma_K = 2 \text{Re} \Gamma_{12} , \qquad (10.19)$$

where we have introduced the subscript K to stress that these formulae apply only to the $K^0 - \bar{K}^0$ system.

The $K_{\rm L}-K_{\rm S}$ mass difference is experimentally measured to be

$$\Delta M_K = M(K_L) - M(K_S) = (3.491 \pm 0.009) \cdot 10^{-15} \,\text{GeV} \,.$$
 (10.20)

In the Standard Model roughly 70% of the measured ΔM_K is described by the real parts of the box diagrams with charm quark and top quark exchanges, wherby the contribution of the charm exchanges is by far dominant. This is related to the smallness of the real parts of the CKM top quark couplings compared with the corresponding charm quark couplings. Thus even if the function $S_0(x_t)$ is by a factor of 1600 larger than $S_0(x_c)$, it cannot compensate for the smallness of the real top quark couplings. Some non-negligible contribution comes from the box diagrams with simultaneous charm and top exchanges. The u-quark contribution is needed only for GIM mechanism but otherwise can be neglected. The remaining 30% of the measured ΔM_K is attributed to long distance contributions which are difficult to estimate [139]. It is a useful exercise to check these statements by using ΔM_K in (10.12) and the expression for M_{12} given in (10.41). Further information with the relevant references can be found in [92].

The situation with $\Delta\Gamma_K$ is rather different. It is fully dominated by long distance effects. Experimentally one has

$$\Delta\Gamma_K = \Gamma(K_{\rm L}) - \Gamma(K_{\rm S}) = -7.4 \cdot 10^{-15} \,\text{GeV}$$
 (10.21)

and consequently $\Delta\Gamma_K \approx -2\Delta M_K$.

With all this information at hand and using the experimentally observed dominance of $\Delta I = 1/2$ transitions in $K \to \pi\pi$, it is possible to derive an important formula for $\bar{\varepsilon}$

$$\bar{\varepsilon} = \frac{i}{1+i} \frac{\operatorname{Im} M_{12}}{\Delta M_K} + \frac{\xi}{1+i} , \qquad \xi = \frac{\operatorname{Im} A_0}{\operatorname{Re} A_0} , \qquad (10.22)$$

with the isospin amplitude A_0 defined below. An explicit derivation of (10.22) can be found in a review by Chau [137]. A recent review by Belusevic [140] is also useful in this respect. Finally I recommend strongly excellent lectures by Yossi Nir [141], where the issues of phase conventions are discussed in detail.

10.3 The First Look at ε and ε'

Let us next move to two important CP violating parameters which can be measured experimentally. The route to them proceeds as follows. It involves the decays $K \to \pi\pi$.

Since a two pion final state is CP even while a three pion final state is CP odd, $K_{\rm S}$ and $K_{\rm L}$ preferably decay to 2π and 3π , respectively via the following CP conserving decay modes:

$$K_{\rm L} \to 3\pi \ ({\rm via} \ {\rm K}_2), \qquad K_{\rm S} \to 2\pi \ ({\rm via} \ {\rm K}_1).$$
 (10.23)

This difference is responsible for the large disparity in their life-times. A factor of 579. However, $K_{\rm L}$ and $K_{\rm S}$ are not CP eigenstates and may decay with small branching fractions as follows:

$$K_{\rm L} \to 2\pi \ ({\rm via} \ {\rm K}_1), \qquad K_{\rm S} \to 3\pi \ ({\rm via} \ {\rm K}_2).$$
 (10.24)

This violation of CP is called *indirect* as it proceeds not via explicit breaking of the CP symmetry in the decay itself but via the admixture of the CP state with opposite CP parity to the dominant one. The measure for this indirect CP violation is defined as

$$\varepsilon = \frac{A(K_{\rm L} \to (\pi\pi)_{I=0})}{A(K_{\rm S} \to (\pi\pi)_{I=0})},\tag{10.25}$$

where ε is, contrary to $\bar{\varepsilon}$ in (10.22), independent of the phase conventions. Following the derivation in [137] one finds

$$\varepsilon = \bar{\varepsilon} + i\xi \ . \tag{10.26}$$

The phase convention dependence of the term $i\xi$ cancells the convention dependence of $\bar{\varepsilon}$. We will write down a nicer formula for ε below.

While *indirect* CP violation reflects the fact that the mass eigenstates are not CP eigenstates, so-called *direct* CP violation is realized via a direct transition of a CP odd to a CP

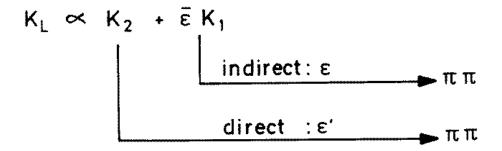


Figure 29: Indirect versus direct CP violation in $K_L \to \pi\pi$.

even state or vice versa (see fig. 29). A measure of such a direct CP violation in $K_L \to \pi\pi$ is characterized by a complex parameter ε' defined as

$$\varepsilon' = \frac{1}{\sqrt{2}} \operatorname{Im}\left(\frac{A_2}{A_0}\right) e^{i\Phi}, \qquad \Phi = \pi/2 + \delta_2 - \delta_0, \tag{10.27}$$

where the isospin amplitudes A_I in $K \to \pi\pi$ decays are introduced through

$$A(K^+ \to \pi^+ \pi^0) = \sqrt{\frac{3}{2}} A_2 e^{i\delta_2}$$
 (10.28)

$$A(K^0 \to \pi^+ \pi^-) = \sqrt{\frac{2}{3}} A_0 e^{i\delta_0} + \sqrt{\frac{1}{3}} A_2 e^{i\delta_2}$$
 (10.29)

$$A(K^0 \to \pi^0 \pi^0) = \sqrt{\frac{2}{3}} A_0 e^{i\delta_0} - 2\sqrt{\frac{1}{3}} A_2 e^{i\delta_2}.$$
 (10.30)

Here the subscript I=0,2 denotes states with isospin 0,2 equivalent to $\Delta I=1/2$ and $\Delta I=3/2$ transitions, respectively, and $\delta_{0,2}$ are the corresponding strong phases. The weak CKM phases are contained in A_0 and A_2 . The strong phases $\delta_{0,2}$ cannot be calculated, at least, at present. They can be extracted from $\pi\pi$ scattering. Then $\Phi \approx \pi/4$.

The isospin amplitudes A_I are complex quantities which depend on phase conventions. On the other hand, ε' measures the difference between the phases of A_2 and A_0 and is a physical quantity.

Experimentally ε and ε' can be found by measuring the ratios

$$\eta_{00} = \frac{A(K_{\rm L} \to \pi^0 \pi^0)}{A(K_{\rm S} \to \pi^0 \pi^0)}, \qquad \eta_{+-} = \frac{A(K_{\rm L} \to \pi^+ \pi^-)}{A(K_{\rm S} \to \pi^+ \pi^-)}.$$
 (10.31)

Indeed, assuming ε and ε' to be small numbers one finds

$$\eta_{00} = \varepsilon - \frac{2\varepsilon'}{1 - \sqrt{\omega}} \simeq \varepsilon - 2\varepsilon', \quad \eta_{+-} = \varepsilon + \frac{\varepsilon'}{1 + \omega/\sqrt{2}} \simeq \varepsilon + \varepsilon'$$
(10.32)

where experimentally $\omega = \text{Re}A_2/\text{Re}A_0 = 0.045$.

In the absence of direct CP violation $\eta_{00} = \eta_{+-}$. The ratio ε'/ε can then be measured through

$$\left|\frac{\eta_{00}}{\eta_{+-}}\right|^2 \simeq 1 - 6 \operatorname{Re}\left(\frac{\varepsilon'}{\varepsilon}\right).$$
 (10.33)

10.4 Basic Formula for ε

With all this information at hand let us derive a formula for ε which can be efficiently used in pheneomenological applications. Using (10.22) and (10.26) we first find the general formula

$$\varepsilon = \frac{\exp(i\pi/4)}{\sqrt{2}\Delta M_K} \left(\operatorname{Im} M_{12} + 2\xi \operatorname{Re} M_{12} \right), \qquad \xi = \frac{\operatorname{Im} A_0}{\operatorname{Re} A_0}. \tag{10.34}$$

The two terms in (10.34) are separately phase convention dependent but there sum is free from this dependence. The off-diagonal element M_{12} in the neutral K-meson mass matrix represents K^0 - \bar{K}^0 mixing. It is given by

$$2m_K M_{12}^* = \langle \bar{K}^0 | \mathcal{H}_{\text{eff}}(\Delta S = 2) | K^0 \rangle,$$
 (10.35)

where $\mathcal{H}_{\text{eff}}(\Delta S = 2)$ is the effective Hamiltonian for the $\Delta S = 2$ transitions. That M_{12}^* and not M_{12} stands on the l.h.s of this formula, is evident from (10.7). The factor $2m_K$ reflects our normalization of external states.

To lowest order in electroweak interactions $\Delta S = 2$ transitions are induced through the box diagrams of fig. 28. Including QCD corrections in the manner analogous to the one already discussed for $\Delta B = 2$ transitions in Section 8.3 one has [90]

$$\mathcal{H}_{\text{eff}}^{\Delta S=2} = \frac{G_{\text{F}}^2}{16\pi^2} M_W^2 \left[\lambda_c^2 \eta_1 S_0(x_c) + \lambda_t^2 \eta_2 S_0(x_t) + 2\lambda_c \lambda_t \eta_3 S_0(x_c, x_t) \right] \times \left[\alpha_s^{(3)}(\mu) \right]^{-2/9} \left[1 + \frac{\alpha_s^{(3)}(\mu)}{4\pi} J_3 \right] Q(\Delta S = 2) + h.c.$$
 (10.36)

where $\lambda_i = V_{is}^* V_{id}$. Here $\mu < \mu_c = \mathcal{O}(m_c)$. In (10.36), the relevant operator

$$Q(\Delta S = 2) = (\bar{s}d)_{V-A}(\bar{s}d)_{V-A}, \tag{10.37}$$

is multiplied by the corresponding coefficient function. This function is decomposed into a charm-, a top- and a mixed charm-top contribution. This form is obtained upon eliminating λ_u by means of the unitarity of the CKM matrix and setting $x_u = 0$. The functions S_0 are given in (3.17)–(3.19).

Short-distance QCD effects are described through the correction factors η_1 , η_2 , η_3 and the explicitly α_s -dependent terms in (10.36). η_2 is the analogue of η_B discussed in Section 8.3. The calculation of η_1 and η_3 is more involved and is discussed in [92, 93]. η_{1-3} are defined in analogy to (8.30). This means that in $\mathcal{O}(\alpha_s)$ they are independent of the renormalization scales and the renormalization scheme for the operator $Q(\Delta S)$. The NLO values of η_i are given as follows [92, 90, 93]:

$$\eta_1 = 1.38 \pm 0.20, \qquad \eta_2 = 0.57 \pm 0.01, \qquad \eta_3 = 0.47 \pm 0.04.$$
(10.38)

The quoted errors reflect the remaining theoretical uncertainties due to leftover μ -dependences at $\mathcal{O}(\alpha_s^2)$ and $\Lambda_{\overline{MS}}$. The factor η_1 plays only a minor role in the analysis of ε but its enhanced value through NLO corrections is essential for the $K_{\rm L} - K_{\rm S}$ mass difference. We refer to [92] for the discussion of ΔM_K .

Defining, in analogy to (8.31), the renormalization group invariant parameter \hat{B}_K by

$$\hat{B}_K = B_K(\mu) \left[\alpha_s^{(3)}(\mu) \right]^{-2/9} \left[1 + \frac{\alpha_s^{(3)}(\mu)}{4\pi} J_3 \right]$$
 (10.39)

$$\langle \bar{K}^0 | (\bar{s}d)_{V-A} (\bar{s}d)_{V-A} | K^0 \rangle \equiv \frac{8}{3} B_K(\mu) F_K^2 m_K^2$$
 (10.40)

and using (10.36) one finds

$$M_{12} = \frac{G_{\rm F}^2}{12\pi^2} F_K^2 \hat{B}_K m_K M_{\rm W}^2 \left[\lambda_c^{*2} \eta_1 S_0(x_c) + \lambda_t^{*2} \eta_2 S_0(x_t) + 2\lambda_c^* \lambda_t^* \eta_3 S_0(x_c, x_t) \right], \tag{10.41}$$

where F_K is the K-meson decay constant and m_K the K-meson mass.

To proceed further we neglect the last term in (10.34) as it constitutes at most a 2% correction to ε . This is justified in view of other uncertainties, in particular those connected with B_K . Inserting (10.41) into (10.34) we find

$$\varepsilon = C_{\varepsilon} \hat{B}_{K} \operatorname{Im} \lambda_{t} \left\{ \operatorname{Re} \lambda_{c} \left[\eta_{1} S_{0}(x_{c}) - \eta_{3} S_{0}(x_{c}, x_{t}) \right] - \operatorname{Re} \lambda_{t} \eta_{2} S_{0}(x_{t}) \right\} \exp(i\pi/4), \qquad (10.42)$$

where we have used the unitarity relation $\text{Im}\lambda_c^* = \text{Im}\lambda_t$ and have neglected $\text{Re}\lambda_t/\text{Re}\lambda_c = \mathcal{O}(\lambda^4)$ in evaluating $\text{Im}(\lambda_c^*\lambda_t^*)$. The numerical constant C_{ε} is given by

$$C_{\varepsilon} = \frac{G_{\rm F}^2 F_K^2 m_K M_{\rm W}^2}{6\sqrt{2}\pi^2 \Delta M_K} = 3.78 \cdot 10^4 \,. \tag{10.43}$$

To this end we have used the experimental value of ΔM_K in (10.20). In principle we could use the theoretical value for ΔM_K but in view of the presence of long distance contributions it is safer to use the experimental value. In this context it should be stressed that the parameter ε being related to CP violation and top quark physics should be dominated by short distance contributions and well approximated by the imaginary parts of the box diagrams. Consequently the only non-perturbative uncertainty in (10.42) resides in \hat{B}_K .

Using the standard parametrization of (2.18) to evaluate $\text{Im}\lambda_i$ and $\text{Re}\lambda_i$, setting the values for s_{12} , s_{13} , s_{23} and m_t in accordance with experiment and taking a value for \hat{B}_K (see below), one can determine the phase δ by comparing (10.42) with the experimental value for ε

$$\varepsilon^{exp} = (2.280 \pm 0.013) \cdot 10^{-3} e^{i\frac{\pi}{4}}.$$
 (10.44)

Once δ has been determined in this manner one can find the apex $(\bar{\varrho}, \bar{\eta})$ of the unitarity triangle in fig. 5 by using

$$\varrho = \frac{s_{13}}{s_{12}s_{23}}\cos\delta, \qquad \eta = \frac{s_{13}}{s_{12}s_{23}}\sin\delta \tag{10.45}$$

and

$$\bar{\varrho} = \varrho(1 - \frac{\lambda^2}{2}), \qquad \bar{\eta} = \eta(1 - \frac{\lambda^2}{2}).$$
 (10.46)

For a given set $(s_{12}, s_{13}, s_{23}, m_t, \hat{B}_K)$ there are two solutions for δ and consequently two solutions for $(\bar{\varrho}, \bar{\eta})$. This will be evident from the analysis of the unitarity triangle discussed in detail below.

Finally we have to say a few words about the non-perturbative parameter \hat{B}_K . There is a long history of evaluating this parameter in various non-perturbative approaches. A short review of older results can be found in [18]. The present status of quenched lattice calculations has been recently reviewed by Gupta [142]. The most accurate result for $B_K(2 \text{ GeV})$ using lattice method is obtained by JLQCD collaboration [143]: $B_K(2 \text{ GeV}) = 0.628 \pm 0.042$. A similar result has been published by Gupta, Kilcup and Sharpe [144] last year. The APE collaboration [146] finds $B_K(2 \text{ GeV}) = 0.66 \pm 0.11$ which is consistent with JLQCD and GKS. In order to convert these values into \hat{B}_K by means of (10.39) one has to face the issue of the choice of the number of flavours f. Fortunately the values for \hat{B}_K for f = 0 and f = 3 corresponding to the JLQCD result, turn out to be very similar: $\hat{B}_K = 0.87 \pm 0.06$ and $\hat{B}_K = 0.84 \pm 0.06$, respectively. The final present lattice value given by Gupta is then

$$(\hat{B}_K)_{\text{Lattice}} = 0.86 \pm 0.06 \pm 0.06$$
 (10.47)

where the second error is attributed to quenching. The corresponding result from APE is $\hat{B}_K = 0.93 \pm 0.16$. On the other hand a recent analysis in the chiral quark model gives surprisingly a value as high as $\hat{B}_K = 1.1 \pm 0.2$ [147]. In our numerical analysis presented below we will use

$$\hat{B}_K = 0.75 \pm 0.15. \tag{10.48}$$

which is in the ball park of various lattice estimates and $\hat{B}_K = 0.70 \pm 0.10$ from the 1/N approach [148, 149]. These values are higher than those found using QCD Hadronic Duality approach ($\hat{B}_K = 0.39 \pm 0.10$) [150] or using the SU(3) symmetry and PCAC ($\hat{B}_K = 1/3$) [151].

As we will see below, $\hat{B}_K \leq 0.75$ requires simultaneously high values of $|V_{ub}/V_{cb}|$ and $|V_{cb}|$ in order to be able to fit the experimental value of ε .

10.5 Basic Formula for B^0 - \bar{B}^0 Mixing

The strength of the $B_{d,s}^0 - \bar{B}_{d,s}^0$ mixings is described by the mass differences

$$\Delta M_{d,s} = M_H^{d,s} - M_L^{d,s} \tag{10.49}$$

with "H" and "L" denoting *Heavy* and *Light* respectively. In contrast to ΔM_K , in this case the long distance contributions are estimated to be very small and $\Delta M_{d,s}$ is very well

approximated by the relevant box diagrams. Moreover, due $m_{u,c} \ll m_t$ only the top sector can contribute significantly to $B_{d,s}^0 - \bar{B}_{d,s}^0$ mixings. The charm sector and the mixed top-charm contributions are entirely negligible. This can be easily verified and is left as an useful exercise.

 $\Delta M_{d,s}$ can be expressed in terms of the off-diagonal element in the neutral B-meson mass matrix by using the formulae developed previously for the K-meson system. One finds

$$\Delta M_q = 2|M_{12}^{(q)}|, \qquad q = d, s.$$
 (10.50)

This formula differs from $\Delta M_K = 2 \text{Re} M_{12}$ because in the B-system $\Gamma_{12} \ll M_{12}$.

Equivalently, the mixing can be described by

$$x_q \equiv \frac{\Delta M_q}{\Gamma_{B_q}},\tag{10.51}$$

where $\Gamma_{B_q} = 1/\tau_{B_q}$ with τ_{B_q} being the corresponding lifetimes. However, working with ΔM_q instead of x_q avoids the experimental errors in lifetimes.

The off-diagonal term M_{12} in the neutral B-meson mass matrix is then given by a formula analogous to (10.35)

$$2m_{B_q}|M_{12}^{(q)}| = |\langle \bar{B}_q^0|\mathcal{H}_{\text{eff}}(\Delta B = 2)|B_q^0\rangle|,$$
 (10.52)

where in the case of $B_d^0 - \bar{B}_d^0$ mixing

$$\mathcal{H}_{\text{eff}}^{\Delta B=2} = \frac{G_{\text{F}}^{2}}{16\pi^{2}} M_{W}^{2} \left(V_{tb}^{*} V_{td}\right)^{2} \eta_{B} S_{0}(x_{t}) \times \left[\alpha_{s}^{(5)}(\mu_{b})\right]^{-6/23} \left[1 + \frac{\alpha_{s}^{(5)}(\mu_{b})}{4\pi} J_{5}\right] Q(\Delta B = 2) + h.c.$$
 (10.53)

Here $\mu_b = \mathcal{O}(m_b)$,

$$Q(\Delta B = 2) = (\bar{b}d)_{V-A}(\bar{b}d)_{V-A}$$
(10.54)

and [90]

$$\eta_B = 0.55 \pm 0.01. \tag{10.55}$$

Finally $J_5 = 1.627$ in the NDR scheme. In the case of $B_s^0 - \bar{B}_s^0$ mixing one should simply replace $d \to s$ in (10.53) and (10.54) with all other quantities unchanged.

We next reapeat what we have done already in Section 8.3. Defining the renormalization group invariant parameters \hat{B}_q by

$$\hat{B}_{B_q} = B_{B_q}(\mu) \left[\alpha_s^{(5)}(\mu) \right]^{-6/23} \left[1 + \frac{\alpha_s^{(5)}(\mu)}{4\pi} J_5 \right]$$
 (10.56)

$$\langle \bar{B}_q^0 | (\bar{b}q)_{V-A} (\bar{b}q)_{V-A} | B_q^0 \rangle \equiv \frac{8}{3} B_{B_q}(\mu) F_{B_q}^2 m_{B_q}^2 , \qquad (10.57)$$

where F_{B_q} is the B_q -meson decay constant and using (10.53) one finds

$$\Delta M_q = \frac{G_F^2}{6\pi^2} \eta_B m_{B_q} (\hat{B}_{B_q} F_{B_q}^2) M_W^2 S_0(x_t) |V_{tq}|^2, \tag{10.58}$$

which implies two useful formulae

$$\Delta M_d = 0.50/\text{ps} \cdot \left[\frac{\sqrt{\hat{B}_{B_d}} F_{B_d}}{200 \,\text{MeV}} \right]^2 \left[\frac{\overline{m}_t(m_t)}{170 \,\text{GeV}} \right]^{1.52} \left[\frac{|V_{td}|}{8.8 \cdot 10^{-3}} \right]^2 \left[\frac{\eta_B}{0.55} \right]$$
(10.59)

and

$$\Delta M_s = 15.1/\text{ps} \cdot \left[\frac{\sqrt{\hat{B}_{B_s}} F_{B_s}}{240 \,\text{MeV}} \right]^2 \left[\frac{\overline{m}_t(m_t)}{170 \,\text{GeV}} \right]^{1.52} \left[\frac{|V_{ts}|}{0.040} \right]^2 \left[\frac{\eta_B}{0.55} \right] . \tag{10.60}$$

There is a vast literature on the calculations of F_{B_d} and \hat{B}_d . The most recent world averages from lattice are [152, 153]

$$F_{B_d} = (175 \pm 25) \,\text{MeV} \,, \qquad \hat{B}_{B_d} = 1.31 \pm 0.03 \,.$$
 (10.61)

This result for F_{B_d} is compatible with the results obtained with the help of QCD sum rules [154]. In our numerical analysis we will use

$$F_{B_d} \sqrt{\hat{B}_{B_d}} = (200 \pm 40) \,\text{MeV}.$$
 (10.62)

The experimental situation on ΔM_d taken from Gibbons [39] is given in table 12.

10.6 Standard Analysis of the Unitarity Triangle

With all these formulae at hand we can now summarize the standard analysis of the unitarity triangle in fig. 5. It proceeds in five steps.

Step 1:

From $b \to c$ transition in inclusive and exclusive B meson decays one finds $|V_{cb}|$ and consequently the scale of the unitarity triangle:

$$|V_{cb}| \implies \lambda |V_{cb}| = \lambda^3 A$$
 (10.63)

Step 2:

From $b \to u$ transition in inclusive and exclusive B meson decays one finds $|V_{ub}/V_{cb}|$ and consequently the side $CA = R_b$ of the unitarity triangle:

$$\left| \frac{V_{ub}}{V_{cb}} \right| \implies R_b = \sqrt{\bar{\varrho}^2 + \bar{\eta}^2} = 4.44 \cdot \left| \frac{V_{ub}}{V_{cb}} \right| \tag{10.64}$$

Step 3:

From experimental value of ε (10.44) and the formula (10.42) one derives, using the approximations (2.37)–(2.39), the constraint

$$\bar{\eta} \left[(1 - \bar{\varrho}) A^2 \eta_2 S_0(x_t) + P_0(\varepsilon) \right] A^2 \hat{B}_K = 0.226,$$
 (10.65)

where

$$P_0(\varepsilon) = \left[\eta_3 S_0(x_c, x_t) - \eta_1 x_c\right] \frac{1}{\lambda^4}, \qquad x_t = \frac{m_t^2}{M_W^2}.$$
 (10.66)

 $P_0(\varepsilon) = 0.31 \pm 0.05$ summarizes the contributions of box diagrams with two charm quark exchanges and the mixed charm-top exchanges. The error in $P_0(\varepsilon)$ is dominated by the uncertainties in η_3 and m_c . However, the $P_0(\varepsilon)$ term contributes only 25% to (10.65) and these uncertainties constitute only a few percent uncertainty in the constraint (10.65). Recalling that m_t and the relevant QCD factors η_2 and η_3 are rather precisely known, we conclude that the main uncertainties in the constraint (10.65) reside in \hat{B}_K and to some extent in A^4 which multiplies the leading term.

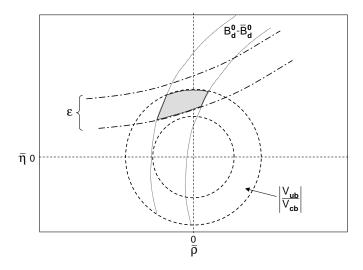


Figure 30: Schematic determination of Unitarity Triangle.

Equation (10.65) specifies a hyperbola in the $(\bar{\varrho}, \bar{\eta})$ plane. This hyperbola intersects the circle found in step 2 in two points which correspond to the two solutions for δ mentioned earlier. This is illustrated in fig. 30. The position of the hyperbola (10.65) in the $(\bar{\varrho}, \bar{\eta})$ plane depends on m_t , $|V_{cb}| = A\lambda^2$ and \hat{B}_K . With decreasing m_t , $|V_{cb}|$ and \hat{B}_K the ε -hyperbola moves away from the origin of the $(\bar{\varrho}, \bar{\eta})$ plane. When the hyperbola and the circle (10.64) touch each other lower bounds consistent with $\varepsilon_K^{\text{exp}}$ can be found [155]:

$$(m_{\rm t})_{\rm min} = M_{\rm W} \left[\frac{1}{2A^2} \left(\frac{1}{A^2 \hat{B}_K R_b} - 1.4 \right) \right]^{0.658}$$
 (10.67)

$$\left| \frac{V_{ub}}{V_{ch}} \right|_{\min} = \frac{\lambda}{1 - \lambda^2 / 2} \left[A^2 \hat{B}_K \left(2x_t^{0.76} A^2 + 1.4 \right) \right]^{-1}$$
 (10.68)

$$(\hat{B}_K)_{\min} = \left[A^2 R_b \left(2x_t^{0.76} A^2 + 1.4 \right) \right]^{-1}. \tag{10.69}$$

Step 4: From the observed $B_d^0 - \bar{B}_d^0$ mixing parametrized by ΔM_d the side $BA = R_t$ of the unitarity triangle can be determined:

$$R_t = \frac{1}{\lambda} \frac{|V_{td}|}{|V_{cb}|} = 1.0 \cdot \left[\frac{|V_{td}|}{8.8 \cdot 10^{-3}} \right] \left[\frac{0.040}{|V_{cb}|} \right]$$
(10.70)

with

$$|V_{td}| = 8.8 \cdot 10^{-3} \left[\frac{200 \,\text{MeV}}{\sqrt{\hat{B}_{B_d}} F_{B_d}} \right] \left[\frac{170 \,GeV}{\overline{m}_t(m_t)} \right]^{0.76} \left[\frac{\Delta M_d}{0.50/\text{ps}} \right]^{0.5} \sqrt{\frac{0.55}{\eta_B}}.$$
 (10.71)

Since m_t , ΔM_d and η_B are already rather precisely known, the main uncertainty in the determination of $|V_{td}|$ from $B_d^0 - \bar{B}_d^0$ mixing comes from $F_{B_d}\sqrt{B_{B_d}}$. Note that R_t suffers from additional uncertainty in $|V_{cb}|$, which is absent in the determination of $|V_{td}|$ this way. The constraint in the $(\bar{\varrho}, \bar{\eta})$ plane coming from this step is illustrated in fig. 30.

Step 5:

The measurement of $B_s^0 - \bar{B}_s^0$ mixing parametrized by ΔM_s together with ΔM_d allows to determine R_t in a different way. Using (10.58) and setting $\Delta M_d^{\rm max} = 0.482/{\rm ps}$ and $|V_{ts}/V_{cb}|^{\rm max} = 0.993$ one finds a useful formula [156]:

$$(R_t)_{\text{max}} = 1.0 \cdot \xi \sqrt{\frac{10.2/ps}{\Delta M_s}}, \qquad \xi = \frac{F_{B_s} \sqrt{\hat{B}_{B_s}}}{F_{B_d} \sqrt{\hat{B}_{B_d}}},$$
 (10.72)

where $\xi = 1$ in the SU(3)-flavour limit. One should note that $m_{\rm t}$ and $|V_{cb}|$ dependences have been eliminated this way and that ξ should in principle contain much smaller theoretical uncertainties than the hadronic matrix elements in ΔM_d and ΔM_s separately. The most recent values relevant for (10.72) are:

$$\Delta M_s > 10.2/ps \text{ (95\% C.L.)}$$
 $\xi = 1.15 \pm 0.05$ (10.73)

The first number is the improved lower bound from ALEPH [157]. The second number comes from quenched lattice calculations summarized in [152] and [153]. A similar result has been obtained using QCD sum rules [158].

The fate of the usefulness of the bound (10.72) depends clearly on both ΔM_s and ξ as well as on the type of the error analysis. We will return to this point soon. For $\xi = 1.2$ the lower bound on ΔM_s in (10.73) implies $R_t \leq 1.20$ which, as we will see, has a moderate impact on the unitarity triangle obtained using the scanning method and the first four steps alone.

Finally, I would like to point out that whereas step 5 can give, in contrast to step 4, the value for R_t free of the $|V_{cb}|$ uncertainty, it does not provide at present a more accurate value of $|V_{td}|$ if the scanning method, discussed below, is used. The point is, that having R_t , one determines $|V_{td}|$ by means of the relation (10.70) which, in contrast to (10.71), depends on $|V_{cb}|$. In fact as we will see below, the inclusion of step 5 has, with $\xi = 1.2$, a visible impact on R_t without essentially any impact on the range of $|V_{td}|$ obtained using the scanning method and the first four steps alone.

10.7 Numerical Results

10.7.1 Input Parameters

The input parameters needed to perform the standard analysis using the first four steps alone are given in table 12. We list here the "present" errors based on what we have discussed above, as well as the "future" errors. The latter are a mere guess, but as we will see in sections 13 and 14, these are the errors one should aim at, in order that the standard analysis could be competitive in the CKM determination with the cleanest rare decays and the CP asymmetries in B-decays.

 $m_{\rm t}$ in table 12 refers to the running current top quark mass normalized at $\mu = m_{\rm t}$: $\overline{m}_{\rm t}(m_{\rm t})$ and is obtained from the value $m_{\rm t}^{Pole} = 175 \pm 6 \, {\rm GeV}$ measured by CDF and D0 by means of the relation.

$$\overline{m}_{t}(m_{t}) = m_{t}^{\text{Pole}} \left[1 - \frac{4}{3} \frac{\alpha_{s}(m_{t})}{\pi} \right]. \tag{10.74}$$

Thus for $m_t = \mathcal{O}(170\,\text{GeV})$, $\overline{m}_t(m_t)$ is typically by 8 GeV smaller than m_t^{Pole} . In principle known $\mathcal{O}(\alpha_s^2)$ corrections to the relation (10.74) could also be included which would decrease the value of $\overline{m}_t(m_t)$ by roughly 1 GeV. Yet this would not be really consistent with the rest of the analysis which does not include the next-to-NLO corrections.

Table 12: Collection of input parameters.

Quantity	Central	Present	Future
$ V_{cb} $	0.040	± 0.003	± 0.001
$ V_{ub}/V_{cb} $	0.080	± 0.020	± 0.005
\hat{B}_{K}	0.75	± 0.15	± 0.05
$\sqrt{\hat{B}_d}F_{B_d}$	$200\mathrm{MeV}$	$\pm 40\mathrm{MeV}$	$\pm 10\mathrm{MeV}$
$m_{ m t}$	$167\mathrm{GeV}$	$\pm 6\mathrm{GeV}$	$\pm 3\mathrm{GeV}$
ΔM_d	0.464 ps^{-1}	$\pm 0.018 \text{ ps}^{-1}$	$\pm 0.006 \text{ ps}^{-1}$

10.7.2 $|V_{ub}/V_{cb}|, |V_{cb}| \text{ and } \varepsilon_K$

The values for $|V_{ub}/V_{cb}|$ and $|V_{cb}|$ in table 12 are not correlated with each other. On the other hand such a correlation is present in the analysis of the CP violating parameter ε which is roughly proportional to the fourth power of $|V_{cb}|$ and linear in $|V_{ub}/V_{cb}|$. It follows that not all values in table 12 are simultaneously consistent with the observed value of ε . This has been emphasized in particular by Herrlich and Nierste [93] and in [17]. Explicitly one has using (10.68):

$$\left| \frac{V_{ub}}{V_{cb}} \right|_{\min} = \frac{0.225}{\hat{B}_K A^2 (2x_t^{0.76} A^2 + 1.4)}.$$
 (10.75)

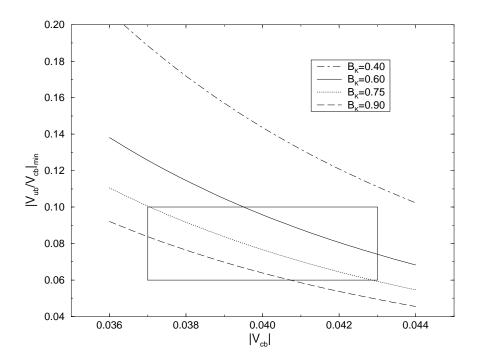


Figure 31: Lower bound on $|V_{ub}/V_{cb}|$ from ε_K .

This bound is shown as a function of $|V_{cb}|$ for different values of \hat{B}_K and $m_t = 173 \,\text{GeV}$ in fig. 31. We observe that simultaneously small values of $|V_{ub}/V_{cb}|$ and $|V_{cb}|$, although still consistent with the ones given in table 12, are not allowed by the size of indirect CP violation observed in $K \to \pi\pi$.

10.7.3 Output of the Standard Analysis

The output of the standard analysis depends to some extent on the error analysis. This should be always remembered in view of the fact that different authors use different procedures. In

Table 13: Present output of the Standard Analysis. $\lambda_t = V_{ts}^* V_{td}$.

Quantity	Scanning	Gaussian	
$ V_{td} /10^{-3}$	6.9 - 11.3	8.6 ± 1.1	
$\mid V_{ts}/V_{cb}\mid$	0.959 - 0.993	0.976 ± 0.010	
$\mid V_{td}/V_{ts}\mid$	0.16 - 0.31	0.213 ± 0.034	
$\sin(2\beta)$	0.36 - 0.80	0.66 ± 0.13	
$\sin(2\alpha)$	-0.76 - 1.0	0.11 ± 0.55	
$\sin(\gamma)$	0.66 - 1.0	0.88 ± 0.10	
$\mathrm{Im}\lambda_t/10^{-4}$	0.86 - 1.71	1.29 ± 0.22	

order to illustrate this I show in tables 13 ("present") and 14 ("future") the results for various quantities of interest using two types of error analyses:

- Scanning: Both the experimentally measured numbers and the theoretical input parameters are scanned independently within the errors given in table 12.
- Gaussian: The experimentally measured numbers and the theoretical input parameters are used with Gaussian errors.

Clearly the "scanning" method is a bit conservative. On the other hand using Gaussian distributions for theoretical input parameters can be certainly questioned. I think that at present the conservative "scanning" method should be preferred, although one certainly would like to have a better method. Interesting new methods have been presented in [159, 160]. They provide more stringent bounds on the apex of the unitarity triangle than presented here. I must admitt that I did not find time yet to analyze these papers to the extend that I could say anything profound about them here. I hope to do it soon. The analysis discussed here has been done in collaboration with Matthias Jamin and Markus Lautenbacher [161].

In figs. 32 and 33 we show the ranges for the upper corner A of the UT in the case of the "present" input and "future" input respectively. The circles correspond to R_t^{max} from (10.72) using $\xi = 1.20$ and $(\Delta M)_s = 10/ps$, 15/ps and 25/ps, respectively. The present bound (10.73) is represented by the first of these circles. This bound has not been used in obtaining the results in tables 13 and 14. Its impact will be analysed separately below. The circles from $B_d^0 - \bar{B}_d^0$ mixing are not shown explicitly for reasons to be explained below. The impact of ΔM_d can however be easily seen by comparing the shaded area with the area one would find by using the lower ε -hyperbola and the R_b -circles alone. The allowed region has a

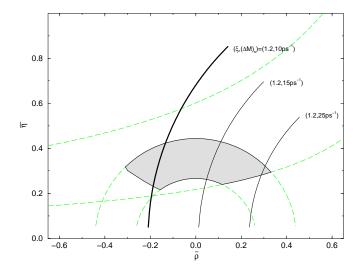


Figure 32: Unitarity Triangle 1998.

typical "banana" shape which can be found in many other analyses [34, 162, 93, 163, 159, 160]. The size of the banana and its position depends on the assumed input parameters and on the error analysis which varies from paper to paper. The results in figs. 32 and 33 correspond to a simple independent scanning of all parameters within one standard deviation. I should remark that the plots in [160] give substantially smaller allowed ranges in the $(\bar{\varrho}, \bar{\eta})$ plane and look more like potatoes than bananas.

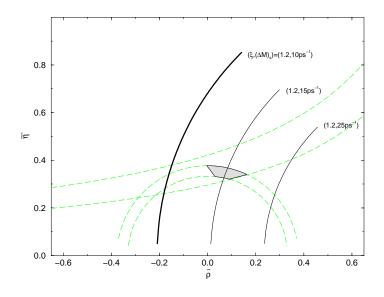


Figure 33: Unitarity Triangle 2008.

As seen in fig. 32 our present knowledge of the unitarity triangle is still rather poor. Fig. 33 demonstrates very clearly that this situation may change dramatically in the future provided the errors in the input parameters will be decreased as anticipated in our "future" scenario.

Comparing the results for $|V_{td}|$ given in table 13 with the ones obtained on the basis of unitarity alone (2.53) we observe that the inclusion of the constraints from ε and ΔM_d had a considerable impact on the allowed range for this CKM matrix element. This impact will be amplified in the future as seen in table 14. An inspection shows that with our input parameters the lower bound on $|V_{td}|$ is governed by ε_K , whereas the upper bound by ΔM_d .

Next we observe that whereas the angle β is rather constrained, the uncertainties in α and γ are huge:

$$35^{\circ} \le \alpha \le 115^{\circ}$$
, $11^{\circ} \le \beta \le 27^{\circ}$, $41^{\circ} \le \gamma \le 134^{\circ}$. (10.76)

The situation will improve when the "future" scenario will be realized:

$$70^{\circ} \le \alpha \le 93^{\circ}$$
, $19^{\circ} \le \beta \le 22^{\circ}$, $65^{\circ} \le \gamma \le 90^{\circ}$. (10.77)

Finally we would like to comment on the impact of the bound on ΔM_s given in (10.73) if the scanning method is used. This impact is still rather small except for the upper limits for $|V_{td}|/|V_{ts}|$ and γ which are lowered in the "scanning" version to 0.27 and 129° respectively. Larger impact of the bound on ΔM_s on various parameters is found by using the methods in [159, 160].

10.7.4 Correlation between ε_K and ΔM_d

Now, why did we omitt the explicit circles from $B_d^0 - \bar{B}_d^0$ mixing in the plots of unitarity triangles above? I have to answer this question because some of my colleagues suspected that a plot similar to the one in fig. 32 and shown already at the Rochester conference in Warsaw was wrong. At first one would expect that the left border of the allowed area coming from $B_d^0 - \bar{B}_d^0$ mixing should have a shape similar to the circles coming from $\Delta M_d/\Delta M_s$ and shown in the figures above. This expectation is correct at fixed values of m_t and $|V_{cb}|$. Yet once these two parameters are varied in the allowed ranges, this is no longer true. In fact one can easily convince oneself that the uncertainties coming from m_t and $|V_{cb}|$ in the analyses of ε_K and ΔM_d cannot be represented simultaneously in the $(\bar{\varrho}, \bar{\eta})$ plane in terms of nice hyperbolas and nice circles. This is simply related to the correlation between ε_K and ΔM_d due to m_t and $|V_{cb}|$. Neglecting this correlation one finds for instance that the most negative value of $\bar{\varrho}$ corresponds to the maximal values of $(m_t, |V_{cb}|)$ in the case of ε_K and to the minimal values of $(m_t, |V_{cb}|)$ in the case of $B_d^0 - \bar{B}_d^0$ which is of course inconsistent. In figs.

32 and 33 we have decided to show the ε_K -hyperbolas. Consequently the impact of $B_d^0 - \bar{B}_d^0$ mixing had to be found numerically and as seen it is not described by a circle. Since m_t is already very well known, this discussion mainly applies to the $|V_{cb}|$ dependence. Finally it should be stressed that similar correlations have to be taken into account in the future when various rare decays discussed in subsequent sections will enter the game of the determination of the unitarity triangle. Needless to say, the radius R_t^{max} determined through (10.72) and shown in the UT plots, being independent of $(m_t, |V_{cb}|)$, is not subject to the correlation in question.

Table 14: Future output of the Standard Analysis. $\lambda_t = V_{ts}^* V_{td}$.

Quantity Scanning		Gaussian
$ V_{td} /10^{-3}$	8.1 - 9.2	8.6 ± 0.3
$\mid V_{ts}/V_{cb}\mid$	0.969 - 0.983	0.976 ± 0.004
$\mid V_{td}/V_{ts}\mid$	0.20 - 0.24	0.215 ± 0.010
$\sin(2\beta)$	0.61 - 0.70	0.67 ± 0.03
$\sin(2\alpha)$	-0.11 - 0.66.0	0.21 ± 0.21
$\sin(\gamma)$	0.90 - 1.0	0.96 ± 0.03
$\mathrm{Im}\lambda_t/10^{-4}$	1.21 - 1.41	1.29 ± 0.06

10.8 Final Remarks

In this section we have completed the determination of the CKM matrix. It is given by the values of $|V_{us}|$, $|V_{cb}|$ and $|V_{ub}|$ in (2.51) and (2.52), the results in table 13 and the unitarity triangle shown in fig. 32. Clearly the accuracy of this determination is not impressive. We have stressed, however, that in ten years from now the standard analysis may give the results shown in table 14 and fig. 33. Moreover a single precise measurement of ΔM_s in the future will have a very important impact on the allowed area in the $(\bar{\varrho}, \bar{\eta})$ plane. Such a measurement should come from SLD and later from LHC.

Having the values of CKM parameters at hand, we can use them to predict various branching ratios of radiative, rare and CP-violating decays. This we will do in the subsequent three sections. We will see there, that the poor knowledge of CKM parameters precludes precise predictions of a number of interesting branching ratios at present. This may change in the next decade as stressed above.

11 ε'/ε in the Standard Model

11.1 Preliminaries

Direct CP violation remains one of the important targets of contemporary particle physics. In this respect the search for direct CP violation in $K \to \pi\pi$ decays plays a special role as already sixteen years have been devoted to this enterprise. In this case, a non-vanishing value of the ratio $\text{Re}(\varepsilon'/\varepsilon)$ defined in (10.27) would give the first signal for direct CP violation ruling out superweak models [166]. The experimental situation of $\text{Re}(\varepsilon'/\varepsilon)$ is, however, unclear at present:

$$\operatorname{Re}(\varepsilon'/\varepsilon) = \begin{cases} (23 \pm 7) \cdot 10^{-4} & [164] \\ (7.4 \pm 5.9) \cdot 10^{-4} & [165]. \end{cases}$$
 (11.1)

While the result of the NA31 collaboration at CERN [164] clearly indicates direct CP violation, the value of E731 at Fermilab [165] is compatible with superweak models in which $\varepsilon'/\varepsilon = 0$. Hopefully, during the next two years the experimental situation concerning ε'/ε will be clarified through the improved measurements by the two collaborations at the 10^{-4} level and by the KLOE experiment at DA Φ NE. A recent discussion of superweak models can be found in [167]. I will not consider them here.

There is no question about that direct CP violation is present in the Standard Model. Yet accidentally it could turn out that it will be difficult to see it in $K \to \pi\pi$ decays. Indeed as we will discuss in detail below, in the Standard Model ε'/ε is governed by QCD penguins and electroweak (EW) penguins. We have met them already in connection with B-decays in Section 8. In spite of being suppressed by α/α_s relative to QCD penguin contributions, electroweak penguin contributions have to be included because of the additional enhancement factor $\text{Re}A_0/\text{Re}A_2 = 22$ (see (11.2)–(11.4)) relative to QCD penguins. With increasing m_t the EW penguins become increasingly important [168, 169] and, entering ε'/ε with the opposite sign to QCD penguins, suppress this ratio for large m_t . For $m_t \approx 200$ GeV the ratio can even be zero [169]. Because of this strong cancellation between two dominant contributions and due to uncertainties related to hadronic matrix elements of the relevant local operators, a precise prediction of ε'/ε is not possible at present. We will discuss this in detail below.

11.2 History of ε'/ε

The first calculations of ε'/ε for $m_{\rm t} \ll M_{\rm W}$ and in the leading order approximation can be found in [170]. For $m_{\rm t} \ll M_{\rm W}$ only QCD penguins play a substantial role. Over the eighties these calculations were refined through the inclusion of isospin braking in the quark masses [171, 172, 173], the inclusion of QED penguin effects for $m_{\rm t} \ll M_{\rm W}$ [175, 171, 172], and through improved estimates of hadronic matrix elements in the framework of the 1/N

approach [176]. This era of ε'/ε culminated in the analyses in [168, 169], where QCD penguins, electroweak penguins (γ and Z^0 penguins) and the relevant box diagrams were included for arbitrary top quark masses. The strong cancellation between QCD penguins and electroweak penguins for $m_t > 150$ GeV found in these papers was confirmed by other authors [177].

All these calculations were done in the leading logarithmic approximation (e.g. one-loop anomalous dimensions of the relevant operators) with the exception of the m_t -dependence which in the analyses [168, 169, 177] has been already included at the NLO level. While such a procedure is not fully consistent, it allowed for the first time to exhibit the strong m_t -dependence of the electroweak penguin contributions, which is not seen in a strict leading logarithmic approximation.

During the nineties considerable progrees has been made by calculating complete NLO corrections to ε' [68, 69, 73, 74, 75]. Together with the NLO corrections to ε and $B^0 - \bar{B}^0$ mixing discussed in the previous section, this allows a complete NLO analysis of ε'/ε including constraints from the observed indirect CP violation (ε) and $B^0_{d,s} - \bar{B}^0_{d,s}$ mixings ($\Delta M_{d,s}$). The improved determination of the V_{ub} and V_{cb} elements of the CKM matrix, the improved estimates of hadronic matrix elements using the lattice approach as well as other non-perturbative approaches and in particular the determination of the top quark mass m_t had of course also an important impact on ε'/ε .

After these general remarks let us discuss ε'/ε in explicit terms. Other reviews of ε'/ε can be found in [178, 179].

11.3 Basic Formulae

The direct CP violation in $K \to \pi\pi$ is described by the parameter ε' defined in (10.27). The latter formula can be rewritten in terms of the real and imaginary parts of the amplitudes $A_0 \equiv A(K \to (\pi\pi)_{I=0})$ and $A_2 \equiv A(K \to (\pi\pi)_{I=2})$ as follows:

$$\varepsilon' = -\frac{\omega}{\sqrt{2}}\xi(1-\Omega)\exp(i\Phi), \qquad (11.2)$$

where

$$\xi = \frac{\operatorname{Im} A_0}{\operatorname{Re} A_0}, \quad \omega = \frac{\operatorname{Re} A_2}{\operatorname{Re} A_0}, \quad \Omega = \frac{1}{\omega} \frac{\operatorname{Im} A_2}{\operatorname{Im} A_0}$$
 (11.3)

and $\Phi \approx \pi/4$. Let us immediately emphasize the most important features of various terms in (11.2):

- Im A_0 is dominated by QCD penguins and is very weakly dependent on m_t .
- Im A_2 increases strongly with m_t and for large m_t is dominated by electroweak penguins. It receives also a sizable contribution from isospin braking $(m_u \neq m_d)$ which conspires with electroweak penguins to cancel substantially the QCD penguin contribution in

 $\text{Im}A_0$. The factor $1/\omega \approx 22$ in Ω giving a large enhancement is to a large extend responsible for this cancellation.

When using (11.2) and (11.3) in phenomenological applications one usually takes $\text{Re}A_0$ and ω from experiment, i.e.

$$ReA_0 = 3.33 \cdot 10^{-7} \text{ GeV}, \qquad ReA_2 = 1.50 \cdot 10^{-8} \text{ GeV}, \qquad \omega = 0.045,$$
 (11.4)

where the last relation reflects the so-called $\Delta I = 1/2$ rule. The main reason for this strategy is the unpleasant fact that until today nobody succeded in fully explaining this rule which to a large extent is believed to originate in the long-distance QCD contributions [180]. On the other hand the imaginary parts of the amplitudes in (11.3) being related to CP violation and the top quark physics should be dominated by short-distance contributions. Therefore $\text{Im}A_0$ and $\text{Im}A_2$ are usually calculated using the effective Hamiltonian for $\Delta S = 1$ transitions:

$$\mathcal{H}_{\text{eff}}(\Delta S = 1) = \frac{G_F}{\sqrt{2}} V_{us}^* V_{ud} \sum_{i=1}^{10} (z_i(\mu) + \tau \ y_i(\mu)) \ Q_i(\mu)$$
 (11.5)

with $\tau = -V_{ts}^* V_{td}/(V_{us}^* V_{ud})$.

The operators Q_i are the analogues of the ones given in (8.35)-(8.37) and (8.105)-(8.108). They are given explicitly as follows:

Current-Current:

$$Q_1 = (\bar{s}_{\alpha} u_{\beta})_{V-A} (\bar{u}_{\beta} d_{\alpha})_{V-A} \qquad Q_2 = (\bar{s} u)_{V-A} (\bar{u} d)_{V-A}$$
(11.6)

QCD-Penguins:

$$Q_3 = (\bar{s}d)_{V-A} \sum_{q=u,d,s} (\bar{q}q)_{V-A} \qquad Q_4 = (\bar{s}_{\alpha}d_{\beta})_{V-A} \sum_{q=u,d,s} (\bar{q}_{\beta}q_{\alpha})_{V-A}$$
(11.7)

$$Q_5 = (\bar{s}d)_{V-A} \sum_{q=u,d,s} (\bar{q}q)_{V+A} \qquad Q_6 = (\bar{s}_{\alpha}d_{\beta})_{V-A} \sum_{q=u,d,s} (\bar{q}_{\beta}q_{\alpha})_{V+A}$$
(11.8)

Electroweak-Penguins:

$$Q_7 = \frac{3}{2} (\bar{s}d)_{V-A} \sum_{q=u,d,s} e_q (\bar{q}q)_{V+A} \qquad Q_8 = \frac{3}{2} (\bar{s}_\alpha d_\beta)_{V-A} \sum_{q=u,d,s} e_q (\bar{q}_\beta q_\alpha)_{V+A}$$
(11.9)

$$Q_9 = \frac{3}{2} (\bar{s}d)_{V-A} \sum_{q=u,d,s} e_q(\bar{q}q)_{V-A} \qquad Q_{10} = \frac{3}{2} (\bar{s}_{\alpha}d_{\beta})_{V-A} \sum_{q=u,d,s} e_q (\bar{q}_{\beta}q_{\alpha})_{V-A}.$$
 (11.10)

Here, e_q denotes the electrical quark charges reflecting the electroweak origin of Q_7, \ldots, Q_{10} . The Wilson coefficient functions $z_i(\mu)$ and $y_i(\mu)$ were calculated including the complete next-to-leading order (NLO) corrections in [68, 69, 73, 74, 75]. The details of these calculations can be found there and in the review [17]. Only the coefficients $y_i(\mu)$ enter the

Table 15: $\Delta S=1$ Wilson coefficients at $\mu=m_{\rm c}=1.3\,{\rm GeV}$ for $m_{\rm t}=170\,{\rm GeV}$ and f=3 effective flavours. $|z_3|,\ldots,|z_{10}|$ are numerically irrelevant relative to $|z_{1,2}|$. $y_1=y_2\equiv 0$.

	$\Lambda_{\overline{\rm MS}}^{(4)} = 245{\rm MeV}$		$\Lambda_{\overline{\rm MS}}^{(4)} = 325{\rm MeV}$		$\Lambda_{\overline{\rm MS}}^{(4)} = 405 {\rm MeV}$				
Scheme	LO	NDR	HV	LO	NDR	HV	LO	NDR	HV
z_1	-0.550	-0.364	-0.438	-0.625	-0.415	-0.507	-0.702	-0.469	-0.585
z_2	1.294	1.184	1.230	1.345	1.216	1.276	1.399	1.251	1.331
y_3	0.029	0.024	0.027	0.034	0.029	0.033	0.039	0.034	0.039
y_4	-0.054	-0.050	-0.052	-0.061	-0.057	-0.060	-0.068	-0.065	-0.068
y_5	0.014	0.007	0.014	0.015	0.005	0.016	0.016	0.002	0.018
y_6	-0.081	-0.073	-0.067	-0.096	-0.089	-0.081	-0.113	-0.109	-0.097
y_7/α	0.032	-0.031	-0.030	0.039	-0.030	-0.028	0.045	-0.029	-0.026
y_8/α	0.100	0.111	0.120	0.121	0.136	0.145	0.145	0.166	0.176
y_9/α	-1.445	-1.437	-1.437	-1.490	-1.479	-1.479	-1.539	-1.528	-1.528
y_{10}/α	0.588	0.477	0.482	0.668	0.547	0.553	0.749	0.624	0.632

evaluation of ε'/ε . Examples of their numerical values are given in table 15. Extensive tables for $y_i(\mu)$ can be found in [17].

Using the Hamiltonian in (11.5) and the experimental values for ε , Re A_0 and ω the ratio ε'/ε can be written as follows:

$$\frac{\varepsilon'}{\varepsilon} = \operatorname{Im} \lambda_t \cdot \left[P^{(1/2)} - P^{(3/2)} \right], \tag{11.11}$$

where

$$P^{(1/2)} = r \sum y_i \langle Q_i \rangle_0 (1 - \Omega_{\eta + \eta'}) ,$$
 (11.12)

$$P^{(3/2)} = \frac{r}{\omega} \sum y_i \langle Q_i \rangle_2 , \qquad (11.13)$$

with

$$r = \frac{G_{\rm F}\omega}{2|\varepsilon|{\rm Re}A_0}$$
, $\langle Q_i\rangle_I \equiv \langle (\pi\pi)_I|Q_i|K\rangle$. (11.14)

One should note that the overall strong phases in ε' and ε cancel in the ratio to an excellent approximation. The sum in (11.12) and (11.13) runs over all contributing operators. $P^{(3/2)}$ is fully dominated by electroweak penguin contributions. $P^{(1/2)}$ on the other hand is governed by QCD penguin contributions which are suppressed by isospin breaking in the quark masses $(m_u \neq m_d)$. The latter effect is described by

$$\Omega_{\eta+\eta'} = \frac{1}{\omega} \frac{(\operatorname{Im} A_2)_{\text{I.B.}}}{\operatorname{Im} A_0} \,. \tag{11.15}$$

For $\Omega_{\eta+\eta'}$ we will take

$$\Omega_{\eta + \eta'} = 0.25 \pm 0.05 \,, \tag{11.16}$$

which is in the ball park of the values obtained in the 1/N approach [172] and in chiral perturbation theory [171, 173]. $\Omega_{\eta+\eta'}$ is independent of $m_{\rm t}$.

The main source of uncertainty in the calculation of ε'/ε are the hadronic matrix elements $\langle Q_i \rangle_I$. They depend generally on the renormalization scale μ and on the scheme used to renormalize the operators Q_i . These two dependences are canceled by those present in the Wilson coefficients $y_i(\mu)$ so that the resulting physical ε'/ε does not (in principle) depend on μ and on the renormalization scheme of the operators. Unfortunately the accuracy of the present non-perturbative methods used to evalutate $\langle Q_i \rangle_I$, like lattice methods, the 1/N expansion, chiral quark models and chiral effective lagrangians, is not sufficient to obtain the required μ and scheme dependences of $\langle Q_i \rangle_I$. A brief review of the existing methods including most recent developments will be given below.

In view of this situation it has been suggested in [73] to determine as many matrix elements $\langle Q_i \rangle_I$ as possible from the leading CP conserving $K \to \pi\pi$ decays, for which the experimental data are summarized in (11.4). To this end it turned out to be very convenient to determine $\langle Q_i \rangle_I$ at the scale $\mu = m_c$. Using the renormalization group evolution one can then find $\langle Q_i \rangle_I$ at any other scale $\mu \neq m_c$. The details of this procedure can be found in [73]. We will briefly summarize the most important results of this work below.

11.4 Hadronic Matrix Elements

11.4.1 Preliminaries

It is customary to express the matrix elements $\langle Q_i \rangle_I$ in terms of non-perturbative parameters $B_i^{(1/2)}$ and $B_i^{(3/2)}$ as follows:

$$\langle Q_i \rangle_0 \equiv B_i^{(1/2)} \langle Q_i \rangle_0^{(\text{vac})}, \qquad \langle Q_i \rangle_2 \equiv B_i^{(3/2)} \langle Q_i \rangle_2^{(\text{vac})}.$$
 (11.17)

The label "vac" stands for the vacuum insertion estimate of the hadronic matrix elements in question. The full list of $\langle Q_i \rangle_I$ is given in [73]. It suffices to give here only a few examples:

$$\langle Q_1 \rangle_0 = -\frac{1}{9} X B_1^{(1/2)}, \qquad (11.18)$$

$$\langle Q_2 \rangle_0 = \frac{5}{9} X B_2^{(1/2)}, \qquad (11.19)$$

$$\langle Q_6 \rangle_0 = -4\sqrt{\frac{3}{2}} \left[\frac{m_{\rm K}^2}{m_{\rm s}(\mu) + m_{\rm d}(\mu)} \right]^2 \frac{F_{\pi}}{\kappa} B_6^{(1/2)},$$
 (11.20)

$$\langle Q_1 \rangle_2 = \langle Q_2 \rangle_2 = \frac{4\sqrt{2}}{9} X B_1^{(3/2)},$$
 (11.21)

$$\langle Q_i \rangle_2 = 0, \qquad i = 3, \dots, 6,$$
 (11.22)

$$\langle Q_8 \rangle_2 = -\left[\frac{\kappa}{2\sqrt{2}} \langle \overline{Q_6} \rangle_0 + \frac{\sqrt{2}}{6} X\right] B_8^{(3/2)}, \qquad (11.23)$$

$$\langle Q_9 \rangle_2 = \langle Q_{10} \rangle_2 = \frac{3}{2} \langle Q_1 \rangle_2,$$
 (11.24)

where

$$\kappa = \frac{F_{\pi}}{F_{K} - F_{\pi}}, \qquad X = \sqrt{\frac{3}{2}} F_{\pi} \left(m_{K}^{2} - m_{\pi}^{2} \right), \tag{11.25}$$

and

$$\langle \overline{Q_6} \rangle_0 = \frac{\langle Q_6 \rangle_0}{B_6^{(1/2)}}.$$
 (11.26)

In the vacuum insertion method $B_i = 1$ independent of μ . In QCD, however, the hadronic parameters B_i generally depend on the renormalization scale μ and the renormalization scheme considered.

11.4.2 $(V-A)\otimes (V-A)$ Operators

Let us now extract some matrix from the data on Re A_0 and Re A_2 in (11.4). To this end we follow [73]. One notes first that in view of the smallness of $\tau = \mathcal{O}(10^{-4})$ entering (11.5), the real amplitudes in (11.4) are governed by the coefficients $z_i(\mu)$. The method of extracting some of the matrix elements from the data as proposed in [73] relies then on the fact that due to the GIM mechanism for $\mu \geq m_c$ the coefficients $z_i(\mu)$ of the penguin operators (i=3....10) vanish at the matching scale μ_c (between the four-quark and three-quark effective theories) in the HV scheme and are negligible in the NDR scheme. However, it should be remembered that the smallness or even vanishing of $z_i(\mu)$ for $\mu \geq m_c$ is characteristic for mass independent renormalization schemes. In other schemes, in which the disparity of m_u and m_c is felt well above $\mu = m_c$, the GIM cancellation is incomplete and $z_i(m_c)$ for penguin operators are larger than in the HV and NDR schemes. Examples of the leading order calculations of this type can be found in [174].

Staying within the NDR and HV schemes, we can however set $z_i(m_c) = 0$ for $i \neq 1, 2$ to find

$$\langle Q_1(m_c)\rangle_2 = \langle Q_2(m_c)\rangle_2 = \frac{10^6 \,\text{GeV}^2}{1.77} \frac{\text{Re}A_2}{z_+(m_c)} = \frac{8.47 \cdot 10^{-3} \,\text{GeV}^3}{z_+(m_c)}$$
 (11.27)

with $z_+ = z_1 + z_2$ and

$$\langle Q_1(m_c)\rangle_0 = \frac{10^6 \,\text{GeV}^2}{1.77} \frac{\text{Re}A_0}{z_1(m_c)} - \frac{z_2(m_c)}{z_1(m_c)} \langle Q_2(m_c)\rangle_0.$$
 (11.28)

These formulae are easy to derive and are left as a useful homework problem.

Comparing next (11.27) with (11.21) one finds immediately

$$B_1^{(3/2)}(m_c) = \frac{0.363}{z_+(m_c)}, (11.29)$$

which using table 15 gives for $m_{\rm c}=1.3\,{\rm GeV}$ and $\Lambda_{\overline{\rm MS}}^{(4)}=325\,{\rm MeV}$

$$B_{1,NDR}^{(3/2)}(m_c) = 0.453, \qquad B_{1,HV}^{(3/2)}(m_c) = 0.472.$$
 (11.30)

The extracted values for $B_1^{(3/2)}$ are by more than a factor of two smaller than the vacuum insertion estimate. They are compatible with the $1/N_c$ value $B_1^{(3/2)}(1 \text{ GeV}) \approx 0.55$ [176] and are somewhat smaller than the lattice result $B_1^{(3/2)}(2 \text{ GeV}) \approx 0.6$ [162]. As analyzed in [73], $B_1^{(3/2)}(\mu)$ decreases slowly with increasing μ . As seen in (11.24), this analysis gives also $\langle Q_9(m_c)\rangle_2$ and $\langle Q_{10}(m_c)\rangle_2$.

In order to extract $B_1^{(1/2)}(m_c)$ and $B_2^{(1/2)}(m_c)$ from (11.28) one can make the very plausible assumption, valid in known non-perturbative approaches, that $\langle Q_-(m_c)\rangle_0 \geq \langle Q_+(m_c)\rangle_0 \geq 0$, where $Q_{\pm}=(Q_2\pm Q_1)/2$. This gives for $\Lambda_{\overline{\rm MS}}^{(4)}=325\,{\rm MeV}$

$$B_{2,NDR}^{(1/2)}(m_c) = 6.6 \pm 1.0, \qquad B_{2,HV}^{(1/2)}(m_c) = 6.2 \pm 1.0.$$
 (11.31)

The extraction of $B_1^{(1/2)}(m_c)$ and of analogous parameters $B_{3,4}^{(1/2)}(m_c)$ are presented in detail in [73]. $B_1^{(1/2)}(m_c)$ depends very sensitively on $B_2^{(1/2)}(m_c)$ and its central value is as high as 15. $B_4^{(1/2)}(m_c)$ is typically by (10–15) % lower than $B_2^{(1/2)}(m_c)$. In any case this analysis shows very large deviations from the results of the vacuum insertion method.

11.4.3 $(V-A)\otimes (V+A)$ Operators

The matrix elements of the $(V-A)\otimes (V+A)$ operators Q_5 – Q_8 cannot be constrained by CP conserving data and one has to rely on existing non-perturbative methods to calculate them. This is rather unfortunate because the QCD penguin operator Q_6 and the electroweak penguin operator Q_8 , having large Wilson coefficients and large hadronic matrix elements, play the dominant role in ε'/ε .

We will now review the present status of B_i factors describing the matrix elements of $Q_5 - Q_8$ operators as obtained in various non-perturbative approaches. We will pay particular attention to the parameters $B_6^{(1/2)}$ and $B_8^{(3/2)}$ which are most important for the evaluation of ε'/ε . We recall that $B_i = 1$ in the vacuum insertion method.

11.4.4 $B_6^{(1/2)}$ and $B_8^{(3/2)}$ from Lattice

We begin with lattice calculations. These have been reviewed recently by Gupta [142] and the APE collaboration [146]. The most reliable results are found for $B_{7,8}^{(3/2)}$. The "modern" quenched estimates for these parameters, which supercede all previously reported values are collected in table 16, which has been taken from Gupta and quenched a bit. The first three calculations use perturbative matching between lattice and continuum, the last one uses non-perturbative matching. Since all three groups agree within perturbative matching and the

non-perturbative matching should be preferred, I conclude (probably naively) that the best quenched lattice values are

$$(B_7^{(3/2)})_{\text{lattice}}(2 \text{ GeV}) = 0.72 \pm 0.05, \qquad (B_8^{(3/2)})_{\text{lattice}}(2 \text{ GeV}) = 1.03 \pm 0.03 \qquad (11.32)$$

where the errors are purely statistical. Concerning the lattice results for $B_{5,6}^{(1/2)}$ the situation is worse. The old results read $B_{5,6}^{(1/2)}(2 \text{ GeV}) = 1.0 \pm 0.2 \text{ [181, 182]}$. More accurate estimates for $B_6^{(1/2)}$ have been recently obtained in [183]: $B_6^{(1/2)}(2 \text{ GeV}) = 0.67 \pm 0.04 \pm 0.05$ (quenched) and $B_6^{(1/2)}(2 \text{ GeV}) = 0.76 \pm 0.03 \pm 0.05$ (f = 2). However, as stressed by Gupta, the systematic errors in this analysis are not really under control. We have to conclude, that there are no solid predictions for $B_{5,6}^{(1/2)}$ from the lattice at present.

Table 16: Lattice results for $B_{7,8}^{(3/2)}(2 \text{ GeV})$ obtained by various groups.

Fermion type	$B_7^{(3/2)}$	$B_8^{(3/2)}$	Matching
Staggered[144]	0.62(3)(6)	0.77(4)(4)	1-loop
Wilson[145]	0.58(2)(7)	0.81(3)(3)	1-loop
Clover[146]	0.58(2)	0.83(2)	1-loop
Clover[146]	0.72(5)	1.03(3)	Non-pert.

11.4.5 $B_6^{(1/2)}$ and $B_8^{(3/2)}$ from the 1/N Approach

The 1/N approach to weak hadronic matrix elements was introduced in [176]. In this approach the 1/N expansion becomes a loop expansion in an effective meson theory. In the strict large N limit only the tree level matrix elements of Q_6 and Q_8 contribute and one finds (11.20) and (11.23) with

$$B_6^{(1/2)} = B_8^{(3/2)} = 1,$$
 (Large – N Limit) (11.33)

while $B_5^{(1/2)} = B_8^{(3/2)} = 0$. The latter fact is not disturbing, however, as the operators Q_5 and Q_7 , having small Wilson coefficients are immaterial for ε'/ε .

Now, $B_6^{(1/2)}$ and $B_8^{(3/2)}$ as given in (11.33) are clearly μ -independent. At first sight this appears as a problem. But in fact it is not! The point is that $Q_{6,8}$ are density×density operators as one can see by writing them with the help of the Fierz reordering as follows

$$Q_6 = -2\sum_{q=u,d,s} \bar{s}(1+\gamma_5)q\bar{q}(1-\gamma_5)d, \qquad Q_8 = -3\sum_{q=u,d,s} e_q \bar{s}(1+\gamma_5)q\bar{q}(1-\gamma_5)d. \quad (11.34)$$

Consequently their μ -dependences are related to the μ -dependence of the quark masses and the tree level factorizable contributions to $\langle Q_6 \rangle_0$ and $\langle Q_8 \rangle_2$ are μ -dependent through the

factor $1/m_{\rm s}^2(\mu)$ as seen in (11.20) and (11.23). This should be contrasted with the matrix elements of $(V-A)\otimes (V-A)$ operators, which are μ -independent in the large-N limit. The μ -dependence of $1/m_{\rm s}^2(\mu)$ in $\langle Q_6\rangle_0$ and $\langle Q_8\rangle_2$ is exactly cancelled in the decay amplitude by the diagonal evolution (no mixing) of the Wilson coefficients $y_6(\mu)$ and $y_8(\mu)$ taken in the large-N limit.

Indeed, the μ -dependence of $1/m_s^2(\mu)$ is governed in LO by $2\gamma_m^{(0)} = 12C_F$. On the other hand, the one-loop anomalous dimensions of $Q_{6,8}$, which govern the diagonal evolution of $y_{6,8}(\mu)$ are given by

$$\gamma_{66}^{(0)} = -2\gamma_m^{(0)} + \frac{2f}{3}, \qquad \gamma_{88}^{(0)} = -2\gamma_m^{(0)}.$$
(11.35)

Since for large N, $\gamma_m^{(0)} \sim \mathcal{O}(N)$, we find indeed $\gamma_{66}^{(0)} = \gamma_{88}^{(0)} = -2\gamma_m^{(0)}$ in the large-N limit [172]. Going back to the respective evolutions of $m_s(\mu)$ and $y_{7,8}(\mu)$ we indeed confirm the cancellation of the μ -dependence in question. This feature is preserved at the two-loop level as discussed in [68]. One can go even further and demonstrate numerically for N=3 that the parameters $B_6^{1/2}$ and $B_8^{3/2}$ depend only very weakly on μ , when $\mu \geq 1$ GeV. In such a numerical renormalization study in [73] the factors $B_6^{(1/2)}$ and $B_8^{(3/2)}$ have been set to unity at $\mu=m_c$. Subsequently the evolution of the matrix elements in the range $1 \text{ GeV} \leq \mu \leq 4 \text{ GeV}$ has been calculated showing that for the NDR scheme $B_{5,6}^{(1/2)}$ and $B_{7,8}^{(3/2)}$ were μ independent within an accuracy of (2–3)%. The μ dependence in the HV scheme has been found to be stronger but still below 10%.

In view of the fact that for $B_6^{(1/2)}=B_8^{(3/2)}=1$ and the known value of $m_{\rm t}$, there is a strong cancellation between gluon and electroweak penguin contributions to ε'/ε , it is important to investigate whether the 1/N corrections significantly affect this cancellation. First attempt in this direction has been made by the Dortmund group [184, 185], which incorporating in part chiral loops found an enhancement of $B_6^{(1/2)}$ and a suppression of $B_8^{(3/2)}$. From [185] $B_6^{(1/2)}=1.3$ and $B_8^{(3/2)}=0.7$ can be extracted.

Recently another Dortmund team [186], in collaboration with Bill Bardeen, performed this time a complete investigation of $\langle Q_6 \rangle_0$ and $\langle Q_8 \rangle_2$ in the twofold expansion in powers of external momenta p, and in 1/N. Their final result gives $\langle Q_6 \rangle_0$ and $\langle Q_8 \rangle_2$ including the orders p^2 and p^0/N . For $\langle Q_8 \rangle_2$ also the term p^0 contributes. Of particular interest are the $\mathcal{O}(p^0/N)$ contributions resulting from non-factorizable chiral loops which are important for the matching between long- and short-distance contributions. The cut-off scale Λ_c in these non-factorizable diagrams is identified with the QCD renormalization scale μ which enters the Wilson coefficients. In contrast to the matrix elements of $Q_{1,2}$ in which the Λ_c dependence was quadratic [176], the Λ_c dependence in the present case is logarithmic which improves the matching considerably. There are several technical and conceptual improvements in [186] over the first attempt in [184, 185] and also over the original approach [176]. Therefore I

strongly recommend to read [186], which is clearly written.

	$\Lambda_c = 0.6 \text{ GeV}$	$\Lambda_c = 0.7 \mathrm{GeV}$	$\Lambda_c = 0.8 \mathrm{GeV}$	$\Lambda_c = 0.9 \mathrm{GeV}$
$B_6^{(1/2)}$	1.10	0.96	0.84	0.72
	(1.30)	(1.19)	(1.09)	(0.99)
$B_8^{(3/2)}$	0.66	0.59	0.52	0.46
	(0.71)	(0.65)	(0.60)	(0.54)

Table 17: Results for $B_{7.8}^{(3/2)}$ obtained in the 1/N approach.

In table 17, taken from [186], we show the values of $B_6^{(1/2)}$ and $B_8^{(3/2)}$ as functions of the cut-off scale Λ_c . The results depend on whether F_{π} or F_K is used in the calculation, the difference being of higher order. The results using F_K are shown in the parentheses. The decrease of both B-factors with $\Lambda_c = \mu$ is qualitatively consistent with their μ -dependence found for $\mu \geq 1$ in [73], but it is much stronger. Clearly one could also expect stronger μ -dependence in the analysis of [73] for $\mu \leq 1$ GeV, but in view of large perturbative corrections for such small scales a meaningful test of the dependence in table 17 cannot be made. We note that for $\Lambda_c = 0.7$ GeV the value of $B_6^{(1/2)}$ is close to unity as in the large-N limit. However, $B_8^{(3/2)}$ is considerably suppressed.

It is difficult to decide which value should be used in phenomenology of ε'/ε . On the one hand, for $\Lambda_c \geq 0.6$ GeV neglected contributions from vector mesons in the loops should be included. On the other hand for $\Lambda_c = \mu = 0.6$ GeV it is difficult to make contact with short distance calculations and with the lattice results which are obtained for $\mu = 2$ GeV. As for $\mu \geq 1$ GeV the parameters in table 17 are expected to be almost μ -independent, let us take the values at $\Lambda_c = 0.9$ GeV as the main result of [186]. Averaging over the F_{π^-} and F_{K^-} -choices we find

$$B_6^{(1/2)} = 0.85 \pm 0.13$$
, $B_8^{(3/2)} = 0.50 \pm 0.04$, $(\Lambda_c = 0.9 \text{ GeV})$ (11.36)

where the errors should not be taken too seriously. The value of $B_6^{(1/2)}$ is compatible with the corresponding lattice results, whereas $B_8^{(3/2)}$ is found to be substantially smaller in the 1/N approach. On the other hand, it will be tempting later on to calculate ε'/ε for the choice $\Lambda_c = 0.6$ GeV which gives instead:

$$B_6^{(1/2)} = 1.2 \pm 0.1$$
, $B_8^{(3/2)} = 0.68 \pm 0.03$, $(\Lambda_c = 0.6 \text{ GeV})$. (11.37)

In view of the large correction to $B_8^{(3/2)}$ one might question the convergence of the 1/N expansion. However, the non-factorizable contributions considered in [186] represent the first

term in a new type of a series absent in the large-N limit and consequently there are no strong reasons for questioning the convergence of the 1/N expansion on the basis of these results. In this context one should also remark that the lattice studies discussed previously use tree level chiral perturbation theory to relate the matrix elements $\langle \pi\pi|Q_i|K\rangle$ to $\langle \pi|Q_i|K\rangle$ which are calculated on the lattice. It is conceivable that including chiral loops in this relation would decrease the value of $B_8^{(3/2)}$.

Finally I would like to express one criticism of the approach in [186] as well as in [176]. It is the lack of any reference to the renormalization scheme dependence which is necessary for a complete matching at the NLO level.

11.4.6 $B_6^{(1/2)}$ and $B_8^{(3/2)}$ from the Chiral Quark Model

Effective Quark Models of QCD can be derived in the framework of the extended Nambu-Jona-Lasinio model of chiral symmetry breaking [187]. For kaon decays and in particular for ε'/ε , an extensive analysis of this model inclusive chiral loops, gluon and $\mathcal{O}(p^4)$ corrections has been performed over the last years by the Trieste group [188, 189]. The crucial parameters in this approach is a mass parameter M and the condensates $\langle \bar{q}q \rangle$ and $\langle \alpha_s GG \rangle$. They can be constrained by imposing the $\Delta I = 1/2$ rule.

Since there exists a recent nice review [179] by the Trieste group of their approach, I will only quote here their estimate of the relevant B_i parameters. They are

$$B_6^{(1/2)} = 1.6 \pm 0.3$$
 $B_8^{(3/2)} = 0.92 \pm 0.02$. (Chiral QM) (11.38)

We observe a rather large enhancement of $B_6^{(1/2)}$, not observed by other groups, and a moderate suppression of $B_8^{(3/2)}$. These parameters correspond roughly to the scale $\mu=0.8\,$ GeV. Looking at the table 17 we may expect a 10% reduction of these values, had the scale $\mu=0.9\,$ GeV been used.

11.4.7 Strategy for $(V - A) \otimes (V + A)$ Operators

We have seen that various approaches differ in their estimates of the most important parameters $B_6^{(1/2)}$ and $B_8^{(3/2)}$. In table 18 we collect the central values from various approaches discussed above. In the case of lattice we have chosen various possible scenarios in view of different results obtained by various groups. Similarly in the case of the 1/N approach we have chosen two sets of B-values corresponding to two values of Λ_c . Even if the B_i factors given in this table are all within say 50% from the vacuum insertion estimate, they give rather different results for central values of ε'/ε as illustrated in the last two columns of this table. How these values have been obtained will be discussed a few pages below.

Table 18: Results for ε'/ε in units of 10^{-4} for three choices of $m_{\rm s}(m_{\rm c})$ and the central values of $B_6^{(1/2)}$ and $B_8^{(3/2)}$ obtained in various approaches. Im $\lambda_t=1.29\cdot 10^{-4}$ and $m_{\rm t}=167\,{\rm GeV}$ have been used.

Approach	$B_6^{(1/2)}$	$B_8^{(3/2)}$	150 MeV	$125~\mathrm{MeV}$	$100~{ m MeV}$
VIA	1.0	1.0	3.2	5.2	8.8
Lattice 1	1.0	0.81	4.2	6.6	10.9
Lattice 2	1.0	1.03	3.0	5.0	8.4
Lattice 3	0.76	0.81	1.7	3.1	5.7
Lattice 4	0.76	1.03	0.6	1.5	3.2
1/N (I)	0.85	0.50	4.3	6.7	11.1
1/N (II)	1.2	0.68	6.9	10.4	16.7
Chiral QM	1.6	0.92	9.7	14.4	22.7

Concerning $B_{7,8}^{(1/2)}$ one can simply set $B_{7,8}^{(1/2)} = 1$ as the matrix elementes $\langle Q_{7,8} \rangle_0$ play only a minor role in the ε'/ε analysis. I should however stress that whereas lattice results are consistent with this choice, this is not the case for the chiral quark model [189] in which values as high as 2.5 are found.

Concerning $B_5^{(1/2)}$ and $B_7^{(3/2)}$ we will simply set them equal to $B_6^{(1/2)}$ and $B_8^{(3/2)}$ respectively. This is consistent with the lattice results and the chiral quark model. There are no results for these parameters from the 1/N approach beyond the large-N limit.

In summary the treatment of $\langle Q_i \rangle_{0,2}$, $i = 5, \dots 8$ in [73, 17, 190] is to set

$$B_{7.8}^{(1/2)}(m_{\rm c}) = 1, \qquad B_5^{(1/2)}(m_{\rm c}) = B_6^{(1/2)}(m_{\rm c}), \qquad B_7^{(3/2)}(m_{\rm c}) = B_8^{(3/2)}(m_{\rm c})$$
 (11.39)

and to treat $B_6^{(1/2)}(m_c)$ and $B_8^{(3/2)}(m_c)$ as free parameters. In particular, in addition to estimates obtained by other groups, we will show below the results for ε'/ε when these parameters are varied in the ranges

$$B_6^{(1/2)}(m_c) = 1.0 \pm 0.2, \qquad B_8^{(3/2)}(m_c) = 1.0 \pm 0.2.$$
 (11.40)

and

$$B_6^{(1/2)}(m_c) = 1.0 \pm 0.2, \qquad B_8^{(3/2)}(m_c) = 0.7 \pm 0.2.$$
 (11.41)

The range (11.40) corresponds to the variation of the B_i parameters in the neighbourhood of the large–N limit. The range (11.41) gives a rough description of the fact that in recent analyses most approaches find $B_8^{(3/2)}$ to be smaller than $B_6^{(1/2)}$. This range will be analyzed at the end of this section.

After this long exposition of B_i parameters let us then incorporate the collected information in the formula for ε'/ε in a manner useful for phenomenological applications.

11.5 An Analytic Formula for ε'/ε

As shown in [191], it is possible to cast the formal expression for ε'/ε in (11.11) into an analytic formula which exhibits the $m_{\rm t}$ dependence together with the dependence on $m_{\rm s}$, $\Lambda_{\overline{\rm MS}}^{(4)}$, $B_6^{(1/2)}$ and $B_8^{(3/2)}$. Such an analytic formula should be useful for those phenomenologists and experimentalists who are not interested in getting involved with the technicalities discussed above.

In order to find an analytic expression for ε'/ε , which exactly reproduces the numerical results based on the formal OPE method, one uses the PBE presented in Section 3.3. The updated analytic formula for ε'/ε of [191] presented in [190] is given as follows:

$$\frac{\varepsilon'}{\varepsilon} = \operatorname{Im} \lambda_t \cdot F(x_t) \,, \tag{11.42}$$

where

$$F(x_t) = P_0 + P_X X_0(x_t) + P_Y Y_0(x_t) + P_Z Z_0(x_t) + P_E E_0(x_t)$$
(11.43)

and

$$\operatorname{Im} \lambda_t = \operatorname{Im} V_{ts}^* V_{td} = |V_{ub}| |V_{cb}| \sin \delta = \eta \lambda^5 A^2$$
(11.44)

in the standard parameterization of the CKM matrix (2.18) and in the Wolfenstein parameterization (2.21), respectively.

The $m_{\rm t}$ -dependent functions in (11.43) are given in (3.14) and (3.26)–(3.28). The coefficients P_i are given in terms of $B_6^{(1/2)} \equiv B_6^{(1/2)}(m_{\rm c}), B_8^{(3/2)} \equiv B_8^{(3/2)}(m_{\rm c})$ and $m_{\rm s}(m_{\rm c})$ as follows:

$$P_i = r_i^{(0)} + R_s \left(r_i^{(6)} B_6^{(1/2)} + r_i^{(8)} B_8^{(3/2)} \right), \tag{11.45}$$

where

$$R_s = \left[\frac{158 \,\text{MeV}}{m_s(m_c) + m_d(m_c)} \right]^2. \tag{11.46}$$

The P_i are renormalization scale and scheme independent. They depend, however, on $\Lambda_{\overline{\rm MS}}^{(4)}$. In table 19 we give the numerical values of $r_i^{(0)}$, $r_i^{(6)}$ and $r_i^{(8)}$ for different values of $\Lambda_{\overline{\rm MS}}^{(4)}$ at $\mu = m_{\rm c}$ in the NDR renormalization scheme. The coefficients $r_i^{(0)}$, $r_i^{(6)}$ and $r_i^{(8)}$ depend only very weakly on $m_{\rm s}(m_{\rm c})$ as the dominant $m_{\rm s}$ dependence has been factored out. The numbers given in table 19 correspond to $m_{\rm s}(m_{\rm c}) = 150$ MeV. However, even for $m_{\rm s}(m_{\rm c}) \approx 100$ MeV, the analytic expressions given here reproduce the numerical calculations of ε'/ε given below to better than 4%. For different scales μ the numerical values in the tables change without modifying the values of the P_i 's as it should be. The values of $B_6^{(1/2)}$ and $B_8^{(3/2)}$ should

also be modified, in principle, but in view of the comments made previously it is a good approximation to keep them μ -independent for $\mu \geq 1$ GeV.

Concerning the scheme dependence only the r_0 coefficients are scheme dependent at the NLO level. Their values in the HV scheme are given in the last row of table 19. The coefficients r_i , i = X, Y, Z, E are on the other hand scheme independent at NLO. This is related to the fact that the m_t dependence in ε'/ε enters first at the NLO level and consequently all coefficients r_i in front of the m_t dependent functions must be scheme independent. Consequently, when changing the renormalization scheme, one is only obliged to change appropriately $B_6^{(1/2)}$ and $B_8^{(3/2)}$ in the formula for P_0 in order to obtain a scheme independence of ε'/ε . In calculating P_i where $i \neq 0$, $B_6^{(1/2)}$ and $B_8^{(3/2)}$ can in fact remain unchanged, because their variation in this part corresponds to higher order contributions to ε'/ε which would have to be taken into account in the next order of perturbation theory.

For similar reasons the NLO analysis of ε'/ε is still insensitive to the precise definition of $m_{\rm t}$. In view of the fact that the NLO calculations needed to extract ${\rm Im}\lambda_t$ (see previous section) have been done with $m_{\rm t}=\overline{m}_t(m_{\rm t})$ we will also use this definition in calculating $F(x_t)$.

Table 19: PBE coefficients for ε'/ε for various $\Lambda_{\overline{\rm MS}}^{(4)}$ in the NDR scheme. The last row gives the r_0 coefficients in the HV scheme.

	$\Lambda_{\overline{\rm MS}}^{(4)}$	$\frac{1}{5} = 245 \text{M}$	/IeV	$\Lambda_{\overline{\rm MS}}^{(4)} = 325{\rm MeV}$			$\Lambda_{\overline{\rm MS}}^{(4)} = 405 \mathrm{MeV}$		
i	$r_{i}^{(0)}$	$r_{i}^{(6)}$	$r_{i}^{(8)}$	$r_{i}^{(0)}$	$r_i^{(6)}$	$r_{i}^{(8)}$	$r_{i}^{(0)}$	$r_i^{(6)}$	$r_{i}^{(8)}$
0	-2.674	6.537	1.111	-2.747	8.043	0.933	-2.814	9.929	0.710
X	0.541	0.011	0	0.517	0.015	0	0.498	0.019	0
Y	0.408	0.049	0	0.383	0.058	0	0.361	0.068	0
Z	0.178	-0.009	-6.468	0.244	-0.011	-7.402	0.320	-0.013	-8.525
E	0.197	-0.790	0.278	0.176	-0.917	0.335	0.154	-1.063	0.402
0	-2.658	5.818	0.839	-2.729	6.998	0.639	-2.795	8.415	0.398

The analytic formulae given above are useful for numerical calculations, but in order to identify the dominant terms in an elegant manner, we follow Gupta [142] and rewrite it as

$$\frac{\varepsilon'}{\varepsilon} = \text{Im}\lambda_t \cdot \left[c_0 + (c_6 B_6^{(1/2)} + c_8 B_8^{(3/2)}) R_s \right]. \tag{11.47}$$

For $m_{\rm t}=167\,$ GeV the values of the coefficients c_i are given in table 20.

The inspection of tables 19 and 20 shows that within a few percent

$$c_6 = r_0^{(6)}, c_8 = r_0^{(8)} + r_Z^{(8)} Z_0(x_t), (11.48)$$

Table 20: The coefficients c_i for various $\Lambda_{\overline{\rm MS}}^{(4)}$ in the NDR and HV schemes and $m_{\rm t}=167\,{\rm GeV}$.

	$\Lambda_{\overline{\rm MS}}^{(4)} = 245{\rm MeV}$		$\Lambda_{\overline{ m MS}}^{(4)} = 3$	$325\mathrm{MeV}$	$\Lambda_{\overline{\rm MS}}^{(4)} = 405 {\rm MeV}$		
Scheme	NDR	HV	NDR	HV	NDR	HV	
c_0	-1.264	-1.248	-1.359	-1.341	-1.430	-1.411	
c_6	6.387	5.668	7.873	6.828	9.735	8.221	
c_8	-3.259	-3.531	-4.063	-4.357	-5.041	-5.353	

whereby c_8 is dominated by the second term. Thus we conclude that the terms involving $r_0^{(6)}$ and $r_Z^{(8)}$ dominate the ratio ε'/ε . Moreover, the function $Z_0(x_t)$ representing a gauge invariant combination of Z^0 - and γ -penguins grows rapidly with m_t and due to $r_Z^{(8)} < 0$ these contributions suppress ε'/ε strongly for large m_t [168, 169] as stressed at the beginning of this section.

11.6 The Status of the Strange Quark Mass

It seems appropriate to summarize now the present status of the value of the strange quark mass. In the case of quenched lattice QCD this has been recently done by Gupta [142]. His final result based on 1997 world data is

$$m_s(2 \,\text{GeV}) = (110 \pm 25) \,\text{MeV}.$$
 (11.49)

It is expected that unquenching will lower this value but it is difficult to tell by how much.

Gupta summarized also the most recent values for $m_s(2 \text{ GeV})$ obtained using QCD sum rules. The older values (in MeV) are 144 ± 21 [192], 137 ± 23 [193] 148 ± 15 [194], whereas the most recent ones are found to be 91 - 116 [195] and 115 ± 22 [196]. On the other hand the following lower bounds on $m_s(2 \text{ GeV})$ have been derived: 118 - 189 [197], 88 ± 9 [198], 104 - 116 [199]. We observe that the QCD sum rule results are consistent with quenched lattice values although generally they are somewhat higher.

We conclude that the error on m_s is still rather large. Therefore it will be useful to present, few pages below, the results for ε'/ε for two values of $m_s(m_c)$:

$$m_{\rm s}(m_{\rm c}) = (150 \pm 20) \text{ MeV}$$
 and $m_{\rm s}(m_{\rm c}) = (125 \pm 20) \text{ MeV}$ (11.50)

corresponding (see table 3) roughly to $m_s(2 \text{ GeV}) = (129 \pm 17) \text{ MeV}$ and $m_s(2 \text{ GeV}) = (107 \pm 17) \text{ MeV}$, respectively.

Finally one should remark that the decomposition of the relevant hadronic matrix elements of penguin operators into a product of B_i factors times $1/m_s^2$, although useful in the

 $1/N_c$ approach, is in principle unnecessary in a brute force method like the lattice approach and in certain methods using effective lagrangians. It is to be expected that the future lattice calculations will directly give the relevant hadronic matrix elements and the issue of m_s in connection with ε'/ε will effectively disappear.

11.7 Numerical Results for ε'/ε

In order to complete the analysis of ε'/ε one needs the value of $\mathrm{Im}\lambda_t$. Since this value has been already determined in section 10.7 (see table 13), we are ready to present the results for ε'/ε . In order to gain some insight in what is going on, let us take the formula (11.47) and insert the central value $\mathrm{Im}\lambda_t=1.29\cdot 10^{-4}$ together with the NDR-values in table 20 for $\Lambda_{\overline{\mathrm{MS}}}^{(4)}=325\,\mathrm{MeV}$. We find then

$$\frac{\varepsilon'}{\varepsilon} = \left[-1.75 + (10.15 \cdot B_6^{(1/2)} - 5.24 \cdot B_8^{(3/2)}) R_s \right] \cdot 10^{-4}$$
 (11.51)

with R_s defined in (11.46).

Our "central" formula (11.51) gives then the values of ε'/ε collected in table 18. We observe that for higher values of m_s the lattice and the 1/N approach (I) give values of ε'/ε in the ball park of a few 10^{-4} . Higher values are obtained for the 1/N approach (II) and in particular in the chiral quark model which even in the first scenario for m_s gives value of ε'/ε close to $\mathcal{O}(10^{-3})$. For smaller values of m_s all approaches give higher values of ε'/ε although only the last two give results consistent with the NA31 value. The results for the 1/N approach (II) are only shown for illustration. A proper analysis of this case would require the calculation of Wilson coefficients for μ well below 1 GeV, which we do not want to do.

When analyzing these numbers some caution is needed. Our "central" formula (11.51) includes certain inputs which are not necessarily the same in all approaches. For instance our value of \hat{B}_K is lower than the values obtained in the lattice and chiral model approaches. Similarly the value c_0 is very much constrained by the incorporation of the $\Delta I = 1/2$ rule which cannot be obtained using VIA. In addition in a given approach c_6 and c_8 may differ somewhat from the ones used. But since they are dominated by the short distance Wilson coefficients these changes cannot be large and we believ that our formula is not too bad and gives some insight in what is going on.

On the other hand, once one begins to vary all input parameters the differences between various approaches wash out to some extend. We note for instance that the coefficients in tables 19 and 20 exhibit a sizable $\Lambda_{\overline{\rm MS}}^{(4)}$ dependence leading to almost linear dependence of ε'/ε on this parameter as pointed out in [73].

Let me than present results of the Munich group based on the input parameters of section 10 and the choice of B_i parameters summarized in (11.40). To this end exact expressions for

 ε'/ε have been used.

For $m_s(m_c) = 150 \pm 20 \,\text{MeV}$ one finds [190]

$$-1.2 \cdot 10^{-4} \le \varepsilon'/\varepsilon \le 16.0 \cdot 10^{-4} \tag{11.52}$$

and

$$\varepsilon'/\varepsilon = (3.6 \pm 3.4) \cdot 10^{-4} \tag{11.53}$$

for the "scanning" method and the "gaussian" method discussed in section 10.7, respectively. Using on the other hand $m_s(m_c) = (125 \pm 20) \,\text{MeV}$ one finds respectively [161]:

$$-0.5 \cdot 10^{-4} \le \varepsilon'/\varepsilon \le 25.2 \cdot 10^{-4} \tag{11.54}$$

and

$$\varepsilon'/\varepsilon = (6.1 \pm 5.2) \cdot 10^{-4}$$
 (11.55)

In [190] the choice $m_{\rm s}(m_{\rm c}) = (100\pm20)\,{\rm MeV}$ has been considered giving $0 \le \varepsilon'/\varepsilon \le 43.0\cdot10^{-4}$ and $\varepsilon'/\varepsilon = (10.4\pm8.3)\cdot10^{-4}$ respectively, but such low values of $m_{\rm s}(m_{\rm c})$ seem now rather improbable.

In table 21 we compare these results with the existing results obtained by various groups. There exists no recent phenomenological analysis from the Dortmund group based on the B_i parameters obtained in [186]. The older result $\varepsilon'/\varepsilon = (9.9 \pm 4.1) \cdot 10^{-4}$ from this group will certainly be superceded by a new analysis which hopefully will be available soon.

We observe that the result for $m_s(m_c) = 150 \pm 20 \,\mathrm{MeV}$ in (11.53) agrees rather well with the 1996 analysis of the Rome group [200]. On the other hand the range in (11.52) shows that for particular choices of the input parameters, values for ε'/ε as high as $16 \cdot 10^{-4}$ cannot be excluded. Such high values are found if simultaneously $|V_{ub}/V_{cb}| = 0.10$, $B_6^{(1/2)} = 1.2$, $B_8^{(3/2)} = 0.8$, $B_K = 0.6$, $m_s(m_c) = 130 \,\mathrm{MeV}$, $\Lambda_{\overline{\mathrm{MS}}}^{(4)} = 405 \,\mathrm{MeV}$ and low values of m_t still consistent with ε_K and the observed $B_d^0 - \bar{B}_d^0$ mixing are chosen. It is, however, evident from the comparison of (11.52) and (11.53) that such high values of ε'/ε and generally values above 10^{-3} are very improbable for $m_s(m_c) = \mathcal{O}(150 \,\mathrm{MeV})$.

We observe that our "gaussian" result for $m_{\rm s}(m_{\rm c})=(125\pm20)\,{\rm MeV}$ agrees well with the E731 value and, as stressed in [190], the decrease of $m_{\rm s}$ even below $m_{\rm s}(m_{\rm c})=100\,{\rm MeV}$ is insufficient to bring the Standard Model in agreement with the NA31 result provided $B_6=B_8=1$. However, for $B_6>B_8$, sufficiently large values of ${\rm Im}\lambda_t$ and $\Lambda_{\overline{\rm MS}}^{(4)}$, and small values of $m_{\rm s}$, the values of ε'/ε in the Standard Model can be as large as $(1-2)\cdot 10^{-3}$ and consistent with the NA31 result. In order to see this explicitly we present in table 22 the values of ε'/ε for three choices of $m_{\rm s}(m_{\rm c})$ and for selective sets of other input parameters keeping $m_{\rm t}=167\,{\rm GeV}$ fixed.

Table 21: Results for ε'/ε in units of 10^{-4} obtained by various groups. The labels (S) and (G) in the last column stand for "Scanning" and "Gaussian" respectively, as discussed in the text.

Reference	$B_6^{(1/2)}$	$B_8^{(3/2)}$	$m_{ m s}(m_{ m c}) [{ m MeV}]$	$\varepsilon'/\varepsilon[10^{-4}]$
Munich [190]	1.0 ± 0.2	1.0 ± 0.2	150 ± 20	$-1.2 \to 16.0 \text{ (S)}$
Munich [190]	1.0 ± 0.2	1.0 ± 0.2	150 ± 20	$3.6 \pm 3.4 \; (G)$
Munich [161]	1.0 ± 0.2	1.0 ± 0.2	125 ± 20	$-0.5 \to 25.2 \text{ (S)}$
Munich [161]	1.0 ± 0.2	1.0 ± 0.2	125 ± 20	$6.1 \pm 5.2 \; (G)$
Rome [200]	1.0 ± 0.2	1.0 ± 0.2	150 ± 20	$4.6 \pm 3.0 \; (G)$
Trieste [179]	1.6 ± 0.3	0.92 ± 0.02	_	$7 \rightarrow 31 \text{ (S)}$
Dubna-DESY [201]	1.0	1.0	_	$-3.0 \to 3.6 \text{ (S)}$

The Trieste group finds generally higher values of ε'/ε , with the central value around $17 \cdot 10^{-4}$ and consequently consistent with the NA31 result. On the basis of table 18 we expect the ε'/ε from the Dortmund group to be below the one from Trieste but generally higher than the results from Munich and Rome for the same value of $m_{\rm s}$.

Finally I should comment on the results of [201] where ε'/ε has been investigated in the framework of an effective chiral lagrangian approach. In this approach the values of $B_6^{(1/2)}$ and $B_8^{(3/2)}$ cannot be calculated and the authors set them to unity in order to obtain the values quoted in table 21. In spite of joined efforts with Bill Bardeen to understand this work and discussions with these authors I failed to appreciate fully this approach. These authors find ε'/ε consistent with zero.

11.8 Summary

The fate of ε'/ε in the Standard Model after the improved measurement of $m_{\rm t}$ and complete NLO calculations of short distance coefficients, depends sensitively on the values of $|V_{ub}/V_{cb}|$, $\Lambda_{\overline{\rm MS}}^{(4)}$ and in particular on $B_6^{(1/2)}$, $B_8^{(3/2)}$ and $m_{\rm s}$. The predictions for ε'/ε obtained by various groups are summarized in table 21. This table and the table 22 show very clearly that any value for ε'/ε in the range

$$0 \le \varepsilon'/\varepsilon \le 3 \cdot 10^{-3} \tag{11.56}$$

is still possible within the Standard Model at present, although most estimates lie below 10^{-3} and in the range of E731 result. Time will show which of the groups came closest to the true prediction. It appears that most calculations give values of $B_6^{(1/2)}$ rather close to unity and $B_8^{(3/2)}$ below one so that the inequality $B_6^{(1/2)} \geq B_8^{(3/2)}$ should be expected to be true. If

Table 22: Values of ε'/ε in units of 10^{-4} for specific values of various input parameters at $m_{\rm t}=167$ GeV.

$\mathrm{Im}\lambda_t[10^{-4}]$	$\Lambda_{\overline{ m MS}}^{(4)}[MeV]$	$B_6^{(1/2)}$	$B_8^{(3/2)}$	$m_{ m s}(m_{ m c}) [{ m MeV}]$	$\varepsilon'/\varepsilon[10^{-4}]$
				100	8.8
1.3	325	1.0	1.0	125	5.2
				150	3.2
				100	11.2
1.3	405	1.0	1.0	125	6.8
				150	4.2
				100	13.8
1.6	405	1.0	1.0	125	8.3
				150	5.2
				100	12.2
1.3	325	1.0	0.7	125	7.5
				150	4.8
				100	15.4
1.3	405	1.0	0.7	125	9.5
				150	6.2
				100	19.0
1.6	405	1.0	0.7	125	11.7
				150	7.6

this feature will survive more precise calculations and $m_{\rm s}(m_{\rm c})$ will be eventually found in the range 125 MeV $\leq m_{\rm s}(m_{\rm c}) \leq$ 150 MeV then ε'/ε within the Standard Model should be somewhere between $5 \cdot 10^{-4}$ and $1 \cdot 10^{-3}$. As an example let us then finally take the range (11.41): $B_6^{(1/2)} = 1.0 \pm 0.2$ and $B_8^{(3/2)} = 0.7 \pm 0.2$. Then the gaussian analysis gives [161]

$$\varepsilon'/\varepsilon = \begin{cases} (5.3 \pm 3.8) \cdot 10^{-4} , & m_{\rm s}(m_{\rm c}) = 150 \pm 20 \,\text{MeV} \\ (8.5 \pm 5.9) \cdot 10^{-4} , & m_{\rm s}(m_{\rm c}) = 125 \pm 20 \,\text{MeV}. \end{cases}$$
(11.57)

In my opinion these results give the best representation of the present status of ε'/ε in the Standard Model.

One prominent physicist once told me that a person who spent fifteen years in a given field should have enough insight into the matters to be able to make predictions even if this is impossible from a scientific point of view. In 1983 I made the first encounter with ε'/ε and if the above was true I should have by now in my head a precise prediction for ε'/ε within

the Standard Model. Clearly I do not have it, but I like to bet. Here is my bet for the ε'/ε in the Standard Model

$$\varepsilon'/\varepsilon = (7\pm 1)\cdot 10^{-4}.\tag{11.58}$$

It is rather close to the central value of the Fermilab result in (10.27). The value in (11.58) corresponds to the average of the values in (11.57) and the error is the one expected from new experiments. Whether the new data will find this value is not really important as there could be new physics invalidating my expectations.

On a more scientific level, let us hope that the future experimental and theoretical results will be sufficiently accurate to be able to see whether $\varepsilon'/\varepsilon \neq 0$, whether the Standard Model agrees with the data or whether some new physics can be discovered in this ratio. In any case the coming years should be very exciting.

12
$$B \rightarrow X_s \gamma$$

12.1 General Remarks

The rare decay $B \to X_s \gamma$ plays an important role in present day phenomenology. The effective Hamiltonian for $B \to X_s \gamma$ at scales $\mu_b = \mathcal{O}(m_b)$ is given by

$$\mathcal{H}_{\text{eff}}(b \to s\gamma) = -\frac{G_F}{\sqrt{2}} V_{ts}^* V_{tb} \left[\sum_{i=1}^6 C_i(\mu_b) Q_i + C_{7\gamma}(\mu_b) Q_{7\gamma} + C_{8G}(\mu_b) Q_{8G} \right] , \qquad (12.1)$$

where in view of $|V_{us}^*V_{ub}/V_{ts}^*V_{tb}| < 0.02$ we have neglected the term proportional to $V_{us}^*V_{ub}$. Here $Q_1....Q_6$ are the usual four-fermion operators whose explicit form is given in (8.35)–(8.37). The remaining two operators, characteristic for this decay, are the magnetic-penguins

$$Q_{7\gamma} = \frac{e}{8\pi^2} m_b \bar{s}_{\alpha} \sigma^{\mu\nu} (1 + \gamma_5) b_{\alpha} F_{\mu\nu}, \qquad Q_{8G} = \frac{g}{8\pi^2} m_b \bar{s}_{\alpha} \sigma^{\mu\nu} (1 + \gamma_5) T^a_{\alpha\beta} b_{\beta} G^a_{\mu\nu}$$
(12.2)

originating in the diagrams of fig. 34. In order to derive the contribution of $Q_{7\gamma}$ to the Hamiltonian in (12.1), in the absence of QCD corrections, one multiplies the vertex in (3.7) by "i" and makes the replacement

$$2i\sigma_{\mu\nu}q^{\nu} \to -\sigma^{\mu\nu}F_{\mu\nu}. \tag{12.3}$$

Analogous procedure gives the contribution of Q_{8G} .

It is the magnetic γ -penguin which plays the crucial role in this decay. However, the role of the dominant current-current operator Q_2 should not be underestimated. Indeed the short distance QCD effects involving in particular the mixing between Q_2 and $Q_{7\gamma}$ are very important in this decay. They are known [202, 203] to enhance $C_{7\gamma}(\mu_b)$ substantially, so that the resulting branching ratio $Br(B \to X_s \gamma)$ turns out to be by a factor of 3 higher than it

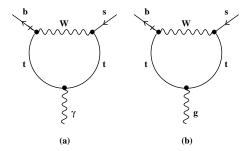


Figure 34: Magnetic Photon (a) and Gluon (b) Penguins.

would be without QCD effects. Since the first analyses in [202, 203] a lot of progress has been made in calculating these important QCD effects beginning with the work in [204, 205]. We will briefly summarize this progress.

A peculiar feature of the renormalization group analysis in $B \to X_s \gamma$ is that the mixing under infinite renormalization between the set $(Q_1...Q_6)$ and the operators $(Q_{7\gamma}, Q_{8G})$ vanishes at the one-loop level. Consequently in order to calculate the coefficients $C_{7\gamma}(\mu_b)$ and $C_{8G}(\mu_b)$ in the leading logarithmic approximation, two-loop calculations of $\mathcal{O}(eg_s^2)$ and $\mathcal{O}(g_s^3)$ are necessary. The corresponding NLO analysis requires the evaluation of the mixing in question at the three-loop level. This peculiar feature caused that the first fully correct calculation of the leading anomalous dimension matrix relevant for this decay has been obtained only in 1993 [206, 207]. It has been confirmed subsequently in [208, 209, 101].

As of 1998 also the NLO corrections to $B \to X_s \gamma$ have been completed. It was a joint effort of many groups. Let us summarize this progress:

- The $O(\alpha_s)$ corrections to $C_{7\gamma}(\mu_W)$ and $C_{8G}(\mu_W)$ have been first calculated in [107] and recently confirmed by several groups [66, 67, 111].
- The two-loop mixing involving the operators $Q_1....Q_6$ and the two-loop mixing in the sector $(Q_{7\gamma}, Q_{8G})$ has been calculated in [53, 48, 68, 73, 74, 75] and [94], respectively. Finally after a heroic effort the three loop mixing between the set $(Q_1...Q_6)$ and the operators $(Q_{7\gamma}, Q_{8G})$ has been completed at the end of 1996 [110]. As a byproduct the authors of [110] confirmed the existing two-loop anomalous dimension matrix in the $Q_1...Q_6$ sector.
- One-loop matrix elements $\langle s\gamma \text{gluon}|Q_i|b\rangle$ have been calculated in [106, 108] and the very difficult two-loop corrections to $\langle s\gamma|Q_i|b\rangle$ have been presented in [109].

We will now discuss all these achievements in explicit terms. In order to appreciate the importance of NLO calculations for this decay it is instructive to discuss first the leading

logarithmic approximation.

12.2 The Decay $B \to X_s \gamma$ in the Leading Log Approximation

12.2.1 Anomalous Dimension Matrix

It is instructive to discuss first the mixing between the sets Q_1, \ldots, Q_6 and $Q_{7\gamma}, Q_{8G}$ in $\hat{\gamma}_s^{(0)}$. To this end I use the work done in colaboration with Misiak, Münz and Pokorski [211]. The point is that this mixing resulting from two-loop diagrams is generally regularization scheme dependent. This is certainly disturbing because the matrix $\hat{\gamma}_s^{(0)}$, being the first term in the expansion for $\hat{\gamma}_s$, is usually scheme independent. As we will show below, there is a simple way to circumvent this difficulty [211].

As noticed in [206, 207] the regularization scheme dependence of $\hat{\gamma}_s^{(0)}$ in the case of $b \to s\gamma$ and $b \to sg$ is signaled in the finite parts of the one-loop matrix elements of Q_1, \ldots, Q_6 for on-shell photons or gluons. They vanish in any 4-dimensional regularization scheme and in the HV scheme but some of them are non-zero in the NDR scheme. One has

$$\langle Q_i \rangle_{\text{one-loop}}^{\gamma} = y_i \langle Q_{7\gamma} \rangle_{\text{tree}}, \qquad i = 1, \dots, 6$$
 (12.4)

and

$$\langle Q_i \rangle_{\text{one-loop}}^G = z_i \langle Q_{8G} \rangle_{\text{tree}}, \qquad i = 1, \dots, 6.$$
 (12.5)

In the HV scheme all the y_i 's and z_i 's vanish, while in the NDR scheme $\vec{y} = (0, 0, 0, 0, -\frac{1}{3}, -1)$ and $\vec{z} = (0, 0, 0, 0, 1, 0)$. This regularization scheme dependence is canceled by a corresponding regularization scheme dependence in $\hat{\gamma}_s^{(0)}$ as first demonstrated in [206, 207]. It should be stressed that the numbers y_i and z_i come from divergent, i.e. purely short-distance parts of the one-loop integrals. So no reference to the spectator-model or to any other model for the matrix elements is necessary here.

In view of all this it is convenient in the leading order to introduce the so-called "effective coefficients" [211] for the operators $Q_{7\gamma}$ and Q_{8G} which are regularization scheme independent. They are given as follows:

$$C_{7\gamma}^{(0)eff}(\mu_b) = C_{7\gamma}^{(0)}(\mu_b) + \sum_{i=1}^{6} y_i C_i^{(0)}(\mu_b)$$
(12.6)

and

$$C_{8G}^{(0)eff}(\mu_b) = C_{8G}^{(0)}(\mu_b) + \sum_{i=1}^{6} z_i C_i^{(0)}(\mu_b).$$
 (12.7)

One can then introduce a scheme-independent vector

$$\vec{C}^{(0)eff}(\mu_b) = \left(C_1^{(0)}(\mu_b), \dots, C_6^{(0)}(\mu_b), C_{7\gamma}^{(0)eff}(\mu_b), C_{8G}^{(0)eff}(\mu_b)\right). \tag{12.8}$$

From the RGE for $\vec{C}^{(0)}(\mu)$ it is straightforward to derive the RGE for $\vec{C}^{(0)eff}(\mu)$. It has the form

$$\mu \frac{d}{d\mu} C_i^{(0)eff}(\mu) = \frac{\alpha_s}{4\pi} \gamma_{ji}^{(0)eff} C_j^{(0)eff}(\mu)$$
 (12.9)

where

$$\gamma_{ji}^{(0)eff} = \begin{cases}
\gamma_{j7}^{(0)} + \sum_{k=1}^{6} y_k \gamma_{jk}^{(0)} - y_j \gamma_{77}^{(0)} - z_j \gamma_{87}^{(0)} & \text{i=7, j=1,...,6} \\
\gamma_{j8}^{(0)} + \sum_{k=1}^{6} z_k \gamma_{jk}^{(0)} - z_j \gamma_{88}^{(0)} & \text{i=8, j=1,...,6} \\
\gamma_{ji}^{(0)} & \text{otherwise.}
\end{cases} (12.10)$$

The matrix $\hat{\gamma}^{(0)eff}$ is a scheme-independent quantity. It equals the matrix which one would directly obtain from two-loop diagrams in the HV scheme. In order to simplify the notation we will omit the label "eff" in the expressions for the elements of this effective one loop anomalous dimension matrix given below and keep it only in the Wilson coefficients of the operators $Q_{7\gamma}$ and Q_{8G} .

We are now ready to give the leading anomalous dimension matrix relevant for the calculation of the $B \to X_s \gamma$ rate in the LO approximation. The 6×6 submatrix of $\hat{\gamma}^{(0)}$ involving the operators Q_1, \ldots, Q_6 is given in (8.51). Here we only give the remaining non-vanishing entries of $\hat{\gamma}^{(0)}$ [206, 207].

The elements $\gamma_{i7}^{(0)}$ with i = 1, ..., 6 are:

$$\gamma_{17}^{(0)} = 0, \qquad \gamma_{27}^{(0)} = \frac{104}{27} C_F$$
(12.11)

$$\gamma_{37}^{(0)} = -\frac{116}{27}C_F$$
 $\gamma_{47}^{(0)} = \left(\frac{104}{27}u - \frac{58}{27}d\right)C_F$
(12.12)

$$\gamma_{57}^{(0)} = \frac{8}{3}C_F$$
 $\gamma_{67}^{(0)} = \left(\frac{50}{27}d - \frac{112}{27}u\right)C_F$
(12.13)

The elements $\gamma_{i8}^{(0)}$ with $i = 1, \dots, 6$ are:

$$\gamma_{18}^{(0)} = 3, \qquad \gamma_{28}^{(0)} = \frac{11}{9}N - \frac{29}{9}\frac{1}{N}$$
(12.14)

$$\gamma_{18}^{(0)} = 3, \qquad \gamma_{28}^{(0)} = \frac{11}{9}N - \frac{29}{9}\frac{1}{N}$$

$$\gamma_{38}^{(0)} = \frac{22}{9}N - \frac{58}{9}\frac{1}{N} + 3f \qquad \gamma_{48}^{(0)} = 6 + \left(\frac{11}{9}N - \frac{29}{9}\frac{1}{N}\right)f \qquad (12.14)$$

$$\gamma_{58}^{(0)} = -2N + \frac{4}{N} - 3f$$

$$\gamma_{68}^{(0)} = -4 - \left(\frac{16}{9}N - \frac{25}{9}\frac{1}{N}\right)f$$
(12.16)

Finally the 2×2 one-loop anomalous dimension matrix in the sector $Q_{7\gamma}, Q_{8G}$ is given by [204]

$$\gamma_{77}^{(0)} = 8C_F \qquad \gamma_{78}^{(0)} = 0
\gamma_{87}^{(0)} = -\frac{8}{3}C_F \qquad \gamma_{88}^{(0)} = 16C_F - 4N$$
(12.17)

12.2.2 Renormalization Group Evolution

The coefficients $C_i(\mu_b)$ in (12.1) can be calculated by using

$$\vec{C}(\mu_b) = \hat{U}_5(\mu_b, \mu_W)\vec{C}(\mu_W) \tag{12.18}$$

Here $\hat{U}_5(\mu_b, \mu_W)$ is the 8×8 evolution matrix which is given in general terms in (6.26) with $\hat{\gamma}$ being this time an 8×8 anomalous dimension matrix. In the leading order $\hat{U}_5(\mu_b, \mu_W)$ is to be replaced by $\hat{U}_5^{(0)}(\mu_b, \mu_W)$ and the initial conditions by $\vec{C}^{(0)}(\mu_W)$ with [204]

$$C_2^{(0)}(\mu_W) = 1 (12.19)$$

$$C_{7\gamma}^{(0)}(\mu_W) = \frac{3x_t^3 - 2x_t^2}{4(x_t - 1)^4} \ln x_t + \frac{-8x_t^3 - 5x_t^2 + 7x_t}{24(x_t - 1)^3} \equiv -\frac{1}{2}D_0'(x_t)$$
(12.20)

$$C_{8G}^{(0)}(\mu_W) = \frac{-3x_t^2}{4(x_t - 1)^4} \ln x_t + \frac{-x_t^3 + 5x_t^2 + 2x_t}{8(x_t - 1)^3} \equiv -\frac{1}{2} E_0'(x_t)$$
 (12.21)

In LO all remaining coefficients are set to zero at $\mu = \mu_W$.

Using the techniques developed in section 5, the leading order results for the Wilson coefficients of all operators entering the effective Hamiltonian in (12.1) can be written in an analytic form. They are [211]

$$C_j^{(0)}(\mu_b) = \sum_{i=1}^8 k_{ji} \eta^{a_i} \qquad (j=1,\dots,6)$$
 (12.22)

$$C_{7\gamma}^{(0)eff}(\mu_b) = \eta^{\frac{16}{23}} C_{7\gamma}^{(0)}(\mu_W) + \frac{8}{3} \left(\eta^{\frac{14}{23}} - \eta^{\frac{16}{23}} \right) C_{8G}^{(0)}(\mu_W) + C_2^{(0)}(\mu_W) \sum_{i=1}^{8} h_i \eta^{a_i}, (12.23)$$

$$C_{8G}^{(0)eff}(\mu_b) = \eta^{\frac{14}{23}} C_{8G}^{(0)}(\mu_W) + C_2^{(0)}(\mu_W) \sum_{i=1}^{8} \bar{h}_i \eta^{a_i}, \qquad (12.24)$$

with

$$\eta = \frac{\alpha_s(\mu_W)}{\alpha_s(\mu_b)}, \tag{12.25}$$

and $C_{7\gamma}^{(0)}(\mu_W)$ and $C_{8G}^{(0)}(\mu_W)$ given in (12.20) and (12.21), respectively. The numbers a_i and k_{ji} have been already given in section 8.4. For convenience we give again the values of a_i together with h_i and \bar{h}_i in table 23.

Let us perform a quick numerical analysis of (12.23) and (12.24). Using the leading μ_b -dependence of α_s :

$$\alpha_s(\mu_b) = \frac{\alpha_s(M_Z)}{1 - \beta_0 \frac{\alpha_s(M_Z)}{2\pi} \ln(M_Z/\mu_b)}$$
(12.26)

one finds the results in table 24.

Two features of these results should be emphasised:

Table 23: Magic Numbers.

i	1	2	3	4	5	6	7	8
a_i	$\frac{14}{23}$	$\frac{16}{23}$	$\frac{6}{23}$	$-\frac{12}{23}$	0.4086	-0.4230	-0.8994	0.1456
h_i	2.2996	-1.0880	$-\frac{3}{7}$	$-\frac{1}{14}$	-0.6494	-0.0380	-0.0185	-0.0057
\bar{h}_i	0.8623	0	0	0	-0.9135	0.0873	-0.0571	0.0209

Table 24: Wilson coefficients $C_{7\gamma}^{(0) \text{eff}}$ and $C_{8G}^{(0) \text{eff}}$ for $m_{\rm t}=170\,{\rm GeV}$ and various values of $\alpha_s^{(5)}(M_{\rm Z})$ and μ .

	$\alpha_s^{(5)}(M_{\rm Z}) = 0.113$		$\alpha_s^{(5)}(M_{\rm Z})$	(z) = 0.118	$\alpha_s^{(5)}(M_{\rm Z}) = 0.123$		
$\mu [{\rm GeV}]$	$C_{7\gamma}^{(0)\mathrm{eff}}$	$C_{8G}^{(0) m eff}$	$C_{7\gamma}^{(0) ext{eff}}$	$C_{8G}^{(0) m eff}$	$C_{7\gamma}^{(0)\mathrm{eff}}$	$C_{8G}^{(0) m eff}$	
2.5	-0.328	-0.155	-0.336	-0.158	-0.344	-0.161	
5.0	-0.295	-0.142	-0.300	-0.144	-0.306	-0.146	
7.5	-0.277	-0.134	-0.282	-0.136	-0.286	-0.138	
10.0	-0.265	-0.130	-0.269	-0.131	-0.273	-0.133	

• The strong enhancement of the coefficient $C_{7\gamma}^{(0) \text{eff}}$ by short distance QCD effects which we illustrate by the relative numerical importance of the three terms in expression (12.23). For instance, for $m_{\rm t} = 170 \, {\rm GeV}$, $\mu_b = 5 \, {\rm GeV}$ and $\alpha_s^{(5)}(M_{\rm Z}) = 0.118$ one obtains

$$C_{7\gamma}^{(0)\text{eff}}(\mu_b) = 0.695 C_{7\gamma}^{(0)}(\mu_W) + 0.085 C_{8G}^{(0)}(\mu_W) - 0.158 C_2^{(0)}(\mu_W)$$

= $0.695 (-0.193) + 0.085 (-0.096) - 0.158 = -0.300.$ (12.27)

In the absence of QCD we would have $C_{7\gamma}^{(0)\text{eff}}(\mu_b) = C_{7\gamma}^{(0)}(\mu_W)$ (in that case one has $\eta = 1$). Therefore, the dominant term in the above expression (the one proportional to $C_2^{(0)}(\mu_W)$) is the additive QCD correction that causes the enormous QCD enhancement of the $B \to X_s \gamma$ rate [202, 203]. It originates solely from the two-loop diagrams. On the other hand, the multiplicative QCD correction (the factor 0.695 above) tends to suppress the rate, but fails in the competition with the additive contributions.

In the case of $C_{8G}^{(0)\text{eff}}$ a similar enhancement is observed

$$C_{8G}^{(0)\text{eff}}(\mu_b) = 0.727 C_{8G}^{(0)}(\mu_W) - 0.074 C_2^{(0)}(\mu_W)$$

= 0.727 (-0.096) - 0.074 = -0.144. (12.28)

• A strong μ_b -dependence of both coefficients as first stressed by Ali and Greub [210] and confirmed in [211]. Since $B \to X_s \gamma$ is dominated by QCD effects, it is not surprising

that this scale-uncertainty in the leading order is particularly large. We will investigate this scale uncertainty in a moment.

12.2.3 Scale Uncertainties at LO

In calculating $Br(B \to X_s \gamma)$ it is customary to use the spectator model in which the inclusive decay $B \to X_s \gamma$ is approximated by the partonic decay $b \to s \gamma$. That is one uses the following approximate equality:

$$\frac{\Gamma(B \to X_s \gamma)}{\Gamma(B \to X_c e \bar{\nu}_e)} \simeq \frac{\Gamma(b \to s \gamma)}{\Gamma(b \to c e \bar{\nu}_e)} \equiv R_{\text{quark}}, \tag{12.29}$$

where the quantities on the r.h.s are calculated in the spectator model corrected for short-distance QCD effects. The normalization to the semileptonic rate is usually introduced in order to reduce the uncertainties due to the CKM matrix elements and factors of m_b^5 in the r.h.s. of (12.29). Additional support for the approximation given above comes from the heavy quark expansions. Indeed the spectator model has been shown to correspond to the leading order approximation of an expansion in $1/m_b$. The first corrections appear at the $\mathcal{O}(1/m_b^2)$ level and will be discussed at the end of this section.

The leading logarithmic calculations [204, 207, 208, 101, 210, 211] can be summarized in a compact form as follows:

$$R_{\text{quark}} = \frac{Br(B \to X_s \gamma)}{Br(B \to X_c e \bar{\nu}_e)} = \frac{|V_{ts}^* V_{tb}|^2}{|V_{cb}|^2} \frac{6\alpha}{\pi f(z)} |C_7^{(0)\text{eff}}(\mu_b)|^2,$$
(12.30)

where

$$f(z) = 1 - 8z + 8z^3 - z^4 - 12z^2 \ln z$$
 with $z = \frac{m_{c,pole}^2}{m_{b,pole}^2}$ (12.31)

is the phase space factor in $Br(B \to X_c e \bar{\nu}_e)$ and $\alpha = e^2/4\pi$. In order to find (12.30) only the tree level matrix element $\langle s\gamma|Q_{7\gamma}|B\rangle$ has to be computed.

There are three scale uncertainties present in (12.30):

- The low energy scale $\mu_b = \mathcal{O}(m_b)$ at which the Wilson Coefficient $C_7^{(0)\text{eff}}(\mu_b)$ is evaluated.
- The high energy scale $\mu_W = \mathcal{O}(M_W)$ at which the full theory is matched with the effective five-quark theory. In LO this scale enters only η . $C_7^{(0)}(\mu_W)$ and $C_8^{(0)}(\mu_W)$ serve in LO as initial conditions to the renormalization group evolution from μ_W down to μ_b . As seen explicitly in (12.20) and (12.21) they do not depend on μ_W .
- The scale $\mu_t = \mathcal{O}(m_t)$ at which the running top quark mass is defined. In LO it enters only x_t :

$$x_t = \frac{\overline{m}_t^2(\mu_t)}{M_W^2}.$$
 (12.32)

As we stressed in connection with $B^0 - \bar{B}^0$ mixing in section 8.3, μ_W and μ_t do not have to be equal. Initially when the top quark and the W-boson are integrated out, it is convenient in the process of matching to keep $\mu_t = \mu_W$. Yet one has always the freedom to redefine the top quark mass and to work with $\overline{m}_t(\mu_t)$ where $\mu_t \neq \mu_W$.

It is evident from the formulae above that in LO the variations of μ_b , μ_W and μ_t remain uncompensated which results in potential theoretical uncertainties in the predicted branching ratio. In the context of phenomenological analyses of $B \to X_s \gamma$, the uncertainty due to μ_b has been discussed [210, 211, 110, 109, 77]. The uncertainties due to μ_W and μ_t have been analyzed first in [77] and recently in [212]. I will follow here my own work with Axel Kwiatkowski and Nicolas Pott [77].

It is customary to estimate the uncertainties due to μ_b by varying it in the range $m_b/2 \le \mu_b \le 2m_b$. Similarly one can vary μ_W and μ_t in the ranges $M_W/2 \le \mu_W \le 2M_W$ and $m_t/2 \le \mu_t \le 2m_t$ respectively. Specifically in our numerical analysis we will consider the ranges

$$2.5 \text{ GeV} \le \mu_b \le 10 \text{ GeV}$$
 (12.33)

and

40 GeV
$$\leq \mu_W \leq 160$$
 GeV 80 GeV $\leq \mu_t \leq 320$ GeV (12.34)

In the LO analysis we use the leading order formula for $\alpha_s(\mu_b)$ in (12.26) with $\alpha_s(M_Z) = 0.118$ and

$$\overline{m}_{t}(\mu_{t}) = \overline{m}_{t}(m_{t}) \left[\frac{\alpha_{s}(\mu_{t})}{\alpha_{s}(m_{t})} \right]^{\frac{4}{\beta_{0}}}.$$
(12.35)

Here $\beta_0 = 23/3$. We set $\overline{m}_t(m_t) = 168$ GeV and $m_t \equiv m_{t,pole} = 176$ GeV.

Varying μ_b , μ_W and μ_t in the ranges (12.33) and (12.34) we find the following uncertainties in the branching ratio [77]:

$$\Delta Br(B \to X_s \gamma) = \begin{cases} \pm 22\% & (\mu_b) \\ \pm 13\% & (\mu_W) \\ \pm 3\% & (\mu_t) \end{cases}$$
 (12.36)

The fact that the μ_W -uncertainty is smaller than the μ_b uncertainty is entirely due to $\alpha_s(\mu_W) < \alpha_s(\mu_b)$. Still this uncertainty is rather disturbing as it introduces an error of approximately $\pm 0.40 \cdot 10^{-4}$ in the branching ratio. The smallness of the μ_t -uncertainty is related to the weak x_t dependence of $C_7^{(0)}(\mu_W)$ and $C_8^{(0)}(\mu_W)$ which in the range of interest can be well approximated by

$$C_7^{(0)}(\mu_W) = -0.122 \ x_t^{0.30} \qquad C_8^{(0)}(\mu_W) = -0.072 \ x_t^{0.19}.$$
 (12.37)

Thus even if $161 \,\text{GeV} \leq \overline{m}_t(\mu_t) \leq 178 \,\text{GeV}$ for μ_t in (12.34), the μ_t uncertainty in $Br(B \to X_s \gamma)$ is small. This should be contrasted with $B_s \to \mu \bar{\mu}$, $K_L \to \pi^0 \nu \bar{\nu}$ and $B_{d,s}^0 - \bar{B}_{d,s}^0$ mixings, where μ_t uncertainties in LO have been found [98, 90] to be $\pm 13\%$, $\pm 10\%$ and $\pm 9\%$ respectively.

A critical analysis of theoretical and experimental uncertainties present in the prediction for $\text{Br}(B \to X_s \gamma)$ based on the formula (12.30) has been made in [211] with the result that the error in the Standard Model prediction in the LO approximation is dominated by the scale ambiguities. The final result of the LO analysis in [211] which omitted the μ_t and μ_W uncertainties was

$$Br(B \to X_s \gamma)_{LO} = (2.8 \pm 0.8) \times 10^{-4}$$
 (12.38)

Similar result has been found in [210].

These finding made it clear already in 1993 that a complete NLO analysis of $B \to X_s \gamma$ was very desirable. Such a complete next-to-leading calculation of $B \to X_s \gamma$ was described in [211] in general terms. As demonstrated formally there, the cancellation of the dominant μ_b -dependence in the leading order can then be achieved. While this formal NLO analysis was very encouraging with respect to the reduction of the μ_b -dependence, it could obviously not provide the actual size of $\text{Br}(B \to X_s \gamma)$ after the inclusion of NLO corrections. Fortunately four years later such a complete NLO analysis exists and the impact of NLO corrections on $\text{Br}(B \to X_s \gamma)$ can be analysed in explicit terms. This is precisely what we will do now.

12.3 $B \rightarrow X_s \gamma$ Beyond Leading Logarithms

12.3.1 Master Formulae

The formula (12.30) modifies after the inclusion of NLO corrections as follows [110]:

$$R_{\text{quark}} = \frac{|V_{ts}^* V_{tb}|^2}{|V_{cb}|^2} \frac{6\alpha}{\pi f(z)} F(|D|^2 + A) , \qquad (12.39)$$

where

$$F = \frac{1}{\kappa(z)} \left(\frac{\overline{m}_b(\mu = m_b)}{m_{b,\text{pole}}} \right)^2 = \frac{1}{\kappa(z)} \left(1 - \frac{8}{3} \frac{\alpha_s(m_b)}{\pi} \right), \tag{12.40}$$

$$D = C_{7\gamma}^{(0)\text{eff}}(\mu_b) + \frac{\alpha_s(\mu_b)}{4\pi} \left\{ C_{7\gamma}^{(1)\text{eff}}(\mu_b) + \sum_{i=1}^8 C_i^{(0)\text{eff}}(\mu_b) \left[r_i + \gamma_{i7}^{(0)\text{eff}} \ln \frac{m_b}{\mu_b} \right] \right\}$$
(12.41)

and A is discussed below.

Let us explain the origin of various new contributions:

• First $\kappa(z)$ is the QCD correction to the semileptonic decay [213]. To a good approximation it is given by [214]

$$\kappa(z) = 1 - \frac{2\alpha_s(\bar{\mu}_b)}{3\pi} \left[(\pi^2 - \frac{31}{4})(1-z)^2 + \frac{3}{2} \right]. \tag{12.42}$$

An exact analytic formula for $\kappa(z)$ can be found in [215]. Here $\bar{\mu}_b = \mathcal{O}(m_b)$ is a scale in the calculation of QCD corrections to the semi-leptonic rate which is generally different from the one used in the $b \to s\gamma$ transition. In this respect we differ from Greub et al. [109] who set $\bar{\mu}_b = \mu_b$.

- The second factor in (12.40) originates as follows. The $B \to X_s \gamma$ rate is proportional to $m_{b,\text{pole}}^3$ present in the two body phase space and to $\overline{m}_b(\mu = m_b)^2$ present in $< s\gamma |Q_{7\gamma}|B>^2$. On the other hand the semileptonic rate is is proportional to $m_{b,\text{pole}}^5$ present in the three body phase space. Thus the m_b^5 factors present in both rates differ by a $\mathcal{O}(\alpha_s)$ correction which has been consistently omitted in the leading logarithmic approximation but has to be included now.
- For similar reason the variable z entering f(z) and $\kappa(z)$ can be more precisely specified at the NLO level to be [109, 110]:

$$z = \frac{m_{c,\text{pole}}}{m_{b,\text{pole}}} = 0.29 \pm 0.02 \tag{12.43}$$

which is obtained from $m_{b,\text{pole}} = 4.8 \pm 0.15 \text{ GeV}$ and $m_{b,\text{pole}} - m_{c,\text{pole}} = 3.40 \text{ GeV}$. This gives

$$\kappa(z) = 0.879 \pm 0.002 \approx 0.88$$
, $f(z) = 0.54 \pm 0.04$. (12.44)

• The amplitude D in (12.41) includes two types of new contributions. The first α_s correction originates in the NLO correction to the Wilson coefficients of $Q_{7\gamma}$:

$$C_{7\gamma}^{\text{eff}}(\mu_b) = C_{7\gamma}^{(0)\text{eff}}(\mu_b) + \frac{\alpha_s(\mu_b)}{4\pi} C_{7\gamma}^{(1)\text{eff}}(\mu_b).$$
 (12.45)

It is this correction which requires the calculation of the three-loop anomalous dimensions [110]. An explicit formula for $C_{7\gamma}^{(1)\text{eff}}(\mu_b)$ has been given for the first time in [110]. We will give a generalization of this formula in a moment.

The two remaining corrections in (12.41) come from one-loop matrix elements $\langle s\gamma|Q_{7\gamma}|B\rangle$ and $\langle s\gamma|Q_{8G}|B\rangle$ and from two-loop matrix elements $\langle s\gamma|Q_i|B\rangle$ of the remaining operators. These two-loop matrix elements have been calculated in [109]. The coefficients of the logarithm are the relevant elements in the leading anomalous dimension matrix. The explicit logarithmic μ_b dependence in the last term in D will play an important role few pages below.

Now $C_{7\gamma}^{(1)\text{eff}}(\mu_b)$ is renormalization scheme dependent. This scheme dependence is cancelled by the one present in the constant terms r_i . Actually ref. [109] does not provide the matrix elements of the QCD-penguin operators and consequently r_i (i = 3 - 6) are unknown. However, the Wilson coefficients of QCD-penguin operators are very small and this omission is most probably immaterial.

• The term A in (12.39) originates from the bremsstrahlung corrections and the necessary virtual corrections needed for the cancellation of the infrared divergences. These have been calculated in [106, 108] and are also considered in [110, 109] in the context of the full analysis. Since the virtual corrections are also present in the terms r_i in D, care must be taken in order to avoid double counting. This is discussed in detail in [110] where an explicit formula for A can be found. It is the equation (32) of [110].

Actually A depends on an explicit lower cut on the photon energy

$$E_{\gamma} > (1 - \delta)E_{\gamma}^{\text{max}} \equiv (1 - \delta)\frac{m_b}{2}.$$
 (12.46)

Moreover A is divergent in the limit $\delta \to 1$. In order to cancel this divergence one would have to consider the sum of $B \to X_s \gamma$ and $b \to X_s$ decay rates. However, the divergence at $\delta \to 1$ is very slow. In order to allow an easy comparison with previous experimental and theoretical publications the authors in [110] choose $\delta = 0.99$. Further details on the δ -dependence can be found in this paper.

• Finally the values of $\alpha_s(\mu_b)$ in all the above formulae are calculated with the use of the NLO expression for the strong coupling constant:

$$\alpha_s(\mu) = \frac{\alpha_s(M_Z)}{v(\mu)} \left[1 - \frac{\beta_1}{\beta_0} \frac{\alpha_s(M_Z)}{4\pi} \frac{\ln v(\mu)}{v(\mu)} \right], \tag{12.47}$$

where

$$v(\mu) = 1 - \beta_0 \frac{\alpha_s(M_Z)}{2\pi} \ln\left(\frac{M_Z}{\mu}\right), \qquad (12.48)$$

$$\beta_0 = \frac{23}{3}$$
 and $\beta_1 = \frac{116}{3}$.

Generalizing the formula (21) of [110] to include μ_t and μ_W dependences one finds [77]

$$C_{7}^{(1)eff}(\mu_{b}) = \eta^{\frac{39}{23}} C_{7}^{(1)eff}(\mu_{W}) + \frac{8}{3} \left(\eta^{\frac{37}{23}} - \eta^{\frac{39}{23}} \right) C_{8}^{(1)eff}(\mu_{W})$$

$$+ \left(\frac{297664}{14283} \eta^{\frac{16}{23}} - \frac{7164416}{357075} \eta^{\frac{14}{23}} + \frac{256868}{14283} \eta^{\frac{37}{23}} - \frac{6698884}{357075} \eta^{\frac{39}{23}} \right) C_{8}^{(0)}(\mu_{W})$$

$$+ \frac{37208}{4761} \left(\eta^{\frac{39}{23}} - \eta^{\frac{16}{23}} \right) C_{7}^{(0)}(\mu_{W}) + \sum_{i=1}^{8} (e_{i} \eta E_{0}(x_{t}) + f_{i} + g_{i} \eta) \eta^{a_{i}}$$

$$+ \Delta C_{7}^{(1)eff}(\mu_{b}),$$

$$(12.49)$$

where in the \overline{MS} scheme

$$C_7^{(1)eff}(\mu_W) = C_7^{(1)eff}(M_W) + 8x_t \frac{\partial C_7^{(0)}(\mu_W)}{\partial x_t} \ln \frac{\mu_t^2}{M_W^2} + \left(\frac{16}{3}C_7^{(0)}(\mu_W) - \frac{16}{9}C_8^{(0)}(\mu_W) + \frac{\gamma_{27}^{(0)\text{eff}}}{2}\right) \ln \frac{\mu_W^2}{M_W^2}$$
(12.50)

$$C_8^{(1)eff}(\mu_W) = C_8^{(1)eff}(M_W) + 8x_t \frac{\partial C_8^{(0)}(\mu_W)}{\partial x_t} \ln \frac{\mu_t^2}{M_W^2} + \left(\frac{14}{3}C_8^{(0)}(\mu_W) + \frac{\gamma_{28}^{(0)eff}}{2}\right) \ln \frac{\mu_W^2}{M_W^2}$$
(12.51)

$$\Delta C_7^{(1)eff}(\mu_b) = \sum_{i=1}^8 \left(\frac{2}{3}e_i + 6l_i\right) \eta^{a_i+1} \ln \frac{\mu_W^2}{M_W^2}$$
 (12.52)

Here $(x = x_t)$

$$C_7^{(1)eff}(M_W) = \frac{-16x^4 - 122x^3 + 80x^2 - 8x}{9(x-1)^4} \operatorname{Li}_2\left(1 - \frac{1}{x}\right) + \frac{6x^4 + 46x^3 - 28x^2}{3(x-1)^5} \ln^2 x + \frac{-102x^5 - 588x^4 - 2262x^3 + 3244x^2 - 1364x + 208}{81(x-1)^5} \ln x + \frac{1646x^4 + 12205x^3 - 10740x^2 + 2509x - 436}{486(x-1)^4}$$

$$C_8^{(1)eff}(M_W) = \frac{-4x^4 + 40x^3 + 41x^2 + x}{6(x-1)^4} \operatorname{Li}_2\left(1 - \frac{1}{x}\right) + \frac{-17x^3 - 31x^2}{2(x-1)^5} \ln^2 x + \frac{-210x^5 + 1086x^4 + 4893x^3 + 2857x^2 - 1994x + 280}{216(x-1)^5} \ln x + \frac{737x^4 - 14102x^3 - 28209x^2 + 610x - 508}{1296(x-1)^4}$$

$$(12.54)$$

and

$$E_0(x) = \frac{x(18 - 11x - x^2)}{12(1 - x)^3} + \frac{x^2(15 - 16x + 4x^2)}{6(1 - x)^4} \ln x - \frac{2}{3} \ln x.$$
 (12.55)

The formulae for $C_{7,8}^{(1)eff}(M_W)$ given above and presented in [110] are obtained from the results in [107, 66, 67, 111] by using the general formulae for the effective coefficient functions in (12.6) and (12.7). For $\mu_W = \mu_t = M_W$ the formulae above reduce to the ones given in [110]. We have put back the superscript "eff" in (12.50) and (12.51) to emphasize that the effective anomalous dimensions should be used here.

Table 25: Magic Numbers.

i	1	2	3	4	5	6	7	8
a_i	$\frac{14}{23}$	$\frac{16}{23}$	$\frac{6}{23}$	$-\frac{12}{23}$	0.4086	-0.4230	-0.8994	0.1456
e_i	$\frac{4661194}{816831}$	$-\frac{8516}{2217}$	0	0	-1.9043	-0.1008	0.1216	0.0183
f_i	-17.3023	8.5027	4.5508	0.7519	2.0040	0.7476	-0.5385	0.0914
g_i	14.8088	-10.8090	-0.8740	0.4218	-2.9347	0.3971	0.1600	0.0225
l_i	0.5784	-0.3921	-0.1429	0.0476	-0.1275	0.0317	0.0078	-0.0031

The numbers e_i - g_i and l_i are given in table 25. These numbers as well as the numerical coefficients in (12.49) can be confirmed easily by using the anomalous dimension matrices in [110] and the techniques developed in section 5.

For completeness we give here some information on the relevant NLO anomalous dimension matrix $\gamma_s^{(1)}$. The 6 × 6 two-loop submatrix of $\gamma_s^{(1)}$ involving the operators Q_1, \ldots, Q_6 is given in (8.52). The two-loop generalization of (12.17) has been calculated in [94]. It is given for both NDR and HV schemes as follows

$$\gamma_{77}^{(1)} = C_F \left(\frac{548}{9} N - 16C_F - \frac{56}{9} f \right)
\gamma_{78}^{(1)} = 0$$

$$\gamma_{87}^{(1)} = C_F \left(-\frac{404}{27} N + \frac{32}{3} C_F + \frac{56}{27} f \right)
\gamma_{88}^{(1)} = -\frac{458}{9} - \frac{12}{N^2} + \frac{214}{9} N^2 + \frac{56}{9} \frac{f}{N} - \frac{13}{9} f N$$
(12.56)

The generalization of (12.11)–(12.16) to next-to-leading order requires three loop calculations. The result can be found in [110].

The constants r_i resulting from the calculations of NLO corrections to decay matrix elements [109] are collected in [110]. It should be stressed that the basis of the operators with i=1-6 used in [110] differs from the standard basis used in the literature [17, 109] and here. The basis used in [110] has been chosen in order to avoid γ_5 problems in the three-loop calculations performed in the NDR scheme. This has to be remembered when using formulae of this paper. In particular the constants r_i calculated in [109] have to be transformed to the basis of [110]. As pointed out this year in [111] and in particular by Kagan and Neubert [216] this transformation made originally in [110] contained some errors. The corrected values of r_i can be found in the hep-version of [110] and in [216]. The numerical analysis given below is based on these new values.

For the discussion below it will be useful to have [206]

$$\gamma_{27}^{(0)\text{eff}} = \frac{416}{81} \qquad \gamma_{28}^{(0)\text{eff}} = \frac{70}{27}$$
(12.57)

which enter (12.50) and (12.51) respectively. They can be obtained from (12.11) and (12.14).

12.3.2 Going Beyond the Spectator Model

In order to calculate the final rate we have to pass from the calculated b-quark decay rates to the B-meson decay rates. Relying on the Heavy Quark Expansion (HQE) calculations one finds [110]

$$Br(B \to X_s \gamma) = Br(B \to X_c e \bar{\nu_e}) \cdot R_{\text{quark}} \left(1 - \frac{\delta_{sl}^{NP}}{m_b^2} + \frac{\delta_{rad}^{NP}}{m_b^2} \right), \tag{12.58}$$

where $\delta_{\rm sl}^{\rm NP}$ and $\delta_{\rm rad}^{\rm NP}$ parametrize nonperturbative corrections to the semileptonic and radiative B-meson decay rates, respectively.

Following [217], one can express $\delta_{\rm sl}^{\rm NP}$ and $\delta_{\rm rad}^{\rm NP}$ in terms of the HQET parameters λ_1 and λ_2

$$\delta_{\rm sl}^{\rm NP} = \frac{1}{2}\lambda_1 + \left(\frac{3}{2} - \frac{6(1-z)^4}{f(z)}\right)\lambda_2. \tag{12.59}$$

$$\delta_{\text{rad}}^{\text{NP}} = \frac{1}{2}\lambda_1 - \frac{9}{2}\lambda_2.$$
 (12.60)

where f(z) is given (12.31).

The value of λ_2 is known from B^*-B mass splitting

$$\lambda_2 = \frac{1}{4}(m_{B^*}^2 - m_B^2) \simeq 0.12 \text{ GeV}^2.$$
 (12.61)

The value of λ_1 is controversial. Fortunately it cancels out in the r.h.s. of (12.58).

The two nonperturbative corrections in (12.58) are both around 4% in magnitude and tend to cancel each other. In effect, they sum up to only around 1%. As stressed in [110], such a small number has to be taken with caution. Indeed, one has to remember that the four-quark operators Q_1 Q_6 have not been included in the calculation of $\delta_{\rm rad}^{\rm NP}$. Contributions from these operators could potentially give one- or two-percent effects. Nevertheless, it seems reasonable to conclude that the total nonperturbative $1/m_b^2$ correction to (12.58) is well below 10%, i.e. it is smaller than the inaccuracy of the perturbative calculation of $R_{\rm quark}$.

In additions to the $1/m_b^2$ corrections one has to consider long distance contributions to $B \to X_s \gamma$. These are not easy to calculate and until recently most estimates were based on phenomenological models. In these model estimates long distance contributions are expected to arise dominantly from transitions $B \to \sum_i V_i + X_s \to \gamma X_s$ where $V_i = J/\psi, \psi', ...$ and are found to be below 10% [218].

A more modern way to estimate these long distance corrections is to use heavy quark expansions treating the charm quark as a heavy quark. As pointed out by Voloshin [219] and also discussed by other authors [220], these non-perturbative corrections originate in the photon coupling to a virtual $c\bar{c}$ loop and their general structure is given by

$$(\Lambda_{\rm QCD}^2/m_{\rm c}^2)(\Lambda_{\rm QCD}m_{\rm b}/m_{\rm c}^2)^n$$

with (n = 0, 1..). The term n = 0 can be estimated reliably. Originally a 3% suppression of the decay rate by this term has been found in [219] Subsequently, however, an overall sign error in this estimate has been pointed out in [221] so that this $1/m_c^2$ correction is positive.

Since $\Lambda_{\rm QCD} m_{\rm b}/m_{\rm c}^2 \approx 0.6$, the terms with n > 0 are not necessarily much smaller. Although the presence of unknown matrix elements in these contributions does not allow a

definite estimate of their actual size, the analyses in [219, 220] find that these contributions are weighted by very small calculable coefficients. Consequently these higher order contributions are expected to be substantially smaller than the n=0 term and the 3% enhancement from $1/m_c^2$ corrections found in [221] appears to be a good estimate of the long distance contributions to the $B \to X_s \gamma$ decay rate. We will include this enhancement in the numerical analysis below.

12.3.3 Numerical Analysis at NLO

Let us investigate how much the uncertainties in (12.36) are reduced after including NLO corrections. We begin this discussion by demonstrating analytically that the μ_b , μ_W and μ_t dependences present in $C_7^{(0)\text{eff}}(\mu_b)$ are indeed cancelled at $\mathcal{O}(\alpha_s)$ by the explicit scale dependent terms in (12.41) and (12.50). The scale dependent terms in (12.51) do not enter this cancellation at this order in α_s in $B \to X_s \gamma$. On the other hand they are responsible for the cancellation of the scale dependences in $C_8^{(0)\text{eff}}(\mu_b)$ relevant for the $b \to s$ gluon transition. Expanding the three terms in (12.23) in α_s and keeping the leading logarithms we find:

$$\eta^{\frac{16}{23}}C_7^{(0)}(\mu_W) = \left(1 + \frac{\alpha_s}{4\pi} \frac{16}{3} \ln \frac{\mu_b^2}{\mu_W^2}\right) C_7^{(0)}(\mu_W) \tag{12.62}$$

$$\frac{8}{3} \left(\eta^{\frac{14}{23}} - \eta^{\frac{16}{23}} \right) C_8^{(0)}(\mu_W) = -\frac{\alpha_s}{4\pi} \frac{16}{9} \ln \frac{\mu_b^2}{\mu_W^2} C_8^{(0)}(\mu_W)$$
 (12.63)

$$\sum_{i=1}^{8} h_i \eta^{a_i} = \frac{\alpha_s}{4\pi} \frac{23}{3} \ln \frac{\mu_b^2}{\mu_W^2} \sum_{i=1}^{8} h_i a_i = \frac{208}{81} \frac{\alpha_s}{4\pi} \ln \frac{\mu_b^2}{\mu_W^2}$$
 (12.64)

respectively. In (12.64) we have used $\sum h_i = 0$. Inserting these expansions into (12.41), we observe that the μ_W dependences in (12.62), (12.63) and (12.64) are precisely cancelled by the three explicit logarithms in (12.50) involving μ_W , respectively. Similarly one can convince oneself that the μ_t -dependence of $C_7^{(0)\text{eff}}(\mu_b)$ is cancelled at $\mathcal{O}(\alpha_s)$ by the $\ln \mu_t^2/M_W^2$ term in (12.50). Finally and most importantly the μ_b dependences in (12.62), (12.63) and (12.64) are cancelled by the explicit logarithms in (12.41) which result from the calculation of the one-loop matrix elements $\langle s\gamma|Q_7|B\rangle$ and $\langle s\gamma|Q_8|B\rangle$ and the two-loop matrix element $\langle s\gamma|Q_2|B\rangle$ as discussed previously. Interestingly the scale dependent term in (12.52) does not contribute to any cancellation of the μ_W dependence at this order in α_s due to the relation

$$\sum_{i=1}^{8} \left(\frac{2}{3} e_i + 6l_i \right) = 0. \tag{12.65}$$

which can be verified by using the table 25.

Clearly there remain small μ_b , μ_W and μ_t dependences in (12.39) which can only be reduced by going beyond the NLO approximation. They constitute the theoretical uncertainty which should be taken into account in estimating the error in the prediction for $Br(B \to X_s \gamma)$. For this reason also the term $\Delta C_7^{(1)eff}(\mu_b)$ in (12.50), originally omitted in [77]), has to be kept as pointed out in [212].

Using the two-loop generalization of (12.35) from Section 4.7 and varying μ_b , μ_W and μ_t in the ranges (12.33) and (12.34) one finds [77] the following respective uncertainties in the branching ratio after the inclusion of NLO corrections:

$$\Delta Br(B \to X_s \gamma) = \begin{cases} \pm 4.3\% & (\mu_b) \\ \pm 1.1\% & (\mu_W) \\ \pm 0.4\% & (\mu_t) \end{cases}$$
 (12.66)

This reduction of the μ_b -uncertainty by roughly a factor of seven relative to $\pm 22\%$ in LO is impressive. The remaining μ_W and μ_t uncertainties are negligible.

Next we would like to comment on the uncertainty due to variation of $\bar{\mu}_b$ in $\kappa(z)$ given in (12.42). In [109] the choice $\bar{\mu}_b = \mu_b$ has been made. Yet in my opinion such a treatment is not really correct, since the scale $\bar{\mu}_b$ in the semi-leptonic decay has nothing to do with the scale μ_b in the renormalization group evolution in the $B \to X_s \gamma$ decay. Varying $\bar{\mu}_b$ in the range $2.5 \,\text{GeV} \le \mu_b \le 10 \,\text{GeV}$ we find

$$\Delta Br(B \to X_s \gamma) = \pm 1.7\% \quad (\bar{\mu}_b) \tag{12.67}$$

Since the μ_b and $\bar{\mu}_b$ uncertainties are uncorrelated we can add them in quadrature finding $\pm 4.6\%$ for the total scale uncertainty due to μ_b and $\bar{\mu}_b$. The addition of the uncertainties in μ_t and μ_W in (12.66) modifies this result slightly and the total scale uncertainty in $Br(B \to X_s \gamma)$ amounts then to

$$\Delta Br(B \rightarrow X_s \gamma) = \pm 4.8\%$$
 (scale) (12.68)

It should be stressed that this pure theoretical uncertainty related to the truncation of the perturbative series should be distinguished from parametric uncertainties related to α_s , the quark masses etc. discussed below.

In our numerical calculations we have included all corrections in the NLO approximation. To work consistently in this order, we have in particular expanded the various factors in (12.39) in α_s and discarded all NNLO terms of order α_s^2 which resulted in the process of multiplication. This treatment is different from [110, 109], where the α_s corrections in (12.40) have not been expanded in the evaluation of (12.39) and therefore some higher order corrections have been kept. Different scenarios of partly incorporating higher order corrections by expanding or not expanding various factors in (12.39) affect the branching ratio by $\Delta Br(B \to X_s \gamma) \approx \pm 3.0\%$. This number indicates that indeed the scale uncertainty in (12.68) realistically estimates the magnitude of yet unknown higher order corrections. The remaining uncertainties are due to the values of the various input parameters. In order to obtain the final result for the branching ratio we have used the parameters given in table 26.

Table 26: Input parameter values and their uncertainties. The masses are given in GeV.

	$\alpha_s(M_Z)$	$m_{t,pole}$	$m_{c,pole}/m_{b,pole}$	$m_{b,pole}$	α_{em}^{-1}	$ V_{ts}^{\star}V_{tb} /V_{cb}$	$Br(B \to X_c e \bar{\nu}_e)$
Central	0.118	176	0.29	4.8	130.3	0.976	0.104
Error	± 0.003	± 6.0	± 0.02	± 0.15	± 2.3	± 0.010	± 0.004

Table 27: Uncertainties in $Br(B \to X_s \gamma)$ due to various sources.

	Scales	$\alpha_s(M_Z)$	$m_{t,pole}$	$m_{c,pole}/m_{b,pole}$	$m_{b,pole}$	α_{em}	CKM angles	$B \to X_c e \bar{\nu}_e$
Ī	$\pm 4.8\%$	$\pm 2.9\%$	$\pm 1.7\%$	$\pm 5.4\%$	$\pm 0.7\%$	$\pm 1.8\%$	$\pm 2.0\%$	$\pm 3.8\%$

Adding all the uncertainties in quadrature we find

$$Br(B \to X_s \gamma) = (3.60 \pm 0.17 \text{ (scale)} \pm 0.28 \text{ (par)}) \times 10^{-4} = (3.60 \pm 0.33) \times 10^{-4}$$
 (12.69)

where we show separately scale and parametric uncertainties. The relative importance of various uncertainties is shown in table 27. Similar results can be found in [110, 212]. We observe that inclusion of NLO corrections increased the value of the LO prediction in (12.38) by roughly 25%. Simultaneously the total error has been decreased by more than a factor of two. The shift upwards is mainly caused by the $\mathcal{O}(\alpha_s)$ corrections to the matrix elements of the contributing operators calculated in [109], not to the Wilson coefficients. One has to remember, however, that this feature is valid in the NDR scheme considered here and may not be true in another renormalization scheme without changing the total result for the decay rate.

We also observe that the parametric uncertainties dominate the theoretical error at present. Once these parametric uncertainties will be reduced in the future the smallness of the scale uncertainties achieved through very involved QCD calculations, in particular in [110, 109, 106, 108, 107, 66, 67], can be better appreciated. This reduction of the theoretical error in the Standard Model prediction for $Br(B\to X_s\gamma)$ could turn out to be very important in the searches for new physics when the experimental data improve.

12.4 Recent Developments

Very recently electroweak $\mathcal{O}(\alpha)$ corrections to R_{quark} have been investigated in an interesting paper by a student of this school, Andrzej Czarnecki, and Bill Marciano [222]. A study of $\mathcal{O}(\alpha)$ corrections to R_{quark} must entail two-loop electroweak contributions to $b \to s\gamma$ as well as one loop corrections to $b \to ce\nu$. A complete calculation of all $\mathcal{O}(\alpha)$ contributions would be a very heroic task, but it is already valuable to identify potentially dominant contributions.

One obvious question is the scale μ in $\alpha_{\rm em} \equiv e^2(\mu)/4\pi$ which is rather arbitrary if corrections $\mathcal{O}(\alpha)$ are not considered. In all recent calculations $m_{\rm b} \leq \mu \leq M_{\rm W}$ has been used, giving $1/\alpha_{\rm em} = 130.3 \pm 2.3$. The inclusion of fermion loop contributions in the photon propagator indicates [222], however, that α renormalized at $q^2 = 0$, i.e $\alpha = 1/137.036$ is more appropriate. This reduces the branching ratio by roughly 5%. The fermion loops in the W-propagator bring a reduction of 2%. Two other reductions, each of roughly 1%, come from short distance photonic corrections to $b \to s\gamma$ and $b \to ce\nu$. The total reduction of $R_{\rm quark}$ quoted in [222] amounts then to $(9 \pm 2)\%$ where the error is a guess-estimate of unknown corrections. With this reduction the branching ratio in (12.69) becomes

$$Br(B \to X_s \gamma) = (3.28 \pm 0.30) \times 10^{-4}$$
 (12.70)

The $\pm 2\%$ error in the estimate of $\mathcal{O}(\alpha)$ corrections is compensated by the fact that α has a negligible error compared to α_{em} in table 26. Personally, I am not yet convinced that the $\mathcal{O}(\alpha)$ reduction is as high as 9%. The reduction of 5% through the change $\alpha_{em} \to \alpha$ appears rather plausible. On the other hand the same sign of three smaller corrections could turn out to be accidental and other corrections, not considered yet, could well cancel them. Some indication for this is given by a very recent analysis of Strumia [223], who performed a complete calculation of two-loop electroweak contributions to $B \to X_s \gamma$ in the large m_t limit, finding them below 1%. In spite of this reservation, the calculation of Czarnecki and Marciano certainly indicates that a reduction of $Br(B\to X_s \gamma)$ through $\mathcal{O}(\alpha)$ corrections by $\mathcal{O}(5\%)$ is certainly possible. A more detailed investigation of this issue would be desirable at some stage in the future.

Finally I would like to mention here a very recent paper of Kagan and Neubert [216] who also made an extensive analysis of $B \to X_s \gamma$. Reanalyzing in detail the issue of the photon-spectrum and of δ in (12.46) and including also QED corrections, Kagan and Neubert arrive at the estimate of $Br(B\to X_s \gamma)$, which in spite of some differences at intermediate stages agrees very well with (12.70). Since the analysis in [216] is very recent, I am not in a position to make any useful comments on it. Certainly of interest is their reanalysis of the extraction of the total decay rate $Br(B\to X_s \gamma)$ from the experimental photon spectrum, which I will briefly mention below.

12.5 Experimental Status

After all this theoretical exposition it is really time to summarize the present data. The branching ratio for $B \to X_s \gamma$ found by the CLEO collaboration already in 1994 [224] is given by

$$Br(B \to X_s \gamma) = (2.32 \pm 0.57 \pm 0.35) \times 10^{-4}$$
 (12.71)

and the very recent preliminary update from CLEO reads [225]

$$Br(B \to X_s \gamma) = (2.50 \pm 0.47 \pm 0.39) \times 10^{-4}$$
. (12.72)

On the other hand the recent ALEPH measurement of the corresponding branching ratio for b-hadrons (mesons and baryons) produced in Z^0 decays reads [226]

$$Br(H_b \to X_s \gamma) = (3.11 \pm 0.80 \pm 0.72) \times 10^{-4}.$$
 (12.73)

which is compatible with the CLEO result. In (12.71)-(12.73) the first error is statistical and the second is systematic. As stressed already by several authors in the literature the measurements in [224] and [226] are quite different and should not be expected to give identical results.

Now, the experimental results given above, are obtained by measuring the high-energy part of the photon spectrum and the extrapolation to the total rate. This requires theoretical understanding of the photon spectrum. Improving recently the analysis of the photon spectrum, Kagan and Neubert [216] find that the result in (12.71) should actually read

$$Br(B \to X_s \gamma) = (2.66 \pm 0.56_{\text{exp}} \pm 0.45_{\text{th}}) \times 10^{-4},$$
 (12.74)

and that the central value in (12.72) should be increased to 2.8. It will be interesting to watch the further development and to have a new official CLEO value including this new insight.

The theoretical estimates in (12.69) and (12.70) are somewhat higher than experimental data. However, within the remaining theoretical and in particular experimental uncertainties, the Standard Model value is compatible with experiment.

12.6 A Look Beyond the Standard Model

The inclusive radiative $B \to X_s \gamma$ decay plays an important role in the indirect searches for physics beyond the Standard Model and places already now rather strong constraints on some new physics scenarios. The possible non-standard contributions can indeed be of the same order of magnitude of the Standard Model loop effects discussed above. This is well illustrated by the simplest of these extensions, where the Higgs sector of the Standard Model is enlarged to include two doublets (Two Higgs Doublet Models, or 2HDM), leading to three

new physical fields, two neutral scalars (CP even and odd) and one charged scalar. In this context, only the charged Higgs H^{\pm} contributes to the Wilson coefficients $C_{7\gamma}$ and C_{8G} . Its interaction with quarks is described by the Lagrangian

$$\mathcal{L} = (2\sqrt{2}G_F)^{1/2} \sum_{i,j=1}^{3} \bar{u}_i \left(A_u m_{u_i} V_{ij} \frac{1 - \gamma_5}{2} - A_d V_{ij} m_{d_i} \frac{1 + \gamma_5}{2} \right) d_j H^+ + \text{h.c.}$$
 (12.75)

Here i, j are generation indices, $m_{u,d}$ are quark masses, and V is the CKM matrix. The fermions may then acquire their masses in two ways: the first possibility, referred to as Model I, is that both up and down quarks get their masses from the same Higgs doublet H_2 , and

$$A_u = A_d = 1/\tan\beta , \qquad (12.76)$$

where $\tan \beta$ is the ratio of the v.e.v. of H_2 and H_1 . In the case of Model II, up quarks get their masses from Yukawa couplings to H_2 , while down quarks get masses from couplings to H_1 , and

$$A_u = -1/A_d = 1/\tan\beta \ . \tag{12.77}$$

Model II is particularly interesting because it is realized in the minimal supersymmetric extension of the SM. The charged-Higgs contributions to $C_{7\gamma}$ and C_{8G} are functions of the top and charged Higgs masses and of $\tan \beta$ whose LO expressions are given in [227]. Recently, the complete NLO corrections to $Br(B \to X_s \gamma)$ in the 2HDM have been computed [111, 212]. Partial results can also be found in [112].

With respect to the Standard Model, in Model II the branching ratio is strongly enhanced for a light charged Higgs and the decoupling at large M_H takes place very slowly. This leads to very stringent bounds on M_H for any particular value of $\tan \beta$. Actually, for $\tan \beta > 1$, the dependence on $\tan \beta$ is very mild and practically saturates for $\tan \beta \geq 2$. Using the current CLEO 95% CL upper bound $Br(B \to X_s \gamma) < 4.2 \times 10^{-4}$ and adopting a conservative approach to evaluate the theoretical uncertainty (scanning), one obtains lower bounds on M_H of $\approx 250 \text{GeV}$, independent of $\tan \beta$ [111, 212]. On the other hand, adding different theoretical errors in quadrature leads to $M_{\rm H} > 370 {\rm GeV}$. Indeed, these bounds are quite sensitive to the errors of the theoretical prediction and to the details of the calculation [111]. For instance including Czarnecki-Marciano $\mathcal{O}(\alpha)$ corrections would weaken them significantly. Improving the calculation to the NLO has also had important effects on these bounds, since the theoretical error is significantly reduced [111]. Finally, it is clear that one of the reasons we are able to obtain such strong bounds on M_H is the poor agreement between the Standard Model prediction and CLEO measurement, and that the situation may drastically change with new experimental results. In the case of a heavy Higgs, a resummation of the leading logarithms of M_H/M_W has been performed in [228].

For what concerns Model I, in that case the charged-Higgs contribution reduces the value of $Br(B \to X_s \gamma)$ and therefore no significant bound can be obtained. On the other hand, it is interesting that new physics effects can bring the prediction for $B \to X_s \gamma$ closer to the CLEO value. A significant effect can only be expected for small $\tan \beta$, since in Model I all charged-Higgs contributions vanish in the limit of large $\tan \beta$, as $\tan^{-2} \beta$. However, in that case the top Yukawa coupling grows and strong limitations come from high energy measurements, in particular of R_b . It can be concluded [111] that the reduction of $Br(B \to X_s \gamma)$ can be at best about 20%.

A more general class of multi-Higgs models, where only one charged Higgs does not decouple and its couplings are arbitrary and may violate CP, has been studied at LO in [229] and more recently at NLO in [212].

In the MSSM, the charged Higgs loops are accompanied by chargino-squark contributions which can partly compensate the effect of the charged Higgs. Therefore the above bounds do not apply to the MSSM, except in some scenarios, like gauge-mediated supersymmetric models [230], where the Higgs contribution is known to dominate over the chargino loops, because the squarks are generally heavy. Indeed, in the supersymmetric limit, there is an exact cancellation of different contributions [231]. In the realistic case of broken supersymmetry, this cancellation is spoiled but, if charginos and stops are light, it may still be partially effective. A complete analysis at LO in the MSSM can be found in [232]. Although no direct limit on M_H can be set, $b \to s \gamma$ has helped in ruling out very large portions of the SUSY parameter space. It can be expected that a NLO analysis would further enhance this exclusion potential.

12.7 Summary and Outlook

The rare decay $B \to X_s \gamma$ plays at present together with $B_{d,s}^0 - \bar{B}_{d,s}^0$ mixing the central role in loop induced transitions in the B-system. On the theoretical side considerable progress has been made recently by calculating NLO corrections, thereby reducing the large μ_b uncertainties present in the leading order. This way the error in the prediction for $Br(B \to X_s \gamma)$ as given in (12.69), and in (12.70) after including QED corrections, has been decreased down to roughly $\pm 10\%$ compared with $\pm (25 - 30)\%$ in the leading order. Since during last two years the central value for $Br(B \to X_s \gamma)$ was changing constantly due to inclusion of various small corrections and different error analyses, it is hard to imagine that the result in (12.70) is the final word on this subject. It will be interesting to see how this value will look like in five years from now.

On the experimental side considerable progress has been made by CLEO [233] in the case of $Br(B_d^0 \to K^*\gamma)$, which we left out due to space limitations. It is very desirable to obtain

now an improved measurement of $Br(B \to X_s \gamma)$. Indeed, in the forthcoming years much more precise measurements of $Br(B \to X_s \gamma)$ are expected from the upgraded CLEO detector, as well as from the B-factories at SLAC and KEK.

Confrontation of these new improved experimental values with the already rather precise theoretical Standard Model estimate may shed some light on whether some physics beyond this model is required to fit the improved data.

More on $B \to X_s \gamma$, in particular on the photon spectrum and the determination of $|V_{td}|/|V_{ts}|$ from $B \to X_{s,d}\gamma$, can be found in [163, 234, 235, 216]. CP violation in $B \to K^* \gamma$ and $B \to \varrho \gamma$ is discussed in [236].

13 Rare K- and B-Decays

13.1 General Remarks

We will now move to discuss the semileptonic rare FCNC transitions $K^+ \to \pi^+ \nu \bar{\nu}$, $K_{\rm L} \to \pi^0 \nu \bar{\nu}$, $B \to X_{s,d} \nu \bar{\nu}$ and $B_{s,d} \to l^+ l^-$ paying particular attantion to the first two decays. The presentation given here overlaps considerably with the ones given in the reviews [17, 18], although there are some differences. In particular the decay $K_{\rm L} \to \mu^+ \mu^-$ will not be considered here in view of space limitations. Some details on this decay, which is not as theoretically clean as the ones discussed here, can be found in the latter reviews and in [237]. On the other hand we will provide certain derivations which cannot be found in [17, 18]. Moreover we discuss briefly two–loop electroweak contributions and make a few remarks on the physics beyond the Standard Model.

The decay modes considered here are very similar in their structure which differs considerably from the one encountered in the decays $K \to \pi\pi$ and $B \to X_s \gamma$ discussed in previous sections. In particular:

- Within the Standard Model all the decays listed above are loop-induced semileptonic FCNC processes determined only by Z^0 -penguin and box diagrams which we encountered already on many occasions in these lectures. Thus, a distinguishing feature of the present class of decays is the absence of a photon penguin contribution. For the decay modes with neutrinos in the final state this is obvious, since the photon does not couple to neutrinos. For the mesons decaying into a charged lepton pair the photon penguin amplitude vanishes due to vector current conservation. Consequently the decays in question are governed by the functions $X_0(x_t)$ and $Y_0(x_t)$ (see (3.26) and (3.27)) which as seen in (3.30) and (3.31) exhibit strong m_t -dependences.
- A particular and very important advantage of these decays is their clean theoretical

character. This is related to the fact that the low energy hadronic matrix elements required are just the matrix elements of quark currents between hadron states, which can be extracted from the leading (non-rare) semileptonic decays. Other long-distance contributions are negligibly small. As a consequence of these features, the scale ambiguities, inherent to perturbative QCD, essentially constitute the only theoretical uncertainties present in the analysis of these decays. These theoretical uncertainties have been considerably reduced through the inclusion of the next-to-leading QCD corrections [65, 98, 91] as we will demonstrate below.

• The investigation of these low energy rare decay processes in conjunction with their theoretical cleanliness, allows to probe, albeit indirectly, high energy scales of the theory and in particular to measure the top quark couplings V_{ts} and V_{td} . Moreover $K_{\rm L} \to \pi^0 \nu \bar{\nu}$ offers a clean determination of the Wolfenstein parameter η and as we will stress below offers the cleanest measurement of ${\rm Im}\lambda_t = {\rm Im}V_{ts}^*V_{td}$ which governs all CP violating K-decays. However, the very fact that these processes are based on higher order electroweak effects implies that their branching ratios are expected to be very small and not easy to access experimentally.

Table 28: Order of magnitude of CKM parameters relevant for the various decays, expressed in powers of the Wolfenstein parameter $\lambda = 0.22$. In the case of $K_L \to \pi^0 \nu \bar{\nu}$, which is CP-violating, only the imaginary parts of $\lambda_{c,t}$ contribute.

	$K^+ \to \pi^+ \nu \bar{\nu}$	$K_{ m L} o \pi^0 u \bar{ u}$	$B \to X_s \nu \bar{\nu}$	$B \to X_d \nu \bar{\nu}$
			$B_s \to l^+ l^-$	$B_d \to l^+ l^-$
λ_c	$\sim \lambda$	$(\mathrm{Im}\lambda_c\sim\lambda^5)$	$\sim \lambda^2$	$\sim \lambda^3$
λ_t	$\sim \lambda^5$	$(\mathrm{Im}\lambda_t \sim \lambda^5)$	$\sim \lambda^2$	$\sim \lambda^3$

The effective Hamiltonians governing the decays $K^+ \to \pi^+ \nu \bar{\nu}$, $K_L \to \pi^0 \nu \bar{\nu}$, $B \to X_{s,d} \nu \bar{\nu}$ and $B \to l^+ l^-$ resulting from the Z^0 -penguin and box-type contributions can all be written in the following general form:

$$\mathcal{H}_{\text{eff}} = \frac{G_{\text{F}}}{\sqrt{2}} \frac{\alpha}{2\pi \sin^2 \Theta_{\text{W}}} \left(\lambda_c F(x_c) + \lambda_t F(x_t) \right) (\bar{n}n')_{V-A} (\bar{r}r)_{V-A} , \qquad (13.1)$$

where n, n' denote down-type quarks $(n, n' = d, s, b \text{ but } n \neq n')$ and r leptons, $r = l, \nu_l$ $(l = e, \mu, \tau)$. The λ_i are products of CKM elements, in the general case $\lambda_i = V_{in}^* V_{in'}$. Furthermore $x_i = m_i^2/M_W^2$. The functions $F(x_i)$ describe the dependence on the internal uptype quark masses m_i (and on lepton masses if necessary) and are understood to include QCD

corrections. They are increasing functions of the quark masses, a property that is particularly important for the top contribution. Since $F(x_c)/F(x_t) \approx \mathcal{O}(10^{-3}) \ll 1$ the top contributions are by far dominant unless there is a partial compensation through the CKM factors λ_i . As seen in table 28 such a partial compensation takes place in $K^+ \to \pi^+ \nu \bar{\nu}$ and consequently in this decay internal charm contribution, albeit smaller than the top contribution, has to be kept. On the other hand in the remaining decays the charm contributions can be safely neglected. Since the charm contributions involve QCD corrections with $\alpha_s(m_c)$, the scale uncertainties in $K^+ \to \pi^+ \nu \bar{\nu}$ are found to be larger than in the remaining decays in which the QCD effects enter only through $\alpha_s(m_t) < \alpha_s(m_c)$. After these general remarks let us enter some details. Other reviews of rare decays can be found in [237, 18].

13.2 The Decay $K^+ \to \pi^+ \nu \bar{\nu}$

13.2.1 The effective Hamiltonian

The effective Hamiltonian for $K^+ \to \pi^+ \nu \bar{\nu}$ can be written as

$$\mathcal{H}_{\text{eff}} = \frac{G_{\text{F}}}{\sqrt{2}} \frac{\alpha}{2\pi \sin^2 \Theta_{\text{W}}} \sum_{l=e,\mu,\tau} \left(V_{cs}^* V_{cd} X_{\text{NL}}^l + V_{ts}^* V_{td} X(x_t) \right) (\bar{s}d)_{V-A} (\bar{\nu}_l \nu_l)_{V-A} \,. \tag{13.2}$$

The index $l=e, \mu, \tau$ denotes the lepton flavour. The dependence on the charged lepton mass resulting from the box-graph is negligible for the top contribution. In the charm sector this is the case only for the electron and the muon but not for the τ -lepton.

We have discussed the top quark contribution already in section 8.2 but there is no harm when we repeat certain things in order to have the most important information about this decay in one place.

The function $X(x_t)$ relevant for the top part is given by

$$X(x_t) = X_0(x_t) + \frac{\alpha_s}{4\pi} X_1(x_t)$$
(13.3)

with the leading contribution $X_0(x)$ given in (3.26) and the QCD correction [98]

$$X_{1}(x_{t}) = -\frac{23x_{t} + 5x_{t}^{2} - 4x_{t}^{3}}{3(1 - x_{t})^{2}} + \frac{x_{t} - 11x_{t}^{2} + x_{t}^{3} + x_{t}^{4}}{(1 - x_{t})^{3}} \ln x_{t}$$

$$+ \frac{8x_{t} + 4x_{t}^{2} + x_{t}^{3} - x_{t}^{4}}{2(1 - x_{t})^{3}} \ln^{2} x_{t} - \frac{4x_{t} - x_{t}^{3}}{(1 - x_{t})^{2}} L_{2}(1 - x_{t})$$

$$+ 8x_{t} \frac{\partial X_{0}(x_{t})}{\partial x_{t}} \ln x_{\mu}, \qquad (13.4)$$

where $x_{\mu} = \mu_t^2/M_W^2$ with $\mu_t = \mathcal{O}(m_t)$ and

$$L_2(1-x) = \int_1^x dt \frac{\ln t}{1-t} \,. \tag{13.5}$$

The μ_t -dependence of the last term in (13.4) cancels to the considered order the μ_t -dependence of the leading term $X_0(x_t(\mu))$. The leftover μ_t -dependence in $X(x_t)$ is tiny and will be given in connection with the discussion of the branching ratio below.

The function X in (13.3) can also be written as

$$X(x_t) = \eta_X \cdot X_0(x_t), \qquad \eta_X = 0.985,$$
 (13.6)

where η_X summarizes the NLO corrections represented by the second term in (13.3). With $m_t \equiv \overline{m}_t(m_t)$ the QCD factor η_X is practically independent of m_t and $\Lambda_{\overline{MS}}$.

The expression corresponding to $X(x_t)$ in the charm sector is the function $X_{\rm NL}^l$. It results from the NLO calculation [91] and is given explicitly in [91, 17]. The inclusion of NLO corrections reduced considerably the large μ_c dependence (with $\mu_c = \mathcal{O}(m_c)$) present in the leading order expressions for the charm contribution [238, 239, 240, 16]. Varying μ_c in the range $1 \, \text{GeV} \le \mu_c \le 3 \, \text{GeV}$ changes $X_{\rm NL}$ by roughly 24% after the inclusion of NLO corrections to be compared with 56% in the leading order. Further details can be found in [91, 17]. The impact of the μ_c uncertainties on the resulting branching ratio $Br(K^+ \to \pi^+ \nu \bar{\nu})$ is discussed below.

The numerical values for $X_{\rm NL}$ for $\mu=m_{\rm c}$ and several values of $\Lambda_{\overline{\rm MS}}^{(4)}$ and $m_{\rm c}(m_{\rm c})$ are given in table 29. The net effect of QCD corrections is to suppress the charm contribution by roughly 30%.

	1					
	$X_{ m NL}^{e}/10^{-4}$			$X_{\rm NL}^{\tau}/10^{-4}$		
$\Lambda_{\overline{\rm MS}}^{(4)} [{ m MeV}] \setminus m_{ m c} [{ m GeV}]$	1.25	1.30	1.35	1.25	1.30	1.35
245	10.32	11.17	12.04	6.94	7.63	8.36
285	10.02	10.86	11.73	6.64	7.32	8.04
325	9.71	10.55	11.41	6.32	7.01	7.72
365	9.38	10.22	11.08	6.00	6.68	7.39
405	9.03	9.87	10.72	5.65	6.33	7.04

Table 29: The functions $X_{\rm NL}^e$ and $X_{\rm NL}^{\tau}$ for various $\Lambda_{\overline{\rm MS}}^{(4)}$ and $m_{\rm c}$.

13.2.2 Deriving the Branching Ratio

The relevant hadronic matrix element of the weak current $(\bar{s}d)_{V-A}$ can be extracted, with the help of isospin symmetry from the leading decay $K^+ \to \pi^0 e^+ \nu$. Consequently the resulting theoretical expression for the branching fraction $Br(K^+ \to \pi^+ \nu \bar{\nu})$ can be related to the experimentally well known quantity $Br(K^+ \to \pi^0 e^+ \nu)$. Let us demonstrate this.

The effective Hamiltonian for the tree level decay $K^+ \to \pi^0 e^+ \nu$ is given by

$$\mathcal{H}_{\text{eff}}(K^+ \to \pi^0 e^+ \nu) = \frac{G_F}{\sqrt{2}} V_{us}^*(\bar{s}u)_{V-A}(\bar{\nu}_e e)_{V-A}.$$
 (13.7)

Using isospin symmetry we have

$$\langle \pi^+ | (\bar{s}d)_{V-A} | K^+ \rangle = \sqrt{2} \langle \pi^0 | (\bar{s}u)_{V-A} | K^+ \rangle.$$
 (13.8)

Consequently neglecting differences in the phase space of these two decays, due to $m_{\pi^+} \neq m_{\pi^0}$ and $m_e \neq 0$, we find

$$\frac{Br(K^+ \to \pi^+ \nu \bar{\nu})}{Br(K^+ \to \pi^0 e^+ \nu)} = \frac{\alpha^2}{|V_{us}|^2 2\pi^2 \sin^4 \Theta_W} \sum_{l=e,\mu,\tau} \left| V_{cs}^* V_{cd} X_{NL}^l + V_{ts}^* V_{td} X(x_t) \right|^2 . \tag{13.9}$$

13.2.3 Basic Phenomenology

We are now ready to present the expression for the branching fraction $Br(K^+ \to \pi^+ \nu \bar{\nu})$ and to collect various formulae relevant for phenomenological applications. Using (13.9) and including isospin breaking corrections one finds

$$Br(K^{+} \to \pi^{+} \nu \bar{\nu}) = \kappa_{+} \cdot \left[\left(\frac{\operatorname{Im} \lambda_{t}}{\lambda^{5}} X(x_{t}) \right)^{2} + \left(\frac{\operatorname{Re} \lambda_{c}}{\lambda} P_{0}(X) + \frac{\operatorname{Re} \lambda_{t}}{\lambda^{5}} X(x_{t}) \right)^{2} \right] , \quad (13.10)$$

$$\kappa_{+} = r_{K^{+}} \frac{3\alpha^{2} Br(K^{+} \to \pi^{0} e^{+} \nu)}{2\pi^{2} \sin^{4} \Theta_{W}} \lambda^{8} = 4.11 \cdot 10^{-11} , \qquad (13.11)$$

where we have used

$$\alpha = \frac{1}{129}, \quad \sin^2 \Theta_W = 0.23, \quad Br(K^+ \to \pi^0 e^+ \nu) = 4.82 \cdot 10^{-2}.$$
 (13.12)

Here $\lambda_i = V_{is}^* V_{id}$ with λ_c being real to a very high accuracy. $r_{K^+} = 0.901$ summarizes isospin breaking corrections in relating $K^+ \to \pi^+ \nu \bar{\nu}$ to $K^+ \to \pi^0 e^+ \nu$. These isospin breaking corrections are due to quark mass effects and electroweak radiative corrections and have been calculated in [241]. Next

$$P_0(X) = \frac{1}{\lambda^4} \left[\frac{2}{3} X_{\text{NL}}^e + \frac{1}{3} X_{\text{NL}}^{\tau} \right]$$
 (13.13)

with the numerical values for $X_{\rm NL}^l$ given in table 29. The corresponding values for $P_0(X)$ as a function of $\Lambda_{\overline{MS}}^{(4)}$ and $m_c \equiv m_c(m_c)$ are collected in table 30. We remark that a negligibly small term $\sim (X_{\rm NL}^e - X_{\rm NL}^{\tau})^2$ has been discarded in (13.10).

Using the improved Wolfenstein parametrization and the approximate formulae (2.37) – (2.39) we can next put (13.10) into a more transparent form [34]:

$$Br(K^+ \to \pi^+ \nu \bar{\nu}) = 4.11 \cdot 10^{-11} A^4 X^2(x_t) \frac{1}{\sigma} \left[(\sigma \bar{\eta})^2 + (\varrho_0 - \bar{\varrho})^2 \right],$$
 (13.14)

Table 30: The function $P_0(X)$ for various $\Lambda_{\overline{\rm MS}}^{(4)}$ and m_c .

	$P_0(X)$		
$\Lambda_{\overline{\mathrm{MS}}}^{(4)}\setminus m_c$	$1.25\mathrm{GeV}$	$1.30\mathrm{GeV}$	$1.35\mathrm{GeV}$
$245\mathrm{MeV}$	0.393	0.426	0.462
$285\mathrm{MeV}$	0.380	0.413	0.448
$325\mathrm{MeV}$	0.366	0.400	0.435
$365\mathrm{MeV}$	0.352	0.386	0.420
$405\mathrm{MeV}$	0.337	0.371	0.405

where

$$\sigma = \left(\frac{1}{1 - \frac{\lambda^2}{2}}\right)^2. \tag{13.15}$$

The measured value of $Br(K^+ \to \pi^+ \nu \bar{\nu})$ then determines an ellipse in the $(\bar{\varrho}, \bar{\eta})$ plane centered at $(\varrho_0, 0)$ with

$$\varrho_0 = 1 + \frac{P_0(X)}{A^2 X(x_t)} \tag{13.16}$$

and having the squared axes

$$\bar{\varrho}_1^2 = r_0^2, \qquad \bar{\eta}_1^2 = \left(\frac{r_0}{\sigma}\right)^2$$
 (13.17)

where

$$r_0^2 = \frac{1}{A^4 X^2(x_t)} \left[\frac{\sigma \cdot Br(K^+ \to \pi^+ \nu \bar{\nu})}{4.11 \cdot 10^{-11}} \right] . \tag{13.18}$$

Note that r_0 depends only on the top contribution. The departure of ϱ_0 from unity measures the relative importance of the internal charm contributions.

The ellipse defined by r_0 , ϱ_0 and σ given above intersects with the circle (2.47). This allows to determine $\bar{\varrho}$ and $\bar{\eta}$ with

$$\bar{\varrho} = \frac{1}{1 - \sigma^2} \left(\varrho_0 - \sqrt{\sigma^2 \varrho_0^2 + (1 - \sigma^2)(r_0^2 - \sigma^2 R_b^2)} \right), \qquad \bar{\eta} = \sqrt{R_b^2 - \bar{\varrho}^2}$$
 (13.19)

and consequently

$$R_t^2 = 1 + R_b^2 - 2\bar{\varrho},\tag{13.20}$$

where $\bar{\eta}$ is assumed to be positive.

In the leading order of the Wolfenstein parametrization

$$\sigma \to 1, \qquad \bar{\eta} \to \eta, \qquad \bar{\varrho} \to \varrho$$
 (13.21)

and $Br(K^+ \to \pi^+ \nu \bar{\nu})$ determines a circle in the (ϱ, η) plane centered at $(\varrho_0, 0)$ and having the radius r_0 of (13.18) with $\sigma = 1$. Formulae (13.19) and (13.20) then simplify to [91]

$$R_t^2 = 1 + R_b^2 + \frac{r_0^2 - R_b^2}{\varrho_0} - \varrho_0, \qquad \varrho = \frac{1}{2} \left(\varrho_0 + \frac{R_b^2 - r_0^2}{\varrho_0} \right). \tag{13.22}$$

Given $\bar{\varrho}$ and $\bar{\eta}$ one can determine V_{td} :

$$V_{td} = A\lambda^3 (1 - \bar{\varrho} - i\bar{\eta}), \qquad |V_{td}| = A\lambda^3 R_t. \tag{13.23}$$

At this point a few remarks are in order:

- The long-distance contributions to $K^+ \to \pi^+ \nu \bar{\nu}$ have been studied in [242] and found to be very small: a few percent of the charm contribution to the amplitude at most, which is savely negligible.
- The determination of $|V_{td}|$ and of the unitarity triangle requires the knowledge of V_{cb} (or A) and of $|V_{ub}/V_{cb}|$. Both values are subject to theoretical uncertainties present in the existing analyses of tree level decays. Whereas the dependence on $|V_{ub}/V_{cb}|$ is rather weak, the very strong dependence of $Br(K^+ \to \pi^+ \nu \bar{\nu})$ on A or V_{cb} makes a precise prediction for this branching ratio difficult at present. We will return to this below.
- The dependence of $Br(K^+ \to \pi^+ \nu \bar{\nu})$ on m_t is also strong. However m_t is known already within $\pm 4\%$ and consequently the related uncertainty in $Br(K^+ \to \pi^+ \nu \bar{\nu})$ is substantially smaller than the corresponding uncertainty due to V_{cb} .
- Once ϱ and η are known precisely from CP asymmetries in B decays, some of the uncertainties present in (13.14) related to $|V_{ub}/V_{cb}|$ (but not to V_{cb}) will be removed.
- A very clean determination of $\sin 2\beta$ without essentially any dependence on m_t and V_{cb} can be made by combining $Br(K^+ \to \pi^+ \nu \bar{\nu})$ with $Br(K_L \to \pi^0 \nu \bar{\nu})$ discussed below.

13.2.4 Numerical Analysis of $K^+ \to \pi^+ \nu \bar{\nu}$

Let us begin the numerical analysis by investigating the uncertainties in the prediction for $Br(K^+ \to \pi^+ \nu \bar{\nu})$ and in the determination of $|V_{td}|$ related to the choice of the renormalization scales μ_t and μ_c in the top part and the charm part, respectively. To this end we will fix the remaining parameters as follows:

$$m_{\rm c} \equiv \overline{m}_{\rm c}(m_{\rm c}) = 1.3 \,\mathrm{GeV}, \qquad m_{\rm t} \equiv \overline{m}_{\rm t}(m_{\rm t}) = 170 \,\mathrm{GeV}$$
 (13.24)

$$V_{cb} = 0.040, |V_{ub}/V_{cb}| = 0.08. (13.25)$$

In the case of $Br(K^+ \to \pi^+ \nu \bar{\nu})$ we need the values of both $\bar{\varrho}$ and $\bar{\eta}$. Therefore in this case we will work with

$$\bar{\rho} = 0, \qquad \bar{\eta} = 0.36 \tag{13.26}$$

rather than with $|V_{ub}/V_{cb}|$. Finally we will set $\Lambda_{\overline{MS}}^{(4)} = 0.325\,\text{GeV}$ and $\Lambda_{\overline{MS}}^{(5)} = 0.225\,\text{GeV}$ for the charm part and top part, respectively. We then vary the scales μ_c and μ_t entering $m_c(\mu_c)$ and $m_t(\mu_t)$, respectively, in the ranges

$$1 \,\text{GeV} \le \mu_c \le 3 \,\text{GeV}, \qquad 100 \,\text{GeV} \le \mu_t \le 300 \,\text{GeV}.$$
 (13.27)

The results of such an analysis are as follows [17]: The uncertainty in $Br(K^+ \to \pi^+ \nu \bar{\nu})$

$$0.68 \cdot 10^{-10} \le Br(K^+ \to \pi^+ \nu \bar{\nu}) \le 1.08 \cdot 10^{-10}$$
(13.28)

present in the leading order is reduced to

$$0.79 \cdot 10^{-10} \le Br(K^+ \to \pi^+ \nu \bar{\nu}) \le 0.92 \cdot 10^{-10}$$
(13.29)

after including NLO corrections. The difference in the numerics compared to [17] results from $r_{K^+} = 1$ used there. Similarly one finds

$$8.24 \cdot 10^{-3} \le |V_{td}| \le 10.97 \cdot 10^{-3}$$
 LO (13.30)

$$9.23 \cdot 10^{-3} \le |V_{td}| \le 10.10 \cdot 10^{-3} \quad \text{NLO},$$
 (13.31)

where $Br(K^+ \to \pi^+ \nu \bar{\nu}) = 0.9 \cdot 10^{-10}$ has been set. We observe that including the full next-to-leading corrections reduces the uncertainty in the determination of $|V_{td}|$ from $\pm 14\%$ (LO) to $\pm 4.6\%$ (NLO) in the present example. The main bulk of this theoretical error stems from the charm sector. Indeed, keeping $\mu_c = m_c$ fixed and varying only μ_t , the uncertainties in the determination of $|V_{td}|$ would shrink to $\pm 4.7\%$ (LO) and $\pm 0.6\%$ (NLO). Similar comments apply to $Br(K^+ \to \pi^+ \nu \bar{\nu})$ where, as seen in (13.28) and (13.29), the theoretical uncertainty due to $\mu_{c,t}$ is reduced from $\pm 22\%$ (LO) to $\pm 7\%$ (NLO).

Finally using the input parameters of table 12 ("present") and performing two types of error analysis one finds [161]

$$Br(K^+ \to \pi^+ \nu \bar{\nu}) = \begin{cases} (9.1 \pm 3.8) \cdot 10^{-11} & \text{Scanning} \\ (8.0 \pm 1.6) \cdot 10^{-11} & \text{Gaussian}, \end{cases}$$
 (13.32)

where the error comes dominantly from the uncertainties in the CKM parameters. The corresponding analysis with the "future" input parameters gives

$$Br(K^{+} \to \pi^{+} \nu \bar{\nu}) = \begin{cases} (8.0 \pm 1.6) \cdot 10^{-11} & \text{Scanning} \\ (7.8 \pm 0.7) \cdot 10^{-11} & \text{Gaussian} , \end{cases}$$
 (13.33)

13.2.5
$$|V_{td}|$$
 from $K^+ \to \pi^+ \nu \bar{\nu}$

Once $Br(K^+ \to \pi^+ \nu \bar{\nu}) \equiv Br(K^+)$ is measured, $|V_{td}|$ can be extracted subject to various uncertainties:

$$\frac{\sigma(|V_{td}|)}{|V_{td}|} = \pm 0.04_{scale} \pm \frac{\sigma(|V_{cb}|)}{|V_{cb}|} \pm 0.7 \frac{\sigma(\bar{m}_c)}{\bar{m}_c} \pm 0.65 \frac{\sigma(Br(K^+))}{Br(K^+)} . \tag{13.34}$$

Taking $\sigma(|V_{cb}|) = 0.002$, $\sigma(\bar{m}_c) = 100 \,\text{MeV}$ and $\sigma(Br(K^+)) = 10\%$ and adding the errors in quadrature we find that $|V_{td}|$ can be determined with an accuracy of $\pm 10\%$ in the present example. This number is increased to $\pm 11\%$ once the uncertainties due to m_t , α_s and $|V_{ub}|/|V_{cb}|$ are taken into account. Clearly this determination can be improved although a determination of $|V_{td}|$ with an accuracy better than $\pm 5\%$ seems rather unrealistic.

13.2.6 Summary and Outlook

The accuracy of the Standard Model prediction for $Br(K^+ \to \pi^+ \nu \bar{\nu})$ has improved considerably during the last five years. Indeed in 1992 ranges like $(5-80) \cdot 10^{-11}$ could be found in the literature. This progress can be traced back to the improved values of m_t and $|V_{cb}|$ and to the inclusion of NLO QCD corrections which considerably reduced the scale uncertainties in the charm sector. I expect that further progress in the determination of CKM parameters via the standard analysis of section 10.7 could reduce the errors in (13.32) by at least a factor of two during the next five years. A numerical example is given in (13.33).

Now, what about the experimental status of this decay? Until August 97 the experimental lower bound on $Br(K^+ \to \pi^+ \nu \bar{\nu})$ was [243]: $Br(K^+ \to \pi^+ \nu \bar{\nu}) < 2.4 \cdot 10^{-9}$. One of the highlights of August 97 was the observation by BNL787 collaboration at Brookhaven [252] of one event consistent with the signature expected for this decay. The branching ratio:

$$Br(K^+ \to \pi^+ \nu \bar{\nu}) = (4.2 + 9.7 - 3.5) \cdot 10^{-10}$$
 (13.35)

has the central value by a factor of 4 above the Standard Model expectation but in view of large errors the result is compatible with the Standard Model. This new result implies that $|V_{td}|$ lies in the range $0.006 < |V_{td}| < 0.06$ which is substantially larger than the range from the standard analysis of section 10. The analysis of additional data on $K^+ \to \pi^+ \nu \bar{\nu}$ present on tape at BNL787 should narrow this range in the near future considerably. In view of the clean character of this decay a measurement of its branching ratio at the level of $2 \cdot 10^{-10}$ would signal the presence of physics beyond the Standard Model. The Standard Model sensitivity is expected to be reached at AGS around the year 2000 [244]. Also Fermilab with the Main Injector could measure this decay [245].

13.3 The Decay $K_{\rm L} \to \pi^0 \nu \bar{\nu}$

13.3.1 The effective Hamiltonian

The effective Hamiltonian for $K_{\rm L} \to \pi^0 \nu \bar{\nu}$ is given as follows:

$$\mathcal{H}_{\text{eff}} = \frac{G_{\text{F}}}{\sqrt{2}} \frac{\alpha}{2\pi \sin^2 \Theta_{\text{W}}} V_{ts}^* V_{td} X(x_t) (\bar{s}d)_{V-A} (\bar{\nu}\nu)_{V-A} + h.c., \qquad (13.36)$$

where the function $X(x_t)$, present already in $K^+ \to \pi^+ \nu \bar{\nu}$, includes NLO corrections and is given in (8.3).

As we will demonstrate shortly, $K_{\rm L} \to \pi^0 \nu \bar{\nu}$ proceeds in the Standard Model almost entirely through direct CP violation [246]. Consequently it is completely dominated by short-distance loop diagrams with top quark exchanges. The charm contribution can be fully neglected and the theoretical uncertainties present in $K^+ \to \pi^+ \nu \bar{\nu}$ due to m_c , μ_c and $\Lambda_{\overline{MS}}$ are absent here. Consequently the rare decay $K_{\rm L} \to \pi^0 \nu \bar{\nu}$ is even cleaner than $K^+ \to \pi^+ \nu \bar{\nu}$ and is very well suited for the determination of the Wolfenstein parameter η and ${\rm Im} \lambda_t$.

Before going into the details it is appropriate to clarify one point [247, 248]. It is usually stated in the literature that the decay $K_{\rm L} \to \pi^0 \nu \bar{\nu}$ is dominated by direct CP violation. Now the standard definition of the direct CP violation (see section 8 of [18]) requires the presence of strong phases which are completely negligible in $K_{\rm L} \to \pi^0 \nu \bar{\nu}$. Consequently the violation of CP symmetry in $K_{\rm L} \to \pi^0 \nu \bar{\nu}$ arises through the interference between $K^0 - \bar{K}^0$ mixing and the decay amplitude. This type of CP violation is often called mixing-induced CP violation. However, as already pointed out by Littenberg [246] and demonstrated explictly in a moment, the contribution of CP violation to $K_{\rm L} \to \pi^0 \nu \bar{\nu}$ via $K^0 - \bar{K}^0$ mixing alone is tiny. It gives $Br(K_{\rm L} \to \pi^0 \nu \bar{\nu}) \approx 5 \cdot 10^{-15}$. Consequently, in this sence, CP violation in $K_{\rm L} \to \pi^0 \nu \bar{\nu}$ with $Br(K_{\rm L} \to \pi^0 \nu \bar{\nu}) = \mathcal{O}(10^{-11})$ is a manifestation of CP violation in the decay and as such deserves the name of direct CP violation. In other words the difference in the magnitude of CP violation in $K_{\rm L} \to \pi^0 \nu \bar{\nu}$ at the expected level would rule out superweak scenarios. More details on this issue can be found in [247, 248, 249].

13.3.2 Deriving the Branching Ratio

Let us derive the basic formula for $Br(K_L \to \pi^0 \nu \bar{\nu})$ in a manner analogous to the one for $Br(K^+ \to \pi^+ \nu \bar{\nu})$. To this end we consider one neutrino flavour and define the complex function:

$$F = \frac{G_F}{\sqrt{2}} \frac{\alpha}{2\pi \sin^2 \Theta_W} V_{ts}^* V_{td} X(x_t). \tag{13.37}$$

Then the effective Hamiltonian in (13.36) can be written as

$$\mathcal{H}_{\text{eff}} = F(\bar{s}d)_{V-A}(\bar{\nu}\nu)_{V-A} + F^*(\bar{d}s)_{V-A}(\bar{\nu}\nu)_{V-A} . \tag{13.38}$$

Now, from (10.8) we have

$$K_L = \frac{1}{\sqrt{2}} [(1 + \bar{\varepsilon})K^0 + (1 - \bar{\varepsilon})\bar{K}^0]$$
 (13.39)

where we have neglected $|\bar{\varepsilon}|^2 \ll 1$. Thus the amplitude for $K_L \to \pi^0 \nu \bar{\nu}$ is given by

$$A(K_L \to \pi^0 \nu \bar{\nu}) = \frac{1}{\sqrt{2}} \left[F(1+\bar{\varepsilon}) \langle \pi^0 | (\bar{s}d)_{V-A} | K^0 \rangle + F^* (1-\bar{\varepsilon}) \langle \pi^0 | (\bar{d}s)_{V-A} | \bar{K}^0 \rangle \right] (\bar{\nu}\nu)_{V-A}.$$
(13.40)

Recalling

$$CP|K^{0}\rangle = -|\bar{K}^{0}\rangle, \qquad C|K^{0}\rangle = |\bar{K}^{0}\rangle$$
 (13.41)

we have

$$\langle \pi^0 | (\bar{d}s)_{V-A} | \bar{K}^0 \rangle = -\langle \pi^0 | (\bar{s}d)_{V-A} | K^0 \rangle,$$
 (13.42)

where the minus sign is crucial for the subsequent steps.

Thus we can write

$$A(K_L \to \pi^0 \nu \bar{\nu}) = \frac{1}{\sqrt{2}} \left[F(1 + \bar{\varepsilon}) - F^*(1 - \bar{\varepsilon}) \right] \langle \pi^0 | (\bar{s}d)_{V-A} | K^0 \rangle (\bar{\nu}\nu)_{V-A}. \tag{13.43}$$

Now the terms $\bar{\varepsilon}$ can be safely neglected in comparision with unity, which implies that the indirect CP violation (CP violation in the $K^0 - \bar{K}^0$ mixing) is negligible in this decay. We have then

$$F(1+\bar{\varepsilon}) - F^*(1-\bar{\varepsilon}) = \frac{G_F}{\sqrt{2}} \frac{\alpha}{\pi \sin^2 \Theta_W} \operatorname{Im}(V_{ts}^* V_{td}) \cdot X(x_t). \tag{13.44}$$

Consequently using isospin relation

$$\langle \pi^0 | (\bar{d}s)_{V-A} | \bar{K}^0 \rangle = \langle \pi^0 | (\bar{s}u)_{V-A} | K^+ \rangle$$
 (13.45)

together with (13.7) and taking into account the difference in the lifetimes of K_L and K^+ we have after summation over three neutrino flavours

$$\frac{Br(K_L \to \pi^0 \nu \bar{\nu})}{Br(K^+ \to \pi^0 e^+ \nu)} = 3 \frac{\tau(K_L)}{\tau(K^+)} \frac{\alpha^2}{|V_{us}|^2 2\pi^2 \sin^4 \Theta_W} \left(\text{Im} \lambda_t \cdot X(x_t) \right)^2$$
(13.46)

where $\lambda_t = V_{ts}^* V_{td}$.

13.3.3 Master Formulae for $Br(K_L \to \pi^0 \nu \bar{\nu})$

Using (13.46) we can write $Br(K_L \to \pi^0 \nu \bar{\nu})$ simply as follows

$$Br(K_{\rm L} \to \pi^0 \nu \bar{\nu}) = \kappa_{\rm L} \cdot \left(\frac{{\rm Im} \lambda_t}{\lambda^5} X(x_t)\right)^2$$
 (13.47)

$$\kappa_{\rm L} = \frac{r_{K_{\rm L}}}{r_{K^+}} \frac{\tau(K_{\rm L})}{\tau(K^+)} \kappa_+ = 1.80 \cdot 10^{-10}$$
(13.48)

with κ_+ given in (13.11) and $r_{K_L} = 0.944$ summarizing isospin breaking corrections in relating $K_L \to \pi^0 \nu \bar{\nu}$ to $K^+ \to \pi^0 e^+ \nu$ [241].

Using the Wolfenstein parametrization we can rewrite (13.47) as

$$Br(K_{\rm L} \to \pi^0 \nu \bar{\nu}) = 1.80 \cdot 10^{-10} \eta^2 A^4 X^2(x_t)$$
 (13.49)

or

$$Br(K_{\rm L} \to \pi^0 \nu \bar{\nu}) = 3.29 \cdot 10^{-5} \eta^2 |V_{cb}|^4 X^2(x_t)$$
 (13.50)

or using

$$X(x_t) = 0.65 \cdot x_t^{0.575} \tag{13.51}$$

as

$$Br(K_{\rm L} \to \pi^0 \nu \bar{\nu}) = 3.0 \cdot 10^{-11} \left[\frac{\eta}{0.39} \right]^2 \left[\frac{\overline{m}_{\rm t}(m_{\rm t})}{170 \text{ GeV}} \right]^{2.3} \left[\frac{|V_{cb}|}{0.040} \right]^4.$$
 (13.52)

The determination of η using $Br(K_L \to \pi^0 \nu \bar{\nu})$ requires the knowledge of V_{cb} and m_t . The very strong dependence on V_{cb} or A makes a precise prediction for this branching ratio difficult at present.

13.3.4 $|V_{cb}|$ and $\mathrm{Im}\lambda_t$ from $K_L \to \pi^0 \nu \bar{\nu}$

It was pointed out in [250] that the strong dependence of $Br(K_L \to \pi^0 \nu \bar{\nu})$ on V_{cb} , together with the clean nature of this decay, can be used to determine this element without any hadronic uncertainties. To this end η and m_t have to be known with sufficient precision in addition to $Br(K_L \to \pi^0 \nu \bar{\nu})$. Inverting (13.52) one finds

$$|V_{cb}| = 40.0 \cdot 10^{-3} \sqrt{\frac{0.39}{\eta}} \left[\frac{170 \,\text{GeV}}{\overline{m}_{t}(m_{t})} \right]^{0.575} \left[\frac{Br(K_{L} \to \pi^{0} \nu \bar{\nu})}{3 \cdot 10^{-11}} \right]^{1/4} . \tag{13.53}$$

We note that the weak dependence of V_{cb} on $Br(K_L \to \pi^0 \nu \bar{\nu})$ allows to achieve a high precision for this CKM element even when $Br(K_L \to \pi^0 \nu \bar{\nu})$ is known with only relatively moderate accuracy, e.g. 10–15%.

With η determined one day from CP asymmetries in B-decays and $m_{\rm t}$ measured very precisely at LHC and NLC, a measurement of $Br(K_{\rm L} \to \pi^0 \nu \bar{\nu})$ with an accuracy of 10% would determine $|V_{cb}|$ with an error of ± 0.001 . A comparision of this determination of $|V_{cb}|$ with the usual one in tree level B-decays would offer an excellent test of the Standard Model and in the case of discrepancy would signal physics beyond it.

On the other hand inverting (13.47) and using (13.51) one finds [249]:

$$\operatorname{Im} \lambda_t = 1.36 \cdot 10^{-4} \left[\frac{170 \,\text{GeV}}{\overline{m}_{\text{t}}(m_{\text{t}})} \right]^{1.15} \left[\frac{Br(K_{\text{L}} \to \pi^0 \nu \bar{\nu})}{3 \cdot 10^{-11}} \right]^{1/2} . \tag{13.54}$$

(13.54) offers the cleanest method to measure $\text{Im}\lambda_t$; even better than the CP asymmetries in B decays discussed briefly in the next section.

13.3.5 Numerical Analysis of $K_{\rm L} \to \pi^0 \nu \bar{\nu}$

The μ_t -uncertainties present in the function $X(x_t)$ have already been discussed in connection with $K^+ \to \pi^+ \nu \bar{\nu}$. After the inclusion of NLO corrections they are so small that they can be neglected for all practical purposes. At the level of $Br(K_L \to \pi^0 \nu \bar{\nu})$ the ambiguity in the choice of μ_t is reduced from $\pm 10\%$ (LO) down to $\pm 1\%$ (NLO), which considerably increases the predictive power of the theory. Varying μ_t according to (13.27) and using the input parameters as in the case of $K^+ \to \pi^+ \nu \bar{\nu}$ we find that the uncertainty in $Br(K_L \to \pi^0 \nu \bar{\nu})$

$$2.53 \cdot 10^{-11} \le Br(K_{\rm L} \to \pi^0 \nu \bar{\nu}) \le 3.08 \cdot 10^{-11} \tag{13.55}$$

present in the leading order is reduced to

$$2.64 \cdot 10^{-11} \le Br(K_{\rm L} \to \pi^0 \nu \bar{\nu}) \le 2.72 \cdot 10^{-11}$$
(13.56)

after including NLO corrections. This means that the theoretical uncertainty in the determination of η amounts to only $\pm 0.7\%$ which is safely negligible.

Using the input parameters of table 12 one finds [161]

$$Br(K_{\rm L} \to \pi^0 \nu \bar{\nu}) = \begin{cases} (2.8 \pm 1.7) \cdot 10^{-11} & \text{Scanning} \\ (2.6 \pm 0.9) \cdot 10^{-11} & \text{Gaussian} \end{cases}$$
(13.57)

where the error comes dominantly from the uncertainties in the CKM parameters. The corresponding analysis with the "future" input parameters gives

$$Br(K_{\rm L} \to \pi^0 \nu \bar{\nu}) = \begin{cases} (2.7 \pm 0.5) \cdot 10^{-11} & \text{Scanning} \\ (2.6 \pm 0.3) \cdot 10^{-11} & \text{Gaussian} \end{cases}$$
 (13.58)

13.3.6 Summary and Outlook

The accuracy of the Standard Model prediction for $Br(K_{\rm L} \to \pi^0 \nu \bar{\nu})$ has improved considerably during the last five years. Indeed in 1992 values as high as $15 \cdot 10^{-11}$ could be found in the literature. This progress can be traced back mainly to the improved values of $m_{\rm t}$ and $|V_{cb}|$ and to some extent to the inclusion of NLO QCD corrections. I expect that further progress in the determination of CKM parameters via the standard analysis of section 9 could reduce the errors in (13.57) by at least a factor of two during the next five years. A numerical example is given in (13.58).

The present upper bound on $Br(K_{\rm L} \to \pi^0 \nu \bar{\nu})$ from FNAL experiment E799 [251] is

$$Br(K_L \to \pi^0 \nu \bar{\nu}) < 1.8 \cdot 10^{-6}$$
 (13.59)

This is about five orders of magnitude above the Standard Model expectation (13.57).

How large could $Br(K_L \to \pi^0 \nu \bar{\nu})$ really be? As shown in [247] one can easily derive by means of isospin symmetry the following model independent bound:

$$Br(K_{\rm L} \to \pi^0 \nu \bar{\nu}) < 4.4 \cdot Br(K^+ \to \pi^+ \nu \bar{\nu})$$
 (13.60)

which through (13.35) gives

$$Br(K_{\rm L} \to \pi^0 \nu \bar{\nu}) < 6.1 \cdot 10^{-9}$$
 (13.61)

This bound is much stronger than the direct experimental bound in (13.59).

Now FNAL-E799 expects to reach the accuracy $\mathcal{O}(10^{-8})$ and a very interesting new experiment at Brookhaven (BNL E926) [244] expects to reach the single event sensitivity $2 \cdot 10^{-12}$ allowing a 10% measurement of the expected branching ratio. There are furthermore plans to measure this gold-plated decay with comparable sensitivity at Fermilab [254] and KEK [255].

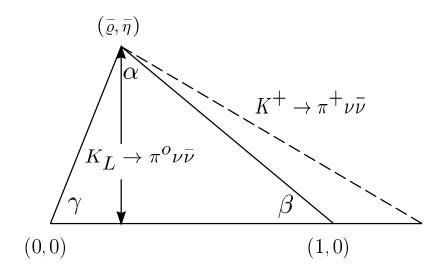


Figure 35: Unitarity triangle from $K \to \pi \nu \bar{\nu}$.

13.4 Unitarity Triangle and $\sin 2\beta$ from $K \to \pi \nu \bar{\nu}$

The measurement of $Br(K^+ \to \pi^+ \nu \bar{\nu})$ and $Br(K_L \to \pi^0 \nu \bar{\nu})$ can determine the unitarity triangle completely, (see fig. 35), provided m_t and V_{cb} are known [85]. Using these two branching ratios simultaneously allows to eliminate $|V_{ub}/V_{cb}|$ from the analysis which removes a considerable uncertainty. Indeed it is evident from (13.10) and (13.47) that, given $Br(K^+ \to \pi^+ \nu \bar{\nu})$ and $Br(K_L \to \pi^0 \nu \bar{\nu})$, one can extract both $Im\lambda_t$ and $Re\lambda_t$. One finds [260, 17]

$$\operatorname{Im} \lambda_t = \lambda^5 \frac{\sqrt{B_2}}{X(x_t)} \qquad \operatorname{Re} \lambda_t = -\lambda^5 \frac{\frac{\operatorname{Re} \lambda_c}{\lambda} P_0(X) + \sqrt{B_1 - B_2}}{X(x_t)}, \qquad (13.62)$$

where we have defined the "reduced" branching ratios

$$B_1 = \frac{Br(K^+ \to \pi^+ \nu \bar{\nu})}{4.11 \cdot 10^{-11}} \qquad B_2 = \frac{Br(K_L \to \pi^0 \nu \bar{\nu})}{1.80 \cdot 10^{-10}}.$$
 (13.63)

Using next the expressions for $\text{Im}\lambda_t$, $\text{Re}\lambda_t$ and $\text{Re}\lambda_c$ given in (2.37)–(2.39) we find

$$\bar{\varrho} = 1 + \frac{P_0(X) - \sqrt{\sigma(B_1 - B_2)}}{A^2 X(x_t)}, \qquad \bar{\eta} = \frac{\sqrt{B_2}}{\sqrt{\sigma} A^2 X(x_t)}$$
 (13.64)

with σ defined in (13.15). An exact treatment of the CKM matrix shows that the formulae (13.64) are rather precise [260]. The error in $\bar{\eta}$ is below 0.1% and $\bar{\varrho}$ may deviate from the exact expression by at most $\Delta \bar{\varrho} = 0.02$ with essentially negligible error for $0 \leq \bar{\varrho} \leq 0.25$.

Using (13.64) one finds subsequently [260]

$$r_s = r_s(B_1, B_2) \equiv \frac{1 - \bar{\varrho}}{\bar{\eta}} = \cot \beta, \qquad \sin 2\beta = \frac{2r_s}{1 + r_s^2}$$
 (13.65)

with

$$r_s(B_1, B_2) = \sqrt{\sigma} \frac{\sqrt{\sigma(B_1 - B_2)} - P_0(X)}{\sqrt{B_2}}$$
 (13.66)

Thus within the approximation of (13.64) $\sin 2\beta$ is independent of V_{cb} (or A) and m_t . An exact treatment of the CKM matrix confirms this finding to a high accuracy. The dependence on V_{cb} and m_t enters only at order $\mathcal{O}(\lambda^2)$ and as a numerical analysis shows this dependence can be fully neglected.

It should be stressed that $\sin 2\beta$ determined this way depends only on two measurable branching ratios and on the function $P_0(X)$ which is completely calculable in perturbation theory. Consequently this determination is free from any hadronic uncertainties and its accuracy can be estimated with a high degree of confidence.

An extensive numerical analysis of the formulae above has been presented in [260, 249]. We summarize the results of the latter paper. Assuming that the branching ratios are known to within $\pm 10\%$

$$Br(K^+ \to \pi^+ \nu \bar{\nu}) = (1.0 \pm 0.1) \cdot 10^{-10}, \qquad Br(K_L \to \pi^0 \nu \bar{\nu}) = (3.0 \pm 0.30) \cdot 10^{-11}$$
 (13.67)

and choosing

$$m_{\rm t} = (170 \pm 3) \,\text{GeV}, \quad P_0(X) = 0.40 \pm 0.06, \quad |V_{cb}| = 0.040 \pm 0.002,$$
 (13.68)

one finds the results given in the second column of table 31. In the third column the results for the choice $|V_{cb}| = 0.040 \pm 0.001$ are shown. It should be remarked that the quoted errors for the input parameter are quite reasonable if one keeps in mind that it will take five years to achieve the accuracy assumed in (13.67). The error in $P_0(X)$ in (13.68) results from the errors (see table 30 and (13.27)) in $\Lambda_{\overline{\rm MS}}^{(4)}$, m_c and μ_c added quadratically. Doubling the error

Table 31: Illustrative example of the determination of CKM parameters from $K \to \pi \nu \bar{\nu}$ for two choices of V_{cb} and other parameters given in the text.

	$ V_{cb} = 0.040 \pm 0.002$	$ V_{cb} = 0.040 \pm 0.001.$
$ V_{td} /10^{-3}$	10.3 ± 1.1	10.3 ± 0.9
$ V_{ub}/V_{cb} $	0.089 ± 0.017	0.089 ± 0.011
$ar{arrho}$	-0.10 ± 0.16	-0.10 ± 0.12
$ar{\eta}$	0.38 ± 0.04	0.38 ± 0.03
$\sin 2\beta$	0.62 ± 0.05	0.62 ± 0.05
$\mathrm{Im}\lambda_t/10^{-4}$	1.37 ± 0.07	1.37 ± 0.07

in m_c would give $P_0(X) = 0.40 \pm 0.09$ and an increase of the errors in $|V_{td}|/10^{-3}$, $\bar{\varrho}$ and $\sin 2\beta$ by at most ± 0.2 , ± 0.02 and ± 0.01 respectively, without any changes in $\bar{\eta}$ and $\text{Im}\lambda_t$.

We observe that respectable determinations of all considered quantities except for $\bar{\varrho}$ can be obtained. Of particular interest are the accurate determinations of $\sin 2\beta$ and of $\text{Im}\lambda_t$. The latter quantity as seen in (13.62) can be obtained from $K_L \to \pi^0 \nu \bar{\nu}$ alone and does not require knowledge of V_{cb} .

As pointed out in [249], $K_L \to \pi^0 \nu \bar{\nu}$ appears to be the best decay to measure $\text{Im} \lambda_t$; even better than the CP asymmetries in B decays discussed in the next section. The importance of measuring accurately $\text{Im} \lambda_t$ is evident. It plays a central role in the phenomenology of CP violation in K decays and is furthermore equivalent to the Jarlskog parameter J_{CP} [261], the invariant measure of CP violation in the Standard Model, $J_{\text{CP}} = \lambda (1 - \lambda^2/2) \text{Im} \lambda_t$.

The accuracy to which $\sin 2\beta$ can be obtained from $K \to \pi \nu \bar{\nu}$ is, in the example discussed above, comparable to the one expected in determining $\sin 2\beta$ from CP asymmetries in B decays prior to LHC experiments. In this case $\sin 2\beta$ is determined best by measuring CP violation in $B_d \to J/\psi K_S$. Using the formula for the corresponding time-integrated CP asymmetry one finds an interesting connection between rare K decays and B physics [260]

$$\frac{2r_s(B_1, B_2)}{1 + r_s^2(B_1, B_2)} = -a_{\rm CP}(B_d \to J/\psi K_{\rm S}) \frac{1 + x_d^2}{x_d}$$
(13.69)

which must be satisfied in the Standard Model. We stress that except for $P_0(X)$ given in table 30 all quantities in (13.69) can be directly measured in experiment and that this relationship is essentially independent of m_t and V_{cb} . Due to very small theoretical uncertainties in (13.69), this relation is particularly suited for tests of CP violation in the Standard Model and offers a powerful tool to probe the physics beyond it. Further comparison between the potential of $K \to \pi \nu \bar{\nu}$ and CP asymmetries in B decays will be given in section 14.

Table 32: Illustrative example of the determination of CKM parameters from $K \to \pi \nu \bar{\nu}$ and from the standard analysis of the unitarity triangle.

	$\sigma(V_{cb}) = \pm 0.002$	$\sigma(V_{cb}) = \pm 0.001$	Present	Future
$\sigma(V_{td})$	±10%	±9%	±24%	±7%
$\sigma(ar{arrho})$	± 0.16	± 0.12	± 0.32	± 0.08
$\sigma(ar{\eta})$	± 0.04	± 0.03	± 0.12	± 0.03
$\sigma(\sin 2\beta)$	± 0.05	±0.05	± 0.22	± 0.05
$\sigma(\mathrm{Im}\lambda_t)$	±5%	±5%	±33%	±8%

Finally we compare the determination of the unitarity triangle by means of $K \to \pi \nu \bar{\nu}$ with the one by means of the standard analysis of the unitarity triangle. The results obtained from $K \to \pi \nu \bar{\nu}$ corresponding to table 31 are given in the second and the third column of table 32. In the fourth and fifth column the corresponding results of the standard analysis of the unitarity triangle are shown. We observe that a considerable progress, when compared with the present analysis of the unitarity triangle, can be achieved through the measurements of $K \to \pi \nu \bar{\nu}$ decays.

13.5 $K \to \pi \nu \bar{\nu}$ Beyond the Standard Model

In view of the very clean character of $K \to \pi \nu \bar{\nu}$, these decays are very suitable for the study of new physics effects. One example is the relation (13.69). Recently several extensive analyses of supersymmetry effects in general supersymmetric models have been presented in [247, 259, 258] where further references can be found. In the MSSM these effects are found to be very small but in certain more general scenarios of supersymmetry enhancements or suppressions of $Br(K^+ \to \pi^+ \nu \bar{\nu})$ and $Br(K_L \to \pi^0 \nu \bar{\nu})$ by factors 2-3 cannot be excluded. Model independent studies of these decays can be found in [247, 258]. The corresponding analyses in various no–supersymmetric extensions of the Standard Model are listed in [256]. In particular, enhancement of $Br(K_L \to \pi^0 \nu \bar{\nu})$ by 1–2 orders of magnitude above the Standard Model expectations is according to [257] still possible in four-generation models.

13.6 The Decays $B \to X_{s,d} \nu \bar{\nu}$

13.6.1 Effective Hamiltonian

The decays $B \to X_{s,d}\nu\bar{\nu}$ are the theoretically cleanest decays in the field of rare B-decays. They are dominated by the same Z^0 -penguin and box diagrams involving top quark exchanges which we encountered already in the case of $K^+ \to \pi^+ \nu \bar{\nu}$ and $K_L \to \pi^0 \nu \bar{\nu}$ except for the appropriate change of the external quark flavours. Since the change of external quark flavours has no impact on the m_t dependence, the latter is fully described by the function $X(x_t)$ in (8.3) which includes the NLO corrections [98]. The charm contribution as discussed at the beginning of this section is fully neglegible here and the resulting effective Hamiltonian is very similar to the one for $K_L \to \pi^0 \nu \bar{\nu}$ given in (13.36). For the decay $B \to X_s \nu \bar{\nu}$ it reads

$$\mathcal{H}_{\text{eff}} = \frac{G_{\text{F}}}{\sqrt{2}} \frac{\alpha}{2\pi \sin^2 \Theta_{\text{W}}} V_{tb}^* V_{ts} X(x_t) (\bar{b}s)_{V-A} (\bar{\nu}\nu)_{V-A} + h.c.$$
 (13.70)

with s replaced by d in the case of $B \to X_d \nu \bar{\nu}$.

The theoretical uncertainties related to the renormalization scale dependence are as in $K_{\rm L} \to \pi^0 \nu \bar{\nu}$ and can be essentially neglected. The same applies to long distance contributions considered in [221]. On the other hand $B \to X_{s,d} \nu \bar{\nu}$ are CP conserving and consequently the relevant branching ratios are sensitive to $|V_{td}|$ and $|V_{ts}|$ as opposed to $Br(K_{\rm L} \to \pi^0 \nu \bar{\nu})$ in which ${\rm Im}(V_{ts}^* V_{td})$ enters. As we will stress below the measurement of both $B \to X_s \nu \bar{\nu}$ and $B \to X_d \nu \bar{\nu}$ offers the cleanest determination of the ratio $|V_{td}|/|V_{ts}|$.

13.6.2 The Branching Ratios

The calculation of the branching fractions for $B \to X_{s,d}\nu\bar{\nu}$ can be done similarly to $B \to X_s\gamma$ in the spectator model corrected for short distance QCD effects. Normalizing as in these latter decays to $Br(B \to X_c e\bar{\nu})$ and summing over three neutrino flavours one finds

$$\frac{Br(B \to X_s \nu \bar{\nu})}{Br(B \to X_c e \bar{\nu})} = \frac{3\alpha^2}{4\pi^2 \sin^4 \Theta_W} \frac{|V_{ts}|^2}{|V_{cb}|^2} \frac{X^2(x_t)}{f(z)} \frac{\bar{\eta}}{\kappa(z)}.$$
 (13.71)

Here f(z) is the phase-space factor for $B \to X_c e \bar{\nu}$ defined already in (12.31) and $\kappa(z)$ is the corresponding QCD correction given in (12.42). The factor $\bar{\eta}$ represents the QCD correction to the matrix element of the $b \to s\nu\bar{\nu}$ transition due to virtual and bremsstrahlung contributions and is given by the well known expression

$$\bar{\eta} = \kappa(0) = 1 + \frac{2\alpha_s(m_b)}{3\pi} \left(\frac{25}{4} - \pi^2\right) \approx 0.83.$$
 (13.72)

In the case of $B \to X_d \nu \bar{\nu}$ one has to replace V_{ts} by V_{td} which results in a decrease of the branching ratio by roughly an order of magnitude.

It should be noted that $Br(B \to X_s \nu \bar{\nu})$ as given in (13.71) is in view of $|V_{ts}/V_{cb}|^2 \approx 0.95 \pm 0.03$ essentially independent of the CKM parameters and the main uncertainty resides in the value of m_t which is already rather precisely known. Setting $Br(B \to X_c e \bar{\nu}) = 10.4\%$, f(z) = 0.54, $\kappa(z) = 0.88$ and using the values in (13.12) we have

$$Br(B \to X_s \nu \bar{\nu}) = 3.7 \cdot 10^{-5} \frac{|V_{ts}|^2}{|V_{cb}|^2} \left[\frac{\overline{m}_t(m_t)}{170 \text{ GeV}} \right]^{2.30}$$
 (13.73)

Taking next, in accordance with (12.44), $\kappa(z) = 0.88$, $f(z) = 0.54 \pm 0.04$ and $Br(B \rightarrow X_c e \bar{\nu}) = (10.4 \pm 0.4)\%$ and using the input parameters of table 12 one finds [161]

$$Br(B \to X_s \nu \bar{\nu}) = \begin{cases} (3.4 \pm 0.7) \cdot 10^{-5} & \text{Scanning} \\ (3.2 \pm 0.4) \cdot 10^{-5} & \text{Gaussian} . \end{cases}$$
 (13.74)

What about the data? One of the high-lights of FCNC-1996 was the upper bound:

$$Br(B \to X_s \nu \bar{\nu}) < 7.7 \cdot 10^{-4} \quad (90\% \text{ C.L.})$$
 (13.75)

obtained for the first time by ALEPH [262]. This is only a factor of 20 above the Standard Model expectation. Even if the actual measurement of this decay is extremly difficult, all efforts should be made to measure it. One should also make attempts to measure $Br(B \to X_d \nu \bar{\nu})$. Indeed

$$\frac{Br(B \to X_d \nu \bar{\nu})}{Br(B \to X_s \nu \bar{\nu})} = \frac{|V_{td}|^2}{|V_{ts}|^2}$$
(13.76)

offers the cleanest direct determination of $|V_{td}|/|V_{ts}|$ as all uncertainties related to m_t , f(z) and $Br(B \to X_c e \bar{\nu})$ cancel out.

13.7 The Decays $B_{s,d} \rightarrow l^+ l^-$

13.7.1 The Effective Hamiltonian

The decays $B_{s,d} \to l^+ l^-$ are after $B \to X_{s,d} \nu \bar{\nu}$ the theoretically cleanest decays in the field of rare B-decays. They are dominated by the Z^0 -penguin and box diagrams involving top quark exchanges which we encountered already in the case of $B \to X_{s,d} \nu \bar{\nu}$ except that due to charged leptons in the final state the charge flow in the internal lepton line present in the box diagram is reversed. This results in a different m_t dependence summarized by the function $Y(x_t)$, the NLO generalization [98] of the function $Y_0(x_t)$ given in (3.27). The charm contributions as discussed at the beginning of this section are fully negligible here and the resulting effective Hamiltonian is given for $B_s \to l^+ l^-$ as follows:

$$\mathcal{H}_{\text{eff}} = -\frac{G_{\text{F}}}{\sqrt{2}} \frac{\alpha}{2\pi \sin^2 \Theta_{\text{W}}} V_{tb}^* V_{ts} Y(x_t) (\bar{b}s)_{V-A} (\bar{l}l)_{V-A} + h.c.$$
 (13.77)

with s replaced by d in the case of $B_d \to l^+l^-$.

The function Y(x) is given by

$$Y(x_t) = Y_0(x_t) + \frac{\alpha_s}{4\pi} Y_1(x_t), \qquad (13.78)$$

where $Y_0(x_t)$ can be found in (3.27) and $Y_1(x_t)$ in (8.133). The leftover μ_t -dependence in $Y(x_t)$ is tiny and amounts to an uncertainty of $\pm 1\%$ at the level of the branching ratio. We

recall that $Y(x_t)$ can also be written as

$$Y(x_t) = \eta_Y \cdot Y_0(x_t), \qquad \eta_Y = 1.026 \pm 0.006,$$
 (13.79)

where η_Y summarizes the NLO corrections. With $m_t \equiv \overline{m}_t(m_t)$ this QCD factor depends only very weakly on m_t . The range in (13.79) corresponds to 150 GeV $\leq m_t \leq$ 190 GeV. The dependence on $\Lambda_{\overline{MS}}$ can be neglected.

13.7.2 The Branching Ratios

The branching ratio for $B_s \to l^+ l^-$ is given by [98]

$$Br(B_s \to l^+ l^-) = \tau(B_s) \frac{G_F^2}{\pi} \left(\frac{\alpha}{4\pi \sin^2 \Theta_W} \right)^2 F_{B_s}^2 m_l^2 m_{B_s} \sqrt{1 - 4 \frac{m_l^2}{m_{B_s}^2}} |V_{tb}^* V_{ts}|^2 Y^2(x_t) \quad (13.80)$$

where B_s denotes the flavour eigenstate (bs) and F_{B_s} is the corresponding decay constant. Using (13.12), (13.79) and (3.31) we find in the case of $B_s \to \mu^+ \mu^-$

$$Br(B_s \to \mu^+ \mu^-) = 3.5 \cdot 10^{-9} \left[\frac{\tau(B_s)}{1.6 \text{ps}} \right] \left[\frac{F_{B_s}}{210 \,\text{MeV}} \right]^2 \left[\frac{|V_{ts}|}{0.040} \right]^2 \left[\frac{\overline{m}_t(m_t)}{170 \,\text{GeV}} \right]^{3.12}.$$
 (13.81)

The main uncertainty in this branching ratio results from the uncertainty in F_{B_s} . Using the input parameters of table 12 together with $\tau(B_s) = 1.6$ ps and $F_{B_s} = (210 \pm 30) \,\text{MeV}$ one finds [161]

$$Br(B_s \to \mu^+ \mu^-) = \begin{cases} (3.6 \pm 1.9) \cdot 10^{-9} & \text{Scanning} \\ (3.4 \pm 1.2) \cdot 10^{-9} & \text{Gaussian.} \end{cases}$$
 (13.82)

For $B_d \to \mu^+ \mu^-$ a similar formula holds with obvious replacements of labels $(s \to d)$. Provided the decay constants F_{B_s} and F_{B_d} will have been calculated reliably by non-perturbative methods or measured in leading leptonic decays one day, the rare processes $B_s \to \mu^+ \mu^-$ and $B_d \to \mu^+ \mu^-$ should offer clean determinations of $|V_{ts}|$ and $|V_{td}|$. In particular the ratio

$$\frac{Br(B_d \to \mu^+ \mu^-)}{Br(B_s \to \mu^+ \mu^-)} = \frac{\tau(B_d)}{\tau(B_s)} \frac{m_{B_d}}{m_{B_s}} \frac{F_{B_d}^2}{F_{B_s}^2} \frac{|V_{td}|^2}{|V_{ts}|^2}$$
(13.83)

having smaller theoretical uncertainties than the separate branching ratios should offer a useful measurement of $|V_{td}|/|V_{ts}|$. Since $Br(B_d \to \mu^+\mu^-) = \mathcal{O}(10^{-10})$ this is, however, a very difficult task. For $B_s \to \tau^+\tau^-$ and $B_s \to e^+e^-$ one expects branching ratios $\mathcal{O}(10^{-6})$ and $\mathcal{O}(10^{-13})$, respectively, with the corresponding branching ratios for B_d -decays by one order of magnitude smaller.

We should also remark that in conjunction with a future measurement of x_s , the branching ratio $Br(B_s \to \mu \bar{\mu})$ could help to determine the non-perturbative parameter B_{B_s} and consequently allow a test of existing non-perturbative methods [263]:

$$B_{B_s} = \left[\frac{x_s}{22.1}\right] \left[\frac{\overline{m}_t(m_t)}{170 \text{ GeV}}\right]^{1.6} \left[\frac{4.2 \cdot 10^{-9}}{Br(B_s \to \mu\bar{\mu})}\right]. \tag{13.84}$$

This test could be of course affected by new physics contributions.

13.7.3 **Outlook**

What about the data?

The bounds on $B_{s,d} \to l\bar{l}$ are still many orders of magnitude away from Standard Model expectations. The best bounds come from CDF [264]. One has:

$$Br(B_s \to \mu^+ \mu^-) \le 2.6 \cdot 10^{-6}$$
 (95%C.L.) (13.85)

and $Br(B_d \to \mu^+ \mu^-) \le 8.6 \cdot 10^{-7}$. CDF should reach in Run II the sensitivity of $1 \cdot 10^{-8}$ and $4 \cdot 10^{-8}$ for $B_d \to \mu \bar{\mu}$ and $B_s \to \mu \bar{\mu}$, respectively. It is hoped that these decays will be observed at LHC-B. The experimental status of $B \to \tau^+ \tau^-$ and its usefulness in tests of the physics beyond the Standard Model is discussed in [265].

13.8 Higher Order Electroweak Effects in Rare Decays

Until now we have considered various penguin and box diagrams contributing to rare decays together with QCD corrections. In none of these contributions the role of the neutral Higgs boson H^0 has been felt. Since the couplings of H^0 to fermions are proportional to fermion masses, contributions of internal H^0 are very strongly suppressed unless H^0 couples at both ends of its propagator to the top. This situation appears first at two-loop level in electroweak interactions. Examples of such diagrams can be constructed from diagrams (a)–(c) in fig. 19 by replacing there the gluon propagator by the H^0 -propagator. Even more important diagrams are obtained by replacing W^{\pm} and the gluon by the fictitious ϕ^{\pm} Higgs exchanges with the appropriate change in internal fermion propagators.

Once the higher order electroweak contributions are considered and one recalls the extensive precision electroweak studies at Z^0 -factories, an obvious question arises. What about the ambiguities in rare meson decays stemming from various possible definitions of electroweak parameters? We have seen in this section that the branching ratios $Br(K_L \to \pi^0 \nu \bar{\nu})$, $Br(K^+ \to \pi^+ \nu \bar{\nu})$, $Br(B \to X_{d,s} \nu \bar{\nu})$ and $Br(B \to l^- l^+)$ all had the following generic structure

$$Br \sim \frac{G_F^2 \alpha^2(M_Z)}{\sin^4 \Theta_W} [F(x_t)]^2,$$
 (13.86)

where we have suppressed the charm contribution to $Br(K^+ \to \pi^+ \nu \bar{\nu})$.

Now, there are several definitions of $\sin^2 \Theta_W$. For instance, $\sin^2 \Theta_W = 0.224$ in the on-shell scheme, whereas the effective $\sin^2 \hat{\Theta}_W|_{\text{eff}} = 0.230$. These two choices result in branching ratios which differ by 5.6% to be compared with uncertainties of 1-2% from QCD after NLO corrections have been taken into account. There is of course also the question of the scale

in α . This is analogous to the recent discussion of two–loop electroweak effects in $B \to X_s \gamma$ presented in section 12.4 and the related issue of $\alpha(\mu)$ there.

Clearly, in order to reduce such uncertainties, one has to consider two-loop electroweak contributions to the rare decays in question. Such an analysis has been performed in [266] in the large m_t -limit. Schematically the formula (13.86) reads now

$$Br \sim \frac{G_F^2 \alpha^2(M_Z)}{\sin^4 \Theta_W} \left[F(x_t) + cG_F m_t^2 \frac{m_t^2}{M_W^2} \right]$$
 (13.87)

where the second term represents two-loop electroweak corrections for large m_t . The scheme dependence of this term cancels in the large m_t limit, the scheme dependence of $\sin^2 \Theta_W$. Moreover the proper scale in α turns out to be M_Z as anticipated (13.86) and in all our calculations before. Evidently the decays in question being governed by short distance penguin and box contributions involve $\alpha(M_Z)$, as opposed to $B \to X_s \gamma$, where due to the on-shell photons $\alpha(m_e)$ matters.

The large $m_{\rm t}$ estimate of the full two-loop electroweak corrections can be only trusted within a factor of two. Yet the residual parameter uncertainties after the inclusion of these corrections turns out to be less than 2%, which is well below the experimental sensitivity in the forseeable future. Similarly for $\sin^2 \hat{\Theta}_W|_{\rm eff} = 0.230$, used previously in our numerical estimates, there is an enhancement of various branching ratios by 1-2% which can also be neglected. It should be stressed that all these effects cancel in the determination of $\sin 2\beta$ from $K \to \pi \nu \bar{\nu}$. Further details can be found in [266].

14 Future Visions

14.1 Preliminaries

Let us next have a look in the future and ask the question how well various parameters of the Standard Model can be determined provided the cleanest decays have been measured to some respectable precision. We have made already such an exercise in section 13.4 using the decays $K_{\rm L} \to \pi^0 \nu \bar{\nu}$ and $K^+ \to \pi^+ \nu \bar{\nu}$. Now we want to make an analogous analysis using CP-asymmetries in *B*-decays. This way we will be able to compare the potentials of the CP asymmetries in determining the parameters of the Standard Model with those of the cleanest rare K-decays: $K_{\rm L} \to \pi^0 \nu \bar{\nu}$ and $K^+ \to \pi^+ \nu \bar{\nu}$. This section is based on [34, 250, 249, 263].

14.2 CP-Asymmetries in B-Decays

CP violation in B-decays is certainly one of the most important targets of B-factories and of dedicated B-experiments at hadron facilities. It is well known that CP violating effects

are expected to occur in a large number of channels at a level attainable at forthcoming experiments. Moreover there exist channels which offer the determination of CKM phases essentially without any hadronic uncertainties. Since extensive reviews on CP violation in B decays can be found in the literature [18, 267, 136], let me concentrate only on the most important points.

The classic determination of α by means of the time dependent CP asymmetry in the decay $B_d^0 \to \pi^+\pi^-$ is affected by the "QCD penguin pollution" which has to be taken care of in order to extract α . The recent CLEO results for penguin dominated decays indicate that this pollution could be substantial as stressed recently in particular in [105]. The most popular strategy to deal with this "penguin problem" is the isospin analysis of Gronau and London [268]. It requires however the measurement of $Br(B^0 \to \pi^0\pi^0)$ which is expected to be below 10^{-6} : a very difficult experimental task. For this reason several, rather involved, strategies [269] have been proposed which avoid the use of $B_d \to \pi^0\pi^0$ in conjunction with $a_{CP}(\pi^+\pi^-,t)$. They are reviewed in [18]. It is to be seen which of these methods will eventually allow us to measure α with a respectable precision. It is however clear that the determination of this angle is a real challenge for both theorists and experimentalists.

The CP-asymmetry in the decay $B_d \to \psi K_S$ allows in the Standard Model a direct measurement of the angle β in the unitarity triangle without any theoretical uncertainties [270]. Of considerable interest [136, 271] is also the pure penguin decay $B_d \to \phi K_S$, which is expected to be sensitive to physics beyond the Standard Model. Comparision of β extracted from $B_d \to \phi K_S$ with the one from $B_d \to \psi K_S$ should be important in this respect. An analogue of $B_d \to \psi K_S$ in B_s -decays is $B_s \to \psi \phi$. The CP asymmetry measures here η [263] in the Wolfenstein parametrization. It is very small, however, and this fact makes it a good place to look for the physics beyond the Standard Model. In particular the CP violation in $B_s^0 - \bar{B}_s^0$ mixing from new sources beyond the Standard Model should be probed in this decay.

The two theoretically cleanest methods for the determination of γ are: i) the full time dependent analysis of $B_s \to D_s^+ K^-$ and $\bar{B}_s \to D_s^- K^+$ [272] and ii) the well known triangle construction due to Gronau and Wyler [273] which uses six decay rates $B^\pm \to D_{CP}^0 K^\pm$, $B^+ \to D^0 K^+$, $\bar{D}^0 K^+$ and $B^- \to D^0 K^-$. Both methods are unaffected by penguin contributions. The first method is experimentally very challenging because of the expected large $B_s^0 - \bar{B}_s^0$ mixing. The second method is problematic because of the small branching ratios of the colour supressed channel $B^+ \to D^0 K^+$ and its charge conjugate, giving a rather squashed triangle and thereby making the extraction of γ very difficult. Variants of the latter method which could be more promising have been proposed in [274, 275]. It appears that these methods will give useful results at later stages of CP-B investigations. In particular the

first method will be feasible only at LHC-B.

All this has been known already for some time and is well documented in the literature [18, 136]. Let us now be more explicit on the most recent developments which deal with the extraction of the angle γ from the decays $B_d^0 \to \pi^- K^+$, $B^+ \to \pi^+ K^0$ and their charge conjugates [276]–[280]. These modes, which have recently been observed by the CLEO collaboration [281], should allow us to obtain direct information on γ at future B-factories (BaBar, BELLE, CLEO III) (for interesting feasibility studies, see [278, 279, 282]). At present, there are only experimental results available for the combined branching ratios of these modes, i.e. averaged over decay and its charge conjugate, suffering from large hadronic uncertainties.

In order to determine the CKM angle γ by using the strategy proposed in [276] (see also [278]), the separate branching ratios for $B_d^0 \to \pi^- K^+$, $B^+ \to \pi^+ K^0$ and their charge conjugates are needed, i.e. the combined branching ratios are not sufficient, and an additional input is required to fix the magnitude of a certain decay amplitude T, which is usually referred to as a "tree" amplitude. Using arguments based on the factorization discussed in section 9, one expects that a future theoretical uncertainty of |T| as small as $\mathcal{O}(10\%)$ may be achievable [278, 279]. Unfortunately detailed studies show, that the properly defined amplitude T is actually not just a colour-allowed "tree" amplitude, where factorization may work reasonably well [283]. It receives also contributions from penguin and annihilation topologies due to certain rescattering effects [280, 284] and consequently the expectations in [278, 279] appear too optimistic. In any case, some model dependence enters in the extracted value of γ by means of these decays.

In this context an interesting method for constraining γ , which does not suffer from a model dependence related to |T|, is the method of Fleischer and Mannel [277]. This method uses only the combined rates for $B^{\pm} \to \pi^{\pm} K$ and $B_d \to \pi^{\mp} K^{\pm}$. Assuming that the final state interactions and electroweak penguin contributions are small, one finds the bound:

$$\sin^2 \gamma \le \frac{Br(B_d \to \pi^{\mp} K^{\pm})}{Br(B^{\pm} \to \pi^{\pm} K)} \equiv R . \tag{14.1}$$

The Fleischer-Mannel bound is of particular interest because the most recent CLEO data give $R=0.65\pm0.40$ [281]. If true, the FM-bound with R<1 would exclude the region around $\bar{\varrho}=0$ in the $(\bar{\varrho},\bar{\eta})$ space putting the " $\gamma=90^\circ$ club" [290] into serious difficulties. It should be stressed that excluding the region around $\bar{\varrho}=0$ would have a profound impact on the unitarity triangle dividing the allowed region for its apex into well separated regions with $\bar{\varrho}<0$ and $\bar{\varrho}>0$. The former could then probably be eliminated by improving the lower bound on ΔM_s leaving only a small allowed area with $\bar{\varrho}>0$. More details on the implications of the FM-bound can be found in [277, 127, 159].

The crucial questions then are, whether R is indeed smaller than unity and whether the

assumptions used to obtain the FM bound can be justified. The first question will hopefully be answered by CLEO and future B factories. Here we concentrate on the second question. Indeed, the theoretical accuracy of the FM bound on γ is limited by rescattering processes of the kind $B^+ \to \{\pi^0 K^+, \pi^0 K^{*+}, \rho^0 K^{*+}, \dots\} \to \pi^+ K^0$ [285]–[288] (for earlier references, see [289]), and by contributions from electroweak penguins [278, 286, 291], which led to considerable interest in the recent literature.

In order to gain some insight into this issue, a completely general parametrization of the $B^+ \to \pi^+ K^0$ and $B_d^0 \to \pi^- K^+$ decay amplitudes was presented in [280], relying only on the isospin symmetry of strong interactions and the phase structure of the Standard Model. This parametrization leads to the following transparent expression for the minimal value of R:

$$R_{\min} = \kappa \sin^2 \gamma + \frac{1}{\kappa} \left(\frac{A_0}{2 \sin \gamma} \right)^2, \tag{14.2}$$

where the "pseudo-asymmetry" A_0 is defined by

$$A_0 \equiv \frac{Br(B_d^0 \to \pi^- K^+) - Br(\overline{B_d^0} \to \pi^+ K^-)}{Br(B^+ \to \pi^+ K^0) + Br(B^- \to \pi^- \overline{K^0})} = A_{\rm CP}(B_d \to \pi^\mp K^\pm) R . \tag{14.3}$$

Rescattering and electroweak penguin effects are included through the parameter κ , which is given by

$$\kappa = \frac{1}{w^2} \left[1 + 2 \left(\epsilon \, w \right) \cos \Delta + \left(\epsilon \, w \right)^2 \right] \tag{14.4}$$

with

$$w \equiv \sqrt{1 + 2\rho \cos\theta \cos\gamma + \rho^2}. \tag{14.5}$$

The parameters ρ and ϵ measure the "strengths" of the rescattering processes and electroweak penguin contributions, respectively, and θ and Δ are CP-conserving strong phases. Simple model estimates typically give values of ρ and ϵ at the level of 1%. However, in a recent attempt to evaluate rescattering processes such as $B^+ \to \{\pi^0 K^+\} \to \pi^+ K^0$, it is found that ρ may be as large as $\mathcal{O}(10\%)$ [287]. A similar feature arises also in a simple model to describe final-state interactions, which assumes elastic rescattering processes and has been proposed in [285, 286]. Also electroweak penguins may play a more important role than naively expected [278, 286, 291], so that ϵ may actually be of $\mathcal{O}(10\%)$.

A detailed study of the impact of these effects on the generalized bound on γ related to (14.2) was performed in [280]. The "original" bound derived in [277] corresponds to $\kappa = 1$ and sets effectively the asymmetry A_0 to zero. As soon as a non-vanishing experimental result for A_0 has been established, also an interval around $\gamma = 0^{\circ}$ and 180° can be ruled out, while the impact on the excluded region around 90° is rather small [280].

An interesting feature of the rescattering effects is that they may lead to sizeable CP violation in the decay $B^+ \to \pi^+ K^0$ [285]–[288], in contrast to simple quark-level estimates,

from which at most a few percent for this CP asymmetry [292] could be expected. This CP asymmetry provides a first step towards the experimental control of rescattering processes [280]. The rescattering effects can be included in the generalized bounds on γ completely by using additional experimental information on the decay $B^+ \to K^+ \overline{K^0}$ and its charge conjugate [280, 293]. Different strategies to constrain rescattering effects have also been considered in [287].

At first sight, an experimental study of $B^+ \to K^+ \overline{K^0}$ appears to be challenging, since model estimates performed at the perturbative quark level give a combined branching ratio $Br(B^{\pm} \to K^{\pm}K) = \mathcal{O}(10^{-6})$, which is one order of magnitude below the present upper limit 2.1×10^{-5} obtained by the CLEO collaboration. However, as was pointed out in [280, 293], rescattering processes may well enhance this branching ratio by $\mathcal{O}(10)$, so that it may be possible to study this mode to obtain insights into final state interactions at future B-factories. Also electroweak penguins can be constrained by using additional information [280], and certainly experiment will tell us one day how important rescattering processes and electroweak penguins in $B \to \pi K$ decays really are. An interesting probe of γ is also provided by $B_s \to K\overline{K}$ decays, which can be combined with their $B_{u,d} \to \pi K$ counterparts through the SU(3) flavour symmetry [294].

Finally I would like to mention a recent interesting paper of Lenz, Nierste and Ostermaier [295], where inclusive direct CP-asymmetries in charmless B^{\pm} -decays including QCD effects have been studied. These asymmetries should offer useful means to constrain the unitarity triangle.

14.3 CP-Asymmetries in *B*-Decays versus $K \to \pi \nu \bar{\nu}$

Let us next compare the potentials of the CP asymmetries in determining the parameters of the Standard Model with those of the cleanest rare K-decays: $K_L \to \pi^0 \nu \bar{\nu}$ and $K^+ \to \pi^+ \nu \bar{\nu}$.

To this end let us assume that the problems with the determination of α will be solved somehow. Since in the usual rescaled unitarity triangle one side is known, it suffices to measure two angles to determine the triangle completely. This means that the measurements of $\sin 2\alpha$ and $\sin 2\beta$ can determine the parameters ϱ and η . As the standard analysis of the unitarity triangle of section 10 shows, $\sin 2\beta$ is expected to be large: $\sin 2\beta = 0.58 \pm 0.22$ implying the time-integrated CP asymmetry $a_{\rm CP}(B_d \to J/\psi K_{\rm S})$ as high as $(30 \pm 10)\%$. The prediction for $\sin 2\alpha$ is very uncertain on the other hand (0.1 ± 0.9) and even a rough measurement of α would have a considerable impact on our knowledge of the unitarity triangle as stressed in [34, 249].

Measuring then $\sin 2\alpha$ and $\sin 2\beta$ from CP asymmetries in B decays allows, in principle,

Table 33: Illustrative example of the determination of CKM parameters from $K \to \pi \nu \bar{\nu}$ and B-decays.

	$K \to \pi u \bar{ u}$	Scenario I	Scenario II
$\sigma(V_{td})$	±10%(9%)	$\pm 5.5\% (3.5\%)$	$\pm 5.0\% (2.5\%)$
$\sigma(ar{arrho})$	$\pm 0.16(0.12)$	±0.03	± 0.01
$\sigma(ar{\eta})$	$\pm 0.04(0.03)$	±0.04	± 0.01
$\sigma(\sin 2\beta)$	± 0.05	±0.06	± 0.02
$\sigma(\mathrm{Im}\lambda_t)$	±5%	±14%(11%)	±10%(6%)

to fix the parameters $\bar{\eta}$ and $\bar{\varrho}$, which can be expressed as [250]

$$\bar{\eta} = \frac{r_{-}(\sin 2\alpha) + r_{+}(\sin 2\beta)}{1 + r_{+}^{2}(\sin 2\beta)}, \qquad \bar{\varrho} = 1 - \bar{\eta}r_{+}(\sin 2\beta), \tag{14.6}$$

where $r_{\pm}(z) = (1 \pm \sqrt{1-z^2})/z$. In general the calculation of $\bar{\varrho}$ and $\bar{\eta}$ from $\sin 2\alpha$ and $\sin 2\beta$ involves discrete ambiguities. As described in [250] they can be resolved by using further information, e.g. bounds on $|V_{ub}/V_{cb}|$, so that eventually the solution (14.6) is singled out.

Let us then consider two scenarios of the measurements of CP asymmetries in $B_d \to \pi^+\pi^$ and $B_d \to J/\psi K_S$, expressed in terms of $\sin 2\alpha$ and $\sin 2\beta$:

$$\sin 2\alpha = 0.40 \pm 0.10$$
, $\sin 2\beta = 0.70 \pm 0.06$ (scenario I) (14.7)

$$\sin 2\alpha = 0.40 \pm 0.04$$
, $\sin 2\beta = 0.70 \pm 0.02$ (scenario II). (14.8)

Scenario I corresponds to the accuracy being aimed for at *B*-factories and HERA-B prior to the LHC era. An improved precision can be anticipated from LHC experiments, which we illustrate with the scenario II.

In table 33 this way of the determination of the Standard Model parameters is compared with the analogous analysis using $K_{\rm L} \to \pi^0 \nu \bar{\nu}$ and $K^+ \to \pi^+ \nu \bar{\nu}$ which has been presented in section 13. We recall that in the latter analysis the following input has been used:

$$|V_{cb}| = 0.040 \pm 0.002(0.001), \qquad m_t = (170 \pm 3) \text{GeV}$$
 (14.9)

$$Br(K_{\rm L} \to \pi^0 \nu \bar{\nu}) = (3.0 \pm 0.3) \cdot 10^{-11} \,, \qquad Br(K^+ \to \pi^+ \nu \bar{\nu}) = (1.0 \pm 0.1) \cdot 10^{-10} \,. \tag{14.10}$$

The value $|V_{cb}| = 0.040 \pm 0.002 (0.001)$ is also used in B physics scenarios I and II respectively.

As can be seen in table 33, the CKM determination using $K \to \pi \nu \bar{\nu}$ is competitive with the one based on CP violation in B decays in scenario I, except for $\bar{\varrho}$ which is less constrained

Table 34: Determination of the CKM matrix from λ , V_{cb} , $K_{\rm L} \to \pi^0 \nu \bar{\nu}$ and $\sin 2\alpha$ from the CP asymmetry in $B_d \to \pi^+ \pi^-$ [249]. Scenario A (B) assumes $V_{cb} = 0.040 \pm 0.002(\pm 0.001)$ and $\sin 2\alpha = 0.4 \pm 0.2(\pm 0.1)$. In both cases we take $Br(K_{\rm L} \to \pi^0 \nu \bar{\nu}) \cdot 10^{11} = 3.0 \pm 0.3$ and $m_{\rm t} = (170 \pm 3) \, {\rm GeV}$.

		A	В
$ar{\eta}$	0.380	± 0.043	± 0.028
$ar{arrho}$	0.070	± 0.058	± 0.031
$\sin 2\beta$	0.700	± 0.077	± 0.049
$ V_{td} /10^{-3}$	8.84	± 0.67	± 0.34
$ V_{ub}/V_{cb} $	0.087	± 0.012	± 0.007

by the rare kaon processes. On the other hand as advertised previously $\text{Im}\lambda_t$ is better determined in $K \to \pi \nu \bar{\nu}$ even if scenario II is considered. The virtue of the comparision of the determinations of various parameters using CP-B asymmetries with the determinations in very clean decays $K \to \pi \nu \bar{\nu}$ is that any substantial deviations from these two determinations would signal new physics beyond the Standard Model. Formula (13.69) is an example of such a comparison.

14.4 Unitarity Triangle from $K_L \to \pi^0 \nu \bar{\nu}$ and $\sin 2\alpha$

Next, results from CP asymmetries in B decays could also be combined with measurements of $K \to \pi \nu \bar{\nu}$. As an illustration we would like to present a scenario [249] where the unitarity triangle is determined by λ , V_{cb} , $\sin 2\alpha$ and $Br(K_L \to \pi^0 \nu \bar{\nu})$. In this case $\bar{\eta}$ follows directly from $Br(K_L \to \pi^0 \nu \bar{\nu})$ (13.52) and $\bar{\varrho}$ is obtained using [250]

$$\bar{\varrho} = \frac{1}{2} - \sqrt{\frac{1}{4} - \bar{\eta}^2 + \bar{\eta}r_-(\sin 2\alpha)}, \qquad (14.11)$$

where $r_{-}(z)$ is defined after (14.6). The advantage of this strategy is that most CKM quantities are not very sensitive to the precise value of $\sin 2\alpha$. Moreover a high accuracy in $\text{Im}\lambda_t$ is automatically guaranteed. As shown in table 34, very respectable results can be expected for other quantities as well with only modest requirements on the accuracy of $\sin 2\alpha$. It is conceivable that theoretical uncertainties due to penguin contributions could eventually be brought under control at least to the level assumed in table 34. As an alternative, $\sin 2\beta$ from $B_d \to J/\psi K_{\rm S}$ could be used as an independent input instead of $\sin 2\alpha$. Unfortunately the combination of $K_{\rm L} \to \pi^0 \nu \bar{\nu}$ and $\sin 2\beta$ tends to yield somewhat less restrictive constraints

on the unitarity triangle [249]. On the other hand it has of course the advantage of being practically free of any theoretical uncertainties.

14.5 Unitarity Triangle and $|V_{cb}|$ from $\sin 2\alpha$, $\sin 2\beta$ and $K_L \to \pi^0 \nu \bar{\nu}$

As proposed in [250], unprecedented precision for all basic CKM parameters could be achieved by combining the cleanest K and B decays. While λ is obtained as usual from $K \to \pi e \nu$, $\bar{\varrho}$ and $\bar{\eta}$ could be determined from $\sin 2\alpha$ and $\sin 2\beta$ as measured in CP violating asymmetries in B decays. Given η , one could take advantage of the very clean nature of $K_{\rm L} \to \pi^0 \nu \bar{\nu}$ to extract A or, equivalently $|V_{cb}|$. As seen in (13.53), this determination benefits further from the very weak dependence of $|V_{cb}|$ on the $K_{\rm L} \to \pi^0 \nu \bar{\nu}$ branching ratio, which is only with a power of 0.25. Moderate accuracy in $Br(K_{\rm L} \to \pi^0 \nu \bar{\nu})$ would thus still give a high precision in $|V_{cb}|$. As an example we take $\sin 2\alpha = 0.40 \pm 0.04$, $\sin 2\beta = 0.70 \pm 0.02$ and $Br(K_{\rm L} \to \pi^0 \nu \bar{\nu}) = (3.0 \pm 0.3) \cdot 10^{-11}$, $m_t = (170 \pm 3)$ GeV. This yields [249]:

$$\bar{\varrho} = 0.07 \pm 0.01, \qquad \bar{\eta} = 0.38 \pm 0.01, \qquad |V_{cb}| = 0.0400 \pm 0.0013, \qquad (14.12)$$

which would be a truly remarkable result. Again the comparision of this determination of $|V_{cb}|$ with the usual one in tree level *B*-decays would offer an excellent test of the Standard Model and in the case of discrepancy would signal physics beyond it.

14.6 Unitarity Triangle from R_t and $\sin 2\beta$

Another strategy is to use the measured value of R_t together with $\sin 2\beta$. Useful measurements of R_t can be achieved using the ratios $Br(B \to X_d \nu \bar{\nu})/Br(B \to X_s \nu \bar{\nu})$, $\Delta M_d/\Delta M_s$, $Br(B_d \to l^+l^-)/Br(B_s \to l^+l^-)$ and $Br(K^+ \to \pi^+\nu \bar{\nu})$. Then (14.6) is replaced by [263]

$$\bar{\eta} = \frac{R_t}{\sqrt{2}} \sqrt{\sin 2\beta \cdot r_-(\sin 2\beta)}, \qquad \bar{\varrho} = 1 - \bar{\eta}r_+(\sin 2\beta). \tag{14.13}$$

The numerical results of this exercise can be found in [263]. Additional strategies involving the angle γ can be found in [34].

15 Summary and Outlook

We are approaching the end of our tour. I hope that some of you enjoyed reading these lectures as much as I did preparing, delivering and finally writing them. The collection of many techniques and formulae should be useful in various phenomenological applications and constitutes hopefully a good introduction to future research. I hope that I have convinced the students that the field of weak decays plays an important role in the deeper understanding

of the Standard Model and particle physics in general. Indeed the field of weak decays and of CP violation is one of the least understood sectors of the Standard Model. Even if the Standard Model is fully consistent with the existing data for weak decay processes, the near future could change this picture dramatically through the advances in experiment and theory. In particular the experimental work done in the next ten years at BNL, CERN, CORNELL, DA Φ NE, DESY, FNAL, KEK, SLAC and eventually LHC will certainly have considerable impact on this field.

Before closing these lectures with a few final messages, I would like to make a list of things we could expect in the next ten years. This list is certainly very biased by my own interests but could be useful anyway. Here we go:

- The error on the CKM elements $|V_{cb}|$ and $|V_{ub}/V_{cb}|$ could be decreased below 0.002 and 0.01, respectively. This progress should come mainly from Cornell, *B*-factories and new theoretical efforts. It would have considerable impact on the unitarity triangle and would improve theoretical predictions for rare and CP-violating decays sensitive to these elements.
- The error on $m_{\rm t}$ should be decreased down to $\pm 3\,{\rm GeV}$ at Tevatron in the Main Injector era and to $\pm 1\,{\rm GeV}$ at LHC.
- The improved measurements of ε'/ε with the accuraccy of $\pm (1-2) \cdot 10^{-4}$ from CERN, FNAL and DA Φ NE should give some insight into the physics of direct CP violation inspite of large theoretical uncertainties. Excluding confidently the superweak models would be an important result. In this respect measurements of CP-violating asymmetries in charged B decays will also play an outstanding role. These experiments can be performed e.g. at CLEO since no time-dependences are needed. The situation concerning hadronic uncertainties is quite similar to ε'/ε . Although these CP asymmetries cannot be calculated reliably, any measured non-vanishing values would unambiguously rule out superweak scenarios. Simultaneously one should hope that some definite progress in calculating relevant hadronic matrix elements will be made.
- More events for $K^+ \to \pi^+ \nu \bar{\nu}$ could in principle be seen at BNL already this or next year. In view of the theoretical cleanliness of this decay an observation of events at the $2 \cdot 10^{-10}$ level would signal physics beyond the Standard Model. A detailed study of this very important decay requires, however, new experimental ideas and new efforts. The new efforts [244, 245] in this direction allow to hope that a measurement of $Br(K^+ \to \pi^+ \nu \bar{\nu})$ with an accuracy of $\pm 10\%$ should be possible before 2005. This would have a very important impact on the unitarity triangle and would constitute an important test of the Standard Model.

- The future improved inclusive B → X_{s,d}γ measurements confronted with improved Standard Model predictions could give the first signals of new physics. It appears that the errors on the input parameters could be lowered further and the theoretical error on Br(B → X_sγ) could be decreased confidently down to ±8% in the next years. The same accuracy in the experimental branching ratio will hopefully come soon from CLEO II and later from KEK and SLAC. This may, however, be insufficient to disentangle new physics contributions although such an accuracy should put important constraints on the physics beyond the Standard Model. It would also be desirable to look for B → X_dγ, but this is clearly a much harder task.
- Similar comments apply to transitions $B \to X_s l^+ l^-$ (not discussed here) which appear to be even more sensitive to new physics contributions than $B \to X_{s,d}\gamma$. An observation of $B \to X_s \mu \bar{\mu}$ is expected from D0 and B-physics dedicated experiments at the beginning of the next decade. The distributions of various kind when measured should be very useful in the tests of the Standard Model and its extensions.
- The theoretical status of K_L → π⁰e⁺e⁻ and of K_L → μμ̄, which we did not cover here, should be improved to confront future data. Experiments at DAΦNE should be very helpful in this respect. The first events of K_L → π⁰e⁺e⁻ should come in the first years of the next decade from KAMI at FNAL. The experimental status of K_L → μμ̄, with the experimental error of ±7% to be decreased soon down to ±1%, is truly impressive.
- The newly approved experiment at BNL to measure $Br(K_L \to \pi^0 \nu \bar{\nu})$ at the $\pm 10\%$ level before 2005 may make a decisive impact on the field of CP violation. In particular $K_L \to \pi^0 \nu \bar{\nu}$ seems to allow the cleanest determination of $\text{Im}\lambda_t$. Taken together with $K^+ \to \pi^+ \nu \bar{\nu}$ a very clean determination of $\sin 2\beta$ can be obtained.
- The measurement of the $B_s^0 \bar{B}_s^0$ mixing and in particular of $B \to X_{s,d} \nu \bar{\nu}$ and $B_{s,d} \to \mu \bar{\mu}$ will take most probably longer time but as stressed in these lectures all efforts should be made to measure these transitions. Considerable progress on $B_s^0 \bar{B}_s^0$ mixing should be expected from HERA-B, SLAC and TEVATRON in the first years of the next decade. LHC-B should measure it to a high precision. With the improved calculations of ξ in (10.72) this will have important impact on the determination of $|V_{td}|$ and on the unitarity triangle.
- Clearly future precise studies of CP violation at SLAC-B, KEK-B, HERA-B, CORNELL, FNAL and LHC-B providing first direct measurements of α , β and γ may totally revolutionize our field. In particular the first signals of new physics could be

found in the $(\bar{\varrho}, \bar{\eta})$ plane. During the recent years several, in some cases quite sophisticated and involved, strategies have been developed to extract these angles with small or even no hadronic uncertainties. Certainly the future will bring additional methods to determine α , β and γ . Obviously it is very desirable to have as many such strategies as possible available in order to overconstrain the unitarity triangle and to resolve certain discrete ambiguities which are a characteristic feature of these methods.

- The forbidden or strongly suppressed transitions such as $D^0 \bar{D}^0$ mixing and $K_L \to \mu e$ are also very important in this respect. Considerable progress in this area should come from the experiments at BNL, FNAL and KEK.
- On the theoretical side, one should hope that the non-perturbative methods will be considerably improved so that various B_i parameters will be calculated with sufficient precision. It is very important that simultaneously with advances in lattice QCD, further efforts are being made in finding efficient analytical tools for calculating QCD effects in the long distance regime. This is, in particular very important in the field of non-leptonic decays, where one should not expect too much from our lattice friends in the coming ten years unless somebody will get a brilliant idea which will revolutionize lattice calculations. The accumulation of data for non-leptonic B and D decays at Cornell, SLAC, KEK and FNAL should teach us more about the role of non-factorizable contributions and in particular about the final state interactions. In this context, in the case of K-decays, important lessons will come from DAΦNE which is an excellent machine for testing chiral perturbation theory and other non-perturbative methods.

In any case the field of weak decays and in particular of the FCNC transitions and of CP violation have a great future and one should expect that they could dominate particle physics in the first part of the next decade. Clearly the next ten years should be very exciting in this field and it is advisable to buy shares before it is too late.

16 Final Messages

The two weeks I have spent in Les Houches in August 1997 will remain in my memory for ever. Therefore I would like to close these lectures by thanking those who contributed most to this happening.

First of all I would like to thank Rajan Gupta and Francois David for inviting me to this school and keeping me busy. In particular I would like to thank Rajan for creating such a pleasent atmosphere and his persistent e-mails reminding me that it is time to finish writing up these lectures.

However my warmest thanks go to the students of this school who made the sixteen hours of my presence in front of the blackboard and the remaining time a real joy. In particular:

- Many thanks to the magnificant seven: Fabien Motsch, Markus Peter, Solveig Skadhauge, Thomas Teubner, Anja Werthenbach, Joerg Westphalen and Stefan Wienzerl for keeping me alive during a two day mountain expedition. Champagne offered after this tour by a very special student of this school, Leung Ka Chun, will never be forgotten.
- The results of our expedition appeared in hep-ph/9708777 under the title "No Loops beyond the Trees in the Splittorff Renormalization Scheme", where further details can be found. Splittorff, the youngest student of the school was the only one of this Les Houches session to climb Mont Blanc. There is nothing exciting in hep-ph/9708777 except one thing: this work will go down in history as yet another Buras et al. paper.
- Many thanks to Luca Girlanda, Nicos Irges and Leszek Motyka for arranging table tennis championships and to Andrzej Czarnecki for giving me Polysporin which allowed me to reach quarter finals where I was slaughtered by a spanish matador (Francisco Guerrero).
- From all these remarks it is clear that I had rather close contacts with the students of this school. Yet my closest companions, day and night, were the washing machine and the dryer both placed next to my room. The lively discussions, in particular at night, in front of my door forced me to work hard on my lectures, except for the last night of my stay when following the advice of the sole experimentalist of the school (Fabien Motsch) I switched off these two important inventions of this century.

I hope that these final comments made it clear why I have enjoyed this school so much. Many thanks to all of you.

Particular thanks go to Markus Lautenbacher for creating many figures and a number of numerical calculations. I would also like to thank Robert Fleischer, Paolo Gambino, Axel Kwiatkowski, Mikolaj Misiak, Nicolas Pott and Luca Silvestrini for helpful discussions during the preparation of these lectures.

This work has been supported by the German Bundesministerium für Bildung and Forschung under contract 06 TM 874 and DFG Project Li 519/2-2.

References

- [1] N. Cabibbo, Phys. Rev. Lett. **10** (1963) 531.
- [2] M. Kobayashi and K. Maskawa, Prog. Theor. Phys. 49 (1973) 652.

- [3] K.G. Wilson, Phys. Rev. 179 (1969) 1499; K.G. Wilson and W. Zimmermann, Comm. Math. Phys. 24 (1972) 87.
- [4] W. Zimmermann, in Proc. 1970 Brandeis Summer Institute in Theor. Phys, (eds. S. Deser, M. Grisaru and H. Pendleton), MIT Press, 1971, p.396; Ann. Phys. 77 (1973) 570.
- [5] E.C.G. Sudarshan and R.E. Marshak, Proc. Padua-Venice Conf. on Mesons and Recently Discovered Particles (1957).
- [6] R.P. Feynman and M. Gell-Mann, Phys. Rev. **109** (1958) 193.
- [7] E. Witten, Nucl. Phys. **B 120** (1977) 189.
- [8] E.C.G. Stueckelberg and A. Petermann, Helv. Phys. Acta 26 (1953) 499; M. Gell-Mann and F.E. Low, Phys. Rev. 95 (1954) 1300; L.V. Ovsyannikov, Dokl. Acad. Nauk SSSR 109 (1956) 1112; K. Symanzik, Comm. Math. Phys. 18 (1970) 227; C.G. Callan Jr, Phys. Rev. D 2 (1970) 1541.
- [9] G. 't Hooft, Nucl. Phys. **B 61** (1973) 455.
- [10] S. Weinberg, Phys. Rev. **D** 8 (1973) 3497.
- [11] D. Zeppenfeld, Z. Phys. C 8 (1981) 77; L.L. Chau, Phys. Rev. D 43 (1991) 2176; M. Gronau, J.L. Rosner and D. London, Phys. Rev. Lett. 73 (1994) 21; O.F. Hernandez, M. Gronau, J.L. Rosner and D. London, Phys. Lett. B 333 (1994) 500, Phys. Rev. D 50 (1994) 4529.
- [12] J. Chay, H. Georgi and B. Grinstein, Phys. Lett. B 247 (1990) 399.
- [13] I.I. Bigi, N.G. Uraltsev and A.I. Vainshtein, Phys. Lett. B 293 (1992) 430 [E: B 297 (1993) 477].
 I.I. Bigi, M.A. Shifman, N.G. Uraltsev and A.I. Vainshtein, Phys. Rev. Lett. 71 (1993) 496;
 B. Blok, L. Koyrakh, M.A. Shifman and A.I. Vainshtein, Phys. Rev. D 49 (1994) 3356 [E: D 50 (1994) 3572].
- [14] A.V. Manohar and M.B. Wise, Phys. Rev. **D** 49 (1994) 1310.
- [15] S.L. Glashow, J. Iliopoulos and L. Maiani Phys. Rev. **D** 2 (1970) 1285.
- [16] G. Buchalla, A.J. Buras and M.K. Harlander, Nucl. Phys. **B 349** (1991) 1.
- [17] G. Buchalla, A.J. Buras and M. Lautenbacher, Rev. Mod. Phys 68 (1996) 1125.
- [18] A.J. Buras and R. Fleischer, hep-ph/9704376, to appear in [19].

- [19] A.J. Buras and M. Lindner, Heavy Flavours II, World Scientific, 1998.
- [20] T. Muta, Foundations of Chromodynamics, World Scientific, 1987.
- [21] M.E. Peskin and D.V. Schroeder, An Introduction to Quantum Field Theory, Addison-Wesley Publishing Company.
- [22] F. Mandl and G. Shaw, Quantum Field Theory, John Wiley & Sons.
- [23] T.-P. Cheng and L.-F. Li, Gauge Theory of Elementary Particle Physics, Clarendon Press, Oxford.
- [24] L.H. Ryder, Quantum Field Theory, Cambridge University Press.
- [25] J.F. Donoghue, E. Golowich and B.R. Holstein, Dynamics of the Standard Model, Cambridge Monographs.
- [26] D. Bailin and A. Love, Introduction to Gauge Field Theory, Adam Hilger, Bristol and Boston.
- [27] S. Pokorski, Gauge Field Theory, Cambridge Monographs.
- [28] S. Weinberg, The Quantum Theory of Fields, Cambridge University Press.
- [29] J.C. Collins, Renormalization, Cambridge University Press.
- [30] L.L. Chau and W.-Y. Keung, Phys. Rev. Lett. **53** (1984) 1802.
- [31] Particle Data Group, Phys. Rev. **D** 54 (1996) 1.
- [32] L. Wolfenstein, Phys. Rev. Lett. **51** (1983) 1945.
- [33] H. Harari and M. Leurer, Phys. Lett. B 181 (1986) 123.
- [34] A.J. Buras, M.E. Lautenbacher and G. Ostermaier, Phys. Rev. D 50 (1994) 3433.
- [35] M. Schmidtler and K.R. Schubert, Z. Phys. C 53 (1992) 347.
- [36] C. Jarlskog and R. Stora, Phys. Lett. **B 208** (1988) 268.
- [37] H. Leutwyler and M. Roos, Z. Physik C25 (1984) 91.
- [38] J.F. Donoghue, B.R. Holstein and S.W. Klimt, Phys. Rev. **D35** (1987) 934.
- [39] L. Gibbons, in proceedings of the 28th International Conference on High Energy Physics, July 1996, Warsaw, Poland, page 183.

- [40] M. Shifman, N.G. Uraltsev and A. Vainshtein, Phys. Rev. D51 (1995) 2217; I. Bigi,
 M. Shifman and N. Uraltsev, Ann. Rev. Nucl. Part. Sci. 47 (1997) 591.
- [41] M. Neubert, Phys. Lett. **B338** (1994) 84; Int. J. Mod. Phys. **A11** (1996) 4173.
- [42] P. Ball, M. Beneke and V.M. Braun, Phys. Rev. **D52** (1995) 3929.
- [43] A. Czarnecki, Phys. Rev. Lett. 76 (1996) 4124. A. Czarnecki and K. Melnikov, Phys. Rev. Lett. 78 (1997) 3630.
- [44] J.P. Alexander et al. (CLEO), CLNS 96/1419, CLEO 96-9 (1996).
- [45] T. Inami and C.S. Lim, Progr. Theor. Phys. **65** (1981) 297.
- [46] G. Burdman, hep-ph/9407378, hep-ph/9508349.
- [47] A.J. Buras, Rev. Mod. Phys **52** (1980) 199.
- [48] A.J. Buras and P.H. Weisz, Nucl. Phys. **B 333** (1990) 66.
- [49] G. 't Hooft and M. Veltman, Nucl. Phys. **B** 44 (1972) 189.
- [50] P. Breitenlohner and D. Maison, Comm. Math. Phys. **52** (1977) 11, 39, 55.
- [51] G. Bonneau, Phys. Lett. **B 94** (1980) 147; Nucl. Phys. **B 177** (1981) 523.
- [52] W. Siegel, Phys. Lett. **B 84** (1979) 193.
- [53] G. Altarelli, G. Curci, G. Martinelli and S. Petrarca, Nucl. Phys. B 187 (1981) 461.
- [54] D. Maison, Phys. Lett. **B 150** (1985) 39.
- [55] H. Nicolai and P.K. Townsend, Phys. Lett. B 93 (1980) 111; P. Majumdar, E.C.
 Poggio and H.J. Schnitzer, Phys. Rev. D 21 (1980) 2203.
- [56] R. Grigjanis, P.J. O'Donnell, M. Sutherland and H. Navelet, Phys. Lett. B213 (1988)355; Phys. Lett. B286 (1992) 413 E.
- [57] M. Misiak, Phys. Lett. **B321** (1994) 113.
- [58] D.A. Akyeampong and R. Delbourgo, Nuovo Cim. 17A (1973) 578, 18A (1973) 94,19A (1974) 219.
- [59] J.G. Körner, N. Nasrallah and K. Schilcher, Phys. Rev. D 41 (1990) 888.
- [60] M. Jamin and M. Lautenbacher, Comput. Phys. Commun. 74 (1993) 265.

- [61] W.A. Bardeen, A.J. Buras, D.W. Duke and T. Muta, Phys. Rev. **D** 18 (1978) 3998.
- [62] D.J. Gross, Methods in Fleld Theory, (eds. R. Balian and J. Zinn-Justin), North-Holland, 1976, p. 141.
- [63] M. Schmelling, in proceedings of the 28th International Conference on High Energy Physics, July 1996, Warsaw, Poland, page 91.
- [64] W.J. Marciano, Phys. Rev. **D** 12 (1975) 3861.
- [65] G. Buchalla and A.J. Buras, Nucl. Phys. **B 398** (1993) 285.
- [66] C. Greub and T. Hurth, Phys. Rev. **D** 56 (1997) 2934.
- [67] A.J. Buras, A. Kwiatkowski and N. Pott, Nucl. Phys. **B 517** (1998) 353.
- [68] A.J. Buras, M. Jamin, M.E. Lautenbacher and P.H. Weisz, Nucl. Phys. B 370 (1992)
 69; Nucl. Phys. B 400 (1993) 37.
- [69] A.J. Buras, M. Jamin and M.E. Lautenbacher, Nucl. Phys. **B** 400 (1993) 75.
- [70] C. Curci and G. Ricciardi, Phys. Rev. **D** 47 (1993) 2991.
- [71] K. Chetyrkin, M. Misiak and Münz, hep-ph/9711280; hep-ph/9711266.
- [72] G. Altarelli and L. Maiani, Phys. Lett. B 52 (1974) 351; M.K. Gaillard and B.W. Lee, Phys. Rev. Lett. 33 (1974) 108.
- [73] A.J. Buras, M. Jamin and M.E. Lautenbacher, Nucl. Phys. B 408 (1993) 209.
- [74] M. Ciuchini, E. Franco, G. Martinelli and L. Reina, Phys. Lett. B 301 (1993) 263.
- [75] M. Ciuchini, E. Franco, G. Martinelli and L. Reina, Nucl. Phys. B 415 (1994) 403.
- [76] N. Tracas and N. Vlachos, Phys. Lett. **B** 115 (1982) 419.
- [77] A.J. Buras, A. Kwiatkowski and N. Pott, Phys. Lett. **B 414** (1997) 157.
- [78] A.J. Buras and M. Münz, Phys. Rev. **D** 52 (1995) 186.
- [79] M.J. Dugan and B. Grinstein, Phys. Lett. **B 256** (1991) 239.
- [80] S. Herrlich and U. Nierste, Nucl. Phys. **B** 445 (1995) 39.
- [81] S. Herrlich, Technical University, PhD Thesis 1994 (in German).
- [82] U. Nierste, Technical University, PhD Thesis 1995.

- [83] R. Fleischer, Zeit. Phys. C 62 (1994) 81.
- [84] A.I. Vainshtein, V.I. Zakharov and M.A. Shifman, JEPT 45 (1977) 670.
- [85] A.J. Buras and M.K. Harlander, A Top Quark Story, in Heavy Flavours, eds. A.J. Buras and M. Lindner, World Scientific, 1992, p.58.
- [86] G. Buchalla, A.J. Buras, M.K. Harlander, M.E. Lautenbacher and C. Salazar, Nucl. Phys. B 355 (1991) 305.
- [87] P. Cho, M. Misiak and D. Wyler, Phys. Rev. D 54 (1996) 3329.
- [88] A. Ali, Th. Mannel and Ch. Greub, Zeit. Phys. C 67 (1995) 417.
- [89] A.J. Buras, Nucl. Phys. **B 434** (1995) 606.
- [90] A.J. Buras, M. Jamin, and P.H. Weisz, Nucl. Phys. B347 (1990) 491;
 J. Urban, F. Krauss, U.Jentschura and G. Soff, Nucl. Phys. B523 (1998) 40.
- [91] G. Buchalla and A.J. Buras, Nucl. Phys. **B 412** (1994) 106.
- [92] S. Herrlich and U. Nierste, Nucl. Phys. **B419** (1994) 292.
- [93] S. Herrlich and U. Nierste, Phys. Rev. **D52** (1995) 6505; Nucl. Phys. **B476** (1996) 27.
- [94] M.Misiak and M. Münz, Phys. Lett. **B344** (1995) 308.
- [95] G. Buchalla, Nucl. Phys. **B 391** (1993) 501.
- [96] E. Bagan, P.Ball, V.M. Braun and P.Gosdzinsky, Nucl. Phys. B 432 (1994) 3; E. Bagan et al., Phys. Lett. B 342 (1995) 362; B 351 (1995) 546.
- [97] M. Jamin and A. Pich, Nucl. Phys. **B425** (1994) 15.
- [98] G. Buchalla and A.J. Buras, Nucl. Phys. **B** 400 (1993) 225.
- [99] G. Buchalla and A.J. Buras, Phys. Lett. **B 336** (1994) 263.
- [100] A. J. Buras, M. E. Lautenbacher, M. Misiak and M. Münz, Nucl. Phys. B423 (1994) 349.
- [101] M. Misiak, Nucl. Phys. B393 (1993) 23; Erratum, Nucl. Phys. B439 (1995) 461.
- [102] N. Pott, hep-ph/9710503.

- [103] J. Krakauer, "Into Thin Air", Villard Books, New York, 1997.
- [104] A.J. Buras and R. Fleischer, Phys. Lett. **B 341** (1995) 379.
- [105] M. Ciuchini, E. Franco, G. Martinelli, and L. Silvestrini, Nucl. Phys. B501 (1997) 271; M. Ciuchini, R. Contino, E. Franco, G. Martinelli, and L. Silvestrini, Nucl. Phys. B512 (1998) 3.
- [106] A. Ali, and C. Greub, Z.Phys. C49 (1991) 431; Phys. Lett. B259 (1991) 182;
 Phys. Lett. B361 (1995) 146.
- [107] K. Adel and Y.P. Yao, Modern Physics Letters A8 (1993) 1679; Phys. Rev. D 49 (1994) 4945.
- [108] N. Pott, Phys. Rev. **D** 54 (1996) 938.
- [109] C. Greub, T. Hurth and D. Wyler, Phys. Lett. B380 (1996) 385; Phys. Rev. D 54 (1996) 3350; C. Greub and T. Hurth, hep-ph/9608449.
- [110] K.G. Chetyrkin, M. Misiak and M. Münz, Phys. Lett. B400 (1997) 206; hep-ph/9612313.
- [111] M. Ciuchini, G. Degrassi, P. Gambino and G.F. Giudice, hep-ph/9710335.
- [112] P. Ciafaloni, A. Romanino, and A. Strumia, hep-ph/9710312.
- [113] A.J. Buras and L. Silvestrini, TUM-HEP-315/98, hep-ph/9806278.
- [114] A. Khodjamirian and R. Rückl, hep-ph/9801443, to appear in [19].
- [115] J. Schwinger, Phys. Rev. Lett. 12 (1964) 630; R.P. Feynman, in Symmetries in Particle Physics, ed. A. Zichichi, Acad. Press 1965, p.167; O. Haan and B. Stech, Nucl. Phys. B 22 (1970) 448.
- [116] D. Fakirov and B. Stech, Nucl. Phys. B 133 (1978) 315; L.L. Chau, Phys. Rep. 95 (1983) 1.
- [117] M. Wirbel, B. Stech and M. Bauer, Z. Phys.C 29 (1985) 637. M. Bauer, B. Stech and M. Wirbel, Z. Phys.C 34 (1987) 103.
- [118] M. Neubert, V. Rieckert, B. Stech and Q.P. Xu, in "Heavy Flavours", eds. A.J. Buras and M. Lindner (World Scientific, Singapore, 1992), p. 286.
- [119] J.D. Bjorken, Nucl. Phys. **B** (Proc. Suppl.) 11 (1989) 325; SLAC-PUB-5389.

- [120] M.J. Dugan and B. Grinstein, Phys. Lett. B 255 (1991) 583.
- [121] M. Neubert and B. Stech, [hep-ph/9705292], to appear in [19]; B. Stech [hep-ph/9706384]; M.Neubert, Nucl. Phys. B (Proc. Suppl.) 64 (1998) 474, [hep-ph/9801269].
- [122] C. Reader and N. Isgur, Phys. Rev. **D** 47 (1993) 1007.
- [123] M. Ciuchini, R. Contino, E. Franco, G. Martinelli, L. Silvestrini, hep-ph/9801420.
- [124] D. Du and Z. Xing, Phys. Lett. B 312 (1993) 199; A. Deandrea et al., Phys. Lett. B 318 (1993) 549, Phys. Lett. B 320 (1994) 170; N.G. Deshpande, B. Dutta, S. Oh, Phys. Rev. D 57 (1998) 5723, hep-ph/9712445.
- [125] H.-Y. Cheng, Phys. Lett. B 335 (1994) 428, Phys. Lett. B 395 (1997) 345; H.-Y.
 Cheng and B. Tseng, [hep-ph/9708211], [hep-ph/9803457].
- [126] J.M. Soares, Phys. Rev. **D** 51 (1995) 3518.
- [127] A. Ali and C. Greub, Phys. Rev. D57 (1998) 2996; A. Ali, J. Chay, C. Greub and P. Ko, Phys. Lett. B 424 (1998) 161.
- [128] A. Ali, G. Kramer and C.-D. Lü, hep-ph/9804363.
- [129] E. Witten, Nucl. Phys. **B** 160 (1979) 57.
- [130] A.J. Buras, J.M. Gérard and R. Rückl, Nucl. Phys. B 268 (1986) 16.
- [131] M. Ciuchini, E. Franco, G. Martinelli, L. Reina and L. Silvestrini, Z.Phys. C68 (1995) 239.
- [132] A.J. Buras and L. Silvestrini, work in progress.
- [133] M.K. Gaillard and B.W. Lee, Phys. Rev. **D10** (1974) 897.
- [134] H. Albrecht et al. (ARGUS), Phys. Lett. B192 (1987) 245; M. Artuso et al. (CLEO), Phys. Rev. Lett. 62 (1989) 2233.
- [135] J.H. Christenson, J.W. Cronin, V.L. Fitch and R. Turlay, Phys. Rev. Lett. 13 (1964) 128.
- [136] R. Fleischer, Int. J. of Mod. Phys. **A12** (1997) 2459.
- [137] L.L. Chau, Physics Reports, 95 (1983) 1.

- [138] A.J. Buras, W. Slominski and H. Steger, Nucl. Phys. **B245** (1984) 369.
- [139] J. Bijnens, J.-M. Gérard and G. Klein, Phys. Lett. **B257** (1991) 191.
- [140] R. Belusevic, KEK preprint 97–264 (1998).
- [141] Y. Nir, SLAC-PUB-5874 (1992).
- [142] R. Gupta, hep-ph/9801412.
- [143] S. Aoki et al., JLQCD collaboration, Phys. Rev. Lett. 80 (1998) 5271.
- [144] G. Kilcup, R. Gupta and S.R. Sharpe, Phys. Rev. **D57** (1998) 1654.
- [145] R. Gupta, T. Bhattacharaya, and S.R. Sharpe, Phys. Rev. **D55** (1997) 4036.
- [146] L. Conti, A. Donini, V. Gimenez, G.Martinelli, M. Talevi and A. Vladikas, hep-lat/9711053.
- [147] S. Bertolini, J.O. Eeg, M. Fabbrichesi and E.I. Lashin, Nucl. Phys. **B514** (1998) 63.
- [148] W.A. Bardeen, A.J. Buras and J.-M. Gérard, Phys. Lett. B211 (1988) 343; J-M. Gérard, Acta Physica Polonica B21 (1990) 257.
- [149] J. Bijnens and J. Prades, Nucl. Phys. **B444** (1995) 523.
- [150] A. Pich and E. de Rafael, Phys. Lett. B158 (1985) 477; J. Prades et al, Z. Phys.
 C51 (1991) 287.
- [151] J.F. Donoghue, E. Golowich and B.R. Holstein, Phys. Lett. **B119** (1982) 412.
- [152] J. Flynn, in proceedings of the 28th International Conference on High Energy Physics, July 1996, Warsaw, Poland, page 335; J.M. Flynn and C.T. Sachrajda, hep-lat/9710057, to appear in [19].
- [153] C. Bernard, hep-ph/9709460.
- [154] E. Bagan, P. Ball, V.M. Braun and H.G. Dosch, Phys. Lett. B278 (1992) 457; M.
 Neubert, Phys. Rev. D45 (1992) 2451 and references therein.
- [155] A.J. Buras, Phys. Lett. **B317** (1993) 449.
- [156] A.J. Buras, hep-ph/9610461.
- [157] The LEP B Oscillations Working Group, LEPBOSC 97/002.3 (August 14, 1997).

- [158] S. Narison, Phys. Lett. **B322** (1994) 247.
- [159] Y. Grossman, Y. Nir, S. Plaszczynski and M. Schune, Nucl. Phys. **B511** (1998) 69.
- [160] P. Paganini, F. Parodi, P. Roudeau and A. Stocchi, hep-ph/9711261; F. Parodi, P. Roudeau and A. Stocchi, hep-ph/9802289.
- [161] A.J. Buras, M.Jamin and M.E. Lautenbacher, 1997, unpublished; A.J. Buras, hep-ph/9711217.
- [162] M. Ciuchini, E. Franco, G. Martinelli, L. Reina and L. Silvestrini, Z. Phys. C68 (1995) 239.
- [163] A. Ali and D. London, Z. Phys. C65 (1995) 431; Nucl. Phys. B (proc. Suppl.) 54A (1997) 297; A. Ali, hep-ph/9801270.
- [164] G. D. Barr et al., Phys. Lett. **B317** (1993) 233.
- [165] L. K. Gibbons et al., Phys. Rev. Lett. **70** (1993) 1203.
- [166] L. Wolfenstein, Phys. Rev. Lett. **13** (1964) 562.
- [167] R. Barbieri, L. Hall, A. Stocchi, and N. Weiner, hep-ph/9712252.
- [168] J. M. Flynn and L. Randall, Phys. Lett. B224 (1989) 221; erratum ibid. Phys. Lett. B235 (1990) 412.
- [169] G. Buchalla, A. J. Buras, and M. K. Harlander, Nucl. Phys. **B337** (1990) 313.
- [170] F.J. Gilman and M.B. Wise, Phys. Lett. B83 (1979) 83; B. Guberina and R.D. Peccei, Nucl. Phys. B163 (1980) 289.
- [171] J.F. Donoghue, E. Golowich, B.R. Holstein and J. Trampetic, Phys. Lett. B179 (1986) 361.
- [172] A. J. Buras and J.-M. Gérard, Phys. Lett. **B192** (1987) 156.
- [173] M. Lusignoli, Nucl. Phys. **B325** (1989) 33.
- [174] M. Ciuchini, E. Franco and R. Onforio, Mod. Phys. Lett. A5 (1990) 2173; W.A. Bardeen, A. J. Buras and J.-M. Gérard, Phys. Lett. B180 (1986) 133; Nucl. Phys. B293 (1987) 787; H.-Y. Cheng, Phys. Rev. D37 (1988) 1908.
- [175] J. Bijnens and M.B. Wise, Phys. Lett. **B137** (1984) 245.

- [176] W. A. Bardeen, A. J. Buras and J.-M. Gérard, Phys. Lett. B180 (1986) 133; Nucl. Phys. B293 (1987) 787; Phys. Lett. B192 (1987) 138.
- [177] E.A. Paschos and Y.L. Wu, Mod. Phys. Lett. A6 (1991) 93; M. Lusignoli, L. Maiani,
 G. Martinelli and L. Reina, Nucl. Phys. B369 (1992) 139.
- [178] B. Winstein and L. Wolfenstein, Rev. Mod. Phys. 65 (1993) 1113.
- [179] S. Bertolini, M. Fabbrichesi and J.O. Eeg, hep-ph/9802405.
- [180] W. A. Bardeen, A. J. Buras and J.-M. Gérard, Phys. Lett. B192 (1987) 138; A. Pich and E. de Rafael, Nucl. Phys. B358 (1991) 311; M. Neubert and B. Stech, Phys. Rev. D 44 (1991) 775; M. Jamin and A. Pich, Nucl. Phys. B425 (1994) 15; J. Kambor, J. Missimer and D. Wyler, Nucl. Phys. B346 (1990) 17; Phys. Lett. B261 (1991) 496; V. Antonelli, S. Bertolini, M. Fabrichesi, and E.I. Lashin, Nucl. Phys. B469 (1996) 181.
- [181] G. W. Kilcup, Nucl. Phys. (Proc. Suppl.) **B20** (1991) 417.
- [182] S. R. Sharpe, Nucl. Phys. (Proc. Suppl.) B20 (1991) 429.
- [183] D. Pekurovsky and G. Kilcup, hep-lat/9709146.
- [184] J. Heinrich, E. A. Paschos, J.-M. Schwarz, and Y. L. Wu, Phys. Lett. B279 (1992) 140.
- [185] E. A. Paschos, review presented at the 27th Lepton-Photon Symposium, Beijing, China (August 1995).
- [186] T. Hambye, G.O. Köhler, E.A. Paschos, P.H. Soldan and W.A. Bardeen, hep-ph/9802300.
- [187] D. Espriu, E. de Rafael and J. Taron, Nucl. Phys. B345 (1990) 22; J. Bijnens, Phys. Rept. 265 (1996) 369.
- [188] S. Bertolini, J.O. Eeg and M. Fabbrichesi, Nucl. Phys. B449 (1995) 197; Nucl. Phys. B476 (1996) 225.
- [189] S. Bertolini, J.O. Eeg, M. Fabbrichesi and E.I. Lashin, Nucl. Phys. **B514** (1998) 93.
- [190] A. J. Buras, M. Jamin, and M. E. Lautenbacher, Phys. Lett. **B389** (1996) 749.
- [191] A. J. Buras and M. E. Lautenbacher, Phys. Lett. B318 (1993) 212.

- [192] S. Narison, Phys. Lett. **B358** (1995) 113.
- [193] M. Jamin and M. Münz, Z. Phys. C66 (1995) 633.
- [194] K. G. Chetyrkin, C. A. Dominguez, D. Pirjol, and K. Schilcher, Phys. Rev. **D51** (1995)
 5090; K. G. Chetyrkin, D. Pirjol, and K. Schilcher, Phys. Lett. **B404** (1997) 337.
- [195] P. Colangelo, F. De Fazio, G. Nardulli, and N. Paver, Phys. Lett. **B408** (1997) 340.
- [196] M. Jamin, Nucl. Phys. B. Proc. Suppl. **64** (1998) 250.
- [197] F.J. Yndurain, hep-ph/9708300.
- [198] H.G. Dosch and S. Narison, Phys. Lett. **B417** (1998) 173.
- [199] L. Lellouch, E. de Rafael, and J. Taron, Phys. Lett. **B414** (1997) 195.
- [200] M. Ciuchini, Nucl. Phys. B. Proc. Suppl. **59** (1997) 149.
- [201] A.A. Belkov, G. Bohm, A.V. Lanyov, A.A. Moshkin, hep-ph/9704354.
- [202] S. Bertolini, F. Borzumati and A. Masiero, Phys. Rev. Lett. 59 (1987) 180.
- [203] N. G. Deshpande, P. Lo, J. Trampetic, G. Eilam and P. Singer Phys. Rev. Lett. 59 (1987) 183.
- [204] B. Grinstein, R. Springer and M.B. Wise, Nucl. Phys. **B339** (1990) 269.
- [205] R. Grigjanis, P.J. O'Donnell, M. Sutherland and H. Navelet, Phys. Lett. B213 (1988)355; Phys. Lett. B286 (1992) 413 E.
- [206] M. Ciuchini, E. Franco, G. Martinelli, L. Reina and L. Silvestrini, Phys. Lett. B316 (1993) 127.
- [207] M. Ciuchini, E. Franco, L. Reina and L. Silvestrini, Nucl. Phys. B421 (1994) 41.
- [208] G. Cella, G. Curci, G. Ricciardi and A. Viceré, Phys. Lett. **B325** (1994) 227.
- [209] G. Cella, G. Curci, G. Ricciardi and A. Viceré, Nucl. Phys. **B431** (1994) 417.
- [210] A. Ali, and C. Greub, Z.Phys. **C60** (1993) 433.
- [211] A. J. Buras, M. Misiak, M. Münz and S. Pokorski, Nucl. Phys. B424 (1994) 374.
- [212] F.M. Borzumati and Ch. Greub, hep-ph/9802391.
- [213] N. Cabibbo and L. Maiani, Phys. Lett. **B79** (1978) 109.

- [214] C.S. Kim and A.D. Martin, Phys. Lett. **B225** (1989) 186.
- [215] Y. Nir, Phys. Lett. **B221** (1989) 184.
- [216] A.L. Kagan and M. Neubert, hep-ph/9805303.
- [217] A.F. Falk, M. Luke and M. Savage, Phys. Rev. **D53** (1996) 2491.
- [218] D. Atwood, B. Blok, and A. Soni, Int. J. Mod. Phys. A11 (1996) 3743; H.-Y. Cheng, Phys. Rev. D51 (1995) 6228; E. Golowich and S. Pakvasa, Phys. Rev. D51 (1995) 1215; G. Ricciardi, Phys. Lett. B355 (1995) 313; A. Khodjamirian, G. Stoll and D. Wyler, Phys. Lett. B358 (1995) 129; G. Eilam, A. Ioannissian and R.R. Mendel, Z. Phys. C71 (1996) 95; G. Eilam, A. Ioannissian, R.R. Mendel and P. Singer, Phys. Rev. D53 (1996) 3629; J.M. Soares, Phys. Rev. D53 (1996) 241; J. Milana, Phys. Rev. D53 (1996) 1403; N.G. Deshpande, X.-G. He and J. Trampetic, Phys. Lett. B367 (1996) 362.
- [219] M.B. Voloshin, Phys. Lett. **B397** (1997) 275.
- [220] A. Khodjamirian, R. Rückl, G. Stoll and D. Wyler, Phys. Lett. B402 (1997) 167; Z. Ligeti, L. Randall and M.B. Wise, Phys. Lett. B402 (1997) 178; A.K. Grant, A.G. Morgan, S. Nussinov and R.D. Peccei, Phys. Rev. D56 (1997) 3151.
- [221] G. Buchalla, G. Isidori and S.-J. Rey, Nucl. Phys. **B511** (1998) 594.
- [222] A. Czarnecki and W.J. Marciano, hep-ph/9804252.
- [223] A. Strumia, hep-ph/9804274.
- [224] M.S. Alam et. al (CLEO), Phys. Rev. Lett. **74** (1995) 2885.
- [225] S. Glenn (CLEO), talk presented at the Meeting of the American Physics Society, Columbus, Ohio, 18-21 March 1998.
- [226] R. Barate et al., (ALEPH), CERN-EP/98-044.
- [227] W.S. Hou and R.S. Willey, Phys. Lett. B202 (1988) 591; B. Grinstein, R. Springer, and M. Wise, Nucl. Phys. B339 (1990) 269.
- [228] H. Anlauf, Nucl. Phys. **B430** (1994) 245.
- [229] P. Krawczyk and S. Pokorski, Nucl. Phys. B364 (1991) 10; Y. Grossmann, Y. Nir, R. Rattazzi, in [19].

- [230] See for instance G.F. Giudice, R. Rattazzi, hep-ph/9801271.
- [231] R. Barbieri and G.F. Giudice, Phys. Lett. **B309** (1993) 86.
- [232] S. Bertolini, F. Borzumati, A. Masiero, and G. Ridolfi, Nucl. Phys. B353 (1991) 591;
 N. Oshimo, Nucl. Phys. B404 (1993) 20; R. Garisto, J.N. Ng, Phys. Lett. B315 (1993) 372;
 M.A. Diaz, Phys. Lett. B304 (1993) 278;
 Y. Okada, Phys. Lett. B315 (1993) 119;
 F. Borzumati, Z. Phys. C63 (1994) 291;
 P. Nath and R. Arnowitt, Phys. Lett. B336 (1994) 395;
 S. Bertolini and F. Vissani, Z. Phys. C67 (1995) 513;
 J. Lopez et al., Phys. Rev. D51 (1995) 147.
- [233] CLEO II, Contribution (PA05-093) to the 28th International Conference on High Energy Physics, July 1996, Warsaw, Poland.
- [234] A. Ali, hep-ph/9606324, hep-ph/9610333.
- [235] I. Bigi et al., Phys. Rev. Lett. 71 (1993) 496; Int. J. Mod. Phys. A9 (1994) 2467; G. Korchemsky and G. Sterman, Phys. Lett. B340 (1994) 96; M. Neubert, Phys. Rev. D49 (1994) 4623; A. Ali and C. Greub, Phys. Lett. B361 (1995) 146; A. Kapustin, Z. Ligeti and H.D. Politzer, Phys. Lett. B357 (1995) 653; R.D. Dikeman, M. Shifman and R.G. Uraltsev, Int. J. Mod. Phys. A11 (1996) 571; N. Pott, Phys. Rev. D 54 (1996) 938.
- [236] C. Greub, H. Simma and D. Wyler, Nucl. Phys. B434 (1995) 39; Erratum-ibid, B444 (1995) 447.
- [237] L. Littenberg and G. Valencia, Ann. Rev. Nucl. Part. Sci. 43 (1993) 729; J.L. Ritchie and S.G. Wojcicki, Rev. Mod. Phys. 65 (1993) 1149; A. Pich, hep-ph/9610243; G. D'Ambrosio and G. Isidori, hep-ph/9611284.
- [238] V.A. Novikov, A.I. Vainshtein, V.I. Zakharov and M.A. Shifman, Phys. Rev. D16, (1977) 223.
- [239] J. Ellis and J.S. Hagelin, Nucl. Phys. **B217** (1983) 189.
- [240] C.O. Dib, I. Dunietz and F.J. Gilman, Mod. Phys. Lett. A6 (1991) 3573.
- [241] W. Marciano and Z. Parsa, Phys. Rev. **D53**, R1 (1996).
- [242] D. Rein and L.M. Sehgal, Phys. Rev. **D39** (1989) 3325; J.S. Hagelin and L.S. Littenberg, Prog. Part. Nucl. Phys. **23** (1989) 1; M. Lu and M.B. Wise, Phys. Lett. **B324** (1994) 461; S. Fajfer, [hep-ph/9602322]; C.Q. Geng, I.J. Hsu and Y.C. Lin, Phys. Rev. **D54** (1996) 877.

- [243] S. Adler et al., Phys. Rev. Lett. **76** (1996) 1421.
- [244] L. Littenberg and J. Sandweiss, eds., AGS2000, Experiments for the 21st Century, BNL 52512.
- [245] P. Cooper, M. Crisler, B. Tschirhart and J. Ritchie (CKM collaboration), EOI for measuring $Br(K^+ \to \pi^+ \nu \bar{\nu})$ at the Main Injector, Fermilab EOI 14, 1996.
- [246] L. Littenberg, Phys. Rev. **D39** (1989) 3322.
- [247] Y. Grossman, Y. Nir and R. Rattazzi, [hep-ph/9701231] in [19].
- [248] G. Buchalla, hep-ph/9612307.
- [249] G. Buchalla and A.J. Buras, Phys. Rev. **D54** (1996) 6782.
- [250] A.J. Buras, Phys. Lett. **B333** (1994) 476.
- [251] T. Nakaya, in proceedings of FCNC97, page 105.
- [252] S. Adler et al., Phys. Rev. Lett. **79**, (1997) 2204.
- [253] L. Littenberg and J. Sandweiss, eds., AGS2000, Experiments for the 21st Century, BNL 52512.
- [254] K. Arisaka et al., KAMI conceptual design report, FNAL, June 1991.
- [255] T. Inagaki, T. Sato and T. Shinkawa, Experiment to search for the decay $K_L \to \pi^0 \nu \bar{\nu}$ at KEK 12 GeV proton synchrotron, 30 Nov. 1991.
- Y. Grossman and Y. Nir, Phys. Lett. B398 (1997) 163; C.E. Carlson, G.D. Dorada and M. Sher, Phys. Rev. D54 (1996) 4393; G. Burdman, hep-ph/9705400; A. Berera, T.W. Kephart and M. Sher, Phys. Rev. D56 (1997) 7457; Gi-Chol Cho, hep-ph/9804327.
- [257] T. Hattori, T. Hasuike and S. Wakaizumi, hep-ph/9804412.
- [258] A.J. Buras, A. Romanino and L. Silvestrini, Nucl. Phys. **B520** (1998) 3.
- [259] Y. Nir and M.P. Worah, hep-ph/9711215.
- [260] G. Buchalla and A.J. Buras, Phys. Lett. **B333** (1994) 221.
- [261] C. Jarlskog, Phys. Rev. Lett. **55**, (1985) 1039; Z. Phys. **C29** (1985) 491.
- [262] ALEPH Collaboration, Contribution (PA10-019) to the 28th International Conference on High Energy Physics, July 1996, Warsaw, Poland.

- [263] A.J. Buras, Nucl. Instr. Meth. **A368** (1995) 1.
- [264] F. Abe et al. (CDF), Phys. Rev. **D57** (1998) R3811.
- [265] Y. Grossman, Z. Ligeti and E. Nardi, Phys. Rev. **D55** (1997) 2768.
- [266] G. Buchalla and A.J. Buras, Phys. Rev. **D57** (1998) 216.
- [267] Y. Nir and H.R. Quinn Ann. Rev. Nucl. Part. Sci. 42 (1992) 211 and in "B Decays", ed S. Stone (World Scientific, 1994), p. 520; I. Dunietz, ibid p.550 and refs. therein.
- [268] M. Gronau and D. London, Phys. Rev. Lett. 65 (1990) 3381.
- [269] A. Snyder and H.R. Quinn, Phys. Rev. D48 (1993) 2139; A.J. Buras and R. Fleischer,
 Phys. Lett. B360 (1995) 138; J.P. Silva and L. Wolfenstein, Phys. Rev. D49 (1995)
 R1151; A.S. Dighe, M. Gronau and J. Rosner, Phys. Rev. D54 (1996) 3309; R.
 Fleischer and T. Mannel, Phys. Lett. B397 (1997) 269; C.S. Kim, D. London and T.
 Yoshikawa, Phys. Rev. D57 (1998) 4010.
- [270] I.I.Y. Bigi and A.I. Sanda, Nucl. Phys. **B193** (1981) 85.
- [271] D. London and A. Soni, Phys. Lett. B407 (1997) 61; Y. Grossman and M.P. Worah,
 Phys. Lett. B395 (1997) 241; M. Ciuchini et al., Phys. Rev. Lett. B79 (1997) 978; R.
 Barbieri and and A. Strumia, Nucl. Phys. B508 (1997) 3.
- [272] R. Aleksan, I. Dunietz and B. Kayser, Z.Phys. C54 (1992) 653;
 R. Fleischer and I. Dunietz, Phys. Lett. B387 (1996) 361.
- [273] M. Gronau and D. Wyler, Phys. Lett. **B265** (1991) 172.
- [274] M. Gronau and D. London, Phys. Lett. B253 (1991) 483. I. Dunietz, Phys. Lett. B270 (1991) 75.
- [275] D. Atwood, I. Dunietz and A. Soni, Phys. Rev. Lett. **B78** (1997) 3257.
- [276] R. Fleischer, Phys. Lett. **B365** (1996) 399.
- [277] R. Fleischer and T. Mannel, Phys. Rev. **D57** (1998) 2752.
- [278] M. Gronau and J.L. Rosner, hep-ph/9711246, hep-ph/9712287.
- [279] F. Würthwein and P. Gaidarev, hep-ph/9712531.
- [280] R. Fleischer, hep-ph/9802433.

- [281] R. Godang et al., hep-ex/9711010.
- [282] The BaBar Physics Book, preprint SLAC-R-504, in preparation.
- [283] J.D. Bjorken, Nucl. Phys. B (Proc. Suppl.) 11 (1989) 325; SLAC-PUB-5389 (1990), published in the proceedings of the SLAC Summer Institute 1990, p. 167.
- [284] A.J. Buras, R. Fleischer and T. Mannel, hep-ph/9711262.
- [285] J.-M. Gérard and J. Weyers, hep-ph/9711469; D. Delépine, J.-M. Gérard, J. Pestieau and J. Weyers, hep-ph/9802361; J.-M. Gérard, J. Pestieau and J. Weyers, hep-ph/9803328.
- [286] M. Neubert, hep-ph/9712224.
- [287] A.F. Falk, A.L. Kagan, Y. Nir and A.A. Petrov, Phys. Rev. **D57** (1998) 4290.
- [288] D. Atwood and A. Soni (1997), hep-ph/9712287, hep-ph/9712252.
- [289] L. Wolfenstein, Phys. Rev. **D52** (1995) 537; J. Donoghue, E. Golowich, A. Petrov and J. Soares, Phys. Rev. Lett. **77** (1996) 2178; B. Blok and I. Halperin, Phys. Lett. **B385** (1996) 324; B. Blok, M. Gronau and J.L. Rosner, Phys. Rev. Lett. **78** (1997) 3999.
- [290] B. Stech, Phys. Lett. **B130** (1983) 189; J. Bjorken, hep-ph/9706524.
- [291] R. Fleischer and T. Mannel, hep-ph/9706261.
- [292] R. Fleischer, Z. Phys. C58 (1993) 483 and C62 (1994) 81; G. Kramer, W.F. Palmer and H. Simma, Z. Phys. C66 (1995) 429.
- [293] R. Fleischer, hep-ph/9804319.
- [294] R. Fleischer, hep-ph/9710331.
- [295] A. Lenz, U. Nierste and G. Ostermaier, hep-ph/9802202; U. Nierste, hep-ph/9805388.